CONSENSUS, CONTROL AND MESSAGE PASSING IN COMPLEX CONTROL SYSTEMS

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Thesis Summary

Real-world emerging systems are characterised by several challenges and high level of complexities, such as stochasticity, nonlinearities, high dimensionality, and systems with coupling. The aim of this thesis is to address these inconveniences in order to develop robust control algorithms for such real engineered emergent systems.

The study in this thesis considered the development of the fully probabilistic (FP) framework that addresses the main challenge of controlling real-world stochastic and uncertain systems. The probabilistic framework characterises the dynamics of the system to be controlled in terms of probability distributions which is a desirable approach to handle the stochasticity of dynamical systems. Nonlinearity of real-world systems on the other hand, hinders the derivation of analytic control solutions, yielding expensive numerical computations. To address this problem, a transformation method has been introduced to the developed FP control framework which facilitated the derivation of an analytic solution despite the nonlinearity of the system dynamics. This method transformed the nonlinear state function to another variant where the nonlinearities are preserved but have now been transformed to a nonlinear affine state function. The inclusion of this novelty allows for the control of more realistic systems which tend to be nonlinear.

Further advancement includes the extension of the developed nonlinear FP control method to control large-scale complex nonlinear systems. This is achieved by decomposing the complex system into small subsystems and then decentrally controlling each individual subsystem by a local controller. Probabilistic message passing is thereafter used to coordinate between the subsystems constituting the complex system, thus achieving the overall objective of the controlled complex system. This decentralised control framework has further been advanced to consider several control objectives, including regulation, tracking and formation control where the subsystems that constitute the overall network rely on the probabilistic message passing approach to interact with each other.

Dedication

I dedicate this accomplishment to my mother and father; آپ میری جان ہو!

I would also like to dedicate this to my grandfather who has sadly left us all but demonstrated constant faith in my ability to pursue a PhD during his life.

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Chapter 1

Introduction

1.1 Overview of the Research Problem

Since the arrival of the modern era of mechanical and technological devices, many systems in the real-world have not just grown in complexity but also dimensionality which raised many challenges to modern control theory and its effectiveness in practice. Countless systems have evolved in nature resulting in complex networks which are composed of a large assemblage of elements that interact with each other. The concerning problems that accompany such large-scaled complex systems are omnipresence, intrinsic nonlinearities, high dimensionality, coupling and the effects of high level of uncertainties [1]. These challenging characteristics make the process of analysing, approximating, modelling and particularly regulating real-world systems increasingly onerous [2], [3], [4]. Thus, traditional control designs have been further developed and new control strategies have been introduced to facilitate the control of these systems [5]. Uncertainties found within real systems were studied profusely to ensure robustness is achieved during the control process [6], [7]. One approach introduced to effectively handle this complexity was stochastic control theory due to its suitability in dealing with uncertainty [8]. The theoretical framework consists of representing systems and their environments by a stochastic model, and minimising the expected value of a cost function. Several stochastic control strategies were introduced, one of which is the fully probabilistic control design method [9]. This method is based on the minimisation of the divergence measure between two probability distributions named the Kullback-Leibler divergence (KLD). FP control design is found to be very efficient in controlling stochastic systems through the derivation of randomised control solutions.

Nonlinear control is one of the most well-researched areas in the field where advanced control strategies are required to assist the complicated nature of the systems which demand more rigorous design specifications [10], [11]. Promising and popular optimal control approaches have been further developed to extend it to nonlinear systems [12], [13], [14], [15]. However, the main limitations of some of these methods is the involvement of online-dynamic optimisation which results in computational delay [11] and the requirement of linearisation techniques such as feedback linearisation. Furthermore, optimal control methods have also been considered for nonlinear stochastic systems [16], [17], [18], but were solved numerically and were hence, unable to provide analytic control solutions.

To address the high dimensionality of complex systems, further control advances in the control literature have proposed control strategies such as decentralised control [19], [20], [21] distributed control [22], [23], pinning control [24], [25], probabilistic control [26], consensus control, [27], and multi-agent control [28], to name a few. Nevertheless, some of these strategies are inadequate due to the construction of the controller as a single-agent which is centralised and thus, means that the global state of the control system must be completely observable and known. Another reason is that some of the aforementioned decentralised control approaches rely only on disconnected and imperfect information for the decision making process. Moreover, these developments are more likely to ignore the uncertainty in a control process, and thus do not prove to be efficient in regards to dealing with the features of complex networks. Therefore, to assist real systems and suit their complex nature accordingly, more precise and intricate control designs are required. An approach that is decentralised, does not assume accessibility to global knowledge and considers the stochastic nature of the considered control system is required to solve these problems.

To re-emphasise, the issue of overlooking some properties of complex dynamical control systems while focusing on controlling large complex systems results in non-optimal decisions being taken leading to a less effective control strategy. Therefore, this thesis will investigate the development of efficient centralised and decentralised probabilistic control algorithms that address many of the aforementioned challenges of real-world dynamical systems.

1.2 Thesis Aims

In this thesis, the aim is to further research and develop the Fully Probabilistic Control Design in order to make it applicable to a wider range of systems which can be found in the real-world. As mentioned before, the FP control design has proven to be an appropriate control strategy to handle stochasticity of dynamical systems. Evidently, disregarding the stochasticity and uncertainty in the dynamics of the controlled system can result into poor performance, thus need to be considered in the derivation of the control law. It is recognised that failure to acknowledge these sources of randomness may lead to systems instability and robustness cannot be achieved. Therefore, this work will look at the development of the FP control method such that it considers uncertain information in control systems.

Moreover, the FP control design was originally proposed in a centralised way restricting its application to large-scaled complex systems. Therefore, exploiting the information structure of such complex systems within a FP control design framework would prove to be rewarding. This involves the process of decentralisation to obtain subsystems that constitute the overall complex system. It is then achievable to understand the behaviour of the complex system by comprehending the behaviour of the simpler subsystems. However, it is key to find a method that allows the subsystems to interact with each other to guarantee the global behaviour of the system is accomplished. In essence, the aim is to meet the global objective of the complex control problem through enforcing local actions at the subsystems that compose the complex system. This forms part of the work in this thesis.

1.3 Thesis Contribution

Further development and exploitation of the FP control design has led to the novel proposed control approaches in this thesis. This contribution will enable the control design to be shaped in a manner that will allow it to perform well for several types of large-scaled complex real-world systems. To be more specific, the contribution of this work falls under

- **The development of the FP controller design for nonlinear stochastic systems.** Many real world systems tend to be governed by nonlinearities. This nonlinearity of the systems, generally hinders the derivation of analytic control solutions of the controller and in particular in the FP control framework. Thus, the extension of these control methods such that the nonlinearity of the systems is considered without the need to linearise their dynamics, has been investigated and developed in this thesis. This contribution ensures that the exact nonlinearities are preserved during the control process, thus, guaranteeing more effective control, and facilitates the derivation of analytic control solutions of the FP control design. To clarify, the main contribution in this part of the work is the derivation of an approximate closed form solution to the optimal control problem for the case of nonlinear dynamics, albeit one that performs well in simulations and is computationally advantageous. This has been achieved mainly following a FP design approach and referred to as the generalised FP control design. This generalised FP control design framework for nonlinear systems is further developed for a number of perturbations to ensure real control problems can be solved in a probabilistic way:
  - **Generalised FP control algorithm designed for nonlinear stochastic systems affected by additive noise.** The analytic solution of the FP control design is facilitated through transformation methods such that the dynamics of the nonlinear systems are expressed to be affine in both the state and control signals (the state and control matrices are multiplied

by the state vector and control input respectively). This novel development of the FP control design is considered for the first time in this work where the system dynamic is assumed to be affected by additive noise.

  - **Generalised FP control algorithm derived for uncertain nonlinear stochastic systems.** The dynamics of real systems tend to be unknown and thus, the underlying physics of the systems need to be estimated. The estimation process however inevitably leads to functional uncertainty where the estimated and actual values will differ from each other. Thus, this work has further developed the FP control approach such that the estimation error between the estimated and actual values is characterised and used in the derivation of the control signal to improve the control accuracy of the considered systems.
  - **Generalised FP control algorithm designed for nonlinear systems with multiplicative noise.** In the real world, systems are also influenced by multiplicative noise. Therefore, a novel algorithm within the FP control design framework has been designed for such systems and has been published in [29]. This work is based on our initial development of the FP control design which considered linear systems affected by multiplicative noises [30]. However, only the development for nonlinear systems will be discussed in this thesis.

- **Decentralised FP control algorithm for nonlinear systems.** To solve large-scale complex control problems, the FP control design has been further developed to consider the decentralised compositions of nonlinear systems to ensure the control algorithm is adapted to the decomposed structure of the network accordingly. The notion of probabilistic message passing has been introduced for information to be exchanged between the subsystems according to the information structure. Rather than deriving a global randomised controller, as was done for the other algorithms for single dynamical systems, multiple controllers are developed to ensure the subsystems are controlled and the global behaviour of the complex network is achieved.

  This novel decentralised advancement involves a number of probabilistic controllers that convey knowledge through probabilistic messages which allows the decomposition of the control of the large complex network to achieve a group of smaller control subproblems. Each of these small subproblems can be dealt with independently, meaning the analysis and implementation of these can now be conducted individually.

- **FP control algorithm for tracking control problem with stochastic reference for nonlinear systems with multiplicative noises.** A fully probabilistic control strategy has been designed for nonlinear systems with multiplicative noises that require the tracking of a stochastic reference model.

- **FP control algorithm for formation control for systems with additive noises.** The decentralised approach for nonlinear complex systems is further developed to extend it to formation control problems. This designed algorithm is based on the tracking error and generated good results when simulated for robotics examples. The control method was able to influence the position of the robots and force them to form the desired pattern.

1.4 Outline of the report

The structure of the thesis is as follows:

- **Chapter 2: Literature Review.** This chapter presents some of the significant challenges found in complex systems and the control approaches that have been introduced to address these challenges in the control literature. The theoretical frameworks that are surveyed in this chapter are: centralised control, distributed control, and decentralised control approaches. Within each framework, some of the proposed control strategies such as the developments of adaptive, optimal and stochastic control, are presented. Furthermore, the Fully Probabilistic control design is surveyed to demonstrate how this methodology can address the shortcomings of the other discussed methods and the extent to which it can be exploited. This exploitation is based on the limitations of the current FP control approaches which are discussed in this chapter.

- **Chapter 3: Fully Probabilistic Control for Nonlinear, Stochastic, and Uncertain Systems.** In this chapter, centralised Fully Probabilistic control designs are developed for nonlinear systems. In these developments, the derived form of the controller was affected by the nature of the stochasticity of the system and the consideration of functional uncertainty. Firstly, a randomised control algorithm is derived for nonlinear stochastic systems that are affected by additive noise. This method is then further extended such that the functional uncertainties of systems are characterised and taken into consideration in the derived optimal nonlinear controllers. Lastly, nonlinear systems affected by multiplicative noises is another class of systems for which the solution to the FP control problem is established. The effectiveness of the centralised controllers for nonlinear systems is demonstrated through simulation examples.

- **Chapter 4: Decentralised FP Control Design for Complex Systems.** Decentralised controllers within a FP control design composition are designed for nonlinear complex systems in this chapter. It discusses how the decomposed subsystems interact with each other to ensure the global objective of the controlled system is achieved. The proposed method is illustrated on a simulation study.

- **Chapter 5: Decentralised FP Control Design for Tracking and Multi-Agent Formation Control.** The decentralised control approach is further utilised and extended to tracking and formation control problems. Firstly, for tracking control, local controllers are designed such that each subsystem follows its corresponding predefined stochastic reference. Secondly, the design of controllers that allow subsystems to communicate with each other while controlling themselves independently to form a certain pattern is demonstrated in this chapter. In addition, the convergence of the developed optimal randomised controller for a formation problem is analysed. Simulation examples are implemented for both control strategies developed in this chapter to illustrate the practicality of these designs.

- **Chapter 6: Conclusion and Direction for Future Work.** This chapter will conclude this thesis by summarising the contributions of the work developed throughout and the novelty of the results achieved. Furthermore, some recommendations for future work is mentioned which could prove to be fruitful for the field of control.

- **Appendix A: Derivation of the FP Control Solution for Nonlinear Systems.** This appendix demonstrates the derivation of the conventional FP control solution for nonlinear systems with additive noises derived in Chapter 3.

- **Appendix B: Derivation of the FP Control Solution for Nonlinear Systems with Functional Uncertainty.** The second fully probabilistic control strategy derived in Chapter 3 which takes functional uncertainties in nonlinear systems into consideration, is proven in this appendix.

- **Appendix C: Derivation of the the FP Control Solution for Nonlinear Systems with Multiplicative Noise.** This appendix provides the proof of the third method developed in Chapter 3 which considers nonlinear systems with multiplicative noises.

- **Appendix D: Derivation of the Decentralised FP Control Design for Nonlinear Systems.** Randomised controllers are developed in Chapter 4 for nonlinear systems with additive noises within a decentralised framework. The derivations of this control strategy are outlined in this appendix.

- **Appendix E: Derivation of the Probabilistic Message Passing Approach.** The control design developed in Chapter 4 and 5 involves the probabilistic message passing approach such that subsystems can communicate with each other. This methodology is key in the design of the decentralised fully probabilistic control design. Therefore, this appendix provides a detailed proof of the probabilistic message passing framework in this appendix.

- **Appendix F: Derivation of the Decentralised FP Control Design for Nonlinear Systems with Multiplicative Noises.** This appendix demonstrates the derivations of the FP control solution for nonlinear systems with multiplicative noises within a decentralised framework.

- **Appendix G: Derivation of the Randomised Controller for Tracking Control of Nonlinear Systems with Multiplicative Noises.** The derivation of the optimal randomised controller for a tracking control problem which has been developed in Chapter 5 is outlined in this appendix. The proof demonstrates how the control solution takes the multiplicative noises that affect the dynamics of the nonlinear systems into consideration.

- **Appendix H: Derivation of the Randomised Controller for Formation Control of Nonlinear Systems with Additive Noises.** This appendix provides the derivations of the local randomised controllers that aim to influence the dynamics of the controlled subsystems such that they form the desired formation. It proves the advancement of the FP control design for formation control problems given in Chapter 5.

- **Appendix I: Convergence Analysis of a Decentralised Nonlinear System with Additive Noises.** This appendix shows the methodology of analysing the convergence of the developed controller for formation problems in Chapter 5 for decentralised nonlinear systems with additive noises.

Chapter 2

Literature review

2.1 Overview of Control Engineering

The behaviour of an engineering system can be affected by an external signal which is sent to ensure the control aim of the system is achieved. This external signal is referred to as the control input to the system. Control systems can be categorised as either being an open-loop or closed-loop. An open-loop control system consists of predefined control inputs for a system that is assumed to be operating in an ideal situation, meaning full knowledge about the system and its operating condition is assumed and uncertainties are assumed to be absent in the system. The knowledge about the system output does not play a role in determining the control inputs. As a result, any changes affecting the system dynamics are not considered by the open-loop system, thus lead to a poor control performance. In contrast, closed-loop systems exploit the information about the output variables of the system by feeding it back which is reflected in the form of a more effective and appropriate control input. This contributes to an accurate control performance where the closed-loop controller compensates for uncertainty in model parameters and measurement noise in the system.

Nevertheless, it is fundamental to seek a mathematical model for the dynamics of a real control problem to ensure the correct strategies are enforced to influence the behaviour of the system to be controlled. The mathematical models used to express the underlying physics of the systems are idealisations which means some model discrepancy is bound to be found. Thus, the challenges that arise with real systems are in the form of uncertain elements from functional uncertainties due to the dynamics being unknown or incomplete, or due to uncertainties from the environment in which the systems are operating [31]. These uncertainties prevent the system from operating as expected if they are not taken into account in the design of the system controller. Therefore, it is vital to have control actions in place to correct the error signal which is the difference between the actual and desired states of the systems. As can be concluded from the previous discussion, closed-loop systems posses the ability to correct the error signals and are therefore more suitable for real-world problems. In addition, feedforward [32] and feedback control [33] are control strategies that take actions to compensate for any effects that fluctuations, uncertainties and noises have on the dynamical systems. Feedfoward control deals with the perturbation before it influences the expected performance and output of the system. Thus, it needs to possess the ability to predict the effect of perturbations on the output of the system. Consequently, when considering feedforward control, it is important to know which external changes to regard as perturbations and how to prevent those from influencing the system. Therefore, acquiring information beforehand about possible changes and disturbances is key for this control method. Unpredicatable perturbations on the other hand can normally be considered by designing feedback control strategies. It can be said that feedback control takes time to correct the output of the plant as it waits for an error or deviation to occur before correcting it. However, it is a very efficient control method in the long term leading to stability as early intervention prevents the system from deviating to a greater extent. Feedback is the solution to automatic control where reliance on human interference is not required. More calculated and informative decisions are made due to the state variables being fedback. On the contrary, feedforward control is a method that aims to make a system error-free, but cannot achieve this in reality due to the existence of uncertainty and external noises which cannot be measured beforehand. To conclude, feedback control is therefore the preferred method as large-scale complex systems in the real world contain many uncertain elements which cannot be predicted in advance [34].

2.2 Centralised Control

A centralised controller requires full knowledge of the global state since it is in charge of managing the entire system by influencing the behaviour of the system to ensure it performs as expected. Many control strategies have been proposed and introduced in the control literature and have proven to be effective. Some of these that have gained recognition in the field of control include stochastic control, optimal control, and adaptive control, to name a few. The aforementioned methods are surveyed in this section due to its increasing popularity and relevance in this thesis. However, as the size of the network increases, a centralised controller will require a tremendous amount of computation and processing power to prevent any performance and response time issues.

2.2.1 Optimal Control

The application of control systems are widespread and can be found in numerous fields. Some of these include vehicle control systems [35], aircraft engine control [36], pharmaceutical manufacturing

[37], and chemical systems [38]. From the aforementioned application domains, the significance of obtaining controllers that operate to the best of their abilities can be realised. In other words, control strategies that deliver optimal results are vital for the satisfactory operation of these systems. On this account, 'optimality' needs to be defined in order to find such a control method. This can be achieved by defining the task that needs to be conducted and specifying a mathematical formulation that determines what can be considered optimal.

Optimal control theory is concerned with designing an optimal controller that finds a variable for a dynamical system to influence the state variables over a given period of time. Given a set of permissible inputs, the aim is to seek a sequence of control inputs that results in an optimal path for the state variables. The optimal response of the system states is determined by a cost function which is expected to be maximised or minimised.

Many techniques have been proposed to solve an optimal problem such as calculus of variations [39] which is implemented to achieve the control trajectory that minimizes the performance index by looking for the minimum value of a functional. A functional is a function of a function, and the reason this optimal control problem deals with functionals is due to the fact that the state and control trajectories are functions of time. In [40], a sequential estimation technique was proposed for nonlinear systems of which the dependency was on the minimum least square measure which involved the implementation of calculus of variations. However, the minimum least square criterion implemented in [40] does not require information about the unknown inputs to the system and measurement errors of the output. Thus, the control problem was redeveloped as a deterministic optimal control problem. In addition, the derivation of the estimator includes linearisation using Taylor series.

The concept of optimisation is theoretically appealing but can be restrictive in terms of its application due to being computationally expensive. Various strategies to solve optimal control problems were introduced including iterative numerical methods such as dynamic programming [41] which is a method that is widely used [42], [43] and has been further explained in Section 2.2.1. Some of these iterative methods face the challenge of having to be used in an offline fashion, which is troublesome for nonlinear systems due to not being able to respond and adjust rapidly to changes in its dynamics.

Achieving optimal control for nonlinear systems is a very challenging task within the field of control as mentioned in [44]. A possibility is to transform the nonlinear optimal control problem into the Hamilton-Jacobi-Bellman partial differential equation [45]. However, the practicality of this approach is restricted due to the difficulties in its solution.

Primbs combined the generalisation of the Lyapunov method called Control Lypunov Functions (CLF) [46] and Model Predictive control (MPC), also called receding horizon control, to achieve nonlinear optimal control. The main aim of CLF is to provide information that is enough to produce a control technique for stability purposes [44]. For MPCs, the goal is to achieve optimum performance [47]. MPCs adopts a moving horizon approach in order to strive for an estimation to the optimal control problem in an online manner. MPCs have demonstrated their effectiveness for the computation of systems which were capable to be computed online [48], [49]. However, the numerous benefits do not imply that there are no concerns regarding stability and the application of it in practice, for which research is still being conducted [50], [51].

Furthermore, the integration of neural network methods with optimal control has also been considered in an attempt to achieve better performance[52], [53], [54].

Dynamic Programming

Dynamic programming (DP) is an optimisation method introduced to assist in solving multi-stage decision problems [41]. It is a technique developed for computational purposes to ensure that the principle of optimality is applied to decision sequences which describe an optimal control policy. The output of the previous decision is used to assist with the decision to be made for future ones. In fact, dynamic programming is based on the principle of optimality.

The principle of optimality considers an optimal policy which is described as follows: regardless of what the initial state and decisions are, the decisions made subsequently must form an optimal policy following the state obtained from the first decision. Hence, the multiple decisions made following the initial state need to be completed in an optimal manner. The principle of optimality reduces a multi-stage problem to a sequence of single-stage problems, thus making the optimisation process computationally efficient. Having constraints on the state and control input results into a reduction of the number of permissible values to be searched which leads to a simplification of the process of establishing the optimal solution. A direct comparison is made between all entries of the permissible control law, thus, ensuring that a global optimal control law is guaranteed [55], [56], [57].

This method can be applied to deterministic systems which are systems where the initial state and decision uniquely determine the next state. It can also be applied to stochastic models where the next state value cannot be determined exactly due to the effect of noise [55]. Furthermore, dynamic programming techniques are also used when implementing control strategies such as Linear-Quadratic Regulators [58] (which is discussed in Section 2.2.1). The main problem with dynamic programming is that it follows intensive search to find the optimal solution. DP implements the principle of optimality in order to decrease the number of computations when finding the optimal control law, yet systems with a high dimension can cause computational restrictions on attaining minimum cost. Thus, the curse of dimensionality [59], [60] which is a set of problems that arise when working with high-dimensional data is another challenge that dynamic programming faces. This problem is a hindrance in the process of solving dynamic optimisation problems by backwards induction due to the high dimensionality of the problem.

Dynamic programming involves the Bellman equation [41] which is a requirement for optimality. A Bellman equation is a recursive equation and is also called a functional equation, as its solution consists of finding the unknown value function. The value function is a function of the state that returns the best value of the control objective. Calculating the value function also allows one to find the policy function which is also a function of the state but gives the optimal action. Originally, it was applied to discrete time control problems but was later applied to continuous time control problems and referred to as the Hamilton-Jacobi-Bellman theory [55]. The problem that arises is that both the computation and memory requirements of dynamic programming grow exponentially with the state dimension [61].

Linear Quadratic Regulator

The Linear Quadratic Regulator (LQR) approach is an optimal control method introduced by R. Kalman in 1960 which was extended from the work by Lyapunov [58]. This approach consists of a linear system with a quadratic cost function and has been discussed abundantly in the control literature and applications [62]. The LQR problem can be solved analytically and the derived optimal controller is described in a state feedback form which is simple and straightforward to implement. The solution of the LQR problem includes the derivation of a Riccati equation which is explained later and again, relies on the aid of dynamic programming to solve this optimisation problem.

Since the focus of this thesis is on the FP control design and includes variants of the Riccati equation as part of the solution of the FP control problem, the LQR is discussed briefly due to its relevance to the researched method in this thesis.

For a deterministic model with a finite time horizon, $H = \{0, 1, ..., N\}$, described by vector $x_k = Ax_{k-1} + Bu_k$, where $x_k \in \mathcal{R}^n$ is the state variable, $u_k \in \mathcal{R}^r$ is the control input, and A and B are the state and control matrices respectively, the state feedback controller which is linear in the states is given by $u_k = -K_k x_{k-1}$ where K_k is the optimal control gain. The optimal controller minimises a quadratic (in the state) performance index $J_{i,N}(x_k) = \frac{1}{2}x_N^T P x_N + \frac{1}{2}\sum_{\tau=i}^{N-1} x_\tau^T Q x_\tau + u_\tau^T R u_\tau$, where $J_{i,N}$ means the cost associated from time i to time N. It is important to note that matrix Q is an $n \times n$ symmetric matrix, i.e. $Q = Q^T$ and positive semidefinite. Furthermore, matrix R is also an $r \times r$ symmetric matrix, but positive definite, i.e. $R = R^T$. Matrix P is an $n \times n$ symmetric positive definite or positive semidefinite matrix. Matrices Q, R, and P establish the significance of the error and the cost of the energies related to the state vector, control vector and the final state, respectively. The optimal control law u_k^*, the optimal control gain K_k, and the Algebraic Riccati Equation solution P_k are given by:

$$u_k^* = K_k x_{k-1}, \tag{2.1}$$

$$K_k = -(B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A, \tag{2.2}$$

$$P_k = A^T P_{k+1} A - A^T P_{k+1} B (B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A + Q. \tag{2.3}$$

This method has been extended to nonlinear systems but required linearisation of the dynamics to allow the application of the LQR approach [63], [64]. However, a nonlinear quadratic regulator (NQR) approach has also been introduced in the literature which is based on the solution of a state dependent Riccati equation [65], [66]. Essentially, the approach is motivated by the LQR formulation for linear systems. The strategy involves the use of parameterisation to adjust the nonlinear system such that it has a linear structure with state-dependent coefficient matrices. This is considered a linearisation approach but is expected to perform better than the aforementioned Jacobian linearisation approach. It updates the linear model approximation at each discrete time step.

The LQR optimal control law is designed for deterministic systems. However, stochastic systems have been considered for the LQR approach in the control literature. Fisher et. al developed a LQR control design for stochastic system which required the stochastic dynamics to be transformed into deterministic dynamics but in higher dimensional state space [67]. However, an optimal control strategy named the Linear Quadratic Gaussian control approach has been developed for linear stochastic systems which is discussed in Section 2.2.2. In addition, Kárnỳ proposed the FP control design which is an optimal control problem for stochastic systems [9]. A control solution was only demonstrated to be derived for linear stochastic systems. A more detailed survey can be found in Section 2.4.

State Dependent Riccati Equation

Nonlinear dynamics can be represented by a linear form which consists of state-dependent coefficients [68]. The dynamics given by $x_k = h(x_{k-1}) + g(x_{k-1})u_k$ can be described by the linear system state equation $x_k = Ax_{k-1} + Bu_k$, where $h(x_{k-1}) = Ax_{k-1}$ and $g(x_{k-1}) = B$. Here, however, A and B would not be constant and are instead state dependent. Once the system state is rewritten in a linear structure with state-dependent coefficients, the SDRE can be solved. The name is derived from the fact that some parts of the Riccati equations are state dependent.

Many methods have been developed which involve State Dependent Riccati equations (SDRE). It has been shown in [69] that solving an algebraic Riccati equation which evolves over time results into a suboptimal solution of the infinite horizon problem.

2.2.2 Stochastic Control

Two concepts that have been widely investigated in control engineering are deterministic [61], [70] and stochastic control theory [71]. Systems where the value of the output is exactly known given the parameters and initial conditions are referred to as deterministic. On the other hand, stochastic control theory deals with dynamical systems that are affected by disturbances and random noise. These systems behave in such a manner that the exact values of the state variables are unknown. Both, deterministic and stochastic approaches aim to design robust controllers to guarantee that control systems perform satisfactorily, or practically optimally, even when knowledge about the system itself is unavailable. Robustness is accomplished with the aid of a control strategy that achieves its purpose of controlling the system dynamics despite the uncertain nature of the system. A great deal of effort was spent on addressing and characterising the uncertainty with many researchers focusing on developing robust control methods in a deterministic fashion [3], [72], [73], while others assumed that the 'actual' system was a constituent of some domain centred around a nominal model on which the control design was based [74], [75]. It was therefore believed that prior information of the disturbance is known [8]. Since the noisy and uncertain behaviour of systems tend to be unknown and unpredictable, this assumption hinders the applicability to real problems. As a result, a coherent strategy to handle the uncertain nature of the plant was suggested by introducing a probabilistic concept to the uncertainty of the model, namely stochastic control theory.

The randomness of systems is of paramount importance in research and builds the foundation of many studies [76], [77]. Hence, the focus of the study in this thesis is on the behaviour of stochastic systems. The randomness found in such systems can be regarded as a mere complexity encountered by systems which would have otherwise been labelled as deterministic. Nevertheless, stochastic systems are a fundamental class of systems due to its practicality which has seen a thriving interest and has considerably developed since many control strategies are concerned with the analysis, design and control of such systems [78]. Standard stochastic control problems require the expected loss function to be minimised with respect to the feedback control strategy [8] and are essentially managed by implementing optimal control approaches.

Controlling a stochastic system is usually difficult as shown in the literature [79], [80] that describes the necessity of having significant memory and computation time. Thus, to deal with this problem, linear stochastic systems with random variables from the Gaussian distribution were mainly considered constraining the objective function to the mean or the variance of the stochastic output of the system [81], [82]. However, in practice, there exist systems with variables or noises that do not belong to the Gaussian distribution. Hence, in those cases, the behaviour of the closed-loop system cannot be described by merely the mean and variance of the output of the system [24]. A few techniques have been proposed to control such systems. In 2009, one proposal involved the development of a stochastic distribution control system [83]. The idea involves the modelling of a controller that enables the probability density function of the output of the system to follow a predetermined ideal probability density function. The second proposed technique involved the control of a closed-loop probability density function [9], [84]. This allows the controller to impact the closed-loop behaviour of the system. And lastly, there is the regulation of the tracking error probability density [85].

A recognised method in the control community is the Linear Quadratic Gaussian (LQG) control approach that aims to find an optimal controller for linear stochastic systems affected by Gaussian noise through the minimisation of a quadratic cost function. The LQG method is a combination of LQR and the Kalman Filter. The Kalman filter is also referred to as a linear-quadratic estimator (LQE). Attempts were made to generalise this approach to make it applicable to nonlinear stochastic systems [86]. However, this approach requires the dynamics of the system to be linearised and a further complication is that optimality in the LQG sense does not necessarily imply robustness [87]. Many proposed methods require linearisation of the system before or during the control process such as discussed by He et al. [88] who attempted to retain the nonlinear characteristic of the system by implementing online linearisation. Consequently, this gap in the literature motivated the research on nonlinear systems. As such, this thesis discusses and presents an approach that preserves the nonlinearities of systems and does not require the linearisation of the dynamics of such systems.

The mathematical approach to unravel the complexity in the form of uncertainties lies within the field of probability theory which is a suitable means of handling stochasticity [89]. This is due to the ability of probabilities to model uncertainty, lack of predictability and complexity. As a result, randomised controllers have been developed to effectively handle uncertain systems with the aim to optimise the expected value of a suitable cost function with respect to the feedback control approach [71].

As mentioned previously, the FP control design proved to be a promising method [9] where Kárnỳ suggested to implement the probabilistic description of the closed-loop in the design model that needs to be controlled. In FP control, a randomised control strategy is designed as opposed to many of the control approaches where a deterministic control strategy is obtained. Since the dynamics of the systems are characterised by probability distributions, it makes sense for the control law to be derived as a distribution as well due to the fact that some statistical perturbations exist in the controller too. This approach is further reviewed in more detail in Section 2.4.

2.2.3 Adaptive Control

Adaptive control is a key branch in control theory that is concerned with the control of the output of a system which consists of uncertainty in the parameters, structure and environment. Adaptive control theory was originally described as an observer for single-input single-output (SISO) states of linear dynamical systems with unknown parameters [90]. This has been extended to multivariable states [91]. The field of adaptive control has evolved rapidly since the 1950's with significant contributions made by Astrom, Landau and others [92], [93], [94]. This theory involves the selection of a suitable controller for an unknown system to identify all the unknown parameters of the system and estimate the state simultaneously. The parameters of the model are estimated using rules that are referred to as adaptive laws.

Within this field, there are two different approaches to adaptive control theory [95], [96]:

1. Indirect adaptive control: This approach considers the approximation of the unknown parameters or state variables of the system which are then used to modify the control parameters [95], [97], [98], [99]. In the literature, these type of systems have also been called self-tuning regulators [92]. Indirect control is still being utilised, as shown in a recent paper that uses it for a class of fractional order systems [100]. The model parameters are approximated using identification laws, and as mentioned before, the control parameters are calculated using the approximated model parameters.

2. Direct adaptive control: In this approach, the control parameters are estimated directly with the objective to minimize some measure that calculates the error between the reference model outputs and the plant [95]. This normally requires the determination of a set of differential/difference equations that characterises the error of the output in terms of a function of the errors of the control parameters. For example, adaptive control laws are established in a form of differential equations which outline the changes in the control parameters and are given as functions of the output error [96].

Based on these two main adaptive control approaches, other forms of adaptive control have also been proposed and discussed in the literature. One variant considers a combined direct and indirect adaptive control strategy, yielding a method that utilises the advantages of the individual techniques combined [101].

In the 1980s, further developments have demonstrated that under certain assumptions regarding the model, it is possible to achieve global stability of the overall system by producing adaptive laws whilst also obtaining an output error that goes to zero [102], [103]. However, since then, the main focus moved towards the robustness properties of adaptive control systems [99]. For the generation of robust adaptive controllers, the following three elements were researched to a great extent:

a) the effect of external disturbances [104],

b) time-varying parameters [105], [106],

c) unmodeled dynamics [107].

In addition, the Multilayer Neural Network [108], [109] have also been combined with adaptive controllers for nonlinear dynamical systems as a method of estimating the dynamics of these systems [110], [111]. It was demonstrated in [110] that the indirect adaptive control approach was preferred for nonlinear control theory.

When the parameters of a plant are described as time-varying or uncertain, adaptive control methods [112], [110] are applied in an online fashion to ensure a good tracking performance. Nonetheless, these techniques do not prove to be efficient when dealing with more complex circumstances such as a multi-variable function describing a plant or if the system demonstrates distinct modes of behaviour during its operation [85]. A number of solutions to this problem have been suggested in [113], [114]. Furthermore, stochastic adaptive control is an effective technique to implement when considering uncertain systems [115], [116].

Adaptive control is one of the most efficient and widely implemented methods for nonlinear dynamical systems that are uncertain as well [117], [118]. Since a high degree of uncertainty exists in complex systems which needs to be controlled, this has become a challenge that today's control theorists are investigating [111]. Thus, it is advantageous to obtain a controller with the ability to adapt itself and adjust the parameters accordingly to deal with this issue. Hence, this branch of control theory seems to be extremely promising and the ongoing research within this field reflects this.

2.3 Large-Scale Complex Systems

The field of control is growing at a fast pace with research being conducted on real-world control systems. Emergent engineering systems are classified as complex systems and are known as complex networks due to their architecture which consists of various nodes interacting with each other at a large-scale. The complexity makes it difficult to analyse, design and implement control techniques. The challenges that surface in complex systems include high level of uncertainties, high dimensionality, and information structure constraints [1], [21]. For large-scale systems, knowledge about the state of the entire system may not be obtainable. Nevertheless, there may be cases where information of the global state is available, but it would be unmanageable to control the system in a centralised way due to the large scalability and highly complex nature which would require more processing power [119]. Thus, to learn more about the overall complex system, it is necessary to consider local information provided by the individual subsystems that constitute the complex system. However, not only the information about the individual subsystems is required, but also information about their interconnections. Many researchers have proposed various control strategies in an attempt to control large-scale complex systems that exhibit the challenges stated above [120], [121], [122], [123]. However, various developed methods seemed to be insufficient for the control of such systems as they would either be managed by a single centralised controller that requires complete knowledge of the global system which is challenging for large-scale systems or that the control algorithm is applied to a decentralised system which consists of incomplete and disconnected knowledge.

Two of the strategies to analyse and control complex systems that are discussed widely in the control literature include decentralised control and distributed control. Although occasionally, the terms distributed and decentralised control have been used interchangeably in the control literature, [124], there exists a fine line between them. Decentralised control can be seen as a subset of distributed control. The key difference lies between the way the control decision is made and how that decision reaches other nodes in the system. Distributed control systems consist of various control loops that are distributed all over the system. The control approach requires some interaction between the subsystems and decisions may still be made in a centralised fashion by a supervisory controller. This is an efficient method as it reduces the cost and increases the reliability by having local controllers and simultaneously ensuring the global objective is met with the supervision of the controller to which the local controllers report. Moreover, when a single processor fails due to outage or experiences a breakdown for instance, both, distributed and decentralised, control systems effectively diminish the impact of this on the network due to the fact that control of other nodes in the network proceed as normal. Only a small part of the network suffers from the technical disturbance.

It is key to consider the attributes of complex systems to realise the different challenges one can face and how to handle them when solving the control problem. The high dimensionality complexity can be resolved by decomposing the complex systems into subsystems and their interconnections. Appropriate local control designs can then be derived for the updated decomposed system which also consider the impact of the interconnections. There are decomposed process designs where the coupled subsystems are strongly connected or designs with weakly coupled subsystems [125]. The aim is to achieve a decomposition such that the sole responsibility of the designed local controllers is to influence the behaviour of the corresponding subsystem it has been derived for. It can be said that decomposition is a precondition for decentralised control in many cases. Numerous design techniques have been proposed to ensure a suitable decomposition of the complex system can be achieved and more efficient and accurate decentralised controllers can be obtained. The notion of the various decomposition strategies is to update the structure to ensure the controllers can be derived in such a manner that they consider the interactions with the other subsystems. One way to obtain the structural decomposition is based on the physical properties of the system [126]. This is however not always achievable since there exist systems where no suitable weak coupling can be found. Therefore, numerical decomposition has been suggested to accomplish the aim of generating more manageable subproblems. Some of these decomposition approaches can be categorised as disjoint decomposition, overlapping decomposition [127], [19], and epsilon decomposition systems [128], to name a few. The disjoint decomposition has been used extensively since it results in disjoint subsystems and interconnections. Furthermore, overlapping decomposition is implemented for large systems where there is some sort of intersection between the subsystems. With overlapping decomposition, the original system that consists of strongly coupled subsystems is expanded into a higher dimensional system but with weakly coupled subsystems. This concept has been applied in many fields to decompose the system in overlapping subsystems such as mechanical systems [129], [129], electric power systems [19], [130], [131], web winding systems [132] and many more.

There has been an increase in studies that focus on incorporating explicit knowledge of the interconnections in the control design and also on distributed control since it considers the communication between subsystems. Most approaches to decomposing the system have been introduced with the aim to augment the system with strong interconnecting subsystems to a system where the subsystems are disjoint or weakly connected. However, this means that the controllers are designed based on incomplete and disconnected knowledge. Consequently, numerous approaches have been exhausted to preserve the communication links between subsystems. In [133], one such attempt was made by Roberson et al. to deal with the decentralised control problem of a platoon of autonomous vehicles. Each vehicle was fitted with an observer to estimate its own state and also the states of the vehicles it communicated with. An observer based state feedback approach was implemented which allowed them to access approximations of information that was not accessible via direct links. They exploited the fact that each vehicle communicates with a common group of neighbouring vehicles which can be referred to as a circulant network. This type of network results into a simplified system representation as it can be transformed to a convenient block diagonal form. The proposed method in [133], however, is limited as it assumes a specific type of network.

2.3.1 Distributed Control

Several distributed control strategies have been proposed to control complex systems. In [134], Kia developed a class of distributed continuous-time coordination algorithms to achieve network optimisation. However, the method requires some conditions to be met such as a connected graph topology and globally Lipschitz gradients. The design parameters are computed offline and complexities such as disturbances have not been considered. Furthermore, a distributed method for formation control was suggested in [135] for multi-robots that communicate and share knowledge with their neighbours. However, the dynamics of the robots are assumed to be homogenous and thus can not be applied to systems that are heterogeneous. Another limitation is the constant velocity that is assumed for the derivation which would prove to be problematic in case the speed changes abruptly resulting in impracticable optimisation.

In [25], a distributed control framework was developed which involved the decomposition of the large-scale complex network into subsystems for which individual probabilistic controllers were derived. Each subsystem took the corresponding interaction with other subsystems into consideration by treating the dynamics of the interacting subsystems as a measurable disturbance. The global aim of the system is realised by the exchange of information between the subsystems which is achieved through message passing. However, this method was solely developed for linear Gaussian systems.

2.3.2 Decentralised Control

The theory of decentralised control is a concept that solves complex large-scale control problems by partitioning the problem into more manageable subproblems [136]. As a result, a single centralised controller is not expected to control the entire plant but instead this task is indirectly fulfilled by multiple independent controllers which are known as decentralised controllers. The control of the interconnected dynamical system is achieved by controlling its subsystems and using knowledge that is solely accessible locally. However, the challenge that needs to be addressed when designing decentralised controllers is that only limited knowledge about the global state of the system is available [119]. Consequently, this would impact the stability of the overall system that is controlled by decentralised controllers since the subsystems do not possess any information about the global state and the behaviour of other subcontrollers.

Decentralisation reflects the information structure characteristics in the solution to the control problem. This is achieved by considering decentralisation in the control law by incorporating knowledge of the states and the command which allows independent implementation of controllers. In addition, the design process can also reflect the objective of decentralisation through the model and the aim of the design process. In that case, the derivation of control solutions are obtained independently and are based on the individual design and model of the subsystems. The concept of decentralising the design process is driven by various reasons which include subsystems that consist of weak coupling, the objectives of subsystems are contradicting, or the system itself is of high dimensionality.

One of the fundamental issues that control theorists and analysts face with decentralised control is the reliability of control systems in the presence of component failure. The solution to improving the reliability of control systems cannot be obtained by the implementation of more reliable and better performing components, but is achieved by considering the way the control systems handle these complications [137]. Each subsystem within interconnecting dynamical systems possesses the possibility of facing failure within the components due to for example a blackout, downtime or partial degradation failure. An extensive amount of studies can be found on obtaining control strategies that tackle this issue which highlights the requirement of considering this in the decentralised control strategy [137], [138], [139], [140].

In a decentralised control system, each subsystem within the network derives a controller to influence its own behaviour without taking into account the behaviour of the other subsystems or their controllers. The decentralised controllers consist of control systems that take place locally. Usually, the subsystems do not posses complete global system knowledge and their controllers are only concerned with controlling the individual subsystem [141]. In case of a technical disturbance, only a small fraction of the network suffers from it. On the other hand, not considering information about other neighbouring subsystems could result in failure to meet the global state objective.

2.4 Fully Probabilistic Control

The similarity between one probability distribution to another probability distribution can be obtained by various divergence measures that have been developed and researched such as Jensen difference divergence [142], Kullback Leibler Divergence (KLD) [143], Jeffreys divergence [144], Kagan's divergence [145] and considerably more. A very popular divergence measure for probability distributions is the KLD which is computationally advantageous and a convenient statistical measure to describe the "distance" or "divergence" between two distributions since many distributions can be expressed as exponential functions. In [9], Kárnỳ proposed the FP control design which is an optimal control method, i.e. minimises a cost function that is based on the KLD. This section is specifically dedicated to the current state-of-the-art that has been produced on the FP control design. The scope to further develop this control method proves to be propitious and is therefore the chosen approach in this thesis.

It was suggested by Kárnỳ [9] to seek a control design in which the derivation of the controller is based on the characterisation of the joint probability distribution of the closed loop control system. Since the considered systems are stochastic in nature, the theory of probabilities is a suitable approach to handle them. This adaption of the description of the closed-loop enabled the shift from the common minimisation of the expected value of a data-dependent cost function to the minimisation of two joint probability distributions using the KLD [143], [146]. The proposed control design involves minimising the discrepancy between the actual joint probability distribution and the ideal joint probability distribution of the closed-loop system. The developed framework is called the Fully Probabilistic Control design which allows the derivation of an explicit form of the optimal controller. The solution of the optimal control law is presented as a probability density function (pdf) [24]. In its original form, the FP control design provided an explicit solution for stochastic systems with additive noise that can be described by any arbitrary probability density function, but was only demonstrated on linear Gaussian systems. Over the years, however, monumental progress has been made on the FP control design. In [147], the design of the FP controller is generalised by extending it to a Gaussian class of linear uncertain stochastic systems of which the dynamics are unknown and the model discrepancy is considered in the design of the control law.

On the other hand, dynamical systems with multiplicative stochastic disturbances have recently received a great amount of attention due to the extension of the application domain to fields such as image processing systems [148], [149] in ultrasound and laser imaging, biological motor systems [150], [151], and aerospace engineering systems to give a few examples. In fact, there are many key problems in chemistry, biology, ecology, economics, physics, and engineering which involve multiplicative noise instead of additive noise [152], [153], [154], [155], [156]. Hence, the consideration of multiplicative noise in real-life engineered dynamical systems is of paramount importance. Unlike additive noise, the second moment of the multiplicative noise is not constant, but rather dependent on the state of the system. Essentially, this creates complications and makes the research more challenging [157]. Consequently, many approaches have been developed in the literature for systems affected by multiplicative noises including the linear matrix inequality (LMI) approach [158], [159], [160] and the Riccati equation [161] to name a few. However, none of the aforementioned considered a fully probabilistic approach. To ensure the FP controller can also be applied to linear systems affected by multiplicative noises, the control framework was further developed in [30].

In addition, FP control design methods have mostly focused on demonstrating the control solutions for linear and quadratic control systems [9], [30] since the derivation of an analytic control solution for nonlinear systems cannot be obtained in a FP framework. This is due to the nonlinearity of the parameters of the probability density functions that characterise the system state and control input. As a solution, nonlinear control problems have been solved numerically within a FP control framework in [17], without the requirement of linearisation of the system dynamics. Furthermore, in [17], the discussed control design is proposed for nonlinear stochastic systems affected by functional uncertainties using the probabilistic dual heuristic programming (DHP) adaptive critic approach.

Despite, the many advancements, a closed form control solution to the FP problem had not been developed for nonlinear systems. The hindrance to the derivation of an analytic solution and how to address it is one of the contribution of the work in this thesis which has appeared in [29] and will be explained in the next chapter. This extension allows the Fully Probabilistic control design to be applied to a wider class of real systems as the physics of many real-world systems are affected by multiplicative noises and governed by nonlinear dynamics [148], [150].

Much work has been done to further extend the FP control design to consider various aspects of stochastic and uncertain systems [147], [18], [162], [30]. The original FP control design method was derived by assuming the dynamics of the system are known [9]. The approach was further developed for systems with unknown dynamics by Herzallah [147] and a generalised randomised controller that considers functional uncertainty was derived. The paper recognises the existence of functional uncertainty which is unavoidable and arises from the poor modelling of complex systems of which the dynamics are unknown. It considers an intelligent control technique that incorporates functional uncertainty in the optimisation of the randomised controller. This method was shown to be effective in improving the performance of the stochastic controlled system which was also assumed to have unknown dynamics, and were thus estimated online. The consideration of functional uncertainty in the design of the randomised controller yielded less transient overshoot due to the development of a cautious controller. The proposed method in [147], however, was demonstrated on linear stochastic systems. Consequently, the work in this thesis considered the extension of this method to systems governed by nonlinearities as can be seen in the discussion of Chapter 3. The proposed adaptive control method in Chapter 3 is more complicated due to the estimation process of the nonlinearities of the system dynamics, increasing the intensity of the estimation error, and making the consideration of functional uncertainty of paramount importance.

The FP control design was originally proposed in a centralised way [9]. It was soon realised by Herzallah et al. that the FP control framework needs to be adapted to ensure large-scale complex systems can be controlled using the FP control design. The pinning control method was suggested for large-scale systems [24]. However, the control strategy involved the design of randomised controllers that pin or control a few nodes in the network, and therefore still follows a centralised architecture. Further development of the FP control design within a decentralised framework was proposed for large-scale complex linear systems in [25]. It decomposes the system into smaller subsystems which makes it more manageable to control. The individual subsystem considers its own dynamics and obtain knowledge about the dynamics of the interacting subsystems as an external observable disturbance. The interaction between the subsystems is achieved via probabilistic message passing. Having information of the other interconnecting subsystems in the form of external observable disturbances as part of the dynamics of the subsystem improves the control reliability. For example, if some part of the system experience failure, the system can still predict information about the other subsystems. Nevertheless, as real complex systems tend to be nonlinear in nature, this probabilistic design approach has been further developed in Chapter 4. In addition, the FP control design has not been demonstrated for a formation control problem. Hence, this gap and its solution are discussed in Chapter 5.

Chapter 3

Fully Probabilistic Control for Nonlinear, Stochastic, and Uncertain Systems

3.1 Introduction

Many engineered dynamical systems in the industry are fraught with a variety of uncertainties which have a direct impact on the dynamics of the systems. These disruptions affecting the performance of the system could come in many forms such as functional uncertainties, noises and disturbances caused by the surrounding environment or operating conditions. The aforementioned complexities have made the designation and derivation process of an optimal controller more challenging. From the literature, it can be established that many branches within control are proposed with the aim to manage uncertainties in order to optimise the performance of the control systems resulting in robustness and allowing near optimal control of real world applications that operate under noises and functional uncertainties.

As such, the focus of this PhD is on stochastic systems since they consider noises and uncertainties in the dynamics. Due to these noises, the exact value of the state cannot be determined at the current time step. However, the probabilistic description of the state can be obtained for which it is then required to derive a controller that handles the probabilistic nature of the system. For the regulation problem of stochastic dynamical systems, a control method was developed following a Fully Probabilistic control framework [9]. This approach considers the full distribution of the stochastic system dynamics for the derivation of randomised controllers. Further advancements to the FP control method consider various aspects of stochastic and uncertain systems [30], [147], [162].

Despite its effectiveness in dealing with stochastic systems, an analytic control solution using the FP control method can be obtained for linear and Gaussian systems only. The nonlinearities of the parameters of the distribution of the system dynamics on the other hand, means that control solutions for nonlinear systems need to be obtained using numerical methods. Additionally, the Fully Probabilistic Design in its original form considers systems with additive noises only [9] which makes the application of it to real-world systems limited as many systems in reality are affected by multiplicative noises as well. Thus, in [30], we considered linear stochastic systems affected by multiplicative noises. The developed method is referred to as the generalised Fully Probabilistic control design as the resulting solution consists of a generalised Riccati equation that has additional terms due to the involvement and consideration of multiplicative noises in the stochastic system.

To address the aforementioned challenges, a novel approach that considers the nonlinearities of the system dynamics is demonstrated in this chapter. Unlike many methods in the literature [88], the proposed approach does not require the system to be linearised and sustains the nonlinear peculiarity of the system. Furthermore, this chapter will exploit different types of noises and uncertainties found in real-world systems and propose a probabilistic controller which takes these inconveniences into consideration.

The structure of this chapter is as follows: Section 3.2 discusses the objectives of the FP control design and briefly describes the conventional FP control design as discussed in [9]. Section 3.3 focuses on the development of an analytic solution of the FP control design for nonlinear systems with additive noises. Section 3.4 describes how to include knowledge of functional uncertainty in the derivation of the randomised controller in a FP control framework. Section 3.5 discusses the design of an optimal randomised controller that considers multiplicative noises which have an impact on the dynamics of nonlinear systems. Section 3.6 concludes this chapter.

3.2 Fully Probabilistic Control Design

A control strategy is implemented to control the state of a dynamical system to a predefined desired state. It is to ensure and guarantee that the system behaves as expected making it crucial for the control problem and the dynamics of the stochastic system to be understood. Since the governing dynamics that describe the states of a dynamical system are usually unknown and affected by noise, the stochasticity of the system only allows the probability distribution of the states to be estimated. Therefore, the objective and formulation of the control problem needs to be adapted accordingly. This signifies that the controller is required to reflect the probabilistic framework in which it is operating.

The Fully Probabilistic Design is an optimal control method that is based on the minimisation of a predefined performance index for systems described by probability distributions. This performance index is derived from the Kullback-Leibler Divergence measure which is the foundation of the FP control design. The KLD, defined in (3.1), measures the distance between the actual and ideal joint probability density functions,

$$\mathcal{D}(f \parallel f^I) = \int f(D) \ln \left(\frac{f(D)}{f^I(D)} \right) dD, \tag{3.1}$$

where $D = \{x_1, \ldots, x_H, u_1, \ldots, u_H\}$, with x and u being the state and control input vector respectively, and H is the control horizon. The FP control design minimises the KLD given in (3.1) by designing a probabilistic controller that brings the actual joint pdf $f(D)$ of the closed-loop system closer to the ideal joint pdf $f^I(D)$ of the closed-loop system.

A stochastic system is characterised with the inputs u_k and measurable states x_k, and expressed in discrete time steps $k = \{1, ..., H\}$. It is assumed that the behaviour of the system state x_k is represented by a known conditional pdf and is given by,

$$s(x_k|u_k, x_{k-1}), \tag{3.2}$$

with x_k being the state of the system at time step k. Furthermore, the probabilistic controller which needs to be derived is described by the probability distribution given by,

$$c(u_k|x_{k-1}), \tag{3.3}$$

where one can notice the dependency of the controller on the state. This is a valid assumption given that the states are directly observable.

To re-emphasise, the objective of the FP control design is to control the joint pdf of the system dynamics and controller to a predefined desired joint pdf. This objective can be achieved by designing a randomised controller $c(u_k|x_{k-1})$ that minimises the discrepancy between the joint distribution of the system state and control input, $f(x_k, u_k|x_{k-1})$ and a predefined ideal joint distribution, $f^I(x_k, u_k|x_{k-1})$. Using the chain rule for probability density functions [163], $f(x_k, u_k|x_{k-1})$ can be factorised as follows,

$$f(x_k, u_k|x_{k-1}) = s(x_k|u_k, x_{k-1})c(u_k|x_{k-1}). \tag{3.4}$$

Similarly, the ideal joint probability distribution of the closed-loop system can be factorised as follows,

$$f^I(x_k, u_k|x_{k-1}) = s^I(x_k|u_k, x_{k-1})c^I(u_k|x_{k-1}), \tag{3.5}$$

where $s^I(x_k|u_k, x_{k-1})$ represents the ideal distribution of the dynamics of the system state, and $c^I(u_k|x_{k-1})$ is the ideal distribution of the controller. The minimisation of the KLD is attained by finding a probabilistic control law, $c(u_k|x_{k-1})$, which regulates the actual closed-loop system, $f(x_k, u_k|x_{k-1})$ stated in (3.4) and brings it as close as possible to the ideal joint probability distribution of the closed-loop system, $f^I(x_k, u_k|x_{k-1})$ as defined in (3.5).

The minimum cost-to-go function that minimises equation (3.1) with respect to the admissible control sequence, u_k, is given by the following recurrence equation [162],

$$-\ln(\gamma(x_{k-1})) = \min_{\{c(u_k|x_{k-1})\}} \int s(x_k|u_k, x_{k-1})c(u_k|x_{k-1}) \Bigg[\underbrace{\ln\left(\frac{s(x_k|u_k, x_{k-1})c(u_k|x_{k-1})}{s^I(x_k|u_k, x_{k-1})c^I(u_k|x_{k-1})}\right)}_{\text{Partial cost}} - \underbrace{\ln(\gamma(x_k))}_{\text{Optimal cost-to-go}} \Bigg] d(x_k, u_k), \tag{3.6}$$

where $-\ln(\gamma(x_{k-1}))$ is the expected minimum cost-to-go function. The equation given in (3.6) equates to the recurrence equation of the dynamic programming solution to the control problem. A more detailed derivation of the cost-to-go function can be found in [162].

3.2.1 General Solution to the FP Control Design Problem

Following the discussion above, it is now possible to obtain a general optimal solution for the FP control problem as shown in Proposition 1.

Proposition 1. The minimisation of the cost-to-go function (3.6) with respect to the control law, $c^*(u_k|x_{k-1})$ results in the designation of the optimal controller. This yields the following optimal randomised controller,

$$c^*(u_k|x_{k-1}) = \frac{c^I(u_k|x_{k-1})\exp[-\beta_1(u_k, x_{k-1}) - \beta_2(u_k, x_{k-1})]}{\gamma(x_{k-1})}, \tag{3.7}$$

where

$$\beta_1(u_k, x_{k-1}) = \int s(x_k|u_k, x_{k-1}) \left(\ln \frac{s(x_k|u_k, x_{k-1})}{s^I(x_k|u_k, x_{k-1})}\right) dx_k, \tag{3.8}$$

$$\beta_2(u_k, x_{k-1}) = -\int s(x_k|u_k, x_{k-1}) \ln(\gamma(x_k)) dx_k, \tag{3.9}$$

$$\gamma(x_{k-1}) = \int c^I(u_k|x_{k-1})\exp[-\beta_1(u_k, x_{k-1}) - \beta_2(u_k, x_{k-1})] du_k. \tag{3.10}$$

Proof. The details of the complete proof of the performance index $-\ln(\gamma(x_{k-1}))$ defined in (3.6) and the optimal control law $c^*(u_k|x_{k-1})$ given in (3.7) - (3.10) can be found in [162], [9]. □

Equation (3.7) provides the general solution of the randomised controller for arbitrary probability density functions of the system dynamics and controllers. However, it is important to note that the randomised optimal control solution only allows an analytic solution to be evaluated for systems that are characterised by Gaussian distributions and are linear as will be demonstrated in the next section.

3.2.2 Solution of FP control for linear systems

This subsection gives a brief insight into the solution of the conventional FP control design for linear dynamical systems which may be used for comparison purposes with the methods developed and explained in the subsequent sections in this chapter.

Consider the discrete time linear stochastic system with additive noise represented by the following model,

$$x_k = Ax_{k-1} + Bu_k + \epsilon_k, \tag{3.11}$$

where $x_k \in \mathcal{R}^n$ is the system state, $u_k \in \mathcal{R}^m$ is the control input, $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$ are the state and control matrices, respectively. The noise defined by ϵ_k is assumed to have a Gaussian distribution with zero mean and covariance Σ. Under the assumption that the noise affecting the system is Gaussian noise, the stochastic linear model in (3.11) can be characterised by the following Gaussian probability density function conditioned on previous state and control input,

$$s(x_k|u_k, x_{k-1}) \sim \mathcal{N}(\bar{x}_k, \Sigma), \tag{3.12}$$

where,

$$\bar{x}_k = Ax_{k-1} + Bu_k, \tag{3.13}$$

is the mean of the system and Σ is the global covariance. Although the distribution of the system dynamics is assumed to be Gaussian in (3.12), any other distribution could have been assumed. If a different distribution to the Gaussian one is assumed though, the analytic solution as will be obtained here might not be possible. The ideal probability distribution of the system state which represents the desired behaviour of the system is given by,

$$s^I(x_k|u_k, x_{k-1}) = \mathcal{N}(0, \Sigma). \tag{3.14}$$

The mean equates to zero since a regulation problem is considered where the aim is to reach state zero. In this section, the covariance matrix Σ of the ideal distribution is assumed to be equal to the covariance matrix of the controlled system and the random noise affecting it. However, this covariance matrix of the ideal distribution is generally a design parameter that can be specified based on prior knowledge we have on the noise affecting the system. For example, if the noise is known to be constant and state independent, the covariance matrix of the ideal distribution can be set to be equal to the one estimated from the data as will be discussed in section 3.3.2. For systems that are affected by state dependent noise, the covariance matrix of the ideal distribution can be specified to be very small, as will be seen later in this chapter and in Chapters 4 and 5 (Sections 4.8.1, 5.3.1), since this noise can be controlled and minimized. Otherwise this covariance of the ideal distribution can be determined from the constraint of the system as discussed in [164]. Furthermore, the ideal distribution of the randomised controller is described by,

$$c^I(u_k|x_{k-1}) = \mathcal{N}(0, \Gamma), \tag{3.15}$$

where the mean is zero and Γ determines the permissible range of control inputs for a given confidence level.

The analytic solution of the FP control problem for linear systems that are described by Gaussian pdfs can be found in the following proposition. It is referred to as the conventional FP control design in this thesis and was originally demonstrated by Kárnỳ [9].

Proposition 2. The randomised control law for the linear system described by the distribution in (3.12) and the ideal pdfs of the system state and controller expressed by (3.14) and (3.15) respectively, is given by,

$$c^*(u_k|x_{k-1}) = \mathcal{N}(\mu_k^*, R_k), \tag{3.16}$$

where,

$$\mu_k^* = -K_k x_{k-1}, \tag{3.17}$$

$$K_k = (\Gamma^{-1} + B^T(\Sigma^{-1} + G_k)B)^{-1} B^T(\Sigma^{-1} + G_k)A, \tag{3.18}$$

$$R_k = \left(\Gamma^{-1} + B^T(\Sigma^{-1} + G_k)B\right)^{-1}. \tag{3.19}$$

The quadratic cost function implemented to obtain the designed randomised controller in (3.16) is given by,

$$-\ln(\gamma(x_k)) = 0.5 x_k^T G_k x_k + 0.5\omega_k, \tag{3.20}$$

with,

$$G_{k-1} = A^T \left\{ (\Sigma^{-1} + G_k) - (\Sigma^{-1} + G_k)B[\Gamma^{-1} + B^T(\Sigma^{-1} + G_k)B]^{-1}B^T(\Sigma^{-1} + G_k) \right\} A, \tag{3.21}$$

$$w_{k-1} = w_k + \text{tr}(G_k\Sigma) + \ln|\Gamma| + \ln\left(B^T(\Sigma^{-1} + G_k)B + \Gamma^{-1}\right), \tag{3.22}$$

where Equation (3.21) represents the discrete time algebraic Riccati equation (DARE) and w_{k-1} in (3.22) is some constant term that has no dependency on the state. The derived optimal controller described by (3.16) is a Gaussian distribution with mean μ_k^* given by (3.17) where the control gain, K_k, is expressed by (3.18) and covariance R_k is defined by (3.19).

3.3 Generalised Fully Probabilistic Design for Nonlinear Systems with Global Variance

The FP control design has been demonstrated for various scenarios, primarily linear and quadratic control systems [9], [147], [30], [165], [25], [166], yet the derivation of a closed form control solution has failed to be demonstrated for nonlinear systems [29]. Research on the FPD for systems governed by nonlinearities has been conducted in the literature, but the proposed methodologies consist of numerical solutions [18], [17]. The hindrance of the achievement of an analytic control solution is due to the nonlinearity of the parameters of the pdfs that describe the system state and control input.

Consequently, this section proposes a novel approach for the derivation of analytic solutions of the randomised controllers for nonlinear systems. The derivation of analytic control solutions is facilitated by the means of transformation methods. This will be shown to be achieved through transforming the nonlinear state function to another variant where the nonlinearities still exist in the state but have now been transformed to a nonlinear affine state function. The introduced novelty allows the FP control design to be extended to more realistic control problems that are characterised by nonlinearities, and does not require linearisation of the systems.

3.3.1 System Description

In real-world, many systems are governed by nonlinear dynamics. As such, a class of nonlinear discrete time dynamical stochastic systems are considered which are described by the following state space model,

$$x_k = \hat{h}(x_{k-1}) + \bar{g}(x_{k-1})u_k + \epsilon_k, \tag{3.23}$$

where $k = 1, ..., H$ denotes the discrete time step, $x_k \in R^n$ describes the state, and $u_k \in R^m$ is the control input to the system. Also, $\epsilon_k \in R^n$ is a Gaussian noise with zero mean and fixed arbitrary covariance, $\bar{\Sigma}$. The nonlinear state vector and control matrix are represented by $\hat{h}(x_{k-1}) \in R^n$ and $\bar{g}(x_{k-1}) \in R^{n\times m}$, respectively.

To re-emphasise, for the nonlinear system (3.23), the presence of the noise ϵ_k means that the previous state and current control input specify the probability distribution of the present state, $s(x_k|u_k, x_{k-1})$ rather than their actual values. To clarify, given the assumption that ϵ_k is a Gaussian noise, the distribution of the present state of the nonlinear system (3.23) can be characterised by a Gaussian distribution with mean given by, $\hat{h}(x_{k-1}) + \bar{g}(x_{k-1})u_k$ and a global covariance matrix given by $\bar{\Sigma}$. However, this nonlinearity of the parameters of the distribution that characterises the dynamics of the nonlinear system (3.23) means that the optimal solution of the randomised controller given in (3.7) can only be obtained using numerical methods, and an analytic or closed form solution of the randomised controller can not be obtained. Therefore, the derivation of the randomised controller will be facilitated by first transforming Equation (3.23) such that it becomes nonlinear affine in the system state to obtain:

$$x_k = \bar{h}(x_{k-1})x_{k-1} + \bar{g}(x_{k-1})u_k + \epsilon_k, \tag{3.24}$$

where $\hat{h}(x_{k-1}) = \bar{h}(x_{k-1})x_{k-1}$.

