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FUZZY DATA ENVELOPMENT ANALYSIS WITH ORDINAL AND INTERVAL DATA

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In this paper, we reformulate the conventional DEA models as an imprecise DEA problem and propose a novel method for evaluating the DMUs when the inputs and outputs are fuzzy and/or ordinal or vary in intervals. For this purpose, we convert all data into interval data. In order to convert each fuzzy number into interval data, we use the nearest weighted interval approximation of fuzzy numbers by applying the weighting function, and we convert each ordinal data into interval one. In this manner, we could convert all data into interval data. The presented models determine the interval efficiencies for DMUs. To rank DMUs based on their associated interval efficiencies, we first apply the Ω-index that is developed for ranking of interval numbers. Then, by introducing an ideal DMU, we rank efficient DMUs to present a complete ranking. Finally, we use one example to illustrate the process and one real application in health care to show the usefulness of the proposed approach. For this evaluation, we consider interval, ordinal, and fuzzy data alongside the precise data to evaluate 38 hospitals selected by OIG. The results reveal the capabilities of the presented method to deal with the imprecise data.

Keywords: Data Envelopment Analysis, efficiency, ranking, fuzzy data, ordinal data, interval data, nearest weighted interval approximation.

1. Introduction

Data Envelopment Analysis (DEA) is a linear programming approach for measuring the relative efficiency of peer decision-making units (DMUs) with multiple inputs and outputs. Since Charnes et al. (1978) first introduced DEA, it has been widely applied to evaluating the relative efficiencies in many applications such as schools, hospitals, banks, etc. (A Emrouznejad and Yang (2018)). The relative efficiency of each decision-making unit is defined as the ratio of the members' weighted sum of outputs to the weighted sum of inputs. The original DEA models assumed that inputs and outputs are measured by exact values on a ratio scale (Charnes et al. (1994)). However, there are many applications that this assumption may not be valid i.e., some or all of the inputs and outputs may be imprecise. The issue of imprecise in DEA appears when there is a notion of uncertainty in data, especially when a set of DMUs contain missing data, ordinal data, interval data, or fuzzy data. Therefore, evaluating the efficiency of a set of DMUs in this situation is worth investigating.

The imprecise data representation with interval and ratio interval data was initially proposed by (Cooper et al. (1999); Cooper et al. (2001)) to study the uncertainty in some applications. Additionally, Cook et al. (1993)

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developed a cone ratio model for incorporating an ordinal factor into the DEA structure. However, in many situations, such as in a manufacturing system, a production process or, a service system, inputs and outputs are volatile and complicated so, it is difficult to measure them accurately. Instead, the data can be given as a fuzzy variable. Many fuzzy approaches have been introduced in the DEA literature. Sengupta (1992) applied the principle of fuzzy set theory to introduce fuzziness in the objective function and the right-hand side vector of the conventional DEA model and developed the tolerance approach that was one of the first fuzzy DEA models. Selecting a suitable ranking method can be applied in choosing the desired criterion in a fuzzy environment. Saeidifar (2011) and Izadikhah et al. (2014) introduced and applied weighted mean concept to rank fuzzy numbers. They proposed a new ranking method for fuzzy numbers, which used a defuzzification of fuzzy numbers and a weighting function. They first, defined a weighted distance measure on fuzzy numbers, and then, by minimizing this distance, they obtained the nearest weighted interval and point approximations of fuzzy numbers.

In this paper, first we convert each ordinal data into interval data. Using the nearest weighted interval approximation of a generalized fuzzy number, we convert each fuzzy number into an interval number. This done by using weighting functions to introduce the nearest weighted interval approximation of generalized fuzzy numbers. Therefore, we will have all data in an interval format. Further, we extend the DEA models to develop a methodology for calculating efficiencies and ranking DMUs in the presence of fuzzy, ordinal, and interval data.

The rest of the paper is organized as follows: Section two gives the literature review. In section three, we review the required background information, including some basic concepts of DEA, basic definitions and notions of fuzzy numbers and possibility space, the nearest weighted interval approximations of fuzzy numbers, efficiency model with interval data and, the Ω-index for ranking of interval numbers. Section four proposes a new method for ranking efficient DMUs by introducing ideal DMUs. An algorithm for interval efficiency and ranking DMUs in the presence of fuzzy, ordinal, and interval data is proposed in section five. Section six presents a comparative study with some other existing DEA methods. In section seven, an illustrative example is used to demonstrate the proposed approach; and an application in hospital efficiency shows the applicability and usefulness of the proposed method. Conclusions and directions for future research are given in section eight.

2. Literature Review

In this section, we review some essential methods that deal with data envelopment analysis models with fuzzy and/or interval and/or ordinal data.

2.1. Fuzzy Data Envelopment Analysis Models

Kahraman and Tolga (1998) improved the Sengupta (1992) method to incorporate uncertainty into the DEA models by defining constraint violations' tolerance levels. This approach fuzzified the inequality or equality signs but, it does not treat fuzzy coefficients directly. Meada et al. (1998) introduced the based approach. This method was further improved by Saati et al. (2002) to solve fuzzy DEA model by parametric programming using α-cut. In this line, many authors tried to develop some new DEA models in fuzzy environment. Kao and Liu (2000) presented a procedure for determining the efficiencies of DMUs with fuzzy observations. Guo and Tanaka (2001) proposed a fuzzy CCR based DEA model to deal with the efficiency evaluation problem with the fuzzy input and output data. Lertworasirikul et al. (2003) developed fuzzy DEA models using imprecise data and fuzzy data. Allahviranloo et al. (2007) presented an input-oriented envelopment fuzzy DEA model for measuring the efficiency of DMUs with a constant return to scale. Wen et al. (2010) reviewed some DEA models under fuzzy environment and presented a fuzzy DEA model based on credibility measures. Zerafat Angiz L. et al. (2010) developed a Fuzzy DEA model to evaluate DMUs under uncertainty under fuzzy environment. Ali Emrouznejad et al. (2011) developed an imprecise data envelopment analysis model to provide two methods for measuring the overall profit MPI when the inputs, outputs, and price vectors are fuzzy or vary in intervals. Kao and Lin (2011) considered the qualitative data as fuzzy numbers and presented a fuzzy DEA model to measure the performance of DMUs. Y.-C. Chen et al. (2013) provided a Fuzzy SBM model to evaluate the risk uncertainty of the Taiwan banking sector.

Agarwal (2014) employed the α-cut approach to develop a fuzzy DEA model to deal with the efficiency measuring and ranking problem with the given fuzzy input and output data. Barak and Dahooei (2018) presented a hybrid method for ranking the airlines' safety based on fuzzy data envelopment analysis and fuzzy multi-attribute decision-making method. Izadikhah and Khoshroo (2018) developed a non-radial DEA model based on a modification of enhanced Russell model in the presence of undesirable data and fuzzy data to evaluate the efficiency scores of 22 barley production farms. X. Zhou et al. (2019) developed a dynamic network DEA model with interval type-2 fuzzy indicators to evaluate 20 sustainable supply chains based on optimistic-pessimistic viewpoints. Heydari et al. (2020) provided a fully fuzzy network DEA-RAM model for evaluating airlines. Peykani et al. (2021) developed a fuzzy network data envelopment analysis for measuring the performance of DMUs in the presence of imprecise and vague data.

2.2. Data Envelopment Analysis Models with Interval or Ordinal Data

After the first attempts of Cooper et al. (1999), Cooper et al. (2001), and Cook et al. (1993) to extend DEA models in the presence of imprecise data, many other authors tried to continue their ideas and develop new DEA models with imprecise data. Despotis and Smirlis (2002) presented radial DEA models for dealing with imprecise data. Y.-M. Wang et al. (2005) proposed a pair of interval DEA models based on interval arithmetic to measure efficiency performance in the interval and/or fuzzy environments. Y.-M. Wang and Yang (2007) developed an interval efficiency such that the upper bound was set to one, and the lower bound was determined through a virtual anti-ideal DMU. Shokouhi et al. (2010) employed a *robust optimization* model to develop an adaptation of the standard DEA under conditions of uncertainty. The input and output parameters were constrained to be within an *uncertainty set*. Toloo and Nalchigar (2011) presented an integrated data envelopment analysis model to identify the most efficient supplier in the presence of both cardinal and ordinal data. Aliakbarpoor and Izadikhah (2012) presented an evaluating model to measure the efficiency performance of DMUs with undesirable and ordinal data. Fathi and Izadikhah (2013) developed a DEA model for evaluating the efficiency of DMUs in the presence of ordinal and fuzzy data.

Hatami-Marbini et al. (2014) presented an evaluation process for measuring the relative efficiencies of a set of DMUs in DEA with interval data and negative data. Toloo et al. (2018) proposed pessimistic and optimistic DEA models to measure the interval efficiencies where some observed inputs, outputs, and dual-role factors have interval structures. Aparicio et al. (2019) developed a DEA efficiency model to measure the impact of imprecision and variability in data on US students and schools participating in PISA. Goker and Karsak (2020) developed an integrated MCDM and DEA method to identify the best performing DMU with the presence of imprecise data. Ebrahimi et al. (2021) presented a pair of DEA models to determine the interval efficiency of DMUs in the presence of interval and weak ordinal data.

3. Preliminaries

This section reviews and justifies the required methods used throughout the paper. To this end, the concepts and models of data envelopment analysis, interval data, ordinal data and converting into interval data, basic concepts of fuzzy numbers, efficiency model with interval data, and the Ω-index for ranking of interval numbers are given.

3.1. Data Envelopment Analysis

Assume that there are n DMUs, where each $DMU_j(j = 1, \ldots, n)$, uses m different inputs, x_{ij} $(i = 1, \ldots, m)$ to produce s different outputs, y_{rj} $(r = 1, \ldots, s)$. Also, we assume that the data set is positive and deterministic. Table 1 reports the used nomenclatures.

Table 1. Nomenclatures

Symbol	Description	Symbol	Description
DMU_o	DMU under evaluation;	x_{ij}^L	lower bound of the i^{th} input of the DMU_j
DMU_j	j^{th} DMU;	x_{ij}^U	upper bound of the i^{th} input of the DMU_j
m	Number of inputs;	y_{rj}^L	lower bound of the r^{th} output of the DMU_j
s	Number of outputs;	y_{rj}^U	upper bound of the r^{th} output of the DMU_j
x_{ij}	i^{th} input of DMU_j	ε_i	small positive number
y_{rj}	r^{th} output of DMU_j	σ_r	small positive number
λ_j	Intensity;	$\hat{x}_{ij}$	Ordinal inputs after transformation
v_i	Weight of i^{th}input;	$\hat{y}_{rj}$	Ordinal outputs after transformation
u_r	Weight of r^{th} output;	θ_o	Input contraction;

3.1.1. Input-oriented CCR model

Model (1) given below is the basic model proposed by Charnes et al. (1978) to evaluate DMUs, an input-oriented CCR model:

$$
\begin{aligned}
&\min \theta_o \\
&s.t. \\
&\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o x_{io}, \quad i = 1, \ldots, m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \ldots, s \\
&\lambda_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
\tag{1}
$$

Dual form of model (1), which known as multiplier form of CCR, is expressed as follows:

$$
\begin{aligned}
&Max \ \sum_{r=1}^{s} u_r y_{ro} \\
&s.t.
\end{aligned}
\tag{2}
$$

$$\sum_{i=1}^{m} v_i x_{io} = 1;$$
$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0; \quad j = 1, \ldots, n.$$
$$u_r, v_i \geq 0; \qquad i = 1, \ldots, m, \ r = 1, \ldots, s.$$

Definition 1. DMU_o is CCR-efficient if θ_o^*= 1 and all slack variables are zero in the alternative optimal solution.

