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Inverse design of extreme waves

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We examine the characteristics of the initial conditions leading to the appearance of freak waves of high amplitude within the framework of the nonlinear Schrödinger equation. We consider several of the most representative shapes of the waves with high amplitude as potential candidates for the rogue waves and obtain sufficient criteria for their parameters. We also estimate the maximum norm of the resulting rogue wave solution. The results pave the way for analysis of the properties of initial conditions that might lead to the occurrence of high amplitude field spikes in the applications governed by the nonlinear Schrödinger equation.

Keywords: extreme waves, rogue waves, freak waves, Peregrine soliton, nonlinear Schrödinger equation.

1. Introduction

The occurrence of large amplitude waves is a generic physical phenomenon. Initially studied mostly in the context of high amplitude rare waves in the ocean [1–7] the research on rogue (or freak) waves expanded to optics [8–14] and other fields [15–18]. There are too many publications in this fast-growing field, and we refer the reader to the papers above and references therein for an overview of recent advances and this area of research. Studying the appearance of rogue waves (RWs) deals with an interesting question: if these events are so rare and (as it was as beautifully formulated in [19]) these waves "appear from nowhere and disappear without a trace", can we predict their emergence? Note, that in the context of optical applications, a seminal nonlinear Schrödinger equation (NLSE) or its generalizations are often used to describe properties of the large amplitude waves. As it was observed in [20–22] RWs in the NLSE have features resembling locally the Peregrine soliton (that formally is a solution with background non-vanishing at the boundaries). Using NLSE as a master model in this Letter we consider the following questions. What are the possible shapes of the resulting extreme pulses and what are the requirements/limitations on their parameters? What initial conditions (initial field distributions) in NLSE lead to the generation of a wave with extremely large amplitude?

2. Basic model and some preliminary estimates

We consider the classical nonlinear Schrödinger equation (NLSE), which describes high-frequency envelope evolution in a variety of physical applications (see, e.g., [23] and references therein) as a generic master model. In the normalized forms, NLSE reads

$$\frac{\partial q}{\partial z}-\frac{i}{2}\frac{\partial^2 q}{\partial t^2}-i\,|q|^2\,q=0. \tag{1}$$

This model has been actively used for analysis of the formation of RWs [24–26] in different physical contexts. To analyze initial conditions leading to the appearance of the high amplitude waves we will apply a back propagation, starting from a rogue wave given by the field $q_{RW}(t) = q(t, z = z_*)$ and solving NLSE backward in z to the point $z = 0$. This determines initial field distributions leading to the generation of RWs. Note that we are interested in a wave that reaches its maximum intensity $P_{RW} = |q_{RW}|^2$ at $z = z_*$.

The necessary condition for the pulse $q(z_*,t) = \sqrt{P(z_*,t)}$, $\exp\,(i\varphi(z_*,t))$ to be RW is that its intensity $P(z_*,t) = |q(z_*,t)|^2$ must have a maximum at $z = z_*$ regardless of t. This is a somewhat restrictive condition as one can imagine local RWs where the pulse is peaked in z only in the vicinity of a particular time point t_* but most known analytical shapes of RWs, such as the Peregrine soliton [20], answer this criterion.

From the NLSE (1) it is easy to show that this is only possible when the RW pulse has a flat phase (which can always be set equal to zero) at $z = z_*$, which means that at the point of the maximum intensity the RW constitutes a transform-limited pulse.

In what follows we assume normalization of the time variable by the parameter T_0 that is a characteristic temporal scale of the extreme wave. However, we keep as a free parameter a characteristic amplitude A of RW. We start with analyzing three types of real high-energy pulses serving as the prototypes of the RWs:

$$q_*(t) = \begin{cases} A\,\exp(-t^2) & \text{Type 1: Gaussian,} \\ A\,\text{sech}\,(t) & \text{Type 2: Hyperbolic secant,} \\ A\,\text{sinc}\,(t)\,\cos(\pi\beta t)\left(1-(2\beta t)^2\right)^{-1} & \text{Type 3: Raised cosine (RC) pulse,} \end{cases} \tag{2}$$

where A is the amplitude of the pulse and $0 \le \beta \le 1$ is the roll-off factor for the raised cosine (we used the value $\beta = 0.2$ throughout). By definition $\text{sinc}\,(t) = \sin(t)/t$, each of these prototype pulses is then propagated backward and forward in distance z using a split-step Fourier method [2] to verify that they indeed constitute the genuine RW and in order to establish the necessary initial conditions at $z = 0$ required to generate these waves. Other shapes, e.g., rational Peregrine soliton, are possible, but here we concentrate mainly on the three above.

