DIRECT DIGITAL CONTROL

OF

INERTIAL GUIDANCE SENSORS

by

SOM NATH PURI

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SUMMARY

This study is concerned with the theoretical design of rebalance control systems for inertial sensors using digital controllers. Specific attention is given to the case where a two-degree-of-freedom gyro is used as an attitude sensor in a strapdown inertial measurement system. Because of the similarity of the dynamical behaviour of gyros and accelerometers, the method can be extended to accelerometers without great difficulty.

A new design method has been developed to yield control algorithms for the digital control elements. Control functions are synthesized to achieve a minimum settling time and eliminate interactions between the gyro control axes. It is shown that the method can be used when filters are included to remove pick-off noise, and can be adapted to take account of finite processing delay in the digital controller.

Sensitivity of the system to control loop gain variations and mismatch in the controller elements is examined. Sensitivity analysis is exploited to allow overall compromise between response speed and the system sensitivity.

The need for high frequency sampling in the control loops imposes restrictions on the execution time in the digital processor. Various methods of reducing the controller complexity are investigated with a view to reducing the computation time. A method of compensation for the effects of rounding coefficients in the control algorithm is developed for the specific problem involving the dry-tuned-gyro. This consideration is extended to identify how the choice of processor is influenced for the gyro rebalance system.

Keywords Accelerometers, direct digital control, gyroscopes, sensitivity analysis, strapdown navigational systems.

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LIST OF PRINCIPAL SYMBOLS

θ_x	angle of rotation of the rotor about x-axis
θ_y	angle of rotation of the rotor about y-axis
I_{xr}	principal moment of inertia of the rotor about x-axis
I_{zr}	principal moment of inertia of the rotor about z-axis
I_{xg}	principal moment of inertia of the gimbal about x-axis
H	angular momentum of the gyro
ω_d	gyroscope motor spin speed
ω_n	gyroscope nutational frequency
P(z)	pulse transfer function matrix of the gyroscope
I	identity matrix
E(z)	matrix of the error z-transforms
R(z)	matrix of reference inputs, z-transforms
$E_1(z)$	matrix of modified error sequence, z-transforms
C(z)	matrix of plant outputs, z-transforms
D(z)	matrix of digital controller discrete time transfer function, z-transforms
f(z)	common closed loop pulse transfer function for all the loops
$\phi(z)$	common error pulse transfer function
det( )	determinant of
Adj( )	adjoint matrix of
s	Laplace operator
z	complex variable defined by a complex z-plane
ρ	input rate
M_o	steady state torque
ω	angular frequency

Ω	notch frequency of the noise filter
ν	3-db rejection bandwidth
k	an integer representing delay in the pulsed transfer function

Other symbols are defined in the text.

CHAPTER 1

INTRODUCTION

1.1 Strapdown system definition and sensor requirements

The basic concept of inertial navigation is that if the acceleration of a vehicle with respect to a known fixed co-ordinate system is measured, the vehicle's velocity and position with respect to the co-ordinate system, may be computed by time integration of the acceleration signal.

Inertial navigation problem in three dimensions is complicated because of co-ordinatised acceleration information. In the case of a stabilized platform, three orthogonally mounted accelerometers are aligned with the reference co-ordinate frame, using either 3 single-degree-of-freedom or 2 two-degree-of-freedom gyroscopes. Such a scheme effectively isolates the 'stable elements' containing the inertial instruments from any rotational motions of the vehicle.

In strapdown navigational systems (Ref. 1.4) the accelerometers and gyros are strapped on to the vehicle frame. Since gyros provide direct measures of the rotational rates of the vehicle, the instantaneous rotation of the vehicle with respect to the reference co-ordinate frame can be computed. This information is then used to resolve the measured acceleration components on to the reference frame for integration into velocity and position.

Strapdown systems are rapidly becoming the preferred way for low cost because of the elimination of the platform gimbal structure and its associated electronics. Halamandaris (Ref. 1.7) and more recently Kirk (Ref. 1.9) in their papers, suggested that with the current state of two-degree-of-freedom dry-tuned gyros with inertial grade accuracy and wide dynamic range, the cost of a strapdown system is further reduced compared to systems using single-degree-of-freedom gyroscopes.

A significant portion of the hardware content and cost of strapdown system is attributed to the servo-electronics required for sensor control. In a strapdown mode the sensor torquing currents are fed through precision resistances to develop voltages which are proportional to the vehicle angular rates, in case of gyroscopes, and to acceleration, in the case of accelerometers. These voltages are then converted into equivalent digital numbers for use in the navigation and attitude equations which are solved by the navigation computer.

1.2 Methods of torque rebalance

In a strapdown mode the inertial sensors convert the signal to be measured into a torque within the feedback loop, by means of compensation. To achieve high accuracy in measurements of angular rates and accelerations, voltage outputs of these sensors must be digitized for further processing in the digital computer. Rahlfs (Ref. 1.11) and Sutherland (Ref. 1.12) pointed out that the integrating digital readout of these sensors by means of pulse rebalance loops is particularly advantageous for high accuracy measurements. Binary pulse width modulated (BPWM) rebalance configuration suggested by Bendett and Blalock (Ref. 1.1 and 1.2), is commonly used compared to simple binary or ternary schemes, mainly because of constant two-level power operation of the torque motor, and at the same time linear behaviour of the servo-loop. It is important to note that all these configurations use analogue control methods of designing rebalance systems.

1.3 General performance requirement

Rebalance systems outlined above have been developed using analogue control schemes (Ref. 1.4 and 1.8), where the requirement for rapid response has not been important. However, in exploiting the maximum capability of a sensor there are advantages in using a digital processor as controller. A digital controller offers the advantages of flexibility in the realization of complex control algorithms and their accurate implementation is not affected by component tolerances. A possibility of time sharing the processor between separate control functions can also reduce hardware requirements. Where rapid response is required the signal sampling rate must be high and for time sharing to be possible it is important to minimize the computation time required for any control algorithm.

A new method for the design of rebalance control system for a dry-tuned gyro using a digital controller has been developed. This digital control method is synthesised to achieve a minimum settling time in the transient response. The need to eliminate interactions between the gyro control axes is also included in the design. The new method enables the controller to have four elements and allows the designer to exploit the sensitivity analysis to show how coefficients in the control functions may be rounded to allow reduced computation time in the processor. Because of the similarity of the dynamical behaviour of gyros and accelerometers the method can be extended to accelerometers.

1.4 Dry-tuned gyro and its performance as a control sensor

This gyro, with its two-degrees of freedom, has a rotor and gimbal assembly suspended on the springs. At the designed rotation speed the spring constants are matched to cancel the inertial torques due to the gimbals, so that the rotor behaves as a free gyro. The basic construction of a single-gimbal tuned gyro is shown in Fig. 1.1. The connection between the rotor and gimbal is provided by an elastic spring, S_2, which permits the rotor to deflect relative to the gimbal about the axis of spring S_2. A second spring, S_1, orthogonal to S_2, connects the gimbal to the drive shaft. The rotor-gimbal assembly is thus free to rotate with respect to the drive shaft about the axis of S_1. The drive motor spins the rotor and gimbal assembly at a high angular velocity relative to the casing.

Operation as a rate gyro is achieved by forming a position control system to align the rotor with the external casing. Signals from the position pick-offs are used to provide feedback control of the torques applied to the rotor. The precession rates are then measured by signals derived from the torque-motor currents. Two of these rebalance control loops are required for each gyro. In the strapdown mode the rebalance loops must be designed for adequate dynamic response, i.e. a short transient settling time or a wide frequency response bandwidth. When the system bandwidth approached the nutational frequency of the gyro it is no longer possible to regard the two rebalanced loops as independent systems. Interaction between the control axes demands that the system must be analysed as a multivariable control system. The design must also aim to counteract the inherent interaction.

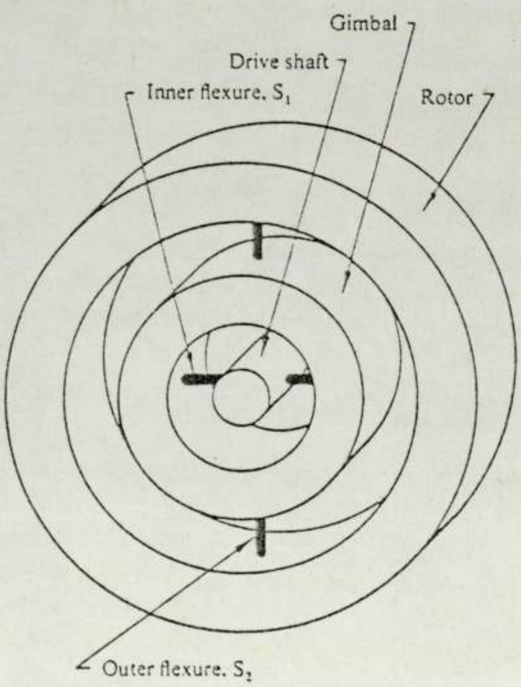


Fig. 1.1 Basic construction of a single-gimbal tuned gyro

1.5 Gyro transfer function and proposed ddc structure

Differential equations for the dry-tuned gyro were developed by Bortz (Ref. 1.3), Craig (Ref. 1.6) and Coffman (Ref. 1.5) and may be written as

$$\begin{aligned} I\ddot{\theta}_x + H\dot{\theta}_y &= M_x \\ I\ddot{\theta}_y - H\dot{\theta}_x &= M_y \end{aligned} \qquad (1.1)$$

where θ_x and θ_y are angles of rotation of the rotor about orthogonal control axes on which torques M_x and M_y are applied. Inertial constant $I = I_{xr} + \frac{I_{xg}}{2}$, involving the principal moments of inertia of the rotor and gimbal along x-axis, and H, the angular momentum can be given as

$$H = (I_{zr} + I_{xg})\omega_d \simeq I_{zr}\omega_d \qquad (1.2)$$

as $I_{zr} >> I_{xg}$. I_{zr} is the principal moment of inertia of the rotor along z-axis. ω_d represent gyro motor spin speed. Laplace transformation of Equation (1.1) gives a transfer function matrix equation

$$\begin{bmatrix} \overline{\theta}_x \\ \overline{\theta}_y \end{bmatrix} = \begin{bmatrix} h_1 & -h_2 \\ h_2 & h_1 \end{bmatrix} \begin{bmatrix} \overline{Q}_x \\ \overline{Q}_y \end{bmatrix} \qquad (1.3)$$

$$h_1 = \frac{1}{(s^2 + \omega_n{}^2)}, \quad h_2 = \frac{\omega_n}{s(s^2 + \omega_n{}^2)}$$

with $\omega_n = \frac{I_{zr}\,\omega_d}{I} \simeq 2\omega_d$

Since $I_{xr} >> I_{xg}$ and $I_{zr} \simeq 2I_{xr}$, and $\overline{Q}_x$, $\overline{Q}_y$ the normalized torques $\overline{M}_x/I$, $\overline{M}_y/I$.

The transfer function in Equation (1.3) may be identified with a block diagram structure for the complete rebalance system as shown in Fig. 1.2.

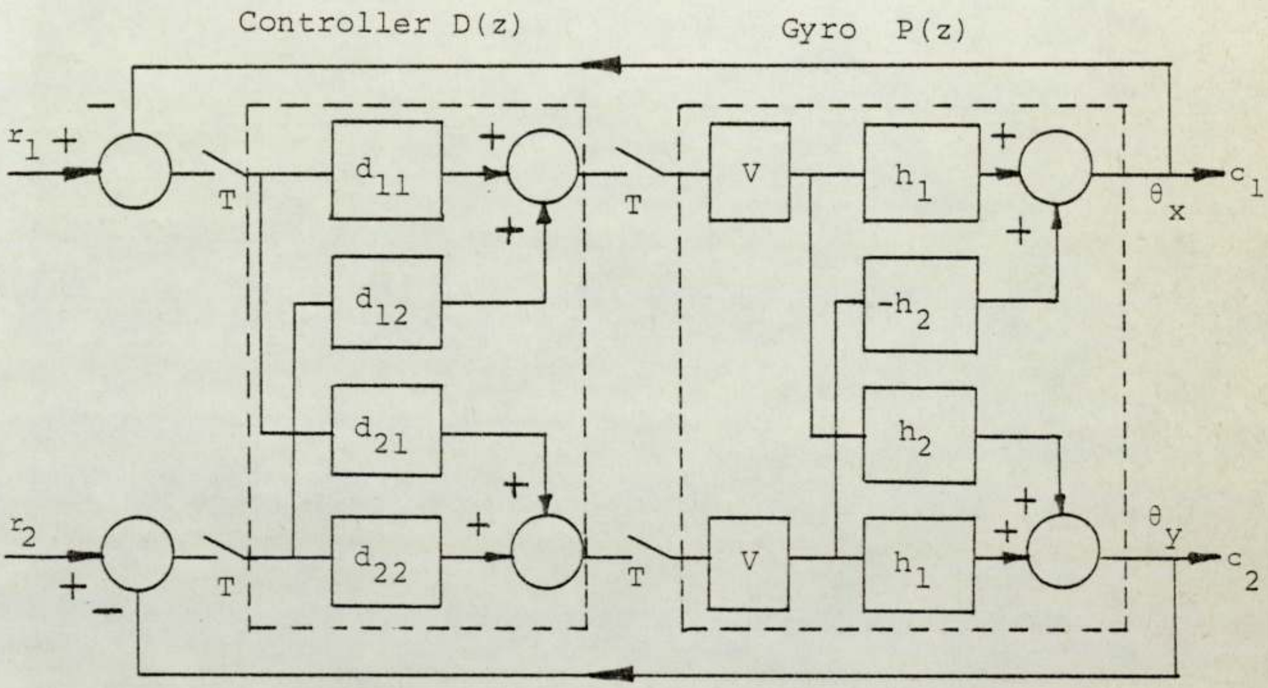


Fig. 1.2 Proposed direct digital control rebalance system

Elements V represent zero-order hold functions. Synchronous sampling switches indicate the effect of analogue-digital signal conversions, and it is assumed at this stage that there is negligible computing delay in the digital processor. Inputs r_1 and r_2 represent the case position angles to which the system responds.

The pulse transfer function of the gyro P(z) for a proposed ddc structure can be found using a standard z-transform table (Ref. 1.10) which gives

$$P(z) = K_1 \begin{bmatrix} N_1 & -N_2 \\ N_2 & N_1 \end{bmatrix} \qquad (1.4)$$

with

$$K_1 = \frac{(T_1-\beta)z^{-1}}{\omega_n{}^3(1-z^{-1})(1-2\alpha z^{-1}+z^{-2})}$$

$$N_1 = c(1-z^{-2})$$

$$N_2 = (1+dz^{-1}+z^{-2})$$

$$\alpha = \mathrm{Cos}(T_1)$$

$$\beta = \mathrm{Sin}(T_1)$$

$$c = \frac{1-\alpha}{T_1-\beta}$$

$$d = \frac{2(\beta-\alpha T_1)}{T_1-\beta}$$

where T_1 is the dimensionless sampling frequency equal to $\frac{2\pi}{b}$

b is the ratio of system sampling frequency to gyro nutational frequency.

CHAPTER 2

EXISTING DESIGN METHODS

In this chapter, two methods reported in the literature for designing multivariable digital control systems in general are discussed.

2.1 Synthesis method due to Nishida

In 1960, Nishida reported a synthesis technique for multivariable Control Systems by means of sampled-data compensations (Ref. 2.2), which is the extension of single loop design of Jury and Schroeder (Ref. 2.1). The system considered consists of a linear model of the plant as shown in Fig. 2.1 and 2.2.

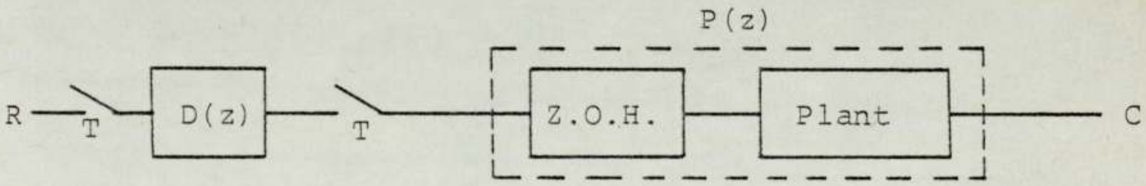


Fig. 2.1 System with series compensator D(z)

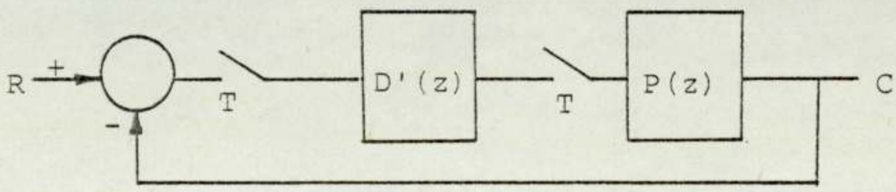


Fig. 2.2 Equivalent feedback system

Nishida approach presents a general design procedure of non-interacting and finite settling multivariable digital control systems where the plant is stable. The method assumes that the number of outputs is equal to that of the inputs and all inputs have transforms of mth order. In this method, the series controller transfer matrix elements are specified by

$$d_{ip}(z) = p_i(z) d_{ip}'(z) \qquad (2.1)$$

where $p_i(z)$ is the lowest order polynomial of z^{-1} which has the zeros at each pole of the elements belonging to the ith column of the transform matrix P(z). And $d_{ip}'(z)$ is any polynomial of an order equal to or higher than (m-1). In this method it is important to note that the plant transfer function including the hold circuit must have at least one matrix element to each row, the mth or higher order pole of which is at s = 0. Otherwise it is necessary to introduce as many integrators as required in front of the corresponding inputs of the plant.

Once Series Controller D(z) is determined the equivalent controller transfer function matrix D'(z) in the feedback system is obtained by

$$D'(z) = [I - D(z)P(z)]^{-1} D(z) \qquad (2.2)$$

This method has several advantages for example a flat response of zero overshoot after a prescribed finite time is obtainable. Also the designer has a large degree of freedom in obtaining the desired response.

There are, however, some disadvantages. Should the overshoot be unacceptable then no method is outlined for its improvements. This problem can be more severe when any element in the plant contains complex conjugate poles very near to the unit circle. Furthermore, when any one or more elements of the plant have an unstable pole, straightforward application of this method is not possible. In this situation Nishida suggested addition of a second minor feedback loop which means one additional controller in the system. He also suggested introduction of an auxiliary controller parallel to hold circuits, where a reduction in settling time by one sampling interval may be necessary.

2.2 Deadbeat response method due to Viswanadham and Deekshatulu

Viswanadham and Deekshatulu (Ref. 2.3) have proposed a synthesis technique for multivariable sampled data control system by the use of state difference equations. The system structure considered is shown in Fig. 2.3.

In this design technique the multivariable process is defined by vector matrix equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad (2.3)$$

where $x(t)$ is nxl state vector, A is the (nxn) coefficient matrix of the process, B is the (nxm) constant input

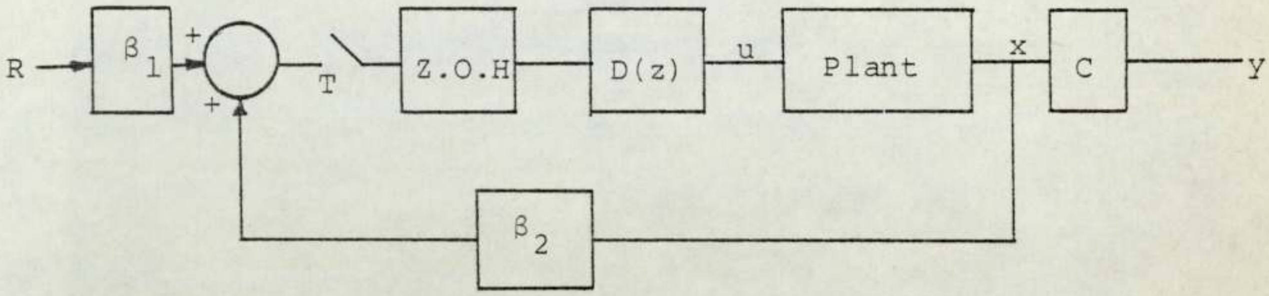


Fig. 2.3 System considered by Viswanadham and Deekshatulu

distribution matrix. u is an mxl control vector. Then

$$y(s) = P(s)\ u(s) \tag{2.4}$$

where

$$P(s) = C[sI-A]^{-1}B \tag{2.5}$$

The solution of the discretised version of Equation (2.3) at sampling instants is (Ref: 2.3)

$$X(kT) = \Phi(kT)X(O) + \sum_{j=1}^{k-1} \Phi\{(k-1-j)T\}D(T)\,u(jT) \tag{2.6}$$

The state vector X(kT) at and after the settling instant is chosen to satisfy the non-interaction condition and the deadbeat response specifications. Once the initial state X(O) and the final desired value X(kT) are chosen, the optional control sequence $\hat{u}^o$ can be determined, then

$$e(kT) = \beta_1 r(kT) + \beta_2 X(kT) \tag{2.7}$$

components of e(kT) are the inputs to the compensators and their inputs are the control sequence; so

$$D_n(z) = \frac{u_n^{\,o}(O) + u_n^{\,o}(T)z^{-1}\ .....}{e_n(O) + e_n(T)z^{-1} + \ .....} \tag{2.8}$$

The main advantage of this method is that it can be extended to the case where nonlinearities exist before the plant.

This approach also has some disadvantages. The achievement of noninteraction and deadbeat response depends on the order of the direct transfer function $P_{ii}(s)$ and also the degree of the input. For instance if the input is a step and $P_{ii}(s)$ is of zero order, then to obtain deadbeat response, the control signal in the ith path should be some constant value, but for noninteraction it should be zero. Hence it is not possible to achieve both. The same happens when the input is a ramp and P_{ii} is of order 1. The transfer function approach does not suffer these limitations, since the contributions from the interacting paths to the outputs are cancelled by a negative signal fed to the outputs through a direct path.

CHAPTER 3

A NEW SYNTHESIS METHOD

3.1 Introduction

The design of single-loop minimum system response time (Ref. 3.1) is extended to cover a new design method for multivariable digital control systems. The objective has been to include those aspects which differ from single-loop design. The method is also extended for the case when settling time is not required to be finite and a method of retarded response offers considerable advantages where speed of response can be traded for sensitivity reduction in a systematic way. The further advantage of this procedure is that sensitivity is reduced without any increase in the complexity of the control algorithms required.

3.2 Basic design method

The system configuration in Fig. 3.1 has a multivariable plant with an mxm pulse-transfer function matrix P(z) and m control loops are formed, each with unity feedback gain. The digital controller has the structure of an mxm array of pulse-transfer functions as represented in D(z).

The error pulse function E(z) is given by

$$E = [I + PD]^{-1}R \qquad (3.1)$$

Also the overall closed-loop response function

$$C = [I + PD]^{-1}PDR \qquad (3.2)$$

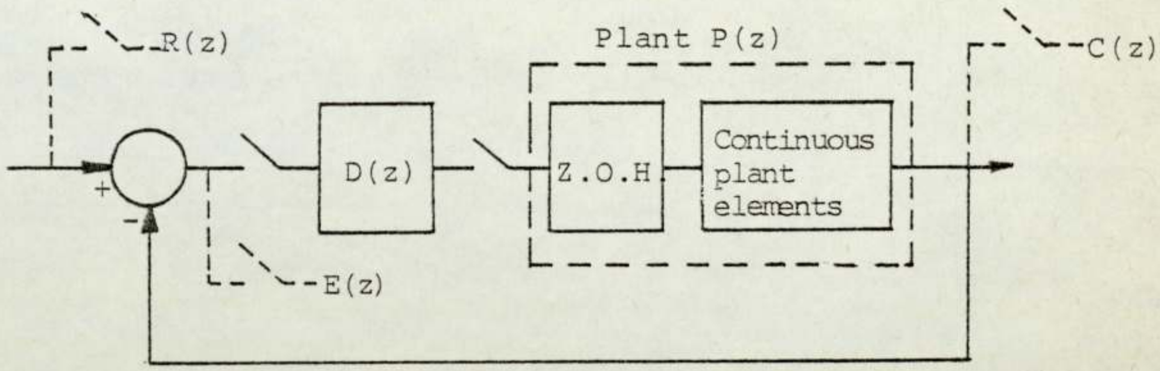


Fig. 3.1 System Configuration

from which we designate the closed-loop transfer function and system error transfer function matrices.

$$I - W = [I + PD]^{-1}PD \tag{3.3}$$

$$W = [I + PD]^{-1} \tag{3.4}$$

A requirement for non-interacting response is that a test signal applied at any single input should produce a response at the corresponding output and have no effect on the other outputs. This implies that I - W must be a diagonal matrix. Furthermore, if the system as a whole is to settle in minimum time, the response time in each loop may be made the same* so that

$$I - W = fI \tag{3.5}$$

where f(z) is the common closed-loop pulse-transfer function for all loops. Following from this we may define

$$W = \phi I \tag{3.6}$$

*It is shown by Steel and Puri (Ref.3.6) that this applies to all stable, minimum phase open-loop system.

where $\phi(z)$ is the common error pulse-transfer function.

The design method proceeds in a similar manner to that used with single-loop systems, by defining the required closed-loop function f and computing the necessary control function matrix D. The relationship giving D follows from Equation (3.3) as

$$D = \frac{f}{\phi} P^{-1} = \frac{f}{\phi} \frac{Adj(P)}{det(P)} \tag{3.7}$$

A minimum prototype response function f may be assigned depending on the form of input R to which the optimum response is required. For example a unit step input applied to any one of the loops leads to an error response $\frac{\phi(z)}{1-z^{-1}}$ in that loop and zero error in all others. The steady-state error will be zero provided that $\phi(z)$ contains a factor $(1-z^{-1})$. We note that $f = 1-\phi$ so that $f=z^{-1}$ for this condition and the response settles in one sample interval. Having assigned f and ϕ as compatible functions they may be substituted in Equation (3.7) to define D. Two conditions need to be observed however, one that det(P) may be zero outside the unit circle in the z-plane, and also that unstable modes in P will present poles in the same region. These conditions have been examined in detail by Steel and Puri, and they also examined the implications of the design policy when some loops have a shorter settling time than others (Refer to paper at back p13

The procedure is the same in principle as that used in single-loop system design (Ref. 3.1), with one exception that the zeros of det(P) and unstable modes of P are the determining features rather than poles and zeros of an individual transfer function element. A significant difference between single-loop design and the multivariable equivalent is that of locating the zeros of det(P). This does not follow in any obvious way from the knowledge of the poles and zeros of the elements of P. Normally the numerator polynomial of det(P) must be formulated explicitly so that its zeros can be found by standard numerical techniques (Ref. 3.2 and 3.3). This later problem may be avoided by a test to see whether there are zeros outside the unit circle, for example using the Jury test (Ref. 3.5). If all the zeros are inside the unit circle in the z-plane, their position need not be accurately known.

The need to allocate zeros to f and ϕ based on the unstable modes of P and zeros of det(P) (Ref. 3.6, p7), leads to a technique where arbitrary coefficients of f and ϕ are found by matching ϕ to (1-f) term by term.