3.2. Interval data

Assume inputs, x_{ij} $(i = 1, \ldots, m)$, and outputs, y_{rj} $(r = 1, \ldots, s)$, are imprecise but located in an interval, where x_{ij}^L and x_{ij}^U are the lower and upper bounds of the i^{th} input of the DMU_j, respectively, and y_{rj}^L , y_{rj}^U are the lower and upper bounds of the r^{th} output of the DMU_j, respectively, that is to say, $x_{ij}^L \leq x_{ij} \leq x_{ij}^U$ and $y_{rj}^L \leq y_{rj} \leq y_{rj}^U$. Note that if $x_{ij}^L = x_{ij}^U$, then the i^{th} input of the DMU_jhas a definite value.

3.3. Ordinal data and converting into interval data

In this section, we consider the transformation of ordinal preference information about the output and input y_{rj} and x_{ij} $(j = 1, \ldots, n)$. For weak ordinal preference information $y_{r1} \geq \ldots \geq y_{rn}$ and $x_{i1} \geq \ldots \geq x_{in}$, we have the following ordinal relationships after scale transformation:

$$1 \geq \hat{y}_{r1} \geq \ldots \geq \hat{y}_{rn} \geq \sigma_r \quad \text{and} \quad 1 \geq \hat{x}_{i1} \geq \ldots \geq \hat{x}_{in} \geq \varepsilon_i \tag{3}$$

Where ε_i is a small positive number reflecting the ratio of the possible minimum of $\langle x_{ij} | j = 1, \ldots, n \rangle$ to its possible maximum and σ_r is a small positive number reflecting the ratio of the possible minimum of $\langle y_{rj} | j = 1, \ldots, n \rangle$ to its possible maximum. The decision-maker can approximately estimate it. It is referred as the ratio parameter for convenience. The resultant permissible interval for each $\hat{x}_{ij}, \hat{y}_{rj}$ is given by:

$\hat{y}_{rj} \in [\sigma_r, 1], \ j = 1, \ldots, n$ and $\hat{x}_{ij} \in [\varepsilon_i, 1], \quad j = 1, \ldots, n$

For strong ordinal preference information:

$$1 \geq \hat{y}_{r1}, \ldots, \hat{y}_{rj} \geq \chi_r \hat{y}_{r,j+1} \ (j = 1, \ldots, n-1) \ \text{and} \ \hat{y}_{rn} \geq \sigma_r \tag{4}$$

$$1 \geq \hat{x}_{i1}, \ldots, \hat{x}_{ij} \geq \eta_i \hat{x}_{i,j+1} \ (j = 1, \ldots, n-1) \ \text{and} \ \hat{x}_{ij} \geq \varepsilon_i \tag{5}$$

Where χ_r and η_i are preference intensity parameters satisfying $\chi_r, \eta_i > 1$, provided by the decision-maker and ε_i,σ_r are the ratio parameters also provided by the decision-maker. The resultant permissible interval for each $\hat{y}_{rj}$ and $\hat{x}_{ij}$ can be derived as follows:

$$\hat{y}_{rj} \in [\sigma_r \chi_r^{n-j}, \chi_r^{1-j}] \quad , j = 1, \ldots, n; \text{ with } \quad \sigma_r \leq \chi_r^{1-n} \tag{6}$$

$$\hat{x}_{ij} \in [\varepsilon_i \eta_i^{n-j}, \eta_i^{1-j}] \quad , j = 1, \ldots, n; \text{ with } \quad \varepsilon_i \leq \eta_i^{1-n} \tag{7}$$

In this way, all the ordinal preference information is converted into interval data.

Remark 1. We will mention each input and output that is definitive transform into interval data as follow:

$$x_{ij} \in [x_{ij}^L, x_{ij}^U], \text{where } x_{ij}^L = x_{ij}^U = x_{ij}$$
$$y_{rj} \in [y_{rj}^L, y_{rj}^U], \text{where } y_{rj}^L = y_{rj}^U = y_{rj} \tag{8}$$

3.4. Basic concepts of Fuzzy numbers

In this subsection, we review some definitions and notions about fuzzy numbers and possibility space and the nearest weighted interval approximations.

3.4.1 Basic definitions and notions about fuzzy numbers and possibility space

Let R be the set of all real numbers. We assume that a fuzzy number A for all $x \in R$ can be expressed as follows:

$$A(x) = \begin{cases} A_L(x), & x \in [a, b] \\ 1, & x \in [b, c] \\ A_R(x), & x \in [c, d] \\ 0, & \text{otherwise.} \end{cases} \quad \text{(I)}$$

Where a, b, c and, d are real numbers such that $a < b \leq c < d$, A_L is a real-valued function that is increasing and right continuous and A_R is a real-valued function that is decreasing and left continuous.

Definition 2. We denote the family of fuzzy numbers by ξ .

Definition 3. A fuzzy number $A = (a, b, c, d)$ is called a trapezoidal fuzzy number if its membership function $A(x)$ has the following form:

$$A(x) = \begin{cases} \dfrac{x-a}{b-a}, & x \in [a, b]; \\ 1, & x \in [b, c]; \\ \dfrac{d-x}{d-c}, & x \in [c, d]; \\ 0, & otherwise. \end{cases} \quad \text{(II)}$$

If b=c, then $A = (a, b, c)$ is a triangular fuzzy number (see Fig. 1).

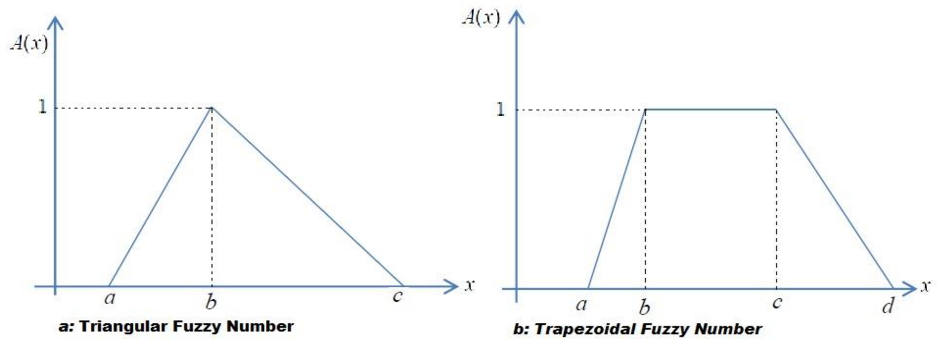


Fig. 1: The membership functions of Fuzzy numbers

3.4.2. The nearest weighted interval approximations

In this subsection, we recall the concept of the nearest weighted interval approximation to a fuzzy number.

Definition 4. (Saeidifar (2011)) A weighting function is a function as f = ($\underline{f}$,$\overline{f}$) : ([0,1], [0,1]) → (R, R) such that the function $\underline{f}$, $\overline{f}$ are nonnegative, monotone increasing and satisfies the following normalization condition:

$$\int_0^1 \underline{f}(\alpha)d\alpha = \int_0^1 \overline{f}(\alpha)d\alpha = 1$$

Remark 2. The function $f(\alpha)$ can be understood as the weight of our interval approximation. The property of monotone increasing of function $f(\alpha)$ means that the higher the cut level is, the more important its weight is in determining the interval approximation of fuzzy numbers. In applications, the function $f(\alpha)$ can be chosen according to the actual situation.

Theorem 1. (Saeidifar (2011)) Let $A \in \xi$ be a fuzzy number with $A_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and $f(\alpha) = (\underline{f}(\alpha), \overline{f}(\alpha))$ be a weighted function. Then, the interval

$$NWIA_f(A) = [NWIA_{\underline{f}}(A), NWIA_{\overline{f}}(A)]$$

is the nearest weighted interval approximation to fuzzy number A.

Obviously, weighted interval approximation synthetically reflects the information on every membership degree. Its advantage is that different α-cut set plays various roles.

Corollary 1. Let *A*= *(a, b, c, d)* be a trapezoidal fuzzy number, and $f(\alpha) = (\underline{f}(\alpha), \overline{f}(\alpha))$ be a weighting function. Then

For $f(\alpha) = (n\alpha^{n-1}, n\alpha^{n-1}), n \in N$ (natural number):

$$NWIA_f(A) = [\tfrac{a+nb}{n+1}, \tfrac{nc+d}{n+1}]. \quad (9)$$

Example 2. Consider a trapezoidal fuzzy number$A = (3,7,8,13)$. Assume $f_1(\alpha) = (2\alpha, 2\alpha)$ and $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$ are two weighting functions. Then the nearest weighted interval to $A$is as follows (see Fig. 2):

$$NWIA_{f_1} = [\frac{17}{3}, \frac{29}{3}],$$
$$NWIA_{f_2} = [\frac{31}{5}, 9].$$

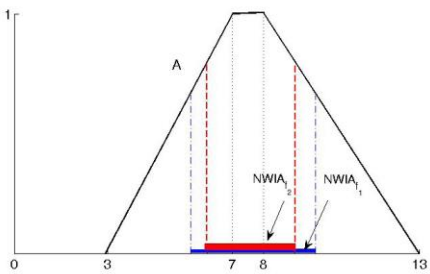


Fig. 2: Fuzzy number (Example 2)

Example 3. Let Abe a fuzzy number with the following membership function

$$\tilde{A} = \begin{cases} 1 - \frac{(x-5)^2}{4}, & 3 \leq x \leq 7; \\ 0, & \text{Otherwise.} \end{cases}$$

Assume $f_1(\alpha) = (2\alpha, 2\alpha)$ and $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$ are two weighting functions. Then the nearest weighted interval to A is as follows (see Fig. 3):

$$NWIA_{f_1} = [\frac{59}{15}, \frac{91}{15}],$$
$$NWIA_{f_2} = [\frac{1319}{315}, \frac{1831}{315}].$$

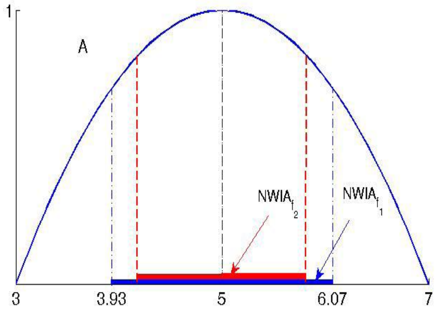


Fig. 3: Fuzzy number (Ex. 3)

3.5. Efficiency model with interval data

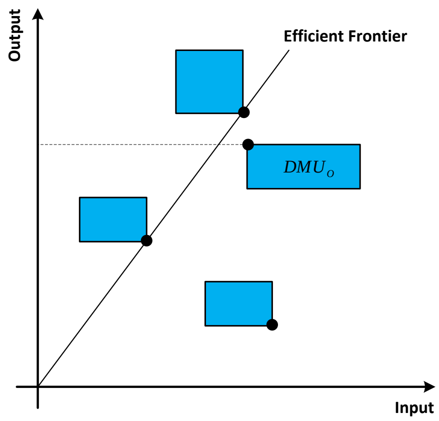
In this section, unlike the original DEA model, we assume further that the levels of inputs and outputs are not known exactly, the true input and output data are known to lie within bounded intervals, i.e., $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$ with upper and lower bounds of the intervals given as constants and assumed strictly positive i.e. $x_{ij}^L > 0$ and $y_{rj}^L > 0$. In this case, the efficiency can be an interval. To deal with such an uncertain situation, the following pair of LP models have been developed to generate the upper and lower bounds of interval efficiency for each DMU (for details, see Despotis and Smirlis (2002)).

