

## Article

# Fractional Conductances of Wires: The S-Matrix Approach

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## Abstract

Quasi-one-dimensional systems with multiple conduction channels are essential for describing a range of physical phenomena. In this paper, we analyse transport in wires where electrons are subject to arbitrary number of strong multi-particle backscattering terms. We present an exact calculation of the system's scattering matrix and derive a formula for the two-terminal conductance. We find the conductance is reduced from its ideal value by a term corresponding to the projection of current fields onto the subspace of integer-valued vectors characterising the gapped channels created by the perturbations. Applying this result, we establish the minimal model required to reproduce the recently observed, yet unexplained, fractional conductance plateaus with even denominators.

**Keywords:** multi-channel Luttinger liquids; transport in coupled-wire systems; fractional conductance

## 1. Introduction

Coupled wire models are a powerful way to model complex systems and allow the analytical methods of 1D quantum physics to be extended to higher dimensions. Starting from a Luttinger liquid model that provides a universal description of low-energy excitations in one spatial dimension [1], multiple copies can be stacked to form a discrete lattice with a continuum direction along the wire. Through coupling different wires by a multi-particle backscattering process, the highly anisotropic original geometry becomes a flexible model that can describe many different phenomena. The versatility occurs through the interactions introducing gaps into the spectrum and freezing out a sector of the field in the strong backscattering limit, resulting in controlling the number of gapped and gapless modes. This approach has proved successful for describing many different phenomena, such as the quantum Hall effect [2], fractional quantum Hall effect [3,4], topological edge modes [5–7], and fractons [8,9]. The transport properties of coupled wire systems, such as the two-terminal conductance in both the integer and fractional quantum Hall effect, provides an experimentally accessible way of classifying the state of the system. Therefore it is important to understand what possible values of conductance can be found from different coupled wire models.

Experimental results of the conductance in quantum point contacts with a shallow constriction [10–16] reveal conductance plateaus upon varying the depth of the constriction. The values of the conductance on these plateaus can be fractions of the quantum of conductance,  $e^2/h$ , with a wide range of different fractions being found. The fractional nature of two-terminal d.c. conductance has been previously studied in two-wire systems in the presence of multi-particle backscattering [17–20]. Notably, fractions with even denominators that appear in experiment (e.g.,  $1/2$  and  $1/6$ ) remain unexplained. In this



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paper we will describe the minimum possible coupled wire model that can account for these experimental conductance values.

To find these models, we introduce a rigorous way to derive the linear conductance through a generic 1D coupled wire model connected up to non-interacting leads with both forward and backward scattering interactions. This is achieved by starting from the ‘scattering’ matrix in terms of the gapped and gapless modes and carefully finding the scattering matrix in terms of the original chiral fields that describe the system far away from the interacting region. The non-local nature of the  $S$ -matrix means that the simple local rotation of fields cannot be performed directly and we must instead use the corresponding transfer matrix.

## 2. Model

Our system contains  $N$  coupled Luttinger liquid wires that are adiabatically connected to two non-interacting leads. Both forward scattering and multi-particle backscattering processes occur inside the interacting region  $[0, L]$ . Bosonised description of this setup starts from assigning two chiral fields,  $\varphi_i^R$  and  $\varphi_i^L$  to each channel,  $i = \{1, \dots, N\}$ . In terms of  $2N$ -dimensional vector  $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^R, \boldsymbol{\varphi}^L)^T$  where the  $N$ -dimensional vectors are  $\boldsymbol{\varphi}^\eta = (\varphi_1^\eta, \dots, \varphi_N^\eta)^T$ , the Lagrangian density can be written as follows:

$$L = -\frac{1}{4\pi} \partial_x \boldsymbol{\varphi} \Sigma \partial_t \boldsymbol{\varphi} - \mathcal{H}, \quad \Sigma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad (1)$$

where the canonical term defines the commutation relations in the operator formulation, and the Hamiltonian,  $\mathcal{H} = \mathcal{H}_{fs} + \mathcal{H}_{bs}$ , contains forward scattering and backscattering terms,

$$\mathcal{H}_{fs} = \partial_x \boldsymbol{\varphi} \hat{V} \partial_x \boldsymbol{\varphi}, \quad \mathcal{H}_{bs} = -\sum_{a=1}^n \Delta_a \cos(\mathbf{k}_a^T \boldsymbol{\varphi}). \quad (2)$$

Outside the wire (interacting section of length  $L$ ), matrix  $\hat{V}$  tends to the diagonal matrix  $\hat{V}_{ij}(x \rightarrow \pm\infty) = \delta_{ij}v_i/4\pi$ , where  $v_i$  is the velocity of the  $i$ -th channel in electrodes. The contacts are assumed to be adiabatic, with the confining potential varying slowly along the wire on the scale of the Fermi wavelength for the electrons originating from the leads. This requirement results in reflectionless contacts so that any particle in the wire will transmit into the leads. The bosonic collective excitations described by a Luttinger liquid represent long-wavelength oscillations and, therefore, spatial dependences of the forward-scattering matrices inside and outside the system do not affect adiabaticity. Nonideal contacts would result in additional terms similar to  $\mathcal{H}_{bs}$  with amplitudes  $\Delta(x)$  being nonzero only in the vicinity of the wire ends. This effect would lead to a non-universal conductance and will not be considered in this paper.