The matrix $\bar{h}$ is defined as follows,

$$\bar{h}(x_{k-1}) = \begin{bmatrix} \bar{h}_{11}(x_{k-1}) & \dots & \bar{h}_{1n}(x_{k-1}) \\ \vdots & \ddots & \vdots \\ \bar{h}_{n1}(x_{k-1}) & \dots & \bar{h}_{nn}(x_{k-1}) \end{bmatrix}. \tag{3.25}$$

The stochastic evolution of the system state defined in Equation (3.24) can then be captured during the control process by estimating its generative distribution, $s(x_k|x_{k-1}, u_k)$, from the observed data as will be explained in the next subsection.

Definition 3.3.1. *Suboptimal solution to the nonlinear FP control design:* The FP control approach for obtaining a suboptimal solution of nonlinear control problems (such as Equation (3.23)) can be obtained using the following definition:

1. Use transformation methods to bring the nonlinear dynamics to the nonlinear affine dynamics. For example, for the formulation in the current section, transform the nonlinear equation in (3.23) to the nonlinear affine dynamics in (3.24).

2. Solve the equations provided in Proposition 1 to obtain the closed form suboptimal solution at each discrete time instant.

3.3.2 System State Estimation

Since the physical description of most real-world systems is unknown, the proposed adaptive control method in this section will utilise neural network techniques to estimate the dynamics of the controlled system. To be specific, the Multilayer Perceptron (MLP) is implemented to predict the probability distribution of the system state variables.

Firstly, it should be noted that the dynamics of the system are unknown. The only data available to us is the measurable state x_k and the input to the system defined as u_k. Furthermore, a prior assumption about the dynamics of the system is that it is governed by affine nonlinearities. This means that in (3.24), the state x_{k-1} is multiplied by a matrix $\bar{h}(x_{k-1})$ whose elements are nonlinear functions of the state and the control input is multiplied by a control matrix $\bar{g}(x_{k-1})$, whose elements are also nonlinear functions of the state. Based on these assumptions and the data available to us, the conditional mean of the system dynamics can be approximated using a MLP neural network.

Throughout this thesis, a three-layer MLP perceptron neural network is assumed. The three layers are the input layer, the hidden layer, and the output layer. The nonlinear activation functions of the hidden layer could be taken to be any of the known activation functions for instance the sigmoid and tanh function. Once the structure of the MLP is defined, this neural network model can be used to provide a prediction for the conditional expectation of the system state as shown in Fig 3.1. The output of the MLP provides an estimation for the conditional expectation of the actual state of the system defined in (3.24) and is given by the following equation,

$$\begin{aligned} \hat{x}_k &= \text{mlp}(x_{k-1}), \\ &= h(x_{k-1})x_{k-1} + g(x_{k-1})u_k, \end{aligned} \tag{3.26}$$

where $h(x_{k-1})$ and $g(x_{k-1})$ are the estimates of the actual state, $\bar{h}(x_{k-1})$, and control, $\bar{g}(x_{k-1})$, matrices respectively. The parameters of the MLP model (3.26) are optimised online at each instant of time by computing the sum of squares error between the actual state values x_k as obtained from (3.24) and the estimated $\hat{x}_k$ as obtained from (3.26). The details of this online optimisation (parameter estimation and control of the system are done simultaneously with each time step) is given in Algorithm 2. However, to improve the convergence property of the neural network model, it is pre-trained offline using some generated data from the system equation as explained in Algorithm 1.

Therefore, once the estimation has been completed, the following stochastic model can be gener-

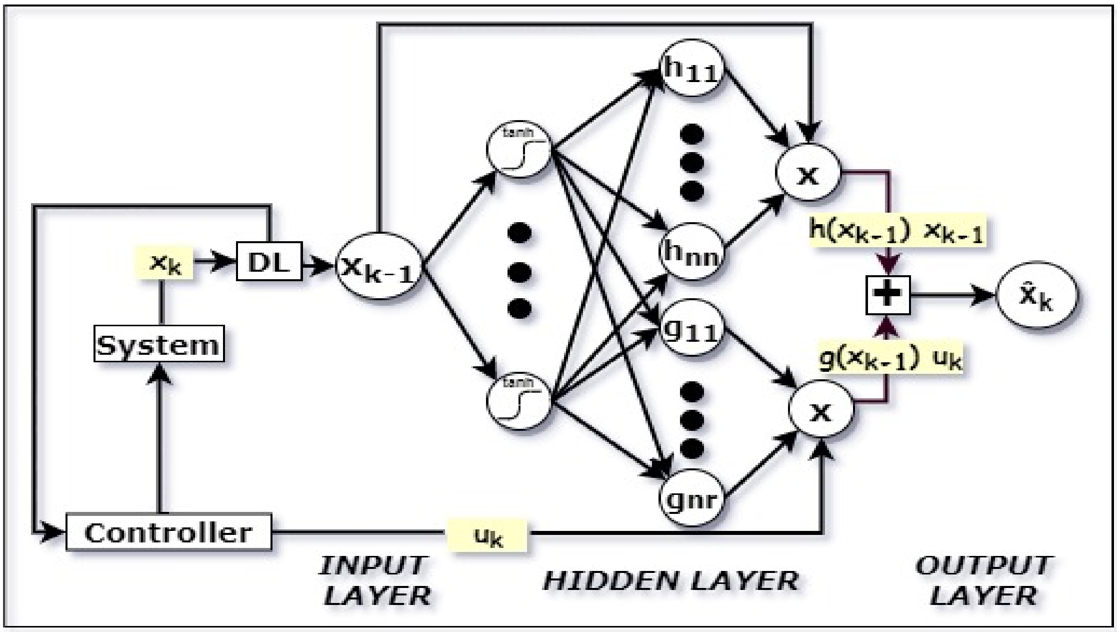

Figure 3.1: The input layer consists of the state vector x_{k-1}, the hidden layer consists of the elements of matrix $h(x_{k-1})$ and $g(x_{k-1})$, and the output layer consists of the state values $\hat{x}_k$. DL is the delay line. In the diagram, an example of an activation function, namely the tanh function, has been given. There are, however, a variety of activation functions that can be selected.

ated,

$$x_k = \hat{x}_k + e_{k-1}, \tag{3.27}$$

where the residual error of the output of the system is characterised by the last term e_{k-1}. It is stated in [162] and [167] that the error e_{k-1} can be shown to be close to Gaussian random noise with zero mean and an input dependent covariance matrix, Σ_k. This is a well-known finding which states that if the approximation model accurately approximates the system behaviour, then the estimation error will be very small and close to a Gaussian noise [168]. This covariance matrix Σ_k indicates the covariance of the error in predicting state x_k and is calculated by considering the residual value of the error between the actual and estimated state values,

$$\tilde{\Sigma}_k = (x_k - \hat{x}_k)(x_k - \hat{x}_k)^T. \tag{3.28}$$

To clarify, $\tilde{\Sigma}_k$ here is input dependent, meaning it can be computed for each state variable. However, using the process outlined in [162], the global covariance Σ_k can be estimated by averaging over all the input values,

$$\Sigma_k = E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right], \tag{3.29}$$

where $E(.)$ represents the expected value. To elucidate, the computation of the global covariance matrix is achieved by adding up the input dependent covariances, $\tilde{\Sigma}_k$ and dividing it by the length of time.

Consequently, according to the universal approximation property of neural network models [168], the conditional distribution of the system state at time k can be represented by a Gaussian distribution,

$$s(x_k|u_k, x_{k-1}) = \mathcal{N}(\hat{x}_k, \Sigma_k), \tag{3.30}$$

where the mean, $\hat{x}_k$ is defined in (3.26) and Σ_k is the global covariance matrix given in (3.29). As mentioned earlier, a Gaussian distribution facilitates the derivation of a closed form solution to the randomised controller.

Algorithm 1 Algorithm for Training MLP

1: **procedure** PRE-TRAINING OF THE MLP FOR THE ESTIMATION OF THE NONLINEAR DYNAMICS AND THE GLOBAL COVARIANCE
2: Generate a vector of length L of control inputs to excite the system equation and cover all operation range
3: Generate a vector of length L of random noise, ϵ_k.
4: Initialise the state at time k = 0 to a certain value, $x(k = 0) = x_0$.
5: for $k = 1 : L$
6: Use Equation (3.23) to calculate the next state value, $\hat{h}(x_{k-1})x_{k-1} + \bar{g}(x_{k-1})u_k + \epsilon_k \leftarrow x_k$.
7: End for
8: Use the generated state vector x_{k-1} that was obtained in *Steps 5-6* along with the control input vector from *Step 2* u_k as input to the neural network model and the non-delayed state vector x_k as output to optimise its parameters. Here, the forward backward algorithm is used to update and optimise the parameters of the neural network model.
9: Forward the generated state data from *Steps 5-6* along with the control input vector from *Step 2* to the optimised neural network model to predict the state values.
10: Calculate the input dependent covariance matrix $\tilde{\Sigma}_k$ using Equation (3.28), $(x_k - \hat{x}_k)^T(x_k - \hat{x}_k) \leftarrow \tilde{\Sigma}_k$.
11: Calculate the global covariance matrix Σ_k using equation (3.29), $E\left[(x_k - \hat{x}_k)^T(x_k - \hat{x}_k)\right] \leftarrow \Sigma_k$.

3.3.3 FP Control Solution for Nonlinear Systems

The advancement of the FP control design is demonstrated in this section for the system state distribution described by (3.30). As discussed previously, the aim of the FP controller is to shape the joint pdf of the closed-loop system such that it converges to the desired pdf of the system. Hence, the following requirement is the provision of the definition of the behaviour of the desired distribution of the system state which in the case of a regulation problem is specified by,

$$s^I(x_k|u_k, x_{k-1}) \sim \mathcal{N}(0, \Sigma_k), \tag{3.31}$$

where the zero mean reflects the regulation objective of making the states converge to zero, and Σ_k denotes the covariance of the ideal distribution of the system which in this case is assumed to be the same as the covariance of the actual distribution given in (3.30). The assumption of equal covariance matrices here is due to the fact that the systems considered in this chapter are assumed to be affected by state and control independent noises. Where the noise is state and control dependent, this assumption can be generalised in a straight forward manner as will be seen in the next chapters. Furthermore, the ideal probability distribution of the controller is described by the following Gaussian distribution with mean zero and ideal covariance Γ which represents the permitted range of optimal control inputs,

$$c^I(u_k|x_{k-1}) \sim \mathcal{N}(0, \Gamma). \tag{3.32}$$

It can be seen that the objective, which in this case is the regulation of the system, is taken into consideration in the ideal distribution of the system $s^I(x_k|x_{k-1}, u_k)$ and the ideal controller distribution $c^I(u_k|x_{k-1})$.

The succeeding theorem outlines the randomised controller obtained from the minimisation of the cost-to-go function given in (3.6).

Theorem 1. The suboptimal control law as described by Definition 3.3.1 for the system state distribution specified by (3.30) and the ideal pdfs of both the state and the controller described by (3.31) and (3.32) is given by,

$$c^*(u_k|x_{k-1}) = \mathcal{N}(u_k^*, R_k), \tag{3.33}$$

where

$$u_k^* = -K_k x_{k-1}, \tag{3.34}$$

$$K_k = R_k\Big[g^T(x_{k-1})(\Sigma_k^{-1} + M_k)h(x_{k-1})\Big], \tag{3.35}$$

$$R_k = \Big[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k)g(x_{k-1})\Big]^{-1}. \tag{3.36}$$

The distribution of the randomised controller in (3.33) is Gaussian with mean u_k^* and covariance R_k. The optimal gain is defined by K_k in (3.35). The performance index for the system state described by the pdf (3.30) is specified by,

$$-\ln(\gamma(x_k)) = 0.5x_k^T M_k x_k + 0.5w_k, \tag{3.37}$$

where,

$$M_{k-1} = h^T(x_{k-1})\Bigg[(\Sigma_k^{-1} + M_k) - (\Sigma_k^{-1} + M_k)g(x_{k-1})\Bigg[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k)g(x_{k-1})\Bigg]^{-1}$$
$$\times g^T(x_{k-1})(\Sigma_k^{-1} + M_k)\Bigg]h(x_{k-1}), \tag{3.38}$$

$$w_{k-1} = w_k + \text{tr}(M_k\Sigma) + \ln|\Gamma| + \ln|g^T(x_{k-1})(\Sigma_k^{-1} + M_k)g_{k-1} + \Gamma^{-1}|. \tag{3.39}$$

Proof. A detailed proof of the randomised optimal controller in (3.33), the performance index in (3.37) and the DARE in Equation (3.38) can be found in Appendix A. □

The DARE is given by (3.38) and the constant term w_{k-1} is specified by (3.39). In comparison to the conventional DARE (3.21) obtained from the conventional FPD (Proposition 2), the derived DARE is a state dependent Riccati equation (SDRE) due to the dependency of the nonlinear parameters of the state and control distributions on previous state values. The SDRE is not new and has been used in the literature to solve quadratic control problems when the system equations are nonlinear [169], [68], [65], [170]. However, to our best knowledge, the SDRE has not been considered in the Fully Probabilistic Control design. As such, it is shown for the first time that the suboptimal solution of the nonlinear FP control design problem for stochastic nonlinear affine systems results in a SDRE. The obtained solution (3.33) will be referred to as the conventional SDRE FP control approach.

3.3.4 Algorithm of Proposed Method in Theorem 1

The optimal controller given by (3.33) requires the evaluation of the Riccati equation solution in (3.38) in order to be implemented. If the probabilistic controller is being employed for an infinite horizon optimal control problem, the solution of the Riccati equation becomes a steady state (SS) solution. It should be noted that the Riccati equation solution is computed backwards in time, i.e. M_k needs to be computed first to determine M_{k-1}. However, in infinite horizon control problems, this is not possible since we do not know the final value of the Riccati equation solution M_k. Therefore, as a solution to this problem, the steady state solution of the Riccati equation can be found as a result of adjusting the time index. To clarify, the way the DARE is evaluated for the employment of the randomised controller derived in (3.33) is presented here. Equation (3.38) is calculated by adjusting the time index such that an increase in k refers to earlier time instants as shown below,

$$M_k = h^T(x_{k-1})\Bigg[(\Sigma_k^{-1} + M_{k-1}) - (\Sigma_k^{-1} + M_{k-1})g(x_{k-1})\Bigg[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_{k-1})g(x_{k-1})\Bigg]^{-1}$$
$$\times g^T(x_{k-1})(\Sigma_k^{-1} + M_{k-1})\Bigg]h(x_{k-1}). \tag{3.40}$$

The idea is to keep iterating the amended DARE (3.40) until a steady state solution is found. The pseudocode in Algorithm 2 summarises the approach of the conventional SDRE FP control design.

Algorithm 2 Pseudo-code of conventional SDRE FP control approach

1: **procedure** IMPLEMENTATION OF THE FP CONTROL DESIGN FOR NONLINEAR SYSTEMS WITH THE GLOBAL COVARIANCE
2: **Initialise:** x_0, M_0 and pre-train the neural network model as discussed in Algorithm 1 (optional).
3: **for** $k = 1 \rightarrow H$ **do**
4: **Estimate** $h(x_{k-1})$, $g(x_{k-1})$ from the neural network model, and compute Σ_k using (3.29).
5: **Calculate** the SS solution of M using (3.40).
6: **Use** the SS value from *Step 5* in (3.35) to find the SS solution of K.
7: **Calculate** u_k^* using equation (3.34).
8: Forward the control signal u_k^* obtained in *Step 7* to the system equation (3.23)
9: **Using** a one step delayed of the new state value x_{k-1} from *Step 8* and the calculated control signal u_k from *Step 7* as input to the neural network model and the new state x_k from *Step 8* as output, retrain the neural network model and update its parameters.
10: **end for**

3.3.5 Simulation

The effectiveness of the control solution given by (3.33) in Theorem 1 for nonlinear systems with global variance Σ_k is verified in this section. As a simulation example, a discrete time model of the driven inverted pendulum is implemented [171] of which a diagram is given in Figure 3.2. The performance of the randomised controller designed in (3.33) is then compared with the nonlinear quadratic regulator SDRE of which the formulations of the suboptimal controller, suboptimal feedback gain and Riccati equation [172] are respectively given by,

$$u_k^* = -K_k x_{k-1}, \tag{3.41}$$

$$K_k = [R + g^T(x_{k-1}) P_k g(x_{k-1}) + R]^{-1} g^T(x_{k-1}) P_k h(x_{k-1}), \tag{3.42}$$

$$\begin{aligned} P_{k-1} = &- h^T(x_{k-1}) P_k g(x_{k-1}) [R + g^T(x_{k-1}) P_k g(x_{k-1})]^{-1} g^T(x_{k-1}) P_k h(x_{k-1}) \\ &+ h^T(x_{k-1}) P_k h(x_{k-1}) + Q, \end{aligned} \tag{3.43}$$

where the weight matrices Q and R determine the significance of the error and the cost of the energies related to the state and control vector, respectively.

The control problem consists of finding the optimal control sequence for the pendulum from the specific initial level to the unstable equilibrium point. With the assumption that the origin corresponds to the unstable equilibrium, the dynamics of the system for the simulation are described in the form of equation (3.24) which is repeated here,

$$x_k = \bar{h}(x_{k-1})x_{k-1} + \bar{g}(x_{k-1})u_k + \epsilon_k, \tag{3.44}$$

where,

$$\bar{h}(x_{k-1}) = \begin{bmatrix} 1 & T_s \\ \frac{T_s g}{L x_{1,k}} \sin(x_{1,k}) & 1 - \frac{T_s \gamma}{ML} \end{bmatrix}, \quad \bar{g}(x_{k-1}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.45}$$

with,

$$T_s = 0.05, \quad M = 0.1, \quad L = 0.1, \quad g = 10, \quad \gamma = 0.05. \tag{3.46}$$

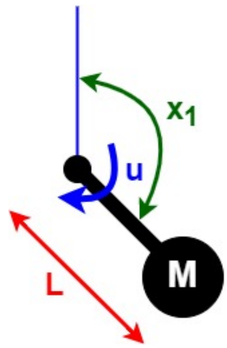

Figure 3.2: Diagram of the inverted pendulum.

However, the original model of the driven inverted pendulum that is used in [171] is deterministic, meaning unaffected by any noises which does not resemble real-world situations. Consequently, the system is simulated the right way as it would operate in the real-world by adding noise to the equation obtained from [171] as shown by (3.44). The noise ϵ_k in (3.44) has zero mean and covariance $\bar{\Sigma} = 0.001\mathcal{I}_{2\times 2}$. The state and control matrices given by $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$, respectively, are unknown and are therefore required to be estimated to obtain $h(x_{k-1})$ and $g(x_{k-1})$ of which the estimation process is explained in Section 3.3.2. Since this simulation example is demonstrated on the method proposed in this section and the NQR SDRE [172], two experiments are required to be implemented.

For comparison purpose, two sets of experiments were conducted. The first set of experiment considers the derivation of the suboptimal control law using the proposed conventional SDRE FP control approach. Here, the covariance Γ of the ideal controller is taken to be 50 to give the controller more freedom which results in a faster rate of convergence. In addition, the covariance of the ideal distribution is taken to be the same as the covariance of the global covariance matrix Σ_k of the actual system state distribution which is estimated as discussed in Section 3.3.2.

In the second set of experiment the suboptimal control law is derived using the NQR SDRE control method. This method requires the specification of the weighting matrices R and Q as can be seen from (3.41) - (3.43). These tuning parameters are chosen to be $R = 0.01$ and $Q = 0.1I_{2\times 2}$ where I is the identity matrix. Multiple experiments are carried out in which the parameters are tuned to obtain a good convergence. Additionally, similar to the proposed conventional SDRE FP control approach, the system equation is assumed to be unknown thus estimated as discussed in Section 3.3.2. The difference here is that only the expected value of the system state estimated using MLP neural network model is required.

The control objective for both approaches is to bring the state values of the pendulum to zero where the initial state is taken to be $x_0 = \begin{bmatrix} 7 & -5 \end{bmatrix}^T$. The results are illustrated in Fig 3.3 - 3.4 where the states have converged towards zero and very closely oscillate around zero for both approaches.

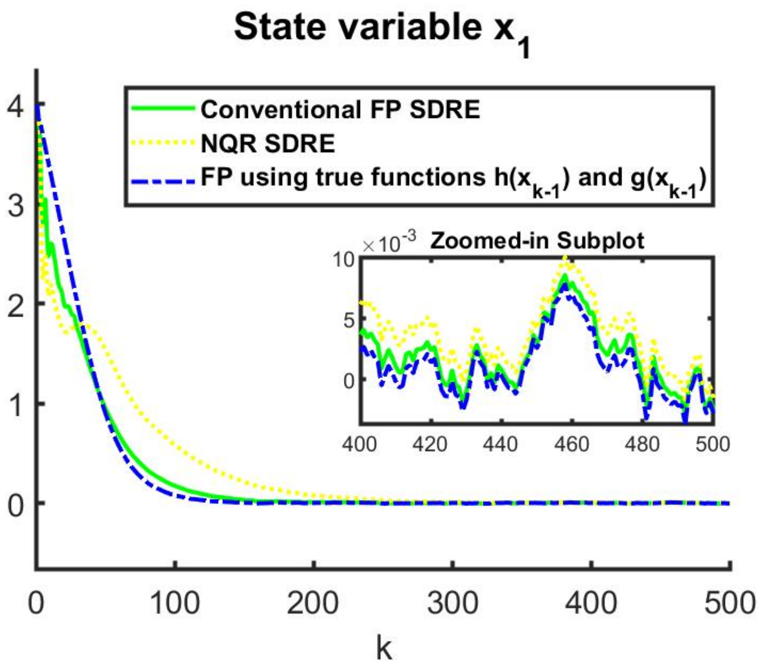

Figure 3.3: Comparison between the conventional SDRE FP control approach, the NQR SDRE control strategy and the SDRE FP control approach which uses the true functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ on state x_1. The subplot demonstrate that the states oscillate around zero.

From the plots 3.3 - 3.4, it can be seen that the states of the pendulum (3.44) have converged faster for the conventional FP SDRE than the NQR SDRE control strategy. In addition, the NQR SDRE demonstrates more oscillations as opposed to the proposed conventional SDRE FP control design which also shows a quicker achievement of the transient response. The plots also demonstrate the performance when the FP control design is implemented but using the true functions, $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ as given by (3.45). From the plots, it can be seen that the proposed conventional SDRE FP control strategy (green), converges slower than the SDRE FP control design (blue) that uses the true functions $h(x_{k-1})$ and $g(x_{k-1})$. The latter is faster due to the fact that the covariance matrix

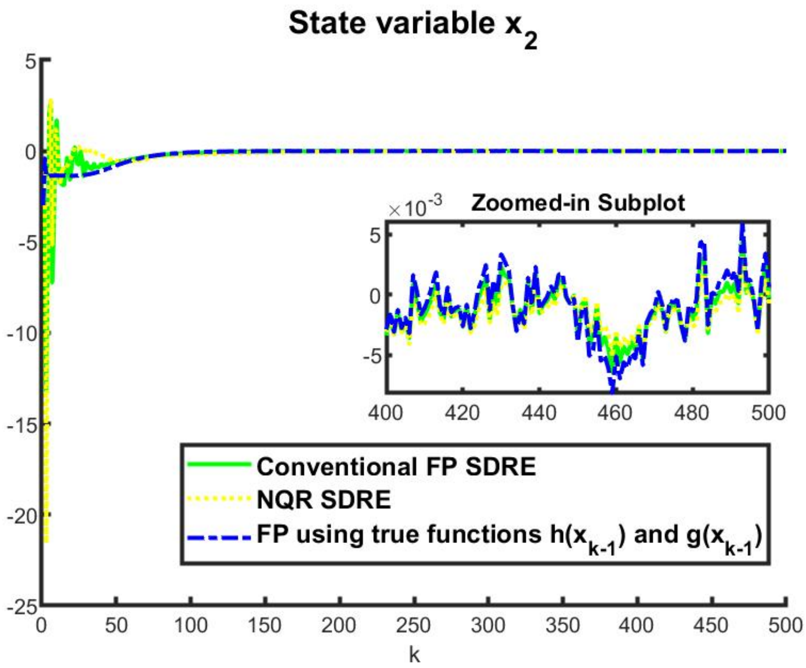

Figure 3.4: Comparison between the conventional SDRE FP control approach, the NQR SDRE control strategy and the SDRE FP control approach which uses the true functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ on state x_2. The subplot demonstrate that the states oscillate around zero.

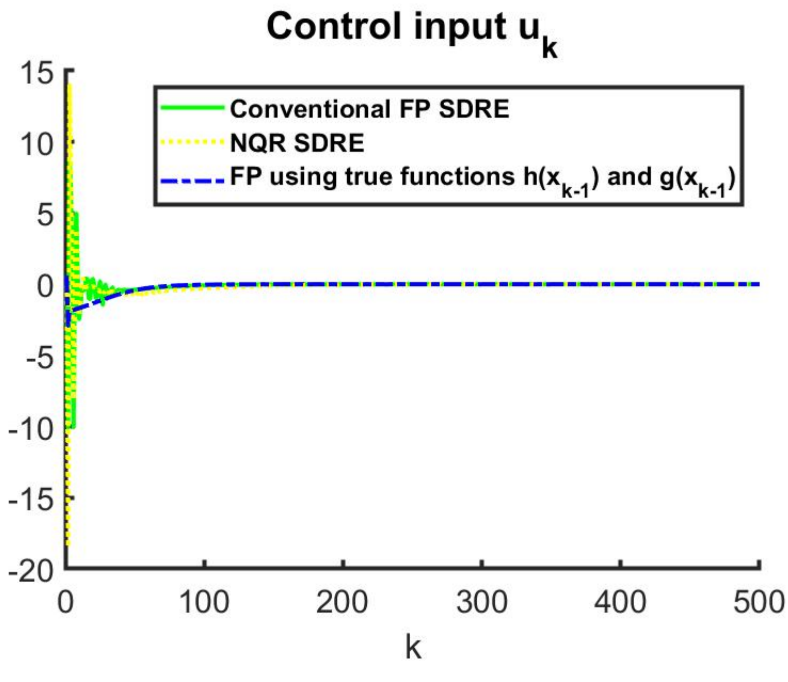

Figure 3.5: Comparison between the Conventional SDRE, the NQR SDRE control strategy and the SDRE FP control approach which uses the true functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ on the control input u_k.

Σ_k is given to be $\Sigma_k = 0.001 \times I_{2\times 2}$ while the estimated covariance matrix of the proposed method converges to $\Sigma_k = \begin{bmatrix} 0.0322 & 0 \\ 0 & 0.0182 \end{bmatrix}$. The penalisation on the states is obtained by finding the inverse of the covariance matrices, namely Σ_k^{-1}. This would give us $\Sigma_k^{-1} = \begin{bmatrix} 31.0634 & 0 \\ 0 & 54.8902 \end{bmatrix}$ for the proposed conventional SDRE FP control method and $\Sigma_k^{-1} = \begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix}$ for the FP control design that uses the true functions $h(x_{k-1})$ and $g(x_{k-1})$. The penalisation is higher on the FP control design with the true functions in comparison to the proposed SDRE FP control method. Higher penalisation means a higher cost is incurred for a slower convergence rate of the states. This explains the quick convergence of the states towards zero when the FP control design using the true functions is implemented. However, as mentioned previously, it is unrealistic to assume that the true functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ are known.

Nevertheless, the conventional FP control design effectively handles the effects of the MLP network approximation errors compared to the NQR SDRE method. The estimation of the global covariance matrix for the conventional FP control method allows for systems uncertainty to be taken into consideration as was explained in Section 3.3.2. Figure 3.5 plots the control inputs u_k obtained from the proposed SDRE FP control strategy with the global covariance, the NQR SDRE approach and the FP control strategy which uses the true functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$. All three computed control inputs u_k converge towards and closely oscillate around zero. In conclusion, the simulation has illustrated the effectiveness of the derived solution to the FP control problem in (3.33).

3.3.6 Selection of Tuning Parameters

In Section 3.3.5, the parameters Q and R were subject to experimentation since they were tuned until the states of the system described by Equation (3.44) converged towards values extremely close to zero. This process makes the method less time efficient due to the requirement of trial and error when tuning the NQR parameters. The advantage of the proposed conventional SDRE FP control design is that the covariance matrix, namely Σ_k is estimated which makes the method more time efficient since no tuning is required.

Furthermore, it should be noted that the inverse of Σ_k and Γ in the FP control method represent the penalisation on the states and control input, respectively. Hence Σ_k^{-1} and Γ^{-1} correspond to Q and R in the NQR control strategy, respectively. Thus, in this section, the simulation in Section 3.3.5 is implemented again, where the Q and R in the NQR method are now updated such that they correspond to the parameters of the SDRE FP control design. In the previous section, the covariance Γ of the ideal controller was set to be 50. Therefore, in this simulation, R is set to be $R = \Gamma^{-1} = 50^{-1} = 0.02$. The global covariance Σ_k in Section 3.3.5 converged to $\Sigma_k = \begin{bmatrix} 0.0322 & 0 \\ 0 & 0.0182 \end{bmatrix}$. Therefore, in this simulation, Q is set to be $Q = \Sigma_k^{-1} = \begin{bmatrix} 31.0634 & 0 \\ 0 & 54.8902 \end{bmatrix}$. The parameters of the proposed SDRE FP control design remain the same as in Section 3.3.5 as well as the initial state values for both the NQR and FP control method, that is $x_0 = \begin{bmatrix} 7 & -5 \end{bmatrix}^T$. The results are displayed in Figures 3.6 - 3.7 which demonstrate that the performance of both the NQR method and the SDRE FP control strategy are comparable when the parameters of the NQR SDRE method correspond to the parameters of the FP control design. In particular, the zoomed-in subplots within Figures 3.6 - 3.7 demonstrate the likeliness of the two control strategies when the parameters are set to be the same. Moreover, the control inputs of both methods have been plotted in Figure 3.8.

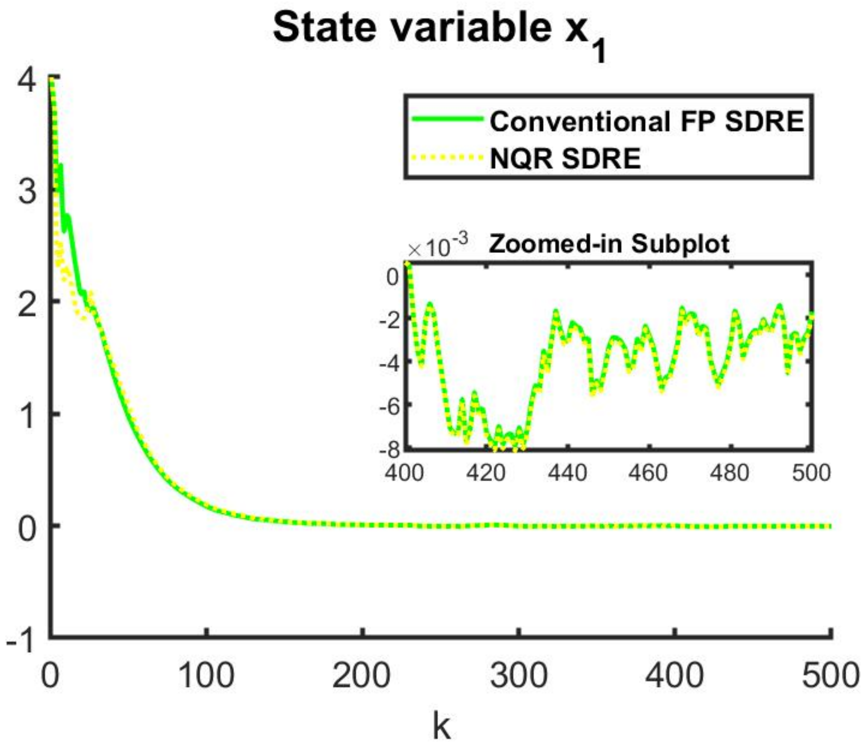

Figure 3.6: Comparison between the conventional SDRE FP control approach and the NQR SDRE control strategy on state x_1. The equivalence of the estimated global covariance matrix Σ_k for the FP control method is taken as the Q value and R is the equivalent of the covariance of the ideal controller Γ. The subplot shows that the states oscillate around zero.

3.4 Generalised Fully Probabilistic Design for Nonlinear Uncertain Systems

The FP control design is further exploited in this section for nonlinear uncertain systems. The dynamics of real world systems are unknown and are hence required to be approximated. However, the modelling of unknown systems governed by nonlinearities inevitably results in some approximation error in the estimated model leading to functional uncertainties and uncertainty from the unknown

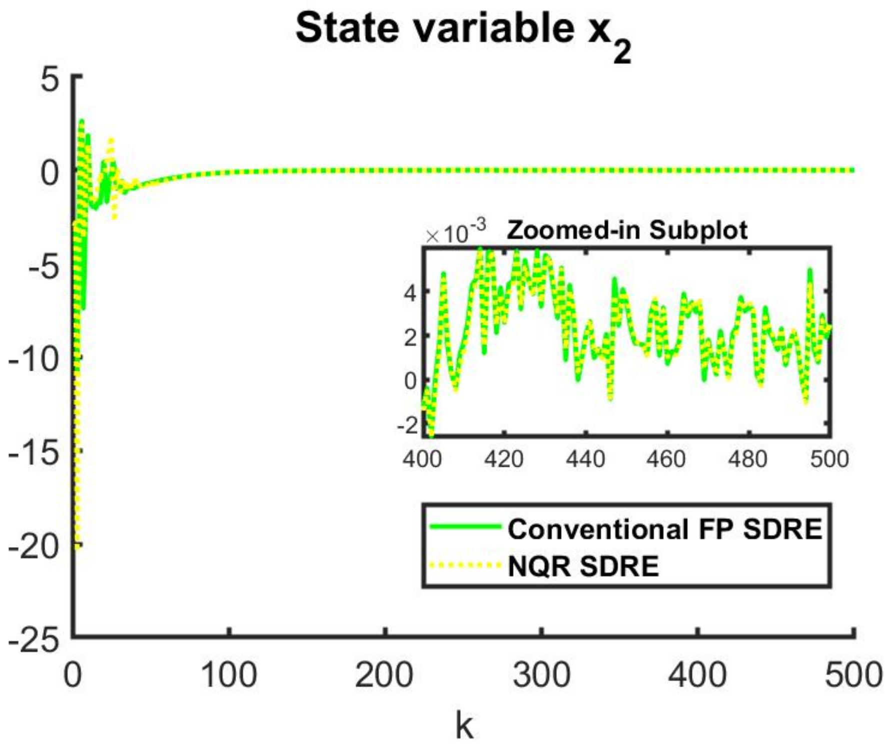

Figure 3.7: Comparison between the conventional SDRE FP control approach and the NQR SDRE control strategy on state x_2. The equivalence of the estimated global covariance matrix Σ_k for the FP control method is taken as the Q value and R is the equivalent of the covariance of the ideal controller Γ. The subplot shows that the states oscillate around zero.

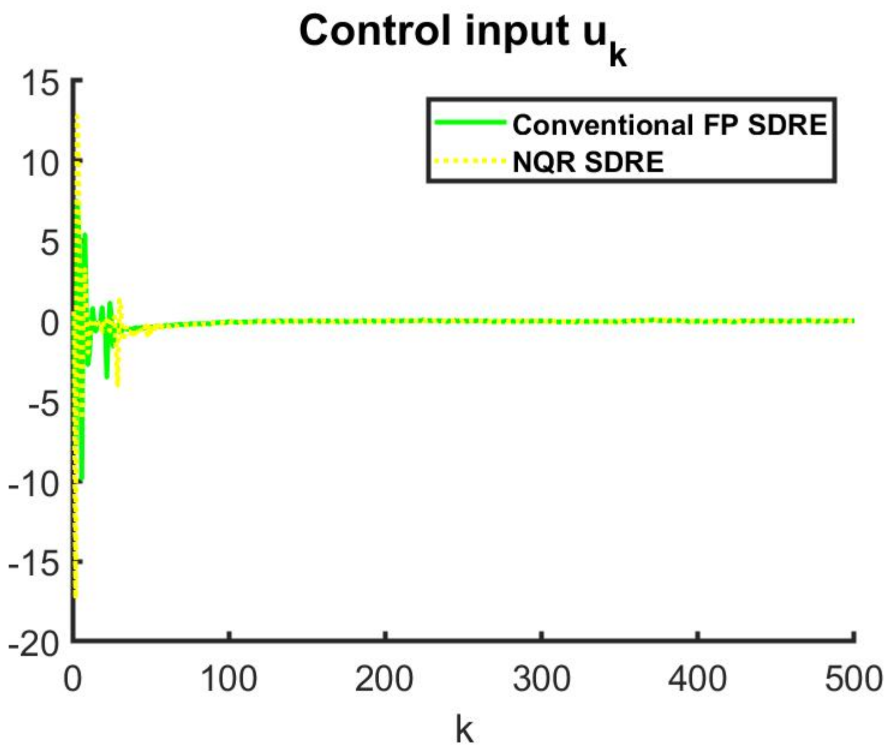

Figure 3.8: Comparison between the Conventional SDRE and the NQR SDRE control strategy on the control input u_k.

parameters. These complications necessitate and increase the requirement for the development of robust cautious controllers. The previous section (Section 3.3) focused on the designation of a fully probabilistic controller for systems governed by nonlinearities. Despite having to estimate the parameters of the distribution of the system state, the controller derived in Section 3.3 did not consider functional uncertainty. Therefore, the designation of the controller in this section considers functional uncertainty to ensure robustness.

3.4.1 System State Estimation

The system for which the controller is derived in this section is governed by the same dynamics as described in Section 3.3.1 and is repeated here for the convenience of the reader,

$$x_k = \bar{h}(x_{k-1})x_{k-1} + \bar{g}(x_{k-1})u_k + \epsilon_k. \tag{3.47}$$

For the estimation of the probability distribution of the system state variables, two Neural Networks that are optimised online, are implemented in this section to provide predictions for the conditional expectation of the system state and the covariance of the estimation error.

To re-emphasise, the functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ need to be approximated to obtain the estimation of the conditional expectation of the system state variables. The same procedure as explained in Section 3.3.2 is followed to achieve this. Again, it is repeated here for the reader's convenience,

$$\begin{aligned}\hat{x}_k =& \text{mlp}(x_{k-1}),\\ =& h(x_{k-1})x_{k-1} + g(x_{k-1})u_k,\end{aligned} \tag{3.48}$$

where $h(x_{k-1})$ and $g(x_{k-1})$ are the approximations of the actual states and control matrices, given by $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$, respectively.

Once the system state estimation has been completed, the following stochastic model can be established,

$$x_k = \hat{x}_k + e(x_{k-1}, u_k), \tag{3.49}$$

where the estimation error $e(x_{k-1}, u_k)$ represents the functional uncertainty of the estimated model at time k which is shown to be close to Gaussian noise [167] with zero mean and an input dependent covariance matrix given by $\tilde{\Sigma}_k = E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$.

This input dependent covariance matrix $\tilde{\Sigma}_k$ can then be estimated using a second Generalised Linear neural network model which takes the state variables and control signal as inputs,

$$\Sigma_k = Dx_{k-1} + Gu_k, \tag{3.50}$$

where D and G are partitioned matrices and are updated online at each instant of time with the aim to minimise the error between the actual covariance matrix, $\tilde{\Sigma}_k$ and the estimated one, Σ_k. Also note that x_{k-1} and u_k should be represented as block matrices in (3.50). The linear structure is assumed for simplification and better clarification of the system uncertainty estimation. To elaborate, the parameters of the GLM network can be obtained using linear optimisation methods (i.e. with the Pseudo-inverse). The state x_{k-1} and the control signal u_k are taken to be the input (I) to the GLM. The actual output (O) of the GLM is the square of the error between the actual state x_k and the estimated state $\hat{x}_k$. The parameters (w) of the GLM are obtained by computing $O = wI \rightarrow w = OI^{\dagger}$, where † is the Pseudo-inverse. The obtained parameters (w) can then be represented as the partitioned matrices D and G.
To re-emphasise, D and G are partitioned matrices that are obtained from the parameters of the GLM to reconstruct the covariance matrix in the correct way. Furthermore, we introduced checks to make sure that the estimated covariances are always positive. To elaborate, whenever the covariance value goes negative, it is replaced by a small positive number.

For a more detailed explanation and visual representation of the estimation process of the conditional distribution of the system, the readers are referred to Figure 3.9.

Following the assumption that the residual error from the estimation process is Gaussian, the distribution of the system state at time k will then be Gaussian and hence represented by,

$$s(x_k|u_k, x_{k-1}) = \mathcal{N}(\hat{x}_k, \Sigma_k), \tag{3.51}$$

where the mean, $\hat{x}_k$ is defined in (3.48) and Σ_k is the covariance matrix given by (3.50). This characterisation of the system state by Gaussian distribution, facilitates the derivation of a closed for control solution as will be seen from further developments. To re-emphasise, the parameters of the estimated pdf of the system state are state and control input dependent. The estimated covariance matrix given in (3.50) characterises the functional uncertainty that results due to the discrepancy between the actual and estimated behaviour of the system dynamics.

To clarify and understand the concept of partitioned matrices, let us consider a two-dimensional system as an example, for which the dimension of the covariance matrix is (2×2). It is assumed that u_k

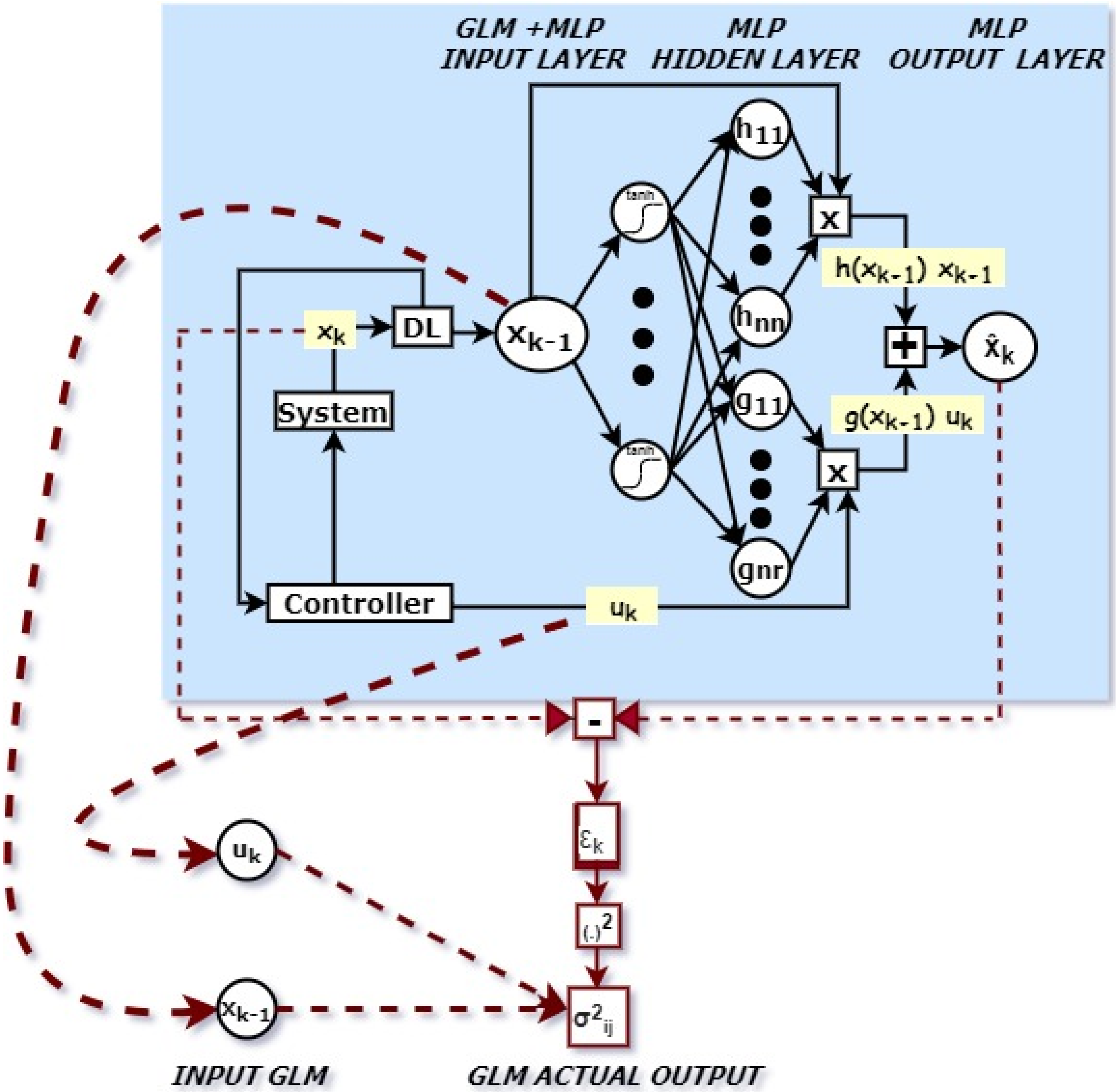

Figure 3.9: The shaded area in blue represents the process of the MLP for the estimation of the system dynamics. The outer part and dashed lines correspond to the GLM process. For both, the MLP and GLM, x_{k-1} is the input layer to the neural network.

is a single input to the system, and is therefore a scalar. Hence, the equation in (3.50) is given by,

$$
\begin{aligned}
\begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{bmatrix} &= \left[\begin{array}{c:c} [d_{11} \quad d_{12}] & [d_{13} \quad d_{14}] \\ \hdashline [d_{21} \quad d_{22}] & [d_{23} \quad d_{24}] \end{array}\right] \left[\begin{bmatrix} x_{1;k-1} \\ x_{2;k-1} \end{bmatrix} \right] + \left[\begin{array}{c:c} g_{11} & g_{12} \\ \hdashline g_{21} & g_{22} \end{array}\right] u_k, \\
&= \left[\begin{array}{c:c} D_{11} & D_{12} \\ \hdashline D_{21} & D_{22} \end{array}\right] x_{k-1} + \left[\begin{array}{c:c} g_{11} & g_{12} \\ \hdashline g_{21} & g_{22} \end{array}\right] u_k, \\
&= Dx_{k-1} + Gu_k
\end{aligned} \tag{3.52}
$$

where d_{ij} are the elements of the partitioned matrix D and g_{ik} are the elements of matrix G with $i = \{1, 2\}$, $j = \{1, 2, 3, 4\}$ and $k = \{1, 2\}$. In this example, D is a partitioned matrix of size (2×2), x_{k-1} is a partitioned matrix of size (1×1), G is a partitioned matrix of size (2×2) and u_k is a partitioned matrix of size (1×1). Note that the dimension of each sub-matrix of the partitioned matrix D corresponds to the dimension of the state vector x_{k-1}, which in the case of this example is 2-dimensional. Likewise, the dimension of each sub-matrix of the partitioned matrix G corresponds to the dimension of the control input u_k which in this case is a scalar.

As can be seen from (3.52), the covariances have been obtained by computing the following,

$$
\begin{aligned}
\sigma_{11}^2 &= d_{11}x_{1;k-1} + d_{12}x_{2;k-1} + g_{11}u_k, \\
\sigma_{12}^2 &= d_{13}x_{1;k-1} + d_{14}x_{2;k-1} + g_{12}u_k, \\
\sigma_{21}^2 &= d_{21}x_{1;k-1} + d_{22}x_{2;k-1} + g_{21}u_k, \\
\sigma_{22}^2 &= d_{23}x_{1;k-1} + d_{24}x_{2;k-1} + g_{22}u_k.
\end{aligned}
$$

3.4.2 FP Control Solution for Nonlinear Systems with Functional Uncertainty

Similar to Section 3.3, the purpose of the developed controller presented in this section is to regulate the system to state zero. Therefore, the ideal distribution of the system is given by,

$$s^I(x_k|u_k, x_{k-1}) = \mathcal{N}(0, \Sigma_2), \tag{3.53}$$

where the zero mean reflects the regulation around zero objective. Note that unlike the method developed in Section 3.3, the covariance of the ideal distribution (3.53) in this section is not the same as the covariance of the actual distribution (3.51). The ideal covariance Σ_2 is assumed to be smaller than the actual covariance Σ_k. This is permissible as the covariance matrix Σ_k is state and control dependent and thus can be driven to a smaller value which is specified by the covariance of the ideal distribution of the system state. Finally, the ideal distribution of the controller is still determined to be Gaussian and given by,

$$c^I(u_k|x_{k-1}) = \mathcal{N}(0, \Gamma), \tag{3.54}$$

where the covariance Γ identifies the permissible range of optimal control inputs.

Given the pdfs of the system state (3.51), and ideal distributions of the system state and control input given by (3.53) and (3.54) respectively, the randomised suboptimal controller is presented in the following theorem.

Theorem 2. The suboptimal control law that minimises the cost-to-go function (3.6) and has been derived based on Definition 3.3.1 and the assumption that the state covariance matrix depends linearly on the system state is specified by the following Gaussian distribution,

$$c^*(u_k|x_{k-1}) = \mathcal{N}(u_k^*, R_k), \tag{3.55}$$

where,

$$u_k^* = -K_k x_{k-1} - P_k, \tag{3.56}$$

$$K_k = R_k\left[g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1})\right], \tag{3.57}$$

$$P_k = R_k\left[\frac{1}{2}g^T(x_{k-1})T_k + \frac{1}{2}\text{tr}(GM_k)\right], \tag{3.58}$$

$$R_k = \left[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})\right]^{-1}. \tag{3.59}$$

The suboptimal controller is specified by a Gaussian distribution with mean u_k^* and covariance R_k. Based on the estimated distribution of the system state given by (3.51), it can be shown that the performance index, $-\ln(\gamma(x_k))$, is described as follows,

$$-\ln(\gamma(x_k)) = 0.5x_k^T M_k x_k + 0.5T_k x_k + 0.5\omega_k, \tag{3.60}$$

where,

$$\begin{aligned} M_{k-1} =& h^T(x_{k-1})\left[(\Sigma_2^{-1} + M_k) - (\Sigma_2^{-1} + M_k)g(x_{k-1})\left[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})\right]^{-1}\right. \\ &\left.\times g^T(x_{k-1})(\Sigma_2^{-1} + M_k)\right]h(x_{k-1}), \end{aligned} \tag{3.61}$$

$$\begin{aligned} T_{k-1} =& T_k h(x_{k-1}) + \text{tr}(DM_k) - 2\left(\frac{1}{2}T_k g(x_{k-1}) + \frac{1}{2}\text{tr}(GM_k)\right) \\ &\times\left[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})\right]^{-1} g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1}), \end{aligned} \tag{3.62}$$

$$\begin{aligned} \omega_{k-1} =& \omega_k - \frac{1}{2}\Big(T_k g(x_{k-1}) + \text{tr}(GM_k)\Big)\left[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})\right]^{-1} \\ &\times\frac{1}{2}\Big(g^T(x_{k-1})T_k + \text{tr}(GM_k)\Big) + \ln|\Gamma| + \ln|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})|. \end{aligned} \tag{3.63}$$

Proof. The proof of Theorem 2 can be found in Appendix B. □

In this theorem, Equation (3.61) is the discrete time algebraic SDRE, T_k in (3.62) is the linear equation and ω_k in (3.63) is the constant term . Note that Equation (3.62) is a key adjustment to the conventional form of the FP control design. This term arises from the consideration of the input and state dependent noise. As such, this equation can be referred to as the equation of cautiousness since it is the part that allows the probabilistic controller to be cautious and take functional uncertainty into consideration. The reason as to why this linear term has appeared in the development of the FP control design in this section is due to the form of the covariance matrix (3.50) which is linear in the state and control input. After observing the form of the suboptimal randomised controller given by Equation (3.55), one can notice that it differs from the controller derived in Section 3.3.3. The mean of the designed controller in this section involves a linear shift given by P_k. This shift exists due to the involvement and consideration of the input and state dependent noise, and is based on the parameter of the noise model (3.50) and the equation of cautiousness (3.62).

3.4.3 Algorithm of Proposed Method in Theorem 2

As outlined in Section 3.3.4, the SDRE defined in (3.61) and in this case, the linear term given by (3.62) need to be computed to implement the fully probabilistic controller given by (3.55). The same approach is followed and hence, the time index is adjusted to obtain the steady state solution of the Riccati equation. Once the SS solution of the Riccati equation has been obtained, it is used in Equation (3.62) of which the time index is again changed such that the solution of the Riccati equation at k is obtained from earlier time instants. Thus, the following is computed to obtain the SS solution of T_k.

$$\begin{aligned} M_k =& h^T(x_{k-1})\Big[(\Sigma_2^{-1} + M_{k-1}) - (\Sigma_2^{-1} + M_{k-1})g(x_{k-1}) \\ &\times \Big[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_{k-1})g(x_{k-1})\Big]^{-1} g^T(x_{k-1})(\Sigma_2^{-1} + M_{k-1})\Big]h(x_{k-1}), \quad (3.64) \\ T_k =& T_{k-1}h(x_{k-1}) + \mathrm{tr}(DM_k) - 2\Big(\frac{1}{2}T_{k-1}g(x_{k-1}) + \frac{1}{2}\mathrm{tr}(GM_k)\Big)\Big[\Gamma^{-1} + g^T(x_{k-1}) \\ &\times (\Sigma_2^{-1} + M_k)g(x_{k-1})\Big]^{-1} g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1}). \quad (3.65) \end{aligned}$$

To re-emphasise, the reversed SDRE (3.64) and reversed equation of cautiousness (3.65) need to be reiterated a certain arbitrary number of times until the steady state solutions are found. The pseudocode in Algorithm 3 summarises the approach of the FP control design developed in this section as a pseudocode.

3.4.4 Simulation

A simulation example is implemented here to demonstrate that the derived randomised controller in (3.55) effectively controls stochastic systems that are characterised by functional uncertainty and achieves the desired control objective. The inverted pendulum on a cart problem which is imple-

Algorithm 3 Pseudo-code of FP control design that considers functional uncertainty

1: **procedure** IMPLEMENTATION OF THE FP CONTROL DESIGN FOR NONLINEAR SYSTEMS THAT CONSIDER FUNCTIONAL UNCERTAINTY
2: **Initialise:** x_0, M_0, T_0 and optionally pre-train the neural network model as discussed in Algorithm 1 (optional).
3: **for** $k = 1 \rightarrow H$ **do**
4: **Estimate** $h(x_{k-1})$, $g(x_{k-1})$ from the neural network model.
5: **Approximate** D and G for Σ_k as according to (3.50).
6: **Calculate** the SS solution of M using (3.64).
7: **Use** the SS solution M from *Step 6* to find SS solution of T using (3.65).
8: **Use** the SS values from *Steps 6-7* in (3.57) - (3.58) to obtain the SS solutions of K and P.
9: **Calculate** u_k^* using *Step 8* in Equation (3.56).
10: **Forward** the control signal u_k^* obtained in *Step 9* to the system equation (3.23).
11: **Using** a one step delayed of the new state value x_{k-1} from *Step 10* and the calculated control signal u_k^* from *Step 9* as input to the neural network model and the new state x_k from *Step 10* as output, retrain the neural network model and update its parameters.
12: **end for**

mented extensively as a simulation example [173], [174] for control problems is used here to evaluate the performance of the proposed nonlinear FP control method. The results are compared with the conventional SDRE FP control approach discussed in Section 3.3 which does not account for functional uncertainty. The discrete time equation of the nonlinear inverted pendulum is described in the form of (3.47) and repeated below,

$$x_k = \bar{h}(x_{k-1})x_{k-1} + \bar{g}(x_{k-1})u_k + \epsilon_k, \tag{3.66}$$

where,

$$\bar{h}(x_{k-1}) = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & a_{22}(x_{k-1}) & a_{23}(x_{k-1}) & a_{24}(x_{k-1}) \\ 0 & 0 & 1 & T \\ 0 & a_{42}(x_{k-1}) & a_{43}(x_{k-1}) & a_{44}(x_{k-1}) \end{bmatrix}, \bar{g}(x_{k-1}) = \begin{bmatrix} 0 \\ b_2(x_{k-1}) \\ 0 \\ b_4(x_{k-1}) \end{bmatrix},$$

and where x_k is four-dimensional such that,

$$x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{bmatrix}. \tag{3.67}$$

The original inverted pendulum system equation that is used in [173] does not involve any noises which is not the case in real-world situations. Therefore, to simulate the system properly as it operates in real-world, the equation obtained from [173] is modified by the addition of noise to it. The noise ϵ_k in (3.66) is Gaussian with zero mean and covariance $\bar{\Sigma} = 0.001\mathcal{I}_{4\times4}$. The matrices $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ are the state and control matrices respectively of which the variables are defined as follows,

$$\begin{aligned}
a_{22}(x_{k-1}) =& 1 + T\frac{-b}{\Omega_2}, \\
a_{23}(x_{k-1}) =& T\frac{m^2L^2g\cos(x_{3,k})\sin(x_{3,k})}{\Omega_2(l+mL^2)}, \\
a_{24}(x_{k-1}) =& T\frac{mL\sin(x_{3,k})}{\Omega_2}(x_{4,k}), \\
a_{42}(x_{k-1}) =& T\frac{mLb\cos(x_{3,k})}{(M+m)\Omega_1}, \\
a_{43}(x_{k-1}) =& -T\frac{mgL\sin(x_{3,k})}{\Omega_1(x_{3,k})}, \\
a_{44}(x_{k-1}) =& 1 - T\frac{m^2L^2\cos(x_{3,k})\sin(x_{3,k})(x_{4,k})}{(M+m)\Omega_1}, \\
b_2(x_{k-1}) =& \frac{T}{\Omega_2}, \\
b_4(x_{k-1}) =& -T\frac{mL\cos(x_{3,k})}{(M+m)\Omega_1},
\end{aligned} \tag{3.68}$$

where,

$$\begin{aligned}
\Omega_1 &= l + mL^2 - \frac{m^2L^2\cos^2(x_{3,k})}{M+m}, \\
\Omega_2 &= M + m - \frac{m^2L^2\cos^2(x_{3,k})}{l+ML^2}.
\end{aligned}$$

The description of the parameters of the system are assumed to be,

$$\begin{aligned}
&M = 0.5\text{kg}, \qquad m = 0.5\text{kg}, \qquad b = 0.1\text{N}.\,\frac{\text{sec}}{\text{m}}, \\
&L = 0.3\text{m, and} \quad l = 0.06\text{kg.m}^2,
\end{aligned} \tag{3.69}$$

where:

- M is the mass of the cart.
- m is the mass of the pendulum.
- b is the friction coefficient between cart and ground.
- L is the length to the pendulum center of mass.
- l is the inertia of the pendulum.

Details on how to find the estimations of matrices $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ to obtain $h(x_{k-1})$ and $g(x_{k-1})$ can be found in Section 3.4.1.

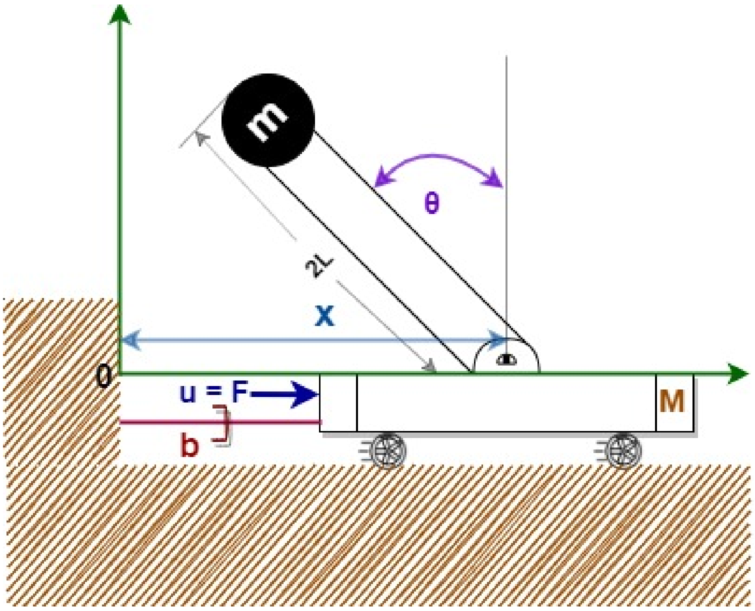

Figure 3.10: A diagram of the inverted pendulum on a cart. The pendulum can be balanced at a specific position following the application of a horizontal force to drive the cart. The pendulum mass, m, is focused at the end of the massless rod. The horizontal displacement on the cart is denoted by x. θ represents the rotational angle of the pendulum. The carriage driven by the Force is u. The friction coefficient of the cart is given by b. The mass of the cart is given by M.

The covariance Γ of the controller is chosen to be 100 to give the controller more freedom and achieve a faster convergence rate. The consideration of functional uncertainty in this section means that the covariance matrix is dependent on the state and control input which was taken into account in the derivation of the proposed method. On the contrary, the method developed in Section 3.3 which is referred to as the conventional SDRE FP control approach, does not consider the functional uncertainty of the system dynamics. There, the ideal covariance matrix Σ_2 is assumed to be the same as the actual global covariance namely Σ_k. In addition, the development in Section 3.3 only requires the evaluation of the SDRE given by equation (3.38). The equation of cautiousness given in (3.62) as well as the additional linear term (3.58) which are required for the evaluation of the randomised controller in this section given by (3.55), do not exist. These additional equations emerge in the proposed method in this section only due to the consideration of the functional uncertainty of the system dynamics as explained earlier.

To clarify, two sets of experiments were conducted. In the first experiment, the conventional FP SDRE developed in Section 3.3 is used to derive the suboptimal randomised controller. In this experiment, the covariance matrix of the ideal distribution of the system state is taken to be equal to the global variance of the actual covariance matrix which is estimated as discussed in Section 3.3.2. In the second experiment, the method proposed in this section that accounts for functional uncertainties is used to derive the suboptimal randomised controller. Here, the covariance matrix, Σ_2, of the ideal distribution of the system state is taken to be,

$$\Sigma_2 = 10^{-2} \times \begin{bmatrix} 0.0001 & 0 & 0 & 0 \\ 0 & 0.0004 & 0 & 0 \\ 0 & 0 & 0.0009 & 0 \\ 0 & 0 & 0 & 0.0013 \end{bmatrix}. \tag{3.70}$$

The ideal covariance matrix Σ_2 is a design parameter which is chosen here to be smaller than the covariance matrix of the noise affecting the system, since the noise is state dependent, thus its effect can be minimised as discussed in Section 3.3.2. For both, the conventional SDRE FP control design and the proposed method, the initial state of the pendulum is taken to be, $x_0 = \begin{bmatrix} -1 & 2.4 & 0.2 & -0.2 \end{bmatrix}$, and the control objective is to bring the four states of the pendulum from their initial values to zero.