3.3 Retarded response design

The design procedure outlined above yields a response which settles in a minimum and finite number of sampling intervals. This is the result of designating the closed-loop response function as a polynomial in z^{-1}. Steel and Puri (p15) have shown that some advantage in reduced sensitivity can be gained by removing the requirement of minimum and finite settling time. The inclusion of a single or more poles in f away from the origin, but inside the unit circle, has the effect of achieving a retarded response. For example a single pole in f will introduce an exponential mode, whereas a complex conjugate pair of poles gives a damped oscillatory mode in the response. The effect of this on the design procedure is that the same denominator must be allocated to ϕ in consequence of equation $f = 1-\phi$. Terms introduced in this way cancel in Equation (3.7), and so do not appear directly in the control functions. The general complexity of the elements in D remains unaltered but coefficients are modified due to changes in the initially undetermined coefficients in f and ϕ.

3.4 Generalised sensitivity relationships

The design method described in Section (3.2) results in a response which settles in minimum number of sampling intervals. This requirement may be relaxed with the advantage of reduced sensitivity. Considerations of the sensitivity of the response to small parameter variations can be judged by the change in stability of the system (Ref. 3.4) which for multivariable system is given by closed-loop characteristic polynomial $\det(I+PD) = 0$. If P is subjected to variations so that

$$P' = P + H \tag{3.8}$$

where H is the deviation matrix such that elements h_{ij} is the variation in p_{ij}. Therefore one must examine the zeros of $\det(I + P'D) = 0$, recognising that under nominal design conditions $I + PD = \frac{1}{\phi}I$ and $D = \frac{f}{\phi}P^{-1}$. Therefore

$$\det(I + P'D) = \det(I + fHP^{-1}) = 0 \tag{3.9}$$

defines the closed-loop modes resulting from a change H; provided the zeros of ϕ do not coincide with any of the roots of determinant.

Small variations anticipated in P may be identified as loop gain changes or movements of the poles and zeros of individual elements. These effects have been examined separately in detail (Ref. 3.6, p17) and results giving sensitivity conditions have been developed which are of considerable generality and significance. This leads to the conclusion that by placing a requirement of finite settling time on the system a design may result which is both sensitive and unnecessary complex in the control algorithm required. A retarded response method based on systematic analytical technique can be used by which speed of response can be traded for sensitivity (Ref. 3.6, p25)

3.5 An assessment of the method

From the discussion in Chapter 2 of the method described by Nishida, it is apparent that the above technique and Nishida's method have some common aspects.

Firstly, both are based on transfer function approach. Secondly, both use a finite settling time design technique to the presentation of the analysis. The proposed method is a minimum settling time design which enables a means of establishing ultimate performance capabilities where high speed of response is required.

A retarded response design method based on analytical technique has been developed by Steel and Puri, by which some of the complex conjugate pair of poles or zeros of det(P) which are inside but very close to the unit circle, need not be assigned to f and ϕ. Sensitivity to parameter variations which will move these poles or zeros on to or outside the unit circle can be avoided by putting constraint on to the movement of root loci at these points. This will result in a system which is less sensitive and at the same time less complex in the controller elements.

Sensitivity effects to loop gain variations have also been identified by Steel and Puri, and methods are proposed for their improvement.

CHAPTER 4

APPLICATION OF THE DESIGN METHOD TO A MODEL OF DRY-TUNED ROTOR GYROSCOPE

4.1 Introduction

In the previous chapter, the brief review of a design method of multivariable digital control algorithms for minimum settling time response was given. In this section the method is applied to a linear model of dry-tuned rotor two-degree-of-freedom gryroscope.

In the rebalance control system of Fig. 1.2 pulse-transfer function matrices P(z) and D(z) are used to decribe the gyro and the digital controller. The objective of this study is to design a digital controller to close the re-balance torque loops such that torque depends on the angular position error of the axis about which that torque results in precession.

4.2 Minimum settling time design

The system design proceeds with the basic result of Equation (3.7) as developed in Chapter 3. In this equation the functions f and ϕ are first of all constrained by the requirement for zero steady state error, which means ϕ must contain $(1-z^{-1})^n$, where n depends on the form of input R, e.g. n = 1 for a step input and n = 2 for a ramp input. Further constraints are due to the need to avoid some cancellations between elements of P and D in the product PD which gives the open loop pulse-transfer function matrix. Cancellations on or outside the unit circle in the z-plane can lead to a sensitive design in which instability will result from a small mismatch in the controller. General rules for avoiding such sensitivity problem can be drawn up assuming an arbitrary form of matrix P(Ref. 3.6). However in the case of the dry-tuned gyro, the inherent symmetry of the dynamic structure, as given by Equation (1.4), leads to some simplifications. All four elements of P have a common pole pair on the unit circle due to the undamped oscillatory characteristics of the gyro. This would normally lead to a pair of zeros in each element of P^{-1} at a corresponding position. But as a result of cross coupling between the gyro axes det(P) is zero at the same point in the z-plane, so that this mode is cancelled from P^{-1}.

Apart from the zeros of det(P) which coincide with the poles of the elements of P there are two other zeros which appear on the unit circle. To avoid sensitivity this pair of zeros must be allocated to f.

Therefore the general method proceeds by designating

$$f = z^{-1} A(z) B(z)$$

and

$$\phi = (1-z^{-1})^n L(z) \qquad (4.1)$$

where A(z) contains the uncancelled pair of zeros of det(P) on the unit circle. Polynomials B(z) and L(z) have undetermined coefficients as necessary to allow equation $f = 1 - \phi$ to be satisfied. Once f and ϕ are explicitly known the necessary control function matrix D can be defined from Equation (3.7).

4.2.1 Class I and class II systems and their comparisons

A class I system is defined as having a finite steady state error in response to a constant rate input change. The class II system gives zero error with a constant rate input and a finite error with a constant acceleration input. Steady state alignment is clearly best in the class II design but comparisons have been made (Ref. 1.5), which showsthat a small constant steady state error in the class I design may be acceptable. It is interesting to compare the design results with the minimum settling time digital controller. The class I design is achieved by designating $n = 1$ in Equation (4.1) and class II with $n = 2$.

The following general features emerge

(i) Settling time following a step input change

Class I: In this case $n = 1$ in Equation (4.1) and to satisfy $f = 1 - \phi$, $B(z) = 1$ and $L(z)$ will be a second order polynomial with two undermine coefficients, hence the settling time = 3T.

Class II: $n = 2$, therefore $B(z)$ and $L(z)$ will contain unknown polynomials of degrees 1 and 2 respectively. In this case settling time = 4T.

(ii) Steady state error calculations

Steady state error along each axis at any sampling instant (qT) consists of terms proportional to the input, the input velocity, the input acceleration and in general still higher derivatives of the input signal (Ref.4.4) and is given by

$$e(qT) = C_o r(qT) + C_1 r'(qT) + \frac{C_2}{2!} r''(qT) + \ldots + \frac{C_m}{m!} r^m(qT) + \ldots \quad (4.2)$$

$$C_m = \left. \frac{d^m \phi^*(s)}{ds^m} \right|_{s=o} = \phi^{*(m)}(O),\ m=0,1,2 \quad (4.3)$$

where $\phi^*(s)$ is the system error pulse-transfer function in terms of the starred transform which is obtainable from $\phi(z)$ by substituting $z = \exp(Ts)$

If $A(z) = (1+\ell z^{-1}+z^{-2})$ represents the uncancelled set of zeros of det(P), it is possible to solve undetermined coefficients of ϕ in term of ℓ for a class I design and the resulting expression gives

$$\phi(z) = (1-z^{-1})(1+ \frac{1+\ell}{2+\ell} z^{-1} + \frac{1}{2+\ell} z^{-2}) \quad (4.4)$$

and corresponding error series along each axis can be written as

$$e(qT) = T(1+\frac{1+\ell}{2+\ell}+\frac{1}{2+\ell}) r'(qT) - \frac{T}{2!}\left[1+3(\frac{1+\ell}{2+\ell})+5(\frac{1}{2+\ell})\right] r''(qT) + \ldots\ldots \quad (4.5)$$

If the gyro is displaced at a constant rate ρ along the axis then the inspection of Equation (4.5) reveals that the steady state error at the sampling instant for class I would be $2T\rho$ whereas it will be zero for class II design.

(iii) Controller complexity

Because of an additional unknown factor in B(z) an extra term is required in the numerator and denominator polynomials for the class II case.

This section has shown how the minimum settling design method can be employed for designing rebalance control system for a dry-tuned gyro and the general features of class I and class II systems. It is now possible to do further comparisons by taking specific value of b, the ratio of sampling frequency to gyro nutational frequency.

4.2.2 Design using a specific gyro model

In this section the design for a specific gyro model is described in which further comparisons for class I and class II systems will be made. If we select

$b = \dfrac{\text{system sampling frequency}}{\text{gyro nutational frequency}} = 5$, Equation (1.4) gives

$$P = K_1 \begin{bmatrix} N_1 & -N_2 \\ N_2 & N_1 \end{bmatrix} \tag{4.6}$$

and

$$P^{-1} = K_2 \begin{bmatrix} N_1 & N_2 \\ -N_2 & N_1 \end{bmatrix}$$

where

$$K_1 = 0.306\, z^{-1}/\omega_n^{\,3}(1-z^{-1})(1-0.6\, z^{-1} + z^{-2})$$

$$K_2 = 0.536\omega_n^{\,3}(1-z^{-1})/z^{-1}(1+1.825\, z^{-1} + z^{-2}) \tag{4.7}$$

$$N_1 = 2.254(1-z^{-2})$$

$$N_2 = (1+3.67\, z^{-1} + z^{-2})$$

The term $(1 + 1.825\, z^{-1} + z^{-2})$ in K_2 has zeros on the unit circle, therefore this must be assigned to f in Equation (3.7). The resulting design factors are as follows:

(i) Closed-loop pulse-transfer function f(z)

class I : $0.2614z^{-1}\ (1+1.825z^{-1}+z^{-2})$

class II : $0.7843z^{-1}(1+1.825z^{-1}+z^{-2})(1-0.667z^{-1})$

(ii) Error-pulse transfer function $\phi(z)$

class I : $(1-z^{-1})(1+0.7386z^{-1} + 0.2614z^{-2})$

class II : $(1-z^{-1})^2(1+1.2157z^{-1} - 0.5229z^{-2})$

(iii) Digital controller D(z)

Control function for class I and class II design is

$$D(z) = K_3 \begin{bmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{bmatrix}$$

where

$$d_1 = 2.254(1-z^{-2})$$

$$d_2 = (1+3.67z^{-1} + z^{-2})$$

Common factor K_3

class I : $\dfrac{0.1402\omega_n^{\,3}}{(1+0.7386z^{-1}+ 0.2614z^{-2})}$

class II : $\dfrac{0.4207\omega_n^{\,3}(1-0.667z^{-1})}{(1-z^{-1})(1+1.2157z^{-1}-0.5229z^{-2})}$

(iv) Steady state error for a tuned gyro with nutational frequency of 480 Hz and for 100 deg/sec. input rate

class I : 0.08°

class II : zero

(v) Maximum torque for constant rate input

class I : 1.2 Mo

class II : 2.5 Mo

where Mo is steady state torque.

Comparison of general and specific features leads to the conclusion that the class I design is preferred since it offers reduced complexity in the control functions and lower torque demands, while the steady state error is acceptably small.

4.3 Sensitivity considerations

The minimum settling time design developed for gyro rebalance control system yields a response which settle in three sampling intervals. It has been suggested (Ref. 4.2) that such systems are potentially sensitive to parameter variations. This is in part due to the cancellations generated between the controller and the plant transfer functions and also due to multiple poles in the closed loop response at the origin of the z-plane. Sensitivity must therefore be examined carefully to ensure that the design will remain satisfactory over a range of parameter changes.

In the case of rebalance control loops the gyro is designed to have an accurately reproducible dynamic characteristics and wide linear range of operation. The pick-off gain is however, one of the less consistent features. This, and the possibility of rounding off coefficients in the digital processor algorithm, will be considered in detail.

4.3.1 Control loop gain variations

A variation in the pick-off gain δ in loop i results in a gain change applied to all the elements of P in row i. When same mode of variation, in this case the gain factor, is present in several elements of P, the component δ is the same in each element, so that the deviation matrix H given in Equation (3.8) has elements $h_{ij} = p_{ij}\delta$ when an element p_{ij} is changed and zero otherwise. Therefore it is useful to write

$$H = \delta K \tag{4.8}$$

where δ is a scalar multiplier and K contains p_{ij} or zero in each element. The general sensitivity condition of Equation (3.9) is now modified to

$$\det(I + f\delta KP^{-1}) = 0 \tag{4.9}$$

Matrix K will contain the elements of P in row i and zero elements elsewhere. We may write K = SP where the matrix S has a element $S_{ii} = 1$ on the diagonal to correspond with row i in P and all other elements are zero. Then the matrix product KP^{-1} in Equation (4.9) reduces to S so that the determinant is satisfied by one equation

$$1 + f\delta = 0 \tag{4.10}$$

The movements of the zeros as a function of δ may be investigated by root locus solution of this equation. When f(z) is a polynomial in z^{-1}, as in the case with a finite settling time design, all its poles are at the origin. The root loci move out from the origin to terminate on the zeros of f(z) or at infinity. Stability limits are reached when δ is large enough to place roots on the unit circle. For the class I system design example Equation (4.10) can be written as

$$0.2614\ z^{-1}\ (1 + 1.825\ z^{-1} + z^{-2}) = -\frac{1}{\delta} \qquad (4.11)$$

The stability limit is reached when Equation (4.11) has a solution on the unit circle as the complex variable z moves along the circumference of the unit circle about the origin of the z-plane. The complex variable z and the frequency ω are related by

$$z = \exp(sT) = \exp(j\omega T) = \cos(\omega T) + j\sin(\omega T) \qquad (4.12)$$

Limiting value of δ can be computed by equating absolute values on either side of Equation (4.11) when angle condition on left hand side approaches 180° and for this design example the value of δ is 2.1. However such large variations will not happen in practice, and it is more meaningful to examine the effect of small variations on the closed loop response by simulation. The programme is discussed in Appendix C. Fig. 4.1 shows the result of a 10% variation in gain in the error response following a unit step change of case position.

It is to be noted from Fig. 4.1 that there is an increase in response settling time when the loop-gain is deviated from its optimum value. Also there is slight overshoot of about 5% for the response corresponding to + 10% gain variation. But it is important to observe that despite the slight overshoot and the greater settling time, the system remains stable.

4.3.2 Controller mismatch

Minimal response time design technique yields the four transfer functions of the control algorithm in the form of

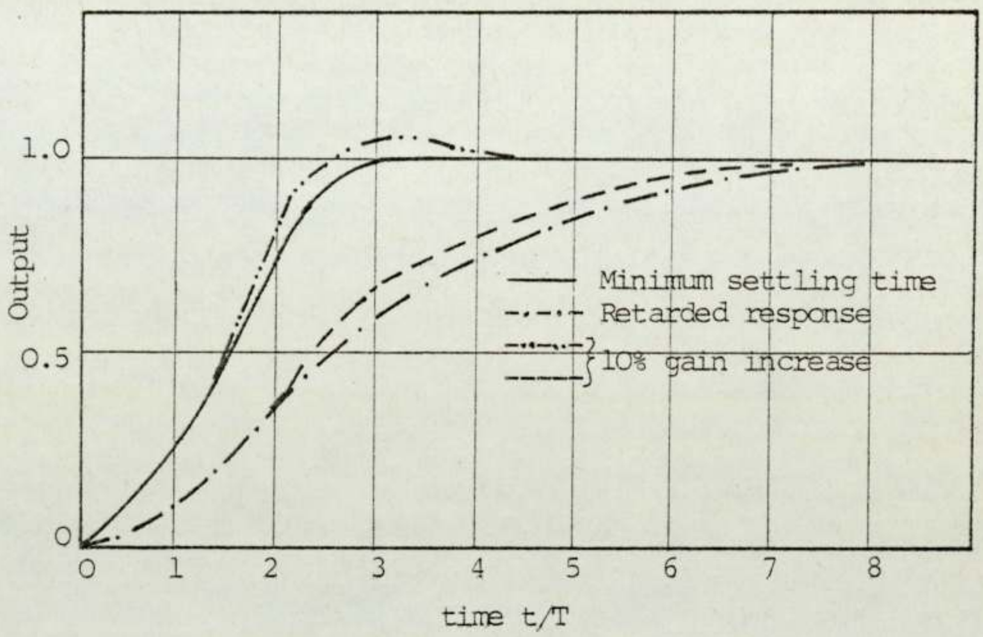


Fig. 4.1 Effect of gain changes

matrix D. It is always desirable to round-off some of the coefficients used in the digital control algorithm as far as possible without effecting the overall response. A solution to this problem will be developed for a gyro balance system.

The general form of digital controller transfer function D is given by Equation (3.7) and when P has the structure as given in Equation (4.6) D becomes

$$D = \frac{f}{(N_1^2 + N_2^2)K_1\phi}\begin{bmatrix} N_1 & N_2 \\ -N_2 & N_1 \end{bmatrix} \tag{4.13}$$

Cancellations are performed in this equation so that the common factor simplifies. For example in the class I design it becomes $0.1402\omega_n^3(1 - z^{-1})/\phi$. We may consider first the possible mismatch in the common factor in realizing D. If $\eta(z)$ is a small change added to ϕ the zeros of det(I + PD) are given by

$$1 - \frac{f}{\phi}\eta = 0 \qquad (4.14)$$

where zeros of ϕ are given as $z = -0.3693 \pm j0.3536$ and are well inside the unit circle, this is not a condition in which root locations are sensitive. The coefficients in the common denominator of D may therefore be rounded without significantly affecting the overall response. In the design example for the class I system the denominator of K_3 as given in Section (4.2.2) may be rounded to $(1+0.75z^{-1} + 0.25z^{-2})$ with negligible effect on the dynamic response.

Mismatch in the numerator polynomials N_1 and N_2 raises a special sensitivity problem in the case of gyro rebalance control system. If $\Delta_1(z)$ and $\Delta_2(z)$ are small changes added to polynomials N_1 and N_2 such that

$$I+PD = \frac{1}{\phi}\begin{bmatrix} 1+\dfrac{f(N_1\Delta_1+N_2\Delta_2)}{(N_1^2+N_2^2)} & \dfrac{f(N_1\Delta_2-N_2\Delta_1)}{(N_1^2+N_2^2)} \\ \dfrac{-f(N_1\Delta_2-N_2\Delta_1)}{(N_1^2+N_2^2)} & 1+\dfrac{f(N_1\Delta_1+N_2\Delta_2)}{(N_1^2+N_2^2)} \end{bmatrix} \qquad (4.15)$$

then the zeros of det(I+PD) are given by

$$\frac{1}{\phi^2}\left[1+\frac{2f(N_1\Delta_1+N_2\Delta_2)}{(N_1^2+N_2^2)}\right] = 0 \text{ (for small } \Delta_1 \text{ and } \Delta_2) \qquad (4.16)$$

The zeros of $(N_1^2+ N_2^2)$ give pole positions at which root locus branches emerge for $\Delta_1 = \Delta_2 = 0$. In the case of tuned-gyro there are two pairs of such zeros located on the unit circle. One pair is cancelled however, by the corresponding zero assigned to f. The remaining pair occur at the nutational frequency and it is significant that root loci from these zeros will enter the region outside the unit circle. When the zeros of Equation (4.16) fall outside the unit circle the system will be unstable. This can be avoided by matching the changes Δ_1 and Δ_2 so that

$$\frac{\Delta_2}{\Delta_1} = - \frac{N_1}{N_2} \tag{4.17}$$

in the region of the z-plane close to the zeros of $(N_1{}^2+N_2{}^2)$ at frequency ω_n. For the tuned gyro $\frac{N_2}{N_1} = -j$ at $z = \exp(j\omega_n T)$.

The polynomials N_1 and N_2 given in the example Equation (4.6) and (4.7) indicate a need to consider rounding the coefficients 3.67 in N_2 with a change $\Delta_2 = \delta_2 z^{-1}$. If a change is also made in the gain coefficient 2.254 of N_1 with $\Delta_1 = \delta_1(1-z^{-2})$ then at $z = \exp(j\omega_n T)$,

$$\frac{\Delta_2}{\Delta_1} = - \frac{\delta_2}{\delta_1} \left(\frac{z^{-1}}{1-z^{-2}}\right) = \frac{\delta_2}{\delta_1} \left[\frac{-1}{2j\sin(\omega_n T)}\right] \tag{4.18}$$

hence Equation (4.18) is satisfied when

$$\frac{\delta_2}{\delta_1} = 2\sin(\omega_n T) \tag{4.19}$$

and the coefficient changes will cancel each other. This relationship assumes small changes in coefficient of z^{-1} in N_2 and gain of $(1-z^{-2})$ in N_1. System response plot of Fig. 4.2 indicates in (i) the result of such a compensating adjustment in which 3.67 in d_2 is rounded to 4; this is compensated by a change in the gain factor 2.254 in d_1 as required by Equation (4.19).

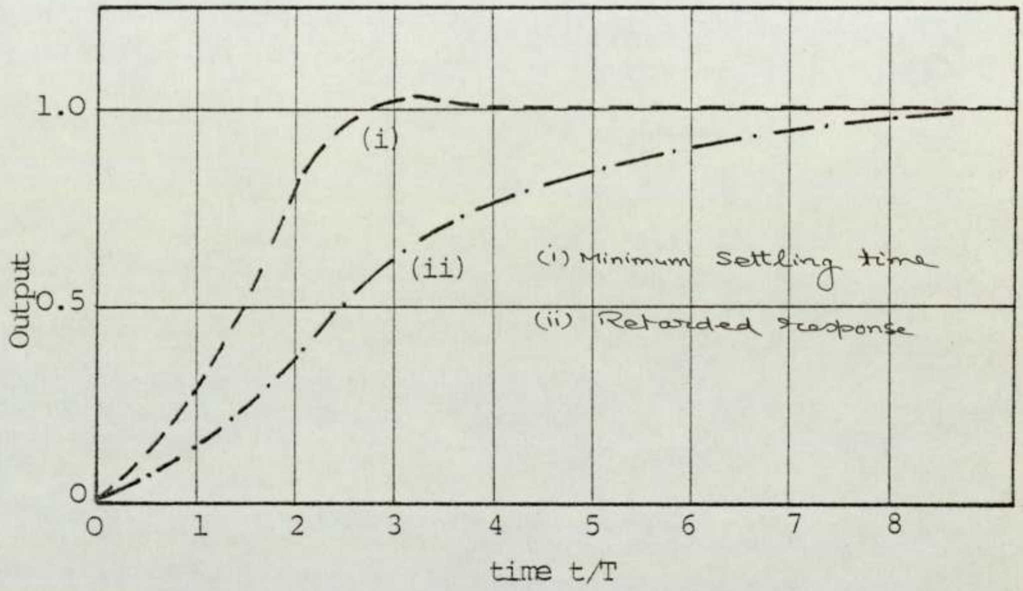


Fig. 4.2 Response with adjusted parameters

It is significant to observe that the system response with adjusted parameters would undoubtedly be looked upon with more favour despite little overshoot and increase in settling time.

Equation (4.19) assumes small changes in coefficient of z^{-1} in N_2 and gain of $(1-z^{-2})$ in N_1. When larger changes are contemplated, further compensation may be necessary to prevent unsatisfactory performance. This can be achieved by retarding the system response.

4.4 Effect of retarded response

In section (3.3) it has been shown that some advantage in reduced sensitivity can be gained by removing the requirement of finite settling time. A term $(1-\nu z^{-1})$ allocated as a pole in f and ϕ results in an overall system response which includes a mode having a time constant τ such that $e^{-T/\tau} = \nu$ and the response will settle exponentially.

With uncertainty in the pick-off gain Equation (4.10) applies and the added pole in f replaces one of the poles at the origin. The reduced multiplicity of the pole leads to reduced sensitivity (Ref. 3.6). In the design example a pole assigned to f at $\nu = 0.6$ results in Equation (4.11) to be modified to

$$\frac{0.1045z^{-1}(1+1.825z^{-1}+z^{-2})}{(1-0.6z^{-1})} = -\frac{1}{\delta} \qquad (4.20)$$

which increases the stability margin from $\delta = 2.1$ of previous value to 4.0. The reduced sensitivity is evident from the response graph of Fig. 4.1 where a 10% change of loop-gain has been introduced.

The variations in the numerator polynomials of D results in root locations given by Equation (4.16). When a simple pole is introduced in f its effect is to alter the angle of departure and the magnitude of the movement of the root loci. In the case of gyro rebalance system the locus from $z=\exp(j\omega_n T)$ is of particular concern. By choosing the pole position the root locus can be set tangential to the unit circle so that a residual mismatch in changes Δ_1 and Δ_2 will have minimum effect on stability. In the design example the locus alignment is achieved with $\nu=0.6$, which corresponds to a retardation with a mode having a time constant of approximately twice the sample interval.

It is also significant to note that the magnitude of the movement of the root loci (Ref. 3.6) is also reduced, for example in this design the value is reduced to one fourth of its original value.

When system response is retarded the common factor K_3 in D for class I design will be

$$K_3 = \frac{0.0561\ \omega_n^{\ 3}}{(1+0.2954z^{-1}+0.1045z^{-2})} \qquad (4.21)$$

and the denominator of which may be rounded to $(1+0.25z^{-1} + 0.125z^{-2})$ with negligible effect on the dynamic response. The response (ii) in Fig. 4.2 shows the further effect of including the retardation factor. This response represents the overall compromise between response speed, sensitivity and controller complexity. The resulting Fig. 4.3 shows a frequency response bandwidth is approximately half the nutational frequency of the gyro.

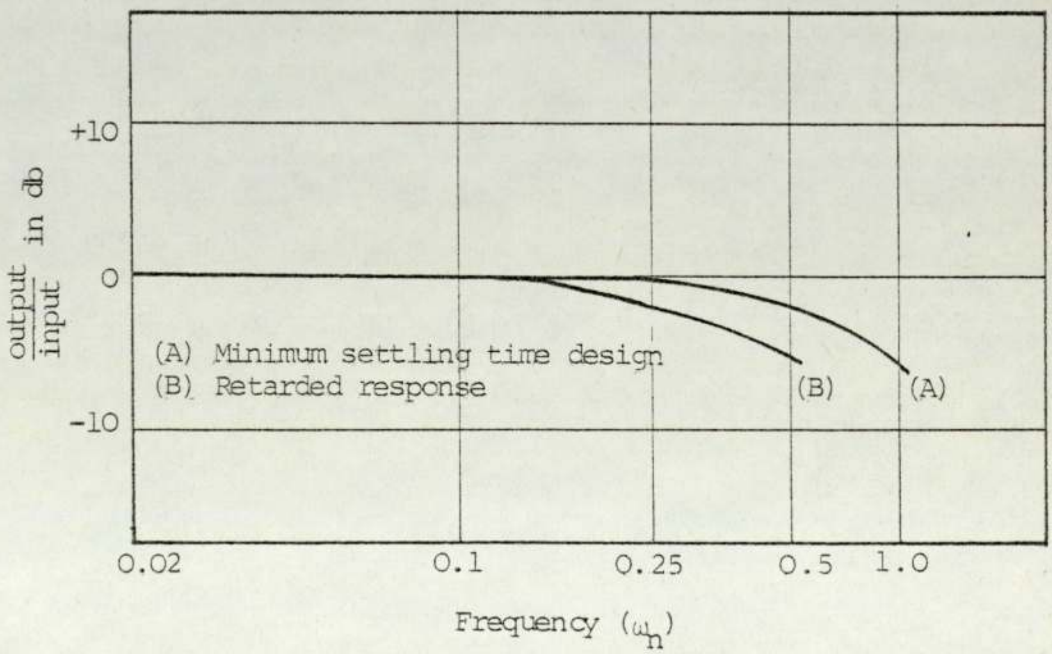


Fig. 4.3 Frequency response plot

4.5 Control loops interaction sensitivity

Because of diagonalization of closed-loop pulse transfer function (Equation 3.5) each input is paired with an output and these input-output pairs do not interact with each other.