The  $n$  vectors  $\mathbf{k}_a$ , called backscattering vectors, are all integer-valued and describe the multi-particle backscattering process. For these to represent possible processes, the number of particles must be conserved in the process. Defining the ‘parity vector’  $\mathbf{t}$  which is 1 for the first  $N$  entries and  $-1$  for the other  $N$  elements, the backscattering vectors must obey the following:

$$\mathbf{t}^T \Sigma \mathbf{k}_a = 0. \quad (3)$$

Assuming strong backscattering limit when all  $\Delta_a \rightarrow \infty$ , we expect that all these terms will gap the system leading to the freeze of corresponding fields, i.e., arguments of cosines. This effect is similar to locking modes in Coulomb dots [21] and gapped phases in the Luttinger liquid wire constriction [3]. This is possible only if the corresponding operators

commute. This so-called Haldane criterion imposes that the backscattering vector satisfy the following:

$$\mathbf{k}_a^T \Sigma \mathbf{k}_b = 0. \quad (4)$$

This bilinear form defines an  $n$ -dimensional vector space of backscattering vectors  $G = \text{Span}\{\mathbf{k}_a\}_{a=1}^n$  that satisfy this condition. In the limit of strong backscattering, which will be considered for the rest of this paper, backscattering Hamiltonian pins all  $\mathbf{k}_a^T \boldsymbol{\varphi}$  sectors of the field to zero modulo  $2\pi$ . Exciting oscillations around this minimum configuration costs energy, resulting in a gap in the spectrum. Therefore it is natural to rotate the problem into a basis that distinguishes between these sectors of the field. We transform the fields via the  $2N \times 2N$  matrix  $M$  as follows:

$$\boldsymbol{\varphi}(x) = M^T \boldsymbol{\phi}(x), \quad (5)$$

such that the commutation relations are unchanged. This enforces that  $M^T \Sigma M = \Sigma$  with the group of possible transformations being the split orthogonal group  $O(N, N)$ . The transformation to the gapped and gapless sectors of the field can be described by the column vectors  $\mathbf{m}_a$  of matrix  $M$ , with the requirement that  $\mathbf{m}_a - \mathbf{m}_{a+N} = \mathbf{k}_a$ . For later calculations, it is useful to define these column vectors as follows:

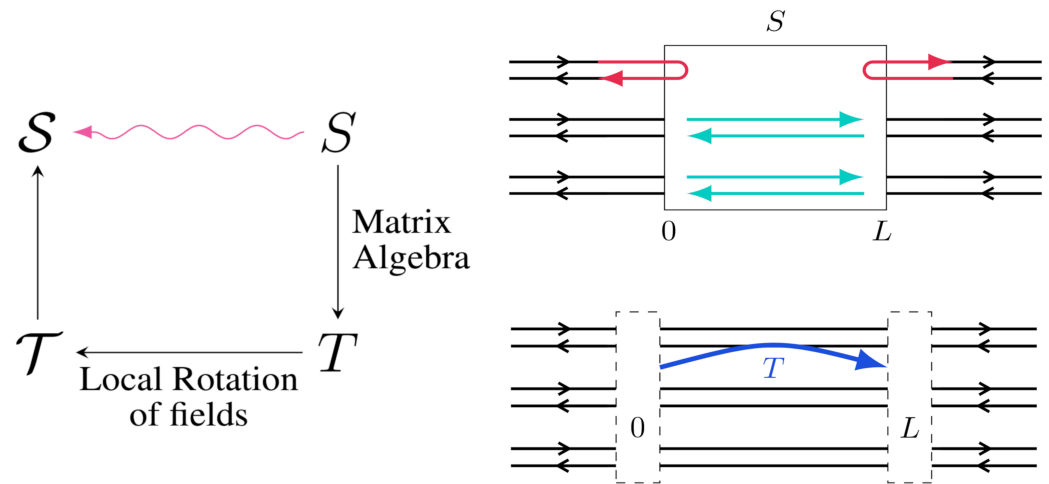
$$\begin{aligned} \mathbf{m}_i - \mathbf{m}_{i+N} &= M \mathbf{T}_i, \\ \mathbf{T}_i &= (0, \dots, 0, \underbrace{1}_i, 0, \dots, 0, \underbrace{-1}_{i+N}, 0, \dots, 0). \end{aligned} \quad (6)$$

With this transformation the arguments of cosines become  $\mathbf{k}_a^T \boldsymbol{\varphi} = \mathbf{k}_a^T M^T \boldsymbol{\phi} = \mathbf{T}_a^T \boldsymbol{\phi} = \phi_a^R - \phi_a^L$ . Note that this definition does not fully specify the matrix, even when there are  $N$  backscattering interactions, as there is freedom in how to define the sum of the two column vectors. This freedom in the choice of matrix  $M$  does not affect the conductance result.

### Scattering and Transfer Matrices

Calculating the conductance through 1D systems requires a description of the contacts and external leads. In both the original quantisation of conductance for a non-interacting system [22] and when there is a region of forward scattering [23–26], the leads solely determine the d.c. conductance. The dominance of the leads suggests that we can consider the transport problem with a scattering matrix that connects the incoming and outgoing original chiral fields,  $\boldsymbol{\varphi}^R$  and  $\boldsymbol{\varphi}^L$ . These fields propagate freely outside the interacting section of the wire but are mixed up inside the wire. However, when incoming excitations are in the energy gap, all the gapped sectors of the field will not be supported inside the wire and must be reflected in the low temperature limit. Therefore, there will be a simple description of the ‘scattering’ matrix in terms of the gapped and gapless modes, as it was performed in [27]. The gapped and gapless modes are not guaranteed to be chiral, so they are not related simply to incoming and outgoing modes, but the equivalent ‘scattering’ matrix still relates the values of the fields on either side of the interacting region.

