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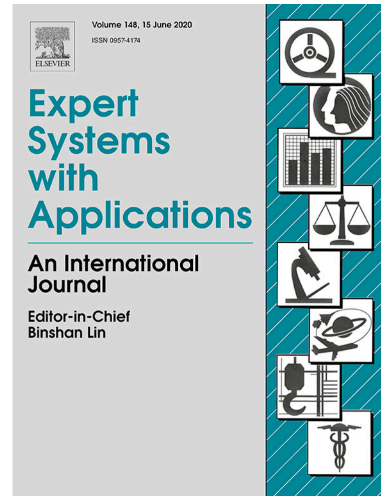

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A Box-Uncertainty in DEA: A Robust Performance Measurement Framework

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Abstract. The problem of assessment of Decision Making Units (DMUs) by using Data Envelopment Analysis (DEA) may not be straightforward due to the data uncertainty. Several studies have been developed to incorporate uncertainty into input/output values in the DEA literature. On the other hand, while traditional DEA models focus more on crisp data, there exist many applications in which data is reported in form of intervals. This paper considers the box-uncertainty in data which means that each input/output value is selected from a symmetric box. This specific type of uncertainty has been addressed as Interval DEA approaches. Our proposed model deals with efficiency evaluation of DMUs with imprecise data in a robust optimization. We assume that inputs and outputs are reported in the form of intervals and propose the robust counterpart problem for the envelopment form of the DEA model. Further, we also develop two ranking methods which have more benefits compared to some existing approaches. An illustrative example is provided to show how the proposed approaches work. An application on hospital efficiency in East Virginia is used to show the usefulness of the proposed approaches.

Keywords. Interval DEA, Box-Uncertainty, Robust optimization techniques, Ranking, Fuzzy DEA, Data Uncertainty.

1. Introduction

The conventional DEA model proposed by Charnes, Cooper and Rhods (1978) (CCR) is a linear programming model which deals with precisely known data where inputs and outputs values are deterministic and exactly known. However, in real-life applications, we may encounter imprecise data due to incomplete or non-attainable information, errors in measurements, unquantifiable variables, or any other source of reason. Imprecise data may lead to some challenges in applying the DEA technique, mostly resulting in a nonlinear DEA model.

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The problem of the evaluation of units with imprecise data has attracted attentions of several scholars. For example, Cooper et al. (1999) developed Imprecise Data Envelopment Analysis (IDEA) method. Their method can be applied in the situation where there exist both imprecisely and exactly-known data in which the IDEA models are transformed into linear programming problems. Kim et al. (1999) proposed a procedure to incorporate partial data into DEA. Their original model was a complicated non-linear model that was transformed into a linear programming problem by applying a linear scale transformation and the variable change technique.

In summary, three different types of approaches can be adopted to model imprecise data in DEA including fuzzy approaches, stochastic methods, and robust optimization-based techniques. Several scholars have modelled imprecise data as fuzzy numbers and incorporated defuzzification methods into DEA. Emrouznejad et al (2014) provided a taxonomy and review of the Fuzzy DEA approaches. Also input/output variables can be considered as random variables, which results in stochastic DEA models. See Olesen and Petersen (2016) for a review on stochastic DEA methods and Peykani et al (2020) for a review on robust DEA methods.

Robust Optimization (RO) is a technique to model optimization problems with uncertain data which aims to determine an optimal solution which is the best for all or the most possible realizations of the uncertain parameters. Ben-Tal and Nemirovski (1998, 1999, 2000) and Bertsimas and Sim (2004) investigated uncertainty in data and proposed different RO approaches to obtain the optimal solution. Wang and Wei (2010) applied Ben-Tal and Nemirovski's approach (2000) in DEA to develop two robust formulations for the multiplier form of the CCR model in the presence of uncertain data. They considered the perturbations on inputs/outputs for the different uncertainty levels and computed the efficiency score of units and provided a ranking for them**.**

Sadjadi and Omrani (2008) proposed the robust formulation of the multiplier form of CCR model based on the RO technique presented by Ben-Tal and Nemirovski (2000). Their proposed model is a non-linear programming problem which shows the drawback of their approach of applying an inappropriate RO technique. They also presented the robust formulation of the multiplier form of the CCR model based on the RO technique proposed by Bertsimas et al. (2004). Unlike, the first model, the second is a linear programming model. Sadjadi and Omrani (2010) proposed a bootstrapped robust model for the multiplier form of the CCR model, based on the approach of Bertsimas et al. (2004), to solve the perturbation and sampling error problems. Sadjadi et al. (2011) proposed an interactive robust model based on Bertsimas et al.'s approach (2004) to find the targets of units according to the DM's preferences.

Omrani (2013) proposed a RO technique, based on the robust approach of Bertsimas et al. (2004), to find the common set of weights in DEA by using the goal programming technique. Their method can be applied to evaluate the absolute efficiency score of units for different values of robustness levels in order to rank them. Ehrgott et al. (2018) used the framework of RO to propose a DEA model in case of data uncertainty. They provided a first-order algorithm to solve their model and showed that the optimal solution of it determined the maximum possible efficiency score of a unit.

Most of the existing methods in the literature apply the RO technique proposed by Ben-Tal and Nemirovski (2000) and Bertsimas et al. (2004) to evaluate DMUs and rank them in the presence of uncertain data. This paper considers the scenario where uncertainty in input/output variables are modelled in the form of intervals. Several Interval DEA approaches have been developed to perform efficiency analysis in DEA. Wang et al. (2005) considered the efficiency assessment of units in the presence of interval and/or fuzzy data. They proposed two linear CCR models to obtain the interval efficiency of DMUs and then applied the interval efficiencies of all units by a minimax regret-based approach to rank units. Wu et al. (2013) proposed a two-phase approach in which the first phase obtains the interval cross-efficiency score of DMUs and the second phase ranks units by applying an improved TOPSIS technique. Khezri et al. (2019) proposed a method for ranking units based on the distances of a unit from the efficiency and inefficiency frontiers and the lower and upper super efficiency scores of it applying a lexicographic order.

Since interval data can be considered as box-uncertainties, RO techniques can be adopted into Interval DEA. This study proposes a robust counterpart problem for the envelopment form of CCR model based on the Ben-Tal and Nemirovski (1999)'s approach. Additionally, the proposed model can be applied to provide two complete ranking of DMUs based on the secondary goal models presented by Wu et al (2015). Salahi et al. (2016) provided an optimistic RO approach to common set of weights in DEA, but their robust counterpart problem for the envelopment form of the CCR model is not formulated correctly. In this study, a robust counterpart problem is formulated, and the drawback of their formulation is addressed.

The rest of this paper unfolds as follows. Section 2 provides the robust counterpart of the envelopment form of the model. Section 3 proposes two RO-based approaches to rank all DMUs based on the secondary goal models proposed by Wu et al. (2015). The results are illustrated by some numerical examples in Section 4. A real application in hospital efficiency is presented in Section 5. Finally, Section 6 concludes the paper and provides direct for future research.

2. The robust formulation of CCR model

This section first proposes the robust counterpart of the envelopment form of the CCR model. Next, the optimistic counterpart of the multiplier form of the CCR model is formulated and then the relationship between the dual of the robust counterpart and the optimistic counterpart of the CCR model is established.

2.1. The robust counterpart of envelopment form of CCR model

Consider a system of n DMUs, denoted by DMU_j, $j = 1,\ldots,n$, where each unit consumes m different inputs to generate s different outputs. The i^{th} input and r^{th} output for DMU_j are denoted by x_{ij} and y_{rj}, respectively, for $i = 1,\ldots,m$ and $r = 1,\ldots,s$. Also, suppose that input and output values are not deterministic for all units and $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$, where the lower and upper bounds are positive and finite values. Assume that DMU_o is the unit under evaluation.

The envelopment form of the CCR model with interval data can be written as follows:

$$\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\theta x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij} \geq 0, \quad i = 1,\ldots,m, \quad x_{ij}^L \leq x_{ij} \leq x_{ij}^U \\
&\sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro} \geq 0, \quad r = 1,\ldots,s, \quad y_{rj}^L \leq y_{rj} \leq y_{rj}^U \qquad (1) \\
&\lambda_j \geq 0, \qquad j = 1,\ldots,n, \\
&\theta \text{ is free}.
\end{aligned}$$

Next theorem provides the formulation of the robust counterpart of the envelopment form of CCR model.