$$\begin{aligned} &\theta_o^U = max \quad \sum_{r=1}^{s} u_r y_{ro}^U \\ &s.t. \\ &\sum_{i=1}^{m} v_i x_{io}^L = 1, \\ &\sum_{r=1}^{s} u_r y_{ro}^U - \sum_{i=1}^{m} v_i x_{io}^L \leq 0, \\ &\sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U \leq 0, j = 1, \ldots, n; j \neq o, \\ &u_r, v_i \geq \varepsilon, \forall r, i \end{aligned} \tag{10}$$

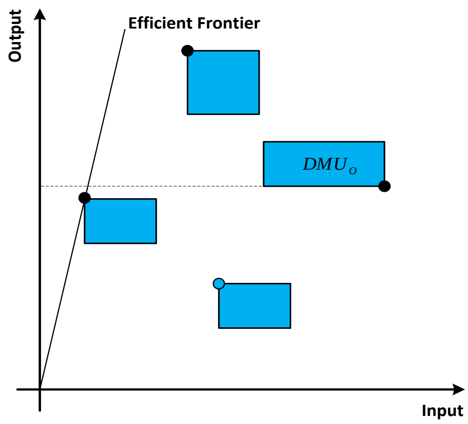
And

$$
\begin{aligned}
&\theta_o^L = max \quad \sum_{r=1}^{s} u_r y_{ro}^L\,, \\
&s.t. \\
&\sum_{i=1}^{m} v_i x_{io}^U = 1, \\
&\sum_{r=1}^{s} u_r y_{ro}^L - \sum_{i=1}^{m} v_i x_{io}^U \le 0, \\
&\sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, j = 1, \dots, n; j \ne o, \\
&u_r, v_i \ge \varepsilon, \forall r, i
\end{aligned} \tag{11}
$$

Therefore, we obtain the interval efficiency $[\theta^L, \theta^U]$ for each DMU. For more details, see illustrative Fig. 4.



(a): Calculating $\boldsymbol{\theta_o^U}$



(b): Calculating $\boldsymbol{\theta_o^L}$

Fig. 4: Comparison between $\boldsymbol{\theta_o^U}$ and $\boldsymbol{\theta_o^L}$

In Fig. 4(a) and Fig. 4(b), there are four DMUs with one input and one output with interval data. From Fig. 4, we can see that $\theta_o^L, \theta_o^U < 1$ and $\theta_o^L < \theta_o^U$.

3.6. The $\boldsymbol{\Omega}$-index for ranking of interval numbers

In assessing the interval efficiency, since the final efficiency score for each DMU is defined as an interval, a practical, yet simple, ranking approach should be devised for comparison and ranking of the interval efficiencies of different DMUs. A few approaches have been developed for ranking interval numbers, but all of them have some shortcomings. Specifically, when interval numbers have identical centers, but different widths, their discrimination is difficult.

One way for denoting the interval $\boldsymbol{A} = [\boldsymbol{a^L}, \boldsymbol{a^R}]$ is as $\boldsymbol{A} = \langle \boldsymbol{m}(\boldsymbol{A}), \boldsymbol{w}(\boldsymbol{A}) \rangle$, where $\boldsymbol{m}(\boldsymbol{A})$ and $\boldsymbol{w}(\boldsymbol{A})$ are the midpoint and the half of the width of the interval A, that is:

$$
\begin{aligned}
\boldsymbol{m}(\boldsymbol{A}) &= \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{a^L} + \boldsymbol{a^R}), \text{ and} \\
\boldsymbol{w}(\boldsymbol{A}) &= \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{a^R} - \boldsymbol{a^L})
\end{aligned}
$$

Then, we can see that $\boldsymbol{a^L} = \boldsymbol{m}(\boldsymbol{A}) - \boldsymbol{w}(\boldsymbol{A})$ and $\boldsymbol{a^R} = \boldsymbol{m}(\boldsymbol{A}) + \boldsymbol{w}(\boldsymbol{A})$ and hence, the interval $\boldsymbol{A} = [\boldsymbol{a^L}, \boldsymbol{a^R}]$, can be stated as $\boldsymbol{A} = [\boldsymbol{m}(\boldsymbol{A}) - \boldsymbol{w}(\boldsymbol{A}), \boldsymbol{m}(\boldsymbol{A}) + \boldsymbol{w}(\boldsymbol{A})]$. As a result, if $\boldsymbol{m}(\boldsymbol{A}) + \boldsymbol{w}(\boldsymbol{A}) \le \boldsymbol{m}(\boldsymbol{B}) - \boldsymbol{w}(\boldsymbol{B})$, we can say that the interval B is completely superior to interval A. The above inequality can be written as follows:

$$\frac{\boldsymbol{m(B)-m(A)}}{\boldsymbol{w(B)+w(A)}} \geq \boldsymbol{1}$$

The above fraction can be selected as a good tool for comparing intervals. The higher the value of the fraction, the greater the difference between two interval numbers.

Definition 6. Let $\angle$ be an extended order relation between intervals $\boldsymbol{A=[a^L,a^R]}$ and $\boldsymbol{B=[b^L,b^R]}$ on the real line R. Then for $\boldsymbol{m(A)\leq m(B)}$; we make an assumption $(\boldsymbol{A\angle B})$ which implies that A is inferior to B (or B is superior to A) in terms of value. Here, the term ''inferior to'' (or ''superior to'') is similar to ''less than'' (or ''greater than'').

Definition 7. Let I be the set of all closed intervals on the real line R. We define an acceptability function $\boldsymbol{\Omega}:\boldsymbol{I}\times\boldsymbol{I}\rightarrow[\boldsymbol{0},\infty)$, where $\boldsymbol{\Omega(A\angle B)}$ or $\boldsymbol{\Omega_{\angle}(A,B)}$ or, in short, $\boldsymbol{\Omega_{\angle}}$, is obtained from:

$$\boldsymbol{\Omega}_{\angle}=\frac{\boldsymbol{m(B)-m(A)}}{\boldsymbol{w(B)+w(A)}}$$

$\boldsymbol{\Omega}_{\angle}$ can be interpreted as ''the degree of acceptability of the assertion that the first interval is inferior to the second interval''. The degree of acceptability of $(\boldsymbol{A\angle B})$ can be further classified according to the relative position of the mean and width of the interval B related to the mean and width of the interval A, as follows:

$$\boldsymbol{\Omega(A\angle B)}\begin{cases}=\boldsymbol{0}, & \text{if } \boldsymbol{m(A)=m(B)},\\ >\boldsymbol{0},<\boldsymbol{1}, & \text{if } \boldsymbol{a^R>b^L} \text{ and } \boldsymbol{m(A)<m(B)},\\ \geq\boldsymbol{1}, & \text{if } \boldsymbol{a^R\leq b^L} \text{ and } \boldsymbol{m(A)<m(B)},\end{cases}$$

If $\boldsymbol{\Omega(A\angle B)=0}$, then the premise ''A is inferior to B'' will not be approved. If $\boldsymbol{\Omega(A\angle B)<1}$, then the interpreter will accept the premise $(\boldsymbol{A\angle B})$ with various grades of satisfaction from zero to one (excluding zero and one). If $\boldsymbol{\Omega(A\angle B)\geq 1}$, then the interpreter is absolutely satisfied with the premise$(\boldsymbol{A\angle B})$, that is, he accepts that $(\boldsymbol{A\angle B})$ is true.

Property 1. For any value judgment, the $\boldsymbol{\Omega}$-index always satisfies the decision-maker; for any two intervals A and B on R, we have one of the following cases:

$\boldsymbol{\Omega(A\angle B)>0}$; or $\boldsymbol{\Omega(B\angle A)>0}$; or $\boldsymbol{\Omega(A\angle B)=\Omega(B\angle A)=0}$.

Property 2. The proposed index is transitive: for any three intervals A, B, and C on R we have:

if $\boldsymbol{\Omega(A\angle B)\geq 0}$ and $\boldsymbol{\Omega(B\angle C)\geq 0}$; then $\boldsymbol{\Omega(A\angle C)\geq 0}$.

But it does not mean that $\boldsymbol{\Omega(A\angle C)\geq max\{\Omega(A\angle B),\Omega(B\angle C)\}}$.

We can say a general statement about the function of the $\boldsymbol{\Omega}$-index: The position of the mean of an interval relative to the mean of another reference interval determines its superiority or inferiority. However, the width of the superior (or inferior) interval relative to the other interval determines the amount of satisfaction the decision-maker feels regarding the superiority or inferiority of the interval related to the reference interval.

Based on the $\boldsymbol{\Omega}$-index it may found more than one efficient DMUs such that their midpoints are equal to one, that lead to $\boldsymbol{\Omega}$-index equal to zero. So, for ranking between them we introduce an ideal DMU $(\boldsymbol{\hat{X},\hat{Y}})$ and calculate the interval efficiency of each efficient DMU against this ideal DMU.

4. Ranking efficient DMUs by introducing ideal DMUs

If there is more than one unit with midpoints equal to one, they have rank one. For ranking between them, we introduce an ideal DMU $(\widehat{\boldsymbol{X}}, \widehat{\boldsymbol{Y}})$ where

$$\begin{cases} \widehat{\boldsymbol{x}}_{\boldsymbol{i}} = \underset{1 \le j \le n}{\boldsymbol{min}}\{\boldsymbol{x}_{\boldsymbol{ij}}^{\boldsymbol{L}}\}, & \boldsymbol{i} = \boldsymbol{1}, \dots, \boldsymbol{m} \\ \widehat{\boldsymbol{y}}_{\boldsymbol{r}} = \underset{1 \le j \le n}{\boldsymbol{max}}\{\boldsymbol{y}_{\boldsymbol{rj}}^{\boldsymbol{U}}\}, & \boldsymbol{r} = \boldsymbol{1}, \dots, \boldsymbol{s} \end{cases}$$

Hence, we calculate the interval efficiency of each efficient DMU against this ideal DMU. This done by solving the following pair of DEA models:

$$\begin{aligned} &\varphi_o^U = max \quad \sum_{r=1}^{s} u_r y_{ro}^U, \\ &s.t. \\ &\sum_{i=1}^{m} v_i x_{io}^L = 1, \\ &\sum_{r=1}^{s} u_r \hat{y}_r - \sum_{i=1}^{m} v_i \hat{x}_i \le 0, \\ &u_r, v_i \ge \varepsilon, \forall r, i \end{aligned} \tag{12}$$

And

$$\begin{aligned} &\varphi_o^L = max \quad \sum_{r=1}^{s} u_r y_{ro}^L, \\ &s.t. \\ &\sum_{i=1}^{m} v_i x_{io}^U = 1, \\ &\sum_{r=1}^{s} u_r \hat{y}_r - \sum_{i=1}^{m} v_i \hat{x}_i \le 0, \\ &u_r, v_i \ge \varepsilon, \forall r, i \end{aligned} \tag{13}$$

Then for each efficient $\boldsymbol{DMU_o}$we obtain the interval $[\boldsymbol{\varphi_o^L}, \boldsymbol{\varphi_o^U}]$, and by calculating the $\boldsymbol{\Omega}$-index we can rank them.