Note that due to the conservation of the integrals $E = \int |q|^2\,dt$ (pulse energy) and the Hamiltonian $2H = \int |q_t|^2\,dt - (1/2)\int |q|^4\,dt$ in Eq. (1), we can estimate a lower bound on the maximum of the peak power that can be achieved for the initial conditions with given $H(0)$ and $E(0)$:

$$P_{\max,*} = \max_{t,z}(|q|^2) = -\frac{4H(z_*)}{P(z_*)}. \tag{3}$$

This gives an estimate of how large a wave can be produced by the prescribed distribution with $H(z_*) = H(0)$ and $E(z_*) = E(0)$. For the proposed pulses this gives [$P_* = A^2$ is the amplitude of the seeding pulse (2)]: $P_{\max,*} = P_*/\sqrt{2} - 2$ (Gauss), $(2/3)(P_* - 1)$ (sech), and $P_{\max,*} = (2/3)P_* - 2\pi^2/3$ for RC (the latter valid for $\beta = 0$ only). In the fiber-optic applications, in order to estimate the typical values let us consider a seed wave pulsewidth $T_* = 10$ ps. Assuming the fiber dispersion coefficient $\beta_2 = -20\ \text{ps}^2/\text{km}$ the dispersion length is given by $L_D = T_*^2/|\beta_2| = 5$ km. The corresponding normalization power $P_0 = 1/(\gamma L_d) \sim 0.16$ W where $\gamma = 1.27\ \text{W}^{-1}\cdot\text{km}^{-1}$ is the nonlinear coefficient.

3. Necessary and sufficient conditions for the seed RW profile

In this section, we shall consider the sufficient condition for the candidate seeding pulse to be a maximum in z. This condition is the standard requirement that $\partial^2 P/\partial z^2 (z = z_*, t) < 0$ for each t (or at least in the central part of the pulse). Substituting $P = q(z,t)q^*(z,t)$ and using NLSE (1) after some tedious but straightforward algebra one obtains

$$\frac{\partial^2 P}{\partial z^2} = \frac{1}{4}\frac{\partial^2 q}{\partial t^2}\frac{\partial^2 q^*}{\partial t^2} - \frac{1}{2}\frac{\partial^2 q}{\partial t^2}(q^*)^2 q - \frac{1}{2}\frac{\partial^2 q^*}{\partial t^2} q^2 q^* - (q^*)^2\left(\frac{\partial q}{\partial t}\right)^2 - 2qq^*\frac{\partial q}{\partial t}\frac{\partial q^*}{\partial t} - \frac{1}{4}q^*\frac{\partial^4 q}{\partial t^4} + \text{c.c}\,.$$

Next, recalling that at the critical point the pulse profile must be real, i.e., $q(z_*,t) = q^*(z_*,t) = q_*(t)$, we arrive at the following criterion:

$$F(t) = \frac{\partial^2 P}{\partial z^2}\Big|_{z=z_*} = \frac{1}{2}\left(\frac{\partial^2 q_*}{\partial t^2}\right)^2 - 2q_*^3\frac{\partial^2 q_*}{\partial t^2} - 6q_*^2\left(\frac{\partial q_*}{\partial t}\right)^2 - \frac{1}{2}q_*(t)\frac{\partial^4 q_*}{\partial t^4} < 0, \tag{4}$$

for the temporal region near the peak of the pulse.

Let us evaluate the F-function for the choices given above. For the Gaussian function one gets:

$$F_G(t) = 4A^2 e^{-4t^2}\left[A^2(1-8t^2) + e^{2t^2}(-1+4t^2)\right].$$

The sign of the function is determined by the sign of the bracket and one can immediately see that no universal maximum is possible as at the wings of the pulse the $F_G(t)$ function becomes positive. At the maximum location one gets $F_G(0) = 4A^2(A^2-1)$ from which we can see that a Gaussian seed can potentially be RW only for small amplitudes $A < 1$ (or for energies less than $E_c = \sqrt{\pi/2} \approx 1.25$) which is much less than the value used in the previous section ($E = 50$).