Theorem 1. *The robust counterpart of the envelopment form of the CCR model in the presence of interval data can be formulated as:*

$$\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&(\theta - \lambda_o) x_{io}^L - \sum_{j=1, j \neq o}^{n} \lambda_j x_{ij}^U \geq 0, \quad i = 1,\ldots,m, \\
&\sum_{j=1, j \neq o}^{n} \lambda_j y_{rj}^L + (\lambda_o - 1) y_{ro}^U \geq 0, \quad r = 1,\ldots,s, \qquad (2) \\
&\lambda_j \geq 0, \qquad j = 1,\ldots,n, \\
&\theta \text{ is free}.
\end{aligned}$$

Proof. Based on the robust counterpart model for linear programming problems, proposed by Ben-Tal and Nemirovski (1999), the robust counterpart of model (1) is as follows:

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\min_{x_{ij}^L \le x_{ij} \le x_{ij}^U} \{\theta x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij}\} \ge 0, \quad i = 1,\ldots,m, \\
&\min_{y_{rj}^L \le y_{rj} \le y_{rj}^U} \{\sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro}\} \ge 0, \quad r = 1,\ldots,s, \qquad (3) \\
&\lambda_j \ge 0, \quad j = 1,\ldots,n, \\
&\theta \text{ is free.}
\end{aligned}
$$

Model (3) can be written as:

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\min_{x_{ij}^L \le x_{ij} \le x_{ij}^U} \{(\theta - \lambda_o) x_{io} - \sum_{j=1, j \ne o}^{n} \lambda_j x_{ij}\} \ge 0, \quad i = 1,\ldots,m, \\
&\min_{y_{rj}^L \le y_{rj} \le y_{rj}^U} \{\sum_{j=1}^{n} \lambda_j y_{rj} + (\lambda_o - 1) y_{ro}\} \ge 0, \quad r = 1,\ldots,s, \qquad (4) \\
&\lambda_j \ge 0, \quad j = 1,\ldots,n, \\
&\theta \text{ is free.}
\end{aligned}
$$

First, we claim that $\lambda_o^* - 1 \le 0$ at the optimality of Model (4). Let's consider the following two cases:

1) DMU_o is an inefficient unit. Therefore $\lambda_o^* = 0$ and so $\lambda_o^* - 1 \le 0$.
2) DMU_o is an efficient unit. If DMU_o is an extreme efficient unit then $\lambda_o^* = 1$ and so $\lambda_o^* - 1 \le 0$. Otherwise DMU_o is not an extreme efficient unit and then $\lambda_o^* < 1$. Therefore, $\lambda_o^* - 1 \le 0$.

Also, we prove that $(\theta_o^* - \lambda_o^*) \ge 0$ at the optimality of Model (4). Since $\theta_o^* \le 1$ and $0 \le \lambda_o^* \le 1$, we have:

1) If $0 < \theta_o^* < 1$, then DMU_o is an inefficient unit. Therefore $\lambda_o^* = 0$ and so $(\theta_O^* - \lambda_o^*) > 0$.
2) If $\theta_o^* = 1$, then DMU_o is an efficient unit. Hence $(\theta_O^* - \lambda_o^*) \ge 0$.

Therfore, the following equations are held:

$$
\min_{x_{ij}^L \le x_{ij} \le x_{ij}^U} \left\{(\theta - \lambda_o) x_{io} - \sum_{j=1, j \ne o}^{n} \lambda_j x_{ij}\right\} = \min_{x_{io}^L \le x_{io} \le x_{io}^U} \{(\theta - \lambda_o) x_{io}\} - \max_{x_{ij}^L \le x_{ij} \le x_{ij}^U} \{\sum_{j=1, j \ne o}^{n} \lambda_j x_{ij}\}
$$

$$
= (\theta - \lambda_o)\, x_{io}^L - \sum_{j=1, j \ne o}^{n} \lambda_j x_{ij}^U.
$$

$$\min_{y_{rj}^L \le y_{rj} \le y_{rj}^U} \left\{ \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} + (\lambda_o - 1) y_{ro} \right\} = \min_{y_{rj}^L \le y_{rj} \le y_{rj}^U} \{ \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \} + \min_{y_{ro}^L \le y_{ro} \le y_{ro}^U} \{ (\lambda_o - 1) y_{ro} \}$$

$$= \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj}^L + (\lambda_o - 1) y_{ro}^U.$$

Consequently, model (4) can be converted into the following model:

$$
\begin{array}{lll}
(\mathrm{R-CCR}) & \min \theta & \\
& \text{s.t.} & \\
& (\theta - \lambda_o) x_{io}^L - \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij}^U \ge 0, & i = 1,\ldots,m, \\
& \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj}^L + (\lambda_o - 1) y_{ro}^U \ge 0, & r = 1,\ldots,s, \quad (5) \\
& \lambda_j \ge 0, & j = 1,\ldots,n.
\end{array}
$$

Model (5) is the robust counterpart of the envelopment form of CCR model and is called (R - CCR) model. ☐

Theorem 1 formulated the robust counterpart of the CCR model in its envelopment form. Salahi et al. (2016) also formulated the robust counterpart of the envelopment form of the CCR model in a different way as follows:

$$
\begin{array}{ll}
\min \theta & \\
\text{s.t.} & \\
\sum_{j=1}^{n} \lambda_j x_{ij}^U + x_{io}^U h_i - x_{io}^L k_i \le 0, & i = 1,\ldots,m, \\
\sum_{j=1}^{n} \lambda_j y_{rj}^L \ge y_{ro}^U, & r = 1,\ldots,s, \quad (6) \\
-h_i + k_i = \theta, & i = 1,\ldots,m, \\
\lambda_j \ge 0, & j = 1,\ldots,n, \\
h_i, k_i \ge 0, & i = 1,\ldots,m.
\end{array}
$$

In what follows, we show that their formulation was not correct:

Salahi et al. (2016) obtained their model (6) based on the following equations:

$$\min \left(\theta x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij} \right) = \min_{x_{io}^L \le x_{io} \le x_{io}^U} \theta x_{io} - \max_{x_{ij}^L \le x_{ij} \le x_{ij}^U} \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io}^U h_i - x_{io}^L k_i - \sum_{j=1}^{n} \lambda_j x_{ij}^U, \quad (7)$$

where

$$-h_i + k_i = \theta, \quad i = 1,\ldots,m.$$

According to the additivity principle of linear programming (LP) theory, there should be no interaction between the decision variables in each constraint or the objective function of the LP model. Both terms in $\theta x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij}$, i.e. θx_{io} and $\sum_{j=1}^{n} \lambda_j x_{ij}$, include the same parameter x_{io}. Given that the additivity principle in LP theory, we should not separately consider θx_{io} and $\lambda_o x_{io}$, when we aim to find the minimum of $\theta x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij}$, for $x_{ij}^L \leq x_{ij} \leq x_{ij}^U$, $j = 1,\ldots,n$. Ignoring this principle in Salahi et al.'s work has caused the parameter x_{io} to take two different values, x_{io}^L and x_{io}^U, at the same time in one constraint, which is not correct. Therefore, equation (7) is not accurate, and so, model (6) was not formulated correctly in Salahi et al.'s approach. We provide a correct robust counterpart of the CCR model in its envelopment form.

In the following theorem, the optimistic counterpart of the multiplier form of CCR model is presented. Further, the relationship between this optimistic counterpart model and the dual of model (5) are established.

Theorem 2. *The optimistic counterpart of the multiplier form of the CCR model is identical to the dual of the R-CCR model in the presence of interval uncertainties.*

Proof. The dual of the (R - CCR) model (5) can be written as follows:

$$
\begin{aligned}
(\mathrm{DR-CCR}) \quad & \max \sum_{r=1}^{s} q_r y_{ro}^U \\
& \text{s.t.} \\
& \sum_{r=1}^{s} q_r y_{rj}^L - \sum_{i=1}^{m} p_i x_{ij}^U \leq 0, \quad j \neq o, \qquad (8) \\
& \sum_{r=1}^{s} q_r y_{ro}^U - \sum_{i=1}^{m} p_i x_{io}^L \leq 0, \\
& \sum_{i=1}^{m} p_i x_{io}^L = 1, \\
& q_r \geq 0 \qquad r = 1,\ldots,s, \\
& p_i \geq 0 \qquad i = 1,\ldots,m,
\end{aligned}
$$

where p_i and q_r are the dual variables corresponding to the constraints $(\theta - \lambda_o) x_{io}^L - \sum_{j=1, j \neq o}^{n} \lambda_j x_{ij}^U \geq 0$ and $\sum_{j=1, j \neq o}^{n} \lambda_j y_{rj}^L + (\lambda_o - 1) y_{ro}^U \geq 0$ for all $i = 1,\ldots,m$ and $r = 1,\ldots,s$, respectively.