The crux of the problem is that our knowledge of the sectors of fields is for local linear combinations of the original chiral fields, but the  $S$ -matrix is a non-local object. In order to change the local basis of the  $S$ -matrix we first must create the appropriate transfer matrix. Then, the local transformation of fields, visualised in Figure 1, can be applied, and the corresponding scattering matrix in terms of the original chiral fields can be found. An important point to note is that transfer matrices cannot be meaningfully defined when there is perfect reflection, so the limit of perfect reflection can only be taken for the  $S$ -matrix.



**Figure 1.** Demonstrating how to relate scattering matrices in different bases through implementing the local rotation of fields on the transfer matrices, which are local objects. The  $S$  matrix relating combinations of fields at  $x = 0, L$  can be determined easily in the gapped and gapless basis but this is a non local object, while the transfer matrix acts on local combinations of fields.

The scattering matrix  $S$  and the transfer matrix  $T$  are defined by a combination of the original chiral fields at two points  $0, L$  as follows:

$$\begin{pmatrix} \varphi^R(L) \\ \varphi^L(L) \end{pmatrix} = T \begin{pmatrix} \varphi^R(0) \\ \varphi^L(0) \end{pmatrix}, \quad \begin{pmatrix} \varphi^R(L) \\ \varphi^L(0) \end{pmatrix} = S \begin{pmatrix} \varphi^R(0) \\ \varphi^L(L) \end{pmatrix}. \quad (7)$$

The two matrices can be related to each other through matrix algebra as follows:

$$S = \begin{pmatrix} \mathcal{T}_{RR} - \mathcal{T}_{RL}\mathcal{T}_{LL}^{-1}\mathcal{T}_{LR} & \mathcal{T}_{RL}\mathcal{T}_{LL}^{-1} \\ \mathcal{T}_{LL}^{-1}\mathcal{T}_{LR} & \mathcal{T}_{LL}^{-1} \end{pmatrix}, \quad T = \begin{pmatrix} \mathcal{T}_{RR} & \mathcal{T}_{RL} \\ \mathcal{T}_{LR} & \mathcal{T}_{LL} \end{pmatrix}, \quad (8)$$

where  $\mathcal{T}_{\eta\eta'}$  are  $N \times N$  matrices that connect the right and left movers at different positions  $x = 0, L$ . The transfer matrix for the rotated fields  $T$  can be found by applying the transformation  $M$  as follows:

$$\phi(L) = (M^T)^{-1} T M^T \phi(0) = T \phi(0), \quad (9)$$

which defines the relation between the transfer matrices  $T = \Sigma M \Sigma T M^T$ . Finally, we can obtain the  $S$ -matrix in the rotated basis through inverting Equation (8). In this basis, for energies below the size of the gap at low temperatures and small voltages, the gapped sectors will fully reflect, and the gapless sectors will perfectly transmit. However, as mentioned earlier, due to the divergence of the transfer matrices, we cannot assume that the  $S$ -matrix perfectly reflects, and therefore we must consider the more general case and take the perfect reflection limit at the end.

To illustrate the process, we start with the one-channel case. In the rotated fields, we can write the scattering matrix as follows:

$$S = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix}, \quad (10)$$

with current conservation, giving the conditions  $r + t = 1 = r' + t'$ . We only consider the situation of complete reflection  $t = t' = 0$  and perfect transmission  $r = r' = 0$ , which means that we can consider  $r = r'$  and  $t = t'$ . The corresponding transfer matrix in this basis can be expressed using  $\rho = r/t$  as follows:

$$T = \begin{pmatrix} 1 - \rho & \rho \\ -\rho & 1 + \rho \end{pmatrix}. \quad (11)$$

Although the perfect transmission limit is well defined as  $\rho \rightarrow 0$ , the  $\rho \rightarrow \infty$  limit diverges. This can be understood as a transfer matrix that becomes reflected and will not relate the fields on either side of the wire, resulting in a singular matrix. The limit  $\rho \rightarrow \infty$  must be associated with  $L\Delta/v \rightarrow \infty$  limit. The non-universal corrections of order  $v/(L\Delta) \ll 1$  require an instanton analysis and lie beyond the scope of our approximation. In this work we focus on the universal values of the fractional conductance at intermediate temperatures, where strong backscattering fully suppresses certain fluctuations. We also do not consider the extreme low-temperature regime, where renormalisation group analysis shows that backscattering becomes irrelevant and the ideal conductance is restored.

Performing any local rotation of chiral fields on the transfer matrix and then finding the equivalent scattering matrix  $S$  for the rotated fields will result in the  $\rho \rightarrow \infty$  limit producing a finite  $S$  matrix with only non-zero off-diagonal elements. Upon implementing current conservation, the off-diagonal elements will be equal to one and match Equation (10). In the one-channel case, this states something obvious, that any linear combinations of chiral fields on either side of the interacting section are completely unrelated to each other in the case of perfect reflection. This freedom of choice in rotation of the chiral fields is a product of the  $\rho \rightarrow \infty$  limit and will manifest again when we include more channels into the description.