For the sech-seed one obtains:

$$F_S(t) = 2A^2(1-A^2)\,\text{sech}^6(t)\,\left(2\cosh(2t) - 3\right).$$

Here again the center of the pulse achieves RW only for the amplitude $A < 1$. At the wings of the pulse the bracket changes sign and one has a local minimum of power. The unit critical amplitude for Gauss and sech appears to be coincidental (see below). Incidentally, the critical amplitude $A = 1$ for sech corresponds to the canonical single soliton solution $q_* = \text{sech}\,(t)$ for which of course the power is stationary in z so that all the derivatives (including of course the first two) $\partial^n P / \partial z^n$ identically vanish everywhere.

For the RC the resulting formula is cumbersome, so we only provide the result for $\beta = 0$, i.e., for a sinc pulse $q_* = A\sin(\pi t)/\pi t$ for $t = 0$:

$$F_R(0) = -\frac{2}{45}A^2\pi^2(-15A^2 + \pi^2).$$

Again there is an upper bound for the amplitudes $A < \pi/\sqrt{15}$ for which this pulse can serve as the RW.

The reason why there is an upper bound for the amplitude follows from the general definition of the F-function (4). It is a non-homogeneous function of the amplitude and at large amplitudes the main contribution comes from the second and the third terms in (4) that scale as A^4. The third term vanishes at the maximum and the second is strictly positive (since at the maximum the second derivative is negative).

Note the fact that since the time is normalized to the target RW duration the amplitude $A \sim 1$ can still lead to significant power levels in the real world units. Recall that the normalization power $P_0 = |\beta_2|/(\gamma T_{RW}^2)$ and is already high (22 dBm) for the target value $T_{RW} = 10$ ps and can jump to 42 dBm for a shorter pulse of 1 ps.

Also note that due to the scaling invariance of the NLSE (1) if $q_0(z,t)$ is a solution (e.g., a pulse with unit amplitude and width) then $\tilde{q}(z,t) = Aq_0(A^2 z, At)$ is also a solution and hence one can always "scale up" the amplitude of RW by means of time contraction.

Figure 1 shows the same spatiotemporal evolution of the 3 seed pulses (2) but this time with the lower, sub-critical values of the amplitude. Note that now the maximum occurs at the right place albeit its magnitude is reduced and the evolution is non-recurrent.

Finally, the expression for the F-function (4) and the criterion for the seed pulse to be a valid RW candidate can be significantly simplified if one considers only the vicinity of the maximum point $t = 0$. Assuming an arbitrary single hump even a real seed pulse $q_*(t)$ one can expand it near the point $t = 0$:

$$q_*(t) \approx A\left[1 + \alpha t^2/2 + \beta t^4/4!\right]$$

where $A = q_*(0) > 0$, α and β are the second and fourth derivatives of the normalized pulse evaluated at zero (note that $\alpha < 0$ for the local maximum). Substituting this polynomial approximation into formula (4) and keeping only the terms up to the order of t^2 which we obtain:

$$F(t) \approx -\frac{A^2}{2}\left(-\alpha^2 + 4\alpha A^2 + \beta\right) + \left(-9\alpha^2 A^4 + \frac{1}{4}\alpha A^2\beta - A^4\beta\right)t^2, \quad |t| \ll 1$$

This is a parabola that must be negative at $t = 0$ which gives a necessary criterion for the seeding pulse to be a valid RW:

$$\beta \geq \alpha^2, \quad A^2 \leq \frac{\beta - \alpha^2}{4|\alpha|}. \qquad (5)$$

Note that when the conditions of Eq. (5) are met the second term in the expansion of $F(t)$ always decreases so at least locally one does have a spatiotemporal maximum.

4. Conclusion

In conclusion, we have studied the possible shapes of occurring RWs in the NLSE systems and arrived at the following necessary and sufficient conditions for any seeding pulse $q_*(t)$ to be a (local) RW:

(i) The pulse must be *real transform limited and symmetric*;

(ii) The "kurtosis" of the normalized pulse, $K = \beta/\alpha^2$ must be greater than one;

(iii) There is a pulse dependent upper bound for the permitted amplitude $A_{\max} = q_*(0)$.

