

A Tracking Error Based Fully Probabilistic Control for Stochastic Discrete Time Systems with Multiplicative Noise

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I. ABSTRACT

This paper proposes the exploitation of the Kullback-Leibler divergence to characterise the uncertainty of the tracking error for general stochastic systems without constraints of certain distributions. The general solution to the fully probabilistic design (FPD) of tracking error control problem is firstly stated. Further development then focuses on the derivation of a randomised controller for a class of linear stochastic Gaussian systems that is affected by multiplicative noise. The derived control solution takes the multiplicative noise of the controlled system into consideration in the derivation of the randomised controller. The proposed FPD of the tracking error of the system dynamics is a more legitimate approach compared to the conventional FPD method. It directly characterises the main objective of systems control. The efficiency of the proposed method is then demonstrated on a flexible beam example where the vibration quenching in flexible beams is shown to be effectively suppressed.

II. INTRODUCTION

In control systems, the tracking error between the system output and a predefined desired output is the most commonly used optimisation signal for the tuning of the parameters of the system controller [Gaudio et al. 2019], [Gerasimov et al. 2019], [Humaidi and Hameed, 2019], [Wu and Du, 2019], [Zhou et al. 2020], [Zhou et al. 2017]. When accompanied with adaptive control [Narendra and Annaswamy, 2005], [Tao, 2003], [Chen and Jiao, 2010], the approach has been particularly proven useful to control systems that are affected by models uncertainty, random noises, and that are operating under changing environment and have unforeseen variations in their overall structure. Despite being adaptive and therefore are expected to deal with the underlying systems uncertainty, many of the aforementioned methods are based on the minimisation of the mean square tracking error to optimise the controller parameters. The minimisation of the mean square tracking error, also known as tracking error variance, on the other hand is based on the assumption of certainty equivalence, therefore, it does not generally yield a good performance. Thus, for more general stochastic systems and for systems with functional and models uncertainty,

the variance of the tracking error cannot be used alone to represent the performance of the closed-loop system [Yue and Wang, 2003], [Herzallah, 2007], [Herzallah and Lowe, 2004], [Herzallah and Lowe, 2006], [Herzallah and Lowe, 2003], [Zhang et al. 2016]. As a result, the Kullback-Leibler divergence [Yu, 2009], [Cliff et al. 2018], [Kulback, 1959] measure has been proposed recently in several control literatures to characterise the uncertainty of the stochastic systems dynamics. This is because the Kullback-Leibler divergence measures the discrepancies between the stochastic system distributions to their desired distributions rather than characterises them by their means or variances.

An efficient control approach, known as FPD, that uses the Kullback-Leibler divergence as a performance measure for designing randomised controllers has been proposed in [Karny, 1996], [Herzallah, 2011]. In this approach, the Kullback-Leibler divergence is used to measure the discrepancy between the joint pdf of the closed loop description of the systems dynamics and an ideal joint pdf. The main advantage of the FPD control approach is that it provides a closed form solution for general description of stochastic systems without constraints of certain distribution. However, although a closed form solution can be obtained, the solution cannot be evaluated analytically due to the multivariate integration involved in the optimisation process. Besides, in its original form the FPD control method considers the design of a randomised controller that shapes the pdf of the system dynamics. Nonetheless, the characterisation of the pdf of the system dynamics can be difficult for many real world systems that work under high levels of uncertainty and stochasticity. Furthermore, in many real engineering systems the controller objective is to make the output of the system dynamics follow a predefined desired output value, thus emphasising the importance of the tracking error rather than the actual system output.

As such, this paper follows an alternative approach where the Kullback-Leibler divergence is defined to be the distance between the pdf of the joint distribution of the tracking error and the randomised controller of the controlled system to an ideal joint distribution function. Therefore, the randomised controller is designed here to reshape the pdf of the tracking error of the controlled system rather than the pdf of its dynamics. Compared with the existing results on the topic and the conventional approach of FPD, this alternative approach has several advantages that have not been reported in the literature. Firstly, the characterisation of the pdf of tracking error of the controlled system is normally easier than the characterisation of the pdf of its dynamics. This is because

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when the stochastic dynamics of the controlled system are estimated accurately, the resulting tracking error of the system will be small and most likely can be characterised by a Gaussian pdf. The aforementioned in turn simplifies the optimisation of the sought randomised controller. Secondly, the ideal distribution of the tracking error can be naturally specified by a zero mean distribution. In particular a Gaussian distribution with zero mean and a pre-specified covariance matrix that determines the allowed fluctuations of the tracking error around its zero mean value would be ideal. Furthermore, the FPD method in its original form considers additive noise only to the system dynamics. Our alternative solution considers stochastic systems with multiplicative noises which represent conditions under which most real world systems operate. Therefore, an additional contribution of the paper is the consideration of the multiplicative noise of the stochastic system in the derivation of the randomised optimal control law. Moreover, the proposed probabilistic minimisation of the tracking error will be shown to be particularly useful for solving the vibration control problem associated with mechanical systems. The vibration control problem is particularly challenging and is relevant to many real world control problems including robotic manipulators, aerospace structures, and biomechanical systems [Simone et al. 2018], [Pappalardo et al. 2016], [Flores and Barbieri, 2006], [Sohn et al. 2009], [Song and Gu, 2007].

To re-emphasise, this alternative solution of the tracking error and the extension of the FPD to stochastic systems with multiplicative noises have not been discussed previously in the literature. Its theoretical development and numerical demonstration will be presented for the first time in this paper.