For multiple channels, the scattering matrix will be given by the following:

$$S = \begin{pmatrix} T & R \\ R' & T' \end{pmatrix} \quad (12)$$

where  $T, R, T', R'$  are all  $N \times N$  matrices with entries  $T = \delta_{ij}t_i, R = \delta_{ij}r_i$  and these describe the reflection and transmission of each channel. Current conservation now becomes the expression  $\sum_i r_i + \sum_i t_i = N$ , but if we assume that there is no scattering between channels, then we will have current conservation in each channel so that  $r_i + t_i = 1$ . Then, the total transfer matrix can be written as follows:

$$T = \begin{pmatrix} 1 - P & P \\ -P & 1 + P \end{pmatrix}, \quad P = \text{diag}(\rho_i), \quad (13)$$

where  $\rho_i$  is the reflection ratio for each channel. For any of the  $\rho_i$  that correspond to gapless modes, these can be safely taken to be 0 as the transfer matrices are well defined. The remaining  $\rho_a$ , where  $a$  now only indexes the gapped sectors, can be set to the same value  $\rho_a = \rho$  as they will all be taken to be infinite. This allows us to simplify the transfer matrix to be  $T = 1 - \rho \Sigma T_a \otimes T_a^T$  where repeated indices now imply summation. Using the transformation in Equation (9) we can find the transfer matrix in the original fields, for  $k_a = (r_a, -l_a)^T$  where  $r_a, l_a$  are  $N$ -dimensional vectors:

$$\mathcal{T} = \mathbb{1}_{2N} - \rho \Sigma k_a \otimes k_a^T = \begin{pmatrix} \mathbb{1}_N - \rho r_a \otimes r_a^T & \rho r_a \otimes l_a^T \\ -\rho l_a \otimes r_a^T & \mathbb{1}_N + \rho l_a \otimes l_a^T \end{pmatrix} \quad (14)$$

The  $\mathcal{T}_{LL}$  matrix can be explicitly inverted to give the following:

$$\mathcal{T}_{LL}^{-1} = \mathbb{1}_N - Q^{ab} l_a \otimes l_b^T, \quad (Q^{-1})_{ab} = -\frac{\delta_{ab}}{\rho} + g_{ab} \quad (15)$$

where  $g_{ab} = \mathbf{l}_a \cdot \mathbf{l}_b = \mathbf{r}_a \cdot \mathbf{r}_b$  with the equality guaranteed from the Haldane criterion in Equation (4). The matrix  $Q^{ab}$  refers to the  $ab$ -th element of the inverse of matrix  $Q^{-1}$ . Therefore, the scattering matrix in the new basis will be as follows:

$$S = \begin{pmatrix} \mathbb{1}_N - Q^{ab} \mathbf{r}_a \otimes \mathbf{r}_b^T & Q^{ab} \mathbf{r}_a \otimes \mathbf{l}_b^T \\ Q^{ab} \mathbf{l}_a \otimes \mathbf{r}_b^T & \mathbb{1}_N - Q^{ab} \mathbf{l}_a \otimes \mathbf{l}_b^T \end{pmatrix}, \quad (16)$$

which defines the transmission and reflection elements in the original basis. In the total reflection limit, we can see that  $Q^{-1} = g$ , which then means that the operators become projectors.

$$S = \begin{pmatrix} 1 - g^{ab} \mathbf{r}_a \otimes \mathbf{r}_b^T & g^{ab} \mathbf{r}_a \otimes \mathbf{l}_b^T \\ g^{ab} \mathbf{l}_a \otimes \mathbf{r}_b^T & 1 - g^{ab} \mathbf{l}_a \otimes \mathbf{l}_b^T \end{pmatrix} = \mathbb{1}_{2N} - 2\Pi_G \quad (17)$$

for  $\Pi_G = (g^{ab}/2) \mathbf{k}_a \otimes \mathbf{k}_b^T$  being the projector onto the space  $G$ .

The d.c. current through the system can be written in terms of the scattering matrix by symmetrising the current on either side of the scattering region. Using the 'parity' vector  $\mathbf{t}$  of Equation (3), the total d.c. current is defined as follows:

$$J = \mathbf{t}^T \begin{pmatrix} \mathbf{O}^R \\ \mathbf{I}^L \end{pmatrix} = \mathbf{t}^T \begin{pmatrix} \mathbf{I}^R \\ \mathbf{O}^L \end{pmatrix}. \quad (18)$$

To show that these two expressions are equivalent, subtract the two vectors and use the particle conservation requirement. This equality implies that the total current is constant across the system. This then allows the total current to be written as a symmetrisation of the previous equation as follows:

$$J = \frac{1}{2} \mathbf{t}^T (S + \mathbb{1}) \mathbf{I} = \frac{1}{2} \mathbf{t}^T (\mathbb{1} - \Pi_G) \mathbf{I}, \quad (19)$$

where  $\mathbf{I} = (\mathbf{I}^R, \mathbf{I}^L)$ . The incoming currents are emitted from the reservoirs at chemical potentials  $\mu_L, \mu_R$  for the left and right reservoirs, respectively. The voltage drop across the system will be the difference between the chemical potentials and therefore the conductance is given by the following:

$$g = N - \frac{1}{2} \mathbf{t}^T \Pi_G \mathbf{t} = \frac{1}{2} \mathbf{t}^T \Pi_G^\perp \mathbf{t}. \quad (20)$$