For example, the kurtosis of a Gaussian pulse is $K_G = 3$ and for sech it is $K_S = 5$ with both having the critical amplitude $A_{\max} = 1$.

The above criterion is quite general and relies only on the form of the NLSE and local properties of the pulse in the vicinity of the maximum. It does not rely on integrability, boundary conditions, or the global characteristics of the pulse

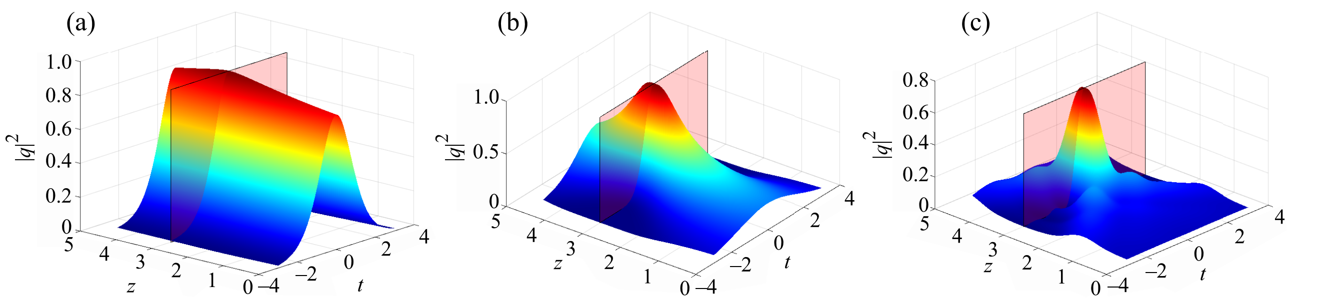

Fig. 1. (Color online) The spatiotemporal intensity evolution obtained numerically for (a) sech, (b) Gauss, and (c) RC target pulses for (sub)critical amplitudes $A = 0.9$ (a), $A = 1$ (b), and $A = 0.8$ (c). The target location z_* of the RW is shown by the transverse plane.

(e.g., the energy, pulse area, etc). Since the NLSE model used in this analysis is universal the obtained results will also be applicable to the variety of the hydrodynamic systems [16] with the obvious substitution $z \to t$ (distance to time) and $t \to x$ (time to distance).