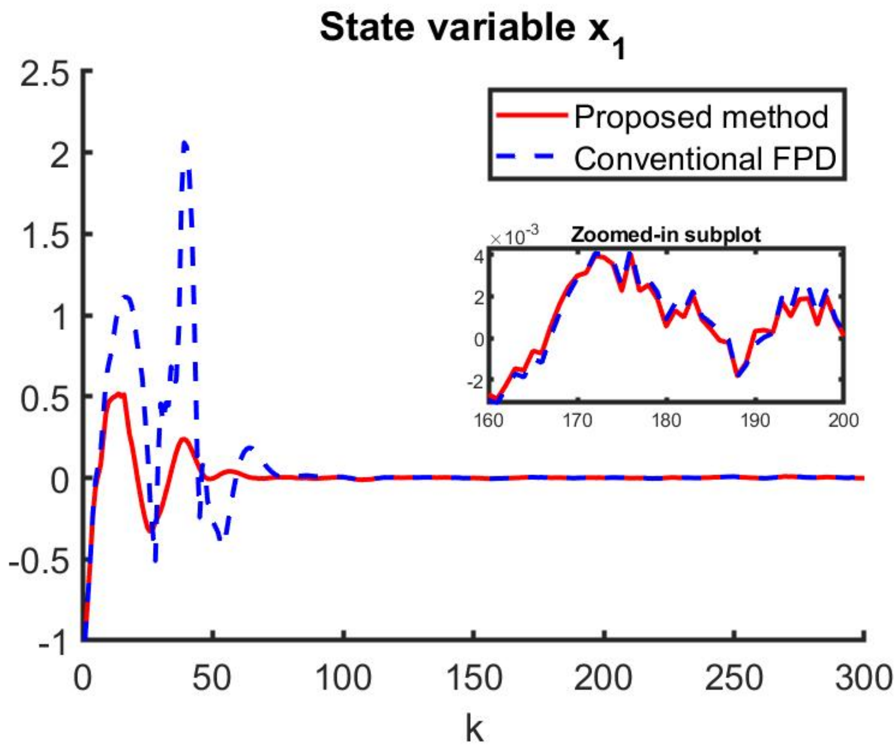

Figure 3.11: Comparison between the proposed method and conventional SDRE FP control design on state x_1.

The results of both experiments are shown in Fig. 3.11 - 3.14 from which it can be seen that the states of the pendulum (3.66) have converged to zero for both the conventional SDRE FP control design and the proposed method in this section. However, compared to the conventional SDRE FP control approach, the states converge faster and with less oscillations using the proposed method in this section which accounts for functional uncertainty and input dependent noises. Hence, the transient response is reached quicker for the proposed FP control design. The consideration of functional uncertainty embedded in P_k affects the transient response of the system and the speed in which the

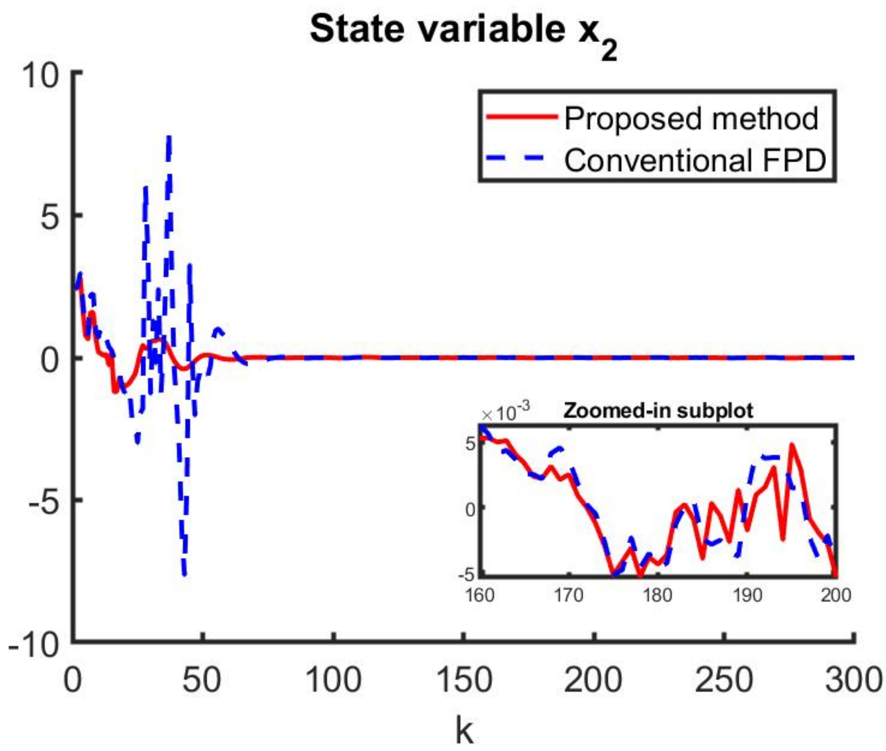

Figure 3.12: Comparison between the proposed method and conventional SDRE FP control design on state x_2.

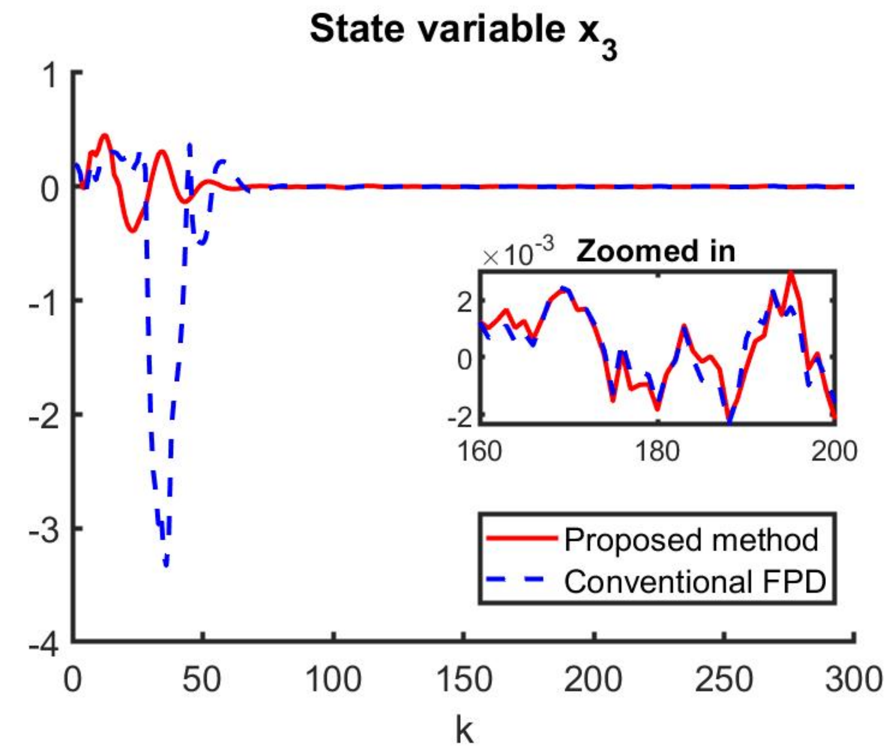

Figure 3.13: Comparison between the proposed method and conventional SDRE FP control design on state variable x_3.

system converges to the steady state value as has also been shown in [147]. Thus, it can be concluded that the converging speed of the proposed design is better. Also, having a controller that takes uncertainties into consideration ensures that the system does not overshoot which can also be clearly

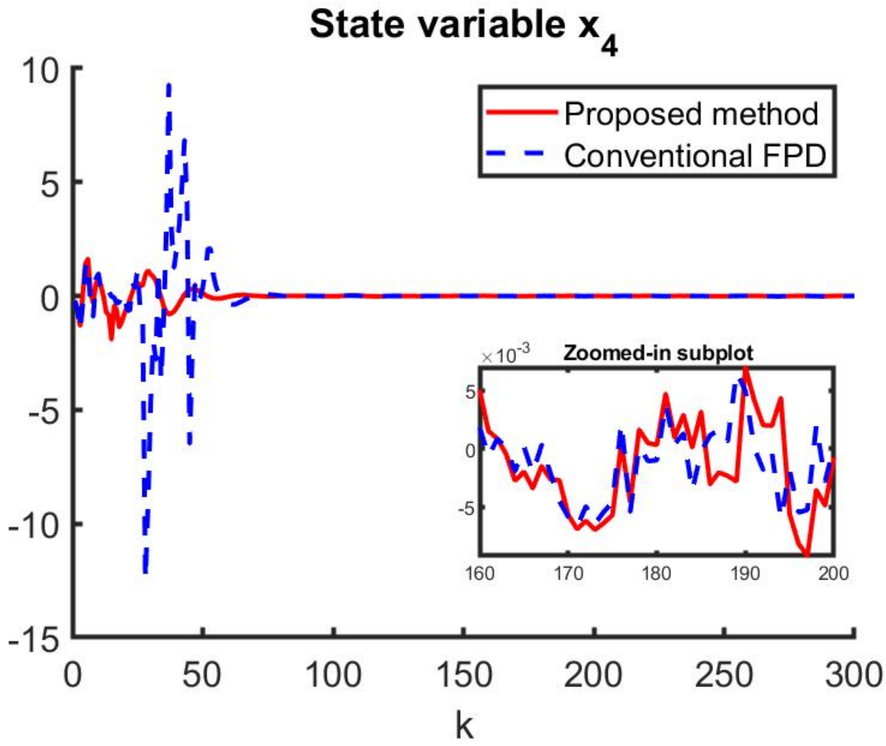

Figure 3.14: Comparison between the proposed method and conventional SDRE FP control design on state variable x_4.

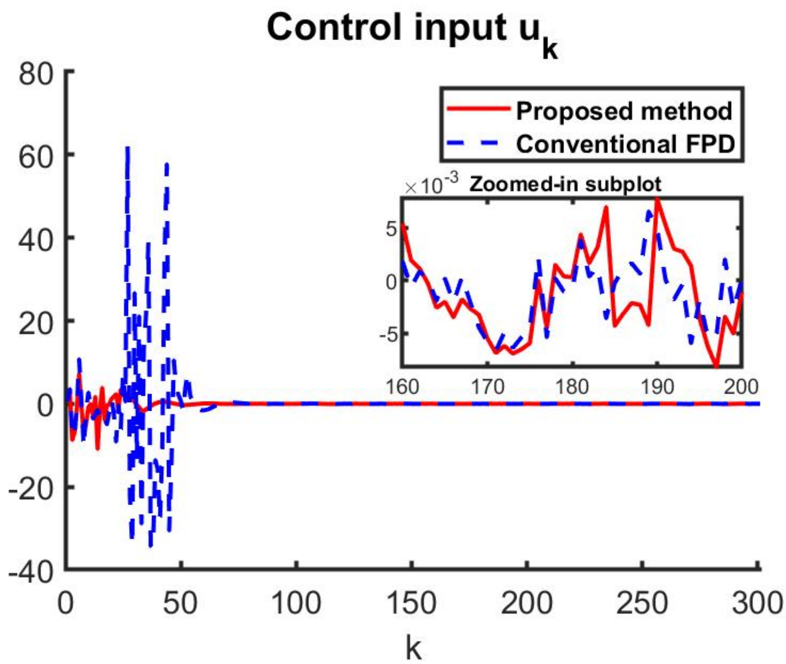

Figure 3.15: Comparison between the proposed FP method that considers functional uncertainties and the conventional SDRE FP control design on the control input u_k.

seen from Fig. 3.11 - 3.14. In addition, Figure 3.15 shows the obtained control sequence for both the proposed SDRE FP control design and the conventional SDRE FP control design.

3.4.5 Simulation on Multiple Noise Sequences

In this section, the previous simulation is repeated multiple times with different noise sequences. This is done with the objective of numerically estimating the distribution of the closed loop system and then demonstrating that the distribution of the closed loop system converges to the predefined ideal distribution. In this section, the simulations for the various noise sequences are repeated for both the proposed SDRE FP control method that takes functional uncertainty into consideration in the derivation of the suboptimal control law, and the conventional SDRE FP control design. The averages, the global averages and the global variances of the states of both the proposed SDRE FP control method and the conventional SDRE FP control design can be found in Table 3.1, 3.2 and 3.3.

For each simulation, the average of the states is calculated over time ($\text{average}_{\text{time}}$). The mean $\bar{x}$ is calculated by finding the average of these averages, i.e. ($\frac{\sum \text{average}_{\text{time}}}{\text{number of simulations}}$). Furthermore, the variance is computed as follows: $\text{var} = \frac{\sum(\text{average}_{\text{time}} - \bar{x})^2}{\text{number of simulations}}$.

Table 3.1: Proposed SDRE FP Control Method

Sim. no. / Average$_{time}$ State no. ($\times10^{-4}$)	1	2	3	4	5	6	7	8
State 1	-8.15	-5.33	-0.67	-0.97	12.4	-18.8	-9.7	-0.68
State 2	6.92	5.09	-0.99	1.9	-11.6	14.7	6.96	-3.48
State 3	-0.83	-0.801	-0.68	0.18	1.05	-2.99	-1.67	-1.1
State 4	7.07	6.06	2.09	-0.41	-9.78	17.2	8.91	1.36

Sim. no. / Average$_{time}$ State no. ($\times10^{-4}$)	9	10	11	12	13	14	15	16
State 1	-0.44	-1.27	-6.93	1.29	7.87	3.08	8.54	4.39
State 2	-0.599	5.94	6.83	-1.57	-6.85	-1.76	-5.29	1.66
State 3	-0.53	0.85	-0.91	0.19	0.46	0.27	1.78	2.5
State 4	2.2	-8.21	7.83	-1.81	-4.79	-2.64	-6.91	-6.6

Sim. no. / Average$_{time}$ State no. ($\times10^{-4}$)	17	18	19	20	21	22	23	24
State 1	4.13	1.15	0.4	3.49	2.22	-5.12	0.48	-0.16
State 2	-2.78	-1.95	1.98	-1.46	-1.86	2.56	2.15	-2.7
State 3	-0.43	-0.25	1.27	0.75	0.38	-1.66	1.2	-0.73
State 4	-2.92	-1.21	-0.12	-2.42	-1.58	6.32	-1.91	0.11

Sim. no. / Average$_{time}$ State no. ($\times10^{-4}$)	25	26	27	28	29	30	31	32
State 1	-0.75	-4.47	-4.47	10.37	-16.95	-18.07	-1	-1.4
State 2	3.69	2.18	3.87	-12.06	13.42	12.12	-0.23	-8.1
State 3	0.85	-3.02	-0.801	0.37	-3.28	-3	0.07	0.32
State 4	-0.15	13.57	5.56	-8.99	16.04	14.38	-0.23	-1.3

Table 3.2: Conventional SDRE FP Control Design

Sim. no. / Average$_{\text{time}}$ State no. ($\times 10^{-4}$)	1	2	3	4	5	6	7	8
States 1	-11.81	-9.65	-0.82	-9.43	18.75	-17.52	-13.11	2.78
State 2	10.51	6.36	-1.23	8.34	-13.56	13.29	7.77	-3.24
State 3	-0.56	-0.71	-0.71	0.89	1.26	-3.2	-1.49	-0.7
State 4	6.33	5.59	1.6	-0.13	-10.25	16.79	8.38	0.85
Sim. no. / Average$_{\text{time}}$ State no. ($\times 10^{-4}$)	9	10	11	12	13	14	15	16
States 1	-0.92	-13.73	-10.64	0.3	10.81	3.74	8.78	-0.11
State 2	-0.29	11.37	7.68	0.44	-7.5	-3.33	-3.27	4.79
State 3	-0.46	-0.29	-0.77	0.47	0.49	0.39	1.84	2.55
State 4	1.83	5.94	6.64	-1.4	-4.66	-2.9	-7.27	-6.21
Sim. no. / Average$_{\text{time}}$ State no. ($\times 10^{-4}$)	17	18	19	20	21	22	23	24
States 1	7.15	2.32	2.18	2.48	4.77	-4.2	2.22	1.53
State 2	-4.75	-3.39	1.19	0.23	-3.02	0.55	-1.3	-2.49
State 3	0.28	-0.44	0.94	0.86	0.42	-1.82	0.47	-0.54
State 4	-2.71	-0.55	0.39	-2.99	-1.62	6.16	-1.77	0.34
Sim. no. / Average$_{\text{time}}$ State no. ($\times 10^{-4}$)	25	26	27	28	29	30	31	32
States 1	-5.85	-0.23	-7.6	18.36	-35.49	-19.39	-59.14	-8.9
State 2	5.79	-3.02	3.64	-14.55	17.5	12.45	-45.2	-87.23
State 3	0.87	-1.22	-0.69	0.22	-3	-2.98	72.3	8.18
State 4	-0.13	2.26	5.16	-8.55	15.59	14.49	-23.8	-93.23

Table 3.3: Global Averages and Global Variances

State number	$\bar{x}$ = Global Average ($\times 10^{-4}$)			Var = Global Variance ($\times 10^{-4}$)		
State number	Proposed	Conventional	Ideal Distr.	Proposed	Conventional	Ideal Distr.
State 1	-1.35	-4.45	0	0.169	0.679	1
State 2	0.76	-2.67	0	0.125	1.11	4
State 3	-0.292	2.28	0	0.00625	0.518	9
State 4	1.48	-2.18	0	0.158	1.05	1.3

To repeat, the ideal probability distribution of the system state is described by a Gaussian distribution with mean of zero and covariance matrix Σ_2 given by

$$\Sigma_2 = 10^{-2} \times \begin{bmatrix} 0.0001 & 0 & 0 & 0 \\ 0 & 0.0004 & 0 & 0 \\ 0 & 0 & 0.0009 & 0 \\ 0 & 0 & 0 & 0.0013 \end{bmatrix}. \tag{3.71}$$

From Table 3.3, it can be realised that the global averages of the states from both the proposed SDRE FP control and the conventional SDRE FP control methods demonstrate close convergence to the mean of the ideal Gaussian distribution of the states, which is zero. Both estimated means and covariance matrices for the conventional FP control design and proposed method can be seen to be converging towards the ideal mean of zero and covariance matrix given in (3.71). The proposed method also demonstrates to work effectively when affected by various noise sequences.

3.5 Generalised Fully Probabilistic Design for Nonlinear Systems with Multiplicative Noises

Among the variety of noises that affects the dynamics of stochastic systems, multiplicative noises are one variant of noises that can be found in systems. The FP control design was further extended to linear systems with multiplicative noises in [30]. This advancement is further developed here for systems governed by nonlinearities that are affected by multiplicative stochastic disturbances which will be discussed and demonstrated in this section.

3.5.1 System State Estimation

The computation of the controller that is designed in this section handles systems that are nonlinear and affected by multiplicative stochastic noises. The dynamics of these systems are assumed to be governed by the following stochastic nonlinear discrete time equation,

$$x_k = \hat{h}(x_{k-1}) + \bar{g}(x_{k-1})u_k + \bar{D}x_{k-1}v_{k-1}. \tag{3.72}$$

The system goes through the same transformation as explained in Section 3.3.1 to obtain,

$$x_k = \bar{h}(x_{k-1})x_{k-1} + \bar{g}(x_{k-1})u_k + \bar{D}x_{k-1}v_{k-1}, \tag{3.73}$$

where $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ are the nonlinear state and control matrices, respectively. The system matrix $\bar{D}$ is multiplied with the scalar noise v_{k-1} with zero mean and variance Q,

$$v_{k-1} \sim \mathcal{N}(0, Q). \tag{3.74}$$

The estimations for the conditional expectation of the system state is obtained by estimating the functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$ as explained in Section 3.3.2. To remind the reader, it is repeated here,

$$\begin{aligned} \hat{x}_k &= \text{mlp}(x_{k-1}), \\ &= h(x_{k-1})x_{k-1} + g(x_{k-1})u_k. \end{aligned} \tag{3.75}$$

The functions $h(x_{k-1})$ and $g(x_{k-1})$ were estimated using the measurable state x_k. However, there does exist some error e_k between the actual value x_k and the estimated $\hat{x}_k$ such that $x_k - \hat{x}_k = e_k = \bar{D}x_{k-1}v_{k-1}$. from which the estimation of matrix $\bar{D}$ defined as D can then be obtained as follows,

$$D = e_k(x_{k-1}v_{k-1})^{\dagger}, \tag{3.76}$$

where $(.)^{\dagger}$ represents the pseudoinverse [175]. In addition, the covariance matrix, $\bar{\Sigma}_k$ which is given by,

$$\begin{aligned}
\bar{\Sigma}_k =&\text{cov}(x_k|u_k, x_{k-1}),\\
=&E\{(x_k - \hat{x}_k), (x_k - \hat{x}_k)^T\},\\
=&E\{\bar{D}x_{k-1}v_{k-1}v_{k-1}^T x_{k-1}^T \bar{D}^T\},\\
=&\bar{D}x_{k-1}Qx_{k-1}^T\bar{D}^T,
\end{aligned} \tag{3.77}$$

can now be estimated due to the achievement of the approximation D of $\bar{D}$.

Hence, the final estimation obtained for the state dependent covariance matrix is specified by,

$$\begin{aligned}
\Sigma_k =&E\{Dx_{k-1}v_{k-1}v_{k-1}^T x_{k-1}^T D^T\},\\
=&Dx_{k-1}Qx_{k-1}^T D^T.
\end{aligned} \tag{3.78}$$

Consequently, since the distribution of the multiplicative noise affecting the system dynamics is assumed to be Gaussian, the distribution of the model described in (3.73) will also be Gaussian,

$$s(x_k|u_k, x_{k-1}) \sim \mathcal{N}(\hat{x}_k, \Sigma_k), \tag{3.79}$$

where the mean $\hat{x}_k$ is given in (3.75) and the covariance is defined in (3.78). This Gaussian assumption on the multiplicative noise and consequently the distribution of the system state facilitate the derivation of a closed form control solution as will be see shortly.

3.5.2 FP Control Solution for Nonlinear Systems with Multiplicative Noises

Having estimated the conditional distribution of the system state, the ideal distribution of the system state and controller needs to be specified next to allow the implementation of the FP control method. The ideal distribution aims to achieve the same objective as outlined in Section 3.4.2 by Equations (3.53) and (3.54). Similar to the additive input dependent noise discussed in Section 3.4, since the multiplicative noise is state dependent, the covariance of the ideal distribution of the system state, Σ_2, can be specified to be smaller than the actual covariance of actual system state.

Theorem 3. Based on Definition 3.3.1, the suboptimal control law that minimises performance index

$-\ln(\gamma(x_{k-1}))$, subject to the probabilistic description of the system dynamics given by (3.79) and ideal distributions of the system and controller defined by (3.53) and (3.54) respectively, is specified as follows,

$$c^*(u_k|x_{k-1}) = \mathcal{N}(u_k^*, R_k), \tag{3.80}$$

where,

$$u_k^* = -K_k x_{k-1}, \tag{3.81}$$

$$K_k = R_k\left[g^T(x_{k-1})({\Sigma_2}^{-1} + S_k)^T h(x_{k-1})\right], \tag{3.82}$$

$$R_k = \left[\Gamma^{-1} + g^T(x_{k-1})({\Sigma_2}^{-1} + S_k)g(x_{k-1})\right]^{-1}. \tag{3.83}$$

The suboptimal control law specified in (3.80) is described by a Gaussian distribution with mean u_k^* in (3.81) and covariance R_k given in (3.83). The term K_k in (3.82) is the suboptimal control gain. The suboptimal performance index for nonlinear systems with multiplicative stochastic disturbances is given by,

$$-\ln(\gamma(x_k)) = 0.5x_k^T S_k x_k + 0.5\omega_k, \tag{3.84}$$

where,

$$\begin{aligned} S_{k-1} = &- h^T(x_{k-1})({\Sigma_2}^{-1} + S_k)g(x_{k-1})\left[\Gamma^{-1} + g^T(x_{k-1})({\Sigma_2}^{-1} + S_k)g(x_{k-1})\right]^{-1} \\ &\times g^T(x_{k-1})({\Sigma_2}^{-1} + S_k)h(x_{k-1}) + h^T(x_{k-1})({\Sigma_2}^{-1} + S_k)h(x_{k-1}) + D^T S_k Q D, \end{aligned} \tag{3.85}$$

$$\omega_{k-1} = \omega_k + \ln|\Gamma| + \ln|g^T(x_{k-1})(\Sigma_2^{-1} + S_k)g(x_{k-1}) + \Gamma^{-1}|. \tag{3.86}$$

Proof. In Appendix C, the proof of Theorem 3 has been demonstrated. □

Here, (3.85) is the discrete time algebraic state dependent Riccati equation due to the dependency of the nonlinear parameters of the state and control distributions on previous state values. Also, compared to the DARE obtained in Sections 3.3.3 and 3.4.2, it has been shown that the derived DARE is a generalised Riccati equation solution that has an additional term, $D^T S_k Q D$, due to the consideration of the multiplicative noise in the stochastic system. The other DAREs given by (3.61) and (3.38) do not have this additional term. Hence, the DARE in (3.85) is referred to as the Generalised SDRE. As a result, a cautious controller is yielded by ensuring it considers the multiplicative noises effects and an analytic control solution that is based on the evaluation of a SDRE. The control solution in

Theorem 3 is referred to as the Generalised FP (GFP) control approach.

3.5.3 Algorithm of Proposed Method in Theorem 3

The employment of the suboptimal control law given by (3.80) requires the evaluation of the generalised state dependent Riccati equation solution specified by (3.85). Following the discussion in Section 3.3.4, the time index of the SDRE has been changed such that,

$$\begin{aligned} S_k = &- h^T(x_{k-1})({\Sigma_2}^{-1} + S_{k-1})g(x_{k-1})\left[\Gamma^{-1} + g^T(x_{k-1})({\Sigma_2}^{-1} + S_{k-1})g(x_{k-1})\right]^{-1} \\ &\times g^T(x_{k-1})({\Sigma_2}^{-1} + S_{k-1})h(x_{k-1}) + h^T(x_{k-1})({\Sigma_2}^{-1} + S_{k-1})h(x_{k-1}) + D^T S_{k-1} Q D. \end{aligned} \tag{3.87}$$

The pseudocode in Algorithm 4 summarises the approach of the FP control design as a pseudo-code.

Algorithm 4 Pseudo-code of Generalised SDRE FP control approach

1: **procedure** IMPLEMENTATION OF THE GENERALISED SDRE FP CONTROL DESIGN FOR NONLINEAR SYSTEMS WITH MULTIPLICATIVE NOISE
2: **Initialise:** x_0, S_0 and optionally pre-train the neural network model as discussed in Algorithm 1 (optional).
3: **for** $k = 1 \rightarrow H$ **do**
4: **Estimate** $h(x_{k-1})$, $g(x_{k-1})$ from the neural network model.
5: **Approximate** D for Σ_k in (3.78).
6: **Calculate** the SS solution of S using (3.87).
7: **Use** the SS values from *Step 6* in (3.82) to obtain the SS solutions of K.
8: **Calculate** u_k^* using *Step 7* in Equation (3.81).
9: **Update** the state using u_k^* from *Step 8*.
10: **Forward** the control signal u_k^* obtained in *Step 9* to the system equation (3.72).
11: **Using** a one step delayed of the new state value x_{k-1} from *Step 10* and the calculated control signal u_k^* from *Step 9* as input to the neural network model and the new state x_k from *Step 10* as output, retrain the neural network model and update its parameters.
12: **end for**

3.5.4 Simulation

To demonstrate the effectiveness of the method developed for nonlinear system that are affected by multiplicative noises, the simulation example implemented in Section 3.4.4 is used here. However, the model in [173], [174] is transformed into a stochastic model with multiplicative noise. The discrete time nonlinear system is given by,

$$x_k = \bar{h}(x_{k-1})x_{k-1} + \bar{g}(x_{k-1})u_k + \bar{D}x_{k-1}v_{k-1}, \tag{3.88}$$

where the state matrix $\bar{h}(x_{k-1})$ and control matrix $\bar{g}(x_{k-1})$ are defined in Section 3.4.4. The matrix $\bar{D}$ is randomly generated and is introduced to the system to be multiplied with the noise term v_{k-1}.

$$\bar{D} = \begin{bmatrix} 0.02 & 0.4 & 0.4 & 0.06 \\ 0.4 & 0.046 & 0.04 & 0.8 \\ -0.4 & 0.08 & -0.24 & 0.026 \\ 0.8 & -0.46 & -0.0028 & 0.04 \end{bmatrix}. \tag{3.89}$$

The noise v_{k-1} is Gaussian with zero mean and variance $Q = 0.09$.

For the simulation, the system is initially assumed to be in state $x_0 = \begin{bmatrix} 0.1 & 0.1 & -0.1 & 0.2 \end{bmatrix}^T$. For the GFP control design for this nonlinear system, $\Gamma = 0.5$. The chosen value for the covariance of the controller delivered good results after experimenting with different values. Nevertheless, to minimise the fluctuations of the system state, it is beneficial to choose a small covariance matrix Σ_2. Thus, for the implementation of the GFP control design for the inverted pendulum and cart problem, the covariance matrix Σ_2 is chosen to be,

$$\Sigma_2 = \begin{bmatrix} 0.004 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.0004 & 0 \\ 0 & 0 & 0 & 0.003 \end{bmatrix}. \tag{3.90}$$

The effectiveness of the designed controller in this section is compared to the conventional SDRE (Section 3.3). Both methods require the estimation of $\bar{h}(x_{k-1})$, $\bar{g}(x_{k-1})$ and $\bar{D}$ to obtain $h(x_{k-1})$, $g(x_{k-1})$, and D respectively. The comparison demonstrated that it is important to take account of the type of noise that is affecting the behaviour of the dynamics of the system when designing the controller. The simulation for the conventional SDRE FP control approach for a system with multiplicative noise is implemented by simply seeing the noise as an added stochastic noise, i.e. $Dx_{k-1}v_{k-1} = \epsilon_k$. It does not account for the multiplicative stochastic behaviour of the system. The ideal covariance matrix used for the conventional SDRE FP control design is the same as the global covariance of the actual system state of which the estimation process is explained in Section 3.3.2.

The control results are demonstrated in Fig. 3.16 - 3.19 which demonstrate the regulation of the states of the system using the conventional SDRE FP control solution (Section 3.3) and the proposed GFP control design method in this section. It can be seen that the states oscillate around zero for both as expected from a regulation problem. However, compared to the conventional SDRE solution, the states converges faster and with less oscillations using the GFP control design. The proposed GFP control design demonstrates a faster transient response. It can therefore be concluded

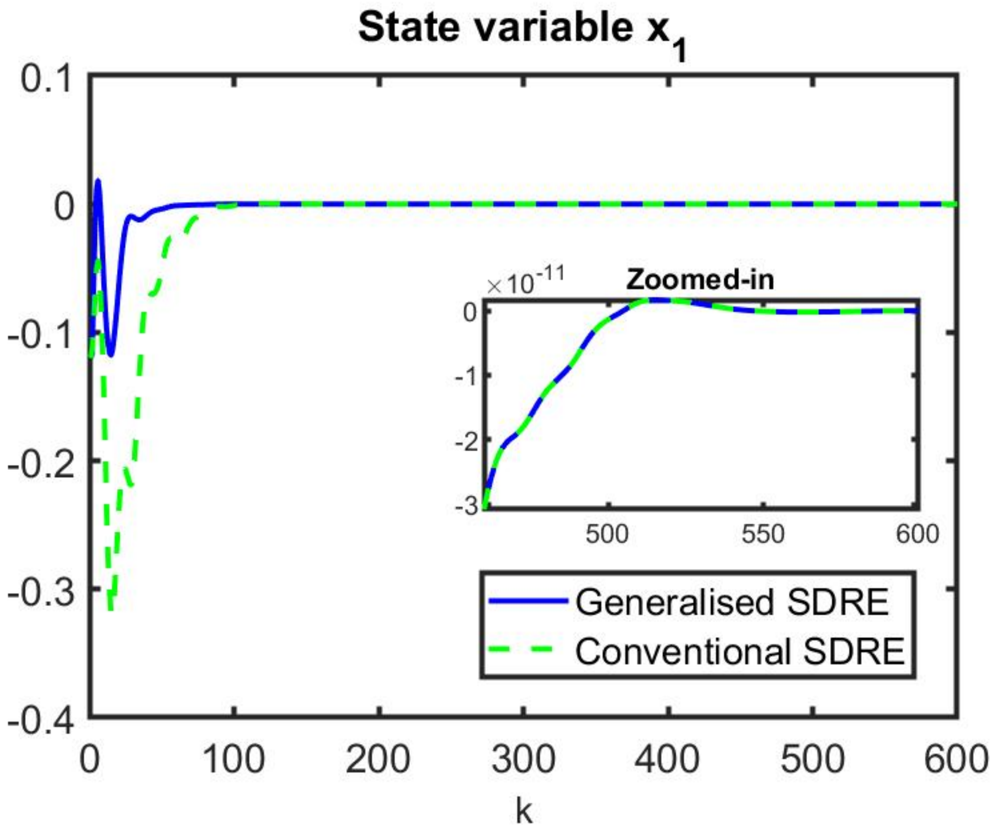

Figure 3.16: Comparison between the Generalised SDRE and Conventional SDRE on state x_1. The zoomed-in subplot demonstrates the states oscillating around zero.

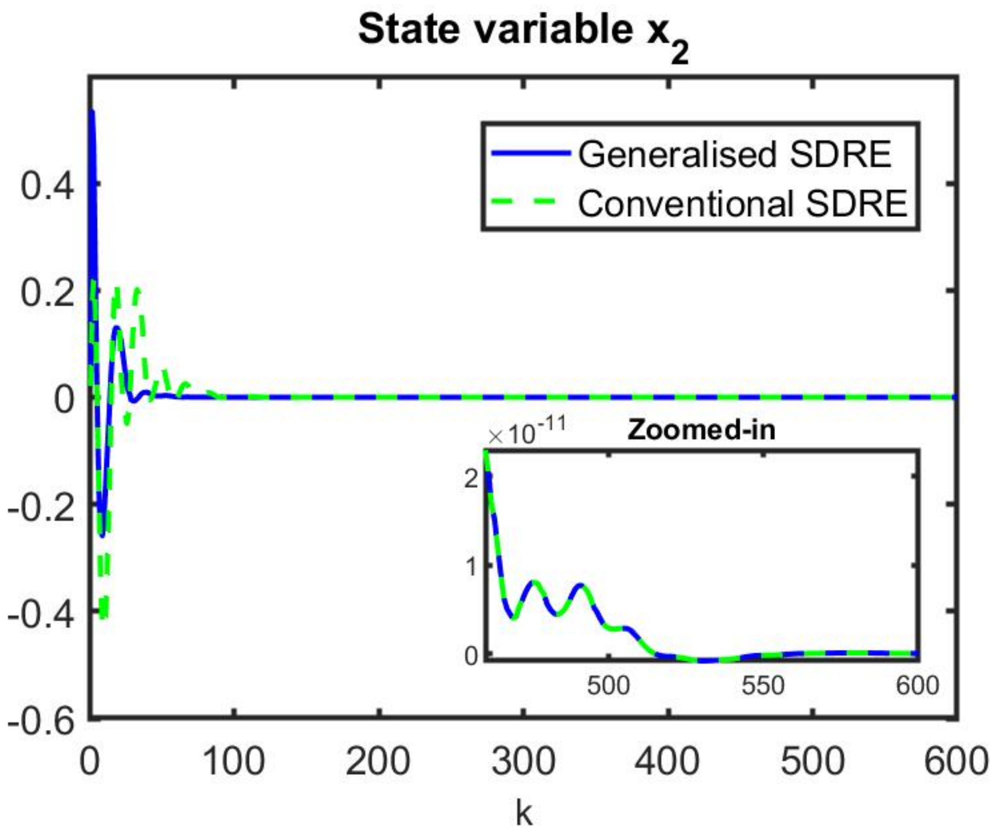

Figure 3.17: Comparison between the Generalised SDRE and Conventional SDRE on state x_2. The zoomed-in subplot demonstrates the states oscillating around zero.

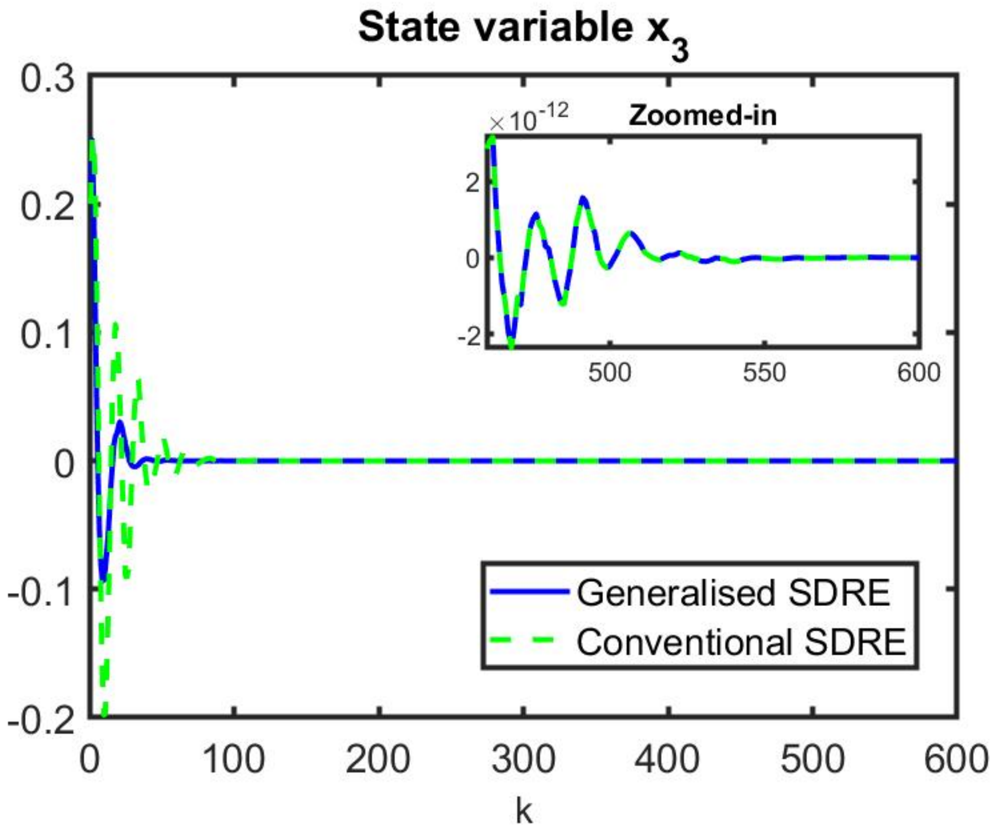

Figure 3.18: Comparison between the Generalised SDRE and Conventional SDRE illustrated on state variable x_3. The zoomed-in subplot demonstrates the states oscillating around zero.

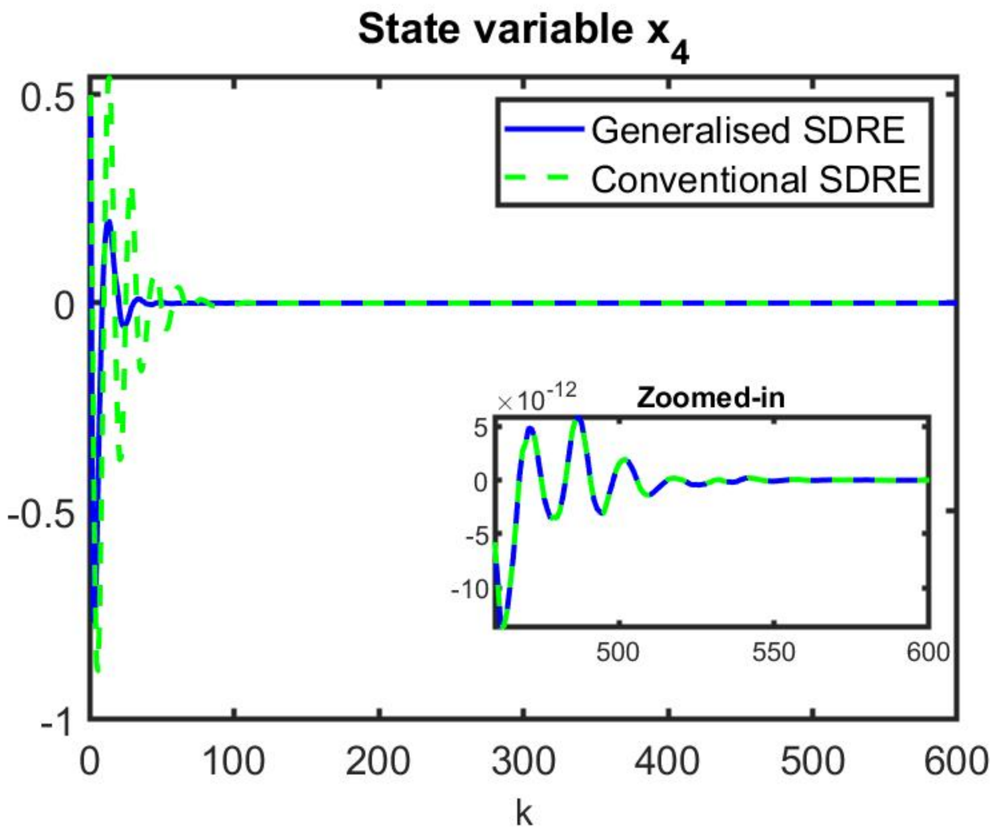

Figure 3.19: Comparison between the Generalised SDRE and Conventional SDRE illustrated on state variable x_4. The zoomed-in subplot demonstrates the states oscillating around zero.

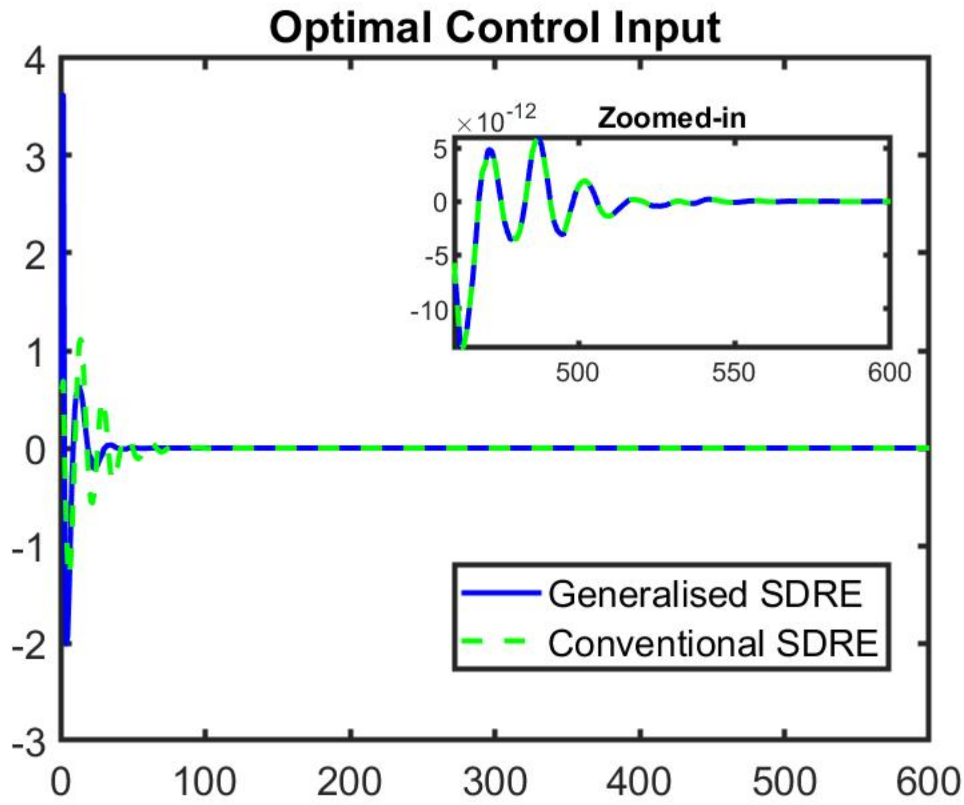

Figure 3.20: Comparison between the Generalised SDRE and Conventional SDRE illustrated on the control inputs u_k. The zoomed-in subplot demonstrates the convergence of the control input towards zero.

that the proposed design considers the dependency of the noise on the states. Furthermore, Figure 3.20 demonstrates the convergence of the control inputs for both the proposed Generalised SDRE FP control design and the conventional SDRE FP control method.

3.6 Conclusion

This chapter discussed the development of an approximate analytic solution for the FP control design of nonlinear systems which has not been considered in the previous literature. Since the control problem involves multiple integrations that need to be solved for the designation of a suboptimal controller, it seemed to be impossible to derive an analytic solution for the controller of nonlinear systems due to the nonlinear dynamics. The discussed approach in this chapter, however, allows the derivation of an approximate analytic solution for control problems for nonlinear systems by a simple transformation of the nonlinear state function of the system dynamics to a nonlinear state function which is affine in the state. The derived solution to the FP control design problem leads to a SDRE due to the nonlinearities found in the mean of the probability distribution of the system dynamics. Furthermore, this chapter has exploited uncertainties and noises that influence the behaviour of nonlinear systems. Firstly, the conventional FP control design was extended to nonlinear systems, where the covariance matrix was estimated as a global covariance. The resulting Riccati equation solution is classified as the State Dependent Riccati equation due to the dependency of the equation on the state variables.

The next developed solution in this chapter takes the system functional uncertainty into consideration in the derivation of the suboptimal randomised controller. It also generated a SDRE and possesses the ability to have a smaller ideal covariance matrix to reduce the fluctuations in the system state. Incorporating these novelties into the FP control design framework resulted in an additional linear term that represents the equation of cautiousness. Also, the suboptimal control law had an extra term which regards the aforementioned equation of cautiousness, resulting into the advancement of a cautious controller. This method was compared with the approach discussed in Section 3.3 where the controller does not account for functional uncertainties. The simulation demonstrated the reduction in overshoots when considering functional uncertainty in the derived suboptimal control law. The transient response was also better than the conventional SDRE FP control design which does not consider functional uncertainties.

Furthermore, a fully probabilistic control strategy was proposed for nonlinear systems with multiplicative noises. The derived solution was a generalised form of the Riccati equation due to the addition of a term which was arisen from considering multiplicative noises in the derivation of the control law. Consequently, a generalised SDRE was derived. Since the covariance matrix is also state and control input dependent for systems with multiplicative noises, it is possible to drive the actual covariance matrix to a smaller ideal covariance matrix for this developed approach. This method was compared to the conventional SDRE derived in Section 3.3 and proved to be an efficient control strategy since considered the dependency of the noise on the states, resulting into a quicker achievement of the transient response.

Chapter 4

Decentralised FP Control Design for Complex Systems

The complexity of real world control systems increases in terms of the dimensionality and scalability of the network for which centralised control strategies fail to deliver good results. The methods developed in Chapter 3 operate in a centralised way and thus are inefficient in controlling large-scale complex systems. The FP control design, however, has been further developed to ensure it can effectively control such systems [166]. Nevertheless, the development is mostly demonstrated on linear Gaussian systems. This chapter demonstrates the decentralised FP control approach on nonlinear systems where the subsystems interact via probabilistic message passing.

4.1 Representation of Subsystems of Interconnected Complex Networks

The approach explained in this chapter is based on large-scale complex stochastic dynamical systems that can be decomposed into N subsystems where the subsystems interact with one another via probabilistic message passing (Section 4.2). Each subsystem is described by probability distributions allowing uncertainties and noises to be considered, which enables the derived control algorithm to be extended to real-world control problems. In this scenario, the decentralised strategy consists of multiple local controllers that are responsible for the control of their corresponding subsystems. As explained in Chapter 2, many methods require the network structure to be adjusted to achieve, for example disjoint or overlapping systems, to ensure communication between subsystems is taken account of in the designation of the controllers. However, the communication strategy introduced in this chapter implements a probabilistic message passing technique to exchange information among the connected subsystems, while preserving the actual structure of the network [26], [27]. The incorporation of probabilistic message passing aids the communication among the subsystems to achieve the global aim of the interconnected network.

To re-emphasise, each node i, where $i \in [1, ..., N]$, in the interconnected complex network is controlled locally by an individual randomised controller of which the distribution is given by $c(u_{k;i}|z_{k-1;i})$, where $k \in \{1, ..., H\}$ is the discrete time step, with H being the control horizon. The set of multivariate control inputs, defined by $u_{k;i}$, ensures the control aim for subsystem i is achieved. The controller distribution depends on the state vector, $z_{k;i} = [x_{k;i}, y_{k;i}]^T$ of the subsystem, where $x_{k;i}$ is the multivariate internal state of subsystem i and where the multivariate observed external state $y_{k;i}$ is received from the neighbouring subsystems by means of probabilistic message passing. Note that the external state $y_{k;i}$ received from neighbouring subsystems has an impact on the control input $u_{k;i}$ of subsystem i due to the dependency of $u_{k;i}$ on both the internal and external states which thereafter influences the internal states of subsystem i. The derived control input $u_{k;i}$ for subsystem i does not affect the states of neighbouring subsystems.

Therefore, the stochastic behaviour of the subsystems of the complex dynamical system is described by

$$\begin{aligned} &s(x_{k;i}, y_{k;i}, u_{k;i}|x_{k-1;i}, ..., x_{0;i}, y_{k-1;i}, ..., y_{0;i}, u_{k-1;i}, ..., u_{0;i}) \\ &= s(x_{k;i}|u_{k;i}, z_{k-1;i})s(y_{k;i}|y_{k-1;i})c(u_{k;i}|z_{k-1;i}). \end{aligned} \tag{4.1}$$

The probabilistic representation of Equation (4.1) is suitable to describe an interconnected stochastic complex dynamical system with coupled nodes. The first distribution in (4.1) given by $s(x_{k;i}|u_{k;i}, z_{k-1;i})$ is the conditional probability distribution of the multivariate internal states of subsystem i. The randomised controller of subsystem i is described by the probability density function $c(u_{k;i}|z_{k-1;i})$, and the multivariate external states of the subsystem are represented by $s(y_{k;i}|y_{k-1;i})$.

From the characteristic of the pdf of the internal states of node i, $s(x_{k;i}|u_{k;i}, z_{k-1;i})$, it can be seen that the internal state dynamics are also influenced by the external states, $y_{k-1;i}$ received through message passing from neighbouring subsystems. This means that the coupling between the node and the neighbouring nodes are taken into consideration in the system dynamics. Conversely, the pdf of the external state of node i is given by $s(y_{k;i}|y_{k-1;i})$ which shows that the dependency is on the previous external state only, namely $y_{k-1;i}$. This is due to the legitimate assumption that holds in this thesis that the control input $u_{k;i}$ does not affect the dynamics of the external state $y_{k-1;i}$, and neither does the internal state $x_{k-1;i}$ have an impact on the behaviour of the external state. To re-emphasise, although it is possible to pass the control input to other subsystems via the probabilistic message passing approach as mentioned in [164], in this thesis, only the states of the subsystems are passed to each other. The control input $u_{k;i}$ of subsystem i does not directly influence the states of the neighbouring subsystems. However, the state of these neighbouring subsystems and the control actions optimised within these neighbouring subsystems will be affected by the external states that are passed to these subsystems. Nevertheless, if the control input $u_{k;i}$ of subsystem i is passed to the neighbouring subsystems following the probabilistic message passing approach, it will then have a direct effect on the states of the neighbouring subsystems as explained in [164].

Furthermore, the proposed decentralised control message passing approach are based on the actual structure of the system. This means that if the control input $u_{k;i}$ applied to subsystem i affects a neighbouring subsystem, then this control input $u_{k;i}$ can be passed through the message passing approach and it will be considered in the model representing the actual system as already explained previously [164]. Nevertheless, in this thesis, the case where the model structure consists of the passing of the control input is not discussed.

The local controllers for the decomposed system follow a FP control method. For this approach, an ideal probability distribution is specified which reflects the control aim of subsystem i and aids to achieve the desired steady state behaviour of the joint pdf of the closed-loop system. The ideal joint pdf is described by

$$\begin{aligned} & s^I(x_{k;i}, y_{k;i}, u_{k;i} | x_{k-1;i}, ..., x_{0;i}, y_{k-1;i}, ..., y_{0;i}, u_{k-1;i}, ..., u_{0;i}) \\ & = s^I(x_{k;i} | u_{k;i}, z_{k-1;i}) s(y_{k;i} | y_{k-1;i}) c^I(u_{k;i} | z_{k-1;i}), \end{aligned} \tag{4.2}$$

where $s^I(x_{k;i}|.)$ and $c^I(u_{k;i}|.)$ represent the ideal distributions of the internal state dynamics and the controller, respectively. The conditional pdf given by $s(y_{k;i}|y_{k-1;i})$ in (4.2) is the same as the pdf of the external state in (4.1) due to the fact that it has been assumed that the external measurable state cannot be influenced or changed in subsystem i. The sole contribution of the inclusion of knowledge about the external states in the dynamics of subsystem i is to inform subsystem i about the state of the neighbouring nodes so that it can receive complete knowledge and information about aspects that have an influential affect on the internal states. To clarify, the randomised local controller designed in this chapter only affects the dynamics of the internal states, not the external. The proposed decentralised control framework has been illustrated in Figure 4.1.

Due to the decomposition, the suboptimal randomised control solution to the FP control design explained in Proposition 1 in Chapter 3 needs to be updated. The Kullback-Leibler Divergence between the actual joint pdf given by (4.1) and the ideal joint pdf defined by (4.2) now needs to be

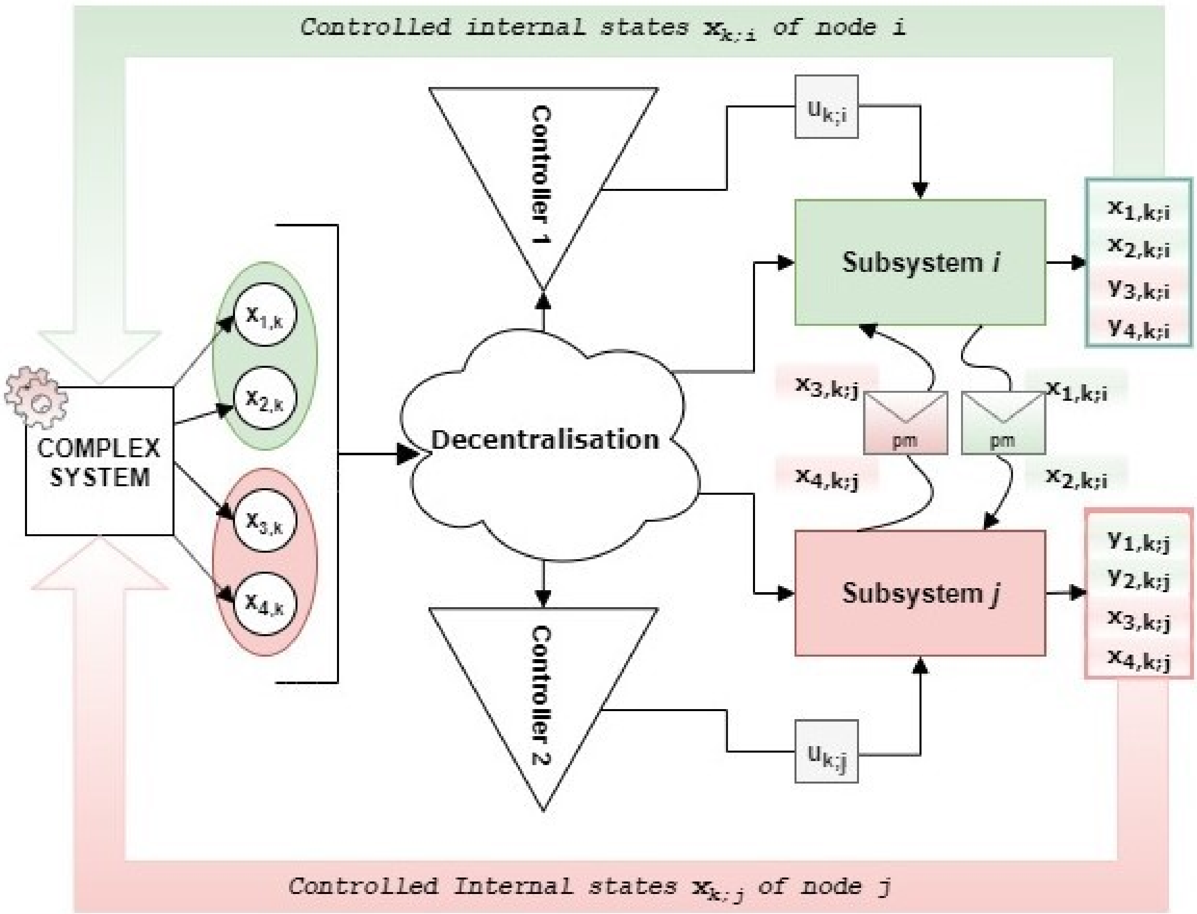

Figure 4.1: The architecture of the proposed decentralised control framework for a complex system that has been decomposed into two subsystems, i and j. Subsystem i controls the states $x_{1,k}$ and $x_{2,k}$ and receives knowledge about the states $x_{3,k}$ and $x_{4,k}$ from node j via probabilistic message passing (pm). On the other hand, node j receives information about the states $x_{1,k}$ and $x_{2,k}$ via probabilistic message passing and controls the states $x_{3,k}$ and $x_{4,k}$. The receiving states to the nodes are registered as external states of the subsystem.

minimised such that,

$$-\ln(\gamma(z_{k-1;i})) = \min_{\{c(u_{k;i}|z_{k-1;i})\}} \sum_{\tau=k}^{H} \int f(Z_{k;i}|Z_{k-1;i}) \ln\left(\frac{f(Z_{k;i}|Z_{k-1;i})}{f^I(Z_{k;i}|Z_{k-1;i})}\right) dZ_{k;i}, \tag{4.3}$$

where $-\ln(\gamma(z_{k-1;i}))$ represents the value function, the joint pdf of the closed-loop system of subsystem i is defined by $f(Z_{k;i}|.) = \prod_{k=1}^{H} s(x_{k;i}|u_{k;i}, z_{k-1;i}) s(y_{k;i}|y_{k-1;i}) c(u_{k;i}|z_{k-1;i})$, and $Z_{k;i} = \{z_{k;i}, ..., z_{H;i}, u_{k;i}, ..., u_{H;i}\}$ is the observed sequence of data.

Proposition 3. The minimum cost-to-go function that derives the optimal controller $c(u_{k;i}|z_{k-1;i})$

can be obtained for the decomposed subsystem i, such that,

$$\ln(\gamma(z_{k-1;i})) = \min_{\{c(u_{k;i}|z_{k-1;i})\}} \int s(x_{k;i}|u_{k;i}, z_{k-1;i}) s(y_{k;i}|y_{k-1;i}) c(u_{k;i}|z_{k-1;i}) \times \Bigg[\underbrace{\ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i}) c(u_{k;i}|z_{k-1;i})}{s^I(x_{k;i}|u_{k;i}, z_{k-1;i}) c^I(u_{k;i}|z_{k-1;i})} \right)}_{\text{Partial cost}} - \underbrace{\ln(\gamma(z_{k;i}))}_{\text{Optimal cost-to-go}} \Bigg] d(z_{k;i}, u_{k;i}). \quad (4.4)$$

Proof. A detailed derivation of the minimum cost-to-go function shown in (4.4) can be found in [25]. □

4.2 Probabilistic Message Passing

The decomposition of large-scale complex systems into a number of subsystems allows local controllers to be designed with the responsibility of achieving the control objective for their individual subsystems. However, it is key to inform the subsystems of the state and objectives of the neighbouring subsystems to achieve the individual as well as the global control objectives. The decentralisation of the complex system integrated with probabilistic message passing simplifies the control problem and circumvents the need of centrally controlling the large network to achieve the global aim.

The message passing strategy can be split into two parts: the passing and the receiving of the probabilistic message. When subsystem i passes a message to subsystem j, subsystem j will receive it as an external multivariate signal. This external state that subsystem j has received influences the dynamics of the internal states of node j but the external signal itself will not get affected or changed by the receiving subsystem j. The sole purpose is to convey a message to the receiving subsystem j about the conditions, states and objectives of other interacting neighbouring subsystems to aid the control of the internal states. As a result, more efficient and accurate local controllers are designed that fulfil the objectives of each individual subsystem as well as the global objective of the interconnected global network.

Once the message has been passed to the receiving subsystem, the receiving subsystem needs to fuse the information together with the prior knowledge they already had to update its external states. Local controllers are responsible for diffusing the message to neighbouring subsystems in order to share the behaviour of the controlled node with each individual interacting subsystem. The strategy of probabilistic message passing is implemented for subsystems to receive diffused messages as external signals.

4.3 Solution to the Decentralised FP Control Design Problem

The following proposition outlines the optimal control solution to the FP control problem in a decentralised framework, subject to the actual and ideal pdfs described by (4.1) and (4.2), respectively.

Proposition 4. The optimal randomised controller that minimises the KLD can be derived from the cost-to-go function (4.4) for the actual joint pdf given by (4.1) and the ideal joint pdf defined in (4.2) such that,

$$c^*(u_{k;i}|z_{k-1;i}) = \frac{c^I(u_{k;i}|z_{k-1;i})\exp(-\beta(u_{k;i}, z_{k-1;i}))}{\gamma(z_{k-1;i})}, \tag{4.5}$$

$$\gamma(z_{k-1;i}) = \int c^I(u_{k;i}|z_{k-1;i})\exp(-\beta(u_{k;i}, z_{k-1;i}))du_{k;i}, \tag{4.6}$$

$$\beta(u_{k;i}, z_{k-1;i}) = \int s(x_{k;i}|u_{k;i}, z_{k-1;i})\left[\ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i})}{s^I(x_{k;i}|u_{k;i}, z_{k-1;i})}\right) - \ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))\right]dx_{k;i}, \tag{4.7}$$

with

$$\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i})) = \int s(y_{k;i}|y_{k-1;i})\ln(\gamma(z_{k;i}))dy_{k;i}. \tag{4.8}$$

Proof. The derivation of the optimal controller given in (4.5) can be found in [25]. □

It should be noted that the control solution given in (4.5) has no limitations regarding the choice of pdfs, the distribution of the system state or the ideal distribution, since a general solution can still be obtained. However, due to the involvement of multiple integrals, an analytic solution for the randomised controller can not be derived except for linear systems with Gaussian distributions. Nevertheless, the same transformation approach as explained in Chapter 3 is followed here to extend the FP control approach within a decentralised framework to nonlinear systems.

4.4 Decentralised Control Approach for Nonlinear Systems with Additive Noises

This section discusses the derivation of the analytic control solution of a randomised controller for complex nonlinear systems with additive noises in a decentralised control framework.

4.4.1 Subsystem Representation of Nonlinear Complex Systems with Additive noises

The following representation of nonlinear discrete time stochastic subsystems is considered here.

$$z_{k;i} = \tilde{F}_i(z_{k-1;i}) + \tilde{g}_i(z_{k-1;i})u_{k;i} + \epsilon_{k;i}, \tag{4.9}$$

where $\tilde{F}_i(z_{k-1;i}) \in \mathbb{R}^n$ and $\tilde{g}_i(z_{k-1;i}) \in \mathbb{R}^{n\times m}$ are the state and control matrices respectively, and are both nonlinear functions of the state $z_{k-1;i}$. The vector $\tilde{F}_i(z_{k-1;i})$ can be written as $\tilde{F}_i(z_{k-1;i}) = \begin{bmatrix} \tilde{f}_i(z_{k-1;i}) \\ \tilde{h}_i(y_{k-1;i}) \end{bmatrix}$, where $\tilde{f}_i(z_{k-1;i})$ is the nonlinear internal state function and where the nonlinear external state function is described by $\tilde{h}_i(y_{k-1;i})$.