All of the details of the forward scattering interactions drop out in the d.c. limit and the conductance is entirely determined by the space created by the backscattering vectors. Note that this can be represented as projecting to the orthogonal complement space to  $G$ . This space  $G^\perp$  can be decomposed into two orthogonal spaces  $\Sigma G$  and  $F$ , where  $\Sigma G = \text{Span}(\Sigma \mathbf{k}_a)_{a=1}^n$  and the free space is what remains,  $F = G^\perp / (\Sigma G)$ . From Equation (4), we can see that the spaces  $G$  and  $\Sigma G$  are orthogonal. Therefore, the projector can be written as  $\Pi_G^\perp = \Pi_{\Sigma G} + \Pi_F$ . However, due to the structure of vectors  $\mathbf{y} \in \Sigma G$ , as they must satisfy  $\mathbf{t} \cdot \mathbf{y} = 0$  due to current conservation, the projector  $\Pi_{\Sigma G}$  does not contribute to the conductance in Equation (21). The conductance can then be written as  $g = \mathbf{t}^T \Pi_F \mathbf{t} / 2$ , the overlap of the parity vector with the projection of the parity vector on the free space.

The conductance can be further simplified if we recollect that all backscattering vectors  $\mathbf{k}_a = (\mathbf{r}_a, -\mathbf{l}_a)$  satisfy the Haldane criterion,  $\mathbf{r}_a \cdot \mathbf{r}_b = \mathbf{l}_a \cdot \mathbf{l}_b = g_{ab}$ , i.e., both sets share the same Gramm matrix  $g_{ab}$ , and the particle number conservation requires  $\mathbf{e} \cdot \mathbf{r}_a = \mathbf{e} \cdot \mathbf{l}_a$  where  $N$ -dimensional vector  $\mathbf{e} = (1, 1, \dots, 1)$ . Then, we can only use the projector on the smaller  $n$ -dimensional subspace of  $N$ -vectors  $\{\mathbf{r}_a\}_{a=1}^n$  as follows:

$$g = N - \mathbf{e} \pi \mathbf{e} = \mathbf{e} \hat{\pi}^\perp \mathbf{e}, \quad (21)$$

where

$$\hat{\tau} = g^{ab} \mathbf{r}_a \otimes \mathbf{r}_b^T, \quad \hat{\tau}^\perp = 1 - \hat{\tau}. \quad (22)$$

Another useful expression is the following:

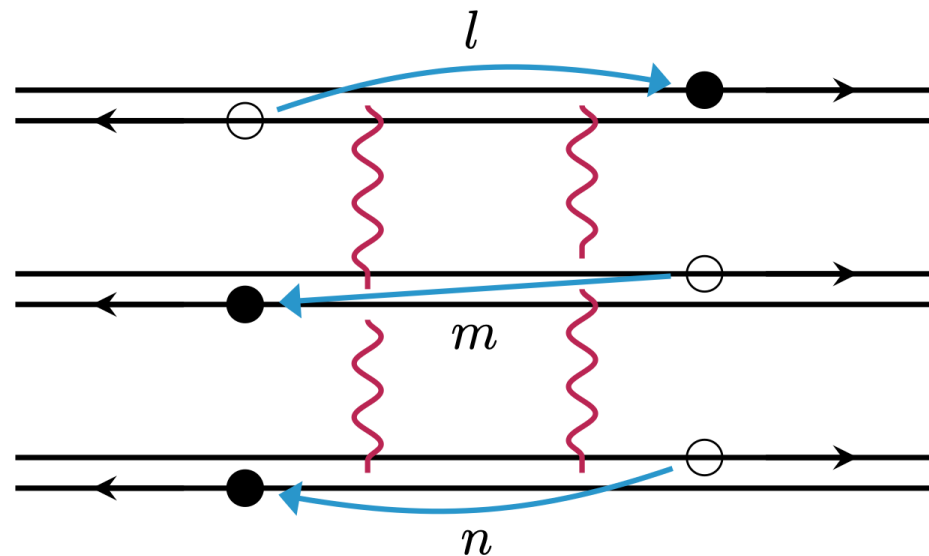
$$g = N - g^{ab} q_a q_b, \quad q_a = \mathbf{e} \cdot \mathbf{r}_a. \quad (23)$$

### 3. Results

To show the complementary approach of calculating the gapped and free projection operators, we will consider two cases in the  $N$  wire system—one backscattering interaction and  $N - 1$  backscattering interactions.

#### 3.1. Single Backscattering Mechanism

When only one backscattering vector is relevant, the projection operator can be simply constructed. The interaction can encompass multiple wires, but the corresponding vector  $\kappa$  must satisfy Equations (3) and (4). An illustration of a possible realisation of this is shown in Figure 2 where  $l$  particles scattering from the left to right movers in one channel is balanced by  $m$  and  $n$  particles scattering in the opposite direction in two other channels.



**Figure 2.** Depicting a possible backscattering interaction that results in one sector of the fields being pinned to zero. The empty holes indicate an annihilation operator has acted on that chiral field, and the filled circles correspond to particles being created. This would be described by  $(n, m, -l, -n, -m, l)^T$ .