1. E. Pelinovsky, *Nonlinear Dynamics of Tsunami Waves*, Applied Physics Institute Press, Gorky (1982).
2. E. Pelinovsky, C. Kharif, and A. Slunyaev, *Rogue waves in the ocean — review and progress*, in: EGU General Assembly Conference Abstracts, *EGU General Assembly*, Vienna (2010), p. 202.
3. C. Kahrif, E. Pelinovsky, and A. Slunyaev, *Rogue Waves in the Ocean*, Springer Nature (2009).
4. A. Hadjihosseini, J. Peinke, and N. Hoffmann, *Stochastic analysis of ocean wave states with and without rogue waves*, *New J. Phys*. **16**, 053037 (2014).
5. E. Didenkulova, *Catalogue of rogue waves occurred in the world ocean from 2011 to 2018 reported by mass media sources*, *Ocean Coast. Manag.* **188**, 105076 (2020).
6. M. Onorato, A. R. Osborne, M. Serio, and S. Bertone, *Freak waves in random oceanic sea states*, *Phys. Rev. Lett.* **86**, 5831 (2001).
7. V. Zakharov, R. Shamin, and A. Yudin, *Energy portrait of rogue waves*, *JETP Lett.* **99**, 514 (2014).
8. D. Solli, C. Ropers, P. Koonath, and B. Jalali, *Optical rogue waves*, *Nature* **450**, 1054 (2007).
9. J. Dudley, G. Genty, and B. Eggleton, *Harnessing and control of optical rogue waves in supercontinuum generation*, *Opt. Exp.* **16**, 3644 (2008).
10. J. Dudley, G. Genty, A. Mussot, A. Chabchoub, and F. Dias, *Rogue waves and analogies in optics and oceanography*, *Nat. Rev. Phys*. **1**, 675 (2019).
11. N. Akhmediev, J. M. Dudley, D. R. Solli, and S. K. Turitsyn, *Recent progress in investigating optical rogue waves*, *J. Optics* **15**, 060201 (2013).
12. P. Suret, R. E. Koussaifi, A. Tikan, C. Evain, S. Randoux, C. Szwaj, and S. Bielawski, *Single-shot observation of optical rogue waves in integrable turbulence using time microscopy*, *Nat. Commun*. **7**, 13136 (2016).
13. S. Residori, U. Bortolozzo, A. Montina, F. Lenzini, and F. Arecchi, *Rogue waves in spatially extended optical systems*, *Fluctuation and Noise Letters* **11**, 1240014 (2012).
14. K. Hammani, B. Kibler, C. Finot, and A. Picozzi, *Emergence of rogue waves from optical turbulence*, *Phys. Lett. A* **374**, 3585 (2010).
15. S. Derevyanko, A. Redyuk, S. Vergeles, and S. Turitsyn, *Invited article: Visualisation of extreme value events in optical communications*, *APL Photonics* **3**, 060801 (2018).
16. M. Tlidi and K. Panajotov, *Two-dimensional dissipative rogue waves due to time-delayed feedback in cavity nonlinear optics*, *Chaos* **27**, 013119 (2017).
17. C. Bonatto, S. D. Prado, F. L. Metz, J. R. Schoffen, R. R. B. Correia, and J. M. Hickmann, *Super rogue wave generation in the linear regime*, *Phys. Rev. E* **102**, 052219 (2020).
18. M. Onorato, S. Residori, U. Bortolozzo, A. Montina, and F. Arecchi, *Rogue waves and their generating mechanisms in different physical contexts*, *Phys. Rep*. **528**, 47 (2013).
19. N. Akhmediev, A. Ankiewicz, and M. Taki, *Waves that appear from nowhere and disappear without a trace*, *Phys. Lett. A* **373**, 675 (2009).
20. A. Tikan, C. Billet, G. El, A. Tovbis, M. Bertola, T. Sylvestre, F. Gustave, S. Randoux, G. Genty, P. Suret, and J. M. Dudley, *Universality of the peregrine soliton in the focusing dynamics of the cubic nonlinear Schrödinger equation*, *Phys. Rev. Lett.* **119**, 033901 (2017).
21. A. Tikan, S. Randoux, G. El, A. Tovbis, F. Copie, and P. Suret, *Local emergence of peregrine solitons: Experiments and theory*, *Fron. Phys.* **8**, 561 (2021).
22. S. Chen, C. Pan, P. Grelu, F. Baronio, and N. Akhmediev, *Fundamental peregrine solitons of ultrastrong amplitude enhancement through self-steepening in vector nonlinear systems*, *Phys. Rev. Lett.* **124**, 113901 (2020).
23. G. P. Agrawal, *Nonlinear Fiber Optics*, Academic Press (2013).
24. S. Randoux, P. Suret, and G. El, *Inverse scattering transform analysis of rogue waves using local periodization procedure*, *Sci. Rep*. **6**, 292238 (2016).
25. B. Yang and J. Yang, *Rogue wave patterns in the nonlinear Schrödinger equation*, *Physica D: Nonlinear Phenomena* **419**, 132850 (2021).
26. D. S. Agafontsev and V. E. Zakharov, *Integrable turbulence and formation of rogue waves*, *Nonlinearity* **28**, 2791 (2015).

Інверсійний дизайн екстремальних хвиль

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Досліджено характеристики початкових умов, що призводять до виникнення екстремальних хвиль великої амплітуди в межах нелінійного рівняння Шредінгера. Розглянуто декілька найбільш репрезентативних форм хвиль з високою амплітудою як потенційних кандидатів на блукаючи хвилі та отримано достатні критерії для їхніх параметрів. Також оцінюється максимальна норма отриманого розв'язку для блукаючої хвилі. Отримані результати відкривають можливості для аналізу властивостей початкових умов, які можуть спричиняти появу високоамплітудних сплесків поля в додатках, що описуються нелінійним рівнянням Шредінгера.

Ключові слова: екстремальні хвилі, блукаючі хвилі, феноменальні хвилі, солітон Перегріна, нелінійне рівняння Шредінгера.