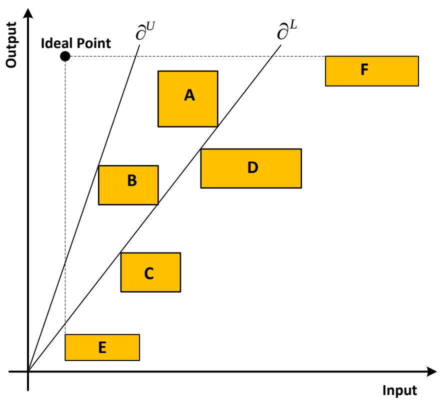


Fig. 5: Ranking with the ideal point

For example, see Fig. 5. In Fig. 5, there are six DMUs with one input and one output with interval data. In Fig. 5 we can see $\boldsymbol{\theta_E^L}, \boldsymbol{\theta_E^U} < \boldsymbol{1}$ and $\boldsymbol{\theta_F^L}, \boldsymbol{\theta_F^U} < \boldsymbol{1}$. Also, $\boldsymbol{\theta_C^L} < \boldsymbol{1}$, $\boldsymbol{\theta_C^U} = \boldsymbol{1}$ and $\boldsymbol{\theta_D^L} < \boldsymbol{1}$, $\boldsymbol{\theta_D^U} = \boldsymbol{1}$. On the other hand, we can see that $\boldsymbol{\theta_B^L} < \boldsymbol{1}$, $\boldsymbol{\theta_B^U} > \boldsymbol{1}$ and $\boldsymbol{\theta_A^L} < \boldsymbol{1}$, $\boldsymbol{\theta_A^U} > \boldsymbol{1}$.

5. An algorithm for interval efficiency and ranking DMUs in the presence of fuzzy, ordinal, and interval data

We assume that there are $\boldsymbol{n}$ homogeneous DMUs, and each $\boldsymbol{DMU_j}$ uses $\boldsymbol{m}$ inputs $\boldsymbol{x_{ij}}$ (i=1,…, m) to produce $\boldsymbol{s}$ outputs $\boldsymbol{y_{rj}}$ (r=1, . . . , s). We also assume that inputs and outputs aren't necessarily deterministic and they may be known as definitive, fuzzy, ordinal, or interval data. We consider five steps to achieve for ranking these DMUs and total results:

Step 1: Firstly, by using the formula (6), (7), and (9), convert all of the data into interval data. Therefore, each input $\boldsymbol{x_{ij}}$ as from $[\boldsymbol{x_{ij}^L}, \boldsymbol{x_{ij}^U}]$ and output $\boldsymbol{y_{rj}}$ as from $[\boldsymbol{y_{rj}^L}, \boldsymbol{y_{rj}^U}]$.

Step 2: By using the formula (10)-(11), we obtain the interval efficiency $[\boldsymbol{\theta^L}, \boldsymbol{\theta^U}]$ for each DMU.

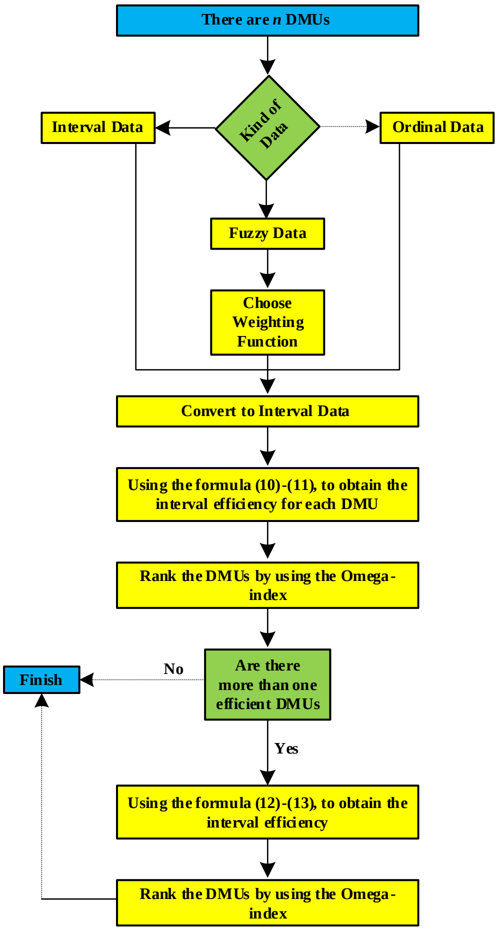


Fig. 6: procedure of the proposed algorithm

Step 3: We can rank the DMUs by using the $\boldsymbol{\Omega}$-index. If there are more than one efficient DMUs, go to step 4.

Step 4: Using the formula (12)-(13), we obtain the interval efficiency $[\boldsymbol{\varphi_o^L}, \boldsymbol{\varphi_o^U}]$ for each efficient DMU.

Step 5: We can rank efficient DMUs by using the $\boldsymbol{\Omega}$-index.

Fig. 6 illustrated the procedure of the proposed algorithm.

6. Comparing with some other existing extended DEA methods

In Table 2, we compare some capabilities of the existing extended DEA models. Some of them, in addition to crisp data, consider interval data, or ordinal data, or fuzzy data.

Table 2: Some capabilities of the existing extended DEA models

No.	Method	**Interval data**	**Ordinal data**	Fuzzy data		Other data types
				α-cut	**Kind of fuzzy number**	
1	Kao and Liu (2000)	-	-	√	Any form	-
2	Agarwal (2014)	-	-	√	Any form	-
3	Y.-C. Chen et al. (2013)	-	-	√	Any form	-
4	Z. Zhou et al. (2012)	-	-	√	Any form	-
5	Zerafat Angiz L. et al. (2010)	-	-	√	TrFN	-
6	Guo and Tanaka (2001)	-	-	√	Any form	-
7	Jahanshahloo et al. (2004)	-	-	√	TFN	-
8	Kao and Lin (2011)	-	√	√	Any form	-
9	Wu and Lee (2010)	-	√	-	-	Stochastic data, Non-discretionary
10	Toloo and Nalchigar (2011)	-	√	-	-	-
11	Despotis and Smirlis (2002)	√	√	-	-	-
12	Zhu (2003)	√	√	-	-	-
13	Y.-M. Wang et al. (2005)	√	√	-	-	-
14	Kao (2006)	√	√	-	-	-
15	Hatami-Marbini et al. (2014)	√	-	-	-	Negative data
16	Proposed Method	√	√	-	Any form	-

To compare our proposed method with some other extended DEA methods, we compare the proposed method with two fuzzy DEA methods, i.e., Kao and Liu (2000) and Agarwal (2014). Therefore, we use the data set taken from Kao and Liu (2000) that also used in Agarwal (2014) for analyzing four DMUs. There are one input and one output. Some of them are fuzzy, and some of them are crisp. Table 3 reports the data and the ranking results of Kao and Liu (2000) and Agarwal (2014).

Table 3: Input and output data for four DMUs

DMU	Input	Output	Kao and Liu ranking	Agarwal ranking
A	(11,12,14)	10	1	1
B	30	(12,13,14,16)	3	2
C	40	11	4	4
D	(45,47,52,55)	(12,15,19,22)	2	3

As is seen, both methods Kao and Liu (2000) and Agarwal (2014) identify DMU_A as the most efficient DMU and DMU_D as the most inefficient DMU. Now, in the proposed method, first, we convert each fuzzy and crisp data into interval ones.

To convert the fuzzy data into interval ones, we use a weighting function $f(\alpha,\alpha) = 3\alpha^2$ in the procedure mentioned in section 3.4.2. Therefore, the interval forms of inputs and outputs of Table 3 are obtained as Table 4.

Using the formula (14)-(15), we obtain the interval efficiency for each DMU as the fourth column of Table 4. Now, by using the concept of midpoint and the width of each obtained interval, we can rank the DMUs by using the Ω-index, as Table 4. The obtained results are similar to the results.

The column related to Ω-index shows, for example,

"Since, $\Omega(B\angle A) = 0.676113 < 1$, then the interpreter will accept the premise $(B\angle A)$ with grade 68 present of satisfaction."

Table 4: The results of the proposed method for data of Table 3

DMU	Interval Input	Interval Output	Interval efficiency	$m(A_i)$	$w(A_i)$	Ω-index	Rank
A	[11.75,12.5]	[10,10]	[0.888,1]	0.944	0.056	0.676113	1
B	[30,30]	[12.75,14.5]	[0.425,1]	0.713	0.287	0.088613	2
C	[40,40]	[11,11]	[0.275,1]	0.638	0.362	-	4
D	[46.5,52.75]	[14.25,19.75]	[0.313,1]	0.657	0.343	0.027048	3

Although, DMU C has crisp data, its performance is evaluated against other DMUs that have interval data. In this regard and with optimistic view point (when it compares against the worst parts of other DMUs, i.e. (x_j^U, y_j^L)), the efficiency score of DMU C is 1 and with pessimistic view point (when it compares against the best parts of other DMUs, i.e. (x_j^L, y_j^U)), the efficiency score of DMU C is 0.275. We can see that the Ω-index for DMU C is not reported in Table 4. According to the Definition 7, $\Omega_\angle$ can be interpreted as ''the degree of acceptability of the assertion that the first interval is inferior to the second interval''. Hare, there is no interval efficiency such that it is inferior to the interval efficiency of DMU C.

7. Applications

7.1. An illustrative example

In this section, we show the ability of the provided approach using a numerical example. We apply the proposed method for evaluating 15 units, which each unit uses four inputs to produce four outputs. The inputs 1 and 4 are completely known, input 2 is of the form of ordinal data and input 3 is of the form of fuzzy data. Also, the outputs 1 and 3 are of the form of interval data, output 2 is of the form of ordinal data, and output 4 is wholly known. The data set for this example are shown in Table 5.