In the case of a gyro rebalance system design it is important to observe how the changes of loop gain and rounding-off coefficients affect the interaction between control axes. When there is a variation in loop gain in one or both loops, the matrix of the open loop pulse-transfer function PD remains diagonalized and therefore does not affect the interaction. Also the changes in the common multiplying factor K_3 in D do not affect this diagonalized pulse-transfer function matrix, therefore interaction is not introduced by loop gain changes or by changes in the denominator of the polynomial.

On the other hand, when the mismatch in the polynomials N_1 and N_2 is considered, the interaction between the control axes is only cancelled when off-diagonal elements of Equation (4.15) are equated to zero, this gives

$$\frac{d_2}{d_1} = \frac{N_2}{N_1} = \frac{\Delta_2}{\Delta_1} \qquad (4.22)$$

Variations in the numerator polynomials have been considered in Equation (4.17). In the gyro equations evaluated at the nutational frequency $\frac{N_1}{N_2} = -\frac{N_2}{N_1}$ so that the condition for the non-interaction is the same as that of cancellation of coefficient changes in this particular case. Some interaction will appear at other frequencies but it is important to note that it is cancelled at the nutational frequency where it has its maximum effect.

4.6 Design with noise filter included

In some dry-tuned gyros design problems have emerged due to pick-off noise at the spin frequency or its harmonics. The elimination of such a noise signal can be accomplished with a notch filter (Ref. 4.1). This filter may be implemented in analogue or digital form. For analogue filter the transfer function is given as

$$g(s) = \frac{s^2+\Omega^2}{s^2+vs+\Omega^2} \tag{4.23}$$

The plot of the magnitude function $|g(j\omega)|$ is sketched in Fig. 4.4

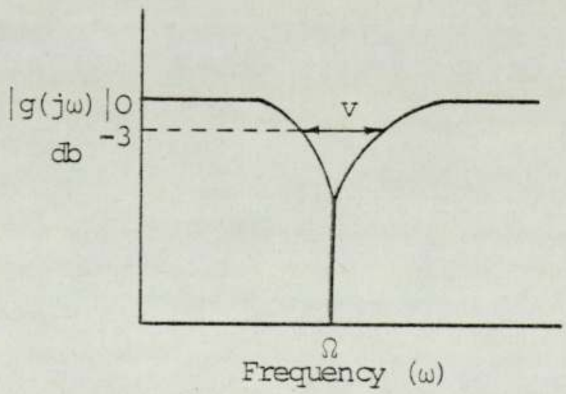


Fig. 4.4 Frequency response characteristics of the notch filter

This shows that $\omega=\Omega$ is the notch frequency at which there is no transmission through the filter. Within the frequency band centered at $\omega=\Omega$ and of width v, signal components are attenuated more than 3-db, the rejection bandwidth.

For the digital notch filter the algorithm

$$g(z) = \frac{(z^{-2}+1)-2a_1z^{-1}+a_2(z^{-2}+1)}{(1-a_1z^{-1}+a_2z^{-2})} \tag{4.24}$$

with

$$a_1 = \frac{2(1-\Omega^2)}{(1-\Omega^2+v)}$$
$$a_2 = \frac{(1+\Omega^2-v)}{(1+\Omega^2+v)} \tag{4.25}$$

gives zero transmission at the notch frequency and unity gain at high and low frequencies.

When the notch filter is implemented in digital form, the design procedure is applied with P^{-1} replaced by $\frac{1}{g}P^{-1}$ in Equation (3.7). The numerator polynomial of g(z) is assigned to f and the denominator to ϕ. This results in the minimum settling time being increased from 3 to 7 sample intervals in the class I design. Also there is an increase in the controller complexity with fourth order polynomials in the numerator and denominator. Sensitivity to parameter changes is increased. The same problem arises when the corresponding analogue filter in the form of Equation (4.23) is implemented before the analogue to digital converter.

A compromise is reached if a non-recursive digital filter is employed. This provides a pair of zeros at the notch frequency which are also allocated to f. The result is a minimum settling time of 5 sample intervals and controller functions with second order numerator and fourth order denominator polynomials. For such a design for class I system f, ϕ and the common factor K_3 in D can be written as

$$f = 0.6841z^{-1}(1+1.825z^{-1}+z^{-2})(1-1.6178z^{-1}+z^{-2})$$

$$\phi = (1-z^{-1})(1+0.3158z^{-1}+0.1741z^{-2}+0.8258z^{-3}+0.6841z^{-4})$$

(4.26)

$$K_3 = \frac{0.367\omega_n^3(1-1.6178z^{-1}+z^{-2})}{(1+0.3158z^{-1}+0.1741z^{-2}+0.8258z^{-3}+0.6841z^{-4})}$$

When such a filter is included in the design the steady state error at sampling instant can be computed from Equation (4.3). For such a design A(z) as given in Equation (4.1) takes the form.

$$A(z)=(1+0.3158z^{-1}+0.1741z^{-2}+0.8258z^{-3}+0.6841z^{-4})$$

(4.27)

and therefore the steady state error for class I design will be 3Tr'(qT). For a nutational frequency of 480 Hz for the gyro and 100 deg/sec. input rate this value of the error is 0.125°. This is due to the fact that when such a filter is included in the design there is increase in the order of the polynomial A(z).

It is significant to note that when such filters are to be included in the design the interaction between gyro control axes is not affected because Equation (4.22) condition still holds good.

In all these cases the overall bandwidth must be less than the notch frequency. This means that attainment of adequate response speed for strapdown applications is considerably impaired by the need for such filtering.

4.7 Processing delay

The execution time of the digital processor implementing the control algorithm may well amount to a significant fraction of the sample interval. This results in an additional delay in the rebalance loops. Fig. 4.5 shows the system response with such delays in the control function for a class I design.

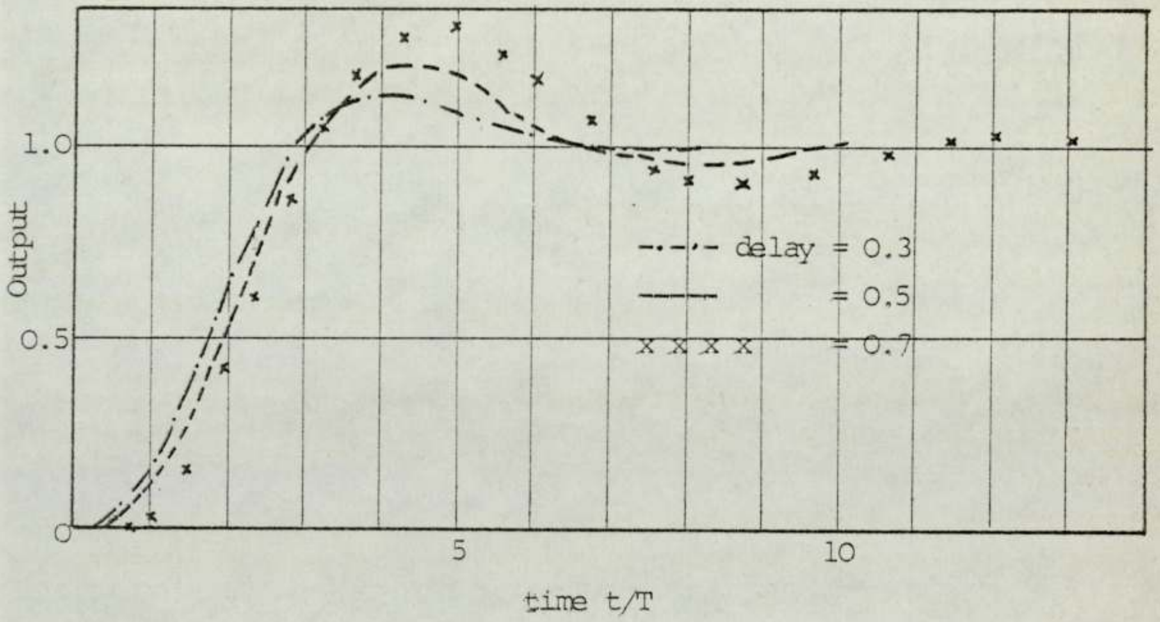


Fig. 4.5 Response with processing delay

From Fig. 4.5 it has been observed that for small computing delays up to 0.5 times the sampling interval, the system response may be acceptable in practice. This is due to three main reasons. Firstly the system remains stable in the presence of such delays. Secondly the output converges to its final value in finite time though it is relatively greater than the minimal response time, and finally, the controller complexity is not affected.

The overall settling time can be improved by taking account of the delay in the design equations. For such a design the analytical procedure outlined here may be implemented with the modified, or delayed z-transform functions (Ref. 1.10). A system of this type can be described by a transfer function $\overline{P}(s)$, each element of which includes a multiplicative term $e^{-\lambda Ts}$ where λT is the processing delay in the system. To be able to design a digital controller for such systems it is necessary to obtain the pulsed-transfer function $P(z,\lambda)$ which takes into account a plant transfer lag in each element where

$$k = \lambda + \Delta \qquad (4.28)$$

The constant k is an integer representing the delay in the pulsed-transfer function, and Δ is a fraction. Since in a given gyro rebalance design the processing delay is a fraction of the sample interval therefore k will always be unity. The system function for a plant of this type can be represented by

$$\overline{P}(s,\lambda) = \overline{P}(s)e^{-\lambda Ts} = e^{-Ts}\overline{P}(s)e^{\Delta Ts} \qquad (4.29)$$

where $\overline{P}(s)$ is the plant transfer function without processing delay. The z-transform for a system function of this type becomes

$$P(z,\lambda) = z^{-1}P(z,-\Delta) \qquad (4.30)$$

The design method now proceeds in a similar manner to that used with conventional multivariable design as given in section (3.2), by defining the required closed loop function f and evaluating the necessary control function D as

$$D = \frac{f}{\phi} P^{-1}(z,\lambda) = \frac{f}{\phi} \frac{Adj(P(z,\lambda))}{det(P(z,\lambda))} = z\frac{f}{\phi} \frac{Adj(P(z,-\Delta))}{det(P(z,-\Delta))} \tag{4.31}$$

When such a delay is introduced in the proposed system, the pulse-transfer function of the gyro as given by Equation (1.4) is modified and new values of K_1, N_1 and N_2 are given by

$$K_1 = \frac{\{\lambda T_1 - \sin(\lambda T_1)\} z^{-1}}{\omega_n{}^3 (1-z^{-1})(1-2\alpha z^{-1}+z^{-2})}$$

$$N_1 = c(1+a_1 z^{-1}+a_2 z^{-1})(1-z^{-1})$$

$$N_2 = (1+d_1 z^{-1}+d_2 z^{-2}+d_3 z^{-3})$$

$$c = \frac{1-\mathrm{Cos}(\lambda T_1)}{\lambda T_1 - \sin(\lambda T_1)}$$

$$a_1 = \frac{\mathrm{Cos}(\lambda T_1)+\mathrm{Cos}(1-\lambda)T_1 - 2\mathrm{Cos}T_1}{1-\mathrm{Cos}\lambda T_1} \tag{4.32}$$

$$a_2 = \frac{1-\mathrm{Cos}(1-\lambda)T_1}{1-\mathrm{Cos}(\lambda T_1)}$$

$$d_1 = \frac{(1-\lambda)T_1 - 2\lambda T_1 \mathrm{Cos}T_1 + 2\sin(\lambda T_1) - \sin(1-\lambda)T_1}{\lambda T_1 - \sin(\lambda T_1)}$$

$$d_2 = \frac{\lambda T_1 - 2(1-\lambda)T_1 \mathrm{Cos}T_1 - \sin(\lambda T_1) + 2\sin(1-\lambda)T_1}{\lambda T_1 - \sin(\lambda T_1)}$$

$$d_3 = \frac{(1-\lambda)T_1 - \sin(1-\lambda)T_1}{\lambda T_1 - \sin(\lambda T_1)}$$

$$\lambda = 1 - \Delta$$

The minimum prototype response function f may be assigned depending on position of zeros of $\det(P(z,\lambda))$ for which the optimum response is desired. For a prescribed computing delay, it is possible to evaluate $P^{-1}(z,\lambda)$. It has been observed that for a known value of the delay λ, there is no cancellation between the elements of $P(z,\lambda)$ which form a common pole pair on the unit circle and the corresponding zeros of $\det(P(z,\lambda))$. Therefore in this design, the common pole pair on the unit circle due to undamped oscillatory characteristics of the gyro must be assigned to polynomial ϕ to avoid sensitivity problem. Also the polynomial $\det(P(z,\lambda))$ which is of sixth order must be evaluated for its roots and any zeros of this which lie on or outside the unit circle in the z-plane must be assigned to response function f. The final design results in the minimum settling time being increased, and also there will be an increase in the conroller complexity. Because of increase in the number of zeros at the origin of the polynomial f, sensitivity to parameter variations is also increased. Because of increased dimensionality with high order polynomials, this will result in the need for more elaborate control algorithms. Therefore it is for this reason that in implementing such a design one must decide about the increased complexity in the system which might result.

4.8 Summary

The method described has enabled the gyro rebalance system to be designed for class I and class II cases. The method of design proposed here has the advantage that it exploits direct analysis in the z-plane and takes account of the behaviour of the rebalance system of a dry-tuned gyro as a multivariable control system. The need to eliminate interaction between the gyro control axes has been included in the design policy.

Comparisons for class I and class II design leads to the conclusion that class I system is preferred, since it offers reduced complexity in the control functions, lower transient torque-motor capability, while the steady state error is acceptably small.

Results giving sensitivity conditions have been developed which are of considerable importance and significance. This involves a differential adjustment of coefficients on the one hand, and an optimal reduction of sensitivity by retardation of the dynamic response on the other.

It has been shown that the design method may be implemented at the expense of increased controller complexity when noise filters are to be included to remove pick-off noise. It is also shown that the method may be adapted when it is necessary to incorporate processing delay but the design will be more complex in terms of increased dimentionality of the controller elements and will be more sensitive to parameter changes. On the other hand a compromise may be reached if the slighly increased overshoot and settling time as shown in Fig. 4.5 is acceptable for a prescribed computing delay.

CHAPTER 5

IMPLEMENTATION OF THE DIGITAL CONTROL ALGORITHM

5.1 Introduction

The design procedure outlined in the previous chapter yields a control algorithm in the form of matrix D, implementation of which most conveniently involves a digital processor. In this chapter considerations are given to the requirements of a processor for such applications. The limited speed of less expensive digital processors dictates that functions must be performed using minimum number of computer instructions. Various methods of reducing the complexity of the control algorithm will be examined, which will allow a more flexible choice of the processor.

5.2 Reduction of control algorithm complexity

Implementation of the four transfer functions of D by a digital processor involves a combination of multiplication, addition and data store operations. It is important to ensure that the processing of each new pair of error samples, together with their conversion from analogue to digital form, can be completed in less than a sample interval. Timing calculations show that with a high sampling frequency required for such applications, there is a need to select the processor carefully particularly where a digital noise filter is to be included.

Multiplication is potentially the most time consuming arithmetic operation involved. There may be need to involve a separate hardware multiplier to overcome the slow operation of software multiplication. Limitations on the choice of processor are considerably relaxed if the number of multiplications can be reduced as far as possible. Various methods of analytical approximation and reduction in processor computing time are investigated to minimize the computation time.

5.2.1 Analytical approximation methods

It is always desirable and sometimes necessary to reduce the order of the control function to allow a more flexible choice of the processor in implementing a given algorithm. Several model reduction techniques have been developed and it has been recognised that the most powerful method for the reduction of higher order transfer function model, is that developed by Chen (Ref. 5.2) for continuous multivariable control systems based on single loop design (Ref. 5.3). In this method one expands the given transfer function into a Cauer-type continued fraction about s=0. This ensures that the model gives the correct steady state response, but the approximation to the transient response may not be good. Furthermore, the stability of the model is not guaranteed even if the original system is stable. Chuang (Ref. 5.4) modified the Chen's method to obtain a more accurate initial transient response, by expanding into a Cauer-type continued fraction about s=0 and s=∞

respectively. Shamash (Ref. 5.12) extended this approach for discrete time systems.

The above analytical methods were used in the case of rebalance control loops of the gyro for reducing the complexity of the controller, but they lead to system stability problems. To overcome this, an optimisation package using simplex method (Ref. 5.7) was used with SLAM simulation, in explicit mode, to minimize the sum of the mean square errors at sampling instants. It was observed that the method consumes a considerable amount of digital simulation time to give optimum values of the reduced order controller coefficients. The digital simulation program is discussed in Appendix D.

5.2.2 Arithmetic simplification

An alternative approach to the problem is to avoid multiplications as far as possible. One method of achieving this is to round-off coefficients to values represented by simple binary operations. In this way whole word multiplications are replaced by a small number of quicker shift and add operations.

The design method has been developed with this possibility in mind. It provides a means of rounding coefficients without adverse effect on the system performance. For example it has been shown in section (4.3.2) that the polynomials $(1+3.67z^{-1}+z^{-2})$ may be rounded to $(1+4z^{-1}+z^{-2})$ and also $(1+0.7386z^{-1} + 0.2614z^{-2})$ is changed to $(1+0.75z^{-1} + 0.25z^{-2})$. When system response is retarded (section 4.4), the factor $(1+0.2954z^{-1} + 0.1045z^{-2})$ can be rounded to $(1+0.25z^{-1} + 0.125z^{-2})$. Multiplications involving the new coefficients 4, 0.75 and 0.25 or 0.25 and 0.125 for the retarded response case may be implemented by simple shift and add operations with significant saving in processing time.

The effect of rounding coefficients in the control algorithm can be further studied by using the simulation program given in Appendix C. Since this method is not based on the iterative design procedure for reducing the controller complexity, a significant saving in digital computer simulation time can be achieved. Moreover, the method of rounding the coefficients described above is ideally suited for this design because of significant saving on the digital processor requirement.

5.3 Digital processor requirement

In the previous section the advantages of rounding coefficients in the control algorithm to reduce the computation time have been identified. This allows a more flexible choice of processor for implementing the digital controller.

In the case of the gyro rebalance system, any component of the rate about the gyro input axis produces a gyroscopic torque which causes the position of the rotor to move about its output axis. Any sensed deviation of the rotor position with respect to the case from its null position is sensed by signal pick-offs which produce a proportional amplitude modulated a.c. signal. Control of the torque motors is achieved through binary pulse width modulated rebalancing schemes, as discussed in Appendix A.

When a digital processor is used as a controller for exploiting the maximum capability of the gyro, both the compensation loops servo functions are mechanised as digital computations. The inputs to the analogue to digital converter are the gyro pick-off signals. The torquing signals are computed digitally using software programming techniques and then converted to analogue signal which can be used to restore the sensors. Digital computations may be performed by time sharing in the main navigational computer, or in a separate special purpose processor.

The arithmetic operations necessary in implementing the programming techniques are multiplication and addition. In the previous sections, the method of rounding the coefficients to minimize the processing time for efficient use of the instruction set are discussed. These are applicable to most processors. There is almost no call for instructions other than load, add, store and data shift in implementing the required control algorithm. This is an important consideration in the selection of a processor for this application.

The basic structure to be realized in the case of rebalance control loops of the gyro for class I design is given in Fig. 5.1, in which e_{11} and e_{22} represent A/D converted pick-off signals and $e_{11}{}^*$ and $e_{22}{}^*$ are the modified error signals after performing digital computations. It is to be noted that Fig. 5.1 is drawn taking into account coefficients rounding in the control algorithm.

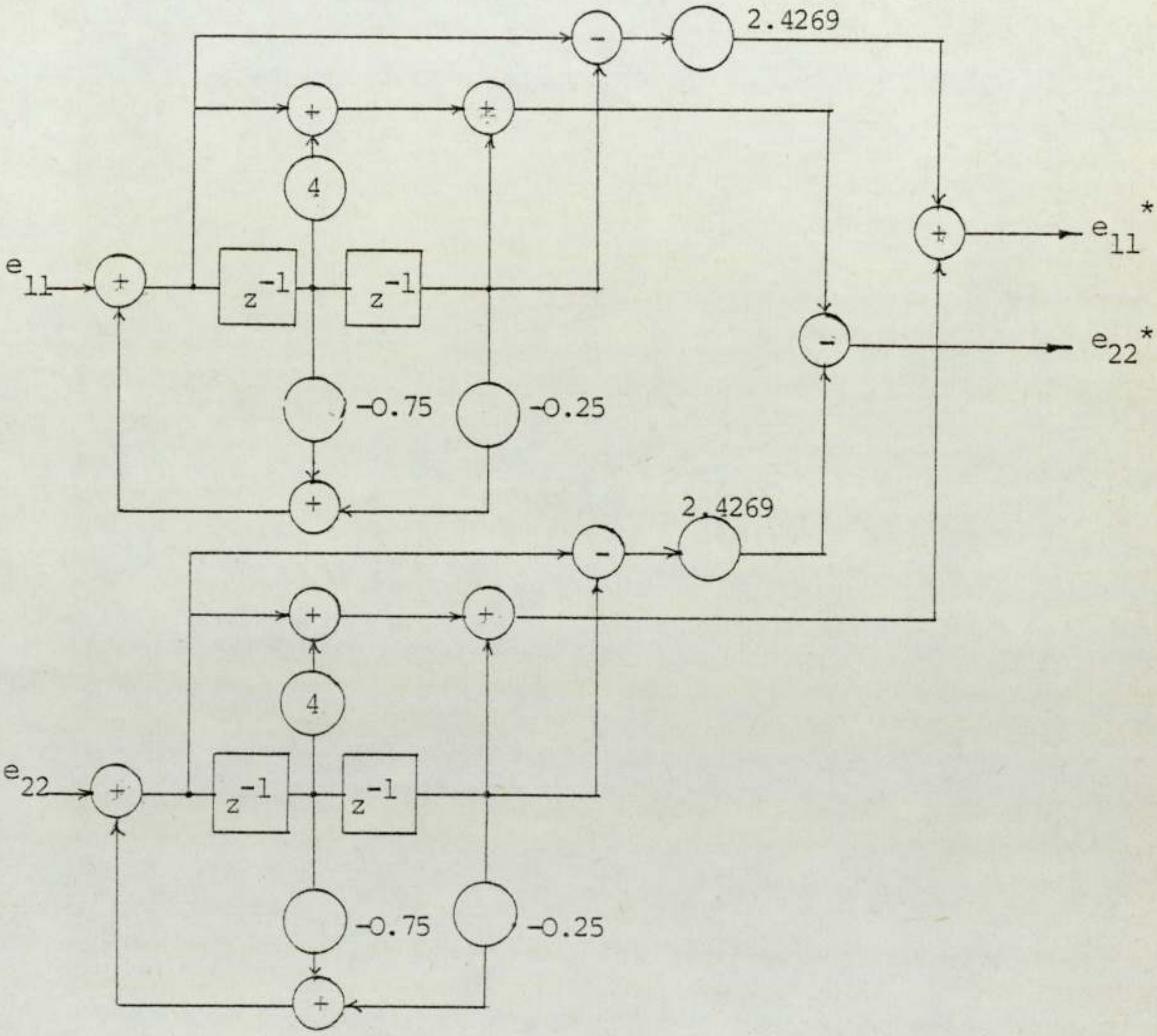


Fig. 5.1 Implementation of control algorithm for class I design

The structure of Fig. 5.1 to be realized is of second order, and the equations which result from this flow graph are

$$e_{11}^*=2.4269(e_{11}-z^{-2}e_{11})+(e_{22}+4z^{-1}e_{22}+z^{-2}e_{22})-\frac{3}{4}z^{-1}e_{11}^*-\frac{1}{4}z^{-2}e_{11}^*$$

$$e_{22}^*= -(e_{11}+4z^{-1}e_{11}+z^{-2}e_{11})+2.4269(e_{22}-z^{-2}e_{22})-\frac{3}{4}z^{-1}e_{22}^*-\frac{1}{4}z^{-2}e_{22}^* \quad (5.1)$$

where e_{11}, e_{22} and e_{11}^*, e_{22}^* are the input and the output sequences corresponding to sampling time intervals. Two multiplications, 12 additions/subtractions and 4 shift/add operations are the main arithmetic instructions required in implementing this design. The realization of such an algorithm, for example on a TMS 9900 microprocessor (Ref. 5.12 and 5.13) using fixed point software multiplication will require about 200 μs. If the sampling frequency as high as 2.4 kHz is selected, which is five times higher than the dominant pole frequency of the gyro, then this implementation time is almost half the sample interval.

5.4 System performance Vs processor word length and speed

It has been shown that a strapdown system employing a digital control for the tuned gyros, the servo-loop functions are mechanized as digital computations and the torquing signals are computed digitally, converted to analogue form, and, after power amplification, used to restore the rotor with the external casing. These same digital torquing signals can be used directly for attitude and navigational computations as well. When a pulse-rebalance loop configuration is desired the torquing signals after digital computations, must be sent to the quantizer where the switchover from negative to positive can be made to occur in synchronism with the sampling frequency.

Selection of the sampling frequency which represents the maximum rate at which the information may be extracted from the rebalance loop, depends upon the minimization of timing errors and improved torquer-current linearity (Ref.1.2).

The quantization frequency is selected based on the dynamic range of operation required in a given application and minimum value of this frequency equals the ratio of the maximum input rate to be rebalancedto the required attitude quantization. The attitude quantization specification is based on overall vehicle attitude determination (Ref. 1.4). Depending upon this information which gives the total number of current pulses required in each limit cycle, one can select the corresponding word length for the processor. It has been observed that in most cases a 8-bits word length processor will be sufficient. Since real time digital processing is to be carried out in each sampling interval, close attention must be paid to the time taken to perform addition, multiplication, load and store data operations, which are the main instructions needed.

A twos-complement number representation, using fixed-point arithmetic may be chosen for this application. Floating point arithmetic, in which the magnitude of the number is represented by a fraction, with separate word to locate the radix point, is more useful in some signal processing applications (Ref. 5.6) where error accumulation due to coefficient round-off leads to intolerably high noise and coefficient sensitivity. Floating point arithmetic employs substantially more memory and the incremental calculations are correspondingly slower. Twos-complement representation have advantages for the execution of arithmetic operation (Ref. 5.10), for example the addition and subtraction of numbers can be performed as though they were unsigned numbers and also when more than two numbers are added, it does not matter if overflow occurs on intermediate summations as long as the final result is in the allowed range.

Sometimes a software multiply instruction may not be available on a processor (Ref. 5.8), and a programme to perform multiplications of two numbers might require a considerable processing time. To overcome this difficulty a fast hardware multiplier unit may sometimes be necessary. The multiplier appears to the processor as two adjacent memory locations (at an address normally reserved for ROM). Memory reference instructions are normally used to access the multiplier. Loading the two numbers in to the registers initiates the multiplication which is normally completed within one processor cycle.