The noise $\epsilon_{k;i}$ belongs to the Gaussian distribution with zero mean and covariance Q_i. The same approach as explained in Chapter 3 to transform the system to a nonlinear system that is affine in the state and control input is followed here to give,

$$z_{k;i} = \bar{F}_i(z_{k-1;i})z_{k-1;i} + \tilde{g}_i(z_{k-1;i})u_{k;i} + \epsilon_{k;i}. \tag{4.10}$$

To clarify the construction of the proposed decomposition which introduces the concept of internal and external variables, Equation (4.10) can be explicitly written in terms of the multivariate internal state $x_{k;i}$ and external state $y_{k;i}$ such that,

$$\begin{aligned} x_{k;i} =& \bar{f}_{1i}(z_{k-1;i})x_{k-1;i} + \sum_{j\in \mathrm{N}_i, j\neq i} c_{ij}x_{k-1;j} + \bar{g}_i(z_{k-1;i})u_{k;i} + \epsilon_{1k;i}, \\ =& \bar{f}_{1i}(z_{k-1;i})x_{k-1;i} + \bar{f}_{2i}(z_{k-1;i})y_{k-1;i} + \bar{g}_i(z_{k-1;i})u_{k;i} + \epsilon_{1k;i}, \\ =& \bar{f}_i(z_{k-1;i})z_{k-1;i} + \bar{g}_i(z_{k-1;i})u_{k;i} + \epsilon_{1k;i}, \end{aligned} \tag{4.11}$$

$$y_{k;i} = \bar{h}_i(y_{k-1})y_{k-1;i} + \epsilon_{2k;i}. \tag{4.12}$$

The noises $\epsilon_{1k;i}$ and $\epsilon_{2k;i}$ have zero mean and covariances $Q_{1;i}$ and $Q_{2;i}$, respectively. The internal state matrix is given by $\bar{f}_{1i}(z_{k-1;i})$ and the control matrix is defined as $\bar{g}_i(z_{k-1;i})$. The matrices c_{ij} represent the coupling strength between the interacting subsystems. The second term of the internal states that consists of the coupling term, namely $\sum_{j\in \mathrm{N}_i, j\neq i} c_{ij}x_{k-1;j}$ where N_i means the neighbouring nodes of subsystem i, has been rewritten to achieve a more compact form such that the states from the neighbouring nodes, $x_{k-1;j}$ enter subsystem i as external states, $y_{k-1;i}$. The elements of matrix $\bar{f}_{2i}(z_{k-1;i})$ are given by c_{ij}, meaning $\bar{f}_{2i}(z_{k-1;i}) = [c_{ij}]_{j\in \mathrm{N}_i, j\neq i}$. Moreover, the vector $y_{k-1;i}$ consists of elements that equate to the states of the neighbouring nodes $x_{k-1;j}$, in other words, $y_{k-1;i} = [x^T_{k-1;j}]^T_{j\in \mathrm{N}_i, j\neq i}$. Lastly, matrix $\bar{f}_i(z_{k-1;i})$ represents $\bar{f}_i(z_{k-1;i}) = \begin{bmatrix} \bar{f}_{1i}(z_{k-1;i}) & \bar{f}_{2i}(z_{k-1;i}) \end{bmatrix}^T$.

In addition, Equation (4.11) - (4.12) can also be written in matrix-vector form such that,

$$\underbrace{\begin{bmatrix} x_{k;i} \\ y_{k;i} \end{bmatrix}}_{z_{k;i}} = \underbrace{\begin{bmatrix} \bar{f}_{1i}(z_{k-1;i}) & \bar{f}_{2i}(z_{k-1;i}) \\ 0 & \bar{h}_i(y_{k-1;i}) \end{bmatrix}}_{=\bar{F}(z_{k-1;i})} \underbrace{\begin{bmatrix} x_{k-1;i} \\ y_{k-1;i} \end{bmatrix}}_{z_{k-1;i}} + \underbrace{\begin{bmatrix} \bar{g}_i(z_{k-1;i}) \\ 0 \end{bmatrix}}_{\tilde{g}_i(z_{k-1;i})} u_{k;i} + \underbrace{\begin{bmatrix} \epsilon_{1k;i} \\ \epsilon_{2k;i} \end{bmatrix}}_{\epsilon_{k;i}}. \tag{4.13}$$

Definition 4.4.1. *Suboptimal solution to the decentralised nonlinear FP control design:* The FP control approach for obtaining a suboptimal solution for decentralised nonlinear control problems (such as Equations (4.9)) can be obtained using the following definition:

1. At each discrete time step, use transformation methods to bring the nonlinear dynamics to the nonlinear affine dynamics. For example, for the formulation in the current section, transform the nonlinear equation in (4.9) to the nonlinear affine dynamics in (4.10).

2. Solve the equations provided in Proposition 4 to obtain the closed form suboptimal solution at each discrete time instant.

The internal state, external state and control matrices described by $\bar{f}_i(z_{k-1;i})$, $\bar{h}_i(y_{k-1})$, and $\bar{g}_i(z_{k-1;i})$ respectively, are unknown and need to be estimated. An MLP neural network is implemented to obtain an approximation for the conditional mean of the system dynamics for the internal states of subsystem i,

$$\begin{aligned} \hat{x}_{k;i} &= \text{mlp}(z_{k-1;i}), \\ &= f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u_{k;i}, \end{aligned} \tag{4.14}$$

where $f_i(z_{k-1;i})$ and $g_i(z_{k-1;i})$ are the estimations of the internal state and control matrices, respectively. To re-emphasise, the internal state matrix can be written as $f_i(z_{k-1;i}) = \begin{bmatrix} f_{1i}(z_{k-1;i}) & f_{2i}(z_{k-1;i}) \end{bmatrix}^T$. Furthermore, an MLP is used to output an estimation for the conditional expectation of the external state variables such that,

$$\begin{aligned} \hat{y}_{k;i} &= \text{mlp}(y_{k-1;i}), \\ &= h_i(y_{k-1;i})y_{k-1;i}, \end{aligned} \tag{4.15}$$

where $h_i(y_{k-1;i})$ is the estimation obtained for the external state matrix, $\bar{h}_i(y_{k-1;i})$. A more detailed approach on how to obtain the MLP estimations can be found in Chapter 3.

Secondly, the global covariance $\Sigma_{x;i}$ can be estimated from the error between the actual internal state $x_{k;i}$ and estimated internal state $\hat{x}_{k;i}$. Similarly, the global covariance of the conditional distribution of the external state, $\Sigma_{y;i}$ can be estimated from the error between the external state received by the probabilistic message passing approach and the estimated external state. The methodology of approximating this can be found in Section 3.3.2.

It is now possible to describe the stochastic behaviour of the dynamics of subsystem i which is characterised by Gaussian probability density function since the noise affecting the system dynamics as specified by Equation (4.13) is Gaussian. This is given by,

$$s(x_{k;i}|u_{k;i}, z_{k-1;i}) \sim \mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i}), \tag{4.16}$$

$$s(y_{k;i}|y_{k-1;i}) \sim \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}). \tag{4.17}$$

The conditional distribution $s(x_{k;i}|u_{k;i}, z_{k-1;i})$ describes the stochastic behaviour of the internal states, whereas $s(y_{k;i}|y_{k-1;i})$ describes the conditional pdf of the external states of subsystem i.

4.4.2 Randomised Suboptimal Controller for Nonlinear Systems

As discussed previously, it is key to specify the desired behaviour of the system state and controller within a FP control design framework to outline the control objective. Since the aim here is to regulate the system states from the initial value to zero, the ideal distributions of the internal and external states are described by,

$$s^I(x_{k;i}|u_{k;i}, z_{k-1;i}) \sim \mathcal{N}(0, \Sigma_{x;i}), \tag{4.18}$$

$$s^I(y_{k;i}|y_{k-1;i}) \sim \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}), \tag{4.19}$$

where $s^I(x_{k;i}|u_{k;i}, z_{k-1;i})$ describes the ideal distribution of the internal states which needs to be reflected in the design of the probabilistic controller. However, the external states are messages received from neighbouring nodes and thus are not supposed to be controlled or influenced by the output of subsystem i. Therefore, the ideal distribution of the measurable external signal $y_{k;i}$ is taken to be the same as the actual distribution of the external signal to subsystem i. To re-emphasise, the external states are the messages passed from neighbouring subsystems, thus, they cannot be influenced by the output of the subsystem they are passed to.

In addition, the ideal pdf of the controller is given by,

$$c^I(u_{k;i}|z_{k-1;i}) \sim \mathcal{N}(0, \Gamma_{k;i}), \tag{4.20}$$

where the mean is assumed to be zero and covariance $\Gamma_{k;i}$ specifies the permissible range of the optimal control inputs for a given confidence level. Given the above conditions and constraints and based on Definition 4.4.1, the next theorem states the form of the suboptimal randomised controller for subsystem i.

Theorem 4. Based on Definition 4.4.1, the suboptimal randomised controller for subsystem i of which the dynamics are characterised by (4.16) and (4.17) and for which the ideal distribution of the subsystem and controller is described by (4.18), (4.19), and (4.20) is given by,

$$c^*(u_{k;i}|z_{k-1;i}) = \mathcal{N}(u^*_{k;i}, \bar{\Gamma}_{k;i}), \tag{4.21}$$

where

$$u^*_{k;i} = -K_{k;i} z_{k-1;i}, \tag{4.22}$$

$$\bar{\Gamma}_{k;i} = \left(\Gamma^{-1}_{k;i} + g_i^T(z_{k-1;i})\tilde{Q}_{k;i} g_i(z_{k-1;i})\right)^{-1}, \tag{4.23}$$

$$K_{k;i} = \bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) \begin{bmatrix} \tilde{Q}_{k;i} f_{1i}(z_{k-1;i}) & \tilde{Q}_{k;i} f_{2i}(z_{k-1;i}) + M_{2,k;i} h_i(y_{k-1;i}) \end{bmatrix}, \tag{4.24}$$

$$\tilde{Q}_{k;i} = (\Sigma^{-1}_{x;i} + M_{1,k;i}). \tag{4.25}$$

The obtained Gaussian distribution of the suboptimal randomised controller (4.21) for subsystem i has mean $u^*_{k;i}$ and covariance given by $\bar{\Gamma}_{k;i}$. The control gain $K_{k;i}$ is described by Equation (4.24). Furthermore,the quadratic performance index for the present control strategy is given by,

$$-\ln(\gamma(z_{k;i})) = \frac{1}{2} z^T_{k;i} M_{k;i} z_{k;i} + \frac{1}{2} V_{k;i}, \tag{4.26}$$

where the matrix $M_{k;i}$ has been partitioned as follows,

$$M_{k;i} = \begin{bmatrix} M_{1,k;i} & M_{2,k;i} \\ M^T_{2,k;i} & M_{3,k;i} \end{bmatrix}, \tag{4.27}$$

and where,

$$M_{1,k-1;i} = - f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i}) + f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i}), \tag{4.28}$$

$$\begin{aligned} M_{2,k-1;i} =& f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) + f_{1i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\ &- f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) \\ &- f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}), \end{aligned} \tag{4.29}$$

$$\begin{aligned} M_{3,k-1;i} =& f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) + 2f_{2i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) + h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i}) \\ &- f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) \\ &- h_i^T(y_{k-1;i})M_{2,k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\ &- 2f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}), \end{aligned} \tag{4.30}$$

$$V_{k-1;i} = V_{k;i} + \text{tr}(\Sigma_{x;i}M_{1,k;i}) + \text{tr}(\Sigma_{y;i}M_{3,k;i}) + \ln|\Gamma_{k;i}| + \ln|\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i})\tilde{Q}_{k;i}g_i(x_{k-1;i})|. \tag{4.31}$$

Proof. The proof of Theorem 5 can be found in Appendix D. □

To further elaborate the idea and understanding of the partitioning of matrix $M_{i;k}$, it can be demonstrated by rewriting the quadratic cost function defined by (4.26) such that $-\ln(\gamma(z_{k;i})) = \frac{1}{2}[x_{k;i}^T M_{1,k;i}x_{k;i} + 2x_{k;i}^T M_{2,k;i}y_{k;i} + y_{k;i}^T M_{3,k;i}y_{k;i} + V_{k;i}]$. From the form of the suboptimal controller given by (4.21) - (4.25), it can be seen that to derive the suboptimal controller, only two of the elements of the full Riccati matrix need to be computed, namely $M_{1,k;i}$ and $M_{2,k;i}$ which are described by Equations (4.28) and (4.29), respectively. The Riccati equation solution, $M_{3,k;i}$ which is quadratic in the external states is not required to be solved. Therefore, the full block of the Riccati matrix (4.27) does not need to be solved which results in a reduction in computational expenses to obtain the control law within the proposed decentralised framework as opposed to the global control solution. The implementation of the suboptimal randomised controller is computationally efficient as a consequence of the decomposition of the Riccati matrix which is especially beneficial when handling large-scale complex systems. In [176], the sequential execution time is shown to scale linearly with the size of the system. However, when it is computed in a decentralised non sequential mode, the time it takes for the system to converge is independent of the actual size of the system. To clarify, if a large system is controlled centrally, the time needed to compute scales linearly, i.e. *time to control node 1* $+$ *time to control node 2* $+ \ldots$. Nevertheless, the decentralised approach allows the control of all the subsystems simultaneously. Hence, parallel computers for the control of a decentralised large system means that the computational time will only be bound to the size of the largest subsystem.

The interaction among interconnecting subsystems is important to achieve the global control objective. The algorithm developed in this chapter uses probabilistic message passing as a communication medium between the subsystems [26], [27]. The next section explains the probabilistic message passing approach in detail.

4.5 Probabilistic Message Passing Algorithm

The decomposition of each subsystem consists of the internal states for which the local controller is responsible and the external state which aids the local controller to complete the information it requires to achieve its local and aid the global control objective.

The approach that has been used to receive messages from neighbouring subsystems to update the external state of the subsystem is probabilistic inference. The stochastic nature of the subsystems allows the complete description of the closed-loop system of each subsystem to be given by the joint pdf of the variables that are linked to each other namely the internal states, external states and the control input of the subsystem under consideration such that,

$$s(x_{k;i}, y_{k;i}, u_{k;i}|z_{k-1;i}). \tag{4.32}$$

The message passing technique involves subsystem i conveying knowledge about its internal states, $x_{k;i}$ to the interacting subsystems. Hence, it is required to find the marginal distribution of the states that need to be passed from subsystem i to other neighbouring subsystems. The process and theorems are outlined and explained using subsystem i and j where node i passes the message to the receiving node j. This is an example implemented to describe the probabilistic message passing approach. To clarify, the passing of messages is not restricted to be unidirectional, it can be bidirectional as well and the same procedure can be followed.

Firstly, the following statement is key to the message passing approach.

Lemma 1. The stochastic formulation of the probabilistic message being conveyed from subsystem i to j is given by,

$$\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) = \int s(x_{k;i}, y_{k;i}, u_{k;i}|z_{k-1;i})dy_{k;i}du_{k;i}, \tag{4.33}$$

which implies that the joint pdf of the variables that compose subsystem i is integrated over the external variable and control input such that the marginal distribution for the internal states of subsystem i

can be found and passed onto subsystem j which receives it as an external signal.

Following the statement in (4.33) provided by Lemma 1, the consequent theorem can be obtained.

Theorem 5. Following the distribution of subsystem i defined in (4.16) - (4.17), its designed optimal controller described by (4.21), and Lemma 1, the knowledge about the internal states of subsystem i to subsystem j is given by the following probabilistic message,

$$\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) = \mathcal{N}(\mu_{x_{k;i}}, \mathfrak{P}_{x_{k;i}}), \tag{4.34}$$

where,

$$\mu_{x_{k;i}} = f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u^*_{k;i}, \tag{4.35}$$

$$\mathfrak{P}_{x_{k;i}} = g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i}) + \Sigma_{x;i}. \tag{4.36}$$

Note that $\Sigma_{x;i}$ in (4.36) is the covariance of the actual pdf of the internal states of subsystem i as given by (4.16). The probabilistic message in (4.34) can be described by any probability distribution and does not necessarily have to be Gaussian. However, if the components of the subsystems and their individual controllers are described by Gaussian distributions, then the distribution of the messages passed (Equation (4.34)) will also result in a Gaussian distribution.

Proof. A detailed proof of this theorem can be found in Appendix E. □

Nevertheless, Theorem 5 is solely the first stage of the probabilistic message passing approach since the message about the internal states of the passing subsystem i (4.34) still needs to be utilised by the receiving subsystem j to update the knowledge of its external state variables. A mathematical representation can be given by,

$$\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) = \mathcal{N}(\mu_{x_{k;i}}, \mathfrak{P}_{x_{k;i}}) \sim y_{k;j} \leftarrow x_{k;i}. \tag{4.37}$$

Using Bayes' rule, prior knowledge about the external states $y_{k;j}$ that subsystem j possesses can be fused with the updated knowledge received from subsystem i as a probabilistic message about the internal states of node i. The following theorem formulates the second stage of the probabilistic message passing approach.

Theorem 6. The prior knowledge that node j has about its external states, namely $s(y_{k;j}|y_{k-1;j}) = \mathcal{N}(y_{k;j}, \Sigma_{y;j})$ can be updated using the notion introduced in (4.37) by fusing it with the probabilistic message (4.34) subsystem i has passed. This is achieved by using Bayes' rule, i.e. the posterior distribution of $\hat{y}_{k;j,new}$ is calculated using Bayes' theorem where prior knowledge about the distribution of $\hat{y}_{k;j,new}$ as defined in Equation (4.17) is combined with the new information from the probabilistic message given by (4.34). Following this, the maximum a posteriori estimation (MAP) estimates $\hat{y}_{k;j,new}$ as the mode of the posterior distribution of this random variable. This gives,

$$s(y_{k;j,new}) = \mathcal{N}(\hat{y}_{k;j,new}, \bar{\Sigma}_{y_{k;j},new}), \tag{4.38}$$

where,

$$\hat{y}_{k;j,new} = \hat{y}_{k;j} + \bar{K}_{k;j}(\mu_{x_{k;i}} - \hat{y}_{k;j}), \tag{4.39}$$

$$\bar{\Sigma}_{y_{k;j},new} = \Sigma_{y;j} - \bar{K}_{k;j}\Sigma_{y;j}, \tag{4.40}$$

with,

$$\bar{K}_{k;j} = \Sigma_{y;j}(\Sigma_{y;j} + \mathfrak{P}_{x_{k;i}})^{-1}. \tag{4.41}$$

Proof. The proof of this theorem can be found in Appendix E. □

4.6 Algorithm of the Decentralised FP Control Approach with Probabilistic Message Passing

There are a number of steps required in the process of implementing the suboptimal randomised controller in a decentralised control framework. The key steps are summarised in Algorithm 5 as a pseudocode. As discussed previously for centralised FP control designs, it is required to find the steady state solutions for both the DARE's $M_{1,k;i}$ and $M_{2,k;i}$ given by (4.28) and (4.29). This is achieved by changing the time index as explained previously.

Algorithm 5 Pseudo-code of randomised controller for decentralised nonlinear systems with additive noises

1: **procedure** IMPLEMENTATION OF FP CONTROL DESIGN FOR DECENTRALISED NONLINEAR SYSTEMS WITH ADDITIVE NOISES
2: **Initialise:** $x_{0;i}, y_{0;i}, M_{1,0;i}, M_{2,0;i}, h_i(y_{k-1;i})$ where $i = \{1, ..., N\}$, N being the number of subsystems and pre-train the neural network model as discussed in Algorithm 1 (optional).
3: **for** $k = 1 \rightarrow H$ **do**
4:     **for** subsystem i:
5:         **Estimate** $f_i(z_{k-1;i})$ and $g_i(z_{k-1;i})$ from the neural network model.
6:         **Calculate** the SS solution of $M_{1;i}$ using (4.28).
7:         $\tilde{Q}_i \leftarrow (\Sigma_{x;i}^{-1} + M_{1;i})$
8:         $\bar{\Gamma}_i = \left(\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i})\tilde{Q}_i g_i(z_{k-1;i})\right)^{-1}$
9:         **Use** *Steps 6-8* in (4.29) to find the SS solution of $M_{2;i}$.
10:         **Use** the SS values from *Steps 6-9* in (4.24) to find the SS solution of K_i.
11:         **Compute** $u_{k;i}^*$ using *Step 10* in Equation (4.22).
12:         **Forward** the control signal u_k^* obtained in *Step 11* to the system equation (4.9).
13:         **Using** a one step delayed of the new state value from *Step 12* and the calculated control signal u_k^* from *Step 11* as input to the neural network model and the new state from *Step 12* as output, retrain the neural network model and update its parameters.
14:     **end for**
15:     **for** subsystem i:
16:         **Compute** $\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i})$ given by (4.34) using (4.35) and (4.36).
17:         **Update** the prior pdf of the external states using (4.38) - (4.41).
18:         **Update** the external state matrices $h_i(y_{k-1;i})$.
19:     **end for**
20: **end for**

4.7 Simulation

The validity of the proposed decentralised FP control approach with probabilistic message passing as a means of communication between the subsystems is demonstrated in the current section. The numerical example implemented to illustrate the effectiveness of the proposed decentralised FP control approach is taken from [177] by Wang et al. The dynamics of the discrete time stochastic system for the simulation in this section are thus given by,

$$x_{k;i} = f_{1i}(z_{k-1;i})x_{k-1;i} + \sum_{j\in \mathrm{N}_i, j\neq i}^{N} c_{ij}x_{k-1;j} + u_{k;i} + \epsilon_{1k;i}, \tag{4.42}$$

where the coupling strength matrix c_{ij} is given by $c_{ij} = \mathcal{L}_{ij}(0.1\mathcal{I}_{2\times 2})$ which links it to the j^{th} state variable. Furthermore, $f_{1i}(z_{k-1;i}) = a(z_{k-1;i}) + \mathcal{L}_{ii}(0.1\mathcal{I}_{2\times 2}) = a(z_{k-1;i}) + c_{ii}$, where,

$$a(z_{k-1;i}) = \begin{bmatrix} -0.5 & \frac{1}{x_{2,k-1;i}}\tanh(0.65x_{1,k-1;i}) - 0.15 \\ 0 & 1.1 - \frac{1}{x_{2,k-1;i}}\tanh(0.95x_{2,k-1;i}) \end{bmatrix}. \tag{4.43}$$

In addition, $\mathcal{L} = [\mathcal{L}_{ij}]_{N\times N}$ is the Laplacian matrix which shows the connections and interactions between the individual subsystems and is given by,

$$\mathcal{L} = \begin{bmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix}. \tag{4.44}$$

The Laplacian matrix, $\mathcal{L}$ represents the decomposition of the complex nonlinear network in three subsystems that need to be regulated to state zero. A key adjustment to the system state equation in [177] is the addition of the noise $\epsilon_{1k;i}$ to obtain a stochastic system as defined in (4.42). This allows us to demonstrate a more realistic control problem which can be found in real-world situations where noises are inevitable. The noise has zero mean and the noise intensity is described by the covariance $Q_i = 0.1\mathcal{I}_{2\times 2}$.

The local controllers designed for this numerical example are responsible for three individual subsystems which are referred to as node $i \in \{\alpha, \chi, \Omega\}$. All nodes equally interact with each other and thus, following the system state equation given by Equation (4.42), the dynamics of the nodes can be described by the distributions,

$$s(x_{k;i}|u_{k;i}, z_{k-1;i}) = \mathcal{N}(f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u_{k;i}, \Sigma_{x;i}), \tag{4.45}$$

where $f_i(z_{k-1;i}) = \begin{bmatrix} f_{1i}(z_{k-1;i}) & f_{2i}(z_{k-1;i}) \end{bmatrix}$ with $f_{2i}(z_{k-1;i}) = [c_{ij}]_{j\in \mathrm{N}_i, j\neq i}$ such that,

$$f_i(z_{k-1;i}) = \begin{bmatrix} -0.5 - 0.02 & \frac{1}{x_{2,k-1;i}}\tanh(0.65x_{1,k-1;i}) - 0.15 & 0.01 & 0 & 0.01 & 0 \\ 0 & 1.1 - \frac{1}{x_{2,k-1;i}}\tanh(0.95x_{2,k-1;i}) - 0.02 & 0 & 0.01 & 0 & 0.01 \end{bmatrix}, \tag{4.46}$$

$$g_i(z_{k-1;i}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4.47}$$

and,

$$s(y_{k;i}|y_{k-1}) = \mathcal{N}(h_i(y_{k-1;i})y_{k-1;i}, \Sigma_{y;i}), \tag{4.48}$$

where,

$$h_i(y_{k-1;i}) = \begin{bmatrix} h_{11;i} & h_{12;i} & 0 & 0 \\ h_{21;i} & h_{22;i} & 0 & 0 \\ 0 & 0 & h_{33;i} & h_{34;i} \\ 0 & 0 & h_{43;i} & h_{44;i} \end{bmatrix}. \tag{4.49}$$

To reaffirm, the pdfs of the system state dynamics are assumed to be unknown apriori and are therefore estimated online as explained in Sections 4.4.1 and 3.3.2. The external state matrix $h_i(y_{k-1;i})$ is initialised randomly at $k = 1$, and is updated accordingly during the probabilistic message passing procedure as explained in Section 4.5. The state vector $z_{k;i}$ for the nodes are given by:

1. node α is given by $z_{k;\alpha} = \begin{bmatrix} z_{1,k;\alpha} & z_{2,k;\alpha} & z_{1,k;\chi} & z_{2,k;\chi} & z_{1,k;\Omega} & z_{2,k;\Omega} \end{bmatrix}^T$, where the first two variables $z_{1,k;\alpha}$ and $z_{2,k;\alpha}$ are the internal states and the remaining four states are the external states. The internal states are initially assumed to be $z_{1,0;\alpha} = 7.5$ and $z_{2,0;\alpha} = -3.5$.
2. node χ is given by $z_{k;\chi} = \begin{bmatrix} z_{1,k;\chi} & z_{2,k;\chi} & z_{1,k;\alpha} & z_{2,k;\alpha} & z_{1,k;\Omega} & z_{2,k;\Omega} \end{bmatrix}^T$, where the first two variables $z_{1,k;\chi}$ and $z_{2,k;\chi}$ are the internal states and the remaining four states are the external states. The internal states are initially assumed to be $z_{1,0;\chi} = -2.7$ and $z_{2,0;\chi} = 5.1$.
3. node Ω is given by $z_{k;\Omega} = \begin{bmatrix} z_{1,k;\Omega} & z_{2,k;\Omega} & z_{1,k;\alpha} & z_{2,k;\alpha} & z_{1,k;\chi} & z_{2,k;\chi} \end{bmatrix}^T$, where the first two variables $z_{1,k;\Omega}$ and $z_{2,k;\Omega}$ are the internal states and the remaining four states are the external states. The internal states are initially assumed to be $z_{1,0;\Omega} = 4.2$ and $z_{2,0;\Omega} = -2.9$.

The covariance of the ideal controllers for the three subsystems is set to be $\Gamma_{k;i} = 5$ which represents the permissible range of control inputs. The covariance of the ideal distribution of the internal states is the same as the global covariance, $\Sigma_{x;i}$ of the actual distribution of the internal states which can be estimated as explained in Section 4.4.1. The three subsystems are identical in this case as can be seen from the parameters and the symmetry of the Laplacian matrix.

In a further experiment, the decentralised FP control approach is compared to the centralised FP control method. The system that is controlled following a centralised approach consists of a state vector of six states, i.e. $x_k = \begin{bmatrix} x_{1,k-1} & x_{2,k-1} & x_{3,k-1} & x_{4,k-1} & x_{5,k-1} & x_{6,k-1} \end{bmatrix}^T$. The global state matrix $A(x_{k-1})$ is given by

$$A(x_{k-1}) = \begin{bmatrix} -0.5 & A_{11} - 0.15 & 0 & 0 & 0 & 0 \\ 0 & 1.1 - A_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & A_{21} - 0.15 & 0 & 0 \\ 0 & 0 & 0 & 1.1 - A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & A_{31} - 0.15 \\ 0 & 0 & 0 & 0 & 0 & 1.1 - A_{32} \end{bmatrix} + L_{coupled}, \quad (4.50)$$

where

$$\begin{aligned} A_{11} &= \tfrac{1}{x_{2,k-1}} \tanh(0.65 x_{1,k-1}) \quad & A_{12} &= \tfrac{1}{x_{2,k-1}} \tanh(0.95 x_{2,k-1}) \\ A_{21} &= \tfrac{1}{x_{4,k-1}} \tanh(0.65 x_{3,k-1}) \quad & A_{22} &= \tfrac{1}{x_{4,k-1}} \tanh(0.95 x_{4,k-1}) \\ A_{31} &= \tfrac{1}{x_{6,k-1}} \tanh(0.65 x_{5,k-1}) \quad & A_{32} &= \tfrac{1}{x_{6,k-1}} \tanh(0.95 x_{6,k-1}) \end{aligned}$$

and $\mathcal{L}_{coupled} = \mathcal{L} \bigotimes \mathcal{I}_{2\times2}$ with $\bigotimes$ being the Kronecker product. The global control matrix is given by

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.51)$$

.

Furthermore, the covariance of the ideal controller is set to be $\Gamma_{global} = 5\mathcal{I}_{3\times3}$. The covariance of the ideal distribution of the states is the same as the global covariance matrix. The initial state values are assumed to be $x_0 = \begin{bmatrix} 7.5 & -3.5 & -2.7 & 5.1 & 4.2 & -2.9 \end{bmatrix}^T$. The results of the system controlled by a centralised controller can be found in Figure 4.3 and the results of the decentralised FP controller can be found in Figure 4.2. As mentioned previously, the decentralised FP control approach allows the control of the subsystems to take place simultaneously, while the execution time of the centralised FP control strategy scales linearly with the size of the system [176]. From Figure (4.2), it can be observed that the internal states of the subsystems closely oscillate around zero. Also, Figure 4.3 demonstrates that the states closely oscillate around zero.

In conclusion, it has hence been validated that the proposed approach is effective since it simplifies the process by decomposing the complex system into smaller sub-problems, allowing the nodes to interact with the neighbouring subsystems via probabilistic message passing while still ensuring that the global objective is achieved using only decentralised local knowledge.

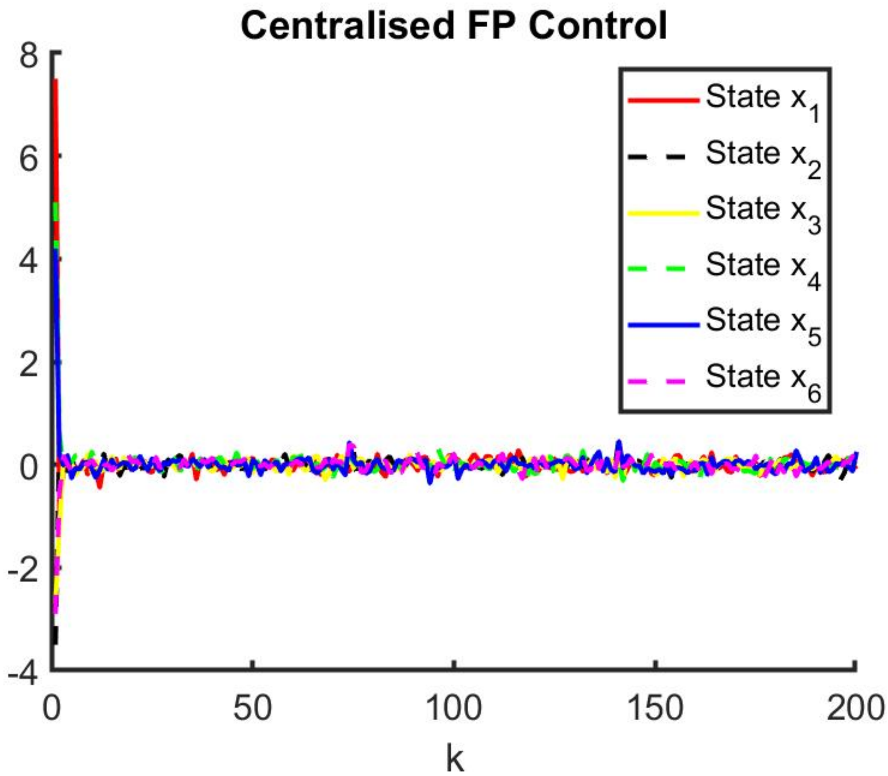

Figure 4.2: The results of the states following a centralised FP control approach are presented in this figure.

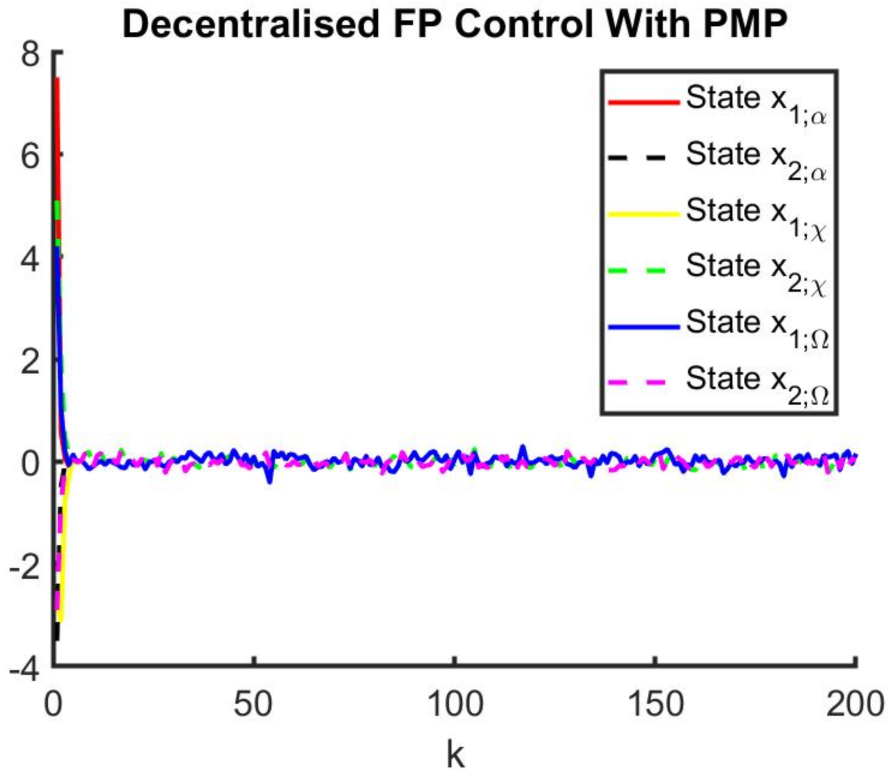

Figure 4.3: The results of the internal states of the nodes following a decentralised FP control approach are presented in this figure.

4.8 Decentralised Control Approach for Nonlinear Systems with Multiplicative Noises

The previously discussed design does not take account of multiplicative noises that affect complex nonlinear systems. Therefore, this sections discusses the design process of FP local controllers for nonlinear subsystems with multiplicative noises.

4.8.1 Subsystem Representation of Nonlinear Complex System with Multiplicative Noises

The FP control design within a decentralised framework has also been further developed for nonlinear systems with multiplicative noises. This allows the randomised controller to be implemented for nonlinear systems affected by a variety of noises.

Following the decentralised framework discussed in Section 4.1 and the transformation of a nonlinear system to a nonlinear system that is affine in the state and control input (Chapter 3), the dynamics of stochastic subsystem i with multiplicative noise is described by,

$$z_{k;i} = \bar{F}_i(z_{k-1;i})z_{k-1;i} + \tilde{g}_i(z_{k-1;i})u_{k;i} + \bar{D}_i z_{k-1;i} \cdot \mathrm{v}_{k-1;i}, \tag{4.52}$$

where $\bar{F}_i(z_{k-1;i})$ and $\tilde{g}(z_{k-1;i})$ are the nonlinear state and control matrices, respectively. The system matrix $\bar{D}_i$ is multiplied by the state vector $z_{k-1;i}$ which is thereafter multiplied by the noise vector $\mathrm{v}_{k-1;i}$ that has zero mean and covariance Q_i, using the Hadamard product.

The matrix-vector form of (4.52) is given by,

$$\underbrace{\begin{bmatrix} x_{k;i} \\ y_{k;i} \end{bmatrix}}_{z_{k;i}} = \underbrace{\begin{bmatrix} \bar{f}_{1i}(z_{k-1;i}) & \bar{f}_{2i}(z_{k-1;i}) \\ 0 & \bar{h}_i(y_{k-1;i}) \end{bmatrix}}_{\bar{F}_i(z_{k-1;i})} \underbrace{\begin{bmatrix} x_{k-1;i} \\ y_{k-1;i} \end{bmatrix}}_{z_{k-1;i}} + \underbrace{\begin{bmatrix} \bar{g}_i(z_{k-1;i}) \\ 0 \end{bmatrix}}_{\tilde{g}_i(z_{k-1;i})} u_{k;i} + \underbrace{\begin{bmatrix} \bar{D}_{1;i} & 0 \\ 0 & \bar{D}_{2;i} \end{bmatrix}}_{\bar{D}_i} \underbrace{\begin{bmatrix} x_{k-1;i} \\ y_{k-1;i} \end{bmatrix}}_{z_{k-1;i}} \cdot \underbrace{\begin{bmatrix} 1_{n\times 1} v_{k-1;i} \\ 1_{m\times 1} \tilde{v}_{k-1;i} \end{bmatrix}}_{\mathrm{v}_{k-1;i}}, \tag{4.53}$$

where $1_{n\times1}$ is an $(n \times 1)$ unit vector which corresponds to the state dimension of the internal variables $x_{k;i}$ and $1_{m\times1}$ is an $(m \times 1)$ unit vector which corresponds to the state dimension of the external variables $y_{k;i}$. Also, it is assumed that the multiplicative noises of the internal and external states are uncorrelated.

As can be read from (4.53), the dynamics of the multivariate internal state $x_{k;i}$ is represented by the following equation,

$$x_{k;i} = \bar{f}_i(z_{k-1;i})z_{k-1;i} + \bar{g}_i(z_{k-1;i})u_{k;i} + \bar{D}_{1;i}x_{k-1;i}v_{k-1;i}, \tag{4.54}$$

where $\bar{f}_i(z_{k-1;i}) = \begin{bmatrix} \bar{f}_{1i}(z_{k-1;i}) & \bar{f}_{2i}(z_{k-1;i}) \end{bmatrix}$ and the scalar noise $v_{k-1;i}$ has zero mean and variance $Q_{1;i}$. In addition, the system state equation of the multivariate external state is given by,

$$y_{k;i} = \bar{h}_i(y_{k-1;i})y_{k-1;i} + \bar{D}_{2;i}y_{k-1;i}\tilde{v}_{k-1;i}, \tag{4.55}$$

where the scalar noise $\tilde{v}_{k-1;i}$ has zero mean and variance $Q_{2;i}$.

As discussed in Sections 3.5.1 and 4.4.1, the state matrices $\bar{f}_i(z_{k-1;i})$ and $\bar{h}_i(y_{k-1;i})$, and the control matrix $\bar{g}_i(z_{k-1;i})$ are estimated using mlp neural networks to obtain $f_i(z_{k-1;i})$, $h_i(y_{k-1;i})$ and $g_i(z_{k-1;i})$ respectively. Previously in Section 3.5.1, the process of estimating the system matrix D has been outlined for nonlinear systems where the concept of internal and external states does not exist. Nevertheless, the same approach to estimate D_i is followed for systems within the decentralised framework. To clarify, the estimation of $D_{1;i}$ of the internal states depends on the internal and external states such that $D_{1;i} = e_{x_{k;i}}(x_{k-1;i}v_{k-1;i})^{\dagger}$, where $e_{x_{k;i}} = x_{k;i} - \hat{x}_{k;i}$. On the other hand, the estimation of $D_{2;i}$ of the external states depends on the external states only, i.e. $D_{2;i} = e_{y_{k;i}}(y_{k-1;i}\tilde{v}_{k-1;i})^{\dagger}$, where $e_{y_{k;i}} = y_{k;i} - \hat{y}_{k;i}$.

The Gaussian probability density functions of the internal and external states of node i can then be obtained,

$$s(x_{k;i}|u_{k;i}, z_{k-1;i}) \sim \mathcal{N}(\hat{x}_{k;i}, R_{x;i}), \tag{4.56}$$

$$s(y_{k;i}|y_{k-1;i}) \sim \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}), \tag{4.57}$$

where $s(x_{k;i}|u_{k;i}, z_{k-1;i})$ is the pdf of the internal states with the mean expressed by,

$$\hat{x}_{k;i} = f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u_{k;i}, \tag{4.58}$$

and covariance,

$$\begin{aligned} R_{x;i} =& E[(x_{k;i} - \hat{x}_{k;i})(x_{k;i} - \hat{x}_{k;i})^T], \\ =& E[(D_{1;i}x_{k-1;i}v_{k-1;i})(D_{1;i}x_{k-1;i}v_{k-1;i})^T], \\ =& E[D_{1;i}x_{k-1;i}v_{k-1;i}v_{k-1;i}^T x_{k-1;i}^T D_{1;i}^T], \\ =& D_{1;i}x_{k-1;i}Q_{1;i}x_{k-1;i}^T D_{1;i}^T. \end{aligned} \tag{4.59}$$

Furthermore, the probabilistic description of the behaviour of the external states of subsystem i is given by $s(y_{k;i}|y_{k-1;i})$ with the mean defined by,

$$\hat{y}_{k;i} = h_i(y_{k-1;i})y_{k-1;i}, \tag{4.60}$$

and covariance matrix expressed by,

$$\Sigma_{y;i} = D_{2;i}y_{k-1;i}Q_{2;i}y_{k-1;i}^T D_{2;i}^T. \tag{4.61}$$

In addition, the ideal distributions of the conditional distributions of the internal and external states of subsystem i are given by,

$$s^I(x_{k;i}|u_{k;i}, z_{k-1;i}) \sim \mathcal{N}(0, \Sigma_{x;i}), \tag{4.62}$$

$$s^I(y_{k;i}|y_{k-1;i}) \sim \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}), \tag{4.63}$$

where the mean of the ideal distribution of the internal variables of subsystem i is zero due to the regulation objective of the control problem, and covariance is $\Sigma_{x;i}$. Unlike the approach discussed previously in Section 4.4.1, the covariance matrix $R_{x;i}$ of the actual distribution of the internal states is different than the covariance matrix $\Sigma_{x;i}$ of the ideal pdf of the internal states of subsystem i. This is due to the state dependency of covariance matrix $R_{x;i}$ on the states which are expected to converge to zero, as can be seen from (4.59). This allows us to set a smaller ideal covariance matrix, $\Sigma_{x;i}$ than the actual covariance $R_{x;i}$.

Since the objective of the control problem is still the same as the previously discussed decentralised FP approach, the ideal distribution of the controller remains the same, namely,

$$c^I(u_{k;i}|z_{k-1;i}) \sim \mathcal{N}(0, \Gamma_{k;i}), \tag{4.64}$$

which is a Gaussian distribution with zero mean and covariance $\Gamma_{k;i}$ to specify the range of permissible control inputs.

4.8.2 Randomised Suboptimal Controller for Nonlinear Systems with Multiplicative Noises

This section presents Theorem 7 to outline the form of the distribution of the randomised controller designed for nonlinear subsystems that are affected by multiplicative stochastic noises.

Theorem 7. Using Proposition 4 and Definition 4.4.1, the suboptimal approximation of the optimal control law for node i that considers multiplicative noises, subject to the dynamics of subsystem i characterised by (4.56) and (4.57), the ideal distribution of the system states given by (4.62) and (4.63), and the ideal pdf of the controller stated in (4.64), can be obtained such that,

$$c^*(u_{k;i}|z_{k-1;i}) = \mathcal{N}(u^*_{k;i}, \bar{\Gamma}_{k;i}), \tag{4.65}$$

where

$$u^*_{k;i} = -K_{k;i} z_{k-1;i}, \tag{4.66}$$

$$\bar{\Gamma}_{k;i} = \left(\Gamma^{-1}_{k;i} + g_i^T(z_{k-1;i})\tilde{Q}_{k;i} g_i(z_{k-1;i})\right)^{-1}, \tag{4.67}$$

$$K_{k;i} = \bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) \begin{bmatrix} \tilde{Q}_{k;i} f_{1i}(z_{k-1;i}) & \tilde{Q}_{k;i} f_{2i}(z_{k-1;i}) + M_{2,k;i} h_i(y_{k-1;i}) \end{bmatrix}, \tag{4.68}$$

$$\tilde{Q}_{k;i} = (\Sigma^{-1}_{x;i} + M_{1,k;i}). \tag{4.69}$$

The suboptimal controller for node i presented by (4.65) has mean $u^*_{k;i}$ and covariance $\bar{\Gamma}_{k;i}$. The control gain $K_{k;i}$ is given by Equation (4.68). Furthermore, the quadratic performance index for the control strategy is given by,

$$-\ln(\gamma(z_{k;i})) = \frac{1}{2} z^T_{k;i} M_{k;i} z_{k;i} + \frac{1}{2} V_{k;i}, \tag{4.70}$$

where matrix $M_{k;i}$ has been partitioned as follows,

$$M_{k;i} = \begin{bmatrix} M_{1,k;i} & M_{2,k;i} \\ M_{2,k;i}^T & M_{3,k;i} \end{bmatrix}^T, \tag{4.71}$$

and where,

$$\begin{aligned} M_{1,k-1;i} =& - f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i}) + f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i}) \\ & + D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i}, \end{aligned} \tag{4.72}$$

$$\begin{aligned} M_{2,k-1;i} =& f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) + f_{1i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\ & - f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) \\ & - f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}), \end{aligned} \tag{4.73}$$

$$\begin{aligned} M_{3,k-1;i} =& f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) + 2f_{2i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) + h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i}) \\ & - f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) \\ & - h_i^T(y_{k-1;i})M_{2,k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\ & - 2f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) + D_{2;i}^T M_{3,k;i} Q_{2;i} D_{2;i}, \end{aligned} \tag{4.74}$$

$$V_{k-1;i} = V_{k;i} + \ln|\Gamma_{k;i}| + \ln|\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i})\tilde{Q}_{k;i}g_i(x_{k-1;i})|. \tag{4.75}$$

Proof. The proof of Theorem 7 can be found in Appendix F. □

If one observes the suboptimal controller designed in Section 4.4 given by Equations (4.21) - (4.25) and compares it to the derived suboptimal controller that considers multiplicative stochastic disturbances described by (4.65) - (4.69), it can be seen that both controllers have the same formulation of the Gaussian distribution including the optimal gains, means and covariance matrices. However, when considering the Riccati equation solution $M_{1,k;i}$, one can notice an additional term, $D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i}$ which arose from the consideration of multiplicative noises. Although, there is an additional term in the Riccati element $M_{3,k;i}$, namely $D_{2;i}^T M_{3,k;i} Q_{2;i} D_{2;i}$, there is no requirement to solve $M_{3,k;i}$ due to the fact that it is not part of the optimal control law. This reduces the computational expenses since solving the full block of $M_{k;i}$ is not required.

The communications between the subsystems follows the probabilistic message passing approach explained in Section 4.2.

4.8.3 Algorithm of the Decentralised FP Control Approach for Nonlinear Subsystems with Multiplicative Noises

The algorithm of the FP control design that considers multiplicative noises in nonlinear subsystems is summarised in Algorithm 6 as a pseudocode of the key steps. The time index is required to be changed to find the steady state solutions of the DARE's $M_{1,k;i}$ and $M_{2,k;i}$ given by (4.72) and (4.73).

Algorithm 6 Pseudo-code of randomised controller for decentralised nonlinear systems with multiplicative noises

1: **procedure** IMPLEMENTATION OF FP CONTROL DESIGN FOR DECENTRALISED NONLINEAR SYSTEMS WITH MULTIPLICATIVE NOISES
2: **Initialise:** $x_{0;i}, y_{0;i}, M_{1,0;i}, M_{2,0;i}, h_i(y_{k-1;i})$ where $i = \{1, ..., N\}$, N being the number of subsystems and pre-train the neural network model as discussed in Algorithm 1 (optional).
3: **for** $k = 1 \rightarrow H$ **do**
4: **for** subsystem i:
5: **Estimate** $f_i(z_{k-1;i})$, $g_i(z_{k-1;i})$, $D_{1;i}$, and $D_{2;i}$ from the neural network model.
6: **Calculate** the SS solution of $M_{1;i}$ using (4.72).
7: $\tilde{Q}_i \leftarrow (\Sigma_{x;i}^{-1} + M_{1;i})$
8: $\bar{\Gamma}_i \leftarrow \left(\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i})\tilde{Q}_i g_i(z_{k-1;i})\right)^{-1}$
9: **Use** *Steps 6-8* in (4.73) to find the SS solution of $M_{2;i}$.
10: **Use** the SS values from *Steps 6-9* in (4.68) to obtain the SS solution of K_i.
11: **Compute** $u_{k;i}^*$ using *Step 10* in Equation (4.66).
12: **Forward** the control signal u_k^* obtained in *Step 11* to the system equation (4.52).
13: **Using** a one step delayed of the new state value from *Step 12* and the calculated control signal u_k^* from *Step 11* as input to the neural network model and the new state from *Step 12* as output, retrain the neural network model and update its parameters.
14: **end for**
15: **for** subsystem i:
16: **Compute** $\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i})$ given by (4.34) using (4.35) and (4.36).
17: **Update** the prior pdf of the external states using (4.38) - (4.41).
18: **Update** the external state matrices $h_i(y_{k-1;i})$.
19: **end for**
20: **end for**

4.9 Simulation

The proposed method discussed in Section 4.8 is illustrated here using the same simulation [177] as Section 4.7. The parameters of the pdfs of the system dynamics stay the same for the three individual subsystems and are also expected to be estimated since there is no apriori knowledge about them. However, instead of additive noises, the simulated example is affected by multiplicative noises, which is the key difference here. The localised controllers are designed in such a way that they consider the multiplicative stochasticity of the subsystems. The matrix $D_{1;i}$ that is multiplied by the noise $v_{k-1;i}$

with zero mean and variance $Q = 1$ is given by,

$$D_{1;i} = \begin{bmatrix} 0.73 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}. \tag{4.76}$$

To re-emphasise, matrix $D_{1;i}$ is unknown and hence is required to be estimated. For the nodes α, χ and Ω, the ideal covariance matrix is chosen to be $\Sigma_{x;i} = 10^{-2}\text{diag}([2.1; 1.1])$. For the proposed method, it is possible to have a smaller ideal covariance $\Sigma_{x;i}$ than the actual covariance $R_{x;i}$ due to the state dependency of the actual covariance matrix. Furthermore, the range of allowable control inputs is described by $\Gamma_{k;i} = 5$.

In addition, in a further experiment, a centralised controller is implemented to control the system. The global state matrix and the global control matrix is taken to be the same as described in Section 4.7 by Equations (4.50) and (4.51), respectively. The ideal covariance matrix is the same as the global covariance matrix and the covariance of the ideal controller is given by $\Gamma_{global} = 5\mathcal{I}_{3\times 3}$. The assumed initial state values remain the same for both the decentralised FP controller and the centralised FP controller as described by Section 4.7.

The results plotted in Figure 4.5 show that the internal states of the subsystems have converged to values extremely close to zero as expected from a regulation problem following a decentralised control strategy. It becomes clear that the addition of the external states to the state vector $z_{k;i}$ is to ensure that node i is aware of the state of the interacting subsystems. The global objective has been achieved by using decentralised controllers that only have access to local information. The approach of probabilistic message passing allowed the three subsystems to communicate with one another. The results of the converged states that are regulated by a controller that follows a decentralised approach are plotted in Figure 4.4. Although both controllers regulate the states, the implementation of decentralised controllers allows the regulation of the states to happen simultaneously resulting in computational efficiency. Controlling the system centrally has meant that the execution time scales linearly with the size of the system [176].

4.10 Conclusion

Many approaches have been introduced for the control of large-scale complex systems in the control literature. However, some methods require full knowledge of the system since a centralised controller was implemented, whereas others developed decentralised controllers that were based on incomplete knowledge of the system. In addition, many technical difficulties can arise and as a result may sever the connection between interacting nodes leading to poorer results since local controllers may fail to

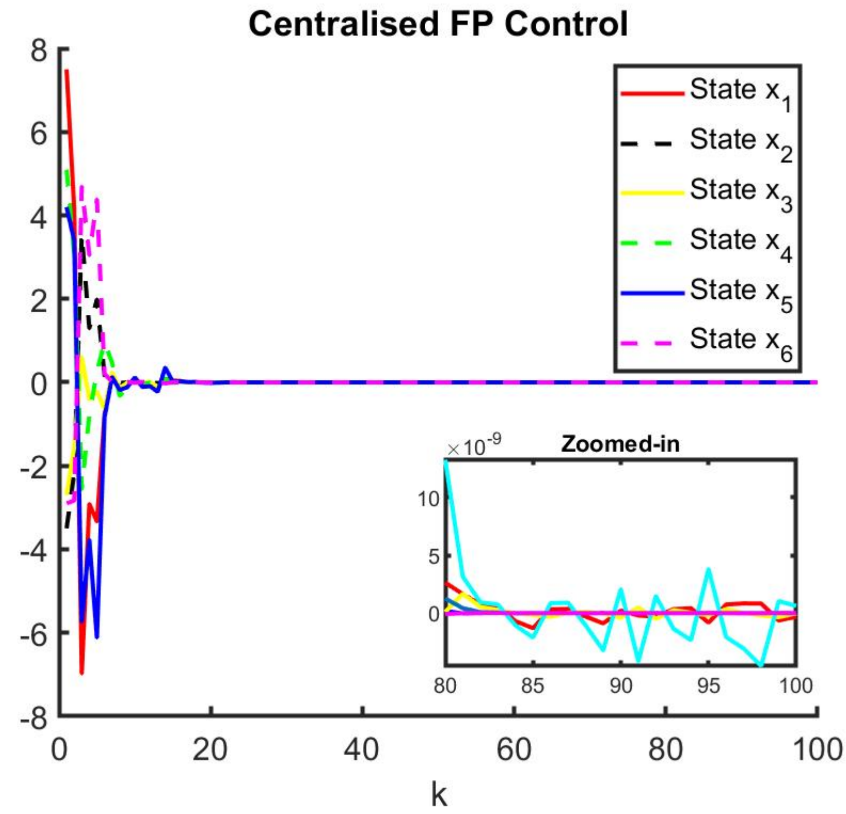

Figure 4.4: The results of the states controlled by a centralised controller are presented in this figure. The zoomed-in plot demonstrates the oscillation of the states around zero.

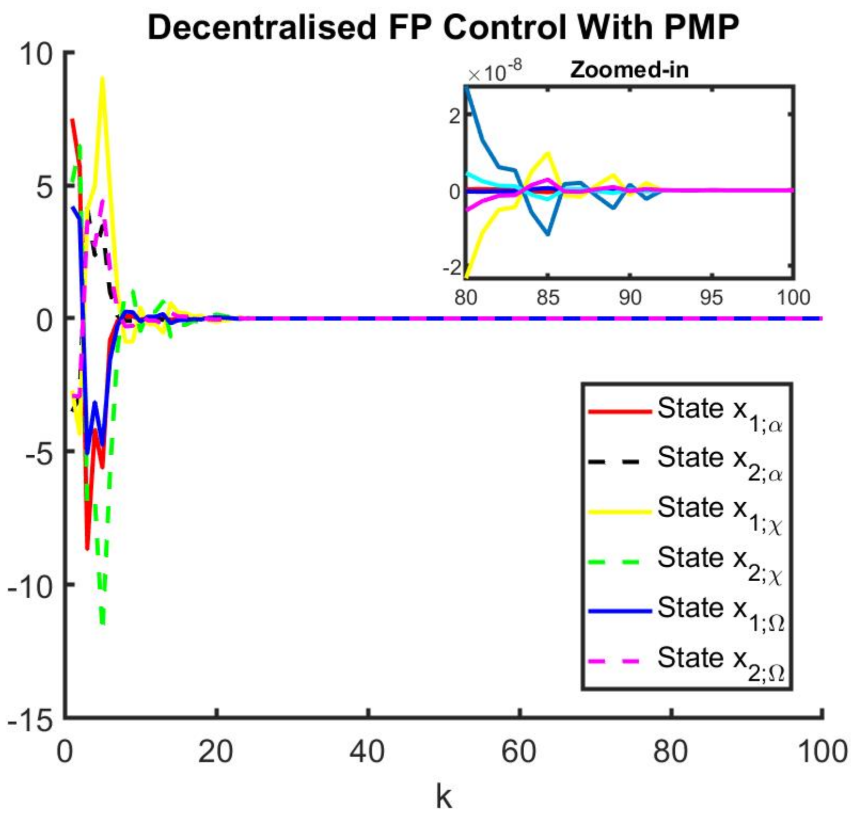

Figure 4.5: The results of states of the subsystems are presented in this figure The zoomed-in plot demonstrates the oscillation of the states around zero.

consider important information.

The proposed method in this chapter has addressed these gaps for nonlinear systems with additive and multiplicative noises in order to circumvent these issues. The large-scale complex network is decomposed into a number of subsystems for which individual local controllers have been developed. These controllers are responsible for the control of the subsystems they are assigned to. Nevertheless, since it is key to consider the communication between the nodes to ensure the global objective of the system is achieved, probabilistic message passing has been introduced to achieve this. Furthermore, the problem of reliability of control systems in the presence of component failure has also been addressed since the state vector of each independent subsystem consists of internal and external state variables. The knowledge received from other neighbouring subsystems via probabilistic message passing is preserved by including it in the state matrix as the dynamics of the external state variables, which are constantly updated. This way, the node always has access to some dynamical information of its neighbouring nodes.

In addition, it has been shown that the proposed method is computationally efficient as the full block of the Riccati matrix is not expected to be computed to obtain the suboptimal controller. Instead, only two elements of the Riccati matrix are required to be solved, resulting in a reduction in computational expenses. The simulations results for the developed decentralised FP control approaches for nonlinear systems (additive and multiplicative) demonstrated the validity and efficiency of the proposed method as it nicely regulated the states to zero.

Chapter 5

Decentralised FP Control Design for Tracking and Multi-Agent Formation Control

5.1 Introduction

The developed FP control strategies introduced in Chapters 3 and 4 (centralised and decentralised) have primarily focused on the control of nonlinear stochastic systems which are required to be regulated to state zero. Nevertheless, many control objectives for real-world dynamical systems involve the tracking of a predefined desired value. This means the output of the system is expected to follow a predetermined desired state value which is achieved by focusing on the tracking error rather than the actual output of the system.

The implementation of this control problem is different than the previously discussed approaches. The regulation problem involves the minimisation of the Kulback-Leibler divergence of the actual and ideal joint pdf of the system state and controller. However, this development required an alternative strategy where the KLD now considers the distance between the actual and ideal joint pdf of the tracking error and the controller. Based on this key modification, a new set of algorithms arise which results into an optimal randomised controller that aims to track a predefined desired value or trajectory. In [178], [179], a tracking error-based FP control design has been studied by Herzallah et al. to design a randomised optimal controller that influences the pdf of the tracking error of the system to be controlled, instead of the pdf of the dynamics of the system. However, the optimal controller proposed by Herzallah is centralised and has only been demonstrated for linear stochastic systems. Therefore, the study discussed in this chapter demonstrates the process of obtaining local randomised controllers within a FP design approach for decentralised nonlinear stochastic systems, thus, allowing it to be implemented for more complex systems which may be found in the real world of control.

The problem of finding a control solution within a fully probabilistic framework that enables the output of the system to track a predefined desired state value is demonstrated in the first part of this chapter. It is important to emphasise that a reference model could be either deterministic or stochastic. The first part of this chapter focuses on solving a tracking control problem where the system output tracks a predefined desired state which is specified by a stochastic model. Additionally, the stochastic system is affected by multiplicative noises which are considered in the development of the tracking control solution. The validity and efficiency of this method is demonstrated with a simulation example.

The FP control design based on the tracking error in a decentralised framework allows control problems with different objectives to be implemented such as the management and control of multi-agent systems. The increasing popularity of the control of multi-agent systems is due to its practicality in a number of applications and the theoretical challenges that are encountered in order to coordinate and control them. These difficulties mostly exist as researchers aim to design decentralised controllers for multi-agent systems that possess incomplete knowledge of the systems or aim to have complete information but require a centralised controller to achieve this. The control of multi-agent systems that has actively been studied is called formation control which aims to coordinate a group of agents to achieve and sustain a formation described by a certain shape. To achieve the control objective, it can be realised that communication between the agents is crucial in order to be aware of the neighbouring interacting nodes. The incorporation of probabilistic message passing in the FP control design plays an important part in formation control. The randomised local controllers developed with the aim of achieving a formation are affected by additive noises. As far as the literature is concerned, formation control within a fully probabilistic framework has not been considered for either linear or nonlinear stochastic systems and is discussed in Section 5.4. Therefore, the formation of a certain shape using the developed method is illustrated by two simulations, one for systems governed by linearities, and the other for nonlinear stochastic systems.

5.2 Fully Probabilistic Control Design for a Tracking Control Problem in a Decentralised Framework

The objective of the FP control design in its original form was to drive the joint pdf of the system state dynamics and the controller to its ideal joint pdf. The behaviour of the closed-loop system of the system to be controlled is completely characterised by the joint probability density function of the dynamical system state and the controller. This description of the closed-loop behaviour, however, may be slightly inconvenient and more complicated if implemented for control problems that require the tracking of a predefined desired state value or trajectory. Therefore, the optimal controller designed for this purpose is reformulated such that it reshapes the joint pdf of the controller and the tracking error rather than the joint pdf of the controller and system state dynamics [178], [179]. The aim of the randomised controller is to ensure the pdf of the tracking error slightly fluctuates around mean zero which enables the system to track the desired state value. Once the pdf of the tracking error has been reshaped to a distribution that fluctuates around zero, the desired state value or trajectory has been achieved. Having small variations around zero means that the tracked trajectory is affected by a low level of uncertainty.

The tracking control solution discussed in [178], [179] is demonstrated for systems that require a central controller. However, the control solution would fail for large-scale complex networks which consist of various nodes interacting with each other. From Chapter 4, it is recognised that the state of the nodes are composed of internal states and external states within the FP control framework. To re-emphasise, the external signals are treated as an external disturbance to the system which is considered for the sole purpose of having knowledge of interacting neighbouring nodes in order to design an optimal local controller. As such, it can be easily deduced that the external states of the system state dynamics of the nodes cannot be controlled or changed by the corresponding local suboptimal controllers. To re-emphasise, the external states are only passed as messages from the neighbouring subsystems and thus, they cannot be influenced by the output of the subsystem they are passed to. The same notion applies to the tracking error control problem where the tracking error is only considered for the internal states of the nodes since the external states are not intended to be controlled.

To be more precise, the state dynamics of node i can be represented by the distribution,

$$s(z_{k;i}|u_{k;i}, z_{k-1;i}) = s(x_{k;i}|u_{k;i}, x_{k-1;i}, y_{k-1;i})s(y_{k;i}|y_{k-1;i}), \tag{5.1}$$

where $z_{k;i} = [x_{k;i}, y_{k;i}]^T$ is the state vector of subsystem i, $x_{k;i}$ is the multivariate internal state of subsystem i and $y_{k;i}$ is the external multivariate state that node i receives from neighbouring subsystems via probabilistic message passing (Chapter 4). Although, state $x_{k;i}$ is observable, the distributions that characterises its dynamics, namely $s(x_{k;i}|u_{k;i}, z_{k-1;i})$ and $s(y_{k;i}|y_{k-1;i})$ need to be estimated. The estimation process of the dynamics of the subsystems has already been discussed in Chapter 4.

The tracking control problem requires the internal state of subsystem i to track the reference model $x_{r,k;i}$ from which the system tracking error $e_{k;i}$ can be obtained,

$$e_{k;i} = x_{k;i} - x_{r,k;i}. \tag{5.2}$$

It is known that the dynamics of the system state can only be represented by a probability distribution as given by (5.1) due to the stochasticity of the system dynamics. Hence, for the purpose of tracking a predefined trajectory, the designed controller is concerned with the reshaping of the distribution of the tracking error. Consequently, knowledge of the pdf of the tracking error is required which is obtained by using the estimated state value $x_{k;i}$. The pdf of the tracking error can be achieved by exploiting the probability distribution of the internal system state $s(x_{k;i}|u_{k;i}, x_{k-1;i}, y_{k-1;i})$ and the definition in (5.2) such that,

$$s_{e_i}(x_{k;i}, x_{r,k;i}) = s(e_{k;i} + x_{r,k;i}|u_{k;i}, e_{k-1;i} + x_{r,k-1;i}, y_{k-1;i}). \tag{5.3}$$

To re-emphasise, for a tracking control problem, the dynamics of the subsystems are described by the pdfs of the tracking error $e_{k;i}$ and the external states $y_{k;i}$ instead of the pdfs of the internal, $x_{k;i}$ and external, $y_{k;i}$ state variables. Therefore, the states of node i are now represented by $n_{k;i} = [e_{k;i}, y_{k;i}]^T$. This means that the behaviour of subsystem i described by (5.1) now conforms to the pdf $s(n_{k;i}|u_{k;i}, n_{k-1;i})$ for a tracking control problem,

$$s(n_{k;i}|u_{k;i}, n_{k-1;i}) = s(e_{k;i}|u_{k;i}, n_{k-1;i})s(y_{k;i}|y_{k-1;i}), \tag{5.4}$$

where $s(e_{k;i}|u_{k;i}, n_{k-1;i})$ and $s(y_{k;i}|y_{k-1;i})$ are the conditional distributions of the tracking error and external states of node i, respectively.