Consider the multiplier form of the CCR model, proposed by Charnes et al. (1978), as follows:

$$\begin{aligned}
&\max \sum_{r=1}^{s} u_{ro} y_{ro} \\
&s.t. \\
&\sum_{r=1}^{s} u_{ro} y_{rj} - \sum_{i=1}^{m} v_{io} x_{ij} \le 0, \quad j \ne o, \qquad (9)\\
&\sum_{r=1}^{s} u_{ro} y_{ro} - \sum_{i=1}^{m} v_{io} x_{io} \le 0, \\
&\sum_{i=1}^{m} v_{io} x_{io} = 1, \\
&u_{ro} \ge 0, \qquad r = 1,\ldots,s, \\
&v_{io} \ge 0, \qquad i = 1,\ldots,m.
\end{aligned}$$

The optimistic counterpart of the multiplier form of the CCR model, based on the optimistic counterpart formulation proposed by Beck and Ben-Tal (2009) is given below:

$$\begin{aligned}
(\mathrm{O}-\mathrm{CCR_m}) \quad & E_o^{op} = \max \sum_{r=1}^{s} u_r y_{ro}^U \\
& \text{s.t.} \\
& \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U \le 0, \quad j \ne o, \qquad (10)\\
& \sum_{r=1}^{s} u_r y_{ro}^U - \sum_{i=1}^{m} v_i x_{io}^L \le 0, \\
& \sum_{i=1}^{m} v_i x_{io}^L = 1, \\
& u_r \ge 0, \qquad r = 1,\ldots,s, \\
& v_i \ge 0, \qquad i = 1,\ldots,m.
\end{aligned}$$

As can be seen above, models (8) and (10) are identical. □

An efficiency analysis for DMUs with interval data can be performed by applying either (R - CCR) or (DR - CCR) models. But what we are interested in is to provide a complete ranking of units. To this end, the next section proposes a full-ranking algorithm.

3. Our proposed RO-based ranking methods

This section provides two approaches to rank all units under the different perspectives in the case of interval input/output parameters. For this purpose, the secondary goal models are integrated to RO techniques as described below.

Regarding the existence of box-uncertainty on inputs/outputs, let's determine the upper efficiency scores of units as a self-evaluation of them in the first step of the proposed methods. This means that, the optimistic counterpart of the multiplier form of the CCR model (10) is solved for each DMU_o, $o \in \{1,...,n\}$, and the optimal solution $\{v_{1o}^*,...,v_{mo}^*, u_{1o}^*, ...,u_{so}^*\}$ is obtained. These optimal value objective functions are denoted by E_o^{op}.

The cross-efficiency evaluation method proposed by Sexton et al. (1986) is one of the most popular methods in DEA for ranking decision making units. This method has several advantages so that many authors have applied it in various cases, see for instance, Doyle and Green (1995), Anderson et al. (2002), Boussofiane et al. (1991), Sun (2002), Ertay and Ruan (2005), Liang et al. (2008), Wu et al. (2015) and Lim et al. (2014). However, the main drawback of the cross- efficiency method that possibly reduces the utility of it is that there may be the non- unique optimal weights and so the cross-efficiency scores of units may not be unique. As a result, some studies have introduced secondary goal models in cross-efficiency evaluation, see Doyle and Green (1994), Liang et al. (2008a), Wang and Chin (2010). Wu et al. (2015) proposed several secondary goal models to determine both desirable and undesirable cross- efficiency scores of all units. Compared with the secondary goal models in the literature, the cross- efficiency scores obtained by their models are always reachable for all decision making units.

In the next step of the proposed methods, two models (11a) and (11b) are suggested to determine the undesirable and desirable cross-efficiency scores of each unit by generalizing the models proposed by Wu et al. (2015) to the case of box-uncertainty for the data.

Table 1. The undesirable and desirable cross- efficiency scores in the case of box-uncertainty

The undesirable cross- efficiency score	The desirable cross- efficiency score
$\min \left(\sum_{r=1}^{s} u_{ro} y_{rj} / \sum_{i=1}^{m} v_{io} x_{ij} : x_{ij}^L \le x_{ij} \le x_{ij}^U, y_{rj}^L \le y_{rj} \le y_{rj}^U\right)$ s.t. $\left(\sum_{r=1}^{s} u_{ro} y_{rk}^L / \sum_{i=1}^{m} v_{io} x_{ik}^U\right) \le 1, \ k = 1,....,n, k \ne o,$ $\left(\sum_{r=1}^{s} u_{ro} y_{ro}^U / \sum_{i=1}^{m} v_{io} x_{io}^L\right) \le 1, \quad (11a)$ $\left(\sum_{r=1}^{s} u_{ro} y_{ro}^U / \sum_{i=1}^{m} v_{io} x_{io}^L\right) = E_o^{op},$ $v_{io} \ge 0, \quad i = 1,...,m,$ $u_{ro} \ge 0, \quad r = 1,...,s.$	$\max \left(\sum_{r=1}^{s} u_{ro} y_{rj} / \sum_{i=1}^{m} v_{io} x_{ij} : x_{ij}^L \le x_{ij} \le x_{ij}^U, y_{rj}^L \le y_{rj} \le y_{rj}^U\right)$ s.t. $\left(\sum_{r=1}^{s} u_{ro} y_{rk}^L / \sum_{i=1}^{m} v_{io} x_{ik}^U\right) \le 1, \ k = 1,....,n, k \ne o,$ $\left(\sum_{r=1}^{s} u_{ro} y_{ro}^U / \sum_{i=1}^{m} v_{io} x_{io}^L\right) \le 1, \quad (11b)$ $\left(\sum_{r=1}^{s} u_{ro} y_{ro}^U / \sum_{i=1}^{m} v_{io} x_{io}^L\right) = E_o^{op},$ $v_{io} \ge 0, \quad i = 1,...,m$ $u_{ro} \ge 0, \quad r = 1,...,s.$

Table 1 reports the suggested models (11a) and (11b). The main goal of model (11a) is to select the weights for inputs and outputs of DMU_o that minimize the cross-efficiency score of DMU_j and keep the upper efficiency score of DMU_o at its optimistic efficiency score E_o^{op}. Similarly, model (11b) aims to select the weights for inputs and outputs of DMU_o that maximize the cross-efficiency score of DMU_j and keep the upper efficiency score of DMU_o at its optimistic efficiency score E_o^{op}.

Models (11a) and (11b) are non-linear programming problems. These models can be converted to their equivalent LP models by applying the Charnes and Cooper (1962) transformation. The resulting LP models are reported in Table 2.

Table 2. The equivalent LP models with models (11a) and (11b).

The LP model equivalent to model (11a)	The LP model equivalent to model (11b)
$\min \left(\sum_{r=1}^{s} u_{ro} y_{rj}; y_{rj}^L \le y_{rj} \le y_{rj}^U\right)$ s.t. $\sum_{r=1}^{s} u_{ro} y_{rk}^L - \sum_{i=1}^{m} v_{io} x_{ik}^U \le 0, \quad k \ne o,$ $\sum_{i=1}^{m} v_{io} x_{ij} \le 1, \quad x_{ij}^L \le x_{ij} \le x_{ij}^U, \quad (12a)$ $\sum_{r=1}^{s} u_{ro} y_{ro}^U - \sum_{i=1}^{m} v_{io} x_{io}^L \le 0,$ $\sum_{r=1}^{s} u_{ro} y_{ro}^U - E_o^{op} \sum_{i=1}^{m} v_{io} x_{io}^L = 0,$ $u_{ro} \ge 0, \quad r = 1,\ldots,s,$ $v_{io} \ge 0, \quad i = 1,\ldots,m.$	$\max \left(\sum_{r=1}^{s} u_{ro} y_{rj}; y_{rj}^L \le y_{rj} \le y_{rj}^U\right)$ s.t. $\sum_{r=1}^{s} u_{ro} y_{rk}^L - \sum_{i=1}^{m} v_{io} x_{ik}^U \le 0, \quad k \ne o,$ $\sum_{i=1}^{m} v_{io} x_{ij} \le 1, \quad x_{ij}^L \le x_{ij} \le x_{ij}^U, \quad (12b)$ $\sum_{r=1}^{s} u_{ro} y_{ro}^U - \sum_{i=1}^{m} v_{io} x_{io}^L \le 0,$ $\sum_{r=1}^{s} u_{ro} y_{ro}^U - E_o^{op} \sum_{i=1}^{m} v_{io} x_{io}^L = 0,$ $u_{ro} \ge 0, \quad r = 1,\ldots,s,$ $v_{io} \ge 0, \quad i = 1,\ldots,m.$

Model (12a) aims to select the input/output weights that generate the maximum possible efficiency score of DMU_o and minimize the cross-efficiency score of DMU_j for any realization of data, simultaneously. Similarly, the main goal of model (12b) is to find the input/output weights for obtaining the maximum possible efficiency score of DMU_o and maximize the cross-efficiency score of DMU_j for any realization of data, simultaneously. In other words, we select the optimal weights of model (10) evaluating DMU_o such that the cross- efficiency score of DMU_j is minimized or maximized for any realization of data.