Table 5. The data set of Example 7.1

DMUs	Input 1	Input 2	Input 3	Output1	Output 2	Output 3
1	217	6	(3,5.5,8)	[48,63]	1	64
2	168	7	$\tilde{x}_{23}$	[24,50]	3	31
3	245	2	$\tilde{x}_{33}$	[25,36]	14	68
4	184	11	(3,7.5,10)	[37,42]	13	75
5	212	14	$\tilde{x}_{53}$	[56,70]	2	43
6	205	12	(5,6,7,8.5)	[58,61]	10	52
7	190	5	$\tilde{x}_{73}$	[67,73]	5	69
8	175	3	$\tilde{x}_{83}$	[32,41]	12	42
9	255	1	$\tilde{x}_{93}$	[25,26]	15	79
10	210	15	$\tilde{x}_{10,3}$	[39,65]	6	50
11	220	8	(6,8.5,11.5)	[37,63]	7	64
12	265	9	(2,5,6,7)	[48,54]	9	59
13	137	10	(6.8,7.8,9.8)	[28,35]	11	67

14	251	4	(6.7,9.7,10.2,12.3)	[35,45]	4	71
15	187	13	(5.7,7.4,9.7)	[42,50]	8	60

In Table 5, we can see that the data of input 3 are fuzzy data in general form. The membership functions of general fuzzy numbers of input 3 in Table 5 are as follows:

$$\tilde{x}_{23} = \begin{cases} 0, & x \le 1 \\ \dfrac{x-1}{0.5}, & 1 \le x \le 1.5 \\ 1, & 1.5 \le x \le 2 \\ e^{-\frac{(x-2)^2}{8}}, & 2 \le x \end{cases} \qquad \tilde{x}_{33} = \begin{cases} x, & 0 \le x \le 1 \\ e^{-\frac{(x-1)^2}{2}}, & 1 \le x \\ 0, & Otherwise \end{cases}$$

$$\tilde{x}_{53} = \begin{cases} \dfrac{1-(x-5)^2}{4}, x \in [3,7] \\ 0, \qquad \text{Otherwise.} \end{cases} \qquad \tilde{x}_{73} = \begin{cases} (x-1)^2, x \in [1,2]; \\ (x-3)^2, x \in [2,3]; \\ 0, \qquad \text{Otherwise.} \end{cases}$$

$$\tilde{x}_{83} = \begin{cases} \dfrac{(x-1)^2}{2}, & x \in [1,3] \\ 1, & x \in [3,4] \\ (5-x)^2, & x \in [4,5] \\ 0, & \text{Otherwise.} \end{cases} \qquad \tilde{x}_{93} = \begin{cases} x^2, & x \in [0,1]; \\ 1, & x \in [1,2]; \\ (3-x)^2, & x \in [2,3]; \\ 0, & \text{Otherwise.} \end{cases}$$

$$\tilde{x}_{10,3} = \begin{cases} \frac{2}{1+(x-1)^2} - 1, & x \in [2,4] \\ 0, & \text{Otherwise.} \end{cases}$$

By using the relations (6), (7) and (13), we can convert the data into interval data as Table 6.

Table 6. Interval form of data set of Example 7.1

	Input 1	Input 2	Input 3	Output 1	Output 2	Output 3
1	[217,217]	[0.112, 0.387]	[4.875, 6.125]	[48,48]	[0.066,0.38]	[64,64]
2	[168,168]	[0.125,0.430]	[1.375, 3.447]	[24,24]	[0.076,0.436]	[31,31]
3	[245,245]	[0.074,0.254]	[0.750, 1.724]	[25,25]	[0.163,0.933]	[68,68]
4	[184,184]	[0.191, 0.656]	[6.375, 8.125]	[37,37]	[0.152,0.871]	[75,75]
5	[212,212]	[0.262, 0.900]	[4.086, 5.914]	[56,56]	[0.071,0.407]	[43,43]
6	[205,205]	[0.212, 0.729]	[5.750, 7.375]	[58,58]	[0.124,0.808]	[52,52]
7	[190,190]	[0.101, 0.348]	[1.857, 2.143]	[67,67]	[0.087,0.501]	[69,69]
8	[175,175]	[0.082, 0.282]	[2.212, 4.143]	[32,32]	[0.142,0.813]	[42,42]
9	[255,255]	[0.066, 0.228]	[0.857, 2.143]	[25,25]	[0.175,1.000]	[79,79]
10	[210,210]	[0.291, 1.000]	[2.644, 3.356]	[39,39]	[0.094,0.537]	[50,50]
11	[220,220]	[0.139, 0.478]	[7.875, 9.250]	[37,37]	[0.1.00,0.575]	[64,64]
12	[265,265]	[0.154, 0.531]	[4.250, 6.250]	[48,54]	[0.115,0.661]	[59,59]
13	[137,137]	[0.172, 0.590]	[7.550, 8.300]	[28,35]	[0.132,0.758]	[67,67]
14	[251,251]	[0.091, 0.313]	[8.95, 10.725]	[35,45]	[0.081,0.468]	[71,71]
15	[187,187]	[0.236,0.810]	[6.80, 7.975]	[42,50]	[0.108,0.616]	[60,60]

To convert the fuzzy data into interval ones, we use a weighting function $f(\alpha,\alpha) = 3\alpha^2$in procedure mentioned in section 3.4.2. Also, by a procedure mentioned in section 3.3, we convert the ordinal data into interval data. Therefore, the interval forms of inputs and outputs of Table 5 are obtained as Table 6.

Using the formula (14)-(15), we obtain the interval efficiency for each DMU, and we can rank the DMUs by using the Ω-index. The results are shown in Table 7.

Table 7. Interval efficiencies, Ω-index values, and ranks of the DMUs for data in Example 7.1

DMU	Interval efficiency	$m(A_i)$	$w(A_i)$	Ω-index	Rank
1	[0.712,1]	0.856	0.144	0.040	8
2	[0.469,1]	0.734	0.266	-	15

3	[0.815,1]	0.908	0.092	0.179	5
4	[0.889,1]	0.944	0.056	0.243	4
5	[0.687,1]	0.844	0.156	0.066	9
6	[0.736,1]	0.868	0.132	0.008	6
7	[1.000,1]	1	0	1	1
8	[0.597,1]	0.798	0.202	0.047	13
9	[0.896,1]	0.948	0.052	0.037	3
10	[0.626,1]	0.813	0.187	0.005	11
11	[0.643,1]	0.822	0.178	0.025	10
12	[0.555,1]	0.778	0.222	0.090	14
13	[1.000,1]	1	0	1	1
14	[0.622,1]	0.811	0.189	0.033	12
15	[0.732,1]	0.866	0.134	0.036	7

We can see DMU 7 and DMU 13 have ranked one. Therefore, we construct the ideal DMU, namely $(\hat{X},\hat{Y})$, as Table 8.

Table 8. Ideal DMU

Input			Output		
I1	**I2**	**I3**	**O1**	**O2**	**O3**
117	0.066	0.75	73	1	79

By using the formula (16)-(17), we obtain the interval efficiency for DMUS 7 and 13, and therefore we can rank these DMUs by using the Ω -index. The results are shown in Table 9.

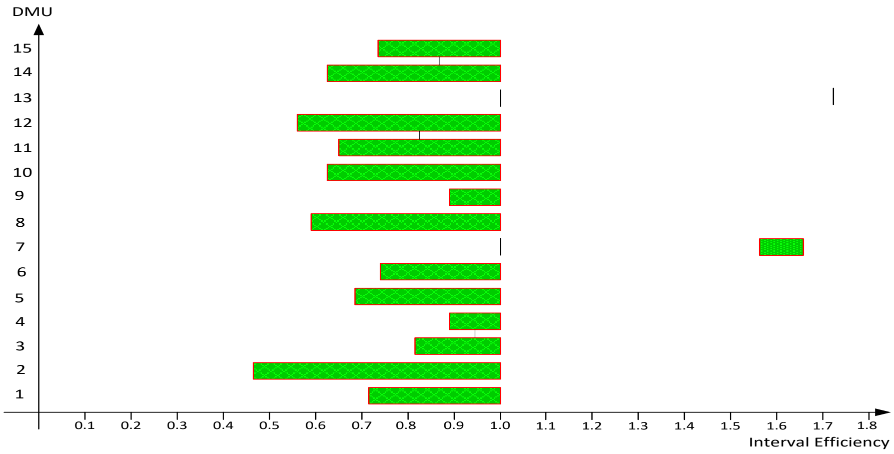


Fig. 7: Interval Efficiency and Ranking

Fig. 7 illustrates the obtained interval efficiencies of these 15 DMUs. The interval efficiencies of Table 8 are given in Fig. 7. From Fig. 7, we can easily see that DMU 13 and DMU 7 are ranked first and second.

Table 9. Interval efficiencies, Ω-index values, and ranks of the DMUs 7 and 13

DMU	Interval efficiency	$m(A_i)$	$w(A_i)$	Ω-index	Rank
7	[0.565,0.653]	0.609	0.044	-	2
13	[0.724,0.724]	0.724	0	2.614	1

From Table 9, as Fig. 7, we can see that DMUs 13 and 7 obtained the first and second ranking order, respectively. The proposed method can solve DEA problems with non-triangular and non-trapezoidal fuzzy numbers, but most of the other fuzzy DEA methods cannot solve these kinds of problems.

7.2 Real-world data: Hospital Data

In this section, we apply our proposed method for evaluating 38 hospitals were selected by OIG to participate in this study. The input and output parameters and their associated values for the 38 hospitals, are presented in Table 10. Information was taken from Tavana et al. (2012).

7.2.1 Data and Information

This study uses three input and three output measures for evaluating these hospitals. Four selected inputs for the DEA analysis are Number of beds, Labor-related expenses and Patient care supplies and other expenses, and Overall hospital ranking in the League table. Three outputs include the Number of outpatient Department visits, Number of inpatient department admissions, and Overall patient satisfaction. Among these data, input 4 is of the form of ordinal data, and output 3 is in the form of fuzzy data, and other data are completely known.

Table 10. The hospital data

	Input Parameters				Output Parameters		
	Number of beds	Labor-related expenses ($)	Patient care supplies and other expenses ($)	Overall hospital ranking in League table	Number of outpatient Department visits	Number of inpatient department admissions	Overall patient satisfaction
1	83	5,428,903	3,142,311	28	42,859	5,274	ML
2	78	6,583,333	4,126,127	15	48,367	5,268	M
3	54	5,495,517	2,177,965	34	55,606	7,302	L
4	80	6,426,532	3,501,847	31	48,879	7,077	MH
5	75	6,782,869	2,894,877	26	46,801	6,593	ML
6	87	5,491,546	2,565,741	35	57,977	7,574	VL
7	58	3,778,001	2,036,342	38	57,787	5,264	MH
8	71	6,999,241	3,036,959	33	70,031	6,090	VL
9	76	7,942,581	3,982,119	22	62,102	5,157	M
10	80	7,473,486	4,741,523	6	52,940	3,476	VH
11	78	6,698,820	3,770,352	16	40,055	5,611	L
12	60	4,293,792	2,110,921	36	56,555	5,586	MH
13	78	7,199,197	3,166,796	27	64,143	5,170	H
14	69	7,608,522	3,400,052	10	48,890	4,456	L
15	80	6,775,716	3,495,441	18	54,330	4,774	M
16	81	8,716,008	3,530,795	8	46,305	6,125	VL
17	77	7,237,227	3,524,780	23	44,564	6,218	M
18	87	7,592,595	4,701,414	1	52,283	3,798	MH
19	49	5,604,079	2,696,243	25	41,782	4,814	VH
20	64	6,721,746	2,760,717	24	38,308	6,418	VL
21	90	5,147,491	2,618,025	32	42,211	5,618	M
22	84	8,416,341	4,086,333	14	41,346	6,705	MH
23	81	6,945,228	4,312,511	13	50,619	4,783	VH
24	81	7,340,542	3,907,518	7	47,010	5,476	M
25	89	9,202,308	4,637,745	12	67,091	5,179	L
26	79	6,861,558	3,445,030	21	60,469	5,515	M
27	86	8,359,115	3,718,448	20	61,267	6,225	VL
28	70	7,636,593	2,870,895	29	47,437	6,843	VH
29	81	7,939,155	4,219,269	5	46,235	5,620	L
30	80	6,310,243	3,439,974	30	63,992	6,538	ML
31	78	6,793,294	4,404,172	3	42,032	4,821	MH
32	86	7,517,868	4,652,596	17	49,402	6,238	H
33	55	6,808,131	2,303,569	37	78,483	5,866	VL
34	80	6,109,813	3,449,003	9	35,649	4,495	M
35	82	7,517,663	4,054,654	11	51,891	4,996	ML
36	79	7,887,497	3,281,593	4	49,168	4,641	VL
37	85	9,046,154	4,696,585	2	50,796	5,160	H
38	85	7,033,971	4,098,183	19	56,017	5,017	L

The importance weights of the qualitative criteria and the ratings are considered as linguistic variables expressed in positive trapezoidal fuzzy numbers (see S.-M. Chen and Lee (2010); Xiao et al. (2012); Ploskas et al. (2017) and Akram and Arshad (2019)), as shown in Table 11.