Following the discussion above, and the attention being shifted to the pdfs of the tracking error $e_{k;i}$ and external states $y_{k;i}$ of node i, the optimal randomised controller can be derived by revisiting and updating the definition of the Kullback-Leibler divergence accordingly. This adjustment is made with the purpose of minimising the divergence between the joint pdf of the tracking error and the controller and a predetermined ideal joint pdf. The redefined KLD that the designed optimal controller minimises is given by,

$$\mathcal{D}(f \parallel f^I) = \int f(\mathbb{D}) \ln\left(\frac{f(\mathbb{D})}{f^I(\mathbb{D})}\right) d\mathbb{D}, \tag{5.5}$$

where,

$$f(\mathbb{D}) = \prod_{k=1}^{H} s(e_{k;i}|u_{k;i}, n_{k-1;i})s(y_{k;i}|y_{k-1;i})c(u_{k;i}|n_{k-1;i}),$$
$$f^I(\mathbb{D}) = \prod_{k=1}^{H} s^I(e_{k;i}|u_{k;i}, n_{k-1;i})s^I(y_{k;i}|y_{k-1;i})c^I(u_{k;i}|n_{k-1;i}),$$

with $\mathbb{D} = (n_{k;i}, ..., n_{H;i}, u_{k;i}, ..., u_{H;i})$ with H being the control horizon. To re-emphasise, for the special case of a regulation problem, the distribution of the system state vector $z_{k;i}$ of subsystem i was composed of the internal $x_{k;i}$ and external states $y_{k;i}$ as shown in (5.1). However, the control solution of a tracking error problem considers the pdf of subsystem i that is now constructed of the tracking error $e_{k;i}$ and external states $y_{k;i}$.

The randomised controller $c(u_{k;i}|n_{k-1;i})$ that minimises the KLD in (5.5) with respect to the control sequence $u_{k;i}$ is given by [162],

$$-\ln(\gamma(n_{k-1;i})) = \min_{\{c(u_{k;i}|n_{k-1;i})\}} \int s(n_{k;i}|u_{k;i}, n_{k-1;i})c(u_{k;i}|n_{k-1;i}) \times \Bigg[\underbrace{\ln\left(\frac{s(n_{k;i}|u_{k;i}, n_{k-1;i})c(u_{k;i}|n_{k-1;i})}{s^I(n_{k;i}|u_{k;i}, n_{k-1;i})c^I(u_{k;i}|n_{k-1;i})}\right)}_{\text{Partial cost}} - \underbrace{\ln(\gamma(n_{k;i}))}_{\text{Optimal cost-to-go}} \Bigg] d(n_{k;i}, u_{k;i}). \tag{5.6}$$

Although the formulation of the minimum cost-to-go function is the same as the conventional approach for decentralised systems discussed in Proposition 3 (Chapter 4), the difference lies in the joint pdf that is considered for the current proposed method, namely the joint pdf of the tracking error term and the external states.

From the minimisation of the recursion equation defined in (5.6), the optimal randomised controller can be obtained as will be outlined in the following proposition.

Proposition 5. The optimal randomised controller that minimises the redefined KLD (5.5) can be derived from the cost-to-go function (5.6) for the tracking error control problem such that,

$$c^*(u_{k;i}|n_{k-1;i}) = \frac{c^I(u_{k;i}|n_{k-1;i}) \exp(-\beta_1(u_{k;i}, n_{k-1;i}) - \beta_2(u_{k;i}, n_{k-1;i}))}{\gamma(n_{k-1;i})}, \tag{5.7}$$

$$\gamma(n_{k-1;i}) = \int c^I(u_{k;i}|n_{k-1;i}) \exp(-\beta_1(u_{k;i}, n_{k-1;i}) - \beta_2(u_{k;i}, n_{k-1;i}))du_{k;i}, \tag{5.8}$$

$$\beta_1(u_{k;i}, n_{k-1;i}) = \int s(e_{k;i}|u_{k;i}, n_{k-1;i}) \left[\ln\left(\frac{s(e_{k;i}|u_{k;i}, n_{k-1;i})}{s^I(e_{k;i}|u_{k;i}, n_{k-1;i})}\right)\right] de_{k;i}, \tag{5.9}$$

$$\beta_2(u_{k;i}, n_{k-1;i}) = -\int s(e_{k;i}|u_{k;i}, n_{k-1;i}) \ln(\tilde{\gamma}(e_{k;i}, y_{k-1;i}))de_{k;i} \tag{5.10}$$

with,

$$\ln(\tilde{\gamma}(e_{k;i}, y_{k-1;i})) = \int s(y_{k;i}|y_{k-1;i}) \ln(\gamma(n_{k;i}))dy_{k;i}. \tag{5.11}$$

Proof. The derivation of the formulation of the optimal controller given by (5.7) - (5.11) can be obtained by following the proof in [25]. □

From the above proposition, the tracking control problem where the state is required to follow a stochastic reference model is discussed in Section 5.3. The proposed method is novel since the fully probabilistic approach has not been demonstrated for the tracking control problem for nonlinear systems in a decentralised framework. The added originality emerges from the consideration of a stochastic reference model that the system state is instructed to track. Also, multiplicative noises are considered for the design of the proposed optimal controller.

In addition, the general solution for the randomised controller outlined in Proposition 5 is derived for a formation control problem in Section 5.4. In the control literature, the FP control design has not been implemented with the purpose of the controlled agents forming a certain shape, i.e. formation control. The developed probabilistic control approach is demonstrated for both linear and nonlinear stochastic systems with additive noises.

While there are no restrictions on the required pdfs, it is a key requirement for the pdfs to be Gaussian in order to derive a closed form control solution for the FP controller using Proposition 5. Hence, in both sections (Sections 5.3, 5.4), the pdfs are specified by the Gaussian distribution though the parameters of the distribution are not restricted to be governed by linear functions. This generalises the solution obtained from the Fully Probabilistic control method and allows its implementation to a broad range of real-world control systems. Otherwise, if the system was treated to be nonlinear in the state and control input, the solution needs to be obtained following a numerical approach where multiple integrations over multiple time steps would have been needed to be evaluated numerically.

5.3 Tracking Error Control with Stochastic Reference

This section discusses the approach of obtaining a randomised controller where the nonlinear system with multiplicative noises is required to track a stochastic reference model.

5.3.1 Problem Formulation

The concept and general solution of the controller explained in Section 5.2 is applied here to nonlinear stochastic systems affected by multiplicative noises. An important aspect that is regarded is the random behaviour of the reference model that the output of the controlled system tracks.

The discrete time stochastic model of subsystem i with multiplicative noises within a decentralised framework has already been considered for a regulation problem in Section 4.8.1 and is now studied for a tracking control problem. To remind the reader, the stochastic dynamics of the multivariate internal state $x_{k;i}$ and external state $y_{k;i}$ of subsystem i with multiplicative noises are repeated below.

$$x_{k;i} = \bar{f}_i(z_{k-1;i})z_{k-1;i} + \bar{g}_i(z_{k-1;i})u_{k;i} + \bar{D}_{1;i}x_{k-1;i}v_{k-1;i}, \tag{5.12}$$

$$y_{k;i} = \bar{h}_i(y_{k-1;i})y_{k-1;i} + \bar{D}_{2;i}y_{k-1;i}\tilde{v}_{k-1;i}, \tag{5.13}$$

where the state matrices of the internal and external states are given by $\bar{f}_i(z_{k-1;i})$ and $\bar{h}_i(y_{k-1;i})$, respectively. The control matrix is represented by $\bar{g}(z_{k-1;i})$ and the system matrices are given by $\bar{D}_{1;i}$ and $\bar{D}_{2;i}$. Furthermore, the internal noise $v_{k-1;i}$ has zero mean and variance $Q_{1;i}$ and the external noise $\tilde{v}_{k-1;i}$ has zero mean and variance $Q_{2;i}$.

Definition 5.3.1. *Suboptimal solution to the decentralised nonlinear FP control design:* The FP control approach for obtaining a suboptimal solution for decentralised nonlinear control problems can be obtained using the following definition:

1. At each discrete time step, use transformation methods to bring the nonlinear dynamics to the nonlinear affine dynamics (as demonstrated by Equations (5.12) - (5.13)).

2. Solve the equations provided in Proposition 5 derived from (5.6) to obtain the closed form suboptimal solution at each discrete time instant.

The parameters of the pdfs of the dynamics of the internal and external states of node i are estimated online using mlp neural networks to obtain the Gaussian distributions, (Section 4.8.1) as follows,

$$s(x_{k;i}|u_{k;i}, z_{k-1;i}) \sim \mathcal{N}(\hat{x}_{k;i}, R_{x;i}), \tag{5.14}$$

$$s(y_{k;i}|y_{k-1;i}) \sim \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}), \tag{5.15}$$

where the means of the multivariate internal $x_{k;i}$ and external $y_{k;i}$ state variables are given by,

$$\hat{x}_{k;i} = f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}, \tag{5.16}$$

$$\hat{y}_{k;i} = h_i(y_{k-1})y_{k-1;i}. \tag{5.17}$$

The covariances of the internal and external states for subsystems affected by multiplicative noises are given by,

$$R_{x;i} = x_{k-1;i}^T D_{1;i}^T Q_{1;i} D_{1;i} x_{k-1;i}, \tag{5.18}$$

and,

$$\Sigma_{y;i} = y_{k-1;i}^T D_{2;i}^T Q_{2;i} D_{2;i} y_{k-1;i}, \tag{5.19}$$

respectively. The same approach as outlined in Section 4.8.1 is followed to obtain the estimations for $D_{1;i}$ and $D_{2;i}$.

To obtain the tracking error as defined by (5.2), the reference model needs to be discussed. In this section, the internal output of subsystem i needs to track a reference model which is stochastic and therefore given by,

$$x_{r,k;i} = \tilde{m}_i(x_{r,k-1;i})x_{r,k-1;i} + \tilde{D}_i x_{r,k-1;i} v_{r,k-1;i}, \tag{5.20}$$

where $\tilde{m}_i(x_{r,k-1;i})$ is the state matrix which is nonlinear in the reference signal $x_{r,k-1;i}$. The Gaussian noise $v_{r,k-1;i}$ has zero mean and variance Q_r and is multiplied by the matrix $\tilde{D}_i$. There are no restrictions imposed on the nature of the model of the reference state and hence, it can be either deterministic or stochastic [180]. Stochastic reference models are relevant to applications where the exact state of the system is not critical or potentially cannot be physically achieved. Therefore, this section will consider the case of stochastic reference model that is affected by multiplicative noise. The case of deterministic reference model will be considered later in Section 5.4. The equation in (5.20) can be expressed as a distribution where the mean $\hat{x}_{r,k;i}$ is given by $\hat{x}_{r,k;i} = \tilde{m}_i(x_{r,k-1;i})x_{r,k-1;i}$ and the covariance is described by $\Sigma_{r,k;i} = x^T_{r,k-1;i}\tilde{D}_i^T Q_r \tilde{D} x_{r,k-1;i}$.

It is now possible to calculate the tracking error $e_{k;i}$ by subtracting the reference $x_{r,k;i}$ from the multivariate internal state $x_{k;i}$ such that the description of the dynamics of the tracking error is given by,

$$\begin{aligned} e_{k;i} &= x_{k;i} - x_{r,k;i}, \\ &= f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + D_{1;i}x_{k-1;i}v_{1,k-1;i} - x_{r,k;i} \\ &= f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i} \\ &\quad + D_{1;i}e_{k-1;i}v_{1,k-1;i} + D_{1;i}x_{r,k-1;i}v_{1,k-1;i} - \tilde{D}_i x_{r,k-1;i}v_{r,k-1;i}, \end{aligned} \tag{5.21}$$

where the substitution $C = [f_{1i}(z_{k-1;i}) - \tilde{m}_i(x_{r,k-1;i})]$ is introduced for notational convenience. Following equations (5.3) and (5.21), the conditional Gaussian distribution of the tracking error with mean $\hat{e}_{k;i}$ and covariance $\Sigma_{e_{k;i}}$ can be determined such that,

$$s(e_{k;i}|u_{k;i}, n_{k-1;i}) \sim \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}), \tag{5.22}$$

where,

$$\begin{aligned}
\hat{e}_{k;i} =& f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i}, \qquad (5.23)\\
\Sigma_{e_{k;i}} =& E[(e_{k;i} - \hat{e}_{k;i})^T(e_{k;i} - \hat{e}_{k;i})],\\
=& E[(D_{1;i}e_{k-1;i}v_{k-1;i} + D_{1;i}x_{r,k-1;i}v_{k-1;i} - \tilde{D}_i x_{r,k-1;i}v_{r,k-1;i})^T\\
& \times (D_{1;i}e_{k-1;i}v_{k-1;i} + D_{1;i}x_{r,k-1;i}v_{k-1;i} - \tilde{D}_i x_{r,k-1;i}v_{r,k-1;i})],\\
=& E[e^T_{k-1;i}D^T_{1;i}v^T_{k-1;i}v_{k-1;i}D_{1;i}e_{k-1;i} + 2e^T_{k-1;i}D^T_{1;i}v^T_{k-1;i}v_{k-1;i}D_{1;i}x_{r,k-1;i}\\
& - 2e^T_{k-1;i}D^T_{1;i}v^T_{k-1;i}v_{r,k-1;i}\tilde{D}_i x_{r,k-1;i} + x^T_{r,k-1;i}D^T_{1;i}v^T_{k-1;i}v_{k-1;i}D_{1;i}x_{r,k-1;i}\\
& - 2x^T_{r,k-1;i}D^T_{1;i}v^T_{k-1;i}v_{r,k-1;i}\tilde{D}_i x_{r,k-1;i} + x^T_{r,k-1;i}\tilde{D}^T_i v^T_{r,k-1;i}v_{r,k-1;i}\tilde{D}_i x_{r,k-1;i}],\\
=& e^T_{k-1;i}D^T_{1;i}Q_{1;i}D_{1;i}e_{k-1;i} + 2e^T_{k-1;i}D^T_{1;i}Q_{1;i}D_{1;i}x_{r,k-1;i} + x^T_{r,k-1;i}D^T_{1;i}Q_{1;i}D_{1;i}x_{r,k-1;i}\\
& + x^T_{r,k-1;i}\tilde{D}^T_i Q_{r;i}\tilde{D}_i x_{r,k-1;i}. \qquad (5.24)
\end{aligned}$$

The above result for $\Sigma_{e_{k;i}}$ is derived with the assumption that the noise of the internal state, $v_{k-1;i}$ and the noise of the reference model, $v_{r,k-1;i}$ are not correlated.

As can be seen from (5.4), the complete description of subsystem i for the purpose of tracking control requires the involvement of the pdf of the external states $y_{k;i}$ as well. Since the pdf of the external states remains unchanged due to the absence of a reference signal, the pdf is still the same as described by (5.15).

5.3.2 Randomised Controller

The derivation of the suboptimal randomised controller for nonlinear subsystems with multiplicative noises defined by the conditional distributions (5.14) - (5.15) considered for a tracking control problem with a stochastic reference model is discussed in this section. The fully probabilistic controller aims to track the desired state trajectory specified by (5.20) by minimising the divergence between the distribution of the tracking error $s(e_{k;i}|u_{k;i}, n_{k-1;i})$ and the pdf of its ideal distribution $s^I(e_{k;i}|u_{k;i}, n_{k-1;i})$. Since the purpose of the control solution is to bring the tracking error to zero, the predefined distribution is assigned to be,

$$s^I(e_{k;i}|u_{k;i}, n_{k-1;i}) \sim \mathcal{N}(0, \Sigma_{2,k;i}), \qquad (5.25)$$

where the mean is zero and the covariance given by $\Sigma_{2,k;i}$ determines the permissible variations of the tracking error around the mean value. The ideal distribution of the controller $c(u_{k;i}|n_{k-1;i})$ is

Gaussian and is given by,

$$c^I(u_{k;i}|n_{k-1;i}) \sim \mathcal{N}(\hat{u}_{k;i}, \Gamma_{k;i}), \tag{5.26}$$

where $\Gamma_{k;i}$ is the covariance of the ideal distribution of the controller and the mean is given by $\hat{u}_{k;i}$. To achieve the control objective of the tracking control problem, and regulate the tracking error around zero, the mean of the ideal distribution of the controller is evaluated as follows,

$$\begin{aligned}
\underbrace{\lim_{k\to\infty}[E\{e_{k;i}\}]}_{=0} = \lim_{k\to\infty}[&\underbrace{E\{f_{1i}(z_{k-1;i})e_{k-1;i}\}}_{=0} + E\{f_{2i}(z_{k-1;i})y_{k-1;i}\} + E\{g_i(z_{k-1;i})u_{k;i}\} \\
&+ E\{Cx_{r,k-1;i}\} + \underbrace{E\{D_{1;i}e_{k-1;i}v_{k-1;i}\}}_{=0} + \underbrace{E\{D_{1;i}x_{r,k-1;i}v_{k-1;i}\}}_{=0} \\
&- \underbrace{E\{\tilde{D}_i x_{r,k-1;i}v_{r,k-1;i}\}}_{=0}], \\
\lim_{k\to\infty}[E\{u_{k;i}\}] =& - g_i^{\dagger}(z_{k-1;i})[f_{2i}(z_{k-1;i})E\{y_{k-1;i}\} + CE\{x_{r,k-1;i}\}] \\
\hat{u}_{k;i} =& - g_i^{\dagger}(z_{k-1;i})\Big[f_{2i}(z_{k-1;i})\hat{y}_{k-1;i} + C\hat{x}_{r,k-1;i}\Big].
\end{aligned} \tag{5.27}$$

The distribution of the randomised controller can now be obtained and is shown in the following theorem.

Theorem 8. Following Definition 5.3.1, the pdf of the tracking error given by (5.22), the ideal distribution of the tracking error provided by (5.25) and the pdf of the ideal controller in (5.26), the suboptimal randomised controller for subsystem i that ensures the internal states of subsystem i follows a predefined desired trajectory and minimises the KLD in (5.5) is given by,

$$c^*(u_{k;i}|n_{k-1;i}) = \mathcal{N}(\mu_{k;i}, \bar{\Gamma}_{k;i}), \tag{5.28}$$

where,

$$\mu_{k;i} = - K_{k;i}n_{k-1;i} - T_{k;i}, \tag{5.29}$$

$$\bar{\Gamma}_{k;i} = \Big(\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i})\tilde{S}_{k;i}g_i(z_{k-1;i})\Big)^{-1}, \tag{5.30}$$

$$K_{k;i} = \bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\Big[\tilde{S}_{k;i}f_{1i}(z_{k-1;i}) \quad \tilde{S}_{k;i}f_{2i}(z_{k-1;i}) + M_{2,k;i}h_i(y_{k-1;i})\Big], \tag{5.31}$$

$$T_{k;i} = \bar{\Gamma}_{k;i}(g_i^T(z_{k-1;i})\tilde{S}_{k;i}Cx_{r,k-1;i} + 0.5g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}), \tag{5.32}$$

$$\tilde{S}_{k;i} = (\Sigma_{2,k;i}^{-1} + M_{1,k;i}). \tag{5.33}$$

The mean of the derived optimal randomised controller is represented by $\mu_{k;i}$ in (5.29) and the covariance $\bar{\Gamma}_{k;i}$ is given by (5.30). The control feedback gain is given by $K_{k;i}$, and $T_{k;i}$ is the linear shift that originated from considering the purpose of the controller, namely tracking control problem.

In addition, the performance index for nonlinear systems with multiplicative stochastic disturbances for a tracking control problem is expressed by the following equation,

$$\ln(\gamma(n_{k;i})) = \frac{1}{2} n_{k;i}^T M_{k;i} n_{k;i} + P_{k;i} n_{k;i} + V_{k;i}, \tag{5.34}$$

where

$$M_{k;i} = \begin{bmatrix} M_{1,k;i} & M_{2,k;i} \\ M_{2,k;i}^T & M_{3,k;i} \end{bmatrix}, \tag{5.35}$$

$$P_{k;i} = \begin{bmatrix} P_{1,k;i} & P_{2,k;i} \end{bmatrix}, \tag{5.36}$$

with,

$$\begin{aligned} M_{1,k-1;i} =& - f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i} g_i(z_{k-1;i})\bar{\Gamma}_{k;i} g_i^T(z_{k-1;i})\tilde{S}_{k;i} f_{1i}(z_{k-1;i}) + f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i} f_{1i}(z_{k-1;i}) \\ & + D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i}, \end{aligned} \tag{5.37}$$

$$\begin{aligned} M_{2,k-1;i} =& \; f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i} f_{2i}(z_{k-1;i}) + f_{1i}^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) \\ & - f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i} g_i(z_{k-1;i})\bar{\Gamma}_{k;i} g_i^T(z_{k-1;i})\tilde{S}_{k;i} f_{2i}(z_{k-1;i}) \\ & - f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i} g_i(z_{k-1;i})\bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}), \end{aligned} \tag{5.38}$$

$$\begin{aligned} M_{3,k-1;i} =& \; f_{2i}^T(z_{k-1;i})\tilde{S}_{k;i} f_{2i}(z_{k-1;i}) + 2 f_{2i}^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) + h_i^T(y_{k-1;i}) M_{3,k;i} h_i(y_{k-1;i}) \\ & - f_{2i}^T(z_{k-1;i})\tilde{S}_{k;i} g_i(z_{k-1;i})\bar{\Gamma}_{k;i} g_i^T(z_{k-1;i})\tilde{S}_{k;i} f_{2i}(z_{k-1;i}) \\ & - h_i^T(y_{k-1;i}) M_{2,k;i} g_i(z_{k-1;i})\bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) \\ & - 2 f_{2i}^T(z_{k-1;i})\tilde{S}_{k;i} g_i(z_{k-1;i})\bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) + D_{2;i}^T M_{3,k;i} Q_{2;i} D_{2;i}, \end{aligned} \tag{5.39}$$

$$\begin{aligned} P_{1,k-1;i} =& \; 2 x_{r,k-1;i}^T C^T \tilde{S}_{k;i} f_{1i}(z_{k-1;i}) + P_{1,k;i} f_{1i}(z_{k-1;i}) + 2 x_{r,k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} \\ & - 2 x_{r,k-1;i}^T C^T \tilde{S}_{k;i} g_i(z_{k-1;i})\bar{\Gamma}_{k;i} g_i^T(z_{k-1;i})\tilde{S}_{k;i} f_{1i}(z_{k-1;i}) \\ & - 2(0.5 g_i^T(z_{k-1;i}) P_{1,k;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i})^T \bar{\Gamma}_{k;i} g_i^T(z_{k-1;i})\tilde{S}_{k;i} f_{1i}(z_{k-1;i}), \end{aligned} \tag{5.40}$$

$$
\begin{aligned}
P_{2,k-1;i} = {} & 2x_{r,k-1;i}^T C^T \tilde{S}_{k;i} f_{2i}(z_{k-1;i}) + 2x_{r,k-1;i}^T C^T M_{2,k;i} h_i(y_{k-1;i}) + P_{1,k;i} f_{2i}(z_{k-1;i}) \\
& + P_{2,k;i} h_i(y_{k-1;i}) - 2x_{r,k-1;i}^T C^T \tilde{S}_{k;i} g_i(z_{k-1;i}) \bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) \tilde{S}_{k;i} f_{2i}(z_{k-1;i}) \\
& - 2(0.5 g_i^T(z_{k-1;i}) P_{1,k;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i})^T \bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) \tilde{S}_{k;i} f_{2i}(z_{k-1;i}) \\
& - 2x_{r,k;i}^T C^T \tilde{S}_{k;i} g_i(z_{k-1;i}) \bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) \\
& - 2(0.5 g_i^T(z_{k-1;i}) P_{1,k;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i})^T \bar{\Gamma}_{k;i} g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}), \qquad (5.41)
\end{aligned}
$$

$$
\begin{aligned}
V_{k-1;i} = {} & V_{k;i} + x_{r,k-1;i}^T C^T \tilde{S}_{k;i} C x_{r,k-1;i} + P_{1,k;i} C x_{r,k-1;i} + x_{r,k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} x_{r,k-1;i} \\
& + x_{r,k-1;i}^T \tilde{D}_i^T M_{1,k;i} Q_{r;i} \tilde{D}_i x_{r,k-1;i} + \hat{u}_{k;i}^T \Gamma_{k;i}^{-1} \hat{u}_{k;i} - x_{r,k-1;i}^T C^T \tilde{S}_{k;i} g_i(z_{k-1;i}) \bar{\Gamma}_{k;i} \\
& \times g_i^T(z_{k-1;i}) \tilde{S}_{k;i} C x_{r,k-1;i} \\
& - 2x_{r,k-1;i}^T C^T \tilde{S}_{k;i} g_i(z_{k-1;i}) \bar{\Gamma}_{k;i} (0.5 g_i^T(z_{k-1;i}) P_{1,k;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i}) \\
& - (0.5 g_i^T(z_{k-1;i}) P_{1,k;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i})^T \bar{\Gamma}_{k;i} (0.5 g_i^T(z_{k-1;i}) P_{1,k;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i}) \\
& + \ln|\Gamma_{k;i}| + \ln|\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i}) \tilde{S}_{k;i} g_i(x_{k-1;i})|, \qquad (5.42)
\end{aligned}
$$

where $M_{1,k;i}$ is the Riccati equation solution of the error $e_{k;i}$, $M_{2,k;i}$ considers the mixed Riccati equation solution of the error $e_{k;i}$ and external variables $y_{k;i}$, $M_{3,k;i}$ is the Riccati equation solution of the external states $y_{k;i}$, $P_{1,k;i}$ is the linear term in the error $e_{k;i}$, $P_{2,k;i}$ is the linear term in the external states $y_{k;i}$ and $V_{k;i}$ is a constant term.

Proof. Appendix G demonstrates the proof of Theorem 8 □

A comparison between the above developed theorem for a tracking control problem and the theorem proposed for a regulation problem (Theorem 7) within the decentralised framework for the FP control design shows that there are many similarities between the two methods. The derived Riccati matrix $M_{k;i}$ in (5.35) and its elements are exactly the same as the Riccati matrix in (4.71) in Theorem 7. However, the difference between the two methods manifests from the additional terms in the solution of the optimal controller in the theorem above. For the tracking control problem, one can realise that the suboptimal control signal consists of a linear shift $T_{k;i}$ which includes the term $P_{1,k;i}$. Hence, to obtain the solution of the probabilistic controller, the Riccati equation solutions $M_{1,k;i}$, $M_{2,k;i}$ and the term $P_{1,k;i}$ needs to be solved. The linear term $P_{1,k;i}$ considers the effect of the reference input $x_{r,k-1;i}$ and the mean of the controller $\hat{u}_{k;i}$ which is vital to achieve the control objective. Again, this method is computationally effective since the computation of the full matrix block $M_{k;i}$ is not required (only $M_{1,k;i}$ and $M_{2,k;i}$) and only the element $P_{1,k;i}$ of the vector $P_{k;i}$ needs to be solved.

It is important to note that the subsystems composing the complex network are required to communicate with their neighbouring subsystems. This is achieved by implementing the probabilistic message passing approach which has been explained in Section 4.5. The messages that are passed to subsystem i are about the internal states of subsystem j, i.e. $x_{k-1;j}$ which node i receives as its external states $y_{k-1;i}$.

5.3.3 Algorithm of the FP Control Solution for a Tracking Control Problem with Stochastic Reference Models

The algorithm of the FP control design that considers multiplicative noises in nonlinear subsystems is summarised in Algorithm 7 as a pseudocode of the key steps.

Algorithm 7 Pseudo-code of randomised controller for tracking problem

1: **procedure** IMPLEMENTATION OF RANDOMISED FP CONTROL DESIGN FOR A TRACKING CONTROL PROBLEM WITH MULTIPLICATIVE NOISES

2: **Initialise:** $x_{0;i}, y_{0;i}, M_{1,0;i}, M_{2,0;i}, P_{1,k;i}, h_i(y_{k-1;i})$ where $i = \{1, ..., N\}$, N being the number of subsystems.

3: **for** $k = 1 \rightarrow H$ **do**

4:     **for** subsystem i:

5:         **Estimate** $f_i(z_{k-1;i})$, $g_i(z_{k-1;i})$, $D_{1;i}$, and $D_{2;i}$.

6:         **Compute** $x_{r,k;i}$.

7:         **Evaluate** $C = f_{1i}(z_{k-1;i}) - \tilde{m}_i(x_{r,k-1;i})$ and $\hat{u}_{k;i}$ in (5.27).

8:         **Calculate** the SS solution of $M_{1;i}$ using (5.37).

9:         $\tilde{S}_i \leftarrow (\Sigma_{2;i}^{-1} + M_{1;i})$

10:         $\bar{\Gamma}_{k;i} \leftarrow \left(\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i})\tilde{S}_i g_i(z_{k-1;i})\right)^{-1,}$

11:         **Use** *Steps 7-10* in (5.38) and (5.40) to find the SS solution of $M_{2;i}$ and $P_{1;i}$.

12:        **Use** *Step 7* and the SS values from *Steps 8-11* in (5.31) - (5.32) to find the SS solutions of K_i and T_i.

13:         **Compute** the error value $e_{k-1;i}$.

14:         **Calculate** $\mu_{k;i}$ using *Steps 12-13* and external states $y_{k-1;i}$ in (5.29).

15:         **Update** the internal states of the subsystems using $\mu_{k;i}$ from *Step 14* .

16:     **end for**

17:     **for** subsystem i:

18:         **Compute** $\mathcal{M}_{j \leftarrow i}(x_{k;i}|z_{k-1;i})$ given by (4.34) using (4.35) and (4.36).

19:         **Update** the prior pdf of the external states using (4.38) - (4.41).

20:         **Update** the external state matrices $h_i(y_{k-1;i})$.

21:     **end for**

22: **end for**

5.3.4 Simulation

This section demonstrates the validity and efficiency of the method proposed for a tracking control problem with a simulation example implemented from [181]. The numerical example consists of a stochastic complex dynamical network which consists of ten identical interacting subsystems. The interaction between the subsystems is represented by the Laplacian matrix, $\mathcal{L}$ which is given in (5.50) below.

In this simulation, the system state equations of the internal and external states are presented according to Equations (5.12) - (5.13) which are repeated here,

$$\begin{aligned} x_{k;i} =& f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + D_{1;i}x_{k-1;i}v_{k-1;i}, \\ y_{k;i} =& h_i(y_{k-1;i})y_{k-1;i} + D_{2;i}y_{k-1;i}\tilde{v}_{k-1;i}, \end{aligned} \tag{5.43}$$

where the internal stochastic noise $v_{k-1;i}$ and external noise $\tilde{v}_{k-1;i}$ have mean zero and variances $Q_{1;i} = 1$ and $Q_{2;i} = 1$, respectively.

The details of the dynamics of the internal state $x_{k;i}$ as well as the dynamics of the external states $y_{k;i}$ of node i are discussed here by formulating (5.43) as Gaussian pdfs. Each subsystem i where $i \in \{1, ..., 10\}$ is described by the nonlinear and Gaussian pdfs given by (5.14) and (5.15) with the parameters,

$$\begin{aligned} \hat{x}_{k;i} =& f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}, \\ \hat{y}_{k;i} =& h_i(y_{k-1;i})y_{k-1;i}, \\ R_{x;i} =& x_{k-1;i}^T D_{1;i}^T Q_{1;i} D_{1;i} x_{k-1;i}, \\ \Sigma_{y;i} =& y_{k-1;i}^T D_{2;i}^T Q_{2;i} D_{2;i} y_{k-1;i}, \end{aligned} \tag{5.44}$$

where $f_{1i}(z_{k-1;i}) = a(z_{k-1;i}) + \mathcal{L}_{ii} I_{2\times 2}$ with,

$$a(z_{k-1;i}) = \begin{bmatrix} -0.5 & 0.25 + \dfrac{1}{x_{2,k-1;i}} \tanh(0.05x_{1,k-1;i}) \\ 0 & 0.85 - \dfrac{1}{x_{2,k-1;i}} \tanh(0.05x_{1,k-1;i} + 0.05x_{2,k-1;i}) \end{bmatrix}, \tag{5.45}$$

and I being the identity matrix. Also, $f_{2i}(z_{k-1;i})y_{k-1;i} = \sum_{j \in \mathrm{N}_i, j \neq i} c_{ij} x_{k-1;j}$ where c_{ij} is the inner-coupling matrix and is given by $c_{ij} = \mathcal{L}_{ij} I_{2\times 2}$ as explained in Chapter 4. To re-emphasise, the elements of matrix $f_{2i}(z_{k-1;i})$ are given by c_{ij}, meaning $f_{2i}(z_{k-1;i}) = [c_{ij}]_{j \in \mathrm{N}_i, j \neq i}$. The vector $y_{k-1;i}$ consists of elements that equate to the states of the neighbouring nodes $x_{k-1;j}$, in other words, $y_{k-1;i} = [x_{k-1;j}^T]_{j \in \mathrm{N}_i, j \neq i}^T$.

Furthermore, the control matrix $g_i(z_{k-1;i})$ is given by,

$$g_i(z_{k-1;i}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.46}$$

For this simulation, the matrices $D_{1;i}$ and $D_{2;i}$ are chosen to be $0.3I_{2\times 2}$. Moreover, the initial state values for the simulation example are taken to be,

$$x_{0;i} = \begin{bmatrix} 5 & -2.4 & 6.5 & -7 & 11.7 & -3.8 & 10.4 & -2.5 & 16.1 & 4.5 \\ -5 & 2.6 & 4 & -1.8 & -25.2 & 3.4 & -22.4 & 14.5 & -19.6 & 2 \end{bmatrix}. \tag{5.47}$$

The parameters of the distributions given by (5.44) and the matrices $D_{1;i}$ and $D_{2;i}$ are unknown and are therefore required to be estimated. This process has been outlined in Section 3.5.1. The external matrix $h_i(y_{k-1;i})$ is initialised randomly at the start and then updated accordingly using the approach outlined in Section 4.5. Since ten subsystems are required to be controlled such that they follow the corresponding desired state trajectories, ten randomised local controllers need to be designed. The dynamics of the reference model are given by,

$$x_{r,k-1;i} = \tilde{m}_i(x_{r,k-1;i})x_{r,k-1;i} + \tilde{D}_i x_{r,k-1;i} v_{r,k-1;i}, \tag{5.48}$$

where,

$$\tilde{m}_i(x_{r,k-1;i}) = \begin{bmatrix} -0.5 & 0.25 + \dfrac{1}{x_{2r,k-1;i}} \tanh(0.05x_{1r,k-1;i}) \\ 0 & 0.85 - \dfrac{1}{x_{2r,k-1;i}} \tanh(0.05x_{1,k-1;i} + 0.05x_{2r,k-1;i}) \end{bmatrix}. \tag{5.49}$$

The initial value of the desired state is given by $x_{r,0;i} = \begin{bmatrix} 2 & -2 \end{bmatrix}$. The noise intensity of $v_{r,k-1;i}$ is described by variance $Q_{r;i} = 1$ where $\tilde{D}_i$ is chosen to be $0.25\mathcal{I}_{2\times 2}$. More importantly, the dynamics of the reference model given in (5.48) characterises the desired states for all ten subsystems. Finally, the Laplacian that describes the connection between the subsystems is given by,

$$\mathcal{L} = \begin{bmatrix} -0.7 & 0.2 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & -0.6 & 0 & 0.25 & 0.25 & 0.05 & 0.05 & 0 & 0 & 0 \\ 0 & 0.1 & -0.35 & 0 & 0 & 0.1 & 0 & 0.1 & 0 & 0.05 \\ 0 & 0.1 & 0 & -0.2 & 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0.05 & 0 & 0 & 0 & -0.25 & 0.1 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.05 & -0.1 & 0 & 0.05 & 0 & 0 \\ 0 & 0.1 & 0 & 0.5 & 0 & 0 & -0.2 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.15 & 0.05 & 0 & 0 & -0.35 & 0 & 0.15 \\ 0 & 0 & 0 & 0.05 & 0.05 & 0 & 0 & 0 & -0.2 & 0 \\ 0.05 & 0.1 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & -0.25 \end{bmatrix}. \tag{5.50}$$

To re-emphasise, the local controllers are derived by shifting the focus on the dynamics of the tracking error which is achieved from the difference between the internal states of node i, $x_{k;i}$ and the reference model $x_{r,k;i}$. As discussed before, the parameters of the conditional distribution of the error terms $e_{k;i}$ are obtained by using the knowledge that is available about the internal dynamics $x_{k;i}$ and the reference model $x_{r,k;i}$. It is then possible to obtain the conditional distribution of the tracking errors $e_{k;i}$ for each subsystem i according to the equations given by (5.22) - (5.24). The ideal covariance matrix of the error is taken to be $\Sigma_{2,k;i} = 0.01 \times \begin{bmatrix} 2.1 & 0 \\ 0 & 1.1 \end{bmatrix}$ for all ten subsystems.

The ideal covariance $\Sigma_{2,k;i}$ is allowed to be smaller than the actual covariance $\Sigma_{e_{k;i}}$ due to the dependency of the actual covariance on the error values which are expected to converge to zero. In addition, the covariance of the controller $\Gamma_{k;i}$ for all nodes is chosen to be $10\mathcal{I}_{2\times2}$. Since communication between the nodes is of paramount importance due to the connections between them, probabilistic message passing is implemented to achieve this (Section 4.5). Given all the information, the randomised controllers were computed using (5.28) for which the results are displayed in Figures 5.1 - 5.2.

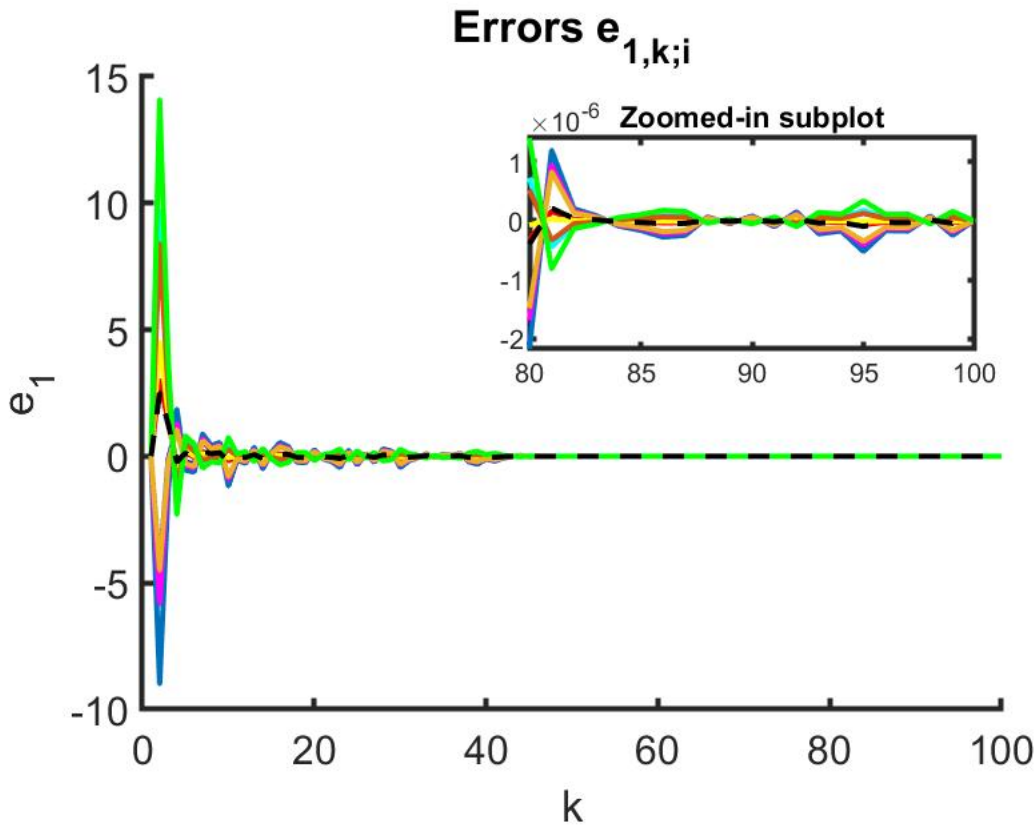

Figure 5.1: The tracking error $e_{1,k;i}$ for all subsystems. The errors closely oscillates around zero.

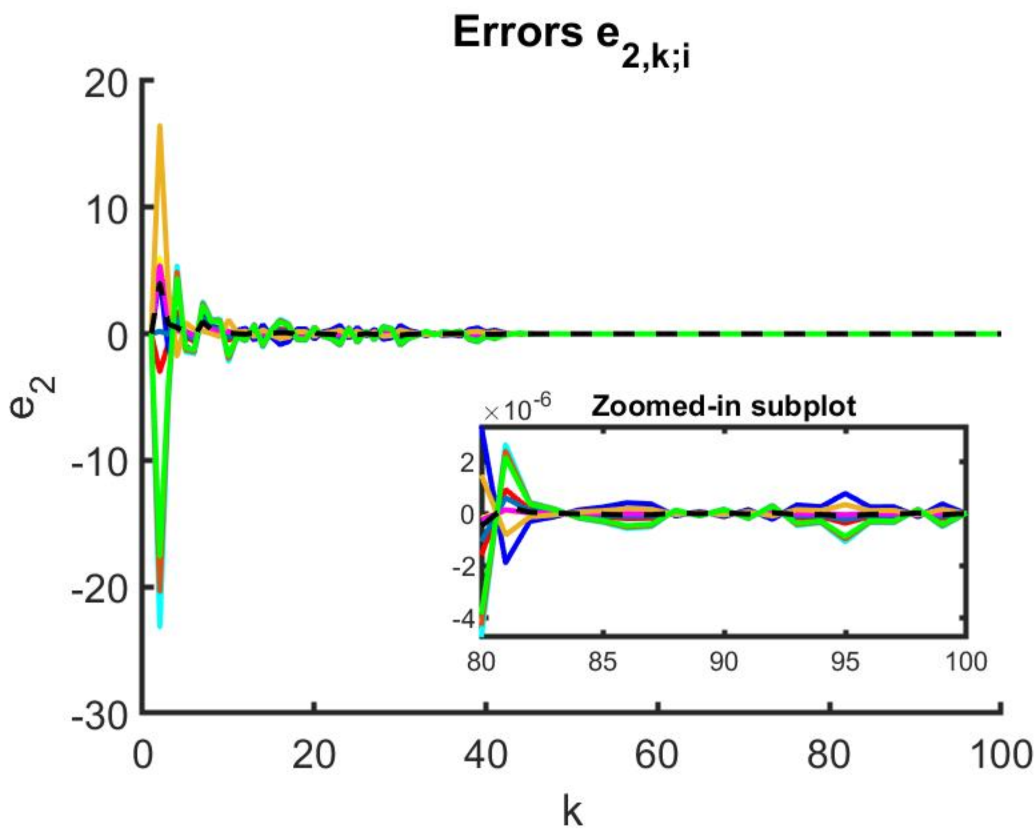

Figure 5.2: The tracking error $e_{2,k;i}$ for all subsystems. The errors closely oscillate around zero.

The plots in Fig 5.1 - 5.2 demonstrate the suboptimal controller influencing the pdf of the error distribution $s(e_{k;i}|u_{k;i}, n_{k-1;i})$ to bring it closer to its ideal pdf and hence reducing the Kullback-Leibler divergence between them. As can be seen, the tracking error has successfully converged to and is oscillating around zero which is the mean value of the ideal pdf of the tracking error. As a result, the internal states of all subsystems have followed and achieved the predefined desired state value, i.e. $x_{i;k} - x_{r,k;i} = e_{k;i} = 0 \rightarrow x_{i;k} = x_{r,k;i}$. Therefore, the results have verified that the FP control design can be successfully implemented for large-scale complex nonlinear systems that require to follow a predefined state reference stochastic model. Its efficiency can be witnessed from the fast convergence of the tracking errors $e_{k;i}$ to zero.

5.4 Formation Control

There are numerous objectives of the implementation of control methods of which one is formation control. Many controllers are required to control multi-agent systems in such a way that they form a certain shape with their states. The control objective of formation has not been considered for the FP control approach for neither linear nor nonlinear systems. This section introduces the concept of formation within the fully probabilistic framework. It is self-evident that formation control involves the control of a number of subsystems that interact with each other. Hence, the approach considered involves decentralised control systems. It is important, however, to highlight that formation control problems consists of multi-agent systems where the coupling between the subsystems does not exist.

5.4.1 Problem Formulation

Following the discussion in Sections 5.2 and 5.3, it is known that the pdf of the system state dynamics need to be considered when the control objective consists of a tracking problem. However, for formation control, the dynamics of subsystem i are not affected by the dynamics of its neighbouring subsystems. In other words, the coupling between the agents does not exist. The multi-agent systems are decomposed using the decentralised framework discussed in Chapter 4 which consists of the concept of internal and external states. The system state equation of stochastic subsystem i is given by,

$$z_{k;i} = \bar{F}_i(z_{k-1;i})z_{k-1;i} + \tilde{g}_i(z_{k-1;i})u_{k;i} + \epsilon_{i;k}, \tag{5.51}$$

where the state matrix of state $z_{k;i}$ is given by $\bar{F}_i(z_{k-1;i})$, the control matrix is described by $\tilde{g}_i(z_{k-1;i})$ and the noise is given by $\epsilon_{i;k}$ which has mean zero and covariance Q_i. The difference between equation (5.51) and equation (4.10) can be realised once (5.51) is written in matrix-vector form,

$$\underbrace{\begin{bmatrix} x_{k;i} \\ y_{k;i} \end{bmatrix}}_{z_{k;i}} = \underbrace{\begin{bmatrix} \bar{f}_{1i}(x_{k-1;i}) & 0 \\ 0 & \bar{h}_i(y_{k-1;i}) \end{bmatrix}}_{= \tilde{F}(z_{k-1;i})} \underbrace{\begin{bmatrix} x_{k-1;i} \\ y_{k-1;i} \end{bmatrix}}_{z_{k-1;i}} + \underbrace{\begin{bmatrix} \bar{g}_i(x_{k-1;i}) \\ 0 \end{bmatrix}}_{\tilde{g}_i(z_{k-1;i})} u_{k;i} + \underbrace{\begin{bmatrix} \epsilon_{1k;i} \\ \epsilon_{2k;i} \end{bmatrix}}_{\epsilon_{k;i}}. \tag{5.52}$$

The state matrix $\bar{f}_{1i}(x_{k-1;i})$ of the internal state $x_{k;i}$ is not anymore a nonlinear function of the previous internal and external states but is now only dependent on the previous internal states $x_{k-1;i}$. Following equation (5.52), the nonlinear dynamics of the internal, $x_{k;i}$ and external, $y_{k;i}$ states for subsystem i with additive noises, where no coupling exists between the subsystems, can be written explicitly as follows,

$$x_{k;i} = f_{1i}(x_{k-1;i})x_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \epsilon_{1k;i}, \tag{5.53}$$

$$y_{k;i} = h_i(y_{k-1;i})y_{k-1;i} + \epsilon_{2k;i}. \tag{5.54}$$

The noises $\epsilon_{1k;i}$ and $\epsilon_{2k;i}$ have zero mean and covariances $Q_{1;i}$ and $Q_{2;i}$, respectively. In addition, comparing the description of the dynamics of the internal states $x_{k;i}$ of subsystem i with the dynamics of the internal states discussed in Section 4.4.1, one can notice that the matrix $f_{2i}(.)$ is equal to zero in (5.53) in the case of formation control. This emphasises the fact that there is no coupling between the agents, meaning neighbouring subsystems of node i do not influence the dynamics of the internal states. However, communication with the neighbouring subsystems is of great significance in order to form the desired formation. Hence, the probabilistic message passing approach is implemented such that node i is able to obtain knowledge about the states of the neighbouring subsystems.

Since the dynamics of internal and external states are unknown, the parameters of subsystem i are estimated online using mlp neural networks following the approach outlined in Section 3.3.2. It is then possible to specify the conditional Gaussian distribution $s(z_{k;i}|u_{k;i}, z_{k-1;i})$ as follows,

$$s(z_{k;i}|u_{k;i}, z_{k-1;i}) \sim s(x_{k;i}|u_{k;i}, x_{k-1;i})s(y_{k;i}|y_{k-1;i}), \tag{5.55}$$

where $s(x_{k;i}|u_{k;i}, x_{k-1;i})$ and $s(y_{k;i}|y_{k-1;i})$ are the Gaussian pdfs of the internal $x_{k;i}$ and external states $y_{k;i}$. The internal state of node i has mean $\hat{x}_{k;i} = f_{1i}(x_{k-1;i})x_{k-1;i} + g_i(x_{k-1;i})u_{k;i}$ and covariance matrix $\Sigma_{x;i}$. The external state has mean value $\hat{y}_{k;i} = h_i(y_{k-1;i})y_{k-1;i}$ and covariance $\Sigma_{y;i}$.

However, formation control requires the subsystems to form a shape which is achieved by following instructions set by the reference model. Note that each agent may track a different reference model since the reference will be determined by the states of its neighbouring subsystems, and thus the control objective would be specific to each individual node. Consequently, the reference signal subsystem i is described as follows,

$$x_{r,k;i} = \tilde{m}_i(x_{k;j}, r_{k;i}), \tag{5.56}$$

where the function $\tilde{m}_i(x_{k;j}, r_{k;i})$ emphasises that the desired reference signal of subsystem i is determined by the states of the neighbouring agents $x_{k;j}$ and an external signal, $r_{k;i}$ that can be specified by each individual agent i according to the control objective it wants to achieve. The states $x_{k;j}$ of the neighbouring agents of agent i enter subsystem i through the defined reference signal in equation (5.56) and are consequently registered as the external states $y_{k;i}$ of node i.

To re-emphasise, for the tracking of a reference state signal, the main focus is on the tracking error between the internal states and the reference signal. Therefore, it is key to determine the pdf of the tracking error. From the reference signal, the tracking error of the system can be obtained as defined by Equation (5.2) such that,

$$\begin{aligned} e_{k;i} =& f_{1i}(x_{k-1;i})x_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \epsilon_{1k;i} - x_{r,k;i} \\ =& f_{1i}(x_{k-1;i})e_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \tilde{x}_{r,k;i} + \epsilon_{1k;i}, \end{aligned} \tag{5.57}$$

where the definition $\tilde{x}_{r,k;i} = (f_{1i}(x_{k-1;i})x_{r,k-1;i} - \tilde{m}_i(x_{k;j}, r_{k;i}))$ has been presented.

The pdf of the tracking error between the internal state of node i and the reference state value can now be determined. The conditional distribution is Gaussian with $\hat{e}_{k;i}$ and $\Sigma_{e_{k;i}}$ as the mean and covariance respectively such that,

$$s(e_{k;i}|u_{k;i}, e_{k-1;i}) \sim \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}), \tag{5.58}$$

where,

$$\hat{e}_{k;i} = f_{1i}(x_{k-1;i})e_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \tilde{x}_{r,k;i}, \tag{5.59}$$

and the covariance matrix $\Sigma_{e_{k;i}}$ is the same as the global covariance matrix $\Sigma_{x;i}$.

5.4.2 Fully Probabilistic Design for Formation Control

This section discusses the control solution of the formation control problem within a decentralised fully probabilistic framework. The controller is required to reshape the pdf of the tracking error between the internal state of subsystem i and its corresponding reference signal. The aim is to shape it such that the tracking error converges to zero, meaning the internal state of node i converges to its corresponding reference state trajectory. Hence, the ideal pdf of subsystem i composed of the tracking error $e_{k;i}$ and the external states $y_{k;i}$ need to be determined which gives,

$$s^I(n_{k;i}|u_{k;i}, n_{k-1;i}) \sim s^I(e_{k;i}|u_{k;i}, e_{k-1;i})s^I(y_{k;i}|y_{k-1;i}) \sim \mathcal{N}(0, \Sigma_{e_{k;i}})\mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}). \tag{5.60}$$

The ideal distribution of the external state $y_{k;i}$ is exactly the same as the actual distribution as we have no intention of controlling it.

Moreover, another requirement is the specification of the ideal controller,

$$c^I(u_{k;i}|e_{k-1;i}) \sim \mathcal{N}(\hat{u}_{k;i}, \Gamma_{k;i}), \tag{5.61}$$

where the permissible range of control inputs is given by the covariance $\Gamma_{k;i}$ and the mean value $\hat{u}_{k;i}$ is described by,

$$\hat{u}_{k;i} = -\, g_i^{\dagger}(x_{k-1;i})\tilde{x}_{r,k;i}. \tag{5.62}$$

The formulation of the randomised suboptimal controller for a formation control problem is given in the following theorem.

Theorem 9. From Definition 5.3.1, the pdfs of the tracking error and external states given by (5.58) and (5.55), their ideal distribution defined by (5.60) and the pdf of the ideal controller in (5.61), the suboptimal randomised controller for subsystem i that minimises the KLD in (5.5) with the objective of formation control is given by,

$$c^*(u_{k;i}|e_{k-1;i}) = \mathcal{N}(\mu_{k;i}, \bar{\Gamma}_{k;i}), \tag{5.63}$$

where,

$$\mu_{k;i} = -\, K_{k;i}n_{k-1;i} - T_{k;i}, \tag{5.64}$$

$$\bar{\Gamma}_{k;i} = \left(\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})\right)^{-1}, \tag{5.65}$$

$$K_{k;i} = \bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})\begin{bmatrix}\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) & M_{2,k;i}h_i(y_{k-1;i})\end{bmatrix}, \tag{5.66}$$

$$T_{k;i} = \bar{\Gamma}_{k;i}(g_i^T(x_{k-1;i})\tilde{S}_{k;i}\tilde{x}_{r,k;i} + 0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}), \tag{5.67}$$

$$\tilde{S}_{k;i} = (\Sigma_{e_{k;i}}^{-1} + M_{1,k;i}). \tag{5.68}$$

The designed randomised controller with the formation control objective is represented by $\mu_{k;i}$ in (5.64) and the covariance $\bar{\Gamma}_{k;i}$ is given by (5.65). The elements $K_{k;i}$ is the control feedback gain and $T_{k;i}$ is the linear shift that ensures the tracking of the reference signal is achieved by the internal state of node i.

Furthermore, the performance index for complex nonlinear subsystems that require the tracking control of a predefined state trajectory is defined by,

$$\ln(\gamma(n_{k;i})) = \frac{1}{2} n_{k;i}^T M_{k;i} n_{k;i} + P_{k;i} n_{k;i} + V_{k;i}, \tag{5.69}$$

where,

$$M_{k;i} = \begin{bmatrix} M_{1,k;i} & M_{2,k;i} \\ M_{2,k;i}^T & M_{3,k;i} \end{bmatrix}, \tag{5.70}$$

$$P_{k;i} = \begin{bmatrix} P_{1,k;i} & P_{2,k;i} \end{bmatrix}, \tag{5.71}$$

with,

$$
\begin{aligned}
M_{1,k-1;i} = & -f_{1i}^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) \\
& + f_{1i}^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i}), && (5.72) \\
M_{2,k-1;i} = & -f_{1i}^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\
& + f_{1i}^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}), && (5.73) \\
M_{3,k-1;i} = & -h_i^T(y_{k-1;i})M_{2,k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\
& + h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i}), && (5.74) \\
P_{1,k-1;i} = & \ 2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) - 2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) \\
& - 2(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) \\
& + P_{1,k;i}f_{1i}(x_{k-1;i}), && (5.75) \\
P_{2,k-1;i} = & \ 2\tilde{x}_{r,k;i}^T M_{2,k;i}h_i(y_{k-1;i}) + P_{2,k;i}h_i(y_{k-1;i}) \\
& - 2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\
& - 2(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}), \\
V_{k-1;i} = & \ V_{k;i} + \tilde{x}_{r,k;i}^T\tilde{S}_{k;i}\tilde{x}_{r,k;i} + P_{1,k;i}\tilde{x}_{r,k;i} + \hat{u}_{k;i}^T\Gamma_{k;i}^{-1}\hat{u}_{k;i} + \text{tr}(M_{1,k;i}\Sigma_{e_{k;i}}) \\
& + \text{tr}(M_{3,k;i}\Sigma_{y;i}) - \tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})\tilde{S}_{k;i}\tilde{x}_{r,k;i} \\
& - 2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}) \\
& - (0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}) \\
& + \ln|\Gamma_{k;i}| + \ln|\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})|, && (5.76)
\end{aligned}
$$

where $M_{1,k;i}$ is the Riccati equation solution of the error $e_{k;i}$, $M_{2,k;i}$ is the mixed Riccati equation solution of the error $e_{k;i}$ and external variables $y_{k;i}$, $M_{3,k;i}$ is the Riccati equation solution of the external states $y_{k;i}$, $P_{1,k;i}$ is the linear term in the error $e_{k;i}$, $P_{2,k;i}$ is the linear term in the external states $y_{k;i}$ and $V_{k;i}$ is a constant term.

The implementation of the suboptimal controller in (5.63) requires the solutions of $M_{1,k;i}$, $M_{2,k;i}$ and $P_{1,k;i}$ described by (5.72), (5.73), and (5.75), respectively. Hence, the full block matrix $M_{k;i}$ and $P_{k;i}$ are not required to be computed resulting in a reduction in computational expenses.

5.4.3 Algorithm of the FP Control Solution for a Formation Control Problem

The algorithm of the FP control design for a formation control problem for nonlinear subsystems with additive noises is summarised in Algorithm 8.

Algorithm 8 Pseudo-code of randomised controller for formation control problem

1: **procedure** IMPLEMENTATION OF RANDOMISED FP CONTROL DESIGN FOR A TRACKING CONTROL PROBLEM WITH MULTIPLICATIVE NOISES
2: **Initialise:** $x_{0;i}, y_{0;i}, M_{1,0;i}, M_{2,0;i}, P_{1,k;i}, h_i(y_{k-1;i})$ where $i = \{1, ..., N\}$, N being the number of subsystems.
3: **for** $k = 1 \rightarrow H$ **do**
4: **for** subsystem i:
5: **Estimate** $f_i(x_{k-1;i})$, $g_i(x_{k-1;i})$.
6: **Compute** $x_{r,k;i}$.
7: **Evaluate** $\tilde{x}_{r,k;i}$ and the mean of the ideal controller, $\hat{u}_{k;i}$ using (5.62).
8: **Calculate** the SS solution of $M_{1;i}$ using (5.72).
9: $\tilde{S}_i \leftarrow (\Sigma_{e_{k;i}}^{-1} + M_{1;i})$
10: $\bar{\Gamma}_{k;i} \leftarrow \left(\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i})\tilde{S}_i g_i(x_{k-1;i})\right)^{-1,}$
11: **Use** *Steps 7-10* in (5.73) and (5.75) to find the SS solution of $M_{2;i}$ and $P_{1;i}$.
12: **Use** *Step 7* and the SS values from *Steps 8-11* in (5.66) - (5.67) to find the SS solutions of K_i and T_i.
13: **Compute** the error value $e_{k-1;i}$.
14: **Calculate** $\mu_{k;i}$ using *Steps 12-13* and external states $y_{k-1;i}$ in (5.64).
15: **Update** the internal states of the subsystems using $\mu_{k;i}$ from *Step 14* .
16: **end for**
17: **for** subsystem i:
18: **Compute** $\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i})$ given by (4.34) using (4.35) and (4.36)
19: **Update** the prior pdf of the external states using (4.38) - (4.41).
20: **Update** the external state matrices $h_i(y_{k-1;i})$.
21: **end for**
22: **end for**

5.4.4 Simulation One: Formation Control for Linear Systems

This section aims to demonstrate the designed FP control solution for a formation control problem on a kinematic problem which consists of N robots that are moving in a plane with the position of each robot given by $x_{k;i} = [x_{1,k;i} \quad x_{2,k;i}]^T$, where $x_{1,k;i}$ and $x_{2,k;i}$ are the positions in the x- and y-axis, respectively, with $i \in \{1, ..., 5\}$. This simulation example is taken from [182] of which the dynamics are linear. Although, the derivation of the optimal randomised controller in Section 5.4 has been demonstrated for nonlinear systems, this can be implemented for linear systems too. To achieve this, the following substitutions can be made to emphasise that the matrices, defined in (5.77) below, are constant with no dependency on the states $x_{k-1;i}$ and $y_{k-1;i}$: $A_{11;i} = f_{1i}(x_{k-1;i})$, $B_i = g_i(x_{k-1;i})$ and $A_{22;i} = h_i(y_{k-1;i})$. Also, in [182], the model is deterministic which is not realistic since stochastic disturbances exist in real-world control problems. Hence, an additive noise has been added to the model given in equation (5.77) for a more realistic picture.

Thus, the kinematic model of each agent is characterised by,

$$x_{k;i} = A_{11;i}x_{k-1;i} + B_i u_{k;i} + \epsilon_{1,k;i}, \tag{5.77}$$

where $A_{11;i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_i = 0.1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $u_{k;i} = [u_{1,k;i} \quad u_{2,k;i}]^T$. The noise $\epsilon_{1,k;i}$ has zero mean and covariance $Q_{1;i} = 0.001\mathcal{I}_{2\times2}$. The equation given in (5.77) describes the internal states of the robots. The dynamics are identical for all five agents/robots. As mentioned previously, it can be observed that the internal dynamics are solely dependent on its internal states for the formation control problem. This is emphasised by having no coupling between the robots. However, the robots are still required to interact with one another as their desired position depends on the position of other neighbouring robots. Denote the neighbouring agents of robot i as N_i and the desired distance of robot i from robot j in a specific formation is described by $r_{ji} = [r_{1,ji} \quad r_{2,ji}]^T$ where $j \in \mathrm{N}_i$. Hence, the reference model for this formation control problem which describes the desired position of each robot i is specified by,

$$x_{r,k;i} = \frac{1}{n_i} \sum_{j \in \mathrm{N}_i, i \neq j} (x_{k;j} + r_{ji}), \tag{5.78}$$

where n_i is the cardinality of N_i, meaning the number of neighbours of agent i. The control objective that robot i wants to achieve is specified by the distance that robot i is expected to be from robot j, where $j \in \mathrm{N}_i$. The explanation in Section 5.4.1 can now be understood better by considering (5.78) in this simulation example which shows that the states $x_{k;j}$ of the neighbouring robots of robot i enter subsystem i through the defined reference signal in equation (5.78) and are then registered as the external states $y_{k;i}$ of node i. Figure 5.3 shows the desired formation of the five robots and the interactions amongst them.

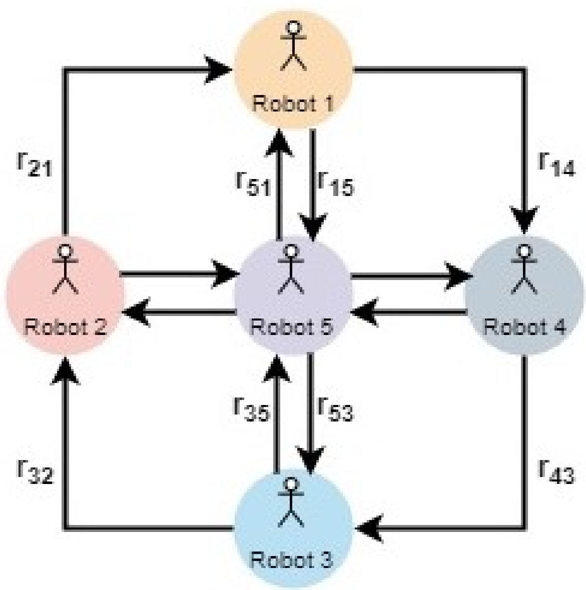

Figure 5.3: This diagram represent the connections between the robots and the desired formation that the controller needs to achieve. The distances from robot i to its neighbouring robots are given by r_{ji}. It is expected for robot 5 to be surrounded by the other robots.