Note that models (12a and 12b) include uncertain parameters. According the robust counterpart model proposed by Ben-Tal and Nemirovski (1999), the robust counterpart of these can be used to find the desired weights for any realization of data. The corresponding models are reported as (13a) and (13b), respectively, in Table 3.

Table 3. The robust counterpart of models (12a) and (12b).

The robust counterpart of model (12a)	The robust counterpart of model (12b)
$\alpha_{oj} = \min \sum_{r=1}^{s} u_{ro} y_{rj}^{U}$ s.t. $\sum_{r=1}^{s} u_{ro} y_{rk}^{L} - \sum_{i=1}^{m} v_{io} x_{ik}^{U} \leq 0, \quad k \neq o,$ $\sum_{i=1}^{m} v_{io} x_{ij}^{U} \leq 1, \quad (13a)$ $\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - \sum_{i=1}^{m} v_{io} x_{io}^{L} \leq 0,$ $\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - E_{o}^{op} \sum_{i=1}^{m} v_{io} x_{io}^{L} = 0,$ $u_{ro} \geq 0, \quad r = 1,\ldots,s$ $v_{io} \geq 0, \quad i = 1,\ldots,m.$	$\beta_{oj} = \max \sum_{r=1}^{s} u_{ro} y_{rj}^{L}$ s.t. $\sum_{r=1}^{s} u_{ro} y_{rk}^{L} - \sum_{i=1}^{m} v_{io} x_{ik}^{U} \leq 0, \quad k \neq o,$ $\sum_{i=1}^{m} v_{io} x_{ij}^{U} \leq 1, \quad (13b)$ $\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - \sum_{i=1}^{m} v_{io} x_{io}^{L} \leq 0,$ $\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - E_{o}^{op} \sum_{i=1}^{m} v_{io} x_{io}^{L} = 0,$ $u_{ro} \geq 0, \quad r = 1,\ldots,s$ $v_{io} \geq 0, \quad i = 1,\ldots,m.$

Model (13a) selects the optimal weight for inputs/outputs of DMU_o in a way that it keeps the efficiency score of DMU_o at its optimistic level and minimizes the cross-efficiency score of DMU_j for any realization of data. On the other hand, model (13b) selects the optimal weights for inputs/outputs of DMU_o such that it keeps the efficiency score of DMU_o at its optimistic level and maximizes the cross-efficiency score of DMU_j for any realization of data.

It should be noted that models (13a) and (13b) are only applied to select the input /output weights of DMU_o in order to minimize or maximize the cross-efficiency score of DMU_j. In other word, these models do not provide the minimum and maximum possible cross-efficiency scores of units.

Now, models (14a) and (14b) are presented to determine the minimum and maximum possible cross-efficiency score of DMU_j applying the optimal weights for DMU_o obtained by models (13a) and (13b), respectively.

Table 4. The minimum and maximum cross-efficiency score.

Minimum cross-efficiency score	Maximum cross-efficiency score

Model (14a)	Model (14b)
$\alpha_{oj}^{min} = \min \sum_{r=1}^{s} u_{ro} y_{rj}^{L}$	$\beta_{oj}^{max} = \max \sum_{r=1}^{s} u_{ro} y_{rj}^{U}$
s.t.	s.t.
$\sum_{r=1}^{s} u_{ro} y_{rk}^{L} - \sum_{i=1}^{m} v_{io} x_{ik}^{U} \leq 0, \quad k \neq o,$	$\sum_{r=1}^{s} u_{ro} y_{rk}^{L} - \sum_{i=1}^{m} v_{io} x_{ik}^{U} \leq 0, \quad k \neq o,$
$\sum_{r=1}^{s} u_{ro} y_{rj}^{U} = \alpha_{oj}, \quad (14a)$	$\sum_{r=1}^{s} u_{ro} y_{rj}^{L} = \beta_{oj}, \quad (14b)$
$\sum_{i=1}^{m} v_{io} x_{ij}^{U} \leq 1,$	$\sum_{i=1}^{m} v_{io} x_{ij}^{U} \leq 1,$
$\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - \sum_{i=1}^{m} v_{io} x_{io}^{L} \leq 0,$	$\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - \sum_{i=1}^{m} v_{io} x_{io}^{L} \leq 0,$
$\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - E_o^{op} \sum_{i=1}^{m} v_{io} x_{io}^{L} = 0,$	$\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - E_o^{op} \sum_{i=1}^{m} v_{io} x_{io}^{L} = 0,$
$u_{ro} \geq 0, \quad r = 1,\dots,s,$	$u_{ro} \geq 0, \quad r = 1,\dots,s,$
$v_{io} \geq 0, \quad i = 1,\dots,m.$	$v_{io} \geq 0, \quad i = 1,\dots,m.$

In Table 4, the values α_{oj} and β_{oj} appearing in the second constraints of models (14a) and (14b) are the optimal values of models (13a) and (13b), respectively. Also $E_o^{op}$is the optimal value of model (10) evaluating DMU_o.

Model (14a) determines the minimum cross-efficiency score of DMU_j while keeping the efficiency score of DMU_o at its optimistic level. Therefore, α_{oj}^{min} is the 'undesirable cross-efficiency score' of DMU_j corresponding to DMU_o.

Model (14b) determines the maximum cross-efficiency score of DMU_j while keeping the efficiency score of DMU_o at its optimistic level. Therefore, β_{oj}^{max} is the 'desirable cross-efficiency score' of DMU_j corresponding to DMU_o.

Theorem 3 proves that the minimum cross- efficiency score of a unit obtained by model (14a) is lower than its maximum cross- efficiency score obtained by model (14b).

Theorem 3. *Let α_{oj}^{min} and β_{oj}^{max} be the undesirable and desirable cross-efficiency scores of DMU_j respectively. Then $\alpha_{oj}^{min} \leq \beta_{oj}^{max}$.*

Proof. Suppose that (u^*, v^*) is an optiaml solution for model (14b). It is clear that (u^*, v^*) is a feasible solution for model (13a), therefore, $\alpha_{oj} \leq \beta_{oj}^{max}$. The two following cases may happen:

Case (a): $\alpha_{oj} = \beta_{oj}^{max}$. In this case, (u^*, v^*) is a feasible solution for model (14a). Hence, $\alpha_{oj}^{min} \leq \sum_{r=1}^{s} u_{ro}^* y_{rj}^L \leq \sum_{r=1}^{s} u_{ro}^* y_{rj}^U = \beta_{oj}^{max}$.

Case (b): $\alpha_{oj} < \beta_{oj}^{max}$. Suppose that $\gamma = \frac{\beta_{oj}^{max}}{\alpha_{oj}} > 1$. In this case, $(\frac{1}{\gamma}u^*, \frac{1}{\gamma}v^*)$ is a fessible solution for model (14a), hence, $\alpha_{oj}^{min} \leq \frac{1}{\gamma}\left(\sum_{r=1}^{s} u_{ro}^* y_{rj}^L\right) \leq \frac{1}{\gamma}\left(\sum_{r=1}^{s} u_{ro}^* y_{rj}^U\right) < \beta_{oj}^{max}$.

Therefore, in both cases, $\alpha_{oj}^{min} \leq \beta_{oj}^{max}$. □

Theorem 4 proves that the maximum cross-efficiency score of a unit obtained by model (14b) is lower than its optimistic efficiency score obtained by model (10).

Theorem 4. *Let* β_{oj}^{max} *be the desirable cross-efficiency scores of* DMU_j *and* E_o^{op} *be its optimistic efficiency score obtained by solving model (10). Then* $\beta_{oj}^{max} \leq E_j^{op}$.

Proof. Assume that $(u_{1o}^*,...,u_{so}^*, v_{1o}^*, ...,v_{mo}^*)$ is an optimal solution for model (14b) evaluating DMU_j. Since $\sum_{i=1}^{m} v_{io}^* x_{ij}^U \leq 1$, the following two cases may happen:

1) $\sum_{i=1}^{m} v_{io}^* x_{ij}^L = 1$. It is clear that, in this case, $(u_{1o}^*,...,u_{so}^*, v_{1o}^*, ...,v_{mo}^*)$ is a feasible solution for model (10). Therefore, $E_j^{op} \geq \beta_{oj}^{max}$.
2) $\alpha = \sum_{i=1}^{m} v_{io}^* x_{ij}^L < 1$. In this case, $\left(\frac{1}{\alpha}u_{1o}^*,...,\frac{1}{\alpha}u_{so}^*, \frac{1}{\alpha}v_{1o}^*,...,\frac{1}{\alpha}v_{so}^*\right)$ is a feasible solution for model (10). Hence, $E_j^{op} \geq \beta_{oj}^{max}$.