A new generation of bipolar bit slice processors provides a means of increasing the speed performance by a factor of about 10 and one such device is by Advanced Micro Devices AM2901 (Ref. 5.2 and 5.9), where the emphasis is on executing the calculations in minimum time. AM2901 is a four-bit processor slice cascadable to any number of bits. Because of the additional hardware associated with these devices, the overall cost of the system is also increased.

The best choice of speed improvements can be anticipated when processors are specially optimised for this particular application.

5.5 Implementation of noise filters

It has been shown that the design method may be implemented when filters are to be included to remove pick-off noise. To implement digital notch filter the transfer function given by Equation (4.24) is characterized by two distinct parameters a_1 and a_2 which can be realized by a digital filter containing only two multipliers with coefficients a_1 and a_2. Hirano, Nishimura and Mitra (Ref. 5.5) have shown that the notch frequency can be changed while keeping the 3-db rejection band and d.c. gain constants just by varying only a_1, and the reject bandwidth by varying a_2.

To implement the noise filter g(z) given by Equation (4.24) can be rewritten as

$$g(z) = 1+ \frac{z^{-2}-a_1 z^{-1}+a_2}{1-a_1 z^{-1}+a_2 z^{-2}} \qquad (5.2)$$

According to Hirano (Ref. 5.5), such a filter can be realized using two multipliers, several two-input adders and few delay elements. A method proposed by Abu-El-Haiza and Peterson (Ref. 5.1) for implementation of digital notch filters based on digital incremental computers is particularly suitable where very low 3-db rejection bandwidth is necessary. This avoids conventional structures of large word lengths because increments of the signals are processed more quickly than the signals themselves. They also proposed that if the inputs to the incremental multipliers are restricted to be ternary, no hardware multipliers would be necessary.

When a compromise is reached for implementing a non-recursive digital filter along with necessary control algorithm, as proposed in section 4.6, there may be a need to involve a separate hardware multiplier to overcome the slow operation of software multiplication.

5.6 Summary

In this chapter the implementation aspect of the digital control algorithm developed in Chapter 4 for a model of a dry-tunedrotor gyro has been considered. Several techniques of reducing the control algorithm complexity were studied. It has been shown that the rounding of coefficients in the control algorithm is the best choice for this application because this allows a more flexible choice of processor in implementing the necessary control functions. Various factors affecting the digital processor requirement such as speed, word length are also discussed to give more insight when implementing such a digital controller in practice. The implementation aspect of digital notch filter is also included.

CHAPTER 6

DISCUSSION AND CONCLUSION

6.1 Finite settling time design

The design method developed is an extension of the technique which has previously been used for single loop systems. In the case of multivariable systems several distinct problems emerge as a result of interaction between control loops. The need to avoid sensitive design conditions in which mismatch between the controller and the plant may produce instability, requires the avoidance of complete pole/ zero cancellations between the controller and the plant in single loop design. It has been shown by Steel and Puri that in multivariable systems it is acceptable for poles of the plant transfer functions to be matched by zeros of the controller functions in certain combinations of elements, so that partial cancellation occurs.

Existing techniques for the design of multivariable digital control systems have been examined. The work of Nishida (Ref. 2.3) indicatss a promising approach to the engineering design of this class of systems. This method has a feature common to that of the proposed design, namely both use transfer function approach.

Nishida's technique is developed based on single loop design method of Jury and Schroeder (Ref. 2.1) but there are three principal areas in which problems arise. Firstly, the method does not apply when the plant contains complex conjugate poles very near to the unit circle. This severely restricts the flexibility of the design. Secondly, should the system overshoot be unacceptable then no method is outlined for its improvement. Finally, when any one or more elements of the plant have an unstable pole, straightforward application of the method is not possible.

The new finite settling time synthesis method described by Steel and Puri (Ref. 3.6) overcomes these problems. The chief advantage of this method is that it guarantees the best possible dynamic performance of the system, and at the same time eliminates interaction. Results giving sensitivity conditions have been developed which are of considerable generality and significance. These lead to the conclusion that by placing a requirement of finite settling time on the system, a design may result which is both sensitive and unnecessarily complex in the control algorithm required.

The retarded response design technique has been used to extend the scope to allow sensitivity to be traded for speed of response. A consequence is that a complex conjugate pair of poles or zeros of det(P) which are inside, but very close to the unit circle, need not be assigned to f or ϕ. This avoids an increase in controller complexity. Sensitivity to parameter changes which will move those poles or zeros on to or outside the unit circle, can be avoided by constraining the displacement of these poles or zeros to either follow a circular path centered at the origin, or alternatively reduce the magnitude of the displacement of the mode. This can be achieved in practice by allocating a single or two poles to f and ϕ and adjusting these pole positions to give minimal sensitivity condition as proposed in the paper by Steel and Puri. This will result in a system which is less sensitive and at the same time of least complexity in the controller elements. The analytical method of sensitivity reduction as given in this paper has the advantage that it avoids the need for iterative processes and gives compact results which the designer, can readily appreciate. It remains however, that the overall effect on the transient response can only be seen by simulation since the detailed effect of pole and zero movements cannot be anticipated in the time domain.

6.1.1 Application to tuned-gyro model

The method has been applied to a model of a two-axis dry-tuned-rotor gyro. The proposed design method has the advantage that it takes account of the behaviour of the rebalance system of a gyro as a multivariable control system. The cross-loop controller elements decouple the response and permit system operation with a closed-loop bandwidth greatly in excess of that produced by Kao and Hung (Ref. 1.8), and Catton (Ref. 1.4), using analogue control methods. This is an important factor to exploit the maximum capability of a gyro as an attitude sensor in a strapdown environment. Furthermore, the digital controller offers the advantages of flexibility in the realization of complex control algorithms and their accurate implementation is not affected by component stability and tolerances.

The design method can also be extended when it is necessary to include notch filters to remove noise interference in the pick off signals, in which case the overall bandwidth will be less than the notch frequency. It is also shown how the controller complexity is affected when processing delay is to be included in the design.

6.1.2 Sensitivity relationships

It has been suggested by several authors (Ref. 1.10 and 4.2), that the minimum settling time design systems are very sensitive because of the multiplicity of the closed-loop poles, all of which are at the origin of the z-plane, this has been shown to give infinite sensitivity with respect to changes in parameters of the system. According to Kuo (Ref. 1.10), the particular measure of sensitivity implies that any arbitrary small variation of a parameter away from its design value results in an infinite percentage movement of the pole.

A more meaningful measure of sensitivity for finite settling response systems is the sensitivity of (some measure of) the time response of the system with respect to system parameter changes. Therefore a study on the sensitivity has been carried out in the case of the gyro rebalance design. Sensitivity of the step response of the optimal system as a function of pick-off gain variation in any one or both loops, and also coefficient variations in the digital control algorithm are investigated in great detail.

It has been observed that, in the case of loop gain variation, the root loci move away from the origin to terminate on the zeros of f(z), or at infinity (Equation 4.12), and it is concluded that the optimal system remains stable for changes in gain up to $\pm$ 210%. Since such large variations in loop gain will never happen, the effect of small variations on the closed-loop response were studied by simulation. The system simulation method as developed in Appendix C may be a very powerful tool for evaluating more meaningful measure of sensitivity for minimum settling response systems. As may be seen from Fig. 4.1 that a 10% gain variation around the optimal value, there is small change in the stability margin of the system. Though there is a slight increase in overshoot and settling time, the resulting system response may be quite acceptable in practice.

In addition to finding the effects of changes in loop-gain variations, a solution to the sensitivity of the system response to mismatch in the controller elements has been developed for the special case of the gyro rebalance control loops. This involves a means of rounding coefficients in the controller elements without adverse effect on the system performance. In this case the sensitivity design is utilized as part of a policy for simplifying control functions to reduce the computation time.

The residual mismatch in numerator polynomials of the controller coefficients results in a pole pair, occur at the nutational frequency and the root loci from these zeros will enter the region outside the unit circle. When a simple pole is added in polynomials f and ϕ, the effect is to alter the angle of departure as well as reduce the magnitude of the root loci movement. This mismatch will have minimum effect on system stability when a pole position is selected such that the root locus can be set tangential to the unit circle and at the same time reduces the magnitude of the root loci movement. In the case of gyro rebalance design, a pole $(1-0.6z^{-1})$ in f and ϕ satisfies both these conditions. It is also shown that such a pole increases the stability margin of the system to loop gain variations from 210% to 400%.

The resulting design method proposed for the solution to the sensitivity problem in the case of gyro rebalance system design involves a differential adjustment of controller coefficients on one hand, and an optimal reduction of sensitivity by retardation of the dynamic response on the other. The final design represents the overall compromise between response speed. sensitivity and the controller complexity of the system.

6.1.3 Implementation of control algorithm

The gyro rebalance system design yields a defined control algorithm which involves four transfer functions to be implemented using a digital processor.

In a given design the real time processing of each new pair of error samples, together with their conversion from analogue to digital form, must be completed in less than a sample interval. The high frequency sampling in the control loops imposes restrictions on the computation time. The number of multiplications involved consumes most of the computing time in implementing such an algorithm. Therefore the design method has been developed to avoid fixed point multiplications as far as possible without adverse effect on the system performance. The method is based on rounding the coefficients in the control function to simple binary values. This will ultimately allow a more flexible choice of processor for implementation. Other important factors associated in selecting the processor are fixed/floating point arithmetic operations, processor speed and word length.

Various methods available for implementing a digital notch filter are considered. For a gyro rebalance system in presence of pick-off noise, one such filter must be included in each loop. It is also shown that when such a filter is included in its non-recursive form in the digital control algorithm, then there may be a need to involve a separate hardware multiplier to overcome the slow operation of the software multiply instruction, due to high coefficient accuracy requirements.

6.1.4 Limitations of the method

The general scope for application of synthesis method to multivariable systems is limited by the analytical complexity involved. Increased dimentionality means that high order polynomials may be involved and in many cases this will result in the need for elaborate control algorithms. The designer must decide where the increased complexity in computation and implementation is worth the improvement in performance which might result. The problem simplifies when the plant transfer function elements, as in the case of tuned-gyro, have common poles.

The synthesis assumed linearity of the control elements which is appropriate to the dry-tuned gyro designed for strapdown applications where linear torquing characteristics over a wide dynamic range are an inherent requirement.

When digital noise filters are to be implemented to remove pick-off noise, there is an increase in the controller complexity, hence one must take into account the finite processing time in the digital processor. In this situation there may be a need to involve a fast hardware multiplier. The controller complexity is also increased when this design is to be implemented in its modified z-transform, to take into account any processing delay which might result.

6.2 GENERAL CONCLUSIONS

A method for the design of rebalance control systems for a dry-tuned gyro using a digital controller has been developed. The synthesis method is based on minimising settling time in the transient response. Since the tuned-gyro has two control loops which interact, therefore a multivariable control technique has been used.

The design method presented enables a controller to be designed at the lowest level of complexity. Cross-coupled control functions are included which ensure flexibility in design as well as improved performance.

Sensitivity of the system to pick-off gain in one or both loops and the possibility of rounding coefficients in the digital controller algorithm is analysed. A solution to the sensitivity problems have been developed. It has been shown that the method of rounding the coefficients to values represented by simple binary operations is the best choice for this application. In this way a significant improvement in operating speed can be obtained because whole word multiplications are replaced by a small number of quicker shift and add operations.

6.3 Further work

There are several important areas in which the finite settling time technique for the synthesis of multivariable digital control algorithms described could benefit from development. For example, it may be possible to extend this approach to a model of 2-axis servo-accelerometer, described in Appendix B. There is also some scope for compromise when non-recursive noise filters are to be included to remove the pick-off noise in some tuned gyros. Further studies may be needed to qualify this design technique when there is a significant processing delay in the system.

In general it could be beneficial to apply this design technique to various inertial sensors in practice, in order to qualify the method in the light of further practical experience. However, the concept of direct digital control of inertial guidance sensors is sound, and has grown more certain with the development of the proposed design method, for simple and efficient control algorithms, necessary in the torque-rebalance loop.

APPENDIX A

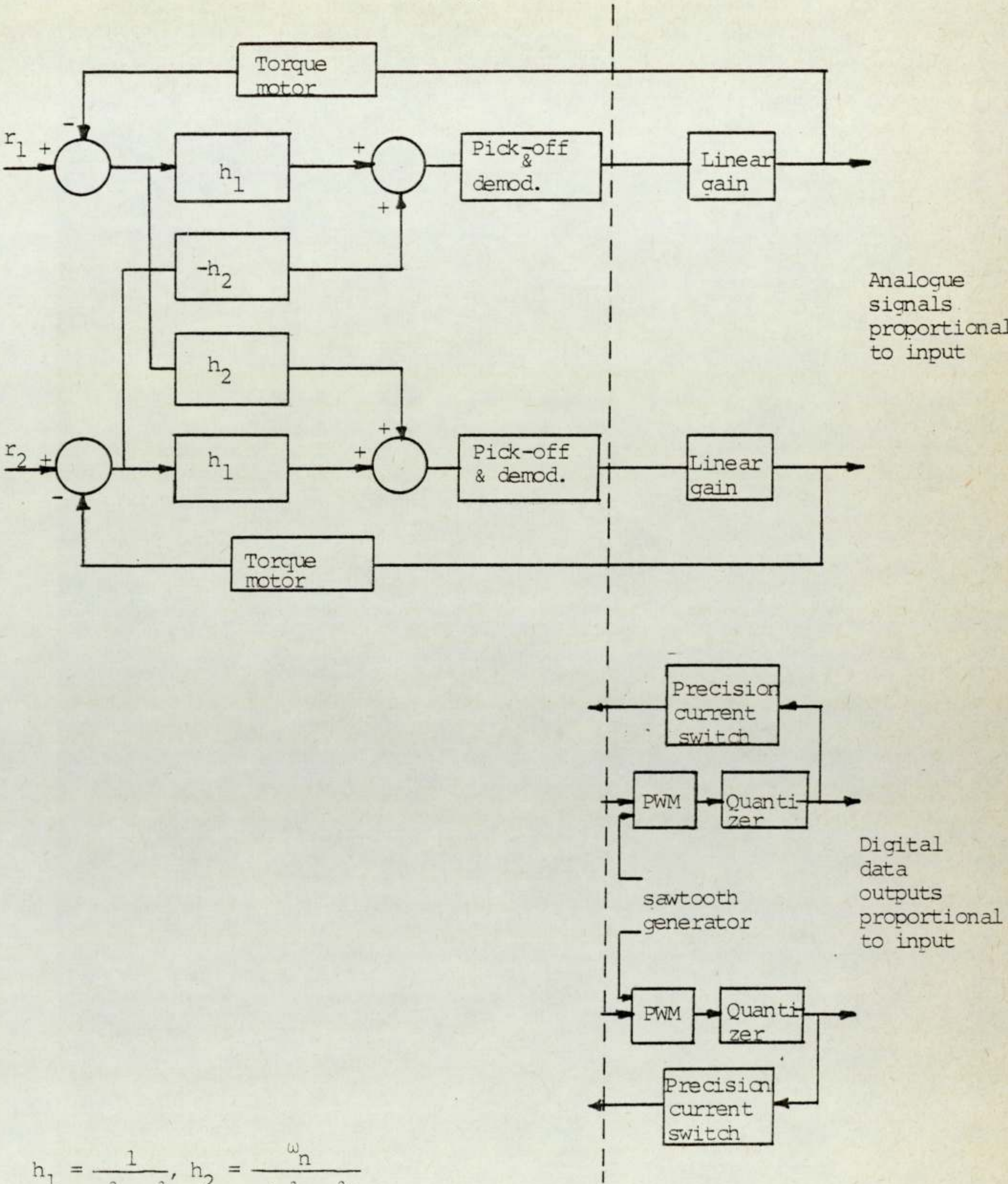
EXISTING ANALOGUE AND PULSE REBALANCE CONTROL SCHEMES

In conventional strapdown system mechanizations (Ref. 1.4 and 1.9) the sensor servo-compensation and control functions are carried out using analogue control methods. In the case of analogue rebalance loops of Fig. A.1 the sensor torquing currents are fed through precision resistances to develop voltages which are proportional to the vehicle angular rates or accelerations. The analogue rebalance loop designs have been proposed by Coffman (Ref. 1.5) comparing both class I and class II systems. These methods are based on linear characteristics of the torque motor.

The pulse rebalance loop configurations are advantageous where inegrating digital readout for high accuracy measurements are required and various such schemes have been developed (Ref. 1.8 and A.1). The binary pulse width modulated (BPWM) mechanization (Fig.A.1) is preferred because in addition to provide direct digital readout it minimizes the requirement on torque motor linearity by restricting operation to two plus-minus torque levels. This assumption has been made in the design method developed in this thesis. In addition BPWM affords the following advantages over other schemes.

(i) Resolution is not limited by the torque motor time constant.

(ii) The torquer operates at a constant power level which minimizes thermal disturbances.



$$h_1 = \frac{1}{s^2+\omega_n^2},\ h_2 = \frac{\omega_n}{s(s^2+\omega_n^2)}$$

Fig. A.1 Analogue and BPWM rebalancing schemes

(iii) The servo loops behave linearly with well-defined performance characteristics.

(iv) The torquing waveform has a fixed fundamental frequency which limits variation in a.c. reaction torques.

The main problems associated with BPWM is that of high power consumption and lower information rate compared to other two schemes.

The essence of this mechanization is the ability to obtain a precise digital measure of the average current fed to the torque motor during each limit cycle period. Knowledge of the restoring torque is directly translatable into vehicle angular motion sensed by the gyroscopes or linear acceleration in case of accelerometers.

APPENDIX B

EXTENSION OF THE METHOD TO SERVO-ACCELEROMETERS

The direct digital control scheme developed for tuned-gyros can also be extended for the accelerometers. Because of the similarity of the dynamical behaviour of accelerometers and mechanical rate gyros, the design problem is further simplified.

Various accelerometers of inertial quality have been developed which can be single axis or two axes ones. In the case of a single axis, any acceleration applied along the sensitive axis tends to move the pendulum from its equilibrium position, and a pick-off provides an a.c. signal of phase and amplitude in relation with the measured deviation. After amplification, demodulation and necessary compensation, this signal is applied to the torque motor to balance the action of the accelerometer and brings the pendulum to its original position. The current going to the torque motor is the measure of input acceleration. Both analogue and pulse rebalance loop system configurations have been developed for single axis accelerometers (Ref. 1.4 and 1.9).

In the case of a two-axes accelerometer (Ref.B.1) based on the development of tuned-suspension gyroscopes, a two-axes suspension carries the torque motor, which consists of magnets and flux return path that establishes a radially oriented field within the airgap. The torque motor is made pendulum relative to both torsional axes of the suspension system.

In the presence of acceleration along any axis perpendicular to the axis of symmetry of the suspension system, the pendulum mass is deflected angularly relative to the accelerometer housing. This deflection is sensed by the pick-offs whose output drives the current through the torquer coils, exerting a moment on the pendulous mass in such a direction as to null the pick-offs. Thus the current through each torque motor is the measure of applied acceleration along corresponding axis.

Most of the strapdown systems developed so far use single axis accelerometers in analogue or pulse rebalance mode. Incoflex two axes accelerometers developed more recently by Russell and Craig, the torquing loop electronics used is an analogue design and a pulse torquing scheme has been proposed. The main advantage of a two axis unit is its low cost per axis.

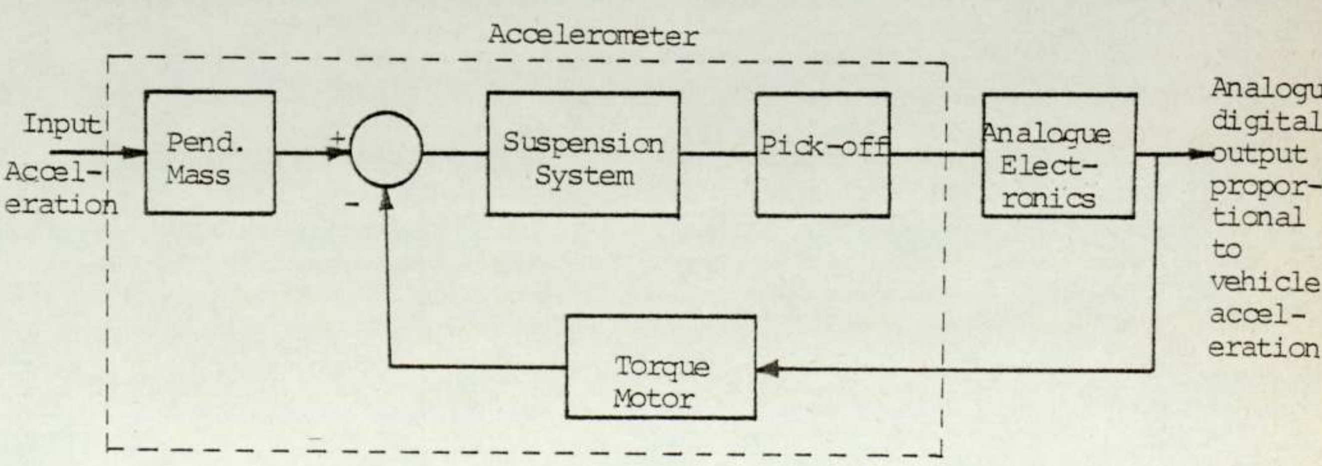


Fig. B.1 Block diagram of a pendululous accelerometer

Once the accelerometer model is known, it is possible to design a rebalance control system using a digital controller in the same way as has been investigated here for the tuned-gyro.

APPENDIX C

COMPUTER SIMULATION OF GYROSCOPE DDC SYSTEM USING 'SLAM'

C.1 Basic description of SLAM

SLAM, the simulation language for analogue modelling (Ref. C.1 and C.2), is a high level language written by staff members of the ICL. The programme provides an application oriented language which allows problem formulation either directly from the system block diagrams or from the system mathematical equations. Included in the programme package is a basic set of functional blocks which allow the representation of a continuous system statement for defining the connections between these blocks (Ref. C.1). One of the greatest advantages of using SLAM from the users point of view is that one is not required to devote considerable time in programming details as in Fortran.

SLAM uses a translatory method of operation, i.e. source programme translated into Fortran, any Fortran statements being passed through without alteration. The resulting Fortran programme may then be compiled by a Fortran compiler loaded and executed.

The basic programming structure may be defined by the programmer and these are

(a) Implicit structure

(b) Explicit structure

A SLAM programme that makes use of the structuring ability of the translater is knwon as an Implicit mode programme. In this mode, all executable statements with the exception of those included in NOSORT blocks (executable statements which do not comply with the rules of sortability in SLAM), are automatically sorted by the translator into initial, dynamic and terminal regions.

A SLAM programme, in which the internal segment structure is explicitly defined by the programmer is known as an Explicit mode programme. This mode is more comprehensive than the Implicit mode, and offers greater programming flexibility and permits the design of programmes better suited to a particular task. For example when analytical approximation method is used for reducing controller complexity, it is best to use SLAM in an Explicit mode for minimizing the sum of the mean square error at each sampling instant.

C.2 Simulation of multivariable digital control systems

Fig. C.1 shows the role of the digital processor in the form of digital controller in multivariable direct digital control systems. The error signals become the input to a properly defined control algorithm. The emphasis in this section is placed on simulating the control algorithm in the form of matrix D(z) along with continuous plant P(s).

The programme listing C.3 gives one method of simulating the gyro system along with the digital control algorithm. The

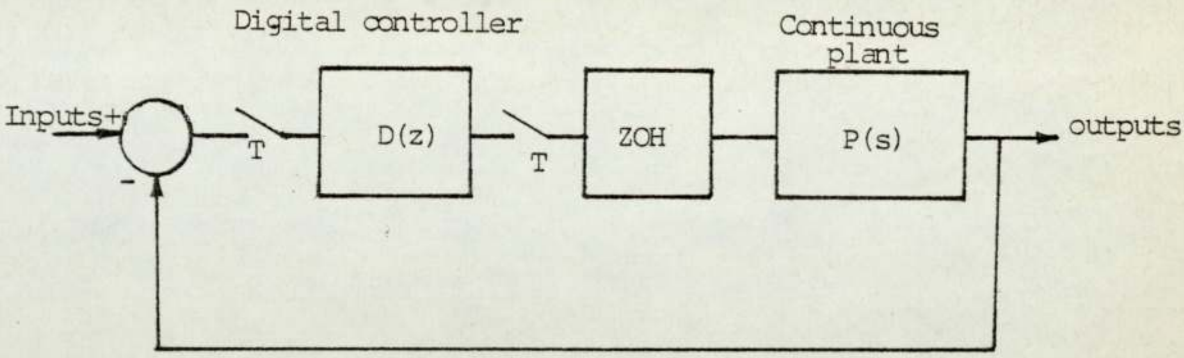


Fig. C.1 Digital control system for simulation

initial region encompasses all those calculations, input/output operations, and initializing procedures that must be performed prior to simulation. Those initializing operations of a more permanent nature (e.g. designation of a particular integration algorithm) should be performed prior to entering the dynamic region. The dyanmic region is that portion of the simulation which takes an active part in the interaction between the digital computer and the external world. It represents all the calculations and I/Θ operations performed at each user-defined discrete value of the independent variable. The terminal region receives control from the dynamic region and returns control to the simulation entry. The terminal region contains the calculations and I/O necessary to properly terminate a single simulation.

The integration system in the dynamic region has two main entries, one for the initialization and one for integration. In addition to setting up initial conditions on the state variables of the integration, the initialization entry also calls the appropriate integration and initialization algorithms into memory from the library and allocates memory for the history information required by these algorithms. The integration entry transfers control to the appropriate algorithm to integrate the specified derivative section over its communication interval.

The simulation of the digital controller algorithm in the form of D(z) is given in a NOSORT block in the programme listing. Statements given in this block are not sorted and are thus executed much the same way as Fortran statements. The IF statement is inserted to ensure that the algorithm will not be executed except at the sampling instants.

In summary this appendix illustrates how one can simulate digital multivariable control system using simulation language SLAM. If the digital controller D(z) is other than minimum settling time design, then the expression for D(z) can be expressed as an algorithm and executed in a NOSORT block in a similar manner, as illustrated in programme listing. Therefore it is concluded that SLAM can be easily adapted to simulate digital control systems and digital filters. The advantage is that the designer can quickly assess the design before making actual hardware implementations.