The vectors of the desired distances from agent i is given by,

$$\boxed{\begin{array}{cccc} r_{14}=[5\ \ -5]^T, & r_{43}=[-5\ \ -5]^T, & r_{32}=[-5\ \ 5]^T, & r_{21}=[5\ \ 5]^T, \\ r_{51}=[0\ \ 5]^T, & r_{52}=[-5\ \ 0]^T, & r_{53}=[0\ \ -5]^T, & r_{54}=[5\ \ 0]^T, \\ r_{15}=-r_{15}, & r_{25}=-r_{52}, & r_{35}=-r_{53}, & r_{45}=-r_{54}. \end{array}}$$

From Figure 5.3, it is apparent which robots interact with one another, which is again specified below for clarification,

$$\begin{aligned} z_{k;1} =& [x_{k;1}\ \ y_{k;2}\ \ y_{k;5}]^T, \\ z_{k;2} =& [x_{k;2}\ \ y_{k;3}\ \ y_{k;5}]^T, \\ z_{k;3} =& [x_{k;3}\ \ y_{k;4}\ \ y_{k;5}]^T, \\ z_{k;4} =& [x_{k;4}\ \ y_{k;1}\ \ y_{k;5}]^T, \\ z_{k;5} =& [x_{k;5}\ \ y_{k;1}\ \ y_{k;2}\ \ y_{k;3}\ \ y_{k;4}]^T. \end{aligned} \tag{5.79}$$

In Equation (5.77), only the dynamics of the internal states of the robots have been described. Since the concept of internal and external states exist in the discussed method, it is also important to take account of the dynamics of the external states which are given by $y_{k;i} = A_{22;i}y_{k;i}$, where the external state matrix $A_{22;i}$ is initially randomly generated and then updated according to the probabilistic message passing approach explained in Section 4.5. The initial positions of the robots are given by,

$$x_{0;1} = [6\ \ -1]^T,\ \ x_{0;2} = [4\ \ 4]^T,\ \ x_{0;3} = [0\ \ 3]^T,\ \ x_{0;4} = [0\ \ 0]^T,\ \ x_{0;5} = [0\ \ 6]^T.$$

The developed fully probabilistic control method for a formation control problem focuses on the $e_{k;i}$ which is obtained from $e_{k;i} = x_{k;i} - x_{r,k;i}$.

To compliment the stochasticity of the error variable, its conditional distribution needs to be determined as described by (5.58). Furthermore, the actual and ideal covariance matrices are the same where the actual global covariance matrix, $\Sigma_{e_{k;i}}$ can be estimated as discussed previously. In addition, the ideal covariance of the controller is $\Gamma_{k;i} = 10I_{2\times 2}$.

The validity and efficiency of the proposed method for formation control problems is illustrated in the plots given by figures 5.4 - 5.6. The figures 5.4 and 5.5 show the error plots of the tracking errors $e_{k;i}$ between the state values $x_{k;i}$ and the reference models $x_{r,k;i}$ of the five robots. It can be seen that the error plots converge to and oscillate around zero very quickly which means that the desired state values are achieved at a very early stage as shown in Figure 5.6. The initial positions of the robots are given by the crosses and the final positioning of the robots are given by the circles. As expected, the final formation demonstrates that the randomised local controllers managed to influence the positions of the robots and obtain a formation where robot 5 is surrounded by the other four robots.

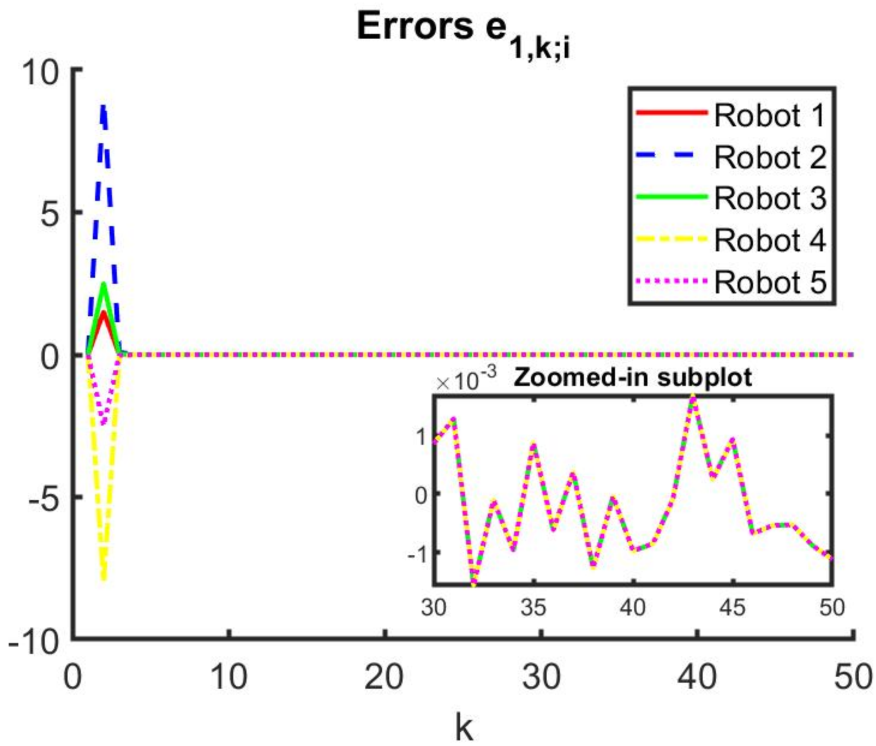

Figure 5.4: The tracking error $e_{1,k;i}$ for all subsystems. The zoomed-in plot demonstrates that the tracking errors oscillate around zero due to the presence of noise.

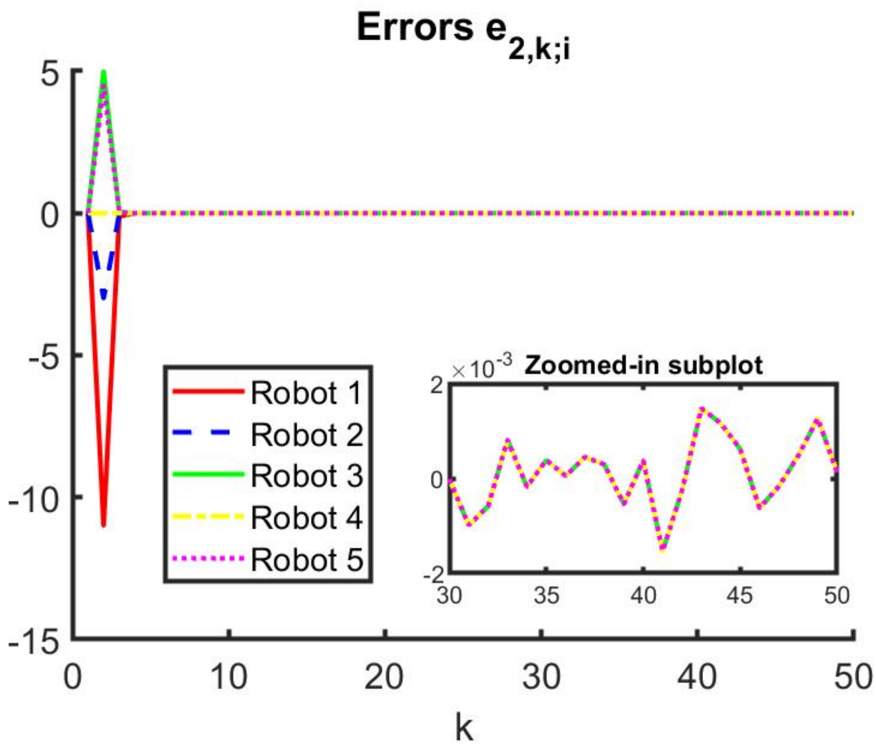

Figure 5.5: The tracking error $e_{2,k;i}$ for all subsystems. The zoomed-in plot demonstrates that the tracking errors oscillate around zero due to the presence of noise.

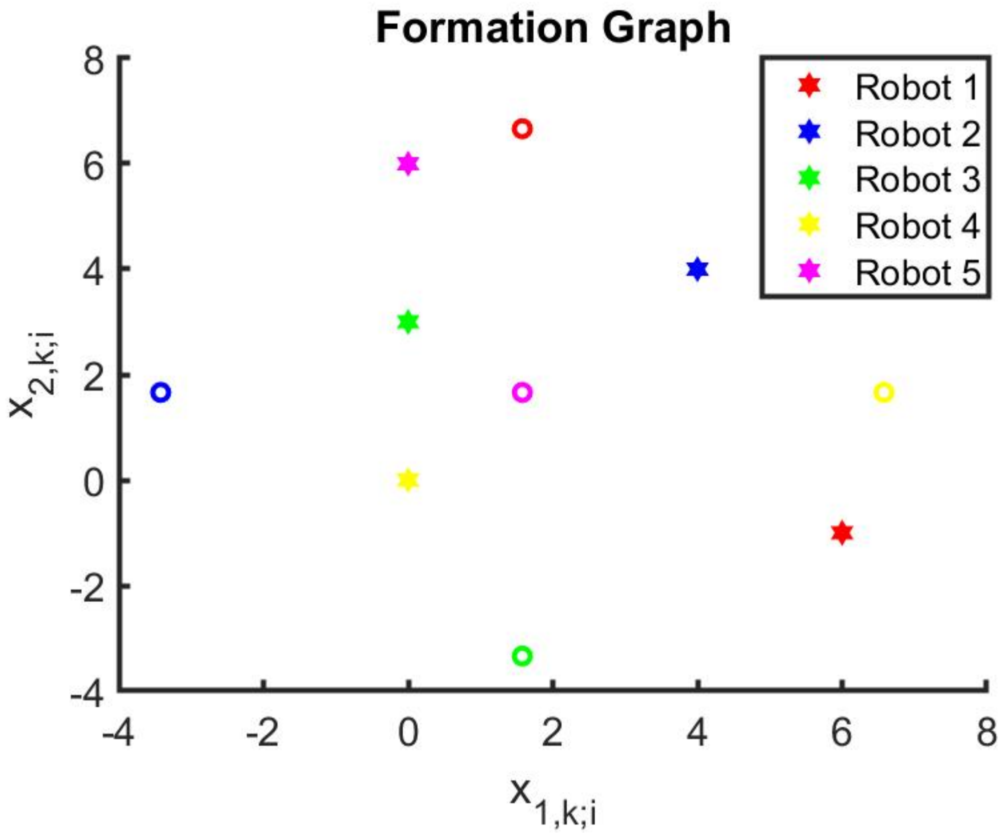

Figure 5.6: The internal states $x_{1,k;i}$ of all subsystems where the dashed black line is the desired trajectory that the states of the nodes need to follow.

5.4.5 Simulation Two: Formation Control for Nonlinear Systems

The fully probabilistic control design derived in Section 5.4 is also demonstrated on a numerical formation control problem for nonlinear stochastic systems. The model is motivated from [183] to which additive noise has been added such that the control problem resembles real-world systems where noises exist. In this example, the pfd of a nonlinear stochastic system is influenced such that the final positions of the agents involved forms a triangle. There are a total of three agents involved for which the parameters of the internal and external conditional distributions, $\mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i})$ and $\mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i})$ respectively, as shown in (5.55), are identical and their mean values are described by,

$$\hat{x}_{k;i} = f_{1i}(x_{k-1;i})x_{k-1;i} + g_i(x_{k-1;i})u_{k;i}, \tag{5.80}$$

where,

$$f_{1i}(x_{k-1;i}) = \begin{bmatrix} 0 & 1 \\ 0 & \dfrac{1}{1+x_{1,k-1;i}^2} \end{bmatrix}, \quad g_i(z_{k-1;i}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{5.81}$$

and,

$$\hat{y}_{k;i} = h_i(y_{k-1;i})y_{k-1;i}, \tag{5.82}$$

where $h_i(y_{k-1;i})$ is initialised at the beginning and updated according to the probabilistic message passing approach as discussed in Section 4.5. It can be seen that the computation of the internal states $x_{k;i}$ only depend on the previous internal state value $x_{k-1;i}$ and not the external states $y_{k-1;i}$, since $f_{2i}(z_{k-1;i}) = 0$. However, each node i still requires communication with its neighbouring subsystems since the desired reference models need to have access to the knowledge of the positions of the neighbouring subsystems. The desired formation and the interaction is illustrated in Figure 5.7.

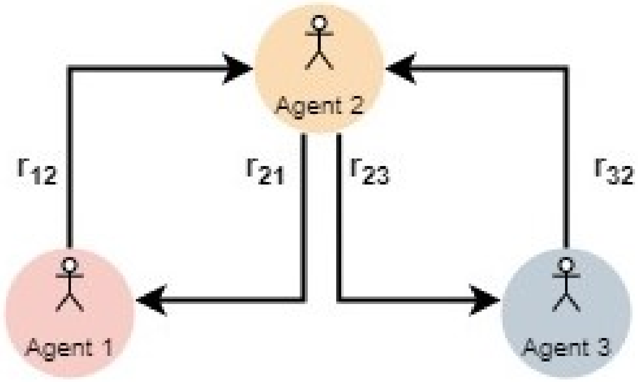

Figure 5.7: This figure represent the desired formation and connections between the agents. The distances from agent i to its neighbouring robots are given by r_{ji}. It is expected to form an equilateral triangle with agent 2 at the top.

The set of neighbouring agents of node i is denoted as N_i and the reference model specifies the desired distance $r_{ji} = [r_{1,ji} \;\; r_{2,ji}]^T$ between robot i to robot j where $j \in \mathrm{N}_i$. The reference model for this formation is then given by,

$$x_{r,k;i} = \frac{1}{n_i} \sum_{j \in \mathrm{N}_i, i \neq j} (x_{k;j} + r_{ji}), \tag{5.83}$$

where n_i is the cardinality of N_i. The desired distances r_{ji} are given by,

$$\boxed{r_{12} = [0.5 \;\; 0.5]^T, \quad r_{32} = [-0.5 \;\; 0.5]^T, \quad r_{21} = -r_{12} \quad r_{23} = -r_{32}.}$$

The system state dynamics of the agents are described as follows,

$$\begin{aligned} z_{k;1} &= [x_{k;1} \;\; y_{k;2}]^T, \\ z_{k;2} &= [x_{k;2} \;\; y_{k;1} \;\; y_{k;2}]^T, \\ z_{k;3} &= [x_{k;3} \;\; y_{k;2}]^T. \end{aligned} \tag{5.84}$$

The positions of the agents are initialised as follows,

$x_{0;1} = [1 \;\; 1]^T, \quad x_{0;2} = [-1 \;\; -1]^T, \quad x_{0;3} = [-3 \;\; -3]^T.$

The proposed method considers the distribution of the tracking error for which the ideal covariance matrix is the same as the actual covariance which is the estimated global covariance $\Sigma_{e_{k;i}}$. The ideal covariance of the controller is determined to be $\Gamma_{k;i} = 10$ for this simulation example.

The plots demonstrate that the randomised controllers successfully managed to reshape the pdfs of the tracking errors of the subsystems such that the tracking errors oscillates closely around zero. Consequently, the internal states of each subsystems converge to the desired state value. The convergence of the errors to and their oscillation around zero can be seen in plots 5.8 - 5.9. The error plots display a fast convergence rate which emphasises the efficiency of the proposed method. The initial and final positions of the agents are illustrated in Figure 5.10 where the hexagrams are the initial positions of the agents and the circles represent the final positions of the agents. The final formation is the desired formation demonstrated in Fig 5.7 which is an equilateral triangle with agent 2 at the top.

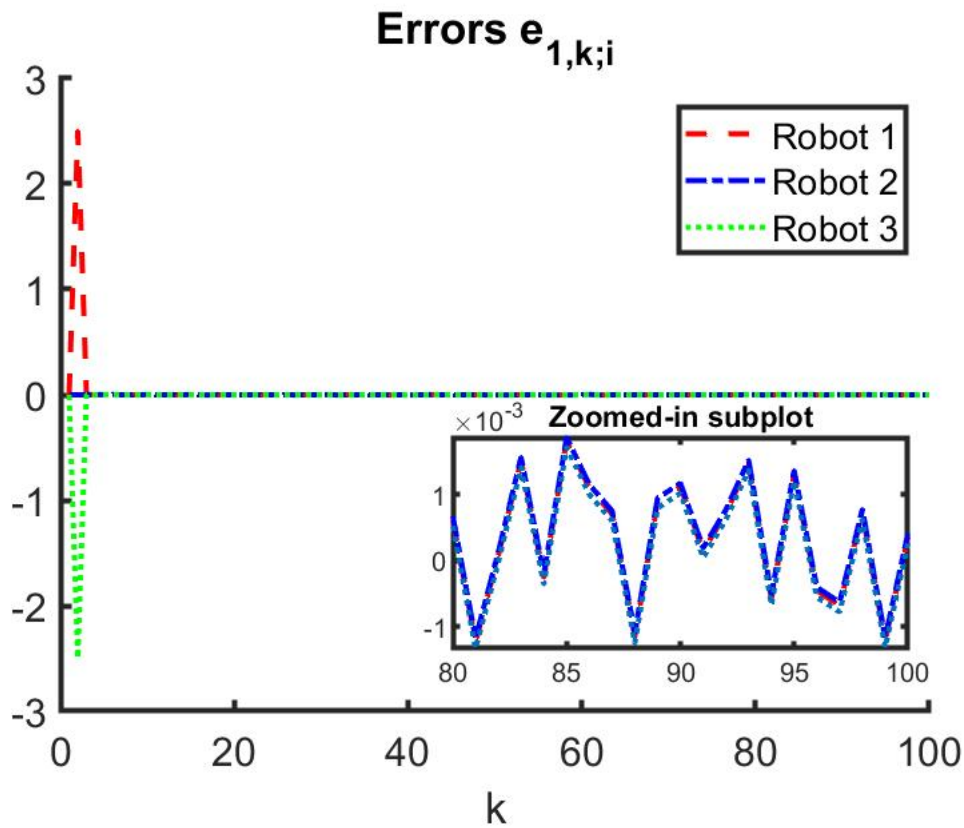

Figure 5.8: The tracking error $e_{1,k;i}$ for all subsystems.

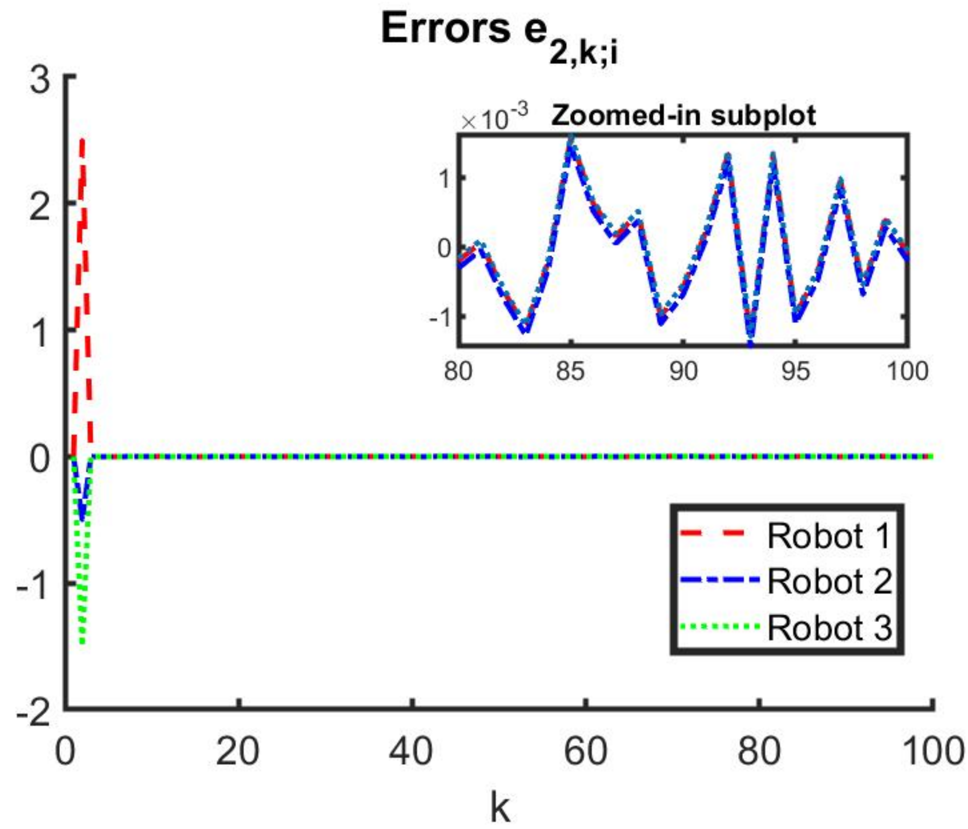

Figure 5.9: The tracking error $e_{2,k;i}$ for all subsystems.

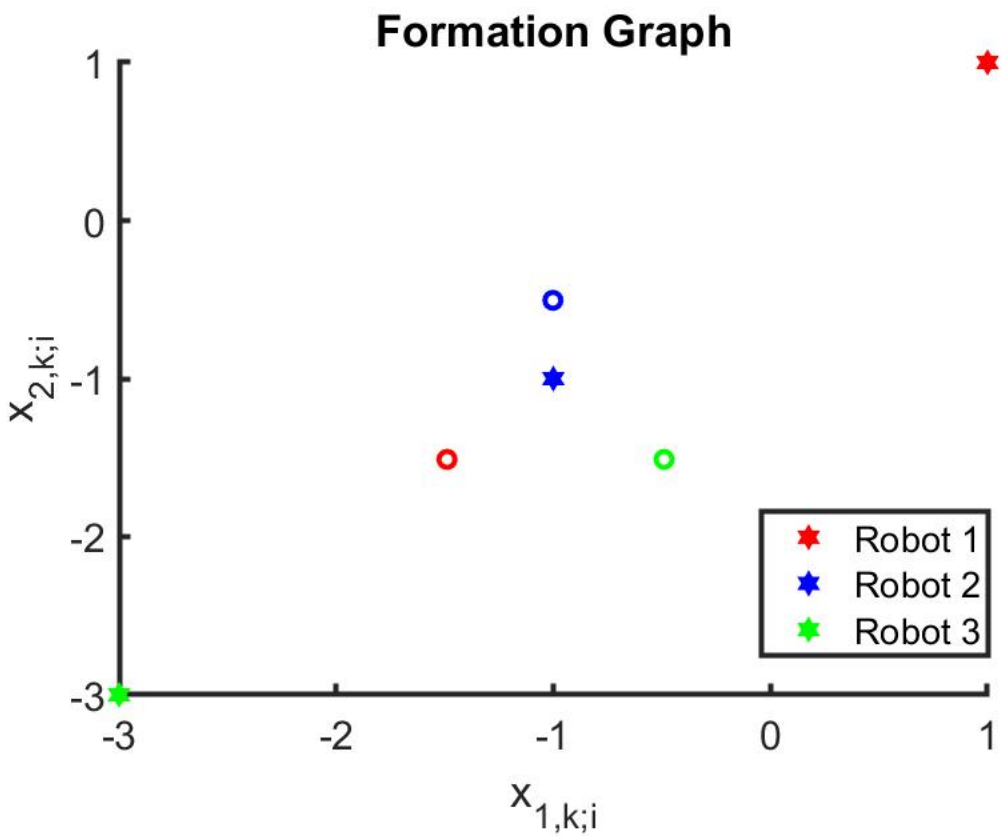

Figure 5.10: The initial positions of the agents are represented by the hexagrams and the final positions are the circles. The desired formation of forming an equilateral triangle has been achieved by the final positions of the agents.

5.5 Convergence Analysis

In this section, the convergence of the developed randomised control strategy in (5.63) for nonlinear systems with additive noises is analysed.

The dynamics of node i considered for the convergence analysis is given by,

$$n_{k;i} = f_i(z_{k-1;i})n_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + \tilde{W}x_{r,k-1;i} + \epsilon_{k;i}, \tag{5.85}$$

where $n_{k;i} = \begin{bmatrix} e_{k;i} \\ y_{k;i} \end{bmatrix}$ and the state matrix is given by $f_i(z_{k-1;i}) = \begin{bmatrix} f_{1i}(x_{k-1;i}) & 0 \\ 0 & h_i(y_{k-1;i}) \end{bmatrix}$ and where $f_{1i}(x_{k-1;i})$ and $h_i(y_{k-1;i})$ are the state matrices for the dynamics of the internal and external variables. Furthermore, the control matrix is given by $g_i(z_{k-1;i}) = \begin{bmatrix} g_i(x_{k-1;i}) \\ 0 \end{bmatrix}$, where $g_i(x_{k-1;i})$ is the control matrix for the internal variables and 0 for the external variables as the aim is not to control them. In this analysis, the reference signal is taken to be $x_{r,k;i} = \tilde{m}(x_{k;j}, r_{k;i}) + \epsilon_{r,k;i}$, where $\epsilon_{r,k;i}$ is some Gaussian noise with zero mean and variance Q_r. Therefore, $\tilde{W} = \begin{bmatrix} f_{1i}(x_{k-1;i}) - \tilde{m}(x_{k;j}) \\ 0 \end{bmatrix}$. In addition, the Gaussian noise, $\epsilon_{k;i} = \begin{bmatrix} \epsilon_{1k;i} + \epsilon_{r,k;i} \\ \epsilon_{2k;i} \end{bmatrix}$, has mean zero and variance $Q_i = \begin{bmatrix} Q_{1;i} \\ Q_{2;i} \end{bmatrix}$. The convergence of the developed controller in (5.63) is analysed and presented by the following theorem.

Theorem 10. The expected value of the error, $e_{k;i}$ is expected to converge to zero, which will make the internal state $x_{k;i}$ converge to the reference signal $x_{r,k;i}$, if there exist a positive definite symmetric matrix M_i which holds the following inequality,

$$D = \begin{bmatrix} D_{11} & 2(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)^T M_i(\tilde{W} - g_i(z_{k-1;i})W_i) & D_{13} \\ * & 2(\tilde{W} - g_i(z_{k-1;i})W_i)^T M_i(\tilde{W} - g_i(z_{k-1;i})W_i) & D_{23} \\ * & * & D_{33} \end{bmatrix} < 0, \tag{5.86}$$

where D is a symmetric matrix with the elements defined as follows,

$$\begin{aligned} D_{11} =& 2(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)^T M_i(f_i(z_{k-1;i}) - g_i(z_{k-;1i})K_i) - 2M_i, \\ D_{13} =& 2(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)^T M_i(I - g_i(z_{k-1;i})Z_i\epsilon^{\dagger}_{k;i}) + (f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)^T M_i h_i \epsilon^{\dagger}_{k;i} \\ & - M_i h_i \epsilon^{\dagger}_{k;i}, \\ D_{23} =& 2(\tilde{W} - g_i(z_{k-1;i})W_i)^T M_i(I - g_i(z_{k-1;i})Z_i\epsilon^{\dagger}_{k;i}) + (\tilde{W} - g_i(z_{k-1;i})W_i)^T M_i h_i \epsilon^{\dagger}_{k;i}, \end{aligned}$$

$$D_{33} = 2(I - g_i(z_{k-1;i}) Z_i \epsilon_{k;i}^{\dagger})^T M_i (I - g_i(z_{k-1;i}) Z_i \epsilon_{k;i}^{\dagger}). \tag{5.87}$$

Proof. The proof of Theorem 10 is given in Appendix I. □

5.6 Conclusion

This chapter discussed the FP control design with regards to control problems that are required to follow a certain predefined desired state trajectory. Although, plenty of work has been conducted on the tracking control problem for the FP control design, the literature has not addressed large-scale complex systems which have been decomposed into smaller subsystems. Hence, randomised local controllers that aim the tracking of a reference signal have been designed in order to expand the horizon of the range of real-world control systems that the FP control design can be applied to. Nonetheless, the development in this chapter also considers tracking control problems for systems that are governed by nonlinearities within a FP control design, which is a novel concept in the fully probabilistic framework. The validity of the proposed method has been verified by the results in the simulation section.

In addition, the tracking error problem can be implemented for various control objectives of which one is formation control. Therefore, the probabilistic approach has been studied further with the objective of controlling multi-agent systems to create a certain formation. The developed control strategy was demonstrated on linear and nonlinear simulation examples where both successfully achieved the desired formation. Moreover, the convergence analysis of the developed controller for a formation problem has been implemented in this chapter.

Chapter 6

Conclusion and Direction for Future Work

This chapter concludes the thesis and explains potential future work that can be implemented for the FP control design which can prove to be promising in the field of control.

6.1 Thesis Conclusion

In today's world, the increasing complexity of real-world control systems is accompanied with challenges such as high level of uncertainties and noises, nonlinearities, high dimensionality and coupling between the systems. The importance of developing control strategies that consider these challenges and as a result facilitate the control of such systems has been emphasised in the control literature.

Therefore, the aim of this thesis involved the analysis of these complexities and the development of appropriate control approaches that handle the aforementioned challenges effectively. Consequently, due to its suitability and efficiency regarding control systems that are affected by noises, the Fully Probabilistic control design has been researched thoroughly and developed further. The stochasticity is taken into account by the control design which results in the derivation of an optimal randomised controller. However, this approach has not been considered and demonstrated on nonlinear systems in the previous literature. The multiple integrations that are required to be evaluated for this control design made it seem impractical to derive an analytic control solution for nonlinear systems due to the nonlinear dynamics. The solution to this problem has been presented in Chapter 3 where a transformation method has been introduced for the transformation of the nonlinear state function of the system dynamics to a nonlinear state function which is affine in the state. Therefore, the first advancement in Chapter 3 is referred to as the conventional SDRE FP control design since it extended the conventional FP control design to nonlinear systems where the covariance matrix was estimated as a global covariance. The involvement of the nonlinearities in the FP control solution resulted in the generation of a SDRE. The efficiency of the proposed randomised controller was demonstrated on a nonlinear inverted pendulum simulation example and compared to the NQR SDRE as well as the FP control design that uses the true functions $\bar{h}(x_{k-1})$ and $\bar{g}(x_{k-1})$. The latter experiment demonstrated that the proposed control design handles the effects of the MLP network approximation effectively. A number of experiments were conducted which showed the benefits of the proposed randomised controller with the main benefit being that the parameters of the control algorithm can be estimated while the NQR SDRE requires the process of trial and error when tuning the NQR parameters until good convergence is obtained.

In addition, the variety of noises and uncertainties that have an impact on the dynamics of nonlinear systems was also discussed in Chapter 3. The second development consisted of an optimal randomised controller that takes functional uncertainties into account when regulating nonlinear systems. This is of paramount importance since realistically, the dynamics of engineered control systems are unknown, and are thus required to be estimated. It is possible to have a smaller ideal covariance matrix to reduce the variations in the system state for the FP control design that considers functional uncertainties. These key considerations led to the generation of a SDRE and an additional linear term which represents the equation of cautiousness. Furthermore, the suboptimal randomised controller can be classified as being cautious, meaning it takes functional uncertainties into consideration, since it includes an additional term which considers the equation of cautiousness. A simulation was presented to demonstrate the performance of the proposed randomised controller against the conventional SDRE FP control design. As expected from a cautious controller, the simulation showed a reduction in overshoots and better transient response than the conventional SDRE FP controller due to the consideration of functional uncertainties in the designed controller.

Thirdly, the fully probabilistic control approach was further studied for nonlinear systems with multiplicative stochastic noises. Although, a SDRE was generated, the solution to this control problem consisted of a generalised form of the Riccati equation because of the inclusion of an extra term which exists due to the controller taking multiplicative noises into account. The covariance matrix for such systems is state and control input dependent which means that it can be driven to a smaller ideal covariance matrix with the support of the derived controller. For nonlinear systems with multiplicative noises, the generalised SDRE FP control design has been derived which considers the dependency of the noise on the states while the conventional SDRE FP controller assume that the variance of the noise is constant.

Chapter 3 explored the FP control design for systems that can be controlled with a centralised controller. However, many real-world systems are too complex and large to be controlled by a single controller. To handle this complexity, the decentralised approach which consists of the decomposition of the network into smaller subsystems, can be followed for which local optimal randomised controllers are derived. Therefore, in Chapter 4, the fully probabilistic control framework is extended to decentralised controllers such that it can be applicable to large-scale complex systems that are governed by nonlinearities and affected by both, additive and multiplicative noises. Moreover, the developed control design addressed the challenges that current state-of-the-art methods faced. Since the smaller subsystems are required to communicate with each other to achieve the global objective, the concept of probabilistic message passing has been integrated in the design of the control strategy. In this thesis, only the marginal distributions of the external states are passed to neighbouring subsystems. However, if the model representing the actual system involves the passing of control inputs to other neighbouring subsystems, this can also be achieved by following the probabilistic message passing approach as explained in [164]. The knowledge that enters the receiving subsystems through probabilistic message passing is preserved by including it in the state matrix as the dynamics of the external variables. This means that the subsystem always has some knowledge about neighbouring subsystems, even if the link between the subsystems may be severed for some time. A core strength of the FP control design within a decentralised framework is the computational efficiency which can be recognised from the fact that the full block of the Riccati matrix is not required to be solved. Since only two elements of the Riccati matrix need to be computed, a reasonable reduction in computational expenses can be achieved. The effectiveness of the decentralised FP control strategies for nonlinear systems with both, additive and multiplicative noises, has been verified by simulating two numerical examples. The results for both control strategies showed a quick and nice convergence of the states to zero, which confirms their validities.

Finally, there are many control objectives that are implemented for real-world systems. The first aim of the controller that has been discussed in Chapter 5 is the requirement of the system state to track a predefined state trajectory. Although, the FP control design has already been considered for a tracking problem, it has not considered the derivation of an analytic solution for nonlinear systems. Furthermore, the fully probabilistic framework has also been extended to large-scale complex systems that have been decomposed into smaller subsystems, where each subsystem is required to track its corresponding reference model. This is also a novel concept within the fully probabilistic framework. Consequently, local randomised controllers have been designed to achieve this control objective. A simulation example has been provided in Chapter 5 to verify the validity of the proposed control design.

Another control objective is formation control which requires multi-agent systems to create a certain formation. Previous literature on the fully probabilistic control design has not dealt with formation control and is therefore introduced for the first time in Chapter 5. The proposed method proved to be successful since the simulation results showed two examples, linear and nonlinear, achieving the desired formation.

6.2 Future Direction of the FP Control Design

The content of this thesis has demonstrated that the fully probabilistic control design is a promising method which can be further developed to consider various aspects of real-world control systems. In Chapter 3, nonlinear systems with additive, multiplicative and functional uncertainties were taken into account when designing the optimal randomised controllers. As such, different variations of noises can be exploited in future work. Among these stochastic disturbances, systems are affected by noises that are described as a multiplication between a nonlinear function of the states and some Gaussian noise. The FP control design can be further developed such that it considers such systems.

Furthermore, the effects of external disturbances on the dynamics of nonlinear systems have not been acknowledged in the design process of the FP controller. A disturbance-observer-based fully probabilistic control approach has been developed and demonstrated on linear systems [184]. This control strategy can be further developed such that it extends to nonlinear stochastic systems.

Chapters 4 and 5 discussed the FP control strategy within a decentralised framework, for a regulation, tracking control, and formation control problem. Although the dynamics were assumed to be unknown and therefore required to be estimated online, the uncertainty from the approximated parameters of the distributions were not regarded. Therefore, it is beneficial to study and design local randomised controllers that consider functional uncertainties.

Moreover, complex systems in the world of control consist of a complicated structure and are therefore more prone to faults in the systems. It is important to consider system performance variations or degradation because of faults, in the design of controllers to ensure control efficiency and reliability. Many fault detection and fault-tolerant control approaches have been developed in the literature [185], [186], [187]. However, the FP control design has yet to see the inclusion of fault detection in its design process and could be a potential route for the extension of the probabilistic framework to a wider range of real-world control systems.

In addition, hybrid or switched systems have seen an increasing popularity due to its practicality and can be found in applications such as robotics, manufacturing, power electronics, air traffic management systems, to name a few [188], [189], [190] [191], [192]. The dynamics of these can be described by an interplay between continuous and discrete dynamics. Furthermore, hybrid control depends on the switching between different models and controllers. The inclusion of these features in the developed framework can provide a more robust control methodology.

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Appendix A

Derivation of the FP Control Solution for Nonlinear Systems

This section demonstrates the derivation of the conventional SDRE FP control solution for nonlinear systems. The optimal randomised controller is obtained by evaluating Proposition 1 in Chapter 3. The initial step consists of the evaluation of the performance index given by (3.10) for which (3.8) and (3.9) need to be computed.

The form of the optimal performance index in (3.37) is justified by backward induction. This means that for the proof, the optimal performance index specified by (3.37) is assumed to be true, and thereafter used in $\beta_2(u_k, x_{k-1})$ which is given by equation (3.9). As a result, the derivation of $\gamma(x_{k-1})$ as stated in equation (3.10) can be obtained.

Firstly, the term $\beta_1(u_k, x_{k-1})$ in (3.8) is evaluated by substituting the actual and ideal distributions of the states given by (3.30) and (3.31) respectively, such that,

$$\begin{aligned}
\beta_1(u_k, x_{k-1}) &= \int s(x_k \,|u_k, x_{k-1}) \left(\ln \frac{s(x_k \,|u_k, x_{k-1})}{s^I(x_k \,|u_k, x_{k-1})} \right) dx_k, \\
&= \int \mathcal{N}(\hat{x}_k, \Sigma_k) \left(\ln \frac{\mathcal{N}(\hat{x}_k, \Sigma_k)}{\mathcal{N}(0, \Sigma_k)} \right) dx_k, \\
&= \int \mathcal{N}(\hat{x}_k, \Sigma_k) \Bigg\{ \ln \left(\frac{1}{2\pi^{\frac{n}{2}} |\Sigma_k|^{\frac{1}{2}}} \right) - \frac{1}{2}(x_k - \hat{x}_k)^T \Sigma_k^{-1} (x_k - \hat{x}_k) \\
&\quad - \ln \left(\frac{1}{2\pi^{\frac{n}{2}} |\Sigma_k|^{\frac{1}{2}}} \right) + \frac{1}{2} x_k^T \Sigma_k^{-1} x_k \Bigg\} dx_k,
\end{aligned}$$

$$
\begin{aligned}
&= \int \mathcal{N}(\hat{x}_k, \Sigma_k)\left(\left[-\frac{1}{2}(x_k - \hat{x}_k)^T \Sigma_k^{-1}(x_k - \hat{x}_k) + \frac{1}{2}x_k^T \Sigma_k^{-1} x_k\right]\right) dx_k, \\
&= \int \mathcal{N}(\hat{x}_k, \Sigma_k)\left(\left[\frac{1}{2}\left(2x_k^T \Sigma_k^{-1}\hat{x}_k - \hat{x}_k^T \Sigma_k^{-1}\hat{x}_k\right)\right]\right) dx_k, \\
&= \frac{1}{2}\hat{x}_k^T \Sigma_k^{-1}\hat{x}_k, \\
&= \frac{1}{2}\Big(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k\Big)^T \Sigma_k^{-1}\Big(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k\Big),
\end{aligned} \tag{A.1}
$$

where equation (3.26) has been used. Similarly, $\beta_2(u_k, x_{k-1})$ as given by equation (3.9) can be evaluated by making the substitution of the assumed form of $-\ln(\gamma(x_k))$ as specified by (3.37) which gives,

$$
\begin{aligned}
\beta_2(u_k, x_{k-1}) &= -\int s(x_k \,|u_k, x_{k-1}\,) \ln(\gamma(x_k)) dx_k, \\
&= \int \mathcal{N}(\hat{x}_k, \Sigma_k)(0.5x_k^T M_k x_k + 0.5w_k) dx_k, \\
&= \underbrace{0.5\int \mathcal{N}(\hat{x}_k, \Sigma_k)(x_k^T M_k x_k) dx_k}_{①} + \underbrace{0.5\int \mathcal{N}(\hat{x}_k, \Sigma_k) w_k dx_k}_{②}.
\end{aligned}
$$

Integral ① is evaluated as follows,

$$
\begin{aligned}
① &= \frac{1}{2}\int \mathcal{N}(\hat{x}_k, \Sigma_k)(x_k^T M_k x_k) dx_k, \\
&= \frac{1}{2}\int \mathcal{N}(\hat{x}_k, \Sigma_k)\left\{(x_k - \hat{x}_k)^T M_k (x_k - \hat{x}_k)\right\} dx_k \\
&\quad + \frac{1}{2}\int \mathcal{N}(\hat{x}_k, \Sigma_k)\left\{2\hat{x}_k^T M_k x_k - \hat{x}_k^T M_k \hat{x}_k\right\} dx_k, \\
&= \frac{1}{2}\left\{\text{tr}(M_k \Sigma_k) + \hat{x}_k^T M_k \hat{x}_k\right\}.
\end{aligned}
$$

From the second integral, the following is obtained, $② = \frac{1}{2}w_k$.

Hence, combining ① and ②, and substituting $\hat{x}_k = h(x_{k-1})x_{k-1} + g(x_{k-1})u_k$ which has been obtained from equation (3.26) gives,

$$
\begin{aligned}
\beta_2(u_k, x_{k-1}) &= \frac{1}{2}\left\{\text{tr}(M_k \Sigma_k) + \hat{x}_k^T M_k \hat{x}_k + w_k\right\}, \\
&= \frac{1}{2}\Bigg\{\Big(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k\Big)^T M_k \Big(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k\Big) \\
&\quad + \text{tr}(M_k \Sigma_k) + w_k\Bigg\}.
\end{aligned} \tag{A.2}
$$

Now, the term $\gamma(x_{k-1})$ defined by (3.10) can be derived using the evaluated forms of $\beta_1(u_k, x_{k-1})$

in (A.1) and $\beta_2(u_k, x_{k-1})$ in (A.2), and the ideal state distribution (3.31) to give,

$$
\begin{aligned}
\gamma(x_{k-1}) &= \int c^I(u_k|x_{k-1}) \exp[-\beta_1(u_k, x_{k-1}) - \beta_2(u_k, x_{k-1})] du_k, \\
&= \int \mathcal{N}(0, \Gamma) \exp\Big\{-0.5\Big[(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k)^T \Sigma_k^{-1} (h(x_{k-1})x_{k-1} \\
&\quad + g(x_{k-1})u_k) + (h(x_{k-1})x_{k-1} + g(x_{k-1})u_k)^T M_k (h(x_{k-1})x_{k-1} + g(x_{k-1})u_k) \\
&\quad + \text{tr}(M_k\Sigma_k) + w_k\Big]\Big\} du_k, \\
&= \int \mathcal{N}(0, \Gamma) \exp\Big\{-0.5\Big[(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k)^T (\Sigma_k^{-1} + M_k)(h(x_{k-1})x_{k-1} \\
&\quad + g(x_{k-1})u_k) + \text{tr}(M_k\Sigma_k) + w_k\Big]\Big\} du_k, \\
&= \int \mathcal{N}(0, \Gamma) \exp\Big\{-0.5\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k)h(x_{k-1})x_{k-1} + 2x_{k-1}^T h^T(x_{k-1}) \\
&\quad \times (\Sigma_k^{-1} + M_k)g(x_{k-1})u_k + u_k^T g^T(x_{k-1})(\Sigma_k^{-1} + M_k)g(x_{k-1})u_k + \text{tr}(M_k\Sigma_k) \\
&\quad + w_k\Big]\Big\} du_k, \\
&= \exp\Big\{-0.5\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k)h(x_{k-1})x_{k-1} + \text{tr}(M_k\Sigma_k) + w_k\Big]\Big\} \\
&\quad \times \int \mathcal{N}(0, \Gamma) \exp\Big\{-0.5\Big[u_k^T g^T(x_{k-1})(\Sigma_k^{-1} + M_k)g(x_{k-1})u_k + 2x_{k-1}^T h^T(x_{k-1}) \\
&\quad \times (\Sigma_k^{-1} + M_k)g(x_{k-1})u_k\Big]\Big\} du_k, \\
&= (2\pi|\Gamma|)^{-\frac{1}{2}} \exp\Big\{-0.5\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k)h(x_{k-1})x_{k-1} + \text{tr}(M_k\Sigma_k) \\
&\quad + w_k\Big]\Big\} \times \int \exp\Big\{-0.5\Big[u_k^T \left[g(x_{k-1})^T(\Sigma_k^{-1} + M_k)g(x_{k-1}) + \Gamma^{-1}\right] u_k \\
&\quad + 2x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k)g(x_{k-1})u_k\Big]\Big\} du_k,
\end{aligned}
\tag{A.3}
$$

The integral in (A.3) can be solved by completing the square with respect to u_k.

The process of completing the square for matrices in general is outlined by Property 1 and is given below.

Property 1:

An expression given by $x^T Ax + x^T b + c$, can be expressed as,

$$(x - d)^T A(x - d) + s,$$

where,

$$d = -\frac{1}{2}A^{-1}b, \qquad s = c - \frac{1}{4}b^T A^{-1}b.$$

Therefore, (A.3) can be rewritten to obtain,

$$\begin{aligned}\gamma(x_{k-1}) =&(2\pi|\Gamma|)^{-\frac{1}{2}}\exp\Big\{-0.5\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}+\text{tr}(M_k\Sigma_k)+w_k\Big]\Big\}\\ &\times\int\exp\Big\{-0.5\Big[\Big((u_k+[g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})+\Gamma^{-1}]^{-1}g^T(x_{k-1})\\ &\times(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}\Big)^T[g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})+\Gamma^{-1}]\\ &\times\Big((u_k+[g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})+\Gamma^{-1}]^{-1}g^T(x_{k-1})(\Sigma_k^{-1}+M_k)\\ &\times h(x_{k-1})x_{k-1}\Big)\Big]\Big\}du_k\times\exp\Big\{-0.5\Big[-\Big(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}\Big)^T\\ &\times[g^T(x_{k-1})(\Sigma^{-1}+M_k)g(x_{k-1})+\Gamma^{-1}]^{-1}\Big(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}\Big)\Big]\Big\}.\end{aligned}\tag{A.4}$$

The following property is used for the simplification of the integral in (A.4),

Property 2:

$$\int\exp\Big(-\frac{1}{2}x^TVx\Big)dx=|2\pi|^{\frac{1}{2}}|V|^{-\frac{1}{2}}.$$

Hence, Equation (A.4) is simplified to obtain,

$$\begin{aligned}\gamma(x_{k-1}) =&|2\pi|^{\frac{1}{2}}|g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})+\Gamma^{-1}|^{-\frac{1}{2}}(2\pi|\Gamma|)^{-\frac{1}{2}}\\ &\times\exp\Big\{-0.5\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}+\text{tr}(M_k\Sigma_k)+w_k\\ &-\Big(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}\Big)^T\Big[g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})+\Gamma^{-1}\Big]^{-1}\\ &\times\Big(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}\Big)\Big]\Big\},\\ =&\exp\Big\{-0.5x_{k-1}^T h^T(x_{k-1})\Big[(\Sigma_k^{-1}+M_k)-(\Sigma_k^{-1}+M_k)g(x_{k-1})[\Gamma^{-1}+g^T(x_{k-1})\\ &\times(\Sigma_k^{-1}+M_k)g(x_{k-1})]^{-1}g^T(x_{k-1})(\Sigma_k^{-1}+M_k)\Big]h(x_{k-1})x_{k-1}-0.5\Big[\text{tr}(M_k\Sigma_k)\\ &+w_k+\ln|\Gamma|+\ln|\Gamma^{-1}+g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})|\Big]\Big\}.\end{aligned}\tag{A.5}$$

From the above, the SDRE equation M_k in (3.38) and the constant term w_k in (3.39) can be found.

The derivation of randomised optimal controller requires the evaluation of the optimal control law defined in (3.7) in Proposition 1,

$$c^*(u_k|x_{k-1})=\frac{c^I(u_{k-1}|x_{k-1})\exp[-\beta_1(u_k,x_{k-1})-\beta_2(u_k,x_{k-1})]\longleftarrow ①}{\gamma(x_{k-1})\longleftarrow ②}.$$

First of all, the numerator is solved to obtain,

$$
\begin{aligned}
① =&c^I(u_k|x_{k-1})\exp[-\beta_1(u_k,x_{k-1})-\beta_2(u_k,x_{k-1})],\\
=&(2\pi|\Gamma|)^{-\frac{1}{2}}\exp\bigg\{-0.5\bigg[u_k^T\Gamma^{-1}u_k+(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)^T\\
&\times(\Sigma_k^{-1}+M_k)(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)+\text{tr}(M_k\Sigma_k)+w_k\bigg]\bigg\},\\
=&(2\pi|\Gamma|)^{-\frac{1}{2}}\exp\bigg\{-0.5\bigg\{x_{k-1}^Th^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}+\text{tr}(M_k\Sigma_k)+w_k\\
&+u_k^T\bigg[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})\bigg]u_k+2u_k^Tg^T(x_{k-1})(\Sigma_k^{-1}+M_k)\\
&\times h(x_{k-1})x_{k-1}\bigg\}\bigg\}. \qquad \text{(A.6)}
\end{aligned}
$$

Equation (A.6) can be further solved by completing the square over u_k which gives,

$$
\begin{aligned}
① =&(2\pi|\Gamma|)^{-\frac{1}{2}}\exp\bigg\{-0.5\bigg\{x_{k-1}^Th^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}+\text{tr}(M_k\Sigma_k)+w_k\bigg\}\bigg\}\\
&\times\exp\bigg\{-0.5\bigg\{\bigg(u_k+[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)\\
&\times h(x_{k-1})x_{k-1})\bigg)^T\bigg[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})\bigg]\\
&\times\bigg(u_k+[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_k^{-1}+M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1})\bigg)\\
&-\bigg(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}\bigg)^T[\Gamma^{-1}+g^T(x_{k-1})(\Sigma^{-1}+M_k)g(x_{k-1})]^{-1}\\
&\times\bigg(g^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}\bigg)\bigg\}\bigg\}. \qquad \text{(A.7)}
\end{aligned}
$$

The denominator ② has already been obtained in (A.5).

Since $\frac{\exp(a)}{\exp(b)} = \exp(a-b)$, we can subtract ② from ① which results in,

$$
\begin{aligned}
& c^*(u_k \mid x_{k-1}) \\
& = (2\pi)^{-\frac{1}{2}} |\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k) g(x_{k-1})|^{\frac{1}{2}} \\
& \times \exp\Big\{-0.5\Big\{ x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1} + \text{tr}(M_k \Sigma_k) + w_k \\
& + \Big(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k) g(x_{k-1})]^{-1} (g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1})\Big)^T \\
& \times \Big[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k) g(x_{k-1})\Big] \\
& \times \Big(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k) g(x_{k-1})]^{-1} (g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1})\Big) \\
& - \Big(g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1}\Big)^T [\Gamma^{-1} + g^T(x_{k-1})(\Sigma^{-1} + M_k) g(x_{k-1})]^{-1} \\
& \times \Big(g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1}\Big) - \Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1} \\
& + \text{tr}(M_k \Sigma_k) + w_k - \Big(g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1}\Big)^T \\
& \times [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k) g(x_{k-1})]^{-1} \Big(g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1}\Big)\Big]\Big\}\Big\}. \quad \text{(A.8)}
\end{aligned}
$$

From (A.8), the following remains,

$$
\begin{aligned}
c^*(u_k | x_{k-1}) = & (2\pi)^{-\frac{1}{2}} |\Gamma^{-1} + g(x_{k-1})^T (\Sigma_k^{-1} + M_k) g(x_{k-1})|^{\frac{1}{2}} \\
& \times \Big\{\exp\Big\{-0.5\Big[\Big(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k) g(x_{k-1})]^{-1} \\
& \times (g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1})\Big)^T \Big[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + M_k) \\
& \times g(x_{k-1})\Big]\Big(u_k + [\Gamma^{-1} + g(x_{k-1})^T (\Sigma_k^{-1} + M_k) g(x_{k-1})]^{-1} \\
& \times (g^T(x_{k-1})(\Sigma_k^{-1} + M_k) h(x_{k-1}) x_{k-1})\Big)\Big]\Big\}\Big\}. \quad \text{(A.9)}
\end{aligned}
$$

This is the form of the derived optimal randomised controller given by (3.33) for nonlinear systems with additive noise as shown in Theorem 1 in Section 3.3.

Appendix B

Derivation of the FP Control Solution for Nonlinear Systems with Functional Uncertainty

The derivation of the FP analytic control solution is discussed here in detail to verify the results in Theorem 2 in Chapter 3. Again, Proposition 1 which is stated in Chapter 3 forms the foundation to the process of deriving the optimal randomised controller.

Firstly, the form of the performance index which is given by $-\ln(\gamma(x_k)) = 0.5x_k^T M_k x_k + 0.5T_k x_k + 0.5\omega_k$ is justified. The terms $\beta_1(u_, x_{k-1})$ in (3.8) and $\beta_2(u_, x_{k-1})$ in (3.9) are required to be solved for which we obtain,

$$
\begin{aligned}
\beta_1(u_k, x_{k-1}) =& \int s(x_k\,|u_k, x_{k-1})\left(\ln\frac{s(x_k\,|u_k, x_{k-1})}{s^I(x_k\,|u_k, x_{k-1})}\right)dx_k,\\
=& \int \mathcal{N}(\hat{x}_k,\ \Sigma_k)\ln\left(\frac{\mathcal{N}(\hat{x}_k,\ \Sigma_k)}{\mathcal{N}(0,\ \Sigma_2)}\right)dx_k,\\
=& \int \mathcal{N}(\hat{x}_k,\ \Sigma_k)\Big\{0.5\Big\{-\ln|\Sigma_k| - (x_k-\hat{x}_k)^T\Sigma_k^{-1}(x_k-\hat{x}_k) + \ln|\Sigma_2|\\
&+ x_k^T\Sigma_2^{-1}x_k\Big\}\Big\}dx_k.\\
=& \int \mathcal{N}(\hat{x}_k,\ \Sigma_k)\Big\{0.5\Big\{-\ln\left(\frac{|\Sigma_k|}{|\Sigma_2|}\right) + x_k^T(\Sigma_2^{-1}-\Sigma_k^{-1})x_k\Big\}\Big\}dx_k\\
&+ \int \mathcal{N}(\hat{x}_k,\ \Sigma_k)\Big\{0.5\Big\{-\hat{x}_k^T\Sigma_k^{-1}\hat{x}_k + 2x_k^T\Sigma_k^{-1}\hat{x}_k\Big\}\Big\}dx_k,\\
=& 0.5\hat{x}_k\Sigma_k^{-1}\hat{x}_k + \int \mathcal{N}(\hat{x}_k,\ \Sigma_k)\left\{0.5\left\{\underbrace{-\ln\left(\frac{|\Sigma_k|}{|\Sigma_2|}\right)}_{①} + \underbrace{x_k^T(\Sigma_2^{-1}-\Sigma_k^{-1})x_k}_{②}\right\}\right\}dx_k.
\end{aligned}
\tag{B.1}
$$

① $\longrightarrow$ To solve $\ln\left(\frac{|\Sigma_k|}{|\Sigma_2|}\right\}$, a very useful identity from [193], namely,

$$\log(\det(H)) = \text{tr}(\log(H)),$$

can be used, given the condition that matrix H is positive definite. As the covariance matrices are positive definite, this rule can be applied to give,

$$\begin{aligned}\log(|\Sigma_k|\,|\Sigma_2|^{-1}) =& \log(\left|\Sigma_k\Sigma_2{}^{-1}\right|),\\ =& \text{tr}(\log(\Sigma_k\Sigma_2{}^{-1})).\end{aligned} \tag{B.2}$$

An important observation is that for the regulation problem considered in the evaluation of the FP control design, the actual covariance of the dynamics of the system is expected to get closer to the covariance of the ideal distribution, i.e. $||\Sigma_k\Sigma_2^{-1}|| \approx I$.

Considering the Maclaurin series expansion for logarithms, it is known that,

$$\log(H) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{(H-I)^j}{j} = (H-I) - \frac{(H-I)^2}{2} + \frac{(H-I)^3}{3}...$$

Exploiting the previous observation namely, $||\Sigma_k\Sigma_2^{-1}|| \approx I$ and the property, given in [194], which states that *if* $||H - I|| \ll 1$*, then the higher order terms in the Maclaurin series expansion for* $\log(H)$ *will become significantly small, and can hence be ignored,* leads to the following deduction of $\text{tr}(\log(\Sigma_k\Sigma_2{}^{-1}))$ in (B.2).

$$\begin{aligned}\text{tr}(\log(\Sigma_k\Sigma_2{}^{-1})) =& \text{tr}(\Sigma_k\Sigma_2{}^{-1} - I),\\ =& \text{tr}(\Sigma_k\Sigma_2{}^{-1}) - n,\end{aligned} \tag{B.3}$$

where n is the dimension of the state x_k.

② $\longrightarrow$ This part can be computed by rewriting $x_k^T(\Sigma_2^{-1}-\Sigma_k^{-1})x_k$ as $(x_k-\hat{x}_k)^T(\Sigma_2^{-1}-\Sigma_k^{-1})(x_k-\hat{x}_k)$

which gives,

$$
\begin{aligned}
=&0.5 \int \mathcal{N}(\hat{x}_k, \Sigma_k) \left[x_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})x_k\right] dx_k, \\
=&0.5 \int \mathcal{N}(\hat{x}_k, \Sigma_k) \Bigg[\underbrace{(x_k - \hat{x}_k)^T(\Sigma_2^{-1} - \Sigma_k^{-1})(x_k - \hat{x}_k)}_{ⓐ} - \underbrace{\hat{x}_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})\hat{x}_k}_{ⓑ} \\
&+ \underbrace{2x_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})\hat{x}_k}_{ⓒ}\Bigg] dx_k,
\end{aligned} \tag{B.4}
$$

where,

$$
\begin{aligned}
ⓐ =&0.5\text{tr}(\Sigma_k(\Sigma_2^{-1} - \Sigma_k^{-1})), \\
=&0.5\text{tr}(\Sigma_k\Sigma_2^{-1} - I), \\
=&0.5\text{tr}(\Sigma_k\Sigma_2^{-1}) - 0.5n,
\end{aligned} \tag{B.5}
$$

and,

$$
ⓑ = -0.5\hat{x}_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})\hat{x}_k, \tag{B.6}
$$

and finally,

$$
ⓒ = \hat{x}_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})\hat{x}_k. \tag{B.7}
$$

Thereafter, (B.3), (B.5), (B.6), and (B.7) can be substituted back into (B.1), which along with the substitution of $\hat{x}_k = h(x_{k-1})x_{k-1} + g(x_{k-1})u_k$ from equation (3.48) gives,

$$
\begin{aligned}
\beta_1(u_k, x_{k-1}) =&\frac{1}{2}\hat{x}_k^T\Sigma_k^{-1}\hat{x}_k + \frac{1}{2}\hat{x}_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})\hat{x}_k + \frac{1}{2}\text{tr}(\Sigma_k\Sigma_2^{-1}) + \frac{1}{2}n - \frac{1}{2}\text{tr}(\Sigma_k\Sigma_2^{-1}) - \frac{1}{2}n, \\
=&\frac{1}{2}\hat{x}_k^T(\Sigma_2^{-1} - \Sigma_k^{-1} + \Sigma_k^{-1})\hat{x}_k, \\
=&\frac{1}{2}(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k)^T\Sigma_2^{-1}(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k).
\end{aligned} \tag{B.8}
$$

Furthermore, $\beta_2(u_k, x_{k-1})$ needs to be evaluated which results in,

$$
\begin{aligned}
\beta_2(u_k, x_{k-1}) =& - \int s(x_k \,|u_k, x_{k-1}\,) \ln(\gamma(x_k)) dx_k, \\
=& \int \mathcal{N}\left(\hat{x}_k, \Sigma_k\right) \left(\frac{1}{2} \left[x_k^T M_k x_k + T_k x_k + \omega_k \right] \right) dx_k, \\
=& \frac{1}{2} \int \mathcal{N}(\hat{x}_k, \Sigma_k) \bigg[(x_k - \hat{x}_k)^T M_k (x_k - \hat{x}_k) + 2 x_k^T M_k \hat{x}_k - \hat{x}_k^T M_k \hat{x}_k \\
& + T_k x_k \bigg] dx_k + \frac{1}{2} \omega_k, \\
=& \frac{1}{2} \text{tr}(\Sigma_k M_k) + \frac{1}{2} \hat{x}_k^T M_k \hat{x}_k + \frac{1}{2} T_k \hat{x}_k + \frac{1}{2} \omega_k, \\
=& \frac{1}{2} (h(x_{k-1}) x_{k-1} + g(x_{k-1}) u_k)^T M_k \left(h(x_{k-1}) x_{k-1} + g(x_{k-1}) u_k \right) + \frac{1}{2} \text{tr}(\Sigma_k M_k) \\
& + \frac{1}{2} T_k \left(h(x_{k-1}) x_{k-1} + g(x_{k-1}) u_k \right) + \frac{1}{2} \omega_k.
\end{aligned} \tag{B.9}
$$

Using (3.50), the term $\text{tr}(\Sigma_k M_k)$ in (B.9) can be written as $\text{tr}(\Sigma_k M_k) = \text{tr}([D x_{k-1} + G u_k] M_k) = \text{tr}(D M_k) x_{k-1} + \text{tr}(G M_k) u_k$. Hence, Equation (B.9) now becomes,

$$
\begin{aligned}
\beta_2(u_k, x_{k-1}) =& \frac{1}{2} (h(x_{k-1}) x_{k-1} + g(x_{k-1}) u_k)^T M_k \left(h(x_{k-1}) x_{k-1} + g(x_{k-1}) u_k \right) \\
& + \frac{1}{2} T_k \left(h(x_{k-1}) x_{k-1} + g(x_{k-1}) u_k \right) + \frac{1}{2} \omega_k + \frac{1}{2} \text{tr}(D M_k) x_{k-1} + \frac{1}{2} \text{tr}(G M_k) u_k.
\end{aligned} \tag{B.10}
$$

The evaluated terms $\beta_1(u_k, x_{k-1})$ and $\beta_2(u_k, x_{k-1})$ can now be substituted in $\gamma(x_{k-1})$ which is given by (3.10) in Proposition 1 such that,

$$
\begin{aligned}
\gamma(x_{k-1}) &= \int c^I(u_k|x_{k-1})\exp[-\beta_1(u_k,x_{k-1})-\beta_2(u_k,x_{k-1})]du_k,\\
&= \int \mathcal{N}(0,\Gamma)\exp\bigg\{-\frac{1}{2}\bigg[(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)^T\Sigma_2^{-1}(h(x_{k-1})x_{k-1}\\
&\quad + g(x_{k-1})u_k) + (h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)^T M_k\,(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)\\
&\quad + T_k\,(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)+\omega_k+\text{tr}(DM_k)x_{k-1}+\text{tr}(GM_k)u_k\bigg]\bigg\}du_k,\\
&=\frac{1}{(2\pi|\Gamma|)^{\frac{1}{2}}}\times\int\exp\bigg\{-0.5\bigg[u_k^T\Gamma^{-1}u_k+x_{k-1}^Th^T(x_{k-1})(\Sigma_2^{-1}+M_k)h(x_{k-1})x_{k-1}\\
&\quad + 2x_{k-1}^Th^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})u_k+u_k^Tg^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})u_k\\
&\quad + \omega_k+\text{tr}(DM_k)x_{k-1}+\text{tr}(GM_k)u_k+T_kh(x_{k-1})x_{k-1}+T_kg(x_{k-1})u_k\bigg]\bigg\}du_k,\\
&=\frac{1}{(2\pi)^{\frac{1}{2}}|\Gamma|^{\frac{1}{2}}}\times\bigg\{\exp\bigg\{-0.5\bigg[x_{k-1}^Th^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}+\omega_k\\
&\quad + \text{tr}(DM_k)x_{k-1}+T_kh(x_{k-1})x_{k-1}\bigg]\bigg\}\bigg\}\\
&\quad \times\int\exp\bigg\{-\frac{1}{2}\bigg[u_k^T[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})]u_k\\
&\quad + 2u_k^T[g^T(x_{k-1})(\Sigma_2^{-1}+M_k)h(x_{k-1})x_{k-1}+\frac{1}{2}(g^T(x_{k-1})T_k^T+\text{tr}(GM_k))]\bigg]\bigg\}du_k.
\end{aligned}
\tag{B.11}
$$

The integral in (B.11) can be further simplified by completing the square. This process has been explained in Appendix A by Property 1 which when applied to (B.11) gives, Equation (B.12)

$$\gamma(x_{k-1}) = \frac{1}{(2\pi)^{\frac{1}{2}}|\Gamma|^{\frac{1}{2}}} \times \Bigg\{\exp\Bigg\{-\frac{1}{2}\Bigg[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k)h(x_{k-1})x_{k-1} + \omega_k$$
$$+ \text{tr}(DM_k)x_{k-1} + T_k h(x_{k-1})x_{k-1}\Bigg]\Bigg\}\Bigg\}$$
$$\times \exp\Bigg\{-\frac{1}{2}\Bigg[-[g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k))]^T$$
$$\times [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]^{-1}[g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1}$$
$$+ \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k))]\Bigg]\Bigg\}$$
$$\times \int \exp\Bigg\{-\frac{1}{2}\Bigg[\Bigg(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1})$$
$$\times (\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k)))\Bigg)^T [\Gamma^{-1} + g^T(x_{k-1})$$
$$\times (\Sigma_2^{-1} + M_k)g(x_{k-1})]\Bigg(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]^{-1}$$
$$\times (g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k)))\Bigg)\Bigg]\Bigg\}du_k. \quad \text{(B.12)}$$

The integral in B.12 can be solved using Property 2 in Appendix A which results in,

$$\gamma(x_{k-1}) = |\Gamma|^{-\frac{1}{2}}|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})|^{-\frac{1}{2}}$$
$$\times \exp\Bigg\{-\frac{1}{2}\Bigg[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + M_k)h(x_{k-1})x_{k-1} + \omega_k + \text{tr}(DM_k)x_{k-1}$$
$$+ T_k h(x_{k-1})x_{k-1}\Bigg]\Bigg\} \times \exp\Bigg\{-\frac{1}{2}\Bigg[-[g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1}$$
$$+ \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k))]^T[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]^{-1}$$
$$\times [g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k))]\Bigg]\Bigg\}. \quad \text{(B.13)}$$

Equation (B.13) can be further expanded to obtain the desired form of the optimal performance index as stated in (3.60) from which the Riccati equation solution M_k, the linear term T_k and the constant term ω_k can be obtained as specified by (3.61), (3.62) and (3.63), respectively.

$$\gamma(x_{k-1}) = \Bigg\{\exp\bigg\{-\frac{1}{2}\bigg[x_{k-1}^T\bigg(h^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1}) - h^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})$$
$$\times [\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})]^{-1}g^T(x_{k-1})(\Sigma_2^{-1}+M_k)h(x_{k-1})\bigg)x_{k-1}$$
$$+\bigg(\text{tr}(DM_k)+T_kh(x_{k-1}) - (g^T(x_{k-1})T_k^T+\text{tr}(GM_k))^T[\Gamma^{-1}+g^T(x_{k-1})$$
$$\times(\Sigma_2^{-1}+M_k)g(x_{k-1})]^{-1}g^T(x_{k-1})(\Sigma_2^{-1}+M_k)h(x_{k-1})\bigg)x_{k-1}$$
$$+\bigg(\omega_k - (0.5[g^T(x_{k-1})T_k^T+\text{tr}(GM_k)])^T[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+M_k)$$
$$\times g(x_{k-1})]^{-1}(0.5[g^T(x_{k-1})T_k^T+\text{tr}(GM_k)]) + \ln|\Gamma| + \ln|\Gamma^{-1}+g^T(x_{k-1})$$
$$\times(\Sigma_2^{-1}+M_k)g(x_{k-1})|\bigg)\bigg]\bigg\}. \tag{B.14}$$

The final equation $\gamma(x_{k-1})$ in (B.14) has justified the form of the performance index as given by (3.84).