Table 11. Linguistic variables and their associated trapezoidal fuzzy numbers

Linguistic variables		Associated trapezoidal fuzzy numbers
Very low	VL	(0, 0, 0.1, 0.2)
Low	L	(0.1, 0.2, 0.2, 0.3)
Medium low	ML	(0.2, 0.3, 0.4, 0.5)
Medium	M	(0.4, 0.5, 0.5, 0.6)
Medium high	MH	(0.5, 0.6, 0.7, 0.8)
High	H	(0.7, 0.8, 0.8, 0.9)
Very high	VH	(0.8, 0.9, 0.9, 1)

Additionally, the rule of conversion between linguistic variables and trapezoidal fuzzy numbers is shown in Fig. 8.

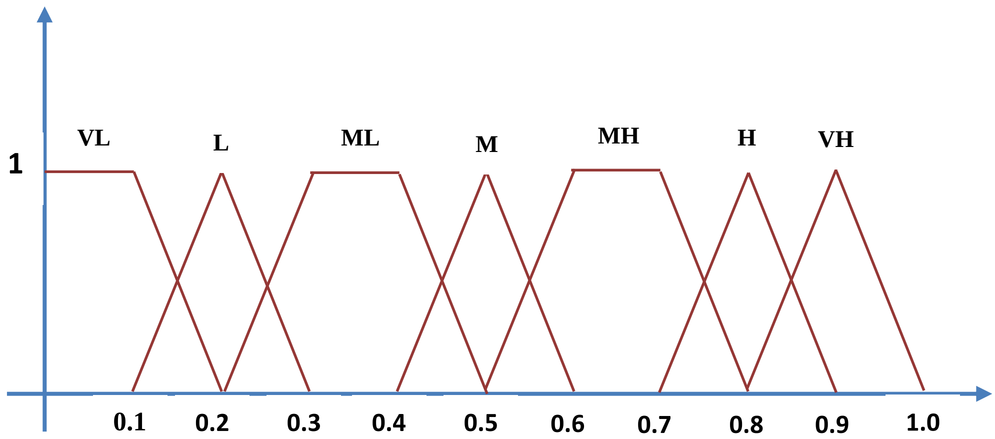


Fig. 8: Linguistic variables as trapezoidal fuzzy numbers

7.2.2. Results of the models

By using the relations (6), (7), and (13), we can convert the data into interval data as Table 12.

Table 12. Interval form of data set of Example 6.2

	Input Parameters				Output Parameters		
	I1	I2	I3	I4	O1	O2	O3
1	[83,83]	[5428903, 5428903]	[3142311, 3142311]	[0.068,0.076]	[42859, 42859]	[5274, 5274]	[0.28,0.42]
2	[78,78]	[6583333, 6583333]	[4126127, 4126127]	[0.235,0.263]	[48367, 48367]	[5268, 5268]	[0.48,0.52]
3	[54,54]	[5495517, 5495517]	[2177965, 2177965]	[0.038,0.043]	[55606, 55606]	[7302, 7302]	[0.18,0.22]
4	[80,80]	[6426532, 6426532]	[3501847, 3501847]	[0.051,0.057]	[48879, 48879]	[7077, 7077]	[0.58,0.72]
5	[75,75]	[6782869, 6782869]	[2894877, 2894877]	[0.082,0.092]	[46801, 46801]	[6593, 6593]	[0.28,0.42]
6	[87,87]	[5491546, 5491546]	[2565741, 2565741]	[0.035,0.039]	[57977, 57977]	[7574, 7574]	[0.00,0.12]
7	[58,58]	[3778001, 3778001]	[2036342, 2036342]	[0.026,0.029]	[57787, 57787]	[5264, 5264]	[0.58,0.72]
8	[71,71]	[6999241, 6999241]	[3036959, 3036959]	[0.042,0.047]	[70031, 70031]	[6090, 6090]	[0.00,0.12]
9	[76,76]	[7942581, 7942581]	[3982119, 3982119]	[0.120,0.135]	[62102, 62102]	[5157, 5157]	[0.48,0.52]
10	[80,80]	[7473486,	[4741523, 4741523]	[0.553,0.621]	[52940, 52940]	[3476,	[0.88,0.92]

		7473486]				3476]	
11	[78,78]	[6698820, 6698820]	[3770352, 3770352]	[0.213,0.239]	[40055, 40055]	[5611, 5611]	[0.18,0.22]
12	[60,60]	[4293792, 4293792]	[2110921, 2110921]	[0.032,0.036]	[56555, 56555]	[5586, 5586]	[0.58,0.72]
13	[78,78]	[7199197, 7199197]	[3166796, 3166796]	[0.075,0.084]	[64143, 64143]	[5170, 5170]	[0.78,0.82]
14	[69,69]	[7608522, 7608522]	[3400052, 3400052]	[0.378,0.424]	[48890, 48890]	[4456, 4456]	[0.18,0.22]
15	[80,80]	[6775716, 6775716]	[3495441, 3495441]	[0.176,0.198]	[54330, 54330]	[4774, 4774]	[0.48,0.52]
16	[81,81]	[8716008, 8716008]	[3530795, 3530795]	[0.457,0.513]	[46305, 46305]	[6125, 6125]	[0.00,0.12]
17	[77,77]	[7237227, 7237227]	[3524780, 3524780]	[0.109,0.123]	[44564, 44564]	[6218, 6218]	[0.48,0.52]
18	[87,87]	[7592595, 7592595]	[4701414, 4701414]	[0.891,1.000]	[52283, 52283]	[3798, 3798]	[0.58,0.72]
19	[49,49]	[5604079, 5604079]	[2696243, 2696243]	[0.090,0.102]	[41782, 41782]	[4814, 4814]	[0.88,0.92]
20	[64,64]	[6721746, 6721746]	[2760717, 2760717]	[0.099,0.112]	[38308, 38308]	[6418, 6418]	[0.00,0.12]
21	[90,90]	[5147491, 5147491]	[2618025, 2618025]	[0.046,0.052]	[42211, 42211]	[5618, 5618]	[0.48,0.52]
22	[84,84]	[8416341, 8416341]	[4086333, 4086333]	[0.258,0.289]	[41346, 41346]	[6705, 6705]	[0.58,0.72]
23	[81,81]	[6945228, 6945228]	[4312511, 4312511]	[0.284,0.319]	[50619, 50619]	[4783, 4783]	[0.88,0.92]
24	[81,81]	[7340542, 7340542]	[3907518, 3907518]	[0.503,0.564]	[47010, 47010]	[5476, 5476]	[0.48,0.52]
25	[89,89]	[9202308, 9202308]	[4637745, 4637745]	[0.312,0.350]	[67091, 67091]	[5179, 5179]	[0.18,0.22]
26	[79,79]	[6861558, 6861558]	[3445030, 3445030]	[0.132,0.149]	[60469, 60469]	[5515, 5515]	[0.48,0.52]
27	[86,86]	[8359115, 8359115]	[3718448, 3718448]	[0.146,0.164]	[61267, 61267]	[6225, 6225]	[0.00,0.12]
28	[70,70]	[7636593, 7636593]	[2870895, 2870895]	[0.062,0.069]	[47437, 47437]	[6843, 6843]	[0.88,0.92]
29	[81,81]	[7939155, 7939155]	[4219269, 4219269]	[0.609,0.683]	[46235, 46235]	[5620, 5620]	[0.18,0.22]
30	[80,80]	[6310243, 6310243]	[3439974, 3439974]	[0.056,0.063]	[63992, 63992]	[6538, 6538]	[0.28,0.42]
31	[78,78]	[6793294, 6793294]	[4404172, 4404172]	[0.737,0.826]	[42032, 42032]	[4821, 4821]	[0.58,0.72]
32	[86,86]	[7517868, 7517868]	[4652596, 4652596]	[0.194,0.218]	[49402, 49402]	[6238, 6238]	[0.78,0.82]
33	[55,55]	[6808131, 6808131]	[2303569, 2303569]	[0.029,0.032]	[78483, 78483]	[5866, 5866]	[0.00,0.12]
34	[80,80]	[6109813, 6109813]	[3449003, 3449003]	[0.416,0.467]	[35649, 35649]	[4495, 4495]	[0.48,0.52]
35	[82,82]	[7517663, 7517663]	[4054654, 4054654]	[0.344,0.386]	[51891, 51891]	[4996, 4996]	[0.28,0.42]
36	[79,79]	[7887497, 7887497]	[3281593, 3281593]	[0.669,0.751]	[49168, 49168]	[4641, 4641]	[0.00,0.12]
37	[85,85]	[9046154, 9046154]	[4696585, 4696585]	[0.810,0.909]	[50796, 50796]	[5160, 5160]	[0.78,0.82]
38	[85,85]	[7033971, 7033971]	[4098183, 4098183]	[0.160,0.179]	[56017, 56017]	[5017, 5017]	[0.18,0.22]

Using the formula (14)-(15), we obtain the interval efficiency for each DMU, and we can rank the DMUs by using the Ω-index. The results are shown in Table 13.

Table 13. Interval efficiencies, Ω-index values, and ranking of the hospital

DMU	Interval efficiency	$m(A_i)$	$w(A_i)$	Ω-index	Rank
1	[0.698,0.700]	0.699	0.001	1	22
2	[0.623,0.657]	0.640	0.017	0.16	25
3	[1.000,1.000]	1	0	1	1
4	[0.810,0.914]	0.862	0.052	0.36	8
5	[0.725,0.776]	0.751	0.026	0.05	14
6	[1.000,1.000]	1	0	1	1
7	[1.000,1.000]	1	0	1	1
8	[0.817,0.817]	0.817	0	14	10
9	[0.699,0.742]	0.720	0.010	0.21	18

10	[0.709,0.787]	0.748	0.039	0.15	15
11	[0.620,0.622]	0.621	0.001	0.2	29
12	[0.960,1.000]	0.980	0.020	1.64	7
13	[0.804,0.862]	0.833	0.029	0.55	9
14	[0.572,0.597]	0.685	0.013	0.85	23
15	[0.615,0.654]	0.634	0.020	0.15	26
16	[0.559,0.559]	0.559	0	2.75	36
17	[0.694,0.724]	0.709	0.015	0.09	20
18	[0.571,0.634]	0.602	0.032	0.16	31
19	[1.000,1.000]	1	0	1	1
20	[0.742,0.742]	0.742	0	0.61	16
21	[0.787,0.791]	0.789	0.002	0.77	11
22	[0.685,0.736]	0.710	0.026	0.02	19
23	[0.721,0.846]	0.784	0.063	0.08	12
24	[0.603,0.632]	0.618	0.014	0.35	30
25	[0.590,0.600]	0.595	0.005	0.64	33
26	[0.690,0.722]	0.706	0.016	0.41	21
27	[0.631,0.631]	0.631	0	0.07	27
28	[0.983,1.000]	0.992	0.008	0.43	6
29	[0.544,0.553]	0.548	0.004	4.75	37
30	[0.779,0.779]	0.779	0	1.08	13
31	[0.602,0.680]	0.641	0.039	0.02	24
32	[0.713,0.749]	0.731	0.018	0.28	17
33	[1.000,1.000]	1	0	1	1
34	[0.554,0.591]	0.572	0.018	0.72	35
35	[0.564,0.598]	0.581	0.017	0.26	34
36	[0.529,0.529]	0.529	0	-	38
37	[0.616,0.644]	0.630	0.014	0.6	28
38	[0.597,0.597]	0.597	0	0.4	32