C.3 Programme listing

```
      REAL ALPHA, BETA, K, MX, MY, QX, QX1, QX2, QY, QY1, QY2, WN
C-- INITIALIZATION
      INPUT T, TD, WN1, B,WN
      ALPHA = COS(T1)
      BETA  = SIN(T1)
      K = (WN*WN) /(T1-BETA)
      C = 1.-ALPHA/(T1-BETA)
      D = (ALPHA*T1-BETA)/(T1-BETA)
      T1 = 2π/B
      B1 = 2*ALPHA + 4*(T1**2*COST1-T1*SINT1+SINT1**2-T1*SINT1*COSTI
     1 (2 + T1**2 -2*COST1-2*T1*SINT1 )
      A1 = (1+B1)/(2.+B1)
      A2 = 1./(2.+B1)
      E11 = 1.0
      E22 = 0.0
      T3  = 0.0
      QX2 = 0.0
      QX1 = 0.0
      QY2 = 0.0
      QY1 = 0.0
      THXO = 0.0
      THYO = 0.0
      E12 = 0.0
      E13 = 0.0
      E23 = 0.0
      E24 = 0.0
      T2  = 0.0
      T4  = 0.0
C-- SIMULATION OF DIGITAL CONTROLLER
      NOSORT (MX, MY, QX, QY = QX1, QX2, QY1, QY2, E11, E12,
     1 E13, E22, E23, E24, THX, THY)
      F1 = TIME/T
      T1 = IFIX(F1)
      IF(T1.EQ.T2.OR.TIME.EQ.TD)GØ TØ 9
      QX2 = QX1
      QX1 = QX
      E13 = E12
      E12 = E11
      E24 = E23
      E23 = E22
      QX2 = QY1
      QY1 = QY
      E11 = 1.-THX
      E22 = - THY
9     QX=K*(A2*C*E11 - A2*C*E13 + A2*E22 + A2*D*E23
     1 + A2*E24) - A1*QX1 -A2*QX2
      QY = K*(-A2*E11 - A2*D*E12 - A2*E13 + A2*C*E22
     1 + A2*C*E24) - A1*QY1 - A2*QY2
      F2 = TIME/T
      T2 = IFIX(F2)
      F4 = TIME-TD/T
```

```
      T4 = IFIX(F4)
      IF(T4.NE.T3.OR.TIME.EQ.TD) GØ TØ 10
      GØ TØ 11
   10 MX = QX
      MY = QY
   11 F3 = (TIME-TD)/T
      T3 = IFIX(F3)
      END
C -- SIMULATION OF CONTINUOUS PLANT
C -- DYNAMIC EQUATIONS
      THX = INTGRL(D1THX,THXO)
      THY = INTGRL(D1THY,THYO)
      D1THX = INTGRL(D2THX, 0.0)
      D1THY = INTGRL(D2THY, 0.0)
      D2THX = MX - WN*D1THY - WN1*D1THX
      D2THY = MY + WN*D1THX - WN1*D1THY
      INTINF
        ALG: RKFS
        CI:CI = 0.417E - 04
        MONITOR : IMON = 2
        RELERR : RLER = 0.005
        END
        ØUTECI TIME, THX, THY, QX, QY, MX, MY
        TERMINATE (TIME. GE.0.0417)
        END
      FINISH
```

APPENDIX D

COMPUTER SIMULATION FOR ANLYTICAL APPROXIMATION METHOD

D.1 Introduction

The purpose of this program is to enable the system designer to apply model reduction technique for optimising a digital controller based on analytical approximation method described in Chapter 5.

The main programme and all subroutines are written in high level language SLAM in its explicit mode. The programme computes and minimizes the sum of the mean square error at each sampling instant with an optimisation package, using simplex method (Ref. D.1). The programme permits the designer to check these values and arrive at optimum controller elements.

D.2 Subroutine outlines

The main subroutines (which are called Segments in explicit mode) are briefly outlined below:

1. FUNCT - Computes value of mean square error and controller elements to terminate a single simulation.
2. MONIT - Used to print out the current values of the parameters to terminate a single simulation.
3. EO4CCF - minimizes mean square error at each sampling instant.

D.3 Programme listing

```
      MASTER DIGSIM
      EXTERNAL FUNCT, MONIT
      REAL TØL, R, F, X(2), SIM(6,5), W1(S), W2(5), W3(5), W4(5), W5(5)
      INTEGER NØUT, N, N1, MAXCAL, IFAIL, I
      TERMINAL
      NØSØRT(X,F=)
      X(1) = 2.254
      X(2) = 4.67
      TØL = SGRT(XO2AAF(R))
      N = 2
      N1 = N + 1
      IFAIL = O
      MAXCAL = 1OO
      CALL EO4CCF(N,X,F,TØL, N1, W1,W2,W3,W4,W5,SIM,FUNCT,MØNIT,
    1 MAXCAL, IFAIL)
      END
      ØUTPUT X,F
      END
      END

      SEGMENT FUNCT(N,XC,FC=)
      REAL MX, MX1, MX2, MY, MY1, MY2, MXO, MYO, K1, L1, L2, FC, XC,ALPHA
    1 BETA, T, WN
      INTEGER N
      ARRAY XC(2)
      INITIAL
      DELTA = 1.
      GAMMA = 1.
      FC1O = O.O
      FC2O = O.O
      CO = 1.E-O3
      DO = 1.E-O3
      C3O = 1.E-O5
      WN = 3O15.94
      T = O.417E-O3
      DX = O.
      ALPHA = CØS(WN*T)
      BETA = SIN(WN*T)
      S1 = BETA/(T*WN)
      S2 = 1.-S1
      S3 = T*WN*S2
      S4 = (1.-ALPHA)/S3
      E1 = S4**2+1
      S5 = T*S2
      S6 = WN/S5
      A1 = S6/E1
      S7 = (S1-ALPHA)*4
      E2 = S7/S2
      B1 = 2.*ALPHA + E2/E1
      K1 = 1./(2.+B1)
      E11 = 1.O
      E22 = O.O
      MXO = O.O
      TO = O.417E-O3
      D3O = 1.E-O5
```

```
      MX1 = 0.0
      MY1 = 0.0
      MX2 = 0.0
      T2  = 0.0
      THXO = 0.0
      THYO = 0.0
      E12 = 0.0
      E13 = 0.0
      E23 = 0.0
      E24 = 0.0
      MY2 = 0.0
      END

      DYNAMIC
      DERIVATIVE
      THX = INTGRL(D1THX, THXO)
      THY = INTGRL(D1THY, THYO)
      D1THX = INTGRL(D2THX, 0.0)
      D1THY = INTGRL(D2THY, 0.0)
      D2THX = MX-WN*D1THY-DX*D1THX
      D2THY = MY+WN*D1THX-DX*D1THY
      C2 = INTGRL(C,CO)
      D2 = INTGRL(D,DO)
      C4 = INTGRL(C3,C3O)
      D4 = INTGRL(D3,D3O)
      FC4 = INTGRL(FC2,FC2O)
      FC = INTGRL(FC1, FC1O)
      NØSØRT(MX,MY,TØRQ1,TØRQ2,C,D,C3,D3,FC1,FC2 = MX1,MY1,
     1 MX2,MY2,E11,E22,E12,E23,E24,THX,THY)
      F1 = TIME/TO
      T1 = IFIX(F1)
      IF(T1.EQ.T2)GØ TØ 9
      MX1 = MX
      MY1 = MY
      E13 = E12
      E12 = E11
      E24 = E23
      E23 = E22
      E11 = 1.-THX
      E22 = -THY
 9    MX = A1*K1*XC(1)*E11 - A1*K1*XC(1)*E13 + A1*K1*E22
     1 + A1*K1*XC(2)*E23 - MX1
      MY =-A1*K1*E11 - A1*K1*XC(2)*E12+A1*K1*XC(1)*E22 .
     1 -A1*K1*XC(1)*E24 - MY1
      C = E11*E11
      D = E22*E22
      C3 = TIME*C
      D3 = TIME*D
      FC1 = C + DELTA*D
      FC2 = C3 + GAMMA*D3
      TØRQ1 = MX*R1
      TØRQ2 = MY*R1
      F2 = TIME/TO
      T2 = IFIX(F2)
      END
```

```
      INTINF
      ALG:RKFS
      CI:CI = 0.417E-04
      MØNITØR : IMON =2
      RELERR : RLER = 0.05
      END
      END
      TERMINATE(TIME.GE.4.17E-03)
      END
      END

      SUBROUTINE MØNIT(FMIN,FMAX,SIM,N,N1,NCALL)
      INTEGER N, N1,NCALL,J,I
      REAL FMIN, FMAX, SIM
      DIMENSIØN SIM(N1,N)
      WRITE (6,1)NCALL,FMIN
      WRITE(6,2)((SIM(I,J),J=1,N),I = 1,N1)
      RETURN

1     FØRMAT(6H AFTER, I5, 30H  FUNCTIØN CALLS, THE VALUE
     1 IS,  E10.8,  14H  WITH SIMPLEX)
2     FØRMAT(3(2E12.8/))
      END
      FINISH
```

REFERENCES

1.1 Bendett, R.M., H.D. Kleine, S.P. Weinstein, and D.A. Zomick: "Test evaluation of a high performance pulse rebalanced gyro for strapdown IMU applications". Navigation, Vol.23, No. 1, 1976.

1.2 Blalock, T.V., E.J. Kennedy and V.A. Thomason: "Development of a digital electronic rebalance loop for a dry-tuned rotor two-degree of freedom gyroscope. Technical report TR-EE/EL-5, University of Tennessee, 1975.

1.3 Bortz, J.E.: "Dynamic errors in a tuned flexure mounted strapdown gyro". NASA-CR-12161, January - July, 1972.

1.4 Catton, G. and J. Michelin: "Strapdown inertial guidance system for tactical missiles". Agard paper from lecture series No.95 on strapdown inertial systems, 1978.

1.5 Coffman, D.E. : "Feasibility study of a digital rebalance loop for a dry-tuned TDF gyro". NASA-CR-144089, May 1974.

1.6 Craig, R.J.D. : "Dynamically tuned gyros in strapdown systems". Agard Conference proceedings No. 116, October, 1972.

1.7 Halamandaris, H. : "Strapdown inertial navigation practical considerations".
NAECON Record, 1973.

1.8 Kao, M.K. and J.C. Hung, : "A pseudo root locus method for the design of a class of two-input two-output systems".
Presented at the proceedings of the 8th annual south eastern symposium of system theory, Knoxville, Tennessee, April 26-27, 1976.

1.9 Kirk, C.W. : "An integrated strapdown guidance and control system for launch vehicle applications".
Presented at the symposium on gyro technology, Bochum, W. Germany, September, 1978.

1.10 Kuo, B.C. : "Analysis and synthesis of sampled-data control systems".
Englewood Cliff, N.J. Prentice Hall, 1963.

1.11 Rahlfs, D. : "Comparative study of pulse rebalance inertial sensors with different dynamics".
Ortung und navigation, No.2-3, 1976.

1.12 Sutherland, A.A., Jr. : "Design of strapdown gyroscopes for a dynamic environment".
NASA-CR-86082, December, 1967.

2.1 Jury, E.I. and W. Schroeder, : "Discrete compensation of sampled-data and continuous control systems".
Trans. AIEE, vol.75, pt.II, 1957.

2.2 Nishida, F. : "Synthesis of multivariable control systems by means of sampled-data compensation". Presented at the first congress of International Federation of Automatic Control, Moscow, 1960.

2.3 Viswanadham, N. and B.L. Deekshatulu. : "Dead beat response in multivariable systems". International Journal of Control, vol.5, No.6, 1968.

3.1 Bergen, A.R. and J.R. Ragazzini. : "Sampled-data processing techniques for feedback control systems". Trans. AIEE, vol. 73, pt. I and II, November, 1954.

3.2 Grant, J.A. and G.D. Hitchins. : "An always convergent minimization technique for the solution of polynomial equations". JIMA, vol.8, 1971.

3.3 Grant, J.A. and G.D. Hitchins. : "Two algorithms for the solution of polynomial equations to limiting machine precision". Computer Journal, vol.18, 1975.

3.4 Hsu, C.H. and C.T. Chen. : "A proof of the stability of multivariable feedback systems". Proc. IEEE, November, 1968.

3.5 Jury, E.I. : "On the roots of a real polynomial inside the unit circle and a stability criteria for linear discrete systems". Presented at the second Congress of International Federation of Automatic Control, Basle, Switzerland, 1963.

3.6 Steel, G.K. and S.N. Puri. : "Minimal response time design for linear multivariable sampled-data control systems".
To be published. Refer to paper at back, p95-139

4.1 Abu-El-Haiza, A.I., M.J. Narasimha, and A.M. Peterson : "A simple hardware implementation of digital notch filters".
Proc. of the 1978 IEEE International Conference on Acoustic, Speech and Signal Processing, Tulsa, USA, 10-12 April, 1978.

4.2 Stanley, F.S. : "Application of continuous system design concepts to the design of sampled-data control systems".
Trans. AIEE, Pt. II, 1959.

4.3 Steel, G.K. and S.N. Puri. : "Direct digital control of dry-tuned rotor gyros".
Presented at the VIII IFAC Symposium on Automatic Control in Space, Oxford, UK, 2-6 July, 1979.
Refer to paper at back, p88-94

4.4 Tou, J.T. : "Optimum design of digital control systems". Academic Press, 1963.

5.1 Abu-El-Haija and A.M. Peterson. : "A structure for digital notch filters".
IEEE Transactions on Acoustics, Speech and Signal Processing, Vol. ASSP-27, Mo.2, April, 1979.

5.2 Chen, C.F. : "Model reduction of multivariable control systems by means of matrix continued fractions".
International Journal of Control, Vol.20, No.2,1974.

5.3 Chen, C.F. and L.S. Shieh. : "A novel approach to linear model simplification".
International Journal of Control, Vol.8, No.6, 1968.

5.4 Chuang, S.C. : "Application of continued fraction method for modelling transfer functions to give more accurate initial transient response".
Electronics Letters, Vol.6, No.26, 31st December,1970.

5.5 Hirano, K., S. Nishimura and S.K. Mitra. : "Design of digital notch filters". IEEE Transactions on Communications, Vol. COM-22, No.7, July, 1974.

5.6 Liu, B. and T. Kaneko. :"Error analysis of digital filters realized with floating point arithmetic".
Proc. IEEE, Vol. 57, No.10, October, 1969.

5.7 Nelder, J.A. and R. Mead. : "A simplex method for function minimization".
Computer Journal, Vol.7, 1965.

5.8 Motorola semiconductor products (1975).
M6800 Application Manual, N.Y.

5.9 Raamot, J. : "Microprocessor applications of integer arithmetic".
Proc. IEEE, Vol. 66, No.2, February, 1978.

5.10 Rabiner, L.R. and B. Gold. : "Theory and application of digital signal processing". Prentice-Hall, N.J. 1975.

5.11 Shamash, Y. : "Continued fraction methods for the reduction of discrete time dynamic systems".
Publication 73/10, Department of Computing and Control, Imperial College, London, February, 1973.

5.12 Texas Instruments, : "Model 990 Computer TMS9900 Microprocessor Assembly Language Programmer's User Guide".
Manual No. 943441-9701, July, 1976.

5.13 Texas Instruments, : "TMS 9900 Microprocessor Data Manual", March 1977.

A.1 Sinkiewicz, J.A. : "Evaluation of selected strapdown inertial instruments and pulse torque loops".
NASA-CR-120427, Vol.1, final report, July, 1974.

B.1 Russell, J. and R.J. Craig. : "Incoflex two axes accelerometer program".
Report No. DA-A043 610, August, 1977.

C.1 SLAM manual 4382 for ICL 1900 series.

C.2 The Sci continuous system simulation language (CSSL), Simulation, December, 1967.

D.1 EO4CCF - Minimizing a general function of N-independent variables by the simplex method.
NAGLIB : 1265/187 : Mk-5 : October, 1975, Computer Centre, Aston University.

DIRECT DIGITAL CONTROL OF DRY-TUNED ROTOR GYROS

G. K. Steel and S. N. Puri

Department of Electrical and Electronic Engineering, University of Aston, Birmingham, U.K.

Abstract. A method of designing the rebalance control system for a dry-tuned gyro using a digital controller is described. Control functions are synthesised to achieve a minimum settling time in the transient response. The need to eliminate interactions between the gyro control axes is included in the design policy.

Sensitivity of the system to mismatch in the controller is examined and methods suggested to improve this aspect of performance. Sensitivity analysis is exploited to show how coefficients in the control functions may be rounded to allow reduced computation time in the processor.

Keywords. Gyroscopes, torque control, inertial navigation, strapdown systems, direct digital control, control system synthesis, sensitivity analysis.

INTRODUCTION

The development of the dry-tuned gyro (Craig, 1972) has contributed significantly to the growing interest in strap-down navigational systems (Kirk, 1978). This has been primarily due to the need to achieve a wide dynamic range in a sensor which is subjected to the vehicle motion rather than that of a stabilized platform. The dry-tuned gyro has been developed to a point where this requirement can be met with inertial grade accuracy and at reduced cost.

This gyro, with two degrees of freedom, has a rotor and gimbal assembly suspended on springs. At the designed rotation speed the spring constants are matched to cancel the inertial torques due to the gimbals, so that the rotor behaves as a free gyro.

Operation as a rate gyro is achieved by forming a position control system to align the rotor with the external case. Signals from position pick-offs are used to provide feedback control of the torques applied to the rotor. The precession rates are then measured by signals derived from the torque motor currents. Two of these 'rebalance' control loops are required for each gyro. In the strap-down mode the rebalance loops must be designed for adequate dynamic response i.e. a short transient settling time or a wide frequency response bandwidth. When the system bandwidth approaches the nutational frequency of the gyro it is no longer possible to regard the two rebalance loops as independent systems. Interaction between the control axes demands that the system must be analysed as a multivariable control system. The design must also aim to counteract the inherent interaction.

Rebalance systems have been constructed using analogue control methods (Coffman, 1974; Blalock, 1975; Kirk, 1978) where the requirement for rapid response has not been important. However, in exploiting the maximum capability of the gyro there are advantages in using a digital processor as controller. A digital controller offers the advantages of flexibility in the realization of complex control algorithms and their accurate implementation is not affected by component tolerances. The possibility of time-sharing the processor between separate control functions can also reduce hardware requirements. Where rapid response is required the signal sampling rate must be high and for time-sharing to be possible it is important to minimise the computation time required for any control algorithm.

In this paper an analytical method is outlined which has been used to design control algorithms giving a minimum settling time and eliminating interaction. The design method is adapted to take account of sensitivity to parameter variations and methods of reducing sensitivity are introduced. Sensitivity reduction is examined as part of a policy for simplifying control functions to reduce computation time.

GYRO TRANSFER FUNCTIONS

Differential equations for the dry-tuned gyro were developed by Bortz (1972), Craig (1972) and Coffman (1974) and may be written as

$$I\ddot{\theta}_x + H\dot{\theta}_y = M_x$$
$$I\ddot{\theta}_y - H\dot{\theta}_x = M_y \qquad (1)$$

where θ_x and θ_y one angles of rotation of the rotor about orthogonal control axes on which torques M_x and M_y are applied. Inertial constant $I = I_{xr} + I_{xg}/2$, involving the principal moments of inertia of the rotor and gimbal, and H is the rotor angular momentum. Laplace transformation of these equations gives a transfer function matrix equation,

$$\begin{bmatrix}\bar{\theta}_x\\ \bar{\theta}_y\end{bmatrix} = \begin{bmatrix}h_1 & -h_2\\ h_2 & h_1\end{bmatrix}\begin{bmatrix}\bar{Q}_x\\ \bar{Q}_y\end{bmatrix} \qquad (2)$$

$h_1 = 1/(s^2+\omega_n^2)$ $h_2 = \omega_n/s(s^2+\omega_n^2)$

with $\omega_n = H/I$ being the nutational frequency in radian/sec. and $\bar{Q}_x$, $\bar{Q}_y$ the normalised torques $\bar{M}_x/I$, $\bar{M}_y/I$.

The transfer functions in Equation (2) may be identified with a block diagram structure for the complete rebalance system as shown in Fig. 1. Elements V represent zero-order hold functions. Synchronous sampling switches indicate the effect of analogue-digital signal conversions and it is assumed at this stage that there is negligible computing delay in the digital processor. Inputs r_1 and r_2 represent the case position angles to which the system responds.

Analysis of the system proceeds by calculating the pulse transfer function of the gyro P(z), using a standard z-transform table (Kuo,1963) which gives

$$P(z) = K_1\begin{bmatrix}N_1 & -N_2\\ N_2 & N_1\end{bmatrix} \qquad (3)$$

$$K_1 = \frac{T(1-\beta/T\omega_n)z^{-1}}{\omega_n(1-z^{-1})(1-2\alpha z^{-1}+z^{-2})}$$

$$N_1 = c(1-z^{-2})$$

$$N_2 = 1+dz^{-1}+z^{-2}$$

with $\alpha = \cos(\omega_n T)$

$$\beta = \sin(\omega_n T)$$

$$c = (1-\alpha)/T\omega_n(1-\beta/T\omega_n)$$

$$d = -2(\alpha-\beta/T\omega_n)/(1-\beta/T\omega_n)$$

ANALYTICAL DESIGN METHOD

In the rebalance control system Fig. 1 pulse transfer function matrices P(z) and D(z) are used to describe the gyro and the digital controller. The error pulse sequence E(z) is given by

$$E = [I + PD]^{-1} R \qquad (4)$$

where I is a unit matrix and R is a column matrix containing the two case position angles r_1 and r_2. Also the overall closed-loop response is given by

$$C = [I + PD]^{-1} PDR \qquad (5)$$

For non-interacting response a case movement in one axis should produce a rotor movement in the corresponding axis and have no effect on the other axis. We therefore designate

$$[I + PD]^{-1} PD = I f(z) \qquad (6)$$

where f(z) is the common closed-loop pulse transfer function of the two loops. Similarly the error pulse transfer function is

$$[I + PD]^{-1} = I\phi(z) \qquad (7)$$

given f and ϕ we may compute the control function D from

$$D = \frac{f}{\phi}P^{-1} = \frac{f}{\phi}\left|\frac{\mathrm{adj}(P)}{\det(P)}\right| \qquad (8)$$

Also f and ϕ must satisfy

$$f = 1 - \phi \qquad (9)$$

The result in Equation (8) is equivalent to that used in the design of single-loop systems for minimum settling time (Bertram, 1956) and has been examined in detail by Steel and Puri (1979) for the general class of multivariable systems.

The functions f and ϕ are first of all constrained by the requirement for zero steady state error. This means that ϕ must contain a factor $(1-z^{-1})^n$, where n depends on the form of input R e.g. n=1 for a step input. Further constraints are due to the need to avoid some cancellations between elements of P and D in the product PD which gives the open loop transfer function matrix. Cancellations on or outside the unit circle in the z-plane can lead to a sensitive design in which instability will result from a small mismatch in the controller. General rules for avoiding such sensitivity problems can be drawn up assuming an aribitrary form of matrix P. However in the case of the dry-tune gyro the inherent symmetry of the dynamic structure leads to some simplifications. All four elements of P have a common pole pair on the unit circle due to the undamped oscillatory characteristic of the gyro. This would normally lead to a pair of zeros in each element of P^{-1} at a corresponding position. But as a result of the cross coupling between the gyro axes det(P) is zero at this same point in the z-plane, so that the mode is cancelled from P^{-1}.

Apart from the zeros of det(P) which coincide with the poles of the elements of P there are two other zeros which appear on the unit circle. To avoid sensitivity this pair of zeros must be allocated to f.

The general method proceeds by designating

$$f = z^{-1} A(z)\ B(z)$$
$$\phi = (1-z^{-1})^n\ C(z) \qquad (10)$$

where A contains the zeros of det(P) on the unit circle. Polynomials B and C have undetermined coefficients as necessary to allow Equation (9) to be satisfied.

Class I and Class II Systems

A class I system is defined as having a finite steady state error in response to a constant rate input change. The class II system gives zero error with a constant rate input and a finite error with a constant acceleration input. Steady state alignment is clearly best in the class II design but comparisons have been made (Coffman, 1974) which show that provided the steady state error is kept small the class I design may be acceptable. It is interesting to compare the design results with the minimum settling time digital controller. The class I design is achieved by designating n = 1 in Equation (10) and the class II with n = 2.

The following general features emerge,

(i) Settling time following a step input change
class I: 3T
class II: 4T

(ii) Steady state error
class I: $2T\rho$ (ρ = rate input)
class II: zero

(iii) Controller complexity
An extra term is required in the enumerator and denominator polynomials for the class II case.

Further comparisons can be made by taking a specific gyro as an example.

Design Example

Nutational frequency: 480 Hz = f_n
Sampling frequency: 5 f_n

$$P = K_1\begin{bmatrix} N_1 & -N_2 \\ N_2 & N_1 \end{bmatrix};\ P^{-1} = K_2\begin{bmatrix} N_1 & N_2 \\ -N_2 & N_1 \end{bmatrix} \qquad (11)$$

$$N_1 = 2.254\ (1-z^{-2})$$
$$N_2 = 1+3.671\ z^{-1} + z^{-2}$$
$$K_1 = z^{-1}/29.66\text{x}10^6(1-z^{-1})(1-0.6z^{-1}+z^{-2})$$
$$K_2 = 4.879\text{x}10^6(1-z^{-1})/z^{-1}(1+1.825\ z^{-1}+ z^{-2})$$

The term $(1+1.825\ z^{-1} + z^{-2})$ in K_2 has zeros on the unit circle and this is assigned to f in Equation (8). The resulting design factors are given in Table 1.

This comparison leads to the conclusion that the class I design is preferred since it offers reduced complexity in the control functions and lower torque demands while the steady state error is acceptably small.

Retarded Response

The above design procedure yields a system response which settles in a minimum and finite number of sample intervals. It will be shown later that some advantage in reduced sensitivity can be gained by removing the requirement of a finite settling time. If a term $(1-\gamma\ z^{-1})$ is allocated as the denominator of f the overall system response will include a mode having a time constant τ such that $e^{-T/\tau} = \gamma$ and will settle exponentially. The effect of this on the design procedure is firstly that the same

Table 1 System Design Factors

Design Factor		Class I	Class II
f(z)		$0.2614\ z^{-1}(1+1.825\ z^{-1}+z^{-2})$	$0.7843\ z^{-1}(1+1.825\ z^{-1} + z^{-2})(1-0.667\ z^{-1})$
ϕ(z)		$(1-z^{-1})(1+0.7387\ z^{-1}+0.2614z^{-2})$	$(1-z^{-1})^2(1+1.2157\ z^{-1}-0.5229\ z^{-2})$
D(z)‡	k_3	$\dfrac{1.276\text{x}10^6}{(1+0.7386\ z^{-1}+0.2614\ z^{-2})}$	$\dfrac{3.827\text{x}10^6(1-0.6667\ z^{-1})}{(1-z^{-1})(1+1.2157\ z^{-1}- 0.5229\ z^{-2})}$
	d_1	$2.254(1-z^{-2})$	$2.254(1-z^{-2})$
	d_2	$1+3.671\ z^{-1}+ z^{-2}$	$(1+3.671\ z^{-1} + z^{-2})$
Steady state error x		0.08°	zero
Maximum Torque*		1.2Mo	2.5 Mo

‡ Control function $D = k_3 \begin{bmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{bmatrix}$

x For 100°/sec input rate

* For constant rate input, where Mo is steady state torque

denominator must be allocated to ϕ in consequence of equation (9). Terms introduced in this way cancel in Equation (8) and so do not appear directly in the control functions. The general complexity of the elements in D remains unaltered but coefficients are modified due to changes in the initially undetermined coefficients in f and ϕ.

SENSITIVITY CONSIDERATIONS

Systems designed for minimum settling time are potentially sensitive to parameter variations (Stanley, 1959). This is in part due to the cancellations generated between the controller and plant transfer functions. Also the design for finite settling time produces multiple poles in the closed loop response at the origin of the z-plane which represents a sensitive condition. Sensitivity must therefore be examined carefully to ensure that the design will remain satisfactory over a range of parameter changes.