We can construct the projection operator onto a vector using  $\Pi_G = \kappa \otimes \kappa / (\kappa^T \kappa)$ . The total conductance through the system becomes,

$$g = N - \frac{(\mathbf{t} \cdot \kappa)^2}{2\kappa^2} = N - \frac{(\mathbf{t} \cdot \mathbf{r})^2}{2\mathbf{r}^2}, \quad (24)$$

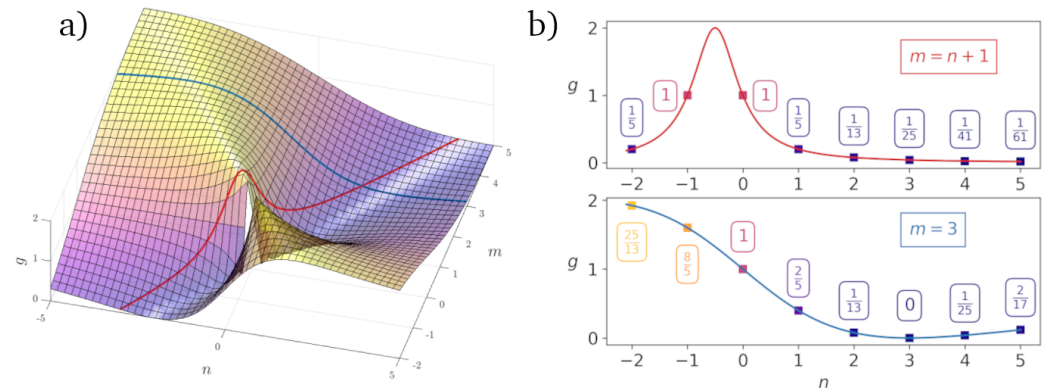
where we have used that for a backscattering vector  $\kappa = \mathbf{r}, -\mathbf{l}$  the Haldane criterion,  $\mathbf{r}^2 = \mathbf{l}^2$ , and particle number conservation,  $\mathbf{r} \cdot \mathbf{r} = \mathbf{l} \cdot \mathbf{l}$ , simplify the expression for the conductance.

In the two-wire case, there are two possibilities that for an arbitrary  $\mathbf{r} = (n_1, n_2)^T$  the corresponding vector  $\mathbf{l}$  can be, either  $\mathbf{l} = (n_1, n_2)^T$  [20] or  $\mathbf{l} = (n_2, n_1)^T$  [27]. Both of these cases give the same two-terminal conductance of

$$g = \frac{(n_1 - n_2)^2}{n_1^2 + n_2^2} \quad (25)$$



which is shown in Figure 3. This matches up with the expression of Reference [20] and a generalisation of Reference [27] where they studied the case  $n_2 = n_1 + 1$ .



**Figure 3.** The possible different fractions that can be obtained from the two-wire case. (a) The surface produced by Equation (25). Notably the conductance is always zero for  $n = m$  and equal to two for  $n = -m$ . (b) The fractional conductance values are shown for integer values of  $n, m$  for two different lines along the surface. The upper plot shows the values when  $m = n + 1$ , while the lower plot is for  $m = 3$ .

### 3.2. Single Gapless Mode

To show how to utilise the free sector description, we now consider an  $N$ -channel system with identical  $N - 1$  interactions between all neighbouring wires described by  $2N$ -dimensional backscattering vectors  $\{k_1, \dots, k_{N-1}\}$  [2,27]. The interaction between the first two wires is  $k_1 = (n_1, n_2, 0, \dots, 0, -n_2, -n_1, 0, \dots, 0)$ . All other interaction vectors are obtained by a shifting non-zero values of  $k_1$ , consequently to the right. An example of such a system with three channels is shown in Figure 4.

There are two ways to approach the construction of the projector in the gapped space. One possibility is to explicitly construct  $\Pi_G$  from all  $n$  perturbation vectors  $k_a$  by putting them as column vectors into the matrix  $\kappa$  and calculating  $\Pi_G = \kappa(\kappa^T \kappa)^{-1} \kappa^T$  (one should keep in mind that  $\kappa$  is a  $2N \times n$  matrix and not a square one). Another option is to construct an orthonormal basis  $e_a$  for the space  $G = \text{Span}\{k_a\}_{a=1}^{N-1}$ . Then, the projection can be written as  $\Pi_G = e_a \otimes e_a$ . However, from Equation (20) we can construct the orthogonal projection onto the free space by finding vectors  $w$  that are orthogonal to all  $k_a$  and  $\Sigma k_a$ . Calling  $q = -n_2/n_1$ , we obtain the following:

$$w^T = (\underbrace{1, q, q^2, \dots, q^{N-1}}_N, \underbrace{-q^{N-1}, \dots, -q^2, -q, -1}_N) \quad (26)$$

The other vector in the free space is given by  $\Sigma w$ , but it does not contribute to the conductance as  $\mathbf{t} \cdot \Sigma w = 0$ . Therefore, we can write the projector onto the free space as  $\Pi_F = w \otimes w^T / w^2$ . Then, using equation  $g = \mathbf{t}^T \Pi_F \mathbf{t} / 2$ , and the fact that the following is true:

$$\mathbf{t} \cdot \mathbf{w} = 2 \sum_{k=0}^{N-1} q^k = \frac{1 - q^N}{1 - q}, \quad \mathbf{w}^2 = 2 \sum_{k=0}^{N-1} q^{2k} = \frac{1 - q^{2N}}{1 - q^2}, \quad (27)$$