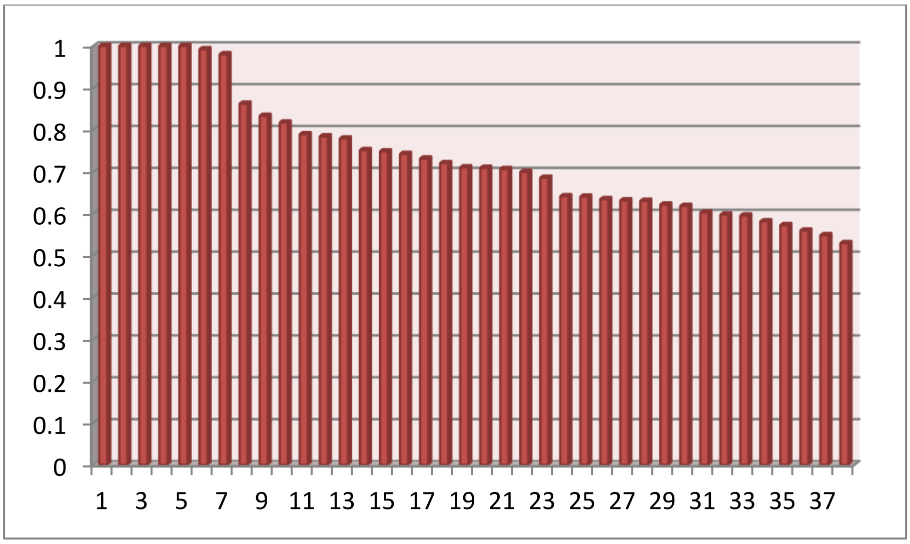


Fig. 9: Ranking DMUs based on the values of $m(A_i)$

From Fig. 9, we can see the ranking DMUs based on the values of $m(A_i)$. From Fig. 9 it is appear that DMUs 3, 6, 7, 19 and 33 are the first ranking order based on the values of $m(A_i)$.

Besides, from Table 13, we can see that, five DMUs 3, 6, 7, 19, and 33 have rank one. Therefore, we construct the ideal DMU, namely $(\hat{X}, \hat{Y})$ as Table 14.

Table 14. Ideal DMU

Input				Output		
I1	I2	I3	I4	O1	O2	O3
49	3778001	2036342	0.026	78483	7574	0.92

By using the formula (16)-(17), we obtain the interval efficiency for DMUS 3,6,7,19 and 33, and therefore we can rank these DMUs by using the Ω -index. The results are shown in Table 15.

Table 15. Interval efficiencies, Ω-index values, and ranks of the DMUs 3,6,7,19,28 and 33.

DMU	Interval efficiency	$m(A_i)$	$w(A_i)$	Ω-index	Rank
3	[0.901,0.901]	0.901	0	0.8	2
6	[0.794,0.794]	0.794	0	1.52	4
7	[0.736,0.782]	0.759	0.023	-	5
19	[0.957,1.000]	0.978	0.022	3.5	1
33	[0.891,0.902]	0.897	0.005	20.6	3

From Table 15, we can see that DMU 19 has the best ranking order. Fig. 10 illustrates the obtained interval efficiencies of these best six DMUs. From Fig. 10 we can easily see that DMU 19 obtain the ranking order one and DMU 3 get the second ranking order. Also, DMU 7 gained the fifth ranking order.

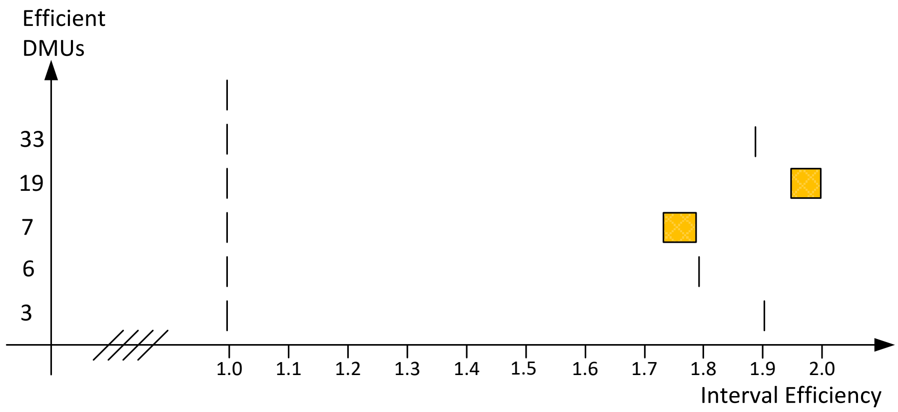


Fig. 10: Interval Efficiency and Ranking for Efficient DMUs

7.2.3 Analysis of linguistic variables

Linguistic variables are the variables whose defining terms are not numerical values but rather sentences or words in a natural or artificial language. This kind of variable can well be represented by triangular or trapezoidal fuzzy numbers. For the study of linguistic variables, readers are referred to Zadeh (1975), Akram and Adeel (2016), Zhang et al. (2017), and Akram et al. (2019). Linguistic variables can be stated as various kinds of fuzzy numbers. S.-M. Chen and Lee (2010), Xiao et al. (2012), Ploskas et al. (2017), and Akram and Arshad (2019) considered linguistic variables as trapezoidal fuzzy numbers. In addition to the trapezoidal fuzzy numbers, Ertuğrul and Güneş (2007) and T.-C. Wang and Chen (2008) applied the triangular fuzzy numbers to interpret the linguistic variables. In this work, the rating of "Overall patient satisfaction" is described using linguistic terms induced by fuzzy linguistic variable, which was expressed in trapezoidal fuzzy numbers. To check the impact of different linguistic variables on the final result, we further run our models by associating triangular fuzzy numbers to linguistic variables as Table 16.

Table 16: Linguistic variables and their associated triangular fuzzy numbers

Linguistic variables	Triangular fuzzy numbers (TFN)
Very low	(0,0,0.1)
Low	(0,0.1,0.3)
Medium low	(0.1,0.3,0.5)
Medium	(0.3,0.5,0.7)
Medium high	(0.5,0.7,0.9)
High	(0.7,0.9,1)
Very high	(0.9,1,1)

The rule of conversion between linguistic variables and triangular numbers is shown in Fig. 11.

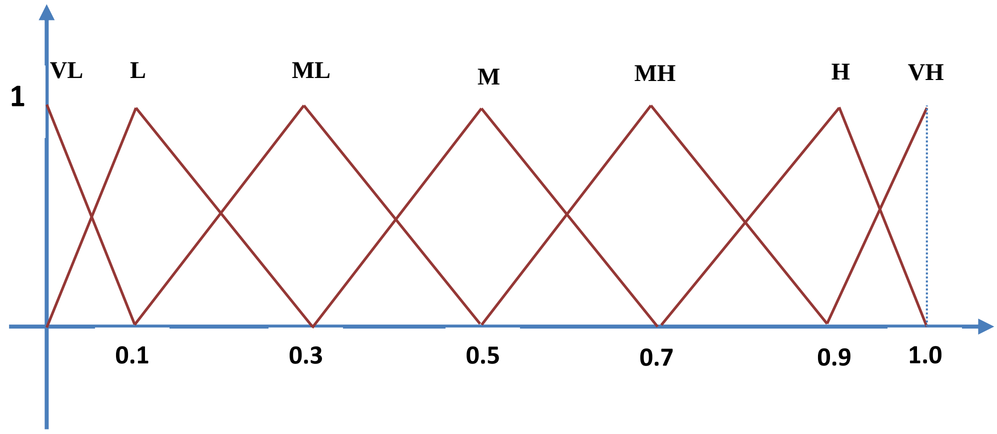


Fig. 11: Linguistic variables as triangular numbers

For this analysis, we calculated the equivalent interval for each triangular number of Table 16 and performed the models once again. The Spearman correlation coefficient between the ranking results of using trapezoidal fuzzy numbers (TrFN) and triangular fuzzy numbers is 0.925. This high correlation indicates that the results are not very sensitive to how to convert the linguistic variables into fuzzy numbers. Fig. 12 illustrates the high Spearman correlation coefficient between two different measures.

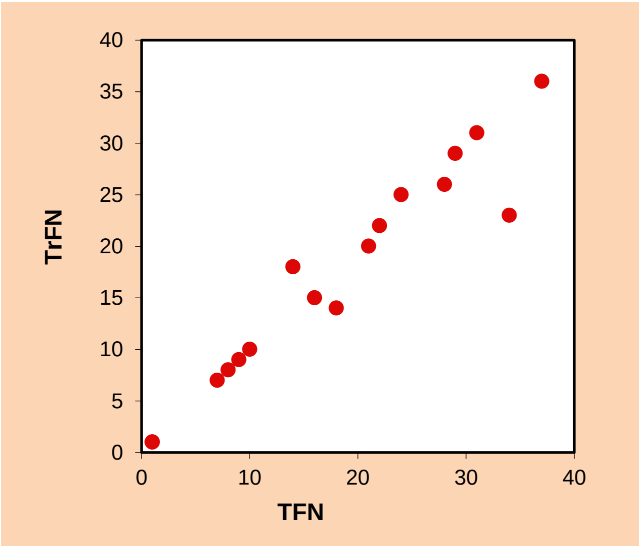


Fig. 12: Spearman correlation coefficient

8. Conclusions

Due to its widely practical used background, DEA has become a pop area of research. To deal with imprecise data, we incorporate fuzzy, ordinal and interval data to quantify imprecise and vague data in DEA models. In this paper, we reformulated the conventional DEA models as an imprecise DEA problem, and proposed a novel method for evaluating the DMUs when the inputs and outputs are fuzzy and/or ordinal, or vary in intervals. For this purpose, we converted all data into interval data. In order to convert each fuzzy number into interval data we used the nearest weighted interval approximation of fuzzy numbers by applying the weighting function, and we converted each ordinal data into interval one. In this manner, we could convert all data into interval data.

After that, we proposed an algorithm for ranking DMUs when we deal with these kinds of data. Finally, we use two numerical examples to illustrate the proposed DEA algorithm and the ranking method.

Also, a case study was provided to demonstrate the efficacy of our proposed method. We tested our proposed method in the real-world by assessing 38 hospitals that were selected by OIG. Therefore, the proposed method was applied to evaluate and rank these hospitals. We used four inputs and three outputs for evaluating the hospitals. Five hospitals became efficient, and by further analysis in the second stage of our proposed algorithm, they were completely ranked. Table 15 and Figures 9 and 10 showed that hospital 19, is the best.

In this paper, we focused on the CCR model, which is based on the constant returns to scale technology. For future works, one can apply variable returns to scale technology on our proposed method and present a new ranking approach based on the BCC model. Also, we can consider the slacks-based measure (SBM) model on our presented method to obtain a new ranking methodology for assessing DMUs. Also, one can extend our presented methodology by considering undesirable data and stochastic data.