In the case of the rebalance control loops the gyro is designed to have an accurately reproducible dynamic characteristic and wide linear range of operation. The pick-off gain is however, one of the less consistent features. This, and the possibility of rounding off coefficients in the digital processor algorithm, will be considered further.

Sensitivity will be studied with reference to the movement of the poles of the closed loop transfer function. It can be shown that the pole positions correspond to the zeros of det(I + PD).

Control Loop Gain Variations

Uncertainty in the pick-off gain, in one or both loops, which represents a fractional change δ in the loop gain, leads to the result that the zeros of det(I + PD) are given by

$$1 + f(z)\ \delta = 0 \qquad (12)$$

The movement of the zeros as a function of δ may be investigated by root-locus solution of this equation. When f(z) is a polynomial in z^{-1}, as is the case with a finite settling time design, all its poles are at the origin. The root loci move out from the origin to terminate on the zeros of f(z) or at infinity. Stability limits are reached when δ is large enough to place roots on the unit circle. For the class I system design example the limiting of value of δ is 2.1. Such large variation will not happen in practice, and it is more meaningful to examine the effect of small variations on the closed-loop response by simulation. Figure 2 shows the results of a 10% increase in gain on the error response following a unit step change of case position.

Controller Mismatch

The general form of the control function D is given by equation (8) and when P has the structure given in equation (11) D becomes

$$D = \frac{f}{\phi K_1 (N_1^2 + N_2^2)} \begin{bmatrix} N_1 & N_2 \\ -N_2 & N_1 \end{bmatrix} \qquad (13)$$

Cancellations are formed in this expression so that the common factor simplifies. For example in the class I design it becomes $\alpha \frac{(1-z^{-1})}{\phi}$ where α is a gain factor. We may consider first the possible mismatch in this common factor in realizing D. If $\eta(z)$ is a small change added to ϕ the zeros of det(I+PD) are given by

$$1 - \frac{f}{\phi}\eta = 0 \qquad (14)$$

When the zeros of ϕ are well inside the unit circle this is not a condition in which root locations are sensitive. The coefficients in the common denominator of D may therefore be rounded without undue detriment to the overall response. In the design example for the class I system the denomination of K_3 given in Table 1 may be rounded to $(1 + 0.75\ z^{-1} + 0.25\ z^{-2})$ with negligible effect on the dynamic response.

Mismatch in the numerator polynomials N_1 and N_2 raises a special sensitivity problem in the case of the gyro rebalance system.

If $\Delta_1(z)$ and $\Delta_2(z)$ are small changes added to N_1 and N_2 the zeros of det(I + PD) are given by

$$\frac{1}{\phi^2}\left[1 + \frac{2f}{(N_1^2 + N_2^2)}(N_1\Delta_1 + N_2\Delta_2)\right] = 0 \qquad (15)$$

The zeros of $(N_1^2 + N_2^2)$ give pole positions at which root locus branches emerge for $\Delta_1 = \Delta_2 = 0$. In the dry-tuned gyro there are two pairs of such zeros located on the unit circle. One pair is cancelled however by the corresponding zero assigned to f. The remaining pair occur at the nutational frequency and it is significant that root loci from these zeros will enter the region outside the unit circle. When the zeros of equation (15) fall outside the unit circle the system will be unstable. This can be avoided by matching the changes Δ_1 and Δ_2 so that

$$\frac{\Delta_2}{\Delta_1} = -\frac{N_1}{N_2} \qquad (16)$$

in the region of the z-plane close to the zeros of $(N_1^2 + N_2^2)$ at frequency ω_n. For the gyro $N_1/N_2 = \pm j$ at $z = \exp(j\omega_n T)$.

The polynomials N_1 and N_2 given in the example equation (11) indicate a need to consider rounding the coefficient 3.671 in N_2 with a change $\Delta_2 = \delta_2 z^{-1}$. If a change is also made in the gain coefficient

2.254 of N_1 with $\Delta_1=\delta_1(1-z^{-2})$ then at $z=\exp(j\omega_n T)$

$$\frac{\Delta_2}{\Delta_1} = -\frac{\delta_2}{\delta_1}\left(\frac{z^{-1}}{1-z^{-2}}\right) = \frac{\delta_2}{\delta_1}\left[\frac{-1}{2j\sin(\omega_n T)}\right] \quad (17)$$

Hence Eq.(16) is satisfied when

$$\delta_2/\delta_1 = 2\sin(\omega_n T) \quad (18)$$

and the coefficient changes will cancel each-other. This relationship assumes small changes are made and where larger changes are contemplated further compensation may be necessary to prevent unsatisfactory performance. This can be achieved by retarding the response.

Effect of Retarded Response

The retardation of the response by adding a pole to f and ϕ has been described earlier and it can now be shown that this results in a reduced sensitivity to parameter variations.

For loop gain variations Eq.(12) applies and the added pole in f replaces one of the poles at the origin. The reduced multiplicity of the pole leads to reduced sensitivity. (Kuo, 1963). In the design example a pole at $\gamma=0.6$ increases the stability margin from $\delta=2.1$ to 4.0. The reduced sensitivity is evident from the response graphs Fig. 2 where a 10% change of loop gain has been introduced.

Variation in the numerator polynomials of D results in root locations given by Eq.(15). When a pole is introduced in f its effect is to alter the angle of departure of the root loci; the locus from $z=\exp(j\omega_n T)$ is of particular concern. By chosing the pole position the root locus can be set tangential to the unit circle so that a residual mismatch in changes Δ_1, Δ_2 will have minimum effect on stability. In the design example the locus alignment is achieved with $\gamma=0.6$, which corresponds to a retardation with a mode having a time constant of approximately twice the sample interval.

The response graphs Fig. 3 indicate in (A) the result of such a compensating adjustment in which 3.671 in d_2 is rounded to 4.0; this is compensated by a change in the gain factor 2.254 in d_1 as required by Eq.(18). The response (B) shows the further effect of including the retardation factor. This response represents the overall compromise between response speed, sensitivity and controller complexity. The resulting frequency response bandwidth is approximately half the nutational frequency of the gyro.

Interaction Sensitivity

It is important to observe how the changes of loop gain and coefficients affect the interaction between control axes. Interactions are cancelled when

$$\frac{d_2}{d_1} = \frac{N_2}{N_1} \quad (19)$$

Changes in the common multiplying factor K_3 in D do not affect this relationship so that interaction is not introduced by loop gain changes, or by changes in the denominator polynomial.

Changes in the numerator polynomials have been constrained by Eq.(16). In the gyro equations, evaluated at the nutational frequency, $N_1/N_2=-N_2/N_1$ so that the condition for no interaction is the same as that for cancellation of coefficient changes in this particular case. Some interaction will appear at other frequencies but it is important that it is cancelled at the nutational frequency where it has maximum effect.

FURTHER DEVELOPMENTS

Noise Filters

In some dry-tuned gyro designs problems have emerged due to pick-off noise at the spin frequency or its harmonics. Noise components at such a discrete frequency can be removed by a notch filter. This may be implemented in analogue or digital form. For the digital filter the algorithm,

$$g(z)=g_1\frac{(1+g_2 z^{-1}+z^{-2})}{(1-g_3 z^{-1})^2}$$

$$\begin{aligned} g_1 &= (\Omega^2+1)/(\Omega+1)^2 \quad (20)\\ g_2 &= 2(\Omega^2-1)/(\Omega^2+1)\\ g_3 &= (1-\Omega)/(1+\Omega)\\ \Omega &= \tan(\omega_o T/2) \end{aligned}$$

ω_o is the notch frequency.

gives zero transmission at the notch frequency and unity gain at high and low frequencies.

The deisgn procedure is applied with P^{-1} replaced by $\frac{1}{g}P^{-1}$ in Eq.(8). The numerator polynomial of g is assigned to f and the denominator to ϕ. This results in the minimum settling time being increased from 3 to 7 sample intervals in the class I design. Also there is an increase in the controller complexity with fourth order polynomials in the numerator and denominator. Sensitivity to parameter changes is increased. The same problem arises when the corresponding analogue filter is implemented before the analogue to digital converter.

A compromise is reached if a non-recursive digital filter is employed. This provides a pair of zeros at the notch frequency which are also allocated to f. The result is a minimum settling time of 5 sample intervals and controller functions with second order numerator and fourth order denominator polynomials.

In all these cases the overall bandwidth must be less than the notch frequency. This means that the attainment of adequate response speed for strap-down applications is considerably impaired by the need for such filtering.

Processing Delay

The execution time of the digital processor implementing the control function may well amount to a significant fraction of the sample interval. This results in an additional delay in the rebalance loops.

The analytical design method outlined here may be implemented with the modified, or delayed, z-transform functions incorporating the processing delay. This results in coefficient changes but no general change in controller complexity. The system response is delayed by the corresponding processing time delay.

IMPLEMENTATION OF DIGITAL CONTROL

The design procedure yields a control algorithm in the form of matrix D. Implementation of the four transfer functions involved in a digital processor involves a combination of multiplication and addition operations. It is important to ensure that the processing of each new pair of error samples, together with their conversion from analogue to digital form, can be completed in less than the sample interval. Timing calculations show that with a sampling frequency as high as 2.4kHz there is a need to select the processor carefully to meet this requirement, particularly where a digital noise filter is to be included.

Multiplication of floating point numbers is potentially the most time consuming arithmetic operation involved. There may be a need to involve a separate hardware multiplier to overcome the slow operation of software multiplication. Limitations on the choice of processor are considerably relaxed if floating point multiplication can be avoided as far as possible. One means of achieving this is to round off coefficients to values represented by simple binary operations. In this way whole word multiplications are replaced by a small number of quicker shift and add operations.

The design method has been developed with this possibility in mind. It provides a means of rounding coefficients without adverse effect on the system performance. For example it has been shown that the polynomial $(1+3.671\ z^{-1}+z^{-2})$ may be rounded to $(1+4\ z^{-1}+z^{-2})$ and also $(1+0.7386\ z^{-1} + 0.2614\ z^{-2})$ is changed to $(1+0.75\ z^{-1} + 0.25\ z^{-2})$. Multiplications involving the new coefficients 4, 0.75 and 0.25 may be implemented by shift and add operations with a significant saving in processing time.

CONCLUSION

The method of design proposed here has the advantage that it exploits direct analysis in the z-plane and takes account of the behaviour of the rebalance system of a dry-tuned gyro as a multivariable control system. The chief advantage of this synthesis technique is that it guarantees the best possible dynamic performance and eliminates interaction. Achievement of such ideal performance can lead to problems of sensitivity to parameter variations. Where it is desirable to round off the coefficients used in the digital control algorithm such sensitivity is a disadvantage. A solution to this problem has been developed for the special case of the gyro rebalance control loops. This involves a differential adjustment of coefficients on the one hand and an optimal reduction of sensitivity by retardation of the dynamic response on the other.

The design method may be implemented when filters are included to remove pick-off noise and may be adapted to take account of finite processing time in the digital controller.

The possible advantages of rounding coefficients in the control algorithm to reduce the computation time have been identified. This will ultimately allow a more flexible choice of processor for implementation of the controller.

The synthesis assumes linearity of the control elements which is appropriate to the dry-tuned gyro designed for strap-down applications where linear torquing characteristics over a wide range are an inherent requirement.

REFERENCES

Bertram, J.E. (1956). Factors in the design of digital controllers for sampled data feedback systems. Trans. AIEE, 75, pt II, 151-159.

Blalock, T.V., E.J. Kennedy and V.A.Thomason. Development of a digital electronic rebalance loop for a dry-tuned rotor two-degree-of freedom gyroscope. Tech. Rep. TR-EE/EL-5. University of Tennessee, September 1975.

Bortz, J.E. Dynamic errors in a tuned flexure mounted strapdown gyro. NASA CR-124161, Jan-July 1972, 62p.

Coffman, D.E. Feasibility study of a digital rebalance loop for a dry-tuned TDF gyro, NASA CR-144089. May 1974, 81p.

Craig, R.J.G. Dynamically tuned gyros in strapdown systems. Agard Conf. proceedings, No. 116, October 1972.

Kirk, C.W. (1978). An integrated strapdown guidance and control system for launch vehicle applications. Presented at the Symposium on gyro technology, Bochum, W. Germany, September 1978.

Kuo, B.C. Analysis and synthesis of sampled-data control systems. Englewood Cliffs, N.J. Prentice-Hall, 1963.
Steel, G.K. and S.N. Puri (1979). Minimal response time design for linear multivariable sampled-data control systems. To be published.
Stanley, F.S. (1959). Application of continuous system design concepts to the design of sampled-data control systems. Trans. AIEE, Pt. II, 74-78.

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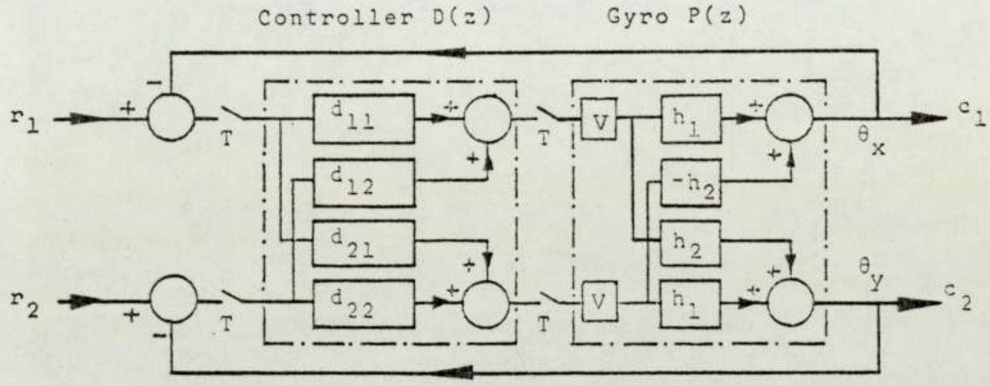

Fig. 1 Rebalance control system

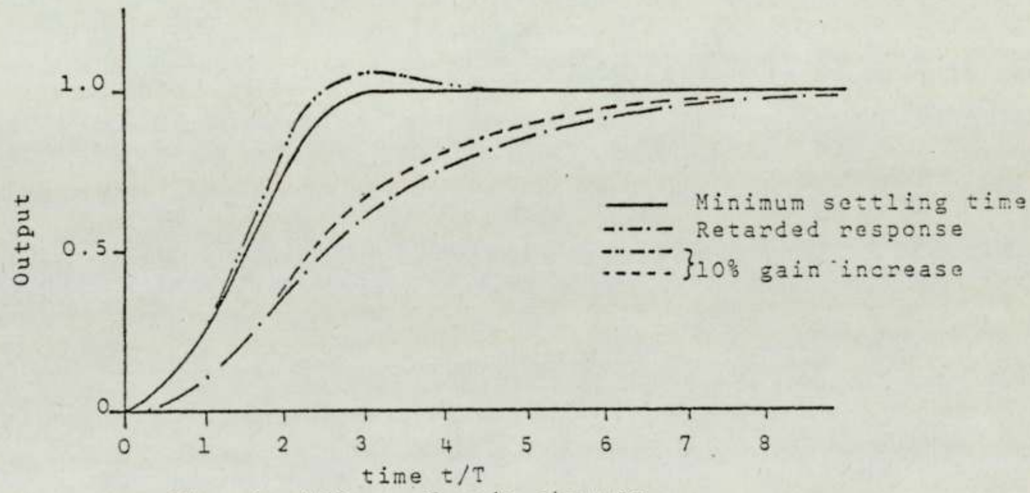

Fig. 2 Effect of gain changes

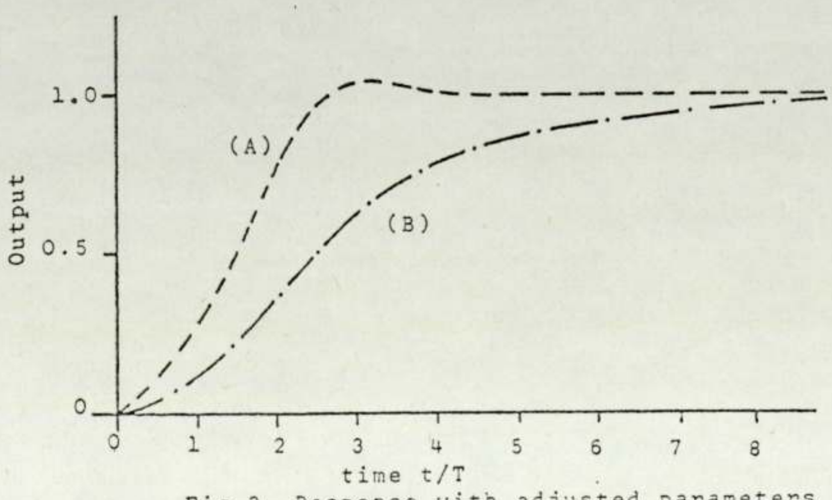

Fig.3 Response with adjusted parameters

Minimal Response Time Design for Linear Multivariable Sampled-data Control Systems.

G.K. Steel Ph.D., D.I.C., M.Inst.M.C., C.Eng., M.I.E.E..

S.N. Puri B.E., M.Tech.

Department of Electrical and Electronic Engineering
The University of Aston in Birmingham
Birmingham
UK

Summary.

The method of minimum, finite settling time design used for single-loop systems is extended to cover multivariable systems. Sensitivity analysis is applied and results of considerable generality are derived which indicate the effects of gain changes and pole or zero movement in the plant transfer functions. A design policy is developed which satisfies the need for low sensitivity in implementation.

Further development evolves a design technique in which speed of response can be systematically traded for sensitivity. At the same time the technique satisfies the need to achieve a digital control algorithm of minimal complexity.

The method of design has been shown to be particularly relevant to systems with lightly damped open-loop modes and an example of this type of system is analysed in detail.

List of Principal Symbols

C, C(z)	Matrix of system output transforms.
D, D(z)	Controller transfer function matrix.
E, E(z)	Matrix of system error transforms.
P, P(z)	Plant transfer function matrix.
f, f(z)	Closed-loop transfer function.
ϕ, $\phi(z)$	Closed-loop error transfer function.
f_{ii}, ϕ_{ii}	Functions f and ϕ for loop i.
f_o, ϕ_o	Minimal polynomials in f and ϕ.
p_{ij}	Element of P.
Δ_{ij}	Cofactor of an element of P.
det(P)	Determinant of P.
s	Laplace transform variable.
z	z-transform variable.
α, α^*	Position of open-loop complex pole pair.
β, β^*	Position of open-loop complex zero pair.
λ	General variational factor.
ρ	Fractional gain change.
δ, δ^*	Displacement of open-loop poles.
η, η^*	Displacement of open-loop zeros.
α, α^*	Position of closed-loop complex pole pair.

1. Introduction.

The analytical technique of designing digital control algorithms for minimum system response time have been long established in the case of single-loop system[1,2]. Given a plant transfer function the analysis computes the pulse-transfer function of a digital controller which will cause the closed-loop system to respond in the minimum number of sample intervals. The solution depends on the form of test input considered, for example a step or ramp function, and the response is tuned to settle in a finite number of sample intervals without overshoot and with zero steady-state error.

The settling time attainable is limited chiefly by the incidence of unstable plant modes, that is poles of plant pulse-transfer function outside the unit circle in the z-plane. Also zeros in the same region have a similar effect. This is because cancellation of poles and zeros outside the unit circle by corresponding zeros and poles in the controller transfer function leads to a sensitive design condition in which mismatch between the controller and the plant will render the closed-loop system unstable. The result of avoiding such cancellation is to extend the settling time, but the response time remains at the minimum practicable value. The settling time may also be increased by the need to avoid unstable modes in the controller. These modes can appear even when a direct cancellation of a zero of the plant transfer function is not involved. It has been shown that by extending the settling time by one sample interval the increased design flexibility allows this to be avoided in at least one example[5].

With these refinements the design results in the ideal response for a given test input. However, as would be expected, the technique has limitations. Most significant among these is the need for accurately known dynamics, a dependence on linearity, potential complexity of the control function and the need for its accurate realization. These limitations arise chiefly from the fact that a finite settling time design is sensitive to parameter variations[11]. For these reasons the design method has not been widely adopted. The design method remains valuable as a means of identifying the ultimate performance capability of the system and may be necessary where a high speed of response is required.

In this paper the design method is extended to cover multi-variable control systems. It assumes feedback of the primary system output variables alone and is based on transfer function analysis. Other work[8,13] in this field has concentrated on state space analysis in which feedback of a complete state vector is a fundamental requirement. Nishida[9] has developed a design method applicable to this class of system based on single-loop design by Jury and Schroeder[6]. It is a transfer function method and is attractive where the requirement for rapid response is not important. However, the use of this method is restricted to systems where all poles and zeros are well inside the unit circle. When any element in the plant contains complex conjugate poles very near to or outisde the unit circle simple application of this method is not possible. The new method developed here overcomes these difficulties and is therefore particularly relevant for systems with lightly damped modes.

Section 2 outlines the main features of the procedure and emphasises those aspects which differ from single-loop system design. Section 3 develops a design procedure for the case when the settling time is not required to be finite. It is shown that such a retarded response offers considerable advantages when seeking a design with the minimum complexity of controller functions. Section 4 leads to a set of very general relationships governing the sensitivity of the system response to mismatch between the controller and the plant. These relationships are used to develop a technique which allows speed of response to be traded for sensitivity in a systematic analytical procedure. The paper concludes with a design example which shows how a design for minimum sensitivity can be achieved.

2. Basic Design Method.

The system configuration shown in FIG.(1) has a multivariable plant with an m x m pulse-transfer function matrix P(z) and m control loops are formed, each with unity feedback gain. The digital controller has the structure of an m x m array of pulse-transfer functions as represented in D(z).

The error response function is given by

$$E = [I + PD]^{-1} R \qquad (1)$$

from which we designate the system error transfer function matrix

$$W = [I + PD]^{-1} \qquad (2)$$

Also the overall closed-loop response function is defined by

$$C = [I + PD]^{-1} PDR \qquad (3)$$

and the closed-loop transfer function matrix is recognised as

$$I - W = [I + PD]^{-1} PD. \qquad (4)$$

A requirement for non-interacting response is that a test signal applied at any single input should produce a response at the corresponding output and have no effect on the other output points. This implies that I - W must be a diagonal matrix. Furthermore for simplicity at this stage it will be assumed that all loops are designed to have the same response* so that

$$I - W = fI \qquad (5)$$

where f(z) is the common closed-loop pulse-transfer function for

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* It will be shown later that this applies to all stable, minimum phase open-loop systems.

all the loops. Following from this we may define

$$W = \phi I \tag{6}$$

where $\phi(z)$ is the common error pulse-transfer function.

The design method proceeds, in a similar manner to that used in single-loop systems(1), by defining the required closed loop function f and computing the necessary control function matrix D. The relationship giving D follows from equation (4) as

$$D = \left(\frac{f}{\phi}\right) P^{-1} = \left(\frac{f}{\phi}\right) \left[\frac{\text{adj}(P)}{\det(P)}\right] \tag{7}$$

A minimum-prototype response function may be assigned to f depending on the form of input R to which the optimum response is required. For example a unit step input applied to any one of the loops leads to an error response $\phi(z)/(1 - z^{-1})$ in that loop and zero error in all others. The steady-state error will be zero provided that $\phi(z)$ contains a factor $(1 - z^{-1})$. Also we note that $f = 1 - \phi$ and when ϕ takes the minimal form $(1 - z^{-1})$, f becomes z^{-1} and the system response settles in one sample interval. Having assigned f and ϕ as compatible functions they may be substituted in equation (7) to obtain D. This basic method must be varied to take account of special conditions in the plant transfer function matrix P, these arise from

(i) Unstable open-loop modes.

(ii) Transfer function zeros outside the unit circle.

(iii) Zeros of det(P) outside the unit circle.

2.1 Unstable Elements.

2.1.1 Design Policy.

In single-loop system design instability of the plant, which places poles of the pulse-transfer function outside the unit circle, requires that corresponding zeros must be assigned to ϕ. If this is not observed zeros are placed in the controller transfer-function which cancel the poles in the plant transfer function. This results in a condition of high sensitivity where a small mismatch between the cancelling terms will lead to instability of the closed-loop system[7]. For similar reasons the cancellation of zeros of the plant transfer function outside the unit circle must be prevented as this leads to an unstable control function.

In the multivariable case it will be shown in Section 4.2 that unstable plant modes must be compensated by allocating corresponding zeros to ϕ. The further effect of this may be to produce unstable control elements which, while unacceptable in the single-loop case, is not always a sensitive condition in multivariable systems. One consequence of this may be that some loops will have a longer settling time than others.

We will first generalise by assuming that all loops do not necessarily have the same settling time.

The general definitions of matrices F and Φ are

$$\begin{aligned} \Phi &= (I + PD)^{-1} \\ F &= (I + PD)^{-1} PD \end{aligned} \qquad (8)$$

where F and Φ remain diagonal to achieve non-interacting control but the diagonal elements are not necessarily equal.

It follows that

$$D = P^{-1} \Phi^{-1} F \qquad (9)$$

and also that Φ^{-1} F, which gives the transmission matrix PD of the forward path, is a diagonal matrix with elements f_{ii}/ϕ_{ii}. The closed-loop natural modes are given by the zeros of det(I + PD) which leads to

$$\prod_{i=1}^{m} (1 + f_{ii}/\phi_{ii}) = 0 \qquad (10)$$

The zeros of each term in this product may be considered separately. We notice that f_{ii}/ϕ_{ii} arises from the matrix product PD with row i of P multiplied by column i of D. Now if any element in row i of P has a pole at $z = \alpha$ this will appear in f_{ii}/ϕ_{ii} unless it is cancelled by a zero in the appropriate element of column i of D. When $z = \alpha$ lies outside the unit circle this is a sensitive cancellation which must be avoided by allocating a zero at $z = \alpha$ in ϕ_{ii}.

The design policy may therefore be stated as follows:

Policy (1)

If one or more elements in row i of P are unstable and have a common pole at $z = \alpha$ on or outside the unit circle a zero must be placed in ϕ_{ii} at $z = \alpha$ as part of the design procedure.

2.1.2 Unstable Control Elements.

The effect of Policy (1) on the control functions D is seen in equation (9). In forming P^{-1} = adj(P)/det(P) with P an m x m matrix we assume that an arbitrary distribution of elements in P carry a simple pole at $z = \alpha$. The multiplicity of the terms $(z - \alpha)$ in P^{-1} is significant and may be assessed as follows:

In forming adj(P) the (m - 1) square minors of P are used to form cofactors. If the highest multiplicity of the pole in any cofactor is n, with $n \leqslant m - 1$, the multiplicity in det(P) is either n or n + 1. Therefore the elements of P^{-1} must be zero or finite at $z = \alpha$.

When a zero is placed in ϕ_{ii} at $z = \alpha$ the elements of column i of P^{-1} are divided by $(z - \alpha)$ in forming D from equation (9). It is significant that any element in column i of P^{-1} which is finite at $z = \alpha$ will lead to a corresponding element of D with a simple pole at this point. When α is outside the unit circle this control element will be unstable. Other elements of D will remain finite or zero at $z = \alpha$.

The presence of unstable control elements produced in this way is not a sensitive condition as complete cancellation between poles in P and zeros in D is prevented as long as $\phi_{ii} = 0$ at $z = \alpha$.