Therefore, in both cases, $E_j^{op} \geq \beta_{oj}^{max}$. □

Now, a cross-efficiency matrix for DMUs with interval data can be constructed based on the desirable and undesirable cross-efficiency scores, as follows:

Table 5. Cross-efficiency matrix

DMU	DMU_1	DMU_2	⋯	DMU_n
DMU_1	$[\alpha_{11}^{min}, \beta_{11}^{max}]$	$[\alpha_{12}^{min}, \beta_{12}^{max}]$	⋯	$[\alpha_{1n}^{min}, \beta_{1n}^{max}]$

DMU_2	$[\alpha_{21}^{min},\beta_{21}^{max}]$	$[\alpha_{22}^{min},\beta_{22}^{max}]$	$\cdots$	$[\alpha_{2n}^{min},\beta_{2n}^{max}]$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
DMU_n	$[\alpha_{n1}^{min},\beta_{n1}^{max}]$	$[\alpha_{n2}^{min},\beta_{n2}^{max}]$	$\cdots$	$[\alpha_{nn}^{min},\beta_{nn}^{max}]$

This paper extended the idea of the weight selection models of Wu et al. (2015) to present two weight selection models under the different perspectives for units with interval data based on their desirable and undesirable cross-efficiency scores β_{oj}^{max} and α_{oj}^{min}. Before presenting the proposed weight selection models, we explain how to construct their constraints.

Regarding β_{oj}^{max} is the maximum cross-efficiency score of DMU_j relative to DMU_o for any realization of data, the following inequality is held:

$$\left(\sum_{r=1}^{s} u_{ro}y_{rj} \Big/ \sum_{i=1}^{m} v_{io}x_{ij}\right) \leq \beta_{oj}^{max}, \; x_{ij}^{L} \leq x_{ij} \leq x_{ij}^{U}, \; y_{rj}^{L} \leq y_{rj} \leq y_{rj}^{U}.$$

Hence,

$$\left(\sum_{r=1}^{s} u_{ro}y_{rj}^{U} \Big/ \sum_{i=1}^{m} v_{io}x_{ij}^{L}\right) \leq \beta_{oj}^{max} \quad (15)$$

Equation (16) is constructed by adding the slack variable to the constraint (15):

$$\left(\sum_{r=1}^{s} u_{ro}y_{rj}^{U} \Big/ \sum_{i=1}^{m} v_{io}x_{ij}^{L}\right) + r_j^1 = \beta_{oj}^{max}. \quad (16)$$

where, r_j^1 is the deviation of $\frac{\sum_{r=1}^{s} u_{ro}y_{rj}^{U}}{\sum_{i=1}^{m} v_{io}x_{ij}^{L}}$ from β_{oj}^{max}. Therefore, if r_j^1 is minimized then the deviation of all possible cross-efficiency scores of DMU_j from β_{oj}^{max} will be minimized. The equation (16) can be linearized as follows:

$$\sum_{r=1}^{s} u_{ro}y_{rj}^{U} - \beta_{oj}^{max}\sum_{i=1}^{m} v_{io}x_{ij}^{L} + s_j^1 = 0,$$

where $s_j^1 = \frac{r_j^1}{\sum_{i=1}^{m} v_{io}x_{ij}^{L}}$.

Similarly, since α_{oj}^{min} is the minimum cross-efficiency score of DMU_j relative to DMU_o for any realization of data, the following inequality is held:

$$\left(\sum_{r=1}^{s} u_{ro} y_{rj} / \sum_{i=1}^{m} v_{io} x_{ij}\right) \geq \alpha_{oj}^{min}, \; x_{ij}^{L} \leq x_{ij} \leq x_{ij}^{U}, \; y_{rj}^{L} \leq y_{rj} \leq y_{rj}^{U}.$$

Hence,

$$\left(\sum_{r=1}^{s} u_{ro} y_{rj}^{L} / \sum_{i=1}^{m} v_{io} x_{ij}^{U}\right) \geq \alpha_{oj}^{min} \quad (17).$$

The following equation (18) is built by adding the slack variable to the constraint (17):

$$\left(\sum_{r=1}^{s} u_{ro} y_{rj}^{L} / \sum_{i=1}^{m} v_{io} x_{ij}^{U}\right) - r_j^2 = \alpha_{oj}^{min}, \quad (18)$$

where r_j^2 is the deviation of $\frac{\sum_{r=1}^{s} u_{ro} y_{rj}^{L}}{\sum_{i=1}^{m} v_{io} x_{ij}^{U}}$ from α_{oj}^{min}. Therefore, if r_j^2 is maximized then the deviation of all possible cross-efficiency scores of DMU_j from α_{oj}^{min} will be maximized. Now, the equation (18) can be linearized as follows:

$$\sum_{r=1}^{s} u_{ro} y_{rj}^{L} - \alpha_{oj}^{min} \sum_{i=1}^{m} v_{io} x_{ij}^{U} - s_j^2 = 0.$$

where $s_j^2 = \frac{r_j^2}{\sum_{i=1}^{m} v_{io} x_{ij}^{U}}$.

Based on the above theory, the following weight selection model is presented:

$$\begin{aligned}
&\min \sum_{j \neq o} (s_j^1 - s_j^2) \\
&s.t. \\
&\sum_{r=1}^{s} u_{ro} y_{ro}^{U} = E_o^{op}, \\
&\sum_{i=1}^{m} v_{io} x_{io}^{L} = 1, \\
&\sum_{r=1}^{s} u_{ro} y_{rk}^{L} - \sum_{i=1}^{m} v_{io} x_{ik}^{U} \leq 0, \quad k \neq o, \qquad (19)
\end{aligned}$$

$$\sum_{r=1}^{s} u_{ro} y_{ro}^{U} - \sum_{i=1}^{m} v_{io} x_{io}^{L} \leq 0,$$
$$\sum_{r=1}^{s} u_{ro} y_{rj}^{U} - \beta_{oj}^{max} \sum_{i=1}^{m} v_{io} x_{ij}^{L} + s_j^1 = 0, \quad j \neq o,$$
$$\sum_{r=1}^{s} u_{ro} y_{rj}^{L} - \alpha_{oj}^{min} \sum_{i=1}^{m} v_{io} x_{ij}^{U} - s_j^2 = 0, \quad j \neq o,$$
$$u_{ro} \geq 0, \quad r = 1,...,s,$$
$$v_{io} \geq 0, \quad i = 1,...,m.$$

where E_o^{op} is the optimistic efficiency score of DMU_o determined by solving model (10), α_{oj}^{min} and β_{oj}^{max} are the undesirable and desirable cross-efficiency scores of DMU_j for any realization of data, obtained by models (14a) and (14b), respectively, and s_j^1 and s_j^2 are the deviations of DMU_j from its desirable and undesirable cross-efficiency scores, respectively. The first and second constraints guarantee that the efficiency score of DMU_o is kept at its optimistic level. The last two constraints emphasize that the cross-efficiency score of DMU_j corresponding to DMU_o, for any realization of data, must be between its desirable cross-efficiency score, β_{oj}^{max}, and undesirable cross-efficiency score, α_{oj}^{min}. Model (19) aims to select the input/output weights for DMU_o such that the deviations of other units from their desirable cross-efficiency score are as small as possible for any realization of data and the deviations of them from their undesirable cross-efficiency score are as large as possible for any realization of data. In other words, model (19) aims to make the cross-efficiency score of units as close as possible to their desirable cross-efficiency scores for any realization of data and as far as possible from their undesirable cross-efficiency scores for any realization of data.

Model (19) is benevolent, because, in this model, DMU_o selects the optimal weights that maximize the cross-efficiency score of the other $(n-1)$ units for any realization of data and keep its efficiency score at its optimistic level for any realization of data.