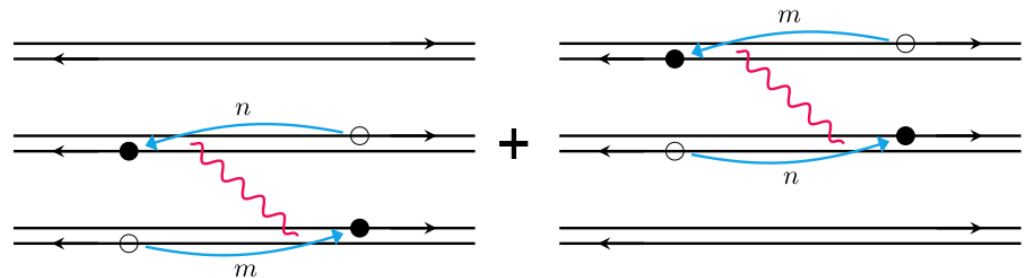
we find the conductance as follows:

$$g = \frac{(\mathbf{t} \cdot \mathbf{w})^2}{2\mathbf{w}^2} = \frac{1 + q}{1 - q} \frac{1 - q^N}{1 + q^N}, \quad (28)$$



in a perfect agreement with the case  $n_1 = n_2 + 1$  considered in Reference [27], and the result for the two-wire system obtained above in Equation (25). When the number of channels  $N \gg 1$ , then the result becomes the following:

$$g_\infty = \left| \frac{n_1 - n_2}{n_1 + n_2} \right|. \quad (29)$$



**Figure 4.** Constructing a single gapless mode from two 2-wire interactions in a 3 wire system. This backscattering would be described by the vectors  $k_1 = (n, -m, 0, -n, m, 0)^T$ ,  $k_2 = (0, n, -m, 0, -n, m)^T$ .

#### 4. Finding Even Denominators

Having applied the conductance equation to specific backscattering interactions, we can now ask the inverse question of what backscattering interaction can produce a specific value of conductance. The channels in the experiment are modes with both right- and left-moving chiralities. Whether spin is counted as a separate channel depends whether spin degeneracy is present or absent. These are all very important issues but we would like to focus on the possible values of fractional conductances without getting in the specifics of the system.

##### 4.1. Generic Analysis

The most puzzling results of the experiments [10] are the fractional conductances with even denominators. For a two-channel system with only one relevant perturbation, as in Equation (25), there can only be conductance with odd denominators. The proof of this can be seen through choosing both  $n_1, n_2$  to have the same parity (in order to guarantee that the denominator is even), with the even parity case being insensitive to an overall factor of 2 and the odd parity case having an odd denominator. Through this, any choice of even  $n_1, n_2$  can be reduced into the case of opposite parity or odd  $n_1, n_2$  covering all possibilities. A similar, but lengthier, procedure of mathematical induction shows that conductances of systems with three and four channels can also have only odd denominators. The proof is based on the fact that a conductance, given by Equation (23), is a square of the projection of vector  $\mathbf{e}$  onto the subspace orthogonal to the one spanned by backscattering vectors  $\{\mathbf{r}_a\}_{a=1}^n$ . Choosing an orthogonal basis  $\mathbf{m}_a \cdot \mathbf{m}_b = \delta_{ab}$  in this orthogonal complement, the projector will be a sum of mutually orthogonal projectors on each of those integer-valued vectors, and the conductance will be a sum:

$$g = \sum_{a=1}^{N-n} \frac{(\mathbf{e} \cdot \mathbf{m}_a)^2}{\mathbf{m}_a^2}. \quad (30)$$

Let us consider a single term in this sum. Each vector  $\mathbf{m}_a$  is an integer-valued, i.e.,  $\mathbf{m}_a = (m_1, m_2, \dots, m_N)$  and the contribution of this term has the following structure:

$$g_a = \frac{(\sum_{i=1}^N m_i)^2}{\sum_{i=1}^N m_i^2}. \quad (31)$$

We are going to prove now that the fraction  $g_a$  cannot have an even denominator unless all vectors are of the dimension  $N \geq 5$ . Please note that we must assume that there is no a common divisor for all  $m_i$ 's since it will be cancelled because both numerator and denominator are homogeneous functions of second degree. To scrutinise the parity of the numerator and denominator, we will represent all integers  $m_i$  as  $m_i = 2k_i + s_i$  where either  $s_i = 1$  or  $s_i = 0$ . The contribution is the written as follows:

$$g_a = \frac{(2k + s)^2}{4q + s}, \quad k = \sum_{i=1}^N k_i, \quad q = \sum_{i=1}^N k_i(k_i + s_i), \quad s = \sum_{i=1}^N s_i. \quad (32)$$

The absence of a common divisor explained above means that we may assume  $s \neq 0$ . For the denominator to be even, we must have  $s$  even. Having  $s = 2c$  with the natural number  $c$  is not enough since the fraction in this case becomes the following:

$$g_a = \frac{2(k + a)^2}{2q + a}, \quad (33)$$

and  $a$  is again must be even for the denominator to be even, i.e.,  $s = 4b$  with  $b$  being another natural number:

$$g_a = \frac{(k + 2b)^2}{q + b}. \quad (34)$$

For the number of channels  $N \leq 4$ , we can achieve maximum value of  $s$  being 4 that corresponds to all  $s_i = 1$  for all  $i = 1, \dots, 4$ . In this case  $b = 1$  and  $q = \sum_{i=1}^4 k_i(k_i + 1)$  is even, so the denominator,  $q + 1$ , is odd. This completes the proof that for  $N \leq 4$  any fraction Equation (32) has odd denominator. Since adding arbitrary number of fractions with odd denominators leads to a single fraction with an odd denominator, this completes the proof that the observed even-valued fractions must be a result of at least 5 channels.