When the controller is implemented in a digital processor there is no difficulty in realizing poles outside the unit circle and a satisfactory design is possible.

2.2 Condition det(P) = 0 Outside the Unit Circle.

When the zeros of all the elements of P are inside the unit circle, i.e. they represent minimum phase transfer functions, poles are introduced into P^{-1} by the zeros of det(P). If these are transferred to D in equation (7) the control elements will be unstable and furthermore this will later be shown to give a sensitive design. Sensitivity can be avoided by allocating the zeros of det(P) to f; the design policy becomes,

Policy (2)

If det(P) = 0 on or outside the unit circle, and these points do not coincide with zeros of the elements of P, all such zeros of det(P) must be allocated to f as part of the design procedure.

Zeros outside the unit circle may appear in the elements of P and, when the same zero is common to several elements, det(P) can become zero at the same point. When these zeros are distributed so that det(P) is not zero at the same point there is no problem of sensitivity if f and ϕ are given their minimal form. In other cases cancellations occur in forming P^{-1} so that a simple zero in elements of P leads to at most a simple pole in the elements of P^{-1}. The design policy in this case can be stated as follows:

Policy (3)

When zeros of det(P) coincide with zeros of the separate elements of P, and lie on or outside the unit circle, corresponding zeros must be assigned to f as part of the design procedure. A simple zero in elements of P is accommodated with a simple zero in f.

One such case of particular interest arises when a zero is common to all elements of one row of P, due for example to an output component in one loop. When this occurs in row i it follows from the results of Section 2.1.1 that it is only necessary to assign a simple zero to f_{ii}. We therefore define a further design policy.

Policy (4)

When all elements of row i of P have a common zero at $z = \alpha$, on or outside the unit circle, a simple zero at $z = \alpha$ must be assigned to f_{ii} as part of the design procedure.

A further aspect is that it may be desirable to apply the above procedures to all the zeros of det(P), including those inside as well as outside the unit circle. This ensures that the output sequences from the controller are of finite duration. These sequences are given by P^{-1} FR which become finite in duration when all the poles of P^{-1} are cancelled in F. It should be noted however that in the multivariable case this can lead to a considerable extension of the settling time and an increased complexity in the controller.

2.3 Effect on Synthesis Procedure.

The need to allocate zeros to f and ϕ based on the unstable modes of P and the zeros of det(P) leads to a change in the computational technique. For example, if a zero is assigned to f at z = a we define

$$f = g\, z^{-1}\, (1 - a\, z^{-1}) \qquad (11)$$

and consequently ϕ must change so that $\phi = 1 - f$. We therefore define

$$\phi = (1 - z^{-1})(1 + b\, z^{-1}) \qquad (12)$$

with b an arbitrary coefficient. The values of b and g are found by matching ϕ to (1 - f) term by term.

When more than one zero is assigned to f the polynomial ϕ is extended so that the degree of z^{-1} in ϕ matches that in f. There are then sufficient undefined coefficients in ϕ for a solution to be possible. The final result is to extend the settling time by one sample interval for each zero allocated to f.

Similarly when zeros are allocated to ϕ extra terms must be placed in f with undefined coefficients. Again the settling time is extended.

This procedure is the same in principle as that used in single-loop systems.

2.4 Loops Having Different Settling Times.

It has been shown above that if an unstable mode is present in any elements of one row of P a cancelling zero need only be placed in ϕ_{ii} corresponding to row i. Similarly if a zero outside the unit circle is present in all elements in row i alone a cancelling zero is placed in f_{ii}. In either case the result is to extend the settling time of loop i leaving the other loops unaffected.

The designer can choose to make all loops have the same settling time, equal to the longest so obtained. To do this the same zeros can be allocated to all f_{ii} and ϕ_{ii} polynomials or an arbitrary choice of zeros may be made to bring all polynomials up to the same degree. This will seldom be of any advantage however as the result will be increased controller complexity and the sensitivity to gain changes will be increased.

It remains however that for stable open-loop systems having no transfer function zeros on or outside the unit circle all loops will have the same minimum settling time. Also when poles outside the unit circle appear in every row of P the same applies. And again, if a common zero on or outside the unit circle appears in every element in one column of P all loops will be affected. This happens in particular when an input actuator gives rise to the pole or zero.

Thus for the majority of cases of practical interest the design procedure will lead to all loops having identical closed loop response.

3. Retarded Response.

The design method outlined in Section 2 yields a response which settles in a finite number of sample intervals. This is the result of designating the closed-loop response function f as a polynomial in z^{-1}. It will be shown later that the requirement for a finite settling time may be relaxed with advantage in reduced sensitivity. If a pole on the positive real axis is introduced in f an exponential mode will appear in the response. This is referred to as "retarding" the response and, while the settling time is theoretically infinite, in practice the steady-state is adequately achieved in a finite number of sample intervals.

Alternatively a second order response component may be introduced by placing a complex pair of poles in f.

The analytical consequences of this policy are outlined as follows for the case where a single pole is introduced. We assume that the open-loop system is stable and write,

$$f = \frac{g\, f_o(z)}{1 - \gamma z^{-1}} \tag{13}$$

where f_o is a designated minimal polynomial of degree n in z^{-1} containing zeros of det(P). The value of γ fixes the position of the pole. Gain factor g takes the value $f_o(1)/(1 - \gamma)$ to satisfy the requirement that f must approach unity for z = 1, giving the steady-state conditions of zero error. It follows that

$$\phi = \frac{\phi_o}{1 - \gamma z^{-1}} \tag{14}$$

where ϕ_o is a polynomial of degree n, equal to that of f_o, with arbitrary coefficients (there being no zeros assigned to ϕ_o when the system is open-loop stable). Now it is required that $f = 1 - \phi$

so that

$$g\, f_o = (1 - \gamma\, z^{-1}) - \phi_o \tag{15}$$

the solution of this is find the n undetermined coefficients in ϕ_o now proceeds by equating coefficients with the same power of z^{-1}.

The above process is subject to some variation when the open-loop system is unstable. Zeros will be assigned to ϕ_o to coincide with poles of elements of P. Depending on how many zeros are so assigned it will be necessary to augment f_o with extra terms to provide the freedom of coefficient adjustment to satisfy equation (15).

It is important to note that in either case the inclusion of a simple pole in f_o and ϕ_o will not affect the degree of the numerator polynomials. Hence the complexity of the control algorithms in D is unchanged when the response is retarded in this way, only coefficient values are changed.

The above procedure can be extended to allow more than one pole in f and ϕ. This can be done without changing the complexity of the control algorithm provided that the total number of poles does not exceed the degree of the polynomials in f_o and ϕ_o. In practice it will be seldom necessary to use more than two extra poles. It will be shown later that the use of a complex pair of poles, giving a damped oscillatory mode in the response, offers advantages in sensitivity reduction compared with a simple pole.

4. Generalised Sensitivity Criteria.

The sensitivity of the closed-loop response to parameter variations can be judged by observing the change in the natural modes. In a multivariable system these are identified by the roots of the closed-loop characteristic polynomial $\det(I + PD) = 0$.

If P is subject to variations we write $P_1 = P + H$ where H is the deviation matrix such that element h_{ij} contains the change in p_{ij}. The zeros of $\det(I + P_1D)$ are found from

$$\det(I + fHP^{-1}) = 0 \qquad (16)$$

since $I + PD = \phi^{-1}$, and $D = P^{-1}f\phi^{-1}$. It is assumed that the zeros of equation (16) do not coincide with the roots of ϕ.

The changes anticipated in P may be identified as simple gain factor changes or in movements of the poles and zeros of individual elements.

A gain change in element p_{ij} is represented as $p'_{ij} = (1 + \rho)p_{ij}$ where ρ is the fractional change in gain. Hence

$$h_{ij} = \rho p_{ij} \qquad (17)$$

A pole or zero movement is most significant when close to the unit circle. We therefore consider the case where a complex pair of poles or zeros change position.

When p_{ij} has complex poles at α, α^*; $p_{ij} = q_{ij}/(z - \alpha)(z - \alpha^*)$ and allowing changes δ and δ^* in α and α^* respectively we have

$$h_{ij} = p_{ij} \frac{(\delta + \delta^*)[z - (\delta\alpha^* + \delta^*\alpha)/(\delta + \delta^*)]}{[z - (\alpha + \delta)][z - (\alpha^* + \delta^*)]} \qquad (18)$$

for small δ and δ^*.

Similarly complex zeros in p_{ij} are represented as

$p_{ij} = q_{ij}(z - \beta)(z - \beta^*)$. Changes η and η^* added to β and β^* then give

$$h_{ij} = p_{ij} \frac{(\eta + \eta^*)\left[z - (\eta\beta^* + \eta^*\beta)/(\eta + \eta^*)\right]}{(z - \beta)(z - \beta^*)} \qquad (19)$$

for small η and η^*.

The three conditions derived above in equations (17), (18) and (19) all have the general form $h_{ij} = p_{ij}\,\lambda$ where λ contains all the variation terms. When several elements in P change in the same manner it is useful to write $H = \lambda K$ where λ is a scalar multiplier and matrix K has $k_{ij} = p_{ij}$ or zero. The element is zero in K when no change is present in p_{ij}.

Equation (16) is now modified to

$$\det(I + f\,\lambda\,K\,P^{-1}) = 0 \qquad (20)$$

and the detailed implications of this can be worked out for each form of variation given in equations (17), (18) and (19).

4.1 Gain Variations.

Two cases of gain variation are considered; firstly where the change affects one element of the plant transfer function matrix and secondly, where there is a systematic change of several elements due to actuator or transducer gain changes.

4.1.1 Change of One Element.

When element p_{ij} is subject to a gain change, matrix K contains p_{ij} as its only non-zero element. On substituting this in equation (20), and noting that from equation (17) $\lambda = \rho$ we get

$$1 + \rho f \frac{p_{ij}}{\det(P)} \Delta_{ij} = 0 \qquad (21)$$

where Δ_{ij} is the cofactor of element p_{ij} in P. Now if det(P) = 0 outside the unit circle, and p_{ij} or Δ_{ij} are not zero at the same point, the root locus solution of equation (21) will indicate a root close to the zeros of det(P) for small variation ρ. The resulting closed loop system will be unstable. This can be avoided if zeros are assigned to f which cancel those of det(P) outside the unit circle. The result of this is stated in Policy (3) above.

This conclusion is modified if there are zeros of the elements p_{ij} or Δ_{ij} outside unit circle which coincide with those of det(P). We must observe how the multiplicity of the zero in p_{ij} Δ_{ij} relates to that in det(P). In forming det(P) summations of terms p_{ij} Δ_{ij} are used and the multiplicity of the zero of det(P) will equal the lowest multiplicity in any p_{ij} Δ_{ij} term. Therefore the ratio p_{ij} Δ_{ij}/det(P) will have no pole at the point in question.

We conclude that a coincidence of zeros of elements of P which makes det(P) = 0 outside the unit circle at the same point does not lead to sensitivity to gain change if f retains its minimal form.

4.1.2 <u>Actuator and Transducer Gain Changes.</u>

A change in transducer gain at output i gives a gain change in all elements of row i in P. Now matrix K will contain elements of P in row i and zero elsewhere. We may write K = SP where matrix S has element $s_{ii} = 1$ on the diagonal and all other elements are zero. Then the product K P^{-1} in equation (20) reduces to S so that we have

$$1 + f\lambda = 0 \qquad (22)$$

A change in actuator gain at input i is equivalent to a change in all elements in column i of P. Thus K will contain the elements of P in column i and zero elsewhere. We may write K = PS with $s_{ii} = 1$ and all other elements of S zero. On substitution into equation (20) the result again reduces to equation (22).

Design for minimum settling time results in f having the form

$$f = g \frac{\prod_{i=1}^{q} (z - a_i)}{z^{\ell}} \qquad q < \ell \qquad (23)$$

and from equation (17) $\lambda = \rho$ so that equation (22) becomes;

$$1 + \rho g \frac{\prod_{i=1}^{q} (-a_i)}{z^{\ell}} = 0 \qquad (24)$$

where for small ρ the solutions are assumed to be close to the origin in the z-plane. The requirement f = 1 for z = 1 fixes g in equation (23) and on substituting this in equation (24) we get

$$|z| = \left| \rho \prod_{i=1}^{q} \left(\frac{-a_i}{1 - a_i}\right) \right|^{\frac{1}{\ell}} \qquad (25)$$

giving for small variations, the magnitude of the displacement of the closed-loop modes away from the origin in the z-plane. This implies a generally sensitive situation since ℓ, being the degree of the polynomial in f serves to determine $|z|$ in terms of the ℓth root of

ρ. For small values of ρ the root is larger in magnitude than ρ and the value of $|z|$ increases substantially as ℓ increases, i.e. as the system settling time increases. Thus the sensitivity of the minimum settling time design is apparent as it places all the poles of f at the origin.

4.1.3 Effect of Retarded Response.

Sensitivity is reduced by retarding the response as described in Section 3. If a pole $(1 - \gamma z^{-1})$ is introduced in f and ϕ. Equation (25) is modified to

$$|z| = \left| \rho \left(\frac{1 - \gamma}{- \gamma}\right) \prod_{i=1}^{q} \left(\frac{-a_i}{1 - a_i}\right) \right|^{\frac{1}{\ell - 1}} \qquad (26)$$

Similarly when a complex pair of poles is introduced

$$|z| = \left| \rho \frac{(1 - \gamma)(1 - \gamma^*)}{\gamma\gamma^*} \prod_{i=1}^{q} \left(\frac{-a_i}{1 - a_i}\right) \right|^{\frac{1}{\ell - 2}} \qquad (27)$$

Equations (26) and (27) show significantly that the introduction of poles in f reduces the degree of the root ℓ to $\ell - 1$ and $\ell - 2$. This considerably reduces the value of $|z|$ when ρ is small.

A further reduction in $|z|$ can be obtained by choosing γ to reduce the factor $\frac{1 - \gamma}{\gamma}$ in equation (26). For this factor to be less than 1 we require $\gamma > 0.5$ and the sensitivity decreases uniformly as the response is retarded more severely with γ approaching 1.

Similarly the factor $(1 - \gamma)(1 - \gamma^*)/\gamma\gamma^*$ in equation (27)reduces the sensitivity when Re $\gamma > 0.5$.

The essential compromise between speed of response of the closed-loop system and sensitivity to gain changes is thus identified.

4.2 Movement of Poles.

The change h_{ij} given in equation (18) applies and is identified as equivalent to $h_{ij} = p_{ij} \lambda$ where λ contains the variation terms. In equation (20) matrix K contains each element p_{ij} involving the pole pair which are assumed to move and zero elements elsewhere. At points in the z-plane close to the pole at $z = \alpha$, $\lim_{z \to \alpha} [(z - \alpha) K] = (z - \alpha)P$ and therefore $\lim_{z \to \alpha} [KP^{-1}] = I$, so that equation (20) reduces to $1 + f\lambda = 0$ and on substituting for λ we get

$$1 + f(\delta + \delta^*) \frac{[z - (\delta\alpha^* + \alpha\delta^*)/(\delta + \delta^*)]}{[z - (\alpha + \delta)][z - (\alpha^* + \delta^*)]} = 0 \qquad (28)$$

valid for small δ. The displacement of the mode can be extracted and becomes

$$\begin{aligned} z - \alpha &= \delta(1 - f(\alpha)) = \delta\, \phi(\alpha) \\ z - \alpha^* &= \delta(1 - f(\alpha^*)) = \delta\, \phi(\alpha^*) \end{aligned} \qquad (29)$$

when $z - \alpha$ and $z - \alpha^*$ are small.

The significance of this is that the system is potentially sensitive to movement of the open-loop poles unless $\phi = 0$ at $z = \alpha$ and α^*. In the controller design equation (7) the poles of elements of P become zeros of D or are absorbed in ϕ. By allocating zeros to ϕ which match the open-loop poles of P we thus ensure that poles of the closed-loop response do not appear adjacent to the open-loop poles when a small change of pole position takes place. It also follows from equation (29) that simple zeros in ϕ at α, α^* are sufficient to compensate for any distribution of the unstable mode among elements of P. We must therefore assign zeros to ϕ corresponding to poles of P on or outside the unit circle to eliminate sensitivity in this critical region of the z-plane.

4.3 Movement of Zeros.

The change h_{ij} given in equation (19) is identified with $h_{ij} = p_{ij}\,\lambda$ where λ contains the variation terms due to η and η^*. In equation (20) matrix K contains elements p_{ij} involving the zero pair which have been assumed to move and zero elements otherwise. Thus at $z = \beta$ and β^*, K = 0. Evaluation of the term KP^{-1}, involved in equation (20), at $z = \beta$ and β^* depends on whether P^{-1} has poles at these points i.e. on whether det(P) has zeros there. Two cases arise.

4.3.1 Case of det(P) $\neq$ 0 at $z = \beta, \beta^*$.

This arises when the zeros $(z - \beta)(z - \beta^*)$ are distributed among the elements of P so that not all separate terms in det(P) contain at least one element with these zeros. In this case no element of P^{-1} has a pole at $z = \beta, \beta^*$ so that KP^{-1} is zero at these points and equation (20) shows that the closed loop system cannot have modes adjacent to β and β^* for changes η and η^* in the open loop zeros.

4.3.2 Case of det(P) = 0 at $z = \beta, \beta^*$.

If element p_{ij} contains a complex zero pair outside the unit circle and this is assumed to move the matrix K contains p_{ij} as its only non-zero element. Equation (20) then reduces to

$$1 + \lambda f \frac{p_{ij}\,\Delta_{ij}}{\det(P)} = 0 \qquad (30)$$

Now λ in this case has poles at β, β^* which cancel the zeros in p_{ij} we are therefore left to consider $\Delta_{ij}/\det(P)$. If the same zeros are present in other elements of P such that det(P) = 0 at β, β^* the cofactor Δ_{ij} may also be zero at the same points so that cancellations occur.

However the multiplicity of the zero in det(P) can at most be one greater than that in any cofactor so that Δ_{ij}/det(P) may have only a simple pole. In this case we must assign to f a simple zero at β, β^* to ensure that a solution of equation (30) does not exist close to these points in the z-plane.

Further insight is obtained by considering a movement of the zero pair taking place in several elements of P at once. For example when the zeros are common to all elements in one row or column of P. In practice this occurs when the zeros are associated with an actuator or transducer component. A similar situation has been considered in Section 4.1.2 where it was shown that the characteristic equation reduced to $1 + f\lambda = 0$. With the form of λ implied by equation (19) we have

$$1 + f(\eta + \eta^*) \left[\frac{z - (\eta\beta^* + \eta^*\beta)/(\eta + \eta^*)}{(z - \beta)(z - \beta^*)}\right] = 0 \qquad (31)$$

and when the displacement of the mode from β, β^* is small this reduces to

$$\begin{aligned} z - \beta &= \eta f(\beta) \\ z - \beta^* &= \eta^* f(\beta^*) \end{aligned} \qquad (32)$$

For zero sensitivity we then clearly require $f(\beta)$ and $f(\beta^*)$ to be zero and this is achieved by allocating zeros at β, β^* to f as part of the design procedure.

The overall conclusion becomes that a sensitivity problem only exists when the zeros are common to elements of P such that det(P) = 0 at the zeros. This is particularly significant when the zeros are close to or outside the unit circle. Sensitivity can be avoided by assigning simple zeros to f at the same point. This principle gives rise to Policy (3) stated above.

4.4 Sensitivity Compensation by Retarded Response.

In the previous section it has been shown that a pole of any element of P on or outside the unit circle must be taken into account by allocating a corresponding zero to ϕ. Also when det(P) has zeros on or outside the unit circle matching zeros must be allocated to f. Failure to observe this rule would create a closed loop system in which a small change in the pole or zero positions and gain would render the system unstable.

When the poles of P or zeros of det(P) fall close inside the unit circle it is not essential to make allocations to ϕ and f. But if this is not done the response will be sensitive to plant variation and may become unstable.

However allocating zeros to ϕ and f to remove sensitivity is not wholly advantageous. The settling time is extended by one sample interval for each extra zero placed in ϕ or f. Also the multiplicity of the poles of f at the origin is increased and this results in greater sensitivity to gain variations. Finally the controller complexity is increased as the elements of D contain polynomials of higher order.

These latter difficulties can be avoided by systematic use of the retarded response as described in Section 3. It was shown in Section 4.1 that the sensitivity to gain variations is improved by allocating poles to f and ϕ. Results will now be obtained to show that the same policy can reduce sensitivity to pole and zero movements and that in these cases optimum positions may be found for the poles allocated to f and ϕ.

4.5 Analytical Method of Sensitivity Compensation.

As an example of the technique for examining the effect of response retardation on sensitivity the method will be demonstrated for the case where a pair of complex conjugate poles are placed in f and ϕ. A similar more restricted method may be used when a single real pole is used but the complex pair are generally more effective.

For the case of an open-loop stable system we define

$$f = g\frac{z^2 \ f_o(z)}{(z - \gamma)(z - \gamma^*)} \qquad (31)$$

where $f_o(z)$ is the polynomial to which all zeros of det(P) on or outside the unit circle have been assigned*. For an open-loop stable system ϕ contains no allocated zeros and there are no undetermined coefficients in f_o. Hence we obtain

$$g = \frac{(1 - \gamma)(1 - \gamma^*)}{f_o(1)} \qquad (32)$$

on recognising that f = 1 at z = 1.

4.5.1 Movement of Poles.

When an open-loop pole pair at α, α^* move to $(\alpha + \delta), (\alpha^* + \delta^*)$ equation (29) gives the displacement of the closed-loop mode. This applies close to $z = \alpha$ and at this point equation (31), together with (32) gives

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* Having regard to cancellations due to zeros of the elements of P which coincide with zeros of det(P).

$$f(\alpha) = \alpha^2 \frac{(1 - \gamma)(1 - \gamma^*)}{(\alpha - \gamma)(\alpha - \gamma^*)} \frac{f_o(\alpha)}{f_o(1)} \qquad (33)$$

and equation (29) becomes

$$z - \alpha = \delta(1 - f(\alpha)) \qquad (34)$$

Now $f(\alpha)$ is a complex number depending on the value of the conjugate pole positions γ, γ^*. Some restrictions can be usefully placed on γ since this determines the form of closed-loop mode introduced to retard the response. If the damping ratio of the mode is specified it may be shown that values of γ must lie on the loci shown in FIG.(2).

Now $f(\alpha)$ can be evaluated using equation (33) for values of γ on one of these loci and a locus plotted as shown in FIG.(3).

It is immediately apparent that the vector $A \angle \theta$ in this diagram represents $(z - \alpha)/\delta$. Both the magnitude and angle of this vector are important. If $A < 1$ the distance $z - \alpha$ is less than the magnitude of open-loop pole displacement δ and based on this an optimum value of γ may be found which minimises the vector length A. This criterion is useful when the direction of the displacement δ is arbitrary.

If the direction of the change δ is defined the sensitivity to that particular change may be further adjusted by noting that the direction of movement $z - \alpha$ is that of δ rotated by angle θ. It may then be appropriate to choose γ so that the displacement $z - \alpha$ moves away from the unit circle or in the limit tangential to a circle through the pole.

If the minimum settling time response is used the movement $z - \alpha$ is obtained using $f(\alpha) = f_o(\alpha)/f_o(1)$ i.e. for $\gamma = 0$ in equation (33). The effectiveness of the retarded response in reducing the movement is obvious on comparison of this value with A obtained above.

4.5.2 Movement of Zeros.

When the zero pair β, β^* move to $(\beta + \eta), (\beta^* + \eta^*)$ the displacement of the closed-loop mode $(z - \beta)$ is given by equation (30). For values of z close to β we get

$$z - \beta = -\eta\ f(\beta) \qquad (35)$$

where, for an open-loop stable system $f(\beta)$ is found using equation (33) with β replacing α. The magnitude and angle of $f(\beta)$ then indicate directly the movement $z - \beta$ relative to the change η. A locus of $f(\beta)$ for values of γ as indicated in FIG.(3) will reveal an optimum choice which minimises the magnitude of $f(\beta)$ or gives an angle which directs $(z - \beta)$ in a preferred direction. A measure of the improvement in sensitivity obtained may be observed by comparison with the value $f_o(\beta)/f_o(1)$ which applies to the minimum settling time design.

The same basic analytical technique may be used when a simple pole is allocated to f. In detail the method is simplified by the fact that the pole may be restricted to lie on the positive real axis inside the unit circle. It can be shown that this necessarily limits the degree of sensitivity compensation which can be achieved i.e. the extent to which $f(\alpha)$ can be brought close to $1 + j0$ in FIG.(3). More flexibility exists with a complex pole pair.

5. Design Example.

The details of a design example using the method of Sections 4 and 5 are given in the Appendix. This example represents a two input/two output system with a common lightly damped resonant mode in all open-loop elements. It is shown that if the minimum settling time design is applied the result is a sensitive system which becomes unstable if the damping ratio of the open-loop mode is reduced. Also the response is considerably affected by a 10% change in loop gain and in particular the settling time is extended from 5 to 9 sample intervals.

The introduction of a retarded response with a damping ratio of 0.5 leads to an optimum choice of response which minimises the magnitude of pole movement under changes of open-loop mode damping. This is seen to considerably improve the sensitivity to damping changes.

It should be noted that the sensitivity reduction is achieved without any increase in the complexity of the control algorithms. The alternative procedure which is to assign the open-loop poles close to the unit circle to ϕ has the significant disadvantage that sensitivity reduction is only gained at the expense of controller complexity.

6. Conclusion.

The synthesis method developed here is an extension of the technique which has previously been used for single-loop systems. In the case of multivariable systems several distinctive problems emerge as a result of the interaction between the control loops.

The need to avoid sensitive design conditions, in which mismatch between the controller and the plant may produce instability, is an important consideration. This requires the removal of complete pole/zero cancellation between the controller and the plant when such poles and zeros are on or outside the unit circle in the z-plane. While this is simple to visualise in the single-loop case the multivariable case required more detailed study taking account of matrix manipulations of the transfer functions involved. It is significant to note that when some transfer function elements of the open-loop system are unstable the avoidance of sensitive cancellations can lead to a requirement for unstable control elements. Also, in contrast to the single-loop case, zeros of the open-loop elements on or outside the unit circle do not necessarily lead to a sensitive minimal design. The zeros of the determinant of the plant pulse transfer function matrix play a similar role in the multivariable problem to that of the zeros of the single-loop plant transfer function. Sensitivity problems arise when the zeros of the determinant lie close to or outside the unit circle.

Analysis of the sensitivity conditions has produced results of considerable general significance. The effect of gain changes in the plant and movements of its poles and zeros have been considered. These results show clearly how sensitivity can be avoided in the design process. However, the achievement of a minimal, finite settling time response in the closed-loop system can lead to a result which is sensitive to gain changes and complex in the control algorithms required.

It has been shown that by designing a closed-loop response which has poles as well as zeros in its transfer function the sensitivity can be improved. A method has been devised for analytically assigning such poles in order to minimise sensitivity. This leads to a design which offers the fastest possible response subject to sensitivity constraints and shows how speed of response can be traded for sensitivity. The further advantage of this procedure is that sensitivity is reduced without any increase in the complexity of the control algorithms required.