The benevolent cross-efficiency score for units with interval data is defined as follows:

Definition 1. Suppose that $(u_{1o}^*,...,u_{so}^*, v_{1o}^*,...,v_{mo}^*)$ is an optimal solution of model (19). The benevolent cross-efficiency score of DMU_j corresponding to DMU_o is defined as follows:

$$E_{oj}^{ben} = \sum_{r=1}^{s} u_{ro}^* y_{rj}^U / \sum_{i=1}^{m} v_{io}^* x_{ij}^L, \quad o,j = 1,...,n. \quad (20)$$

Definition 2. The benevolent cross-efficiency score of DMU_j is defined as:

$$E_j^{ben} = \frac{1}{n}\sum_{o=1}^{n} E_{oj}^{ben}.$$

Model (19) can be transformed into an aggressive model as follows:

$$\begin{aligned}
&\max \sum_{j \neq o} (s_j^1 - s_j^2) \\
&\text{s.t.} \\
&\sum_{r=1}^{s} u_{ro} y_{ro}^U = E_o^{op}, \\
&\sum_{i=1}^{m} v_{io} x_{io}^L = 1, \\
&\sum_{r=1}^{s} u_{ro} y_{rk}^L - \sum_{i=1}^{m} v_{io} x_{ik}^U \leq 0, && k \neq o, \qquad (21) \\
&\sum_{r=1}^{s} u_{ro} y_{ro}^U - \sum_{i=1}^{m} v_{io} x_{io}^L \leq 0, \\
&\sum_{r=1}^{s} u_{ro} y_{rj}^U - \beta_{oj}^{max} \sum_{i=1}^{m} v_{io} x_{ij}^L + s_j^1 = 0, && j \neq o, \\
&\sum_{r=1}^{s} u_{ro} y_{rj}^L - \alpha_{oj}^{min} \sum_{i=1}^{m} v_{io} x_{ij}^U - s_j^2 = 0, && j \neq o, \\
&u_{ro} \geq 0, && r = 1,\dots,s, \\
&v_{io} \geq 0, && i = 1,\dots,m.
\end{aligned}$$

Model (21) selects the optimal weights for inputs/outputs for DMU_o that minimize the cross-efficiency score of the other $(n-1)$ units for any realization of data and keep the efficiency score of DMU_o at its optimistic level for any realization of data. Hence it is an aggressive model.

Similarly, the aggressive cross-efficiency score for units with interval data is defined as follows:

Definition 3. Suppose that $(u_{1o}^*,\dots,u_{so}^*, v_{1o}^*,\dots,v_{mo}^*)$ is an optimal solution of model (21). The aggressive cross-efficiency score of DMU_j corresponding to DMU_o is defined as follows:

$$E_{oj}^{agg} = \sum_{r=1}^{s} u_{ro}^* y_{rj}^L \Big/ \sum_{i=1}^{m} v_{io}^* x_{ij}^U, \quad o,j = 1,\dots,n. \quad (22)$$

Definition 4. The aggressive cross-efficiency score of DMU_j is defined as:

$$E_j^{agg} = \frac{1}{n}\sum_{o=1}^{n} E_{oj}^{agg}.$$

In the following, the proposed approaches for ranking DMUs with box-uncertainty under both benevolent and aggressive perspectives is summarized in Algorithm 1:

Algorithm

Step 1: Solve model (10) and obtain E_o^{op} for all $o \in \{1,...,n\}$.

Step 2: Solve models (13a), (13b), (14a) and (14b) and obtain α_{oj}^{min} and β_{oj}^{max} for all $o,j \in \{1,...,n\}$.

Step 3: Solve model (19) and obtain E_{oj}^{ben} and E_j^{ben} according to Definition 1 and Definition 2, for all $o,j \in \{1,...,n\}$ and determine the rank of units under the benevolent perspective by computing the values E_j^{ben}, $j = 1,...,n$.

Step 4: Solve model (21) and obtain E_{oj}^{agg} and E_j^{agg} according to Definition 3 and Definition 4, for all $o,j \in \{1,...,n\}$ and determine the rank of units under the aggressive perspective by computing the values E_j^{agg}, $j = 1,...,n$.

The next section illustrates how Algorithm 1 provides a full ranking of DMUs with box-uncertainty from both benevolent and aggressive perspectives.

4. An Illustrative Example

This section considers two numerical examples to show the discrimination power of the proposed ranking methods in case of interval data.

Table 6. The data of seven manufacturing industries.

DMU	I_1^L	I_1^U	I_2^L	I_2^U	O_1^L	O_1^U	E_o^{op}
1	564403	621755	674111	743281	806549	866063	1.000
2	614371	669665	685943	742345	917507	985424	1.000
3	762203	798427	762207	805677	1117142	1195562	1.000
4	862016	937044	779894	846496	1206179	1261031	1.000
5	1016898	1082662	799714	877137	1381315	1462543	1.000
6	1164350	1267970	807172	889416	1497679	1652787	1.000
7	1731916	1816008	818590	895746	1702249	1812655	1.000

Consider the data of seven manufacturing industries with two inputs (Capital (I_1) and Labor (I_2)) and one outputs (the Gross output value (O_1)) reported by Wang et al. (2005). The input/output values are reported in Table 6.

The proposed algorithm is applied to rank seven manufacturing industries in this example. In steps 1 and 2 of the algorithm, models (10), (13a), (13b), (14a) and (14b) are solved and the interval cross-efficiency matrix is constructed and reported in Table 7. The j^{th} row of Table 7 shows the cross- efficiency interval DMU_j with respect to all units.

Table 7. The interval cross-efficiency matrix.

DMU	1	2	3	4	5	6	7
1	[0.845, 0.907]	[0.771, 0.868]	[0.716, 0.887]	[0.708, 0.952]	[0.607, 0.968]	[0.530, 0.980]	[0.490, 0.907]
2	[0.912, 1.000]	[0.873, 1.000]	[0.885, 0.955]	[0.876, 1.000]	[0.769, 1.000]	[0.677, 1.000]	[0.625, 1.000]
3	[0.892, 0.996]	[0.854, 0.922]	[0.804, 0.937]	[0.796, 1.000]	[0.690, 1.000]	[0.604, 1.000]	[0.558, 0.996]
4	[0.838, 1.000]	[0.802, 0.971]	[0.820, 0.929]	[0.880, 0.922]	[0.786, 0.936]	[0.696, 0.948]	[0.643, 1.000]
5	[0.831, 1.000]	[0.795, 1.000]	[0.812, 1.000]	[0.872, 1.000]	[0.863, 0.939]	[0.769, 0.951]	[0.711, 1.000]
6	[0.769, 1.000]	[0.736, 1.000]	[0.753, 1.000]	[0.808, 1.000]	[0.821, 1.000]	[0.823, 1.000]	[0.760, 1.000]
7	[0.610, 0.941]	[0.584, 1.000]	[0.597, 1.000]	[0.641, 1.000]	[0.652, 1.000]	[0.660, 1.000]	[0.857, 1.000]

The cross- efficiency intervals of DMU_1 is shown in Figure 1. The horizontal axis shows the units and the vertical axis shows the cross- efficiency interval of DMU_1 relative to all units. The narrowest cross-efficiency interval is for DMU_1 and the widest cross-efficiency interval is for DMU_6.

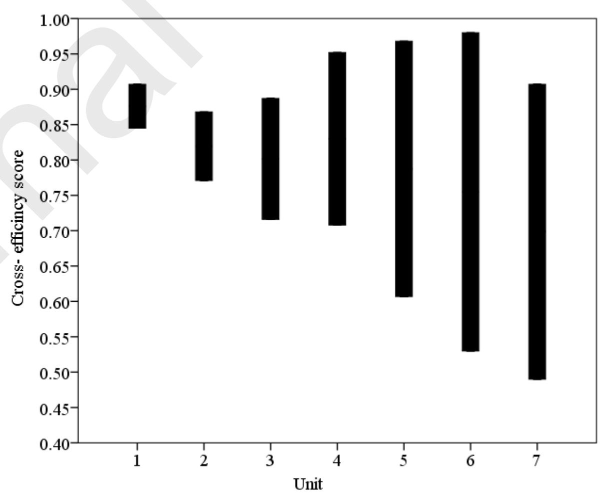

Figure 1. The cross- efficiency intervals for DMU_1.

Next, models (19) and (21) should be solved to construct the benevolent and aggressive cross-efficiency matrices reported in Table 8 and Table 9, respectively. The j^{th} row of Tables 8 and 9 show the benevolent cross-efficiency scores and the aggressive cross- efficiency scores of DMU_j with respect to all units, respectively.

Table 8. The benevolent cross- efficiency matrix.

DMU	1	2	3	4	5	6	7
1	1.000	0.957	0.978	1.000	1.000	1.000	0.580
2	1.000	1.000	1.000	1.000	1.000	1.000	0.648
3	1.000	0.978	1.000	1.000	1.000	1.000	0.708
4	0.953	0.912	0.932	1.000	1.000	1.000	0.729
5	0.937	0.897	0.917	0.983	1.000	1.000	0.825
6	0.925	0.885	0.905	0.971	0.987	1.000	0.924
7	0.682	0.653	0.667	0.716	0.728	0.737	1.000

Table 9. The aggressive cross- efficiency matrix.

DMU	1	2	3	4	5	6	7
1	0.845	0.809	0.827	0.887	0.902	0.914	0.490
2	0.892	0.854	0.873	0.937	0.953	0.965	0.558
3	0.912	0.873	0.892	0.957	0.973	0.986	0.625
4	0.838	0.802	0.820	0.880	0.895	0.906	0.643
5	0.831	0.795	0.812	0.872	0.886	0.898	0.711
6	0.769	0.736	0.753	0.808	0.821	0.832	0.760
7	0.610	0.584	0.597	0.641	0.652	0.660	0.857

Figure 2 shows the benevolent cross-efficiency scores of all DMUs summarized in Table 8. The horizontal axis shows the units and the vertical axis shows the benevolent cross-efficiency scores of a unit relative to all DMUs. As can be seen, the maximum benevolent cross-efficiency score of each unit is achieved by the unit itself.