In addition, the optimal randomised controller is derived by computing the optimal control law defined in (3.7) in Proposition 1 in Chapter 3 such that,

$$c^*(u_k|x_{k-1}) = \frac{c^I(u_{k-1}|x_{k-1})\exp[-\beta_1(u_k,x_{k-1}) - \beta_2(u_k,x_{k-1})] \longleftarrow ①}{\gamma(x_{k-1}) \longleftarrow ②}.$$

Instead of computing the numerator and denominator separately and then making the division between them (as was shown in Appendix A), it is possible to simplify the process resulting in a reduction in computational time.

From careful observation, the following can be deduced. The optimal control law is given by,

$$c^*(u_k|x_{k-1}) = \frac{c^I(u_{k-1}|x_{k-1})\exp[-\beta_1(u_k,x_{k-1}) - \beta_2(u_k,x_{k-1})]}{\int c^I(u_k|x_{k-1})\exp[-\beta_1(u_k,x_{k-1}) - \beta_2(u_k,x_{k-1})]du_k \longleftarrow \gamma(x_{k-1})}, \tag{B.15}$$

from which it can be seen that the numerator and denominator are the same apart from the fact that the denominator is integrated over u_k.

Since the denominator has already been obtained in (B.14), let us focus on the numerator. The numerator is derived by making the following substitutions: $c^I(u_k|x_{k-1}) = \mathcal{N}(0,\Gamma)$, and $\beta_1(u_k,x_{k-1})$ given by (B.8) and $\beta_2(u_k,x_{k-1})$ specified by (B.10), and thereafter, completing the square over u_k

which results in,

$$
\begin{aligned}
① &= \frac{1}{(2\pi|\Gamma|)^{\frac{1}{2}}}\\
\exp(f) &= \begin{cases} \times\Big\{\exp\Big\{-\frac{1}{2}\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1}+M_k)h(x_{k-1})x_{k-1}+\omega_k+\text{tr}(DM_k)x_{k-1} \\ +T_k h(x_{k-1})x_{k-1}\Big]\Big\}\Big\} \end{cases}\\
\exp(s) &= \begin{cases} \times\exp\Big\{-\frac{1}{2}\Big[-[g^T(x_{k-1})(\Sigma_2^{-1}+M_k)h(x_{k-1})x_{k-1}+\frac{1}{2}(g^T(x_{k-1})T_k^T+\text{tr}(GM_k))]^T \\ \times[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})]^{-1}[g^T(x_{k-1})(\Sigma_2^{-1}+M_k)h(x_{k-1})x_{k-1} \\ +\frac{1}{2}(g^T(x_{k-1})T_k^T+\text{tr}(GM_k))]\Big]\Big\} \end{cases}\\
\exp(Y) &= \begin{cases} \times\exp\Big\{-\frac{1}{2}\Big[\Big(u_k+[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1}) \\ \times(\Sigma_2^{-1}+M_k)h(x_{k-1})x_{k-1}+\frac{1}{2}(g^T(x_{k-1})T_k^T+\text{tr}(GM_k)))\Big)^T[\Gamma^{-1}+g^T(x_{k-1}) \\ \times(\Sigma_2^{-1}+M_k)g(x_{k-1})]\Big(u_k+[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1}) \\ \times(\Sigma_2^{-1}+M_k)h(x_{k-1})x_{k-1}+\frac{1}{2}(g^T(x_{k-1})T_k^T+\text{tr}(GM_k)))\Big)\Big]\Big\}, \end{cases}
\end{aligned}
$$

where the substitutions $\exp(f)$, $\exp(s)$ and $\exp(Y)$ are made such that,

$$① = \frac{1}{(2\pi|\Gamma|)^{\frac{1}{2}}}\exp(f)\exp(s)\exp(Y). \tag{B.16}$$

The factors $\exp(Y)$ and $\exp(s)$ arose from computing the square over u_k and $\exp(Y)$ is the exponential that depends on u_k.

Although the final form of the denominator has already been derived in (B.14), it is worth going a few steps back to Equation (B.12) (highlighted), where the integration over u_k has not been carried out yet. It is possible then to rewrite Equation (B.12) in terms of $\exp(f)$, $\exp(s)$ and $\exp(Y)$ such that,

$$② = \gamma(x_{k-1}) = \frac{1}{(2\pi|\Gamma|)^{\frac{1}{2}}}\exp(f)\exp(s)\int\exp(Y)du_k. \tag{B.17}$$

Finally, from the derived results in (B.16) and (B.17), the optimal randomised controller is simplified to,

$$c^*(u_k|x_{k-1}) = \frac{①}{②} = \frac{\cancel{(2\pi|\Gamma|)^{-\frac{1}{2}}}\ \cancel{\exp(f)}\cancel{\exp(s)}\exp(Y)}{\cancel{(2\pi|\Gamma|)^{-\frac{1}{2}}}\ \cancel{\exp(f)}\cancel{\exp(s)}\int\exp(Y)du_k}, \tag{B.18}$$

and can now be represented by Definition 1 as follows,

Definition 1: Updated optimal control law $c^*(u_k|x_{k-1})$.

$$c^*(u_k|x_{k-1}) = \frac{\exp(Y)}{\int \exp(Y) du_k},$$

where,

$$\begin{aligned}\exp(Y) = \exp\Bigg\{-\frac{1}{2}\Bigg[\Bigg(& u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1})(\Sigma_2^{-1} + M_k) \\ & \times h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k))\Bigg)^T [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k) \\ & \times g(x_{k-1})]\Bigg(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1})(\Sigma_2^{-1} + M_k) \\ & \times h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k))\Bigg)\Bigg]\Bigg\}, \end{aligned} \tag{B.19}$$

and $\int \exp(Y)du_k$ is solved using Property 2 such that,

$$\int \exp(Y)du_k = |2\pi|^{\frac{1}{2}}|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})|^{-\frac{1}{2}}. \tag{B.20}$$

Hence, for nonlinear systems with functional uncertainties, the optimal controller is given by,

$$\begin{aligned} c^*(u_k|x_{k-1}) = & |2\pi|^{-\frac{1}{2}}|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})|^{\frac{1}{2}} \\ & \times \exp\Bigg\{-\frac{1}{2}\Bigg[\Bigg(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]^{-1}(g^T(x_{k-1}) \\ & \times (\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T + \text{tr}(GM_k))\Bigg)^T \\ & \times [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k)g(x_{k-1})]\Bigg(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + M_k) \\ & \times g(x_{k-1})]^{-1}(g^T(x_{k-1})(\Sigma_2^{-1} + M_k)h(x_{k-1})x_{k-1} + \frac{1}{2}(g^T(x_{k-1})T_k^T \\ & + \text{tr}(GM_k))\Bigg)\Bigg]\Bigg\}. \end{aligned} \tag{B.21}$$

From (B.21), the mean and variance of the Gaussian distribution of the optimal randomised controller can be obtained as specified by (3.55) in Theorem 2 in Section 3.4.

Appendix C

Derivation of the FP Control Solution for Nonlinear Systems with Multiplicative Noise

The designation of an optimal controller for nonlinear systems with multiplicative stochastic disturbances is demonstrated in this appendix. The first step towards achieving the optimal control solution to the FP design consists of the evaluation of the optimal performance index given by (3.10) in Proposition 1 in Chapter 3.

The term $\gamma(x_{k-1})$ is based on the terms $\beta_1(u_k, x_{k-1})$ and $\beta_2(u_k, x_{k-1})$. Hence, the term $\beta_1(u_k, x_{k-1})$ is computed first using equations (3.79) and (3.53) such that,

$$
\begin{aligned}
\beta_1(u_k, x_{k-1}) = &\int s(x_k\,|u_k, x_{k-1})\left(\ln \frac{s(x_k\,|u_k, x_{k-1})}{s^I(x_k\,|u_k, x_{k-1})}\right) dx_k,\\
= &\int \mathcal{N}(\hat{x}_k,\ \Sigma_k) \ln\left(\frac{\mathcal{N}(\hat{x}_k,\ \Sigma_k)}{\mathcal{N}(0,\ \Sigma_2)}\right) dx_k,\\
=&0.5\left[\hat{x}_k^T\Sigma_k^{-1}\hat{x}_k + \int \mathcal{N}(\hat{x}_k,\ \Sigma_k)\left\{-\ln\left(\frac{|\Sigma_k|}{|\Sigma_2|}\right) + x_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})x_k\right\} dx_k\right],\\
=&0.5\hat{x}_k^T\Sigma_k^{-1}\hat{x}_k + 0.5\text{tr}(\Sigma_k\Sigma_2^{-1}) + 0.5n - 0.5\text{tr}(\Sigma_k\Sigma_2^{-1}) - 0.5n\\
&\quad + 0.5\hat{x}_k^T(\Sigma_2^{-1} - \Sigma_k^{-1})\hat{x}_k,\\
= &\ 0.5\hat{x}_k^T(\Sigma_2^{-1} - \Sigma_k^{-1} + \Sigma_k^{-1})\hat{x}_k,
\end{aligned}
$$

where (3.75) is substituted in $\hat{x}_k$ to give,

$$
=0.5(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k)^T\Sigma_2^{-1}(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k). \tag{C.1}
$$

If the reader wants a more detailed derivation of the above, please refer to the derivation of $\beta_1(u_k, x_{k-1})$ in Appendix B.

Similary, $\beta_2(u_k, x_{k-1})$ can be evaluated using equation (3.84) as follows,

$$\begin{aligned}
\beta_2(u_k, x_{k-1}) &= -\int s(x_k|u_k, x_{k-1}) \ln(\gamma(x_k)) dx_k, \\
&= \int \mathcal{N}(\hat{x}_k, \Sigma_k) \left[0.5\left(x_k^T S_k x_k + \omega_k\right)\right] dx_k, \\
&= 0.5 \int \left\{ \mathcal{N}(\hat{x}_k, \Sigma_k) \left[(x_k - \hat{x}_k)^T S_k (x_k - \hat{x}_k) + 2x_k^T S_k \hat{x}_k - \hat{x}_k^T S_k \hat{x}_k \right] dx_k \right\} \\
&\quad + 0.5\omega_k, \\
&= 0.5\text{tr}(\Sigma_k S_k) + 0.5\hat{x}_k^T S_k \hat{x}_k + 0.5\omega_k, \\
&= 0.5(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k)^T S_k \left(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k\right) \\
&\quad + 0.5\text{tr}(\Sigma_k S_k) + 0.5\omega_k.
\end{aligned} \tag{C.2}$$

Using (3.78), the term $\text{tr}(\Sigma_k S_k)$ can be further evaluated to give,

$$\begin{aligned}
\text{tr}(\Sigma_k S_k) &= \text{tr}(S_k D x_{k-1} Q x_{k-1}^T D^T), \\
&= x_{k-1}^T D^T S_k Q D x_{k-1}.
\end{aligned}$$

Hence,

$$\beta_2(u_k, x_{k-1}) = 0.5(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k)^T S_k \left(h(x_{k-1})x_{k-1} + g(x_{k-1})u_k\right) + 0.5\hat{\omega}_k. \tag{C.3}$$

The $\hat{\omega}_k$ in (C.3) equates to,

$$\hat{\omega}_k = x_{k-1}^T D^T S_k Q D x_{k-1} + \omega_k. \tag{C.4}$$

Now that $\beta_1(u_k, x_{k-1})$ and $\beta_2(u_k, x_{k-1})$ have been computed, $\gamma(x_{k-1})$ which is given by (3.10) in

Proposition 1 can be evaluated,

$$
\begin{aligned}
\gamma(x_{k-1}) &= \int c^I(u_k|x_{k-1})\exp[-\beta_1(u_k,x_{k-1})-\beta_2(u_k,x_{k-1})]du_k,\\
&= \int \mathcal{N}(0,\Gamma)\exp\Big[-0.5(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)^T\Sigma_2^{-1}(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)\\
&\quad -0.5(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)^T S_k\,(h(x_{k-1})x_{k-1}+g(x_{k-1})u_k)-0.5\hat{\omega}_k\Big]du_k,\\
&= \frac{1}{(2\pi)^{\frac{1}{2}}|\Gamma|^{\frac{1}{2}}}\times\int \exp\Big\{-0.5\Big[u_k^T\Gamma^{-1}u_k + x_{k-1}^T h^T(x_{k-1})(\Sigma_2^{-1}+S_k)h(x_{k-1})x_{k-1}\\
&\quad +2x_{k-1}^T h^T(x_{k-1})(\Sigma_2^{-1}+S_k)g(x_{k-1})u_k + u_k^T g^T(x_{k-1})(\Sigma_2^{-1}+S_k)g(x_{k-1})u_k+\hat{\omega}_k\Big]\Big\}du_k,\\
&= \frac{1}{(2\pi)^{\frac{1}{2}}|\Gamma|^{\frac{1}{2}}}\times\Big\{\exp\Big\{-0.5\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1}+S_k)h(x_{k-1})x_{k-1}+\hat{\omega}_k\Big]\Big\}\Big\}\\
&\quad \times\int \exp\Big\{-0.5\Big[u_k^T[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+S_k)g(x_{k-1})]u_k\\
&\quad +2u_k^T g^T(x_{k-1})(\Sigma_2^{-1}+S_k)h(x_{k-1})x_{k-1}\Big]\Big\}du_k.
\end{aligned}
\tag{C.5}
$$

The integral in (C.5) can be evaluated by completing the square over u_k as shown by Property 1 in Appendix A which gives,

Table A:

$$
\begin{aligned}
\gamma(x_{k-1}) = \frac{1}{(2\pi)^{\frac{1}{2}}|\Gamma|^{\frac{1}{2}}} &\times \exp\Big\{-0.5\Big[x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1}+S_k)h(x_{k-1})x_{k-1}\\
&+\hat{\omega}_k\Big]\Big\}\\
&\times\exp\Big\{0.5\Big(g^T(x_{k-1})(\Sigma_k^{-1}+S_k)h(x_{k-1})x_{k-1}\Big)^T[\Gamma^{-1}+g^T(x_{k-1})\\
&\times(\Sigma_k^{-1}+S_k)g(x_{k-1})]^{-1}\Big(g^T(x_{k-1})(\Sigma_k^{-1}+S_k)h(x_{k-1})x_{k-1}\Big)\Big\}
\end{aligned}
$$

$$
\int \exp(Y) = \begin{cases}
\times\int\exp\Big\{-0.5\Big\{\Big(u_k+[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+S_k)g(x_{k-1})]^{-1}\\
\times(g^T(x_{k-1})(\Sigma_k^{-1}+S_k)h(x_{k-1})x_{k-1})\Big)^T[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+S_k)\\
\times g(x_{k-1})]\Big(u_k+\big[\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+S_k)g(x_{k-1})\big]^{-1}\\
\times(g^T(x_{k-1})(\Sigma_k^{-1}+S_k)h(x_{k-1})x_{k-1})\Big)\Big\}du_k.
\end{cases}
$$

The integral $\int \exp{(Y)}$ in *Table A* can be solved using Property 2 in Appendix A which simplifies it to,

$$
\int \exp(Y)du_k = |2\pi|^{\frac{1}{2}}|\Gamma^{-1}+g^T(x_{k-1})(\Sigma_2^{-1}+S_k)g(x_{k-1})|^{-\frac{1}{2}}. \tag{C.6}
$$

Hence, $\gamma(x_{k-1})$ is simplified to,

$$\begin{aligned}\gamma(x_{k-1}) =& |\Gamma|^{-\frac{1}{2}}|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + S_k)g(x_{k-1})|^{-\frac{1}{2}} \\ &\times \exp\bigg\{0.5(g^T(x_{k-1})(\Sigma_k^{-1} + S_k)h(x_{k-1})x_{k-1})^T[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + S_k) \\ &\times g(x_{k-1})]^{-1}(g^T(x_{k-1})(\Sigma_k^{-1} + S_k)h(x_{k-1})x_{k-1}) - 0.5x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + S_k) \\ &\times h(x_{k-1})x_{k-1} - 0.5\hat{\omega}_k\bigg\}, \\ =& \exp\bigg\{0.5x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + S_k)g(x_{k-1})\bigg(\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + S_k)g(x_{k-1})\bigg)^{-1} \\ &\times g^T(x_{k-1})(\Sigma_k^{-1} + S_k)h(x_{k-1})x_{k-1} - 0.5x_{k-1}^T h^T(x_{k-1})(\Sigma_k^{-1} + S_k)h(x_{k-1})x_{k-1} \\ &-0.5\hat{\omega}_k - 0.5\ln|\Gamma| - 0.5\ln|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_k^{-1} + S_k)g(x_{k-1})|\bigg\}.\end{aligned}$$

By making the substitution of $\hat{\omega}_k = x_{k-1}^T D^T S_k Q D x_{k-1} + \omega_k$, the following can be obtained,

$$\begin{aligned}\gamma(x_{k-1}) =& \exp\bigg\{-0.5x_{k-1}^T\bigg\{-h^T(x_{k-1})(\Sigma_2^{-1} + S_k)g(x_{k-1})[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + S_k) \\ &\times g(x_{k-1})]^{-1}g^T(x_{k-1})(\Sigma_2^{-1} + S_k)^T h(x_{k-1}) + h^T(x_{k-1})(\Sigma_2^{-1} + S_k)h(x_{k-1}) \\ &- D^T S_k Q D\bigg\}x_{k-1} - 0.5\bigg\{\omega_k + \ln(|\Gamma|) + \ln|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + S_k) \\ &\times g(x_{k-1})|\bigg\}\bigg\}.\end{aligned} \tag{C.7}$$

From the above, the generalised SDRE in (3.85) and the constant term in (3.86) can be obtained. It is now possible to derive the optimal control law which is defined in (3.7) in Proposition 1 and repeated below,

$$c^*(u_k|x_{k-1}) = \frac{c^I(u_{k-1}|x_{k-1})\exp[-\beta_1(u_k, x_{k-1}) - \beta_2(u_k, x_{k-1})] \longleftarrow ①}{\gamma(x_{k-1}) \longleftarrow ②}.$$

However, using the result in Definition 1 in Appendix B, the process of finding the optimal controller has been tremendously simplified since now $c^*(u_k|x_{k-1}) = \frac{\exp(Y)}{\int \exp(Y)du_k}$. The exponential $\exp(Y)$ has already been found *Table A* and its integral over u_k has been solved in (C.6). Therefore, the final control solution of the FP control design for nonlinear systems with functional uncertainties is given by,

$$
\begin{aligned}
c^*(u_k|x_{k-1}) =& |2\pi|^{-\frac{1}{2}}|\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + S_k)g(x_{k-1})|^{\frac{1}{2}} \\
& \times \exp\bigg\{-0.5\bigg\{\bigg(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + S_k)g(x_{k-1})]^{-1}(g^T(x_{k-1}) \\
& \times (\Sigma_2^{-1} + S_k))h(x_{k-1})x_{k-1})\bigg)^T[\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + S_k)g(x_{k-1})] \\
& \times \bigg(u_k + [\Gamma^{-1} + g^T(x_{k-1})(\Sigma_2^{-1} + S_k)g(x_{k-1})]^{-1}(g^T(x_{k-1})(\Sigma_2^{-1} + S_k) \\
& \times h(x_{k-1})x_{k-1})\bigg)\bigg\}\bigg\},
\end{aligned} \tag{C.8}
$$

which concludes the proof for Theorem 3.

Appendix D

Derivation of the Decentralised FP Control Design for Nonlinear Systems

The derivation of the decentralised FP control solution given by (4.21) for nonlinear stochastic subsystems with additive noises is discussed in this appendix. Theorem 4 is proven using Proposition 4 in Chapter 4 for nonlinear systems within a decentralised framework. This is achieved in two parts. The first part contributes to the justification of the form of $-\ln(\gamma(z_{k;i}))$ in (4.26) and the Riccati equations in Theorem 4. The second half of the proof focuses on the proof of the randomised optimal controller $c^{*}(u_{k;i}|z_{k-1;i})$ given by equation (4.21).

D.1 Optimal Performance Index, $-\ln(\gamma(z_{k;i}))$

The first part is proven by using proof by induction, meaning the form of the optimal performance index defined in equation (4.26) is assumed to be true and then substituted in $\beta(u_{k;i}, z_{k-1;i})$.

In Proposition 4, the optimal performance index is given by $\gamma(z_{k-1;i})$ in (4.6) for which the term $\beta(u_{k;i}, z_{k-1;i})$ in (4.7) needs to be solved.

However, the term is evaluated by splitting it into two parts as follows,

$$\begin{aligned}\beta(u_{k;i}, z_{k-1;i}) &= \int s(x_{k;i}|u_{k;i}, z_{k-1;i})\left[\ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i})}{s^{I}(x_{k;i}|u_{k;i}, z_{k-1;i})}\right) - \ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))\right]dx_{k;i},\\ &= \underbrace{\int s(x_{k;i}|u_{k;i}, z_{k-1;i})\ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i})}{s^{I}(x_{k;i}|u_{k;i}, z_{k-1;i})}\right)dx_{k;i}}_{①}\\ &\quad \underbrace{-\int s(x_{k;i}|u_{k;i}, z_{k-1;i})\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))dx_{k;i}}_{②}\,. \end{aligned} \tag{D.1}$$

First of all, the first integral ① is solved for which the actual and ideal distributions of the internal states given by (4.16) and (4.18) respectively, need to be substituted. This gives,

$$
\begin{aligned}
① =& \int s(x_{k;i}|u_{k;i}, z_{k-1;i}) \ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i})}{s^I(x_{k;i}|u_{k;i}, z_{k-1;i})}\right) dx_{k;i}, \\
=& \int \mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i}) \Big[\ln(2\pi|\Sigma_{x;i}|)^{-\frac{1}{2}} - \frac{1}{2}(x_{k;i} - \hat{x}_{k;i})^T \Sigma_{x;i}^{-1}(x_{k;i} - \hat{x}_{k;i}) - \ln(2\pi|\Sigma_{x;i}|)^{-\frac{1}{2}} \\
& + \frac{1}{2} x_{k;i}^T \Sigma_{x;i}^{-1} x_{k;i}^{-1} \Big] dx_{k;i}, \\
=& \int \mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i}) \left(x_{k;i}^T \Sigma_{x;i}^{-1} \hat{x}_{k;i} - \frac{1}{2} \hat{x}_{k;i}^T \Sigma_{x;i}^{-1} \hat{x}_{k;i} \right) dx_{k;i}, \\
=& \frac{1}{2} \hat{x}_{k;i}^T \Sigma_{x;i}^{-1} \hat{x}_{k;i}, \\
=& (f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i})^T \Sigma_{x;i}^{-1} (f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i}), \qquad \text{(D.2)}
\end{aligned}
$$

where we used equation (4.14). Secondly, ② is evaluated for which $\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))$ given by (4.8) needs to be computed first. To achieve this, the actual distribution of the external states defined in (4.17) and the assumed form of the performance index in (4.26) are substituted. However, the performance index is written explicitly rather than in matrix format such that $-\ln(\gamma(z_{k;i})) = \frac{1}{2} z_{k;i}^T M_{k;i} z_{k;i} + \frac{1}{2} V_{k;i} = \frac{1}{2}(x_{k;i}^T M_{1,k;i} x_{k;i} + 2x_{k;i}^T M_{2,k;i} y_{k;i} + y_{k;i}^T M_{3,k;i} y_{k;i} + V_{k;i})$. Hence, we obtain,

$$
\begin{aligned}
& - \ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i})) \\
&= - \int s(y_{k;i}|y_{k-1;i}) \ln(\gamma(z_{k;i})) dy_{k;i}, \\
&= \int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}) \left\{ \frac{1}{2} \Big[x_{k;i}^T M_{1,k;i} x_{k;i} + 2x_{k;i}^T M_{2,k;i} y_{k;i} + y_{k;i}^T M_{3,k;i} y_{k;i} + V_{k;i} \Big] \right\} dy_{k;i}, \\
&= \frac{1}{2} x_{k;i}^T M_{1,k;i} x_{k;i} + x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \frac{1}{2} V_{k;i} + \frac{1}{2} \int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}) \Big[(y_{k;i} - \hat{y}_{k;i})^T M_{3,k;i} \\
& \quad \times (y_{k;i} - \hat{y}_{k;i}) + 2\hat{y}_{k;i}^T M_{3,k;i} y_{k;i} - \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} \Big] dy_{k;i}, \\
&= \frac{1}{2} x_{k;i}^T M_{1,k;i} x_{k;i} + x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \frac{1}{2} \left(\operatorname{tr}(M_{3,k;i} \Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + V_{k;i} \right). \qquad \text{(D.3)}
\end{aligned}
$$

Substituting (D.3) into ② gives,

$$
\begin{aligned}
② = &\int s(x_{k;i}|u_{k;i}, z_{k-1;i}) \bigg[\frac{1}{2} x_{k;i}^T M_{1,k;i} x_{k;i} + x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \frac{1}{2}\bigg(\text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i} \\
&+ \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i}\bigg)\bigg] dx_{k;i}, \\
=& \frac{1}{2} \int \mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i}) \bigg[x_{k;i}^T M_{1,k;i} x_{k;i} + 2x_{k;i}^T M_{2,k;i} \hat{y}_{k;i}\bigg] dx_{k;i} + \frac{1}{2}\bigg(\text{tr}(M_{3,k;i}\Sigma_{y;i}) \\
&+ \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + V_{k;i}\bigg), \\
=& \frac{1}{2} \int \mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i}) \bigg[(x_{k;i} - \hat{x}_{k;i})^T M_{1,k;i} (x_{k;i} - \hat{x}_{k;i}) + 2\hat{x}_{k;i}^T M_{1,k;i} x_{k;i} - \hat{x}_{k;i}^T M_{1,k;i} \hat{x}_{k;i} \\
&+ 2x_{k;i}^T M_{2,k;i} \hat{y}_{k;i}\bigg] dx_{k;i} + \frac{1}{2}\bigg(\text{tr}(M_{3,k;i}\Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + V_{k;i}\bigg), \\
=& \frac{1}{2}\bigg(\hat{x}_{k;i}^T M_{1,k;i} \hat{x}_{k;i} + 2\hat{x}_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg), \\
=& \frac{1}{2}\bigg((f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i})^T M_{1,k;i} (f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i}) \\
&+ 2(f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i})^T M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} \\
&+ y_{k-1;i}^T h_i^T(y_{k-1;i}) M_{3,k;i} h_i(y_{k-1;i}) y_{k-1;i} + \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg), \quad \text{(D.4)}
\end{aligned}
$$

where we used (4.14) and (4.15). Now that ① and ② have been evaluated in (D.2) and (D.4), respectively, the term $\beta(u_{k;i}, z_{k-1;i})$ can be obtained as follows,

$$
\begin{aligned}
\beta(u_{k;i}, z_{k-1;i}) =& \frac{1}{2}\bigg((f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i})^T \tilde{Q}_{k;i} (f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i}) \\
&+2(f_i(z_{k-1;i}) z_{k-1;i} + g_i(z_{k-1;i}) u_{k;i})^T M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} \\
&+y_{k-1;i}^T h_i^T(y_{k-1;i}) M_{3,k;i} h_i(y_{k-1;i}) y_{k-1;i} + \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg),
\end{aligned}
\quad \text{(D.5)}
$$

where $\tilde{Q}_{k;i} = (M_{1,k;i} + \Sigma_{k;i}^{-1})$.

Following the evaluation of $\beta(u_{k;i}, z_{k-1;i})$ in (D.5), and using $c^I(u_{k;i}|z_{k-1;i}) = \mathcal{N}(0, \Gamma_{k;i})$, the term $\gamma(z_{k-1;i})$ in (4.6) can be solved. Also, $f_i(z_{k-1;i}) z_{k-1;i}$ is written in terms of the internal and external states to separate the elements for the Riccati equation solutions in the upcoming derivations such that, $f_i(z_{k-1;i})z_{k-1;i} = f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}$. Therefore, we obtain,

$$\begin{aligned}
\gamma(z_{k-1;i}) &= \int c^I(u_{k;i}|z_{k-1;i}) \exp(-\beta(u_{k;i}, z_{k-1;i}))du_{k;i},\\
&= \int \mathcal{N}(0, \Gamma_{k;i}) \exp(-\beta(u_{k;i}, z_{k-1;i}))du_{k;i}\\
&= |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \int \exp\bigg\{ -\frac{1}{2}\bigg[u^T_{k;i}\Gamma^{-1}_{k;i}u_{k;i} + (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}\\
&+ g_i(z_{k-1;i})u_{k;i})^T\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i})\\
&+ 2(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\\
&+ y^T_{k-1;i}h^T_i(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i})\\
&+ V_{k;i}\bigg]\bigg\}du_{k;i},\\
&= |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i}\\
&+ f_{2i}(z_{k-1;i})y_{k-1;i}) + 2(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\\
&+ y^T_{k-1;i}h^T_i(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg]\bigg\}\\
&\times \int \exp\bigg\{-\frac{1}{2}\bigg[u^T_{k;i}[\Gamma^{-1}_{k;i} + g^T_i(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})]u_{k;i} + 2u^T_{k;i}[g^T_i(z_{k-1;i})\\
&\times \tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + g^T_i(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}]\bigg]\bigg\}.
\end{aligned} \tag{D.6}$$

The integral in (D.6) can be further evaluated by completing the square over $u_{k;i}$ which has been explained in Appendix A, Property 1.

Let us define $\bar{\Gamma}_{k;i} = [\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})]^{-1}$. This gives the following,

Table B:

$$\gamma(z_{k-1;i}) = |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T\tilde{Q}_{k;i}$$
$$\times (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + 2(f_{1i}(z_{k-1;i})x_{k-1;i}$$
$$+ f_{2i}(z_{k-1;i})y_{k-1;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}$$
$$\times h_i(y_{k-1;i})y_{k-1;i} + \mathrm{tr}(M_{1,k;i}\Sigma_{x;i}) + \mathrm{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg]\bigg\}$$
$$\times \exp\bigg\{\frac{-1}{2}\bigg[-\bigg(g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})$$
$$\times y_{k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\bigg)^T\bar{\Gamma}_{k;i}\bigg(g_i^T(z_{k-1;i})\tilde{Q}_{k;i}$$
$$\times (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i}$$
$$\times h_i(y_{k-1;i})y_{k-1;i}\bigg)\bigg]\bigg\}$$

$$\int \exp(Y)du_{k;i} = \begin{cases} \times \int \exp\bigg\{-\frac{1}{2}\bigg[\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} \\ +f_{2i}(z_{k-1;i})y_{k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}]\bigg)^T\bar{\Gamma}_{k;i}^{-1} \\ \times\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) \\ +g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}]\bigg)\bigg]\bigg\}du_{k;i}. \end{cases}$$

The integral $\int \exp(Y)du_{k;i}$ can be evaluated using Property 2 in Appendix A such that,

$$\int \exp(Y)du_{k;i} = |2\pi|^{\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}}. \tag{D.7}$$

Hence, we have,

$$
\begin{aligned}
\gamma(z_{k-1;i}) =& |\Gamma_{k;i}|^{-\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}} \exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i}) \\
& \times x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + 2(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T M_{2,k;i} \\
& \times h_i(y_{k-1;i})y_{k-1;i} + y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + \text{tr}(M_{1,k;i}\Sigma_{x;i}) \\
& + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg]\bigg\} \\
& \times \exp\bigg\{-\frac{1}{2}\bigg[-\bigg(g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) \\
& + g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\bigg)^T\bar{\Gamma}_{k;i}\bigg(g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} \\
& + f_{2i}(z_{k-1;i})y_{k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\bigg)\bigg]\bigg\}, \\
=& \exp\bigg\{-\frac{1}{2}\bigg[x_{k-1;i}^T\bigg(-f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i}) \\
& + f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i})\bigg)x_{k-1;i} + 2x_{k-1;i}^T\bigg(f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) \\
& + f_{1i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) - f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) \\
& - f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}),\bigg)y_{k-1;i} \\
& + y_{k-1;i}^T\bigg(f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) + 2f_{2i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) + h_i^T(y_{k-1;i}) \\
& \times M_{3,k;i}h_i(y_{k-1;i}) - f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i}) \\
& - h_i^T(y_{k-1;i})M_{2,k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\
& - 2f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\bigg)y_{k-1;i} \\
& + \bigg(V_{k;i} + \text{tr}(\Sigma_{x;i}M_{1,k;i}) + \text{tr}(\Sigma_{y;i}M_{3,k;i}) + \ln|\Gamma_{k;i}| + \ln|\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i}) \\
& \times \tilde{Q}_{k;i}g_i(x_{k-1;i})|\bigg)\bigg]\bigg\}, \qquad \text{(D.8)}
\end{aligned}
$$

which justifies the form of the performance index (4.26) and the Riccati equation solutions and constant term given by (4.28) - (4.31).

D.2 Optimal Randomised Controller, $c^*(u_{k;i}|z_{k-1;i})$

The optimal randomised controller is obtained by evaluating the optimal control law in (4.5) given by Proposition 4 in Chapter 4.

Using the same logic as was explained in Appendix B in Definition 1, only $\frac{\exp(Y)}{\int \exp(Y) du_{k;i}}$ needs to be computed to find $c^*(u_{k;i}|z_{k-1;i})$. The exponential $\exp(Y)$ is obtained from *Table B* and

$\int \exp(Y) du_{k;i}$ is solved in (D.7). Thus, the optimal randomised controller is given by

$$\begin{aligned} c^*(u_{k;i}|z_{k-1;i}) =&(2\pi)^{-\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{-\frac{1}{2}} \int \exp\bigg\{-\frac{1}{2}\bigg[\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} \\ &+ f_{2i}(z_{k-1;i})y_{k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}]\bigg)^T \bar{\Gamma}_{k;i}^{-1} \\ &\times \bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) \\ &+ g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}]\bigg)\bigg]\bigg\} du_{k;i}, \end{aligned} \tag{D.9}$$

which is the desired form as given by (4.21) in Theorem 4 in Chapter 4.

Appendix E

Derivation of the Probabilistic Message Passing Approach

This chapter explains the derivation of the probabilistic message passing approach as outlined in Theorem 5 and Theorem 6. The evaluation of the two theorems require the implementation of two identities, namely the Woodbury Identity [195] and the Push-through Identity [196].

The Woodbury Identity is defined as follows,

Property 3:

$$(\hat{\mathbf{A}} + \hat{\mathbf{U}}\hat{\mathbf{C}}\hat{\mathbf{V}})^{-1} = \hat{\mathbf{A}}^{-1} - \hat{\mathbf{A}}^{-1}\hat{\mathbf{U}}(\hat{\mathbf{C}}^{-1} + \hat{\mathbf{V}}\hat{\mathbf{A}}^{-1}\hat{\mathbf{U}})^{-1}\hat{\mathbf{V}}\hat{\mathbf{A}}^{-1}.$$

The push-through identity is given by,

Property 4:

$$(\mathbf{\Gamma}^{-1} + \mathbf{B}^T\mathbf{\Sigma}^{-1}\mathbf{B})\mathbf{B}^T = \mathbf{\Gamma}\mathbf{B}^T(\mathbf{\Sigma} + \mathbf{B}\mathbf{\Gamma}\mathbf{B}^T)^{-1}\mathbf{\Sigma}.$$

E.1 Proof of Theorem 5

Theorem 5 is proven by evaluating the integral given by (4.33) in Lemma 1. The integral can be further evaluated by applying the chain rule for probabilities [163] to the joint pdf to obtain,

$$\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) = \int s(x_{k;i}|z_{k-1;i}, u_{k;i})s(y_{k;i}|y_{k-1;i})c(u_{k;i}|z_{k-1;i})dy_{k;i}du_{k;i}. \tag{E.1}$$

The factor $s(y_{k;i}|y_{k-1;i})$ in (E.1) can be integrated over $y_{k;i}$ which gives a normalisation constant. As a consequence, we have,

$$\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) = \int s(x_{k;i}|z_{k-1;i}, u_{k;i})c(u_{k;i}|z_{k-1;i})du_{k;i}. \tag{E.2}$$

This can be evaluated by substituting (4.16), (4.14) and (4.21) into (E.2) to obtain,

$$\begin{aligned}
&\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) \\
&= \int \exp\bigg\{-\frac{1}{2}\bigg[\Big(x_{k;i} - (f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u_{k;i})\Big)^T \Sigma_{x;i}^{-1} \\
&\times \Big(x_{k;i} - (f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u_{k;i})\Big) + \Big(u_{k;i} - u^*_{k;i}\Big)^T \bar{\Gamma}_{k;i}^{-1}\Big(u_{k;i} - u^*_{k;i}\Big)\bigg]\bigg\}du_{k;i}.
\end{aligned} \tag{E.3}$$

The terms that have no dependency on $u_{k;i}$ can be taken out of the integral such that,

$$\begin{aligned}
&\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) \\
&= \exp\bigg\{-\frac{1}{2}\bigg[(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i})^T\Sigma_{x;i}^{-1}(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + u^{*T}_{k;i}\bar{\Gamma}_{k;i}^{-1}u^*_{k;i}\bigg]\bigg\} \\
&\times \int \exp\bigg\{-\frac{1}{2}\bigg[u^T_{k;i}(g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1}g_i(z_{k-1;i}) + \bar{\Gamma}_{k;i}^{-1})u_{k;i} - 2u_{k;i}\Big(g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1} \\
&\times (x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + \bar{\Gamma}_{k;i}^{-1}u^*_{k;i}\Big)\bigg]\bigg\}du_{k;i}.
\end{aligned} \tag{E.4}$$

The integral can be further evaluated by completing the square over $u_{k;i}$ in (E.4), as shown by *Property 1* in Appendix A, gives,

$$\begin{aligned}
&\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) \\
&= \exp\bigg\{-\frac{1}{2}\bigg[(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i})^T{\Sigma_{x;i}}^{-1}(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + u^*_{k;i}\bar{\Gamma}_{k;i}^{-1}u^*_{k;i} \\
&- (g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1}(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + \bar{\Gamma}_{k;i}^{-1}u^*_{k;i})^T(g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1}g_i(z_{k-1;i}) \\
&+ \bar{\Gamma}_{k;i}^{-1})^{-1}(g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1}(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + \bar{\Gamma}_{k;i}^{-1}u^*_{k;i})\bigg]\bigg\} \\
&\times \int \exp\bigg\{-\frac{1}{2}\bigg[\Big(u_{k;i} + (g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1}g_i(z_{k-1;i}) + \bar{\Gamma}_{k;i}^{-1})^{-1}(g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1} \\
&\times (x_{k;i} - f_i(z_{k-1;i})z_{k-1;i})^T + \bar{\Gamma}_{k;i}^{-1}u^*_{k;i})\Big)^T(g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1}g_i(z_{k-1;i}) + \bar{\Gamma}_{k;i}^{-1})\Big(u_{k;i} + (g_i^T(z_{k-1;i}) \\
&\times \Sigma_{x;i}^{-1}g_i(z_{k-1;i}) + \bar{\Gamma}_{k;i}^{-1})^{-1}(g_i^T(z_{k-1;i})\Sigma_{x;i}^{-1}(x_k - f_i(z_{k-1;i})z_{k-1;i})^T + \bar{\Gamma}_{k;i}^{-1}u^*_{k;i})\Big)\bigg]\bigg\}du_{k;i}.
\end{aligned} \tag{E.5}$$

The integral can be solved using *Property 2* in Appendix A which results in a normalisation constant which is disregarded in this proof for the purpose of simplification. Hence, we have,

$$\begin{aligned}
&\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) \\
&= \exp\Bigg\{-\frac{1}{2}\Bigg[(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i})^T \Sigma_{x;i}{}^{-1}(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + u^*_{k;i}\bar{\Gamma}^{-1}_{k;i}u^*_{k;i} \\
&- (g_i^T(z_{k-1;i})\Sigma^{-1}_{x;i}(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + \bar{\Gamma}^{-1}_{k;i}u^*_{k;i})^T(g_i^T(z_{k-1;i})\Sigma^{-1}_{x;i}g_i(z_{k-1;i}) \\
&+ \bar{\Gamma}^{-1}_{k;i})^{-1}(g_i^T(z_{k-1;i})\Sigma^{-1}_{x;i}(x_{k;i} - f_i(z_{k-1;i})z_{k-1;i}) + \bar{\Gamma}^{-1}_{k;i}u^*_{k;i})\Bigg]\Bigg\}. \qquad \text{(E.6)}
\end{aligned}$$

Using the Woodbury Identity [195] defined in *Property 3*, and the push-through Identity [196] stated in *Property 4*, Equation (E.6) can be further evaluated to obtain the following,

$$\begin{aligned}
&\mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) \\
&= \exp\Bigg\{-\frac{1}{2}\Bigg[\Bigg(x_{k;i} - (f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u^*_{k;i})\Bigg)^T\Bigg[g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i}) + \Sigma_{x;i}\Bigg]^{-1} \\
&\times\Bigg(x_{k;i} - (f_i(z_{k-1;i})z_{k-1;i} + g_i(z_{k-1;i})u^*_{k;i})\Bigg)\Bigg]\Bigg\}, \qquad \text{(E.7)}
\end{aligned}$$

which concludes the proof.

E.2 Proof of Theorem 6

Fusing the message that has been passed from node i given by (4.34) with the prior knowledge that subsystem j already possesses about its external states, $s(y_{k;j}|y_{k-1;j}) = \mathcal{N}(\hat{y}_{k;j}, \Sigma_{y;j})$, can be achieved by using the Bayes' rule. This is obtained from the Bayes' theorem by using the MAP estimate. This gives,

$$\begin{aligned}
s(y_{k;j;new}) =& \mathcal{M}_{j\leftarrow i}(x_{k;i}|z_{k-1;i}) \times s(y_{k;i}|y_{k-1;i}) \\
=& \exp\Bigg\{-\frac{1}{2}(y_{k;j} - \mu_{x_{k;i}})^T\mathfrak{P}^{-1}_{x_{k;i}}(y_{k;j} - \mu_{x_{k;i}}) - \frac{1}{2}(y_{k;j} - \hat{y}_{k;j})^T\Sigma^{-1}_{y;j}(y_{k;j} - \hat{y}_{k;j})\Bigg\} \\
=& \exp\Bigg\{-\frac{1}{2}y^T_{k;j}(\mathfrak{P}^{-1}_{x_{k;i}} + \Sigma^{-1}_{y;j})y_{k;j} + y^T_{k;j}(\mathfrak{P}^{-1}_{x_{k;i}}\mu_{x_{k;i}} + \Sigma^{-1}_{y;j}\hat{y}_{k;j}) - \frac{1}{2}\mu^T_{x_{k;i}}\mathfrak{P}^{-1}_{x_{k;i}}\mu_{x_{k;i}} \\
&- \frac{1}{2}\hat{y}^T_{k;j}\Sigma^{-1}_{y;j}\hat{y}_{k;j}\Bigg\}. \qquad \text{(E.8)}
\end{aligned}$$

Completing the square allows Equation (E.8) to be further evaluated to obtain,

$$
\begin{aligned}
s(y_{k;j;new}) = \exp\Big\{ & -\frac{1}{2}(y_{k;j} - \hat{y}_{k;j,new})^T \bar{\Sigma}_{y_{k;j},new}(y_{k;j} - \hat{y}_{k;j,new}) \\
& + (\mathfrak{P}_{x_{k;i}}^{-1}\mu_{x_{k;i}} + \Sigma_{y;j}^{-1}\hat{y}_{k;j})^T \bar{\Sigma}_{y_{k;j},new}(\mathfrak{P}_{x_{k;i}}^{-1}\mu_{x_{k;i}} + \Sigma_{y;j}^{-1}\hat{y}_{k;j}) \\
& - \frac{1}{2}\mu_{x_{k;i}}^T \mathfrak{P}_{x_{k;i}}^{-1}\mu_{x_{k;i}} - \frac{1}{2}\hat{y}_{k;j}^T \Sigma_{y;j}^{-1}\hat{y}_{k;j} \Big\},
\end{aligned}
\tag{E.9}
$$

where

$$
\hat{y}_{k;j,new} = (\mathfrak{P}_{x_{k;i}}^{-1} + \Sigma_{y;j}^{-1})^{-1}(\mathfrak{P}_{x_{k;i}}^{-1}\mu_{x_{k;i}} + \Sigma_{y;j}^{-1}\hat{y}_{k;j}), \tag{E.10}
$$

$$
\bar{\Sigma}_{y_{k;j},new} = (\mathfrak{P}_{x_{k;i}}^{-1} + \Sigma_{y;j}^{-1})^{-1}. \tag{E.11}
$$

Equations (E.10) and (E.11) can be further evaluated using the Woodbury identity, and the substitution of $\bar{K}_{k;j} = \Sigma_{y;j}(\Sigma_{y;j} + \mathfrak{P}_{x_{k;i}})^{-1}$ allows the claimed form given by (4.38) - (4.41) to be achieved.

Appendix F

Derivation of the Decentralised FP Control Design for Nonlinear Systems with Multiplicative Noises

Theorem 7 states the decentralised optimal controller for nonlinear systems with multiplicative noises within a decentralised framework. To prove this, Proposition 4 in Chapter 4 for nonlinear systems within a decentralised framework is considered.

The proof is split into two parts. The first part contributes to the justification of the form of $-\ln(\gamma(z_{k;i}))$ and the Riccati equation solutions in Theorem 7. The second half proves the form of the randomised optimal controller $c^*(u_{k;i}|z_{k-1;i})$ given in equation (4.65).

F.1 Optimal Performance Index, $-\ln(\gamma(z_{k;i}))$

Using proof by induction with the assumption that the form of the quadratic performance index in (4.70) is true, the forms of the Riccati equation solutions (4.72) - (4.74) and the quadratic cost functions (4.70) are justified.

In Proposition 4 in Chapter 4, the term $\beta(u_{k;i}, z_{k-1;i})$ defined in equation (4.7) needs to be solved in order to derive the optimal performance index given by $\gamma(z_{k-1;i})$ as stated in equation (4.6).

Similar to Appendix D, the term $\beta(u_{k;i}, z_{k-1;i})$ is evaluated by splitting it into two parts as follows,

$$\beta(u_{k;i}, z_{k-1;i}) = \int s(x_{k;i}|u_{k;i}, z_{k-1;i})\left[\ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i})}{s^I(x_{k;i}|u_{k;i}, z_{k-1;i})}\right) - \ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))\right]dx_{k;i},$$
$$= \underbrace{\int s(x_{k;i}|u_{k;i}, z_{k-1;i})\ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i})}{s^I(x_{k;i}|u_{k;i}, z_{k-1;i})}\right)dx_{k;i}}_{①}$$
$$\underbrace{-\int s(x_{k;i}|u_{k;i}, z_{k-1;i})\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))dx_{k;i}}_{②}. \tag{F.1}$$

First of all, the first integral ① is solved for which the actual and ideal distributions of the internal states given by (4.56) and (4.62) need to be substituted. This gives,

$$① = \int s(x_{k;i}|u_{k;i}, z_{k-1;i})\ln\left(\frac{s(x_{k;i}|u_{k;i}, z_{k-1;i})}{s^I(x_{k;i}|u_{k;i}, z_{k-1;i})}\right)dx_{k;i},$$
$$= \int \mathcal{N}(\hat{x}_{k;i}, R_{x;i})\Big[-\frac{1}{2}\ln(2\pi|R_{x;i}|) - \frac{1}{2}(x_{k;i} - \hat{x}_{k;i})^T R_{x;i}^{-1}(x_{k;i} - \hat{x}_{k;i}) + \frac{1}{2}\ln(2\pi|\Sigma_{x;i}|)$$
$$+ \frac{1}{2}x_{k;i}^T\Sigma_{x;i}^{-1}x_{k;i}\Big]dx_{k;i},$$
$$= \int \mathcal{N}(\hat{x}_{k;i}, R_{x;i})\Big\{\frac{1}{2}\Big[\ln\left(\frac{|\Sigma_{x;i}|}{|R_{x;i}|}\right) - x_{k;i}^T R_{x;i}^{-1}x_{k;i} + x_{k;i}^T\Sigma_{x;i}^{-1}x_{k;i} + 2x_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i}$$
$$- \hat{x}_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i}\Big]\Big\}dx_{k;i},$$
$$= \int \mathcal{N}(\hat{x}_{k;i}, R_{x;i})\Big\{\frac{1}{2}\Big[\ln\left(\frac{|\Sigma_{x;i}|}{|R_{x;i}|}\right) + x_{k;i}^T(\Sigma_{x;i}^{-1} - R_{x;i}^{-1})x_{k;i} + 2x_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i}$$
$$- \hat{x}_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i}\Big]\Big\}dx_{k;i},$$
$$= \frac{1}{2}\hat{x}_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i} + \int \mathcal{N}(\hat{x}_{k;i}, R_{x;i})\Big\{\frac{1}{2}\Big[-\ln\left(\frac{|R_{x;i}|}{|\Sigma_{x;i}|}\right) + (x_{k;i} - \hat{x}_{k;i})^T(\Sigma_{x;i}^{-1} - R_{x;i}^{-1})$$
$$\times (x_{k;i} - \hat{x}_{k;i}) + 2x_{k;i}^T(\Sigma_{x;i}^{-1} - R_{x;i}^{-1})\hat{x}_{k;i} - \hat{x}_{k;i}^T(\Sigma_{x;i}^{-1} - R_{x;i}^{-1})\hat{x}_{k;i}\Big]\Big\}dx_{k;i},$$
$$= \frac{1}{2}\Big\{\hat{x}_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i} - \text{tr}(R_{x;i}\Sigma_{x;i}^{-1}) + n + \text{tr}(R_{x;i}[\Sigma_{x;i}^{-1} - R_{x;i}^{-1}]) + \hat{x}_{k;i}^T(\Sigma_{x;i}^{-1} - R_{x;i}^{-1})\hat{x}_{k;i}\Big\},$$
$$= \frac{1}{2}\Big\{\cancel{\hat{x}_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i}} - \cancel{\text{tr}(R_{x;i}\Sigma_{x;i}^{-1})} + \cancel{n} + \cancel{\text{tr}(R_{x;i}\Sigma_{x;i}^{-1})} - \cancel{n} + \hat{x}_{k;i}^T\Sigma_{x;i}^{-1}\hat{x}_{k;i} - \cancel{\hat{x}_{k;i}^T R_{x;i}^{-1}\hat{x}_{k;i}}\Big\},$$
$$= \frac{1}{2}\hat{x}_{k;i}^T\Sigma_{x;i}^{-1}\hat{x}_{k;i},$$
$$= (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i})^T\Sigma_{x;i}^{-1}$$
$$\times (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}), \tag{F.2}$$

where equation (4.58) has been used.

Secondly, ② is evaluated for which $\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))$ defined in (4.8) needs to be solved first. The actual distribution of the external states defined in (4.57) and the assumed form of the performance index $-\ln(\gamma(z_{k;i})) = \frac{1}{2}(x_{k;i}^T M_{1,k;i}x_{k;i} + 2x_{k;i}^T M_{2,k;i}y_{k;i} + y_{k;i}^T M_{3,k;i}y_{k;i} + V_{k;i})$ given by (4.70) are substituted to solve $\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i}))$ which gives,

$$\begin{aligned}
-\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i})) =& -\int s(y_{k;i}|y_{k-1;i}) \ln(\gamma(z_{k;i})) dy_{k;i}, \\
=& \int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}) \Big\{ \frac{1}{2} \Big[x_{k;i}^T M_{1,k;i} x_{k;i} + 2 x_{k;i}^T M_{2,k;i} y_{k;i} + y_{k;i}^T M_{3,k;i} y_{k;i} \\
& + V_{k;i} \Big] \Big\} dy_{k;i}, \\
=& \frac{1}{2} x_{k;i}^T M_{1,k;i} x_{k;i} + x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \frac{1}{2} V_{k;i} + \frac{1}{2} \int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}) \Big[(y_{k;i} - \hat{y}_{k;i})^T \\
& \times M_{3,k;i} (y_{k;i} - \hat{y}_{k;i}) + 2 \hat{y}_{k;i}^T M_{3,k;i} y_{k;i} - \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} \Big] dy_{k;i}, \\
=& \frac{1}{2} x_{k;i}^T M_{1,k;i} x_{k;i} + x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \frac{1}{2} \Big(\mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + V_{k;i} \Big).
\end{aligned} \tag{F.3}$$

Substituting (F.3) into ② gives,

$$\begin{aligned}
② =& \int s(x_{k;i}|u_{k;i}, z_{k-1;i}) \Big[\frac{1}{2} x_{k;i}^T M_{1,k;i} x_{k;i} + x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \frac{1}{2} \Big(\mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) \\
& + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + V_{k;i} \Big) \Big] dx_{k;i}, \\
=& \frac{1}{2} \int \mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i}) \Big[x_{k;i}^T M_{1,k;i} x_{k;i} + 2 x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} \Big] dx_{k;i} + \frac{1}{2} \Big(\mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) \\
& + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + V_{k;i} \Big), \\
=& \frac{1}{2} \int \mathcal{N}(\hat{x}_{k;i}, \Sigma_{x;i}) \Big[(x_{k;i} - \hat{x}_{k;i})^T M_{1,k;i} (x_{k;i} - \hat{x}_{k;i}) + 2 \hat{x}_{k;i}^T M_{1,k;i} x_{k;i} - \hat{x}_{k;i}^T M_{1,k;i} \hat{x}_{k;i} \\
& + 2 x_{k;i}^T M_{2,k;i} \hat{y}_{k;i} \Big] dx_{k;i} + \frac{1}{2} \Big(\mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + V_{k;i} \Big), \\
=& \frac{1}{2} \Big(\hat{x}_{k;i}^T M_{1,k;i} \hat{x}_{k;i} + 2 \hat{x}_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + \mathrm{tr}(M_{1,k;i} \Sigma_{x;i}) + \mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) + V_{k;i} \Big), \\
=& \frac{1}{2} \Big((f_{1i}(z_{k-1;i}) x_{k-1;i} + f_{2i}(z_{k-1;i}) y_{k-1;i} + g_i(z_{k-1;i}) u_{k;i})^T M_{1,k;i} (f_{1i}(z_{k-1;i}) x_{k-1;i} \\
& + f_{2i}(z_{k-1;i}) y_{k-1;i} + g_i(z_{k-1;i}) u_{k;i}) + 2 (f_{1i}(z_{k-1;i}) x_{k-1;i} + f_{2i}(z_{k-1;i}) y_{k-1;i} \\
& + g_i(z_{k-1;i}) u_{k;i})^T M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} + y_{k-1;i}^T h_i^T(y_{k-1;i}) M_{3,k;i} h_i(y_{k-1;i}) y_{k-1;i} \\
& + \mathrm{tr}(M_{1,k;i} \Sigma_{x;i}) + \mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) + V_{k;i} \Big),
\end{aligned} \tag{F.4}$$

where the equations (4.58) and (4.60) have been used. Using equations (4.59) and (4.61), the traces $\mathrm{tr}(M_{1,k;i} \Sigma_{x;i})$ and $\mathrm{tr}(M_{3,k;i} \Sigma_{y;i})$ can be further evaluated to obtain,

$$\begin{aligned}
\mathrm{tr}(M_{1,k;i} \Sigma_{x;i}) =& x_{k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} x_{k-1;i}, \\
\mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) =& y_{k-1;i}^T D_{2;i}^T M_{3,k;i} Q_{2;i} D_{2;i} y_{k-1;i}.
\end{aligned} \tag{F.5}$$

Therefore, substituting the evaluated terms ①, ② and (F.5) into (F.1), the term $\beta(u_{k;i}, z_{k-1;i})$ can be obtained as follows,

$$\begin{aligned}\beta(u_{k;i}, z_{k-1;i}) =& \frac{1}{2}\bigg((f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i})^T \tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} \\ &+ f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}) + 2(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} \\ &+ g_i(z_{k-1;i})u_{k;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} \\ &+ y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + \hat{\omega}_{k;i}\bigg),\end{aligned} \tag{F.6}$$

where $\tilde{Q}_{k;i} = (M_{1,k;i} + \Sigma_{x;i}^{-1})$ and,

$$\hat{\omega}_{k;i} = x_{k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} x_{k-1;i} + y_{k-1;i}^T D_{2;i}^T M_{3,k;i} Q_{2;i} D_{2;i} y_{k-1;i} + V_{k;i}. \tag{F.7}$$

Following the evaluation of $\beta(u_{k;i}, z_{k-1;i})$, and using $c^I(u_{k;i}|z_{k-1;i}) = \mathcal{N}(0, \Gamma_{k;i})$ from (4.64), the term $\gamma(z_{k-1;i})$ in (4.6) can be solved as follows,

$$\begin{aligned}\gamma(z_{k-1;i}) =& \int c^I(u_{k;i}|z_{k-1;i}) \exp(-\beta(u_{k;i}, z_{k-1;i}))du_{k;i}, \\ =& \int \mathcal{N}(0, \Gamma_{k;i}) \exp(-\beta(u_{k;i}, z_{k-1;i}))du_{k;i} \\ =& |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \int \exp\bigg\{ -\frac{1}{2}\bigg[u_{k;i}^T \Gamma_{k;i}^{-1} u_{k;i} + (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} \\ &+ g_i(z_{k-1;i})u_{k;i})^T \tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}) \\ &+ 2(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} \\ &+ y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + \hat{\omega}_{k;i}\bigg]\bigg\} du_{k;i}, \\ =& |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \exp\bigg\{ -\frac{1}{2}\bigg[(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T \tilde{Q}_{k;i}(f_{1i}(z_{k-1;i}) \\ &\times x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + 2(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T M_{2,k;i} \\ &\times h_i(y_{k-1;i})y_{k-1;i} + y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + \hat{\omega}_{k;i}\bigg]\bigg\} \\ &\times \int \exp\bigg\{ -\frac{1}{2}\bigg[u_{k;i}^T[\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})]u_{k;i} \\ &+ 2u_{k;i}^T[g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i} \\ &\times h_i(y_{k-1;i})y_{k-1;i}]\bigg]\bigg\}.\end{aligned} \tag{F.8}$$

The integral in (F.8) can be further evaluated by completing the square over $u_{k;i}$ which has been explained by Property 1 in Appendix A. Let us define $\bar{\Gamma}_{k;i} = [\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})]^{-1}$. This gives the following,

Table C:

$$\gamma(z_{k-1;i}) = |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \exp\Bigg\{-\frac{1}{2}\Bigg[(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i})^T\tilde{Q}_{k;i}$$

$$\times (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + 2(f_{1i}(z_{k-1;i})x_{k-1;i}$$

$$+ f_{2i}(z_{k-1;i})y_{k-1;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + y^T_{k-1;i}h^T_i(y_{k-1;i})M_{3,k;i}$$

$$\times h_i(y_{k-1;i})y_{k-1;i} + \hat{\omega}_{k;i}\Bigg]\Bigg\}$$

$$\times \exp\Bigg\{-\frac{1}{2}\Bigg[-\Bigg(g^T_i(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})$$

$$\times y_{k-1;i}) + g^T_i(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\Bigg)^T\bar{\Gamma}_{k;i}\Bigg(g^T_i(z_{k-1;i})\tilde{Q}_{k;i}$$

$$\times (f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) + g^T_i(z_{k-1;i})M_{2,k;i}$$

$$\times h_i(y_{k-1;i})y_{k-1;i}\Bigg)\Bigg]\Bigg\}$$

$$\int \exp(Y)du_{k;i} = \begin{cases} \times \int \exp\Bigg\{-\frac{1}{2}\Bigg[\Bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g^T_i(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} \\ +f_{2i}(z_{k-1;i})y_{k-1;i}) + g^T_i(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}]\Bigg)^T\bar{\Gamma}^{-1}_{k;i} \\ \times\Bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g^T_i(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}) \\ +g^T_i(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}]\Bigg)\Bigg]\Bigg\}du_{k;i}. \end{cases}$$

The integral $\int \exp(Y)du_{k;i}$ in *Table C* can be evaluated using Property 2 in Appendix A such that,

$$\int \exp(Y)du_{k;i} = |2\pi|^{\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}}. \tag{F.9}$$

After expanding the brackets and making the substitution of $\hat{\omega}_{k;i}$ as given by (F.7), the final term

$\gamma(z_{k-1;i})$ is obtained,

$$\begin{aligned}\gamma(z_{k-1;i}) =&|\Gamma_{k;i}|^{-\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}}\exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(z_{k-1;i})x_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i})^T\tilde{Q}_{k;i}\\&\times(f_{1i}(z_{k-1;i})x_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i})+2(f_{1i}(z_{k-1;i})x_{k-1;i}+f_{2i}(z_{k-1;i})\\&\times y_{k-1;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}+y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i}\\&+x_{k-1;i}^T D_{1;i}^T M_{1,k;i}Q_{1;i}D_{1;i}x_{k-1;i}+y_{k-1;i}^T D_{2;i}^T M_{3,k;i}Q_{2;i}D_{2;i}y_{k-1;i}+V_{k;i}\bigg]\bigg\}\\&\times\exp\bigg\{-\frac{1}{2}\bigg[-\bigg(g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i})\\&+g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\bigg)^T\bar{\Gamma}_{k;i}\bigg(g_i^T(z_{k-1;i})\tilde{Q}_{k;i}(f_{1i}(z_{k-1;i})x_{k-1;i}\\&+f_{2i}(z_{k-1;i})y_{k-1;i})+g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\bigg)\bigg]\bigg\},\\=&\exp\bigg\{-\frac{1}{2}\bigg[x_{k-1;i}^T\bigg(-f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i})\\&+f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{1i}(z_{k-1;i})+D_{1;i}^T M_{1,k;i}Q_{1;i}D_{1;i}\bigg)x_{k-1;i}+2x_{k-1;i}^T\bigg(f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}\\&\times f_{2i}(z_{k-1;i})+f_{1i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})-f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}\\&\times g_i^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i})-f_{1i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}\\&\times h_i(y_{k-1;i})\bigg)y_{k-1;i}+y_{k-1;i}^T\bigg(f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}f_{2i}(z_{k-1;i})+2f_{2i}^T(z_{k-1;i})M_{2,k;i}\\&\times h_i(y_{k-1;i})+h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})-f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\\&\times\tilde{Q}_{k;i}f_{2i}(z_{k-1;i})-h_i^T(y_{k-1;i})M_{2,k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\\&-2f_{2i}^T(z_{k-1;i})\tilde{Q}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\\&+D_{2;i}^T M_{3,k;i}Q_{2;i}D_{2;i}\bigg)y_{k-1;i}+\bigg(V_{k;i}+\ln|\Gamma_{k;i}|+\ln|\Gamma_{k;i}^{-1}+g_i^T(x_{k-1;i})\\&\times\tilde{Q}_{k;i}g_i(x_{k-1;i})|\bigg)\bigg]\bigg\},\end{aligned}\tag{F.10}$$

which justifies the form of the performance index (4.70) and the Riccati equation solutions and constant term given by equations (4.72) - (4.75).