For  $N = 5$ , we may have  $b = q = 1$  and  $k = -1$  to obtain  $g_a = 1/2$ . This correspond to the projector onto the vector  $\mathbf{r} = (1, 1, 1, 1, -2)$ . This would be a conductance for 5 channel and 4 backscattering terms with all backscattering vectors orthogonal to the vector  $\mathbf{r}$ . More examples can be found below.

#### 4.2. Specific Examples

The smallest number of channels, and therefore the minimal coupled wire model, necessary for an even denominator in the presence of one relevant perturbation is five ( $N = 5$ ). Surprisingly, choosing the perturbation that involves the smallest nontrivial number of quasiparticles in all five channels  $\mathbf{r}_1 = (2, 1, 1, 1, 1)$  provides the  $1/2$  conductance as follows:

$$g = N - \frac{(2 + 1 + 1 + 1 + 1)^2}{2^2 + 1^2 + 1^2 + 1^2 + 1^2} = \frac{1}{2}. \quad (35)$$

To discuss possible values of conductance in systems with many channels in the presence of  $n$  relevant perturbations, it is easiest to first consider the case where all backscattering vectors  $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ , where  $\mathbf{q}_a = (\mathbf{r}_a, -\mathbf{l}_a)$ , are orthogonal to each other. Using the Haldane criterion and particle conservation, the conductance can be written in terms of only  $N$ -dimensional vectors  $\mathbf{r}_a = (n_1^a, n_2^a, \dots, n_N^a)^T$  as follows:

$$g = N - \sum_{a=1}^n \frac{(\sum_{j=1}^N n_j^a)^2}{\sum_{j=1}^N (n_j^a)^2}. \quad (36)$$

This simplified expression shows how the total conductance is the difference between the number of channels (which is the conductance of the system in the absence of perturbations) and the conductance “stolen” by the gapped channels. To have  $g$  be an even denominator fraction, at least one of the terms in the  $a$  sum must be a fraction with an even denominator; therefore, we can restrict our attention to terms with this form.

According to the experiments, the wires have spinful channels due to the lack of an external magnetic field, and therefore, a coupled wire model that describes the appearance of even denominator fractions must have an even number of channels. For the  $N = 6$  wire system, both the  $1/2$  and  $1/6$  conductance fractions can be obtained with orthogonal backscattering interactions. The two perturbations  $\mathbf{r}_1 = (0, 0, 0, 0, 0, 1)$  and  $\mathbf{r}_2 = (2, 1, 1, 1, 1, 0)$  lead to the following:

$$\begin{aligned} g &= 6 - \frac{(0+1)^2}{0^2+1^2} - \frac{(2+1+1+1+1+0)^2}{2^2+1^2+1^2+1^2+1^2+0^2} \\ &= 6 - 1 - \frac{9}{2} = \frac{1}{2}. \end{aligned} \quad (37)$$

With the addition of another orthogonal perturbation,  $\mathbf{r}_3 = (-1, 0, 0, 1, 1, 0)$ , we can find the following:

$$g = \frac{1}{2} - \frac{(-1+1+1+0)^2}{(-1)^2+0^2+1^2+1^2} = \frac{1}{6}. \quad (38)$$

Very recent experiments have shown a fractional conductance plateau of  $1/4$  [28] and the minimal coupled wire model needed to obtain this result exactly has  $N = 12$  channels with three mutually orthogonal perturbations.

Although we have only discussed the case when all the backscattering vectors are orthogonal, this analysis holds when the vectors are linearly independent. For any non-collinear vectors, we can construct orthogonal vectors through Gram-Schmidt and choose a scaling of the new orthogonal vector such that every element has integer values. The projection operator is independent of a basis chosen to span the space  $G$  and therefore the conductance for any backscattering operators can be represented by Equation (36) with the appropriate choice of  $\mathbf{r}_a$ .

## 5. Discussion

In constructing the minimal coupled wire model required to match experimental measurements of the fractional conductance plateaux, we have made no attempt to provide a microscopic reasoning for the occurrence of the backscattering interactions. It is important to note that there are multiple different choices of backscattering operators that can produce the same conductance result due to the projection operator that appears in the conductance. However, they will differ by the number of particles involved in the multi-particle backscattering, with higher-order processes becoming increasingly unlikely. The fact that at least 6 (three spin-degenerate) channels are required to obtain an even denominator fractional conductance, with more channels for higher multiples of 2 in the denominator, means that any coupled wire model must explain the appearance of extra channels.

Finally, it is possible to approach even-denominator fractions using results with odd denominators. The difference between the approximation and the actual value comes at the

expense of higher-order multi-particle backscattering. For example, in the four-channel case with three two-wire perturbations, where we can use Equation (28); if  $q = -3$  by  $n_2 = 3$ , and  $n_1 = 1$ , then the conductance is  $g = 20/41 \approx 0.488$ . Choosing instead  $n_2 = 25, n_1 = 8$  produces a closer result of  $g \approx 0.504$  but requires more particles to be involved in the backscattering, resulting in the process being less likely to be relevant.

Future work is to examine more transport characteristics of these systems, such as shot noise, in order to determine the nature of the quasiparticles in the gapped sector.

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