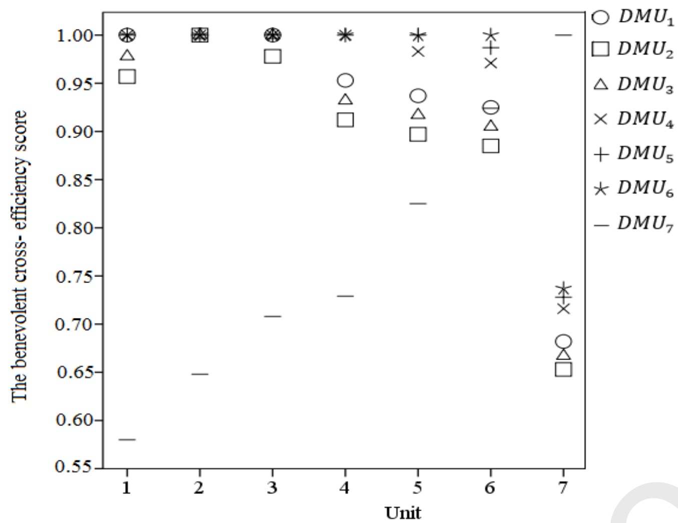

Figure 2. The benevolent cross-efficiency scores of units.

Figure 3 shows the aggressive cross-efficiency scores of all DMUs summarized in Table 9. The horizontal axis shows the units and the vertical axis shows the aggressive cross-efficiency scores of a unit relative to all DMUs.

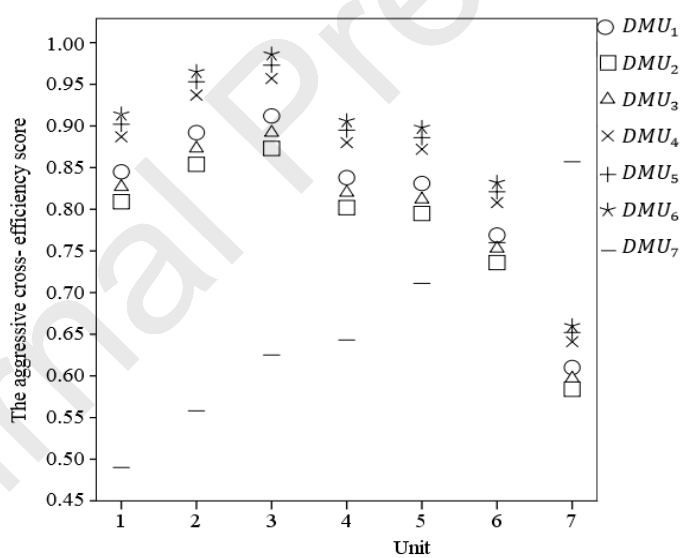

Figure 3. The aggressive cross-efficiency scores of units.

Finally, the benevolent and aggressive cross-efficiency scores of units and the rank of DMUs are shown in Table 10. The last column of Table 10 shows that, in this example, the rank of units determined by Wang et al. (2005)'s method is completely identical to ours in the case of benevolent perspective and slightly different from ours in regarding the ranks of DMU_1 and DMU_5 in the case of aggressive perspective. Consequently, at least for this example, the ranking results of Wang et al. (2005)'s method are the same as the proposed benevolent method.

Table 10. The benevolent and aggressive cross-efficiency scores of units.

DMU	The proposed method				The method of Wang et al. (2005)
	E_j^{ben}	Rank	E_j^{agg}	Rank	Rank
1	0.959	3	0.810	5	3
2	0.997	2	0.862	2	2
3	1.000	1	0.888	1	1
4	0.939	4	0.826	4	4
5	0.937	5	0.829	3	5
6	0.942	6	0.783	6	6
7	0.740	7	0.657	7	7

5. An Appliction in Hospital Effiency

Hospitals, just like other organizations, use some resources to produce some services, which is an output for healthcare organizations. The hospital efficiency assessment is important for two main reasons: first, the efficiency evaluation of state-owned organizations, such as hospitals, can determine the sustainability of national progress. Second, one can evaluate the effect of a healthcare system reform in the planning phase by examining annual changes in the efficiency scores of hospitals. DEA has been widely used for evaluating the healthcare centers and hospitals. For example, Chang (1998) combined DEA and regression analysis to assess the efficiency of hospitals in Taiwan over five years, 1990 to 1994. Field and Emrouznejad (2003) evaluated both technical and scale efficiency of 22 neonatal care centers in Scotland by applying DEA models. Kirigia et al. (2008) considered the technical and scale efficiency of healthcare centers to evaluate changes in productivity. Gholami et al. (2015) considered the hospital efficiency and the hospital quality with Information Technology (IT). For more studies about hospital efficiency, see Sherman (1984), Hollingsworth et al. (1999), Kirigia et al. (2002), Hollingsworth (2003, 2008), Mulumba et al. (2017), Stefko et al. (2018).

Most of the existing studies about hospital efficiency in the literature consider the situation that all inputs and outputs are crisp data. However, this assumption can be violated due to the existence of uncertainty in inputs, such as the clinic size (number of beds), number of doctors and nurses, or the existence of uncertainty in outputs, such as the total number of immunization, and patient days. For example, data may be reported as fuzzy, stochastic or interval form. Dotoli et al. (2015) proposed a novel cross-efficiency fuzzy DEA technique for evaluating the healthcare systems in a region of southern Italy. Sang et al. (2018)

proposed a model based on improved fuzzy DEA to evaluate the medical health resource. Hatefi and Haeri (2019) proposed an approach to assess the efficiency of hospitals based on fuzzy DEA.

Table 11. The input/ output data.

DMU	Lower and upper bound of inputs						Lower and upper bound of outputs				Efficiency
	I_1^L	I_1^U	I_2^L	I_2^U	I_3^L	I_3^U	O_1^L	O_1^U	O_2^L	O_2^U	E_o^{op}
1	83	83	5,288,837	5,558,654	3,075,694	3,279,944	40,725	43,986	5,081	5,298	0.7501
2	78	78	6,503,675	6,743,308	3,967,271	4,179,354	47,143	50,292	5,008	5,330	0.6485
3	54	54	5,387,805	5,642,797	2,105,003	2,240,473	53,448	56,557	7,082	7,470	1.0000
4	80	80	6,175,255	6,501,080	3,426,207	3,520,057	48,581	50,570	6,863	7,276	0.9013
5	75	75	6,498,667	6,836,454	2,879,534	2,917,747	44,873	47,620	6,415	6,655	0.7968
6	87	87	5,461,342	5,584,902	2,520,584	2,583,701	57,635	59,995	7,254	7,753	1.0000
7	58	58	3,740,977	3,926,099	1,992,561	2,039,397	56,071	58,613	5,243	5,335	1.0000
8	71	71	6,757,767	7,318,406	3,020,256	3,060,040	66,571	72,741	6,024	6,317	0.8967
9	76	76	7,876,658	8,095,079	3,940,705	4,066,142	59,748	64,474	5,003	5,390	0.7038
10	80	80	7,173,052	7,604,272	4,514,404	4,820,706	52,014	55,158	3,454	3,503	0.6125
11	78	78	6,658,627	6,868,300	3,653,094	3,861,595	38,453	41,709	5,382	5,878	0.6853
12	60	60	4,229,814	4,378,809	2,072,924	2,166,649	54,298	58,580	5,541	5,736	1.0000
13	78	78	6,952,265	7,451,889	3,082,876	3,264,333	62,482	65,791	5,004	5,314	0.7519
14	69	69	7,416,026	7,985,144	3,339,871	3,445,613	47,125	50,777	4,313	4,477	0.6126
15	80	80	6,552,117	6,974,244	3,393,025	3,622,326	52,619	54,520	4,745	4,995	0.6496
16	81	81	8,638,436	9,046,345	3,379,677	3,606,001	44,958	46,861	6,027	6,255	0.5888
17	77	77	7,187,290	7,274,137	3,468,031	3,563,553	44,417	45,513	6,130	6,364	0.6957
18	87	87	7,406,576	7,945,651	4,509,126	4,712,227	51,525	52,968	3,705	3,932	0.5571
19	49	49	5,514,974	5,730,731	2,635,308	2,793,308	41,724	42,697	4,642	4,984	0.8104
20	64	64	6,460,942	6,801,063	2,714,061	2,863,140	36,871	39,354	6,354	6,664	0.8184
21	90	90	4,959,608	5,157,271	2,616,716	2,645,252	41,202	44,275	5,426	5,646	0.8525
22	84	84	8,229,498	8,659,573	3,897,136	4,234,667	40,254	42,545	6,388	6,848	0.6579
23	81	81	6,827,159	7,034,127	4,241,786	4,315,961	48,807	50,978	4,737	4,840	0.6015
24	81	81	7,013,154	7,588,652	3,747,310	4,100,940	46,375	47,541	5,439	5,740	0.6366
25	89	89	9,100,162	9,341,263	4,415,597	4,722,616	66,447	68,527	5,139	5,406	0.6329
26	79	79	6,744,912	7,007,023	3,290,348	3,602,468	57,530	63,105	5,373	5,692	0.7437
27	86	86	8,317,319	8,395,895	3,618,050	3,846,734	59,864	61,929	6,109	6,303	0.6676
28	70	70	7,260,109	7,644,993	2,759,217	2,919,987	45,796	49,614	6,645	6,916	0.7930
29	81	81	7,543,785	8,074,915	4,160,199	4,361,458	45,976	47,160	5,411	5,631	0.5896
30	80	80	6,262,916	6,462,320	3,312,351	3,607,501	62,987	66,193	6,482	6,812	0.8430
31	78	78	6,496,427	6,870,738	4,305,519	4,586,064	40,178	42,641	4,628	4,950	0.5894
32	86	86	7,486,293	7,713,333	4,420,897	4,752,162	47,095	51,605	6,229	6,427	0.6683
33	55	55	6,699,201	7,120,624	2,230,776	2,352,635	77,737	79,299	5,678	5,974	1.0000
34	80	80	5,815,931	6,140,362	3,441,760	3,583,859	35,275	36,572	4,378	4,677	0.6094
35	82	82	7,413,919	7,540,968	3,908,686	4,221,706	50,687	54,408	4,769	5,001	0.6075
36	79	79	7,717,127	7,924,568	3,272,733	3,353,132	46,877	49,684	4,588	4,676	0.5618
37	85	85	8,737,680	9,054,296	4,651,967	4,832,316	49,993	50,984	5,131	5,317	0.5465
38	85	85	7,019,200	7,328,694	3,927,289	4,280,962	54,124	57,401	5,005	5,057	0.6316