F.2 Optimal Randomised Controller, $c^*(u_{k;i}|z_{k-1;i})$

The optimal randomised controller is obtained by evaluating the optimal control law in (4.5) given by Proposition 4 in Chapter 4.

Using the same strategy as was explained in Appendix B in Definition 1, only $\frac{\exp(Y)}{\int\exp(Y)du_{k;i}}$ needs to be computed to find $c^*(u_{k;i}|z_{k-1;i})$. The exponential $\exp(Y)$ is obtained from *Table C* and

$\int \exp(Y) du_{k;i}$ is solved in (F.9). Thus, the optimal randomised controller is given by,

$$\begin{aligned} c^*(u_{k;i}|z_{k-1;i}) =& |2\pi|^{-\frac{1}{2}} |\bar{\Gamma}_{k;i}|^{-\frac{1}{2}} \exp\bigg\{ -\frac{1}{2} \bigg[\bigg(u_{k;i} + \bar{\Gamma}_{k;i} [g_i^T(z_{k-1;i}) \tilde{Q}_{k;i} (f_{1i}(z_{k-1;i}) x_{k-1;i} \\ & + f_{2i}(z_{k-1;i}) y_{k-1;i}) + g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i}] \bigg)^T \bar{\Gamma}_{k;i}^{-1} \\ & \times \bigg(u_{k;i} + \bar{\Gamma}_{k;i} [g_i^T(z_{k-1;i}) \tilde{Q}_{k;i} (f_{1i}(z_{k-1;i}) x_{k-1;i} + f_{2i}(z_{k-1;i}) y_{k-1;i}) \\ & + g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i}] \bigg) \bigg] \bigg\}, \end{aligned} \tag{F.11}$$

which is the desired form as given by (4.65) in Theorem 7, in Chapter 4.

Appendix G

Derivation of the Randomised Controller for Tracking Control of Nonlinear Systems with Multiplicative Noises

The derivation of the FP control design for a tracking control objective is obtained following Proposition 5 in Chapter 5. Theorem 8 describes the local optimal randomised controller for nonlinear systems with multiplicative noises for subsystem i.

The proof is split into two parts. The first part contributes to the justification of the form of $-\ln(\gamma(n_{k;i}))$ as defined in equation (5.34) and the Riccati equation solutions in Theorem 8. The second half verifies the form of the randomised optimal controller $c^*(u_{k;i}|n_{k-1;i})$ given by (5.28).

G.1 Optimal Performance Index, $-\ln(\gamma(n_{k;i}))$

For this proof, the form of the quadratic performance index in (5.34) is assumed to be true, for which the terms $\beta_1(u_{k;i}, n_{k-1;i})$ and $\beta_2(u_{k;i}, n_{k-1;i})$ are evaluated. These terms are then substituted in (5.8) given by Proposition 5 in Chapter 5 to obtain $\gamma(n_{k-1;i})$ with the expectation to justify the Riccati equations (5.37) - (5.39), the linear terms (5.40) - (5.41) and the quadratic cost function (5.34).

Hence, the first step consists of computing $\beta_1(u_{k;i}, n_{k-1;i})$ defined in (5.9) by substituting the actual and ideal distributions of the tracking error $e_{k;i}$ given by (5.22) - (5.25), respectively. This gives,

$$
\begin{aligned}
\beta_1(u_{k;i}, n_{k-1;i}) &= \int s(e_{k;i}|u_{k;i}, n_{k-1;i}) \ln\left(\frac{s(e_{k;i}|u_{k;i}, n_{k-1;i})}{s^I(e_{k;i}|u_{k;i}, n_{k-1;i})}\right) de_{k;i}, \\
&= \int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}) \bigg[-\frac{1}{2}\ln(2\pi|\Sigma_{e_{k;i}}|) - \frac{1}{2}(e_{k;i} - \hat{e}_{k;i})^T \Sigma_{e_{k;i}}^{-1}(e_{k;i} - \hat{e}_{k;i}) \\
&\quad + \frac{1}{2}\ln(2\pi|\Sigma_{2,k;i}|) + \frac{1}{2} e_{k;i}^T \Sigma_{2,k;i}^{-1} e_{k;i} \bigg] de_{k;i}, \\
&= \int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}) \bigg\{ \frac{1}{2} \bigg[\ln\left(\frac{|\Sigma_{2,k;i}|}{|\Sigma_{e_{k;i}}|}\right) - e_{k;i}^T \Sigma_{e_{k;i}}^{-1} e_{k;i} + e_{k;i}^T \Sigma_{2,k;i}^{-1} e_{k;i} \\
&\quad + 2 e_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i} - \hat{e}_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i} \bigg] \bigg\} de_{k;i}, \\
&= \int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}) \bigg\{ -\frac{1}{2} \bigg[\ln\left(\frac{|\Sigma_{e_{k;i}}|}{|\Sigma_{2,k;i}|}\right) + e_{k;i}^T(\Sigma_{e_{k;i}}^{-1} - \Sigma_{2,k;i}^{-1}) e_{k;i} - 2 e_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i} \\
&\quad + \hat{e}_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i} \bigg] \bigg\} de_{k;i}, \\
&= \frac{1}{2} \hat{e}_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i} + \int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}) \bigg\{ -\frac{1}{2} \bigg[\ln\left(\frac{|\Sigma_{e_{k;i}}|}{|\Sigma_{2,k;i}|}\right) + (e_{k;i} - \hat{e}_{k;i})^T \\
&\quad \times (\Sigma_{e_{k;i}}^{-1} - \Sigma_{2,k;i}^{-1})(e_{k;i} - \hat{e}_{k;i}) + 2 e_{k;i}^T (\Sigma_{e_{k;i}}^{-1} - \Sigma_{2,k;i}^{-1}) \hat{e}_{k;i} \\
&\quad - \hat{e}_{k;i}^T (\Sigma_{e_{k;i}}^{-1} - \Sigma_{2,k;i}^{-1}) \hat{e}_{k;i} \bigg] \bigg\} de_{k;i}, \\
&= \frac{1}{2} \bigg\{ \hat{e}_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i} - \mathrm{tr}(\Sigma_{e_{k;i}} \Sigma_{2,k;i}^{-1}) + n - \mathrm{tr}(\Sigma_{e_{k;i}}[\Sigma_{e_{k;i}}^{-1} - \Sigma_{2,k;i}^{-1}]) \\
&\quad - \hat{e}_{k;i}^T (\Sigma_{e_{k;i}}^{-1} - \Sigma_{2,k;i}^{-1}) \hat{e}_{k;i} \bigg\}, \\
&= \frac{1}{2} \bigg\{ \cancel{\hat{e}_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i}} - \cancel{\mathrm{tr}(\Sigma_{e_{k;i}} \Sigma_{2,k;i}^{-1})} + \cancel{n} - \cancel{n} + \cancel{\mathrm{tr}(\Sigma_{e_{k;i}} \Sigma_{2,k;i}^{-1})} \\
&\quad + \hat{e}_{k;i}^T \Sigma_{2,k;i}^{-1} \hat{e}_{k;i} - \cancel{\hat{e}_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i}} \bigg\}, \\
&= \frac{1}{2} \hat{e}_{k;i}^T \Sigma_{2,k;i}^{-1} \hat{e}_{k;i}, \\
&= (f_{1i}(z_{k-1;i}) e_{k-1;i} + f_{2i}(z_{k-1;i}) y_{k-1;i} + g_i(z_{k-1;i}) u_{k;i} + C x_{r,k-1;i})^T \Sigma_{2,k;i}^{-1} \\
&\quad \times (f_{1i}(z_{k-1;i}) e_{k-1;i} + f_{2i}(z_{k-1;i}) y_{k-1;i} + g_i(z_{k-1;i}) u_{k;i} + C x_{r,k-1;i}), \qquad \text{(G.1)}
\end{aligned}
$$

where equation (5.23) has been used.

Thereafter, $\beta_2(u_{k;i}, n_{k-1;i})$ defined in (5.10) is evaluated for which $\ln(\tilde{\gamma}(e_{k;i}, y_{k-1;i}))$ defined in (5.11) needs to be solved first. The actual distribution of the external states defined in (5.15) and the assumed form of the performance index, namely $-\ln(\gamma(n_{k;i})) = \frac{1}{2}(e_{k;i}^T M_{1,k;i} e_{k;i} + 2 e_{k;i}^T M_{2,k;i} y_{k;i} + y_{k;i}^T M_{3,k;i} y_{k;i} + P_{1,k;i} e_{k;i} + P_{2,k;i} y_{k;i} + V_{k;i})$ given by (5.34) are substituted to solve $\ln(\tilde{\gamma}(e_{k;i}, y_{k-1;i}))$

as follows,

$$
\begin{aligned}
-\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i})) =& -\int s(y_{k;i}|y_{k-1;i})\ln(\gamma(n_{k;i}))dy_{k;i},\\
=&\int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i})\bigg\{\frac{1}{2}\bigg[e_{k;i}^T M_{1,k;i}e_{k;i} + 2e_{k;i}^T M_{2,k;i}y_{k;i} + y_{k;i}^T M_{3,k;i}y_{k;i}\\
&+ P_{1,k;i}e_{k;i} + P_{2,k;i}y_{k;i} + V_{k;i}\bigg]\bigg\}dy_{k;i},\\
=&\frac{1}{2}\bigg(e_{k;i}^T M_{1,k;i}e_{k;i} + P_{1,k;i}e_{k;i} + 2e_{k;i}^T M_{2,k;i}\hat{y}_{k;i} + V_{k;i}\bigg) + \frac{1}{2}\int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i})\\
&\times\bigg[(y_{k;i}-\hat{y}_{k;i})^T M_{3,k;i}(y_{k;i}-\hat{y}_{k;i}) + 2\hat{y}_{k;i}^T M_{3,k;i}y_{k;i} - \hat{y}_{k;i}^T M_{3,k;i}\hat{y}_{k;i}\\
&+ P_{2,k;i}y_{k;i}\bigg]dy_{k;i},\\
=&\frac{1}{2}\bigg(e_{k;i}^T M_{1,k;i}e_{k;i} + P_{1,k;i}e_{k;i} + 2e_{k;i}^T M_{2,k;i}\hat{y}_{k;i} + V_{k;i} + \text{tr}(M_{3,k;i}\Sigma_{y;i})\\
&+ \hat{y}_{k;i}^T M_{3,k;i}\hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i}\bigg).
\end{aligned}
\tag{G.2}
$$

Substituting (G.2) into $\beta_2(u_{k;i}, n_{k-1;i})$ gives,

$$
\begin{aligned}
\beta_2(u_{k;i}, n_{k-1;i}) =& \int s(e_{k;i}|u_{k;i}, n_{k-1;i})\bigg[\frac{1}{2}\bigg(e_{k;i}^T M_{1,k;i}e_{k;i} + P_{1,k;i}e_{k;i} + 2e_{k;i}^T M_{2,k;i}\hat{y}_{k;i}\\
&+ V_{k;i} + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i}\hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i}\bigg)\bigg]de_{k;i},\\
=&\frac{1}{2}\int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}})\bigg[e_{k;i}^T M_{1,k;i}e_{k;i} + P_{1,k;i}e_{k;i} + 2e_{k;i}^T M_{2,k;i}\hat{y}_{k;i}\bigg]de_{k;i}\\
&+ \frac{1}{2}\bigg(\text{tr}(M_{3,k;i}\Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i}\hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i} + V_{k;i}\bigg),\\
=&\frac{1}{2}\int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}})\bigg[(e_{k;i}-\hat{e}_{k;i})^T M_{1,k;i}(e_{k;i}-\hat{e}_{k;i}) + 2\hat{e}_{k;i}^T M_{1,k;i}e_{k;i}\\
&- \hat{e}_{k;i}^T M_{1,k;i}\hat{e}_{k;i} + P_{1,k;i}e_{k;i} + 2e_{k;i}^T M_{2,k;i}\hat{y}_{k;i}\bigg]de_{k;i} + \frac{1}{2}\bigg(\text{tr}(M_{3,k;i}\Sigma_{y;i})\\
&+ \hat{y}_{k;i}^T M_{3,k;i}\hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i} + V_{k;i}\bigg),\\
=&\frac{1}{2}\bigg(\hat{e}_{k;i}^T M_{1,k;i}\hat{e}_{k;i} + 2\hat{e}_{k;i}^T M_{2,k;i}\hat{y}_{k;i} + P_{1,k;i}\hat{e}_{k;i} + \hat{y}_{k;i}^T M_{3,k;i}\hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i}\\
&+ \text{tr}(M_{1,k;i}\Sigma_{e_{k;i}}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
\beta_2(u_{k;i}, n_{k-1;i}) =& \frac{1}{2}\Bigg((f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i})^T \\
& \times M_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i}) \\
& + 2(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i})^T \\
& \times M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} \\
& + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i}) + y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} \\
& + P_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \mathrm{tr}(M_{1,k;i}\Sigma_{x;i}) + \mathrm{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i} \Bigg), \qquad \text{(G.3)}
\end{aligned}
$$

where we used equations (5.23) and (5.17). In addition, using equations (5.24) and (5.19), the traces $\mathrm{tr}(M_{1,k;i}\Sigma_{e_{k;i}})$ and $\mathrm{tr}(M_{3,k;i}\Sigma_{y;i})$ can be further evaluated to obtain,

$$
\begin{aligned}
\mathrm{tr}(M_{1,k;i}\Sigma_{x;i}) =& \mathrm{tr}\Bigg(M_{1,k;i}\Bigg(e_{k-1;i}^T D_{1;i}^T Q_{1;i} D_{1;i} e_{k-1;i} + 2e_{k-1;i}^T D_{1;i}^T Q_{1;i} D_{1;i} x_{r,k-1;i} \\
& + x_{r,k-1;i}^T D_{1;i}^T Q_{1;i} D_{1;i} x_{r,k-1;i} + x_{r,k-1;i}^T \tilde{D}^T Q_{r;i} \tilde{D} x_{r,k-1;i} \Bigg)\Bigg), \\
=& e_{k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} e_{k-1;i} + 2e_{k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} x_{r,k-1;i} \\
& + x_{r,k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} x_{r,k-1;i} + x_{r,k-1;i}^T \tilde{D}_i^T M_{1,k;i} Q_{r;i} \tilde{D}_i x_{r,k-1;i} \qquad \text{(G.4)}
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathrm{tr}(M_{3,k;i}\Sigma_{y;i}) =& \mathrm{tr}(M_{3,k;i}(y_{k-1;i}^T D_{2;i}^T Q_{2;i} D_{2;i} y_{k-1;i})), \\
=& y_{k-1;i}^T D_{2;i}^T M_{3,k;i} Q_{2;i} D_{2;i} y_{k-1;i}. \qquad \text{(G.5)}
\end{aligned}
$$

Therefore, the final term $\beta_2(u_{k;i}, n_{k-1;i})$ is found to be,

$$
\begin{aligned}
\beta_2(u_{k;i}, n_{k-1;i}) =& \frac{1}{2}\Bigg((f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i})^T \\
& \times M_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i}) \\
& + 2(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i})^T \\
& \times M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} \\
& + g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i}) + y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} \\
& + P_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \hat{\omega}_{k;i} \Bigg), \qquad \text{(G.6)}
\end{aligned}
$$

where,

$$\hat{\omega}_{k;i} = e_{k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} e_{k-1;i} + 2e_{k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} x_{r,k-1;i}$$
$$+ x_{r,k-1;i}^T D_{1;i}^T M_{1,k;i} Q_{1;i} D_{1;i} x_{r,k-1;i} + x_{r,k-1;i}^T \tilde{D}_i^T M_{1,k;i} Q_{r;i} \tilde{D}_i x_{r,k-1;i}$$
$$+ y_{k-1;i}^T D_{2;i}^T M_{3,k;i} Q_{2;i} D_{2;i} y_{k-1;i} + V_{k;i}. \tag{G.7}$$

Following the evaluation of $\beta_1(u_{k;i}, n_{k-1;i})$ in (G.1) and $\beta_2(u_{k;i}, n_{k-1;i})$ in (G.6), and using $c^I(u_{k;i}|n_{k-1;i}) = \mathcal{N}(\hat{u}_{k;i}, \Gamma_{k;i})$ from (5.26), the term $\gamma(n_{k-1;i})$ defined in (5.8) can be solved as follows,

$$\gamma(n_{k-1;i}) = \int c^I(u_{k;i}|n_{k-1;i}) \exp(-\beta_1(u_{k;i}, n_{k-1;i}) - \beta_2(u_{k;i}, n_{k-1;i}))du_{k;i},$$
$$= |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \int \exp\Bigg\{ -\frac{1}{2}\Bigg((f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}$$
$$+ Cx_{r,k-1;i})^T \tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}$$
$$+ Cx_{r,k-1;i}) + 2(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + g_i(z_{k-1;i})u_{k;i}$$
$$+ Cx_{r,k-1;i})^T M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} + P_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i}$$
$$+ g_i(z_{k-1;i})u_{k;i} + Cx_{r,k-1;i}) + y_{k-1;i}^T h_i^T(y_{k-1;i}) M_{3,k;i} h_i(y_{k-1;i}) y_{k-1;i} + P_{2,k;i}$$
$$\times h_i(y_{k-1;i}) y_{k-1;i} + \hat{\omega}_{k;i} + \hat{u}_{k;i}^T \Gamma_{k;i}^{-1} \hat{u}_{k;i} - 2u_{k;i}^T \Gamma_{k;i}^{-1} \hat{u}_{k;i} + u_{k;i}^T \Gamma_{k;i}^{-1} u_{k;i} \Bigg) \Bigg\} du_{k;i},$$
$$= |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \exp\Bigg\{ -\frac{1}{2}\Bigg[(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i})^T \tilde{S}_{k;i}$$
$$\times (f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + 2(f_{1i}(z_{k-1;i})e_{k-1;i}$$
$$+ f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i})^T M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} + P_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i}$$
$$+ f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + y_{k-1;i}^T h_i^T(y_{k-1;i}) M_{3,k;i} h_i(y_{k-1;i}) y_{k-1;i}$$
$$+ P_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} + \hat{\omega}_{k;i} + \hat{u}_{k;i}^T \Gamma_{k;i}^{-1} \hat{u}_{k;i} \Bigg] \Bigg\}$$
$$\times \int \exp\Bigg\{ -\frac{1}{2}\Bigg[u_{k;i}^T [\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i}) \tilde{S}_{k;i} g_i(z_{k-1;i})] u_{k;i}$$
$$+ 2u_{k;i}^T [g_i^T(z_{k-1;i}) \tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i})$$
$$+ g_i^T(z_{k-1;i}) M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} + \frac{1}{2} g_i^T(z_{k-1;i}) P_{1,k;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i}] \Bigg] \Bigg\}, \tag{G.8}$$

where $\tilde{S}_{k;i} = (M_{1,k;i} + \Sigma_{2,k;i}^{-1})$. The integral in (G.8) can be further evaluated by completing the square over $u_{k;i}$ which has been explained in Appendix A, Property 1. The definition $\bar{\Gamma}_{k;i} = [\Gamma_{k;i}^{-1} + g_i^T(z_{k-1;i}) \tilde{S}_{k;i} g_i(z_{k-1;i})]^{-1}$ is introduced for notational convenience. This results into the following,

Table D:

$$\gamma(z_{k-1;i}) = |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i})^T \tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + 2(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i})^T M_{2,k;i} \times h_i(y_{k-1;i})y_{k-1;i} + P_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + y^T_{k-1;i}h^T_i(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{2,k;i} \times h_i(y_{k-1;i})y_{k-1;i} + \hat{\omega}_{k;i} + \hat{u}^T_{k;i}\Gamma^{-1}_{k;i}\hat{u}_{k;i}\bigg]\bigg\} \times \exp\bigg\{-\frac{1}{2}\bigg[-\bigg(g^T_i(z_{k-1;i})\tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + g^T_i(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g^T_i(z_{k-1;i})P^T_{1,k;i} - \Gamma^{-1}_{k;i}\hat{u}_{k;i}\bigg)^T \bar{\Gamma}_{k;i}\bigg(g^T_i(z_{k-1;i})\tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i}) \times y_{k-1;i} + Cx_{r,k-1;i}) + g^T_i(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g^T_i(z_{k-1;i})P^T_{1,k;i} - \Gamma^{-1}_{k;i}\hat{u}_{k;i}\bigg)\bigg]\bigg\}$$

$$\int \exp(Y)du_{k;i} = \begin{cases} \times \int \exp\bigg\{-\frac{1}{2}\bigg[\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g^T_i(z_{k-1;i})\tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} \\ +f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + g^T_i(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} \\ +\frac{1}{2}g^T_i(z_{k-1;i})P^T_{1,k;i} - \Gamma^{-1}_{k;i}\hat{u}_{k;i}]\bigg)^T \bar{\Gamma}^{-1}_{k;i}\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g^T_i(z_{k-1;i})\tilde{S}_{k;i} \\ \times(f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + g^T_i(z_{k-1;i}) \\ \times M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g^T_i(z_{k-1;i})P^T_{1,k;i} - \Gamma^{-1}_{k;i}\hat{u}_{k;i}]\bigg)\bigg]\bigg\}du_{k;i}. \end{cases} \tag{G.9}$$

The integral $\int \exp(Y)du_{k;i}$ in *Table D* can be evaluated using Property 2 in Appendix A such that,

$$\int \exp(Y)du_{k;i} = |2\pi|^{\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}}. \tag{G.10}$$

After expanding the brackets and making the substitution of $\hat{\omega}_{k;i}$ as given by (G.7), the final term

$\gamma(n_{k-1;i})$ is obtained as follows,

$$\begin{aligned}\gamma(n_{k-1;i}) =&|\Gamma_{k;i}|^{-\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}}\exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(z_{k-1;i})e_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i}+Cx_{r,k-1;i})^T\\
&\times\tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i}+Cx_{r,k-1;i})\\
&+2(f_{1i}(z_{k-1;i})e_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i}+Cx_{r,k-1;i})^TM_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\\
&+P_{1,k;i}(f_{1i}(z_{k-1;i})e_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i}+Cx_{r,k-1;i})\\
&+y_{k-1;i}^Th_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i}+P_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}+\hat{u}_{k;i}^T\Gamma_{k;i}^{-1}\hat{u}_{k;i}\\
&+e_{k-1;i}^TD_{1;i}^TM_{1,k;i}Q_{1;i}D_{1;i}e_{k-1;i}+2e_{k-1;i}^TD_{1;i}^TM_{1,k;i}Q_{1;i}D_{1;i}x_{r,k-1;i}\\
&+x_{r,k-1;i}^TD_{1;i}^TM_{1,k;i}Q_{1;i}D_{1;i}x_{r,k-1;i}+x_{r,k-1;i}^T\tilde{D}_i^TM_{1,k;i}Q_{r;i}\tilde{D}_ix_{r,k-1;i}\\
&+y_{k-1;i}^TD_{2;i}^TM_{3,k;i}Q_{2;i}D_{2;i}y_{k-1;i}+V_{k;i}\bigg]\bigg\}\\
&\times\exp\bigg\{-\frac{1}{2}\bigg[-\bigg(g_i^T(z_{k-1;i})\tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i}+Cx_{r,k-1;i})\\
&+g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}+\frac{1}{2}g_i^T(z_{k-1;i})P_{1,k;i}^T-\Gamma_{k;i}^{-1}\hat{u}_{k;i}\bigg)^T\\
&\times\bar{\Gamma}_{k;i}\bigg(g_i^T(z_{k-1;i})\tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i}+f_{2i}(z_{k-1;i})y_{k-1;i}+Cx_{r,k-1;i})\\
&+g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}+\frac{1}{2}g_i^T(z_{k-1;i})P_{1,k;i}^T-\Gamma_{k;i}^{-1}\hat{u}_{k;i}\bigg)\bigg]\bigg\},\end{aligned}$$

which can be further evaluated to obtain,

$$
\begin{aligned}
&= \exp\bigg\{-\frac{1}{2}\bigg[e_{k-1;i}^T\bigg(-f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{S}_{k;i}f_{1i}(z_{k-1;i})\\
&+ f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i}f_{1i}(z_{k-1;i}) + D_{1;i}^T M_{1,k;i}Q_{1;i}D_{1;i}\bigg)e_{k-1;i}\\
&+ 2e_{k-1;i}^T\bigg(f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i}f_{2i}(z_{k-1;i}) + f_{1i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\\
&- f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{S}_{k;i}f_{2i}(z_{k-1;i})\\
&- f_{1i}^T(z_{k-1;i})\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\bigg)y_{k-1;i}\\
&+ y_{k-1;i}^T\bigg(f_{2i}^T(z_{k-1;i})\tilde{S}_{k;i}f_{2i}(z_{k-1;i}) + 2f_{2i}^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\\
&+ h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i}) - f_{2i}^T(z_{k-1;i})\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\\
&\times \tilde{S}_{k;i}f_{2i}(z_{k-1;i}) - h_i^T(y_{k-1;i})M_{2,k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\\
&- 2f_{2i}^T(z_{k-1;i})\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) + D_{2;i}^T M_{3,k;i}Q_{2;i}D_{2;i}\bigg)y_{k-1;i}\\
&+ \bigg(2x_{r,k-1;i}^T C^T\tilde{S}_{k;i}f_{1i}(z_{k-1;i}) + P_{1,k;i}f_{1i}(z_{k-1;i}) + 2x_{r,k-1;i}^T D_{1;i}^T M_{1,k;i}Q_{1;i}D_{1;i}\\
&- 2x_{r,k-1;i}^T C^T\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{S}_{k;i}f_{1i}(z_{k-1;i})\\
&- 2(0.5g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{S}_{k;i}f_{1i}(z_{k-1;i})\bigg)e_{k-1;i}\\
&+ \bigg(2x_{r,k-1;i}^T C^T\tilde{S}_{k;i}f_{2i}(z_{k-1;i}) + 2x_{r,k-1;i}^T C^T M_{2,k;i}h_i(y_{k-1;i}) + P_{1,k;i}f_{2i}(z_{k-1;i})\\
&+ P_{2,k;i}h_i(y_{k-1;i}) - 2x_{r,k-1;i}^T C^T\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{S}_{k;i}f_{2i}(z_{k-1;i})\\
&- 2(0.5g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\tilde{S}_{k;i}f_{2i}(z_{k-1;i})\\
&- 2x_{r,k;i}^T C^T\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\\
&- 2(0.5g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\bigg)y_{k-1;i}\\
&+ \bigg(V_{k;i} + x_{r,k-1;i}^T C^T\tilde{S}_{k;i}Cx_{r,k-1;i} + P_{1,k;i}Cx_{r,k-1;i} + x_{r,k-1;i}^T D_{1;i}^T M_{1,k;i}Q_{1;i}D_{1;i}x_{r,k-1;i}\\
&+ x_{r,k-1;i}^T\tilde{D}_i^T M_{1,k;i}Q_{r;i}\tilde{D}_i x_{r,k-1;i} + \hat{u}_{k;i}^T\Gamma_{k;i}^{-1}\hat{u}_{k;i} - x_{r,k-1;i}^T C^T\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(z_{k-1;i})\\
&\times \tilde{S}_{k;i}Cx_{r,k-1;i} - 2x_{r,k-1;i}^T C^T\tilde{S}_{k;i}g_i(z_{k-1;i})\bar{\Gamma}_{k;i}(0.5g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})\\
&- (0.5g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}(0.5g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})\\
&+ \ln|\Gamma_{k;i}| + \ln|\Gamma_{k;i}^{-1} + g_i^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})|\bigg)\bigg]\bigg\}, \qquad \text{(G.11)}
\end{aligned}
$$

which justifies the form of the performance index given by (5.34) and the Riccati equation solutions (5.37) - (5.39), the linear terms (5.40) - (5.41) .

G.2 Optimal Randomised Controller, $c^*(u_{k;i}|n_{k-1;i})$

The optimal randomised controller is obtained by evaluating the optimal control law in (5.7) given by Proposition 5 in Chapter 5.

Using the same approach as was explained in Appendix B in Definition 1, only $\frac{\exp(Y)}{\int \exp(Y) du_{k;i}}$ needs to be computed to find the local randomised controller $c^*(u_{k;i}|n_{k-1;i})$ for node i. The exponential $\exp(Y)$ is obtained from *Table D* and $\int \exp(Y) du_{k;i}$ is solved in (G.10). Thus, the optimal randomised controller is given by,

$$
\begin{aligned}
c^*(u_{k;i}|z_{k-1;i}) =& |2\pi|^{-\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{-\frac{1}{2}} \exp\bigg\{-\frac{1}{2}\bigg[\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(z_{k-1;i})\tilde{S}_{k;i}(f_{1i}(z_{k-1;i})e_{k-1;i} \\
& + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} \\
& + \frac{1}{2}g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}]\bigg)^T \bar{\Gamma}_{k;i}^{-1}\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(z_{k-1;i})\tilde{S}_{k;i} \\
& \times (f_{1i}(z_{k-1;i})e_{k-1;i} + f_{2i}(z_{k-1;i})y_{k-1;i} + Cx_{r,k-1;i}) + g_i^T(z_{k-1;i})M_{2,k;i} \\
& \times h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g_i^T(z_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}]\bigg)\bigg]\bigg\}, \qquad \text{(G.12)}
\end{aligned}
$$

which is the desired form as given by (5.28) in Theorem 8.

Appendix H

Derivation of the Randomised Controller for Formation Control of Nonlinear Systems with Additive Noises

For the derivation of local randomised controllers with the objective of formation control, Proposition 5 in Chapter 5 is considered. Theorem 9 in Chapter 5 describes the local optimal randomised controller in (5.63) for nonlinear systems with additive noises for subsystem i where the objective is formation control.

The first part in this section verifies the form of $-\ln(\gamma(n_{k;i}))$ in (5.69) and the Riccati equation solutions in Theorem 9. The second half proves the stated randomised optimal controller $c^*(u_{k;i}|n_{k-1;i})$ given by (5.63) in Theorem 9.

H.1 Optimal Performance Index, $-\ln(\gamma(n_{k;i}))$

It is assumed that the form of the quadratic cost function in (5.69) is true, depending on which the terms $\beta_1(u_{k;i}, n_{k-1;i})$ and $\beta_2(u_{k;i}, n_{k-1;i})$ are evaluated. These terms are then substituted in (5.8) given by Proposition 5 in Chapter 5 to justify Theorem 9.

Firstly, $\beta_1(u_{k;i}, n_{k-1;i})$ given by (5.9) is evaluated by substituting the actual and ideal distributions of the tracking error $e_{k;i}$ given by (5.58) - (5.60), respectively such that,

$$\begin{aligned}
\beta_1(u_{k;i}, n_{k-1;i}) &= \int s(e_{k;i}|u_{k;i}, e_{k-1;i}) \ln\left(\frac{s(e_{k;i}|u_{k;i}, e_{k-1;i})}{s^I(e_{k;i}|u_{k;i}, e_{k-1;i})}\right) de_{k;i}, \\
&= \int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}) \Bigg[-\frac{1}{2}\ln(2\pi|\Sigma_{e_{k;i}}|) - \frac{1}{2}(e_{k;i} - \hat{e}_{k;i})^T \Sigma_{e_{k;i}}^{-1} (e_{k;i} - \hat{e}_{k;i}) \\
&\quad + \frac{1}{2}\ln(2\pi|\Sigma_{e_{k;i}}|) + \frac{1}{2} e_{k;i}^T \Sigma_{e_{k;i}}^{-1} e_{k;i} \Bigg] de_{k;i}, \\
&= \frac{1}{2}\hat{e}_{k;i}^T \Sigma_{e_{k;i}}^{-1} \hat{e}_{k;i}, \\
&= (f_{1i}(x_{k-1;i}) e_{k-1;i} + g_i(x_{k-1;i}) u_{k;i} + \tilde{x}_{r,k;i})^T \Sigma_{e_{k;i}}^{-1} \\
&\quad \times (f_{1i}(x_{k-1;i}) e_{k-1;i} + g_i(x_{k-1;i}) u_{k;i} + \tilde{x}_{r,k;i}),
\end{aligned} \tag{H.1}$$

where equation (5.59) has been used.

Furthermore, $\beta_2(u_{k;i}, n_{k-1;i})$ defined in (5.10) is evaluated which requires the computation of $\ln(\tilde{\gamma}(e_{k;i}, y_{k-1;i}))$ defined in (5.11). The actual distribution of the external states defined in (5.55) and the assumed form of the performance index, namely,
$-\ln(\gamma(n_{k;i})) = \frac{1}{2}(e_{k;i}^T M_{1,k;i} e_{k;i} + 2e_{k;i}^T M_{2,k;i} y_{k;i} + y_{k;i}^T M_{3,k;i} y_{k;i} + P_{1,k;i} e_{k;i} + P_{2,k;i} y_{k;i} + V_{k;i})$ which is given by (5.69) are substituted to solve $\ln(\tilde{\gamma}(e_{k;i}, y_{k-1;i}))$ as follows,

$$\begin{aligned}
-\ln(\tilde{\gamma}(x_{k;i}, y_{k-1;i})) &= -\int s(y_{k;i}|y_{k-1;i}) \ln(\gamma(n_{k;i})) dy_{k;i}, \\
&= \int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}) \Bigg\{ \frac{1}{2} \Bigg[e_{k;i}^T M_{1,k;i} e_{k;i} + 2e_{k;i}^T M_{2,k;i} y_{k;i} + y_{k;i}^T M_{3,k;i} y_{k;i} \\
&\quad + P_{1,k;i} e_{k;i} + P_{2,k;i} y_{k;i} + V_{k;i} \Bigg] \Bigg\} dy_{k;i}, \\
&= \frac{1}{2}\Bigg(e_{k;i}^T M_{1,k;i} e_{k;i} + P_{1,k;i} e_{k;i} + 2e_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + V_{k;i} \Bigg) + \int \mathcal{N}(\hat{y}_{k;i}, \Sigma_{y;i}) \\
&\quad \times \Bigg\{ \frac{1}{2} \Bigg[(y_{k;i} - \hat{y}_{k;i})^T M_{3,k;i} (y_{k;i} - \hat{y}_{k;i}) + 2\hat{y}_{k;i}^T M_{3,k;i} y_{k;i} - \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} \\
&\quad + P_{2,k;i} y_{k;i} \Bigg] \Bigg\} dy_{k;i}, \\
&= \frac{1}{2}\Bigg(e_{k;i}^T M_{1,k;i} e_{k;i} + P_{1,k;i} e_{k;i} + 2e_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + V_{k;i} + \mathrm{tr}(M_{3,k;i} \Sigma_{y;i}) \\
&\quad + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + P_{2,k;i} \hat{y}_{k;i} \Bigg).
\end{aligned} \tag{H.2}$$

Substituting (H.2) into $\beta_2(u_{k;i}, n_{k-1;i})$ gives,

$$
\begin{aligned}
\beta_2(u_{k;i}, n_{k-1;i}) = \int s(e_{k;i}|u_{k;i}, e_{k-1;i}) \Bigg[\frac{1}{2}\Bigg(& e_{k;i}^T M_{1,k;i} e_{k;i} + P_{1,k;i} e_{k;i} + 2e_{k;i}^T M_{2,k;i} \hat{y}_{k;i} \\
& + V_{k;i} + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i}\Bigg)\Bigg] de_{k;i}, \\
= & \frac{1}{2}\int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}) \Big[e_{k;i}^T M_{1,k;i} e_{k;i} + P_{1,k;i} e_{k;i} + 2e_{k;i}^T M_{2,k;i} \hat{y}_{k;i} \Big] de_{k;i} \\
& + \frac{1}{2}\Bigg(\text{tr}(M_{3,k;i}\Sigma_{y;i}) + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i} + V_{k;i} \Bigg), \\
= & \frac{1}{2}\int \mathcal{N}(\hat{e}_{k;i}, \Sigma_{e_{k;i}}) \Big[(e_{k;i} - \hat{e}_{k;i})^T M_{1,k;i} (e_{k;i} - \hat{e}_{k;i}) + 2\hat{e}_{k;i}^T M_{1,k;i} e_{k;i} \\
& - \hat{e}_{k;i}^T M_{1,k;i} \hat{e}_{k;i} + P_{1,k;i} e_{k;i} + 2e_{k;i}^T M_{2,k;i} \hat{y}_{k;i} \Big] de_{k;i} + \frac{1}{2}\Bigg(\text{tr}(M_{3,k;i}\Sigma_{y;i}) \\
& + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} + P_{2,k;i}\hat{y}_{k;i} + V_{k;i} \Bigg), \\
= & \frac{1}{2}\Bigg(\hat{e}_{k;i}^T M_{1,k;i} \hat{e}_{k;i} + 2\hat{e}_{k;i}^T M_{2,k;i} \hat{y}_{k;i} + P_{1,k;i}\hat{e}_{k;i} + \hat{y}_{k;i}^T M_{3,k;i} \hat{y}_{k;i} \\
& + \text{tr}(M_{1,k;i}\Sigma_{e_{k;i}}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + P_{2,k;i}\hat{y}_{k;i} + V_{k;i} \Bigg), \\
= & \frac{1}{2}\Bigg((f_{1i}(x_{k-1;i}) e_{k-1;i} + g_i(x_{k-1;i}) u_{k;i} + \tilde{x}_{r,k;i})^T M_{1,k;i} (f_{1i}(x_{k-1;i}) e_{k-1;i} \\
& + g_i(x_{k-1;i}) u_{k;i} + \tilde{x}_{r,k;i}) + 2(f_{1i}(x_{k-1;i}) e_{k-1;i} + g_i(x_{k-1;i}) u_{k;i} + \tilde{x}_{r,k;i})^T \\
& \times M_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} + P_{1,k;i}(f_{1i}(x_{k-1;i}) e_{k-1;i} + g_i(x_{k-1;i}) u_{k;i} + \tilde{x}_{r,k;i}) \\
& + P_{2,k;i} h_i(y_{k-1;i}) y_{k-1;i} + y_{k-1;i}^T h_i^T(y_{k-1;i}) M_{3,k;i} h_i(y_{k-1;i}) y_{k-1;i} \\
& + \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i} \Bigg), \qquad \text{(H.3)}
\end{aligned}
$$

where we used equations (5.59) and (5.55). Since $\beta_1(u_{k;i}, n_{k-1;i})$ in (H.1) and $\beta_2(u_{k;i}, n_{k-1;i})$ in (H.3) have been derived, and since we know that $c^I(u_{k;i}|z_{k-1;i}) = \mathcal{N}(\hat{u}_{k;i}, \Gamma_{k;i})$ from (5.61), the term $\gamma(n_{k-1;i})$ given in (5.8) can be evaluated such that,

$$
\begin{aligned}
\gamma(n_{k-1;i}) = &\int c^I(u_{k;i}|n_{k-1;i})\exp(-\beta_1(u_{k;i},n_{k-1;i}) - \beta_2(u_{k;i},n_{k-1;i}))du_{k;i},\\
=&|2\pi\Gamma_{k;i}|^{-\frac{1}{2}}\int \exp\bigg\{-\frac{1}{2}\bigg((f_{1i}(x_{k-1;i})e_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \tilde{x}_{r,k;i})^T\\
&\times \tilde{S}_{k;i}(f_{1i}(x_{k-1;i})e_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \tilde{x}_{r,k;i})\\
&+ 2(f_{1i}(x_{k-1;i})e_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \tilde{x}_{r,k;i})^T M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\\
&+ P_{1,k;i}(f_{1i}(x_{k-1;i})e_{k-1;i} + g_i(x_{k-1;i})u_{k;i} + \tilde{x}_{r,k;i}) + P_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\\
&+ y^T_{k-1;i}h^T_i(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + \mathrm{tr}(M_{1,k;i}\Sigma_{x;i})\\
&+ \mathrm{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i} + \hat{u}^T_{k;i}\Gamma^{-1}_{k;i}\hat{u}_{k;i} - 2u^T_{k;i}\Gamma^{-1}_{k;i}\hat{u}_{k;i} + u^T_{k;i}\Gamma^{-1}_{k;i}u_{k;i}\bigg)\bigg\}du_{k;i},\\
=&|2\pi\Gamma_{k;i}|^{-\frac{1}{2}}\exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i})^T\tilde{S}_{k;i}\\
&\times (f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i}) + 2(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i})^T\\
&\times M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{1,k;i}(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i})\\
&+ y^T_{k-1;i}h^T_i(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}\\
&+ \mathrm{tr}(M_{1,k;i}\Sigma_{x;i}) + \mathrm{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i} + \hat{u}^T_{k;i}\Gamma^{-1}_{k;i}\hat{u}_{k;i}\bigg]\bigg\}\\
&\times\int\exp\bigg\{-\frac{1}{2}\bigg[u^T_{k;i}[\Gamma^{-1}_{k;i} + g^T_i(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})]u_{k;i}\\
&+ 2u^T_{k;i}[g^T_i(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})x_{k-1;i} + \tilde{x}_{r,k;i}) + g^T_i(x_{k-1;i})\\
&\times M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g^T_i(x_{k-1;i})P^T_{1,k;i} - \Gamma^{-1}_{k;i}\hat{u}_{k;i}]\bigg]\bigg\},
\end{aligned}
\tag{H.4}
$$

where $\tilde{S}_{k;i} = (M_{1,k;i} + \Sigma^{-1}_{e_{k;i}})$.

The integral in (H.4) can be further evaluated by completing the square over $u_{k;i}$ which has been explained in Appendix A, Property 1. The definition $\bar{\Gamma}_{k;i} = [\Gamma^{-1}_{k;i} + g^T_i(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})]^{-1}$ is introduced for notational convenience. This results in the following,

Table E:

$$
\begin{aligned}
\gamma(x_{k-1;i}) = |2\pi\Gamma_{k;i}|^{-\frac{1}{2}} \exp\bigg\{-\frac{1}{2}\bigg[&(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i})^T \tilde{S}_{k;i} \\
&\times (f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i}) + 2(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i})^T \\
&\times M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{1,k;i}(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i}) \\
&+ y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} \\
&+ \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i} + \hat{u}_{k;i}^T\Gamma_{k;i}^{-1}\hat{u}_{k;i}\bigg]\bigg\} \\
&\times \exp\bigg\{-\frac{1}{2}\bigg[-\bigg(g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})x_{k-1;i} + \tilde{x}_{r,k;i}) \\
&+ g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}\bigg)^T \\
&\times \bar{\Gamma}_{k;i}\bigg(g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})x_{k-1;i} + \tilde{x}_{r,k;i}) + g_i^T(x_{k-1;i}) \\
&\times M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}\bigg)\bigg]\bigg\}
\end{aligned}
$$

$$
\int \exp(Y)du_{k;i} = \begin{cases}
\times \int \exp\bigg\{-\frac{1}{2}\bigg[\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})x_{k-1;i} \\
+\tilde{x}_{r,k;i}) + g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T \\
-\Gamma_{k;i}^{-1}\hat{u}_{k;i}]\bigg)^T \bar{\Gamma}_{k;i}^{-1}\bigg(u_{k;i} + \bar{\Gamma}_{k;i}[g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})x_{k-1;i} \\
+\tilde{x}_{r,k;i}) + g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T \\
-\Gamma_{k;i}^{-1}\hat{u}_{k;i}]\bigg)\bigg]\bigg\}du_{k;i}.
\end{cases}
\tag{H.5}
$$

The integral $\int \exp(Y)du_{k;i}$ in *Table E* can be evaluated using Property 2 in Appendix A such that,

$$
\int \exp(Y)du_{k;i} = |2\pi|^{\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}}. \tag{H.6}
$$

After expanding the brackets, the final term $\gamma(n_{k-1;i})$ is obtained,

$$
\begin{aligned}
\gamma(n_{k-1;i}) =& |\Gamma_{k;i}|^{-\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{\frac{1}{2}} \exp\bigg\{-\frac{1}{2}\bigg[(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i})^T \tilde{S}_{k;i} \\
& \times (f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i}) + 2(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i})^T M_{2,k;i} \\
& \times h_i(y_{k-1;i})y_{k-1;i} + P_{1,k;i}(f_{1i}(x_{k-1;i})e_{k-1;i} + \tilde{x}_{r,k;i}) \\
& + y_{k-1;i}^T h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i})y_{k-1;i} + P_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} \\
& + \hat{u}_{k;i}^T\Gamma_{k;i}^{-1}\hat{u}_{k;i} + \text{tr}(M_{1,k;i}\Sigma_{x;i}) + \text{tr}(M_{3,k;i}\Sigma_{y;i}) + V_{k;i}\bigg]\bigg\} \\
& \times \exp\bigg\{-\frac{1}{2}\bigg[-\bigg(g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})x_{k-1;i} + \tilde{x}_{r,k;i}) \\
& + g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}\bigg)^T \\
& \times \bar{\Gamma}_{k;i}\bigg(g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})x_{k-1;i} + \tilde{x}_{r,k;i}) \\
& + g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i} + \frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i}\bigg)\bigg]\bigg\}, \\
=& \exp\bigg\{-\frac{1}{2}\bigg[e_{k-1;i}^T\bigg(-f_{1i}^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) \\
& + f_{1i}^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i})\bigg)e_{k-1;i} + 2e_{k-1;i}^T\bigg(f_{1i}^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\
& - f_{1i}^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}),\bigg)y_{k-1;i} \\
& + y_{k-1;i}^T\bigg(h_i^T(y_{k-1;i})M_{3,k;i}h_i(y_{k-1;i}) - h_i^T(y_{k-1;i})M_{2,k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i} \\
& \times g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\bigg)y_{k-1;i} \\
& + \bigg(2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) + P_{1,k;i}f_{1i}(x_{k-1;i}) - 2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i} \\
& \times g_i^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i}) - 2(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i} \\
& \times g_i^T(x_{k-1;i})\tilde{S}_{k;i}f_{1i}(x_{k-1;i})\bigg)e_{k-1;i} \\
& + \bigg(2\tilde{x}_{r,k;i}^T M_{2,k;i}h_i(y_{k-1;i}) + P_{2,k;i}h_i(y_{k-1;i}) \\
& - 2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i}) \\
& - 2(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T - \Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})\bigg)y_{k-1;i}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(V_{k;i}+\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}\tilde{x}_{r,k;i}+P_{1,k;i}\tilde{x}_{r,k;i}+\hat{u}_{k;i}^T\Gamma_{k;i}^{-1}\hat{u}_{k;i}\\
&-\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}g_i^T(x_{k-1;i})\tilde{S}_{k;i}\tilde{x}_{r,k;i}-2\tilde{x}_{r,k;i}^T\tilde{S}_{k;i}g_i(x_{k-1;i})\bar{\Gamma}_{k;i}\\
&\times(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T-\Gamma_{k;i}^{-1}\hat{u}_{k;i})-(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T-\Gamma_{k;i}^{-1}\hat{u}_{k;i})^T\bar{\Gamma}_{k;i}\\
&\times(0.5g_i^T(x_{k-1;i})P_{1,k;i}^T-\Gamma_{k;i}^{-1}\hat{u}_{k;i})+\text{tr}(M_{1,k;i}\Sigma_{x;i})+\text{tr}(M_{3,k;i}\Sigma_{y;i})\\
&+\ln|\Gamma_{k;i}|+\ln|\Gamma_{k;i}^{-1}+g_i^T(x_{k-1;i})\tilde{S}_{k;i}g_i(x_{k-1;i})|\Bigg)\Bigg]\Bigg\},
\end{aligned}
\tag{H.7}
$$

which verifies the form of the performance index given by (5.69) and the Riccati equation solutions (5.72) - (5.74), and the linear terms (5.75) - (5.76).

H.2 Proof: Optimal Randomised Controller $c^*(u_{k;i}|n_{k-1;i})$

The local optimal randomised controller for node i for formation control is obtained by evaluating the optimal control law in (5.7) given by Proposition 5 in Chapter 5.

Using the same approach as was explained in Appendix B in Definition 1, only $\frac{\exp(Y)}{\int\exp(Y)du_{k;i}}$ needs to be computed to find the local randomised controller $c^*(u_{k;i}|n_{k-1;i})$ for node i. The exponential $\exp(Y)$ is obtained from *Table E* and $\int\exp(Y)du_{k;i}$ is solved in (H.6). Thus, the optimal randomised controller for a formation control problem is given by,

$$
\begin{aligned}
c^*(u_{k;i}|z_{k-1;i})=&|2\pi|^{-\frac{1}{2}}|\bar{\Gamma}_{k;i}|^{-\frac{1}{2}}\exp\Bigg\{-\frac{1}{2}\Bigg[\Bigg(u_{k;i}+\bar{\Gamma}_{k;i}[g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})e_{k-1;i}\\
&+\tilde{x}_{r,k;i})+g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}+\frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T\\
&-\Gamma_{k;i}^{-1}\hat{u}_{k;i}]\Bigg)^T\bar{\Gamma}_{k;i}^{-1}\Bigg(u_{k;i}+\bar{\Gamma}_{k;i}[g_i^T(x_{k-1;i})\tilde{S}_{k;i}(f_{1i}(x_{k-1;i})e_{k-1;i}+\tilde{x}_{r,k;i})\\
&+g_i^T(x_{k-1;i})M_{2,k;i}h_i(y_{k-1;i})y_{k-1;i}+\frac{1}{2}g_i^T(x_{k-1;i})P_{1,k;i}^T-\Gamma_{k;i}^{-1}\hat{u}_{k;i}]\Bigg)\Bigg]\Bigg\},
\end{aligned}
\tag{H.8}
$$

which is the Gaussian distribution of the controller as described by (5.63) in Theorem 9.

Appendix I

Convergence Analysis of a Decentralised Nonlinear System with Additive Noises

This appendix provides the proof of Theorem 10. The equation in (5.85) can be explicitly written as,

$$\begin{aligned} e_{k;i} =& f_{1i}(x_{k-1;i})e_k + g_i(x_{k-1;i})u_{k;i} + (f_{1i} - \tilde{m}(x_{k;j}))x_{r,k-1;i} + \epsilon_{1k;i} + \epsilon_{r,k;i}, \\ =& f_{1i}(x_{k-1;i})e_k + g_i(x_{k-1;i})u_{k;i} + \tilde{W}x_{r,k-1;i} + \epsilon_{1k;i} + \epsilon_{r,k;i}, \quad \text{(I.1)} \\ y_{k;i} =& h_i(y_{k-1;i})y_{k-1;i} + \epsilon_{2k;i}. \quad \text{(I.2)} \end{aligned}$$

A two-dimensional example is presented to examine the dimensionality of the matrices and vectors and ensuring they align with each other. We have,

$$\begin{aligned} n_k = \begin{bmatrix} e_{1,k} \\ e_{2,k} \\ y_{1,k} \\ y_{2,k} \end{bmatrix} = & \begin{bmatrix} f_{1i,1}(x_{k-1;i}) & f_{1i,2}(x_{k-1;i}) & 0 & 0 \\ f_{1i,3}(x_{k-1;i}) & f_{1i,4}(x_{k-1;i}) & 0 & 0 \\ 0 & 0 & h_{i,1}(y_{k-1;i}) & h_{i,2}(y_{k-1;i}) \\ 0 & 0 & h_{i,3}(y_{k-1;i}) & h_{i,4}(y_{k-1;i}) \end{bmatrix} \begin{bmatrix} e_{1,k-1;i} \\ e_{2,k-1;i} \\ y_{1,k-1;i} \\ y_{2,k-1;i} \end{bmatrix} \\ & + \begin{bmatrix} g_{1i}(x_{k-1;i}) \\ g_{2i}(x_{k-1;i}) \\ 0 \\ 0 \end{bmatrix} u_{k;i} + \begin{bmatrix} (f_{1i,1}(x_{k-1;i}) - \tilde{m}_1(x_{k;j})) & (f_{1i,2}(x_{k-1;i}) - \tilde{m}_2(x_{k;j})) \\ (f_{1i,3}(x_{k-1;i}) - \tilde{m}_3(x_{k;j})) & (f_{1i,4}(x_{k-1;i}) - \tilde{m}_4(x_{k;j})) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ & \times \begin{bmatrix} x_{r,1,k-1;i} \\ x_{r,2,k-1;i} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1,1,k;i} + \epsilon_{r,1,k;i} \\ \epsilon_{1,2,k;i} + \epsilon_{r,2,k;i} \\ \epsilon_{2,1,k;i} \\ \epsilon_{2,2,k;i} \end{bmatrix}. \quad \text{(I.3)} \end{aligned}$$

For the convergence analysis, the dynamics given in equation (5.85) are considered. Using the derived controller (5.63) and assuming that the steady state solution of the optimal controller gain has been reached, the following can be states,

$$\mu_{k;i} = - K_i n_{k-1;i} - W_i x_{r,k-1;i} - Z_i, \tag{I.4}$$

where,

$$K_i = \bar{\Gamma}_{k;i} g_i^T(x_{k-1;i}) \begin{bmatrix} \tilde{S}_i f_{1i}(x_{k-1;i}) & M_{2;i} h_i(y_{k-1;i}) \end{bmatrix}, \tag{I.5}$$

$$W_i = \bar{\Gamma}_{k;i} g_i^T(x_{k-1;i}) \tilde{S}_i \tilde{W}, \tag{I.6}$$

$$Z_i = \bar{\Gamma}_{k;i} (0.5 g_i^T(x_{k-1;i}) P_{1;i}^T - \Gamma_{k;i}^{-1} \hat{u}_{k;i}), \tag{I.7}$$

and where the definitions of $M_{1;i}$, $M_{2;i}$, and $P_{1;i}$ can be found in (5.72), (5.73), and (5.75), respectively.

Let us define a Lyapunov function $\hat{V}_k$, which is positive definite, as follows,

$$\begin{aligned} \hat{V}_{k-1;i} =& (n_{k-1;i} + h_i)^T M_i (n_{k-1;i} + h_i) + n_{k-1;i}^T M_i n_{k-1;i}, \\ =& 2 n_{k-1;i}^T M_i n_{k-1;i} + 2 h_i^T M_i n_{k-1;i} + h_i^T M_i h_i, \end{aligned} \tag{I.8}$$

where h_i is some constant term and since it is assumed that steady state solution has been reached for M_i and constant h_i, there is no time dependency.

The derivative of a Lyapunov function (I.8) is expected to be negative definite, and given by,

$$
\begin{aligned}
&\triangle\hat{V}_{k-1;i} = \hat{V}_{k;i} - \hat{V}_{k-1;i},\\
=&(n_{k;i}+h_i)^T M_i (n_{k;i}+h_i) + n_{k;i}^T M_i n_{k;i} - (n_{k-1;i}+h_i)^T M_i (n_{k-1;i}+h_i) - n_{k-1;i}^T M_i n_{k-1;i},\\
=&(f_i(z_{k-1;i})n_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + \epsilon_{k;i} + \tilde{W}x_{r,k-1;i} + h_i)^T M_i\\
\times&(f_i(z_{k-1;i})n_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + \epsilon_{k;i} + \tilde{W}x_{r,k-1;i} + h_i)\\
+&(f_i(z_{k-1;i})n_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + \epsilon_{k;i} + \tilde{W}x_{r,k-1;i})^T M_i\\
\times&(f_i(z_{k-1;i})n_{k-1;i} + g_i(z_{k-1;i})u_{k;i} + \epsilon_{k;i} + \tilde{W}x_{r,k-1;i})\\
-&2n_{k-1;i}^T M_i n_{k-1;i} - 2n_{k-1;i}^T M_i h_i - h_i^T M_i h_i,\\
&\textit{using the fact that } u_{k;i} = \mu_{k;i} = -K_i n_{k-1;i} - W_i x_{r,k-1;i} - Z_i,\\
=&[f_i(z_{k-1;i})n_{k-1;i} - g_i(z_{k-1;i})K_i n_{k-1;i} - g_i(z_{k-1;i})W_i x_{r,k-1;i} - g_i(z_{k-1;i})Z_i + \epsilon_{k;i}\\
+&\tilde{W}x_{r,k-1;i} + h_i]^T M_i [f_i(z_{k-1;i})n_{k-1;i} - g_i(z_{k-1;i})K_i n_{k-1;i} - g_i(z_{k-1;i})W_i x_{r,k-1;i}\\
-&g_i(z_{k-1;i})Z_i + \epsilon_{k;i} + \tilde{W}x_{r,k-1;i} + h_i]\\
+&[f_i(z_{k-1;i})n_{k-1;i} - g_i(z_{k-1;i})K_i n_{k-1;i} - g_i(z_{k-1;i})W_i x_{r,k-1;i} - g_i(z_{k-1;i})Z_i + \epsilon_{k;i}\\
&+\tilde{W}x_{r,k-1;i}]^T M_i [f_i(z_{k-1;i})n_{k-1;i} - g_i(z_{k-1;i})K_i n_{k-1;i} - g_i(z_{k-1;i})W_i x_{r,k-1;i}\\
-&g_i(z_{k-1;i})Z_i + \epsilon_{k;i} + \tilde{W}x_{r,k-1;i}] - 2n_{k-1;i}^T M_i n_{k-1;i} - 2n_{k-1;i}^T M_i h_i - h_i^T M_i h_i,\\
=&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} - g_i(z_{k-1;i})Z_i + \epsilon_{k;i} + h_i\right]^T M_i\\
\times&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} - g_i(z_{k-1;i})Z_i + \epsilon_{k;i} + h_i\right]\\
+&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} - g_i(z_{k-1;i})Z_i + \epsilon_{k;i}\right]^T M_i\\
\times&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} - g_i(z_{k-1;i})Z_i + \epsilon_{k;i}\right]\\
-&2n_{k-1;i}^T M_i n_{k-1;i} - 2n_{k-1;i}^T M_i h_i - h_i^T M_i h_i,\\
=&2\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} - g_i(z_{k-1;i})Z_i\epsilon_{k;i}^{\dagger}\epsilon_{k;i} + \epsilon_{k;i}\right]^T M_i\\
\times&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} - g_i(z_{k-1;i})Z_i\epsilon_{k;i}^{\dagger}\epsilon_{k;i} + \epsilon_{k;i}\right]\\
+&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} - g_i(z_{k-1;i})Z_i\epsilon_{k;i}^{\dagger}\epsilon_{k;i} + \epsilon_{k;i}\right]^T M_i h_i\\
+&h_i^T M_i h_i - 2n_{k-1;i}^T M_i n_{k-1;i} - 2n_{k-1;i}^T M_i h_i - h_i^T M_i h_i,\\
=&2\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} + [I - g_i(z_{k-1;i})Z_i\epsilon_{k;i}^{\dagger}]\epsilon_{k;i}\right]^T M_i\\
\times&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} + [I - g_i(z_{k-1;i})Z_i\epsilon_{k;i}^{\dagger}]\epsilon_{k;i}\right]\\
+&\left[(f_i(z_{k-1;i}) - g_i(z_{k-1;i})K_i)n_{k-1;i} + (\tilde{W} - g_i(z_{k-1;i})W_i)x_{r,k-1;i} + [I - g_i(z_{k-1;i})Z_i\epsilon_{k;i}^{\dagger}]\epsilon_{k;i}\right]^T M_i h_i\epsilon_{k;i}^{\dagger}\epsilon_{k;i}\\
-&2n_{k-1;i}^T M_i n_{k-1;i} - 2n_{k-1;i}^T M_i h_i\epsilon_{k;i}^{\dagger}\epsilon_{k;i},\\
=&n_{k-1;i}^T D n_{k-1;i}, \qquad \text{(I.9)}
\end{aligned}
$$

where $\epsilon^{\dagger}_{k;i}$ is the pseudoinverse of $\epsilon_{k;i}$ and is defined by $\epsilon^{\dagger}_{k;i} = (\epsilon^T_{k;i}\epsilon_{k;i})^{-1}\epsilon^T_{k;i}$. The vector $m_{k-1;i}$ is defined as,

$$m_{k-1;i} = \begin{bmatrix} n_{k-1;i} \\ x_{r,k-1;i} \\ \epsilon_{k;i} \end{bmatrix}, \tag{I.10}$$

and matrix D is defined by (5.86) - (5.87).

As the derivative of the defined Lyapunov function (I.9) is negative definite, it can be said that $D < 0$.

To satisfy this, let us define a small positive number σ, such that $0 < \sigma < \lambda_{max}(M_i)$. The existence of this number σ allows the following inequality to hold,

$$D < -\sigma I, \tag{I.11}$$

where $\lambda_{max}(M_i)$ is the maximum eigenvalue of matrix M_i.

The expectation, denoted as $E[.]$, of the derivative of the Lyapunov function can be described by the following inequality,

$$E[\triangle \hat{V}_{k-1;i}] \leq -\sigma E[||m_{k-1;i}||^2] \leq -\sigma E[||e_{k-1;i}||^2]. \tag{I.12}$$

Based on the definition of the Lyapunov function (I.8), the following can be said,

$$\lambda_{min}(M_i)E[||e_{k-1;i}||^2] \leq E[\hat{V}_{k-1;i}] \leq \lambda_{max}(M_i)E[||e_{k-1;i}||^2]. \tag{I.13}$$

When we combine inequalities (I.12) and (I.13), we obtain,

$$E[\hat{V}_{k;i}] - E[\hat{V}_{k-1;i}] \leq -\frac{\sigma}{\lambda_{max}(M_i)}E[\hat{V}_{k-1;i}], \tag{I.14}$$

which yields,

$$E[\hat{V}_{k;i}] \leq \theta E[\hat{V}_{k-1;i}], \tag{I.15}$$

where $\theta = 1 - \frac{\sigma}{\lambda_{max}(M_i)}$. From (I.15), we can easily obtain,

$$E[\hat{V}_{k-1;i}] \leq \theta^{k-1}E[\hat{V}_1], \tag{I.16}$$

which concludes that $\lim_{k \to \infty} E[\hat{V}_{k-1;i}] = 0$.