References

Agarwal, S. (2014). Efficiency Measure by Fuzzy Data Envelopment Analysis Model. *Fuzzy Information and Engineering*, *6*(1), 59–70.

Akram, M., & Adeel, A. (2016). $$\varvec{m}$$m-Polar Fuzzy Labeling Graphs with Application. *Mathematics in Computer Science*, *10*(3), 387–402.

Akram, M., Adeel, A., & Alcantud, J. C. R. (2019). Group decision-making methods based on hesitant N-soft sets. *Expert Systems with Applications*, *115*, 95–105.

Akram, M., & Arshad, M. (2019). A Novel Trapezoidal Bipolar Fuzzy TOPSIS Method for Group Decision-Making. *Group Decision and Negotiation*, *28*(3), 565–584.

Aliakbarpoor, Z., & Izadikhah, M. (2012). Evaluation and ranking DMUs in the presence of both undesirable and ordinal factors in data envelopment analysis. *International Journal of Automation and Computing*, *9*(6).

Allahviranloo, T., Hosseinzade Lotfi, F., & Adabitabar Firozja, M. (2007). Fuzzy Efficiency Measure with Fuzzy Production Possibility Set. *Applications and Applied Mathematics (AAM): An International Journal*, *2*(2), 152 – 166.

Aparicio, J., Cordero, J. M., & Ortiz, L. (2019). Measuring efficiency in education: The influence of imprecision and variability in data on DEA estimates. *Socio-Economic Planning Sciences*, *68*, 100698.

Barak, S., & Dahooei, J. H. (2018). A novel hybrid fuzzy DEA-Fuzzy MADM method for airlines safety evaluation. *Journal of Air Transport Management*, *73*, 134–149.

Charnes, A., Cooper, W. W., Lewin, A. Y., & Seiford, L. M. (Eds. . (1994). *Data Envelopment Analysis: Theory, Methodology, and Applications*. Springer Netherlands.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, *2*(6), 429–444.

Chen, S.-M., & Lee, L.-W. (2010). Fuzzy multiple attributes group decision-making based on the interval type-2 TOPSIS method. *Expert Systems with Applications*, *37*(4), 2790–2798.

Chen, Y.-C., Chiu, Y.-H., Huang, C.-W., & Tu, C. H. (2013). The analysis of bank business performance and market risk—Applying Fuzzy DEA. *Economic Modelling*, *32*(0), 225–232.

Cook, W. D., Kress, M., & Seiford, L. M. (1993). On the Use of Ordinal Data in Data Envelopment Analysis. *The Journal of the Operational Research Society*, *44*(2), 133–140.

Cooper, W. W., Park, K. S., & Yu, G. (1999). IDEA and AR-IDEA: Models for Dealing with Imprecise Data in DEA. *Management Science*, *45*(4), 597–607.

Cooper, W. W., Park, K. S., & Yu, G. (2001). IDEA (Imprecise Data Envelopment Analysis) with CMDs (Column

Maximum Decision Making Units). *The Journal of the Operational Research Society*, *52*(2), 176–181.

Despotis, D. K., & Smirlis, Y. G. (2002). Data envelopment analysis with imprecise data. *European Journal of Operational Research*, *140*(1), 24–36.

Ebrahimi, B., Dellnitz, A., Kleine, A., & Tavana, M. (2021). A novel method for solving data envelopment analysis problems with weak ordinal data using robust measures. *Expert Systems with Applications*, *164*, 113835.

Emrouznejad, A, & Yang, G. L. (2018). A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *Socio-Economic Planning Sciences*, *61*, 4–8.

Emrouznejad, Ali, Rostamy-Malkhalifeh, M., Hatami-Marbini, A., Tavana, M., & Aghayi, N. (2011). An overall profit Malmquist productivity index with fuzzy and interval data. *Mathematical and Computer Modelling*, *54*(11–12), 2827–2838.

Ertuğrul, İ., & Güneş, M. (2007). Fuzzy Multi-criteria Decision Making Method for Machine Selection. In P. W. Melin P., Castillo O., Ramírez E.G., Kacprzyk J. (Ed.), *Analysis and Design of Intelligent Systems using Soft Computing Techniques. Advances in Soft Computing* (p. Vol. 41). Springer, Berlin, Heidelberg.

Fathi, N., & Izadikhah, M. (2013). Evaluation Decision Units in The Presence of Ordinal and Fuzzy Data. *Journal of Basic and Applied Scientific Research*, *3*(3), 1177–1184.

Goker, N., & Karsak, E. E. (2020). Two-stage common weight DEA-Based approach for performance evaluation with imprecise data. *Socio-Economic Planning Sciences*, 100943.

Guo, P., & Tanaka, H. (2001). Fuzzy DEA: a perceptual evaluation method. *Fuzzy Sets and Systems*, *119*(1), 149–160.

Hatami-Marbini, A., Emrouznejad, A., & Agrell, P. J. (2014). Interval data without sign restrictions in DEA. *Applied Mathematical Modelling*, *38*(7–8), 2028–2036.

Heydari, C., Omrani, H., & Taghizadeh, R. (2020). A fully fuzzy network DEA-Range Adjusted Measure model for evaluating airlines efficiency: A case of Iran. *Journal of Air Transport Management*, *89*, 101923.

Izadikhah, M., & Khoshroo, A. (2018). Energy management in crop production using a novel Fuzzy Data Envelopment Analysis model. *RAIRO - Operations Research*, *52*(2), 595–617.

Izadikhah, M., Saeidifar, A., & Roostaee, R. (2014). Extending TOPSIS in fuzzy environment by using the nearest weighted interval approximation of fuzzy numbers. *Journal of Intelligent & Fuzzy Systems*, *27*, 2725–2736.

Jahanshahloo, G. R., Soleimani-damaneh, M., & Nasrabadi, E. (2004). Measure of efficiency in DEA with fuzzy input–output levels: a methodology for assessing, ranking and imposing of weights restrictions. *Applied Mathematics and Computation*, *156*(1), 175–187.

Kahraman, C., & Tolga, E. (1998). Data envelopment analysis using fuzzy concept. In *Multiple-Valued Logic, 1998. Proceedings. 1998 28th IEEE International Symposium on* (pp. 338–343).

Kao, C. (2006). Interval efficiency measures in data envelopment analysis with imprecise data. *European Journal of Operational Research*, *174*(2), 1087–1099.

Kao, C., & Lin, P.-H. (2011). Qualitative factors in data envelopment analysis: A fuzzy number approach. *European Journal of Operational Research*, *211*(3), 586–593.

Kao, C., & Liu, S.-T. (2000). Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Systems*, *113*(3), 427–437.

Lertworasirikul, S., Fang, S.-C., A. Joines, J., & L.W. Nuttle, H. (2003). Fuzzy data envelopment analysis (DEA): a possibility approach. *Fuzzy Sets and Systems*, *139*(2), 379–394.

Meada, Y., Entani, T., & Tanaka, H. (1998). Fuzzy DEA with interval efficiency. *Proceedings of 6th European Congress on Intelligent Techniques and Soft Computing. EUFIT '98*. Aachen, Germany, Verlag Mainz.

Peykani, P., Mohammadi, E., & Emrouznejad, A. (2021). An adjustable fuzzy chance-constrained network DEA approach with application to ranking investment firms. *Expert Systems with Applications*, *166*, 113938.

Ploskas, N., Papathanasiou, J., & Tsaples, G. (2017). *Implementation of an Extended Fuzzy VIKOR Method Based on Triangular and Trapezoidal Fuzzy Linguistic Variables and Alternative Deffuzification Techniques*. Springer International Publishing.

Saati, S. M., Memariani, A., & Jahanshahloo, G. R. (2002). Efficiency Analysis and Ranking of DMUs with Fuzzy Data. *Fuzzy Optimization and Decision Making*, *1*(3), 255–267.

Saeidifar, A. (2011). Application of weighting functions to the ranking of fuzzy numbers. *Computers & Mathematics with Applications*, *62*(5), 2246–2258.

Sengupta, J. K. (1992). A fuzzy systems approach in data envelopment analysis. *Computers & Mathematics with Applications*, *24*(8–9), 259–266.

Shokouhi, A. H., Hatami-Marbini, A., Tavana, M., & Saati, S. (2010). A robust optimization approach for imprecise data envelopment analysis. *Computers & Industrial Engineering*, *59*(3), 387–397.

Tavana, M., Hatami-Marbini, A., & Emrouznejad, A. (2012). Productivity Growth and Efficiency Measurements in Fuzzy Environments with an Application to Health Care. *Int. J. Fuzzy Syst. Appl.*, *2*(2), 1–35.

Toloo, M., Keshavarz, E., & Hatami-Marbini, A. (2018). Dual-role factors for imprecise data envelopment analysis. *Omega (United Kingdom)*, *77*, 15–31.

Toloo, M., & Nalchigar, S. (2011). A new DEA method for supplier selection in presence of both cardinal and ordinal data. *Expert Systems with Applications*, *38*(12), 14726–14731.

Wang, T.-C., & Chen, Y.-H. (2008). Applying fuzzy linguistic preference relations to the improvement of consistency of fuzzy AHP. *Information Sciences*, *178*(19), 3755–3765.

Wang, Y.-M., Greatbanks, R., & Yang, J.-B. (2005). Interval efficiency assessment using data envelopment analysis. *Fuzzy Sets and Systems*, *153*(3), 347–370.

Wang, Y.-M., & Yang, J.-B. (2007). Measuring the performances of decision-making units using interval efficiencies. *Journal of Computational and Applied Mathematics*, *198*(1), 253–267.

Wen, M., You, C., & Kang, R. (2010). A new ranking method to fuzzy data envelopment analysis. *Comput. Math. Appl.*, *59*(11), 3398–3404.

Wu, D., & Lee, C.-G. (2010). Stochastic DEA with ordinal data applied to a multi-attribute pricing problem. *European Journal of Operational Research*, *207*(3), 1679–1688.

Xiao, Z., Xia, S., Gong, K., & Li, D. (2012). The trapezoidal fuzzy soft set and its application in MCDM. *Applied Mathematical Modelling*, *36*(12), 5844–5855.

Zadeh, L. A. (1975). The Concept of a Linguistic Variable and its Application to Approximate Reasoning-I. *Information Sciences*, *8*, 199–249.

Zerafat Angiz L., M., Emrouznejad, A., & Mustafa, A. (2010). Fuzzy assessment of performance of a decision making units using DEA: A non-radial approach. *Expert Systems with Applications*, *37*(7), 5153–5157.

Zhang, Y., Li, K. W., & Wang, Z.-J. (2017). Prioritization and Aggregation of Intuitionistic Preference Relations: A Multiplicative-Transitivity-Based Transformation from Intuitionistic Judgment Data to Priority Weights. *Group Decision and Negotiation*, *26*(2), 409–436.

Zhou, X., Wang, Y., Chai, J., Wang, L., Wang, S., & Lev, B. (2019). Sustainable supply chain evaluation: A dynamic double frontier network DEA model with interval type-2 fuzzy data. *Information Sciences*, *504*, 394–421.

Zhou, Z., Zhao, L., Lui, S., & Ma, C. (2012). A generalized fuzzy DEA/AR performance assessment model. *Mathematical and Computer Modelling*, *55*(11–12), 2117–2128.

Zhu, J. (2003). Efficiency evaluation with strong ordinal input and output measures. *European Journal of Operational Research*, *146*(3), 477–485.