Jacobs (2001) compared the rank of hospitals from the cost indices with the rank of them obtained by DEA and stochastic frontier analysis (SFA). Mateus et al. (2015) proposed a method to obtain the efficiency levels of hospitals with cross-sectional and panel data using SFA and DEA. Li et al. (2019) proposed a method to assess the nonhomogeneous hospitals based on stochastic DEA. Mahdiyan et al.

(2018) proposed an approach based on stochastic DEA to assess the hospitals of Iran. Also, Rabbani et al. (2016) proposed a bootstrap interval DEA to measure the efficiency of hospitals in Iran.

This example considers 38 hospitals selected by Inspector General (OIG) – East Virginia Department of Health and Human Services (Hatami-Marbini et al. (2012)). The lower and upper bounds of input and output values for each hospital are presented in Tables 11. As shown in this table, number of beds (I_1), labor-related expenses (I_2) and patient care supplies and other expenses (I_3)) are selected as the inputs to the DEA model. The number of outpatient department visit (O_1) and number of inpatient department admissions (O_2) selected as output variables in the DAE model. The definition of fields is given below:

- Labor-related Expenses: medical doctors' compensation, nonmedical doctors' salaries, fringe benefits, and non-payroll labor.
- Patient care supplies and other expenses include: food and food service supplies, medical supplies, drugs, and other supplies and expenses.
- Inpatient care is for patients whose condition requires admission to a hospital and
- Outpatient care is for patients who may need clinical services although do not necessarily need to be admitted to the hospital.

The last column of Table 11 shows the optimistic efficiency score of units obtained by model (10). Next, models (13a), (13b), (14a) and (14b) are solved to determine α_{oj}^{min} and β_{oj}^{max} as the minimum and maximum cross-efficiency scores of DMU_j relative to DMU_o and then, the cross-efficiency matrix is constructed as explained in Table 5.

Finally, the weight selection models (19) and (21) should be solved to determine the benevolent and aggressive cross-efficiency scores of units. The results are summarized in Table 12.

According to the proposed benevolent method, the only efficient hospital is hospital 3 with the rank of 1. The proposed aggressive method does not assign an efficiency score of 1 to any hospital due to its pessimistic nature. Hospital 3 has also the rank of 1 in the aggressive method. The second place in the proposed ranking methods belongs to hospital 7. Hospital 12 has the third place in the aggressive and benevolent methods. Also, hospitals 33 and 6 have the fourth and fifth places, respectively, in both aggressive and benevolent methods. . The rank of other units in the proposed benevolent approach can be equal to, smaller/greater than the rank of the proposed aggressive approach. For example, hospital 1 has the rank of 16 in the benevolent method and the rank of 18 in the aggressive method We have the similar situation for the 8th place. Hospital 5 has 12th place in the benevolent method and 10th place in the aggressive method.

The proposed methods determine a full ranking of all units in this example which shows the discrimination power of the proposed methods.

Table 12. The results of the proposed method

DMU	The proposed method			
	Benevolent		Aggressive	
	E_j^{ben}	Rank	E_j^{agg}	Rank
1	0.6309	16	0.5790	18
2	0.5974	21	0.5476	22
3	**1.0000**	**1**	**0.9593**	**1**
4	0.7571	8	0.6971	8
5	0.7108	12	0.6613	10
6	0.8935	5	0.8313	5
7	**0.9878**	**2**	**0.9588**	**2**
8	0.8037	6	0.7213	7
9	0.6106	19	0.5565	20
10	0.4608	37	0.4286	37
11	0.5931	23	0.5323	24
12	0.9781	3	0.9092	3
13	0.6627	15	0.6009	15
14	0.5388	32	0.4893	33
15	0.5952	22	0.5477	21
16	0.5569	27	0.5185	26
17	0.6246	17	0.5985	16
18	0.4551	38	0.4186	38
19	0.7264	10	0.6708	9
20	0.7163	11	0.6564	12
21	0.6724	14	0.6269	13
22	0.5724	25	0.5167	27
23	0.5415	31	0.5162	28
24	0.5843	24	0.5317	25
25	0.5434	30	0.5109	29
26	0.6784	13	0.6129	14
27	0.6150	18	0.5884	17
28	0.7179	9	0.6574	11
29	0.5478	28	0.5078	30
30	0.7944	7	0.7372	6
31	0.5322	33	0.4817	34
32	0.6011	20	0.5576	19
33	**0.9265**	**4**	**0.8600**	**4**
34	0.5212	34	0.4773	35
35	0.5453	29	0.5065	31
36	0.5112	35	0.4853	32
37	0.4894	36	0.4646	36
38	0.5692	26	0.5352	23

The correlation between the proposed methods (benevolent and aggressive) with Salahi et al (2016) can be investigated. For this purpose, we determine the Spearman's rank order correlation coefficient between the proposed benevolent approach and the aggressive approach derived from the corresponding numerical efficiency scores. The correlation between the proposed benevolent method and the proposed aggressive method is 0.9923, the correlation between the proposed benevolent method and the method of Salahi et al. (2016) is 0.8929 and the correlation between the proposed aggressive method and the method of Salahi et al. (2016) is 0.9515. It can be seen that all three methods have a high correlation at least in this example. Regarding Table 12, the main advantage of the proposed method in two cases of benevolent and aggressive, is their high discrimination power for distinguishing between the decision-making units. The correlation between the proposed ranking methods shows that there is no significant difference between the obtained ranks by the proposed benevolent and aggressive methods and we can apply any of these approaches as the powerful ranking methods, to suit our point of view.

6. Conclusion

This paper considered box-uncertainty in DEA models where each input/output variable varies in an interval. A robust optimization framework was proposed for performance measurement and ranking of DMUs with interval data. A correct formulation was presented for the counterpart of the envelopment form of the CCR model with interval data based on the Ben-Tal and Nemirovski's approach (1999). We proved the relationship between the dual of the robust counterpart of the envelopment of CCR model and the optimistic counterpart of the multiplier form of CCR model based on the approach of Beck and Ben-Tal (2009). Also, two new approaches were proposed for ranking DMUs with interval data in DEA applying the robust optimization techniques. The suggested methods extend the secondary goal models proposed by Wu et al. (2015) to the case of interval data with using the robust counterpart and the optimistic counterpart of LPs.

Several RO-based approaches have been developed in DEA models but they mostly did not provide a full ranking of DMUs. The novelty of the proposed approaches is that we propose a correct counterpart for the envelopment form of the CCR model. Salahi et al. (2016) also attempted to propose the counterpart for the CCR model in its envelopment form.. Their formulation is not correct as we proved in this paper. Also, the proposed approaches fully ranks 38 hospitals in East Virginia.

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