In the majority of cases the design process will result in all loops having the same closed-loop response. Exceptions to this can occur when the open-loop system has unstable elements or transmission zeros outside the unit circle.

The general scope of the application of this design method to multivariable systems is limited. First of all the analytical complexity increases rapidly with the number of loops involved. This may also lead to the need for elaborate control algorithms involving high order polynomials. The problem simplifies significantly when the transfer function elements have common poles and zeros. Systems such as missile flight control systems in which the dynamics are closely integrated exhibit this property. Such closely integrated dynamic systems frequently exhibit lightly damped modes and in this context sensitivity considerations and the method of sensitivity compensation developed here is most important. A further requirement is that the system must be linear over an adequate dynamic range and the system dynamics well defined. It is unlikely that many process control configurations will meet this requirement but useful results have been obtained in the case of gyroscope control systems where dynamic precision is a fundamental requirement[12]. With these limitations the method is useful in establishing ultimate performance capabilities against which sub-optimal designs may be judged.

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8. References.

1. Bergen, A.R., and Ragazzini, J.R.: 'Sampled-data processing techniques for feedback control systems', Trans. AIEE, 1954, 73, (2), pp. 237-247.

2. Bertram, J.E.: 'Factors in the design of digital controllers for sampled-data feedback systems', Trans. AIEE, Application and Industry, 1956, 75, (2), pp. 151-159.

3. CO2AEF, NAGLIB: 1271/728: MK5: October 1975,: 'Roots of real polynomial equation, using the method of Grant and Hitchins'.

4. Grant, J.A., and Hitchins, G.D.: ' Two algorithms for the solution of polynomial equations to limiting machine precision'. Computer Journal, 1975, 18, pp. 258-264.

5. Hyde, J.P.: 'On stable digital compensators for minimum settling time of sampled-data control systems', IEEE Trans., 1965, AC-10, pp. 115-117.

6. Jury, E.I., and Schroeder, W.: 'Discrete Compensation of sampled-data and continuous control systems', Trans. AIEE, 1957, 75, (2), pp. 317-325.

7. Kuo, B.C.: 'Analysis and synthesis of sampled-data control systems', (Englewood Cliffs, N.J., Prentice Hall, 1963).

8. Leden, B.: 'Output deadbeat control - a geometric approach', Int. J. of Control, 1977, pp. 493-507.

9. Nishida, F.: 'Synthesis of multivariable control systems by means of sample-data compensations', presented at the first

Congress of International Federation of Automatic Control', Moscow 1960.

10. SLAM Manual 4382 for ICL 1900 series.

11. Stanley, F.S.: 'Application of continuous system design concepts to the design of sampled-data systems', Trans. AIEE, 1959, pp. 74-78.

12. Steel, G.K., and Puri, S.N.: 'Direct digital control of dry-tuned rotor gyros', presented at the eighth International Federation of Automatic Control Symposium on Automatic Control in Space, Oxford, July 1979.

13. Viwanadham, N., and Deekshatulu, B.L.: 'Deadbeat response in multivariable system', Int. J. Control, 1968, pp. 589-594.

List of Captions

APPENDIX

Design Example

A1. Plant Transfer Function.

Consider a 2 input - 2 output plant with transfer function matrix.

$$\bar{P}(s) = \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} \\ \bar{p}_{21} & \bar{p}_{22} \end{bmatrix}$$

$$\text{where } \bar{p}_{11} = \bar{p}_{22} = \frac{4\ \omega_o}{s(s^2 + 2\zeta\ \omega_o s + \omega_o{}^2)} \qquad \text{A1}$$

$$\bar{p}_{12} = -\bar{p}_{21} = \frac{1}{(s^2 + 2\zeta\ \omega_o s + \omega_o{}^2)}$$

The quadratic factor contributes a lightly damped mode when ζ is small with undamped natural frequency ω_o. We chose a sampling frequency of 5 times the undamped natural frequency so that the dimensionless sampling interval $T\ \omega_o = 2\pi/5$ and a value $\zeta = 0.05$ for the damping ratio.

The z-transform of the elements of the pulse-transfer-function matrix, $P(z)$, including the zero-order hold operations, are found to be

$$p_{11} = p_{22} = \frac{1.18}{\omega_o{}^3}\ \frac{z^{-1}(1 + 3.572\ z^{-1} + 0.933\ z^{-2})}{(1 - z^{-1})(1 - 0.5812\ z^{-1} + 0.8818\ z^{-2})}$$

$$p_{12} = -p_{21} = \frac{0.6643}{\omega_o{}^3}\ \frac{z^{-1}(1 + 0.956\ z^{-1})}{(1 - 0.5812\ z^{-1} + 0.8818\ z^{-2})} \qquad \text{A2}$$

All these elements have a common pole pair just inside the unit circle at $z = 0.939\angle{\pm 72^\circ}$ due to the lightly damped mode. On forming $P^{-1} = Q$ the elements of Q become

$$q_{11} = q_{22} = F(1 + 3.572\ z^{-1} + 0.933\ z^{-1})$$

$$q_{12} = -q_{21} = 0.56\ F(1 - z^{-1})(1 + 0.956\ z^{-1}) \qquad \text{A3}$$

$$F = \frac{0.645\ \omega_o^{\ 3}(1 - z^{-1})(1 - 0.5812z^{-1} + 0.8818\ z^{-2})}{(1 + 5.416\ z^{-1} + 10.67\ z^{-2} + 5.09\ z^{-3} + 0.881\ z^{-4})}$$

A2. Minimum Settling Time Design.

The forth order polynomial in the denominator of F provides the zeros of det(P) and these are found by numerical solution[3,4] to be.

(a) $z = -0.294 \pm j\ 0.166$

(b) $z = -2.4138 \pm j\ 1.37$

The pair (a) lie inside the unit circle while (b) lie outside. The zeros (b) must be assigned to f in accordance with Policy (2) to avoid sensitivity. By further assigning the zeros (a) to f the controller output sequence is of finite duration and the system will settle completely in five sample intervals.

Thus we define the minimal form of f to be

$$f = 0.0434\ z^{-1}\ (1 + 5.416\ z^{-1} + 10.67\ z^{-2} + 5.09\ z^{-3} + 0.881\ z^{-4}) \qquad \text{A4}$$

and get

$$\phi = (1 - z^{-1})(1 + 0.957\ z^{-1} + 0.722\ z^{-2} + 0.2589\ z^{-3} + 0.0382z^{-4}) \qquad \text{A5}$$

where the additional coefficients in ϕ have been computed to satisfy $1 - f = \phi$.

The controller functions are calculated from $D = P^{-1}\ f/\phi$ so that

$$d_{11} = d_{22} = H\left[1 + 3.572\ z^{-1} + 0.933\ z^{-2}\right] \qquad \text{A6}$$

$$- d_{12} = d_{21} = H\left[0.560(1 - z^{-1})(1 + 0.956\ z^{-1})\right]$$

$$H = \frac{0.028\ \omega_o^{\ 3}\ (1 - 0.5812\ z^{-1} + 0.8818\ z^{-2})}{1 + 0.957\ z^{-1} + 0.722\ z^{-2} + 0.2589\ z^{-3} + 0.0389\ z^{-4}} \qquad \text{A6}$$

The resulting response of the system to a unit step change of one input is obtained by a digital simulation of the system and appears as shown on FIG.(5). There is no interaction with the other output. On the same diagram the effects of 10% change of gain in one loop are indicated.

The effect of changing the damping of the open-loop mode to one third of its designed value is shown to produce instability as indicated on FIG.(6).

A3. Retarded Response.

The method described in Section 4.5 will be implemented. To do this, we construct $f(\alpha)$ as defined in equation (33) with f_o corresponding to f given in equation (A4) and $z = \alpha = 0.939\ \underline{/72^o}$ fixing the pole position. When $f(\alpha)$ is evaluated for values of γ on the locus FIG.(2), corresponding to a mode of damping ratio 0.5, we obtain the design locus FIG.(6). From this diagram the minimum value of $1 - f(\alpha)$ is found to be $0.912\ \underline{/-9.8^o}$ for $\gamma = 0.64 + j\ 0.37$. With this value f is given by

$$f = \frac{0.01156\ z^{-1}(1 + 5.416\ z^{-1} + 10.67\ z^{-2} + 5.09\ z^{-3} + 0.881\ z^{-4})}{(1 - \gamma\ z^{-1})(1 - \gamma^*\ z^{-1})}$$

and $\qquad$ A7

$$\phi = \frac{(1 - z^{-1})(1 - 0.291\ z^{-1} + 0.192\ z^{-2} + 0.069\ z^{-3} + 0.01\ z^{-4})}{(1 - \gamma\ z^{-1})(1 - \gamma^*\ z^{-1})}$$

The corresponding control functions are unchanged apart from the common factor H which is of the same order of complexity but has some different numerical coefficients i.e.

$$H = \frac{0.00745\ \omega_o^{\ 3}(1 - 0.5812\ z^{-1} + 0.8818\ z^{-2})}{(1 - 0.291\ z^{-1} + 0.192\ z^{-2} + 0.069\ z^{-3} + 0.01\ z^{-4})}$$

Simulation of the resulting system gives the response shown on FIG.(5) to a unit step applied at one input. There is again no interaction with the other output.

The effect of 10% gain change in one loop is also shown on the graph FIG.(5). The reduction of sensitivity expected as a result of retarding the response is not immediately apparent. Sensitivity of the movement of poles of f away from the origin has been used as the basis of the theoretical assessment. This has the effect of extending the settling time. It is difficult to form a basis for comparison between the two cases seen here except to observe that the settling time is proportionately less affected in the case of the retarded response.

Changing the damping ratio of the open-loop mode to a third of its design value produces the response shown in FIG.(6). The system is now seen to be considerably less sensitive than was the minimum settling time design. This may be observed to result from the reduction in the sensitivity factor $(z - \alpha)/\delta = 1 - f(\alpha)$. Without the poles added to f the value of $f(\alpha)$ is found to be $0.65\ \underline{/142.2^o}$ so that for the minimum settling time design $(z - \alpha)/\delta$ $1.565\ \underline{/-14.7^o}$. This is to be compared with the value $0.912\ \underline{/-9.8^o}$ found on FIG.(6).

The fact that the magnitude of $(z - \alpha)/\delta$ is greater than unity in the case of the minimum settling time design means that the closed-loop poles move further then the open-loop poles and cross the unit circle while the open-loop damping is still finite. This cannot happen in the retarded case, as the magnitude is now less than unity, and the system will remain stable even when the damping is taken to zero.

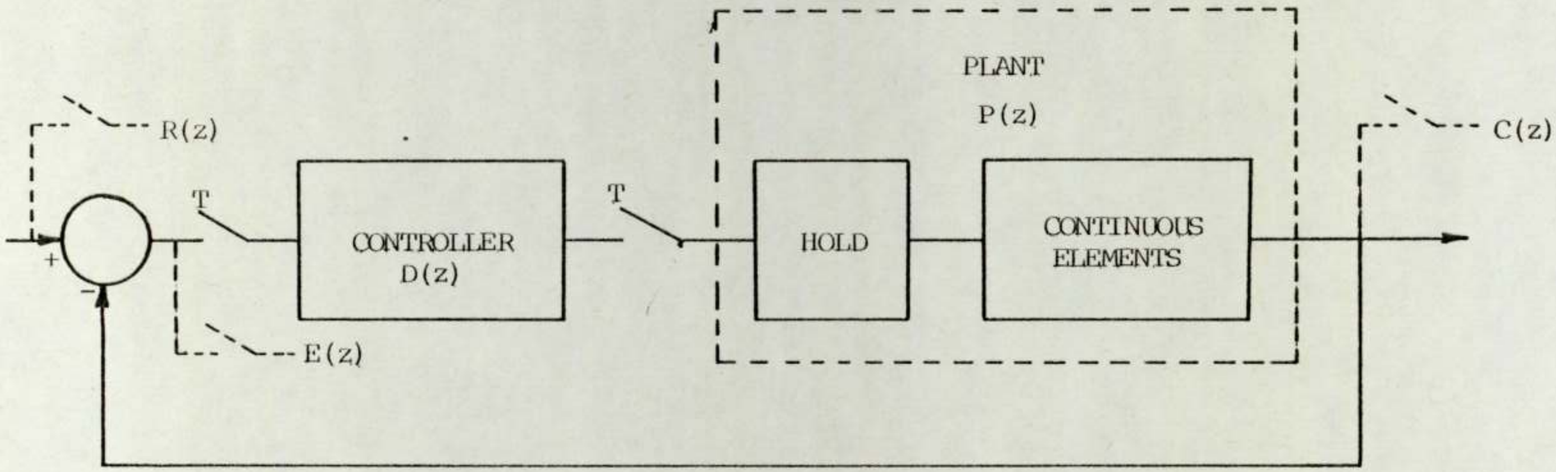

FIG. (1) System configuration

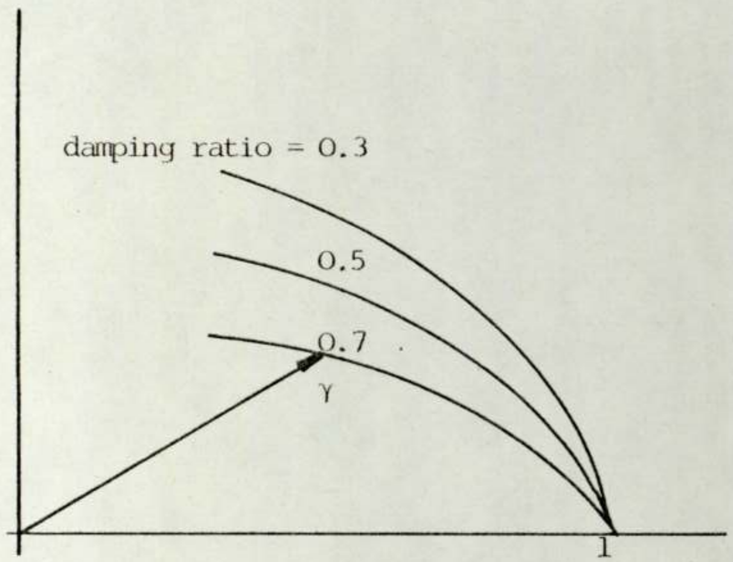

FIG. (2) Values of γ for constant damping ratio

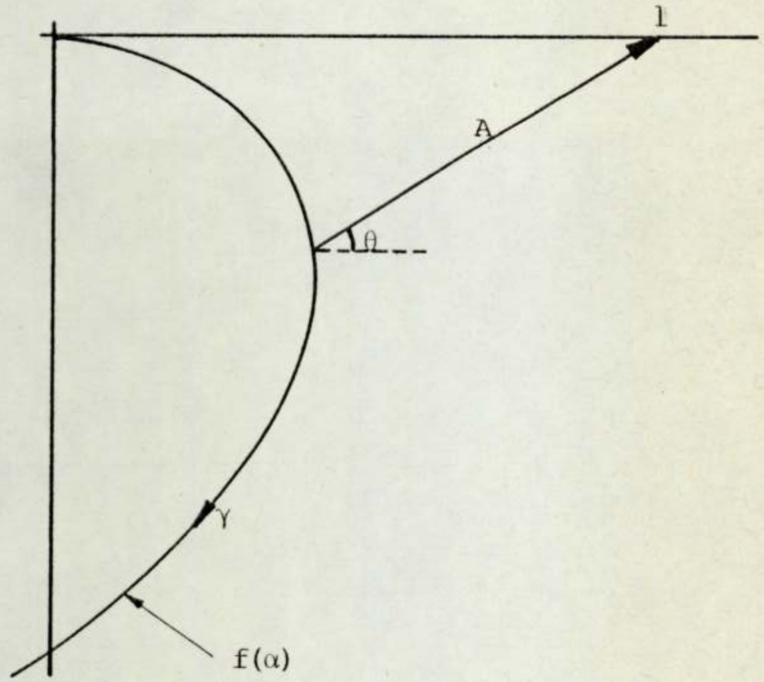

FIG. (3) Locus of $f(\alpha)$

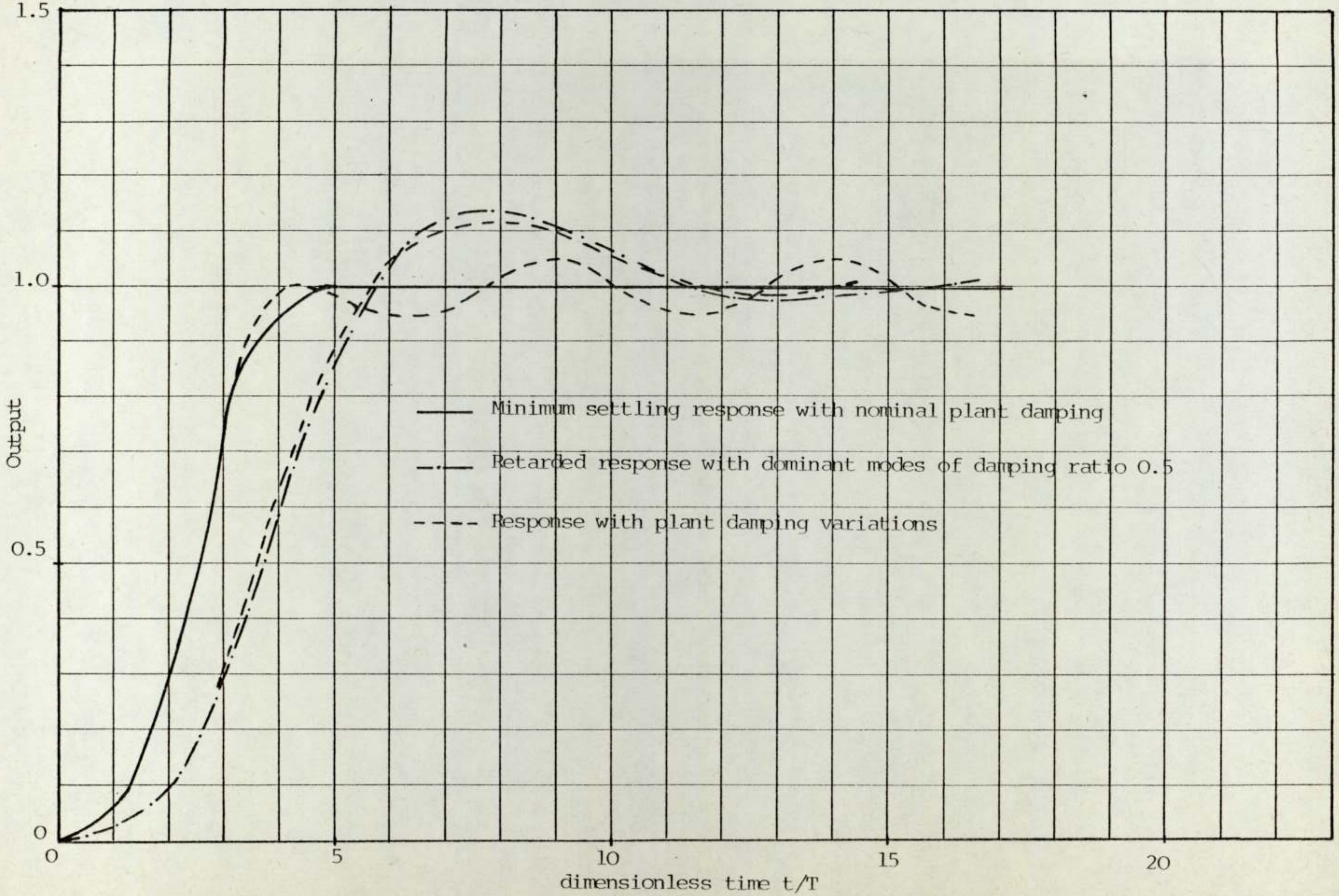

FIG.(4) Effect of plant damping variations

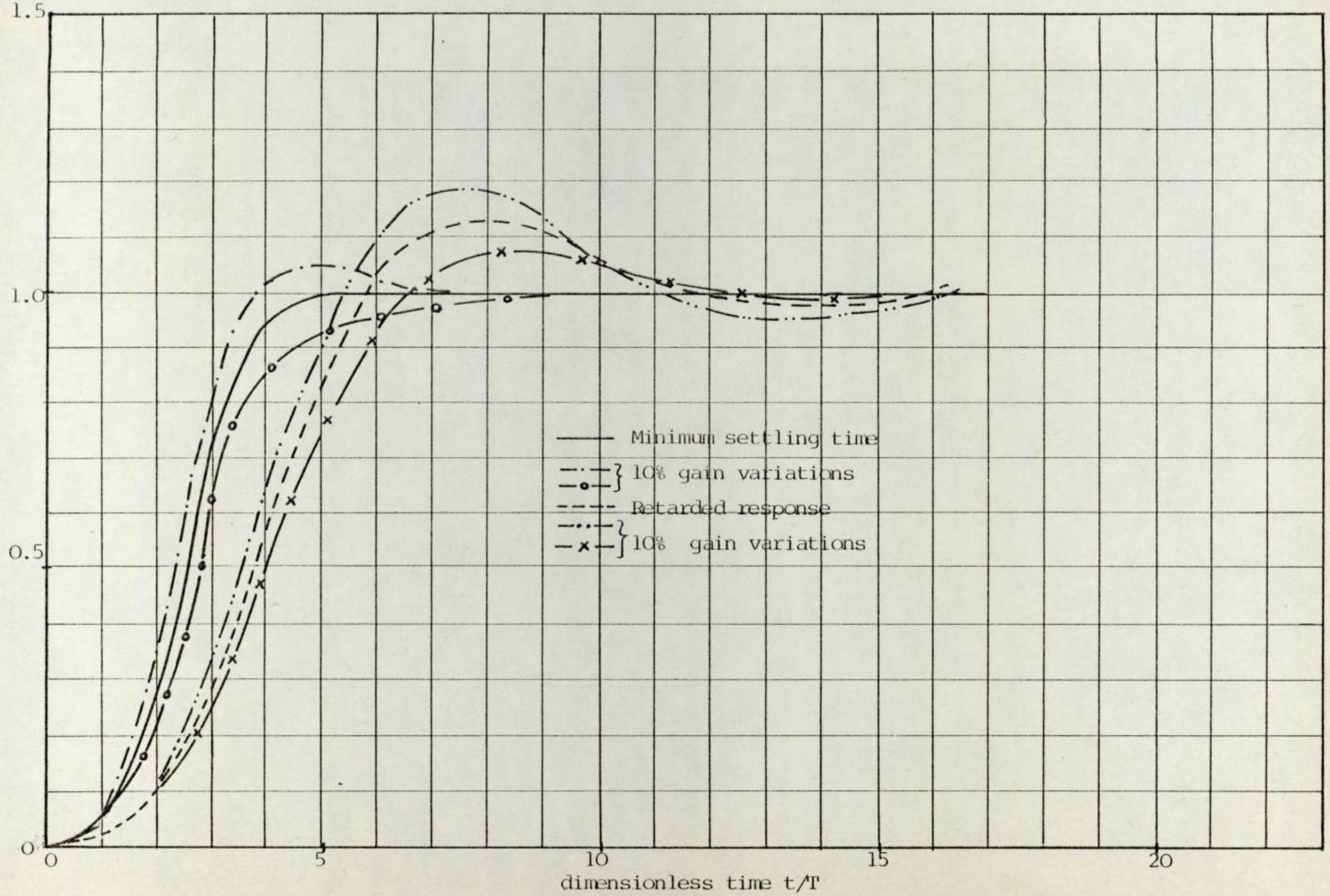

FIG. (5) Effect of gain changes

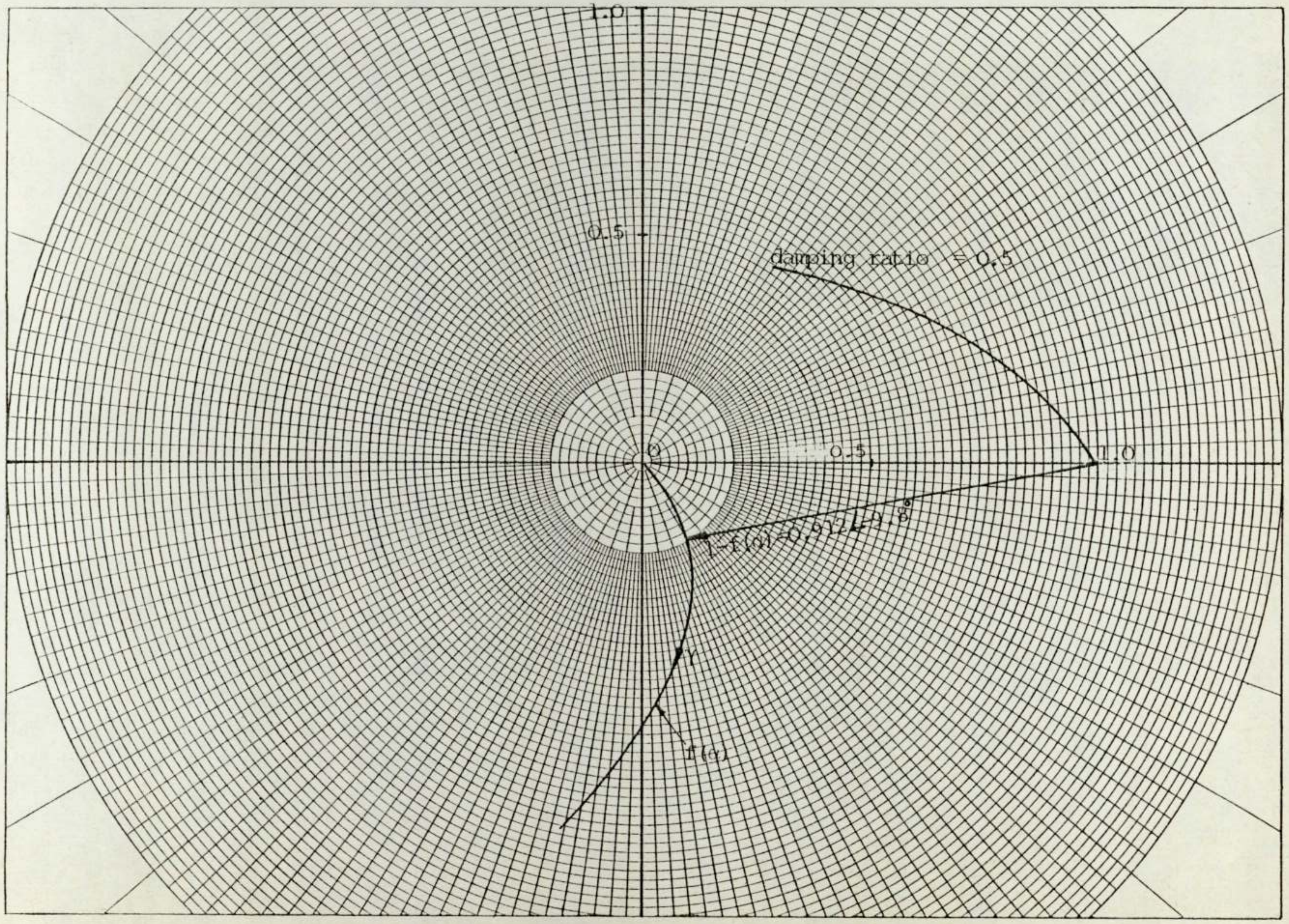

FIG. (6) Design locus of f(α)