III. PROBLEM STATEMENT

In the original formulation of the Fully Probabilistic Design, the aim is to derive a randomised controller that shapes the joint probability density function of the stochastic system dynamics and the controller. This joint probability density function of the controller and the dynamics of the stochastic system represents the complete description of the closed loop behaviour of the controlled system. However, in some control applications the system is required to track a predefined desired trajectory. Thus for these control applications, it would be more convenient to design the controller such that it reshapes the pdf of the tracking error as opposed to the original formulation of reshaping the pdf of the system dynamics. For the system to be able to track the desired signal, the controller should be designed such that the pdf of the tracking error is centred around zero with small variations. This objective of achieving a narrow distribution of the tracking error centred around zero error state, implies that the system has tracked the desired trajectory and at the same time indicates that the uncertainty in the tracked trajectory is small. To be more specific, assume that the stochastic system can be described at each time instant, k by the following conditional pdf,

$$s(x_k|x_{k-1}, u_{k-1}), \quad (1)$$

where $x_k \in \mathfrak{R}^n$ is the system state, and $u_k \in \mathfrak{R}^m$ is the system input. Define the reference state that the system will be required to track as $x_r \in \mathfrak{R}^n$, then the system tracking error should be given by,

$$e_k = x_k - x_r. \quad (2)$$

Because the considered system in this paper is stochastic and subject to random forces and functional uncertainties, only the probability density function of the state values defined in (1) can be specified. On the other hand, since the objective in this paper is to design a randomised controller that shapes the pdf of the tracking error as a result of the requirements that the system state tracks a desired set point, the pdf of the tracking error needs to be assumed to be known which may be an unrealistic assumption for many real world control problems. However, the density function of the tracking error can be obtained from the density function of the system dynamics using probability theory as follows,

$$s_e(x_k, x_r) = s(e_k + x_r | e_{k-1} + x_r, u_{k-1}). \quad (3)$$

In general $s(x_k|\cdot)$ is not known in reality, thus needs to be estimated online utilising the observed data of the controlled system. The estimation process of this pdf will be explained in Section IV-B.

Once the pdf of the tracking error is estimated, the randomised controller can be derived by redefining the Kulback Leibler divergence such that the discrepancy between the joint pdf of the tracking error and the controller and a predefined ideal joint pdf is minimised,

$$\mathcal{D}(f||f^I) = \int f(\mathbb{D}) \ln \left(\frac{f(\mathbb{D})}{f^I(\mathbb{D})} \right) d(\mathbb{D}), \quad (4)$$

where $f(\mathbb{D}) = \prod_{k=1}^H s(e_k | e_{k-1}, u_{k-1}) c(u_{k-1} | e_{k-1})$, $f^I(\mathbb{D}) = \prod_{k=1}^H s^I(e_k | e_{k-1}, u_{k-1}) c^I(u_{k-1} | e_{k-1})$, $\mathbb{D} = (e_0, \dots, e_H, u_0, \dots, u_{H-1})$, and H is the control horizon. Following the same approach of the original FPD, the minimisation of the Kullback Leibler divergence defined in (4) can be achieved by recursively solving the backward recurrence equation that is given in the following proposition.

Proposition 1: The optimal randomised controller $c(u_{k-1}|e_{k-1})$ can be obtained by recursively solving the following recurrence equation [Herzallah, 2011],

$$\begin{aligned} -\ln(\gamma(e_{k-1})) &= \min_{c(u_{k-1}|e_{k-1})} \int s(e_k | u_{k-1}, e_{k-1}) c(u_{k-1} | e_{k-1}) \\ &\times \left[\underbrace{\ln \left(\frac{s(e_k | u_{k-1}, e_{k-1}) c(u_{k-1} | e_{k-1})}{I s(e_k | u_{k-1}, e_{k-1})^I c(u_{k-1} | e_{k-1})^I} \right)}_{\text{=partial cost} \implies U(e_k, u_{k-1})} \right. \\ &\left. - \underbrace{\ln(\gamma(e_k))}_{\text{optimal cost-to-go}} \right] d(e_k, u_{k-1}). \end{aligned} \quad (5)$$

Proof: The derivation of the above result can be found in [Herzallah, 2011].

The optimal randomised controller that minimises the recurrence equation specified in (5) can then be shown to

be given as specified in the following proposition.

Proposition 2: The pdf of the optimal randomised controller that minimises the cost to go function (5) is given by,

$$c(u_{k-1} | e_{k-1}) = \frac{c^I(u_{k-1} | e_{k-1}) \exp[-\beta_1(u_{k-1}, e_{k-1}) - \beta_2(u_{k-1}, e_{k-1})]}{\gamma(e_{k-1})}, \quad (6)$$

where

$$\begin{aligned} \gamma(e_{k-1}) &= \int c^I(u_{k-1} | e_{k-1}) \exp[-\beta_1(u_{k-1}, e_{k-1}) \\ &\quad - \beta_2(u_{k-1}, e_{k-1})] du_{k-1} \\ \beta_1(u_{k-1}, e_{k-1}) &= \int s(e_k | u_{k-1}, e_{k-1}) \left[\ln \frac{s(e_k | u_{k-1}, e_{k-1})}{s^I(e_k | u_{k-1}, e_{k-1})} \right] de_k \\ \beta_2(u_{k-1}, e_{k-1}) &= - \int s(e_k | u_{k-1}, e_{k-1}) \ln(\gamma(e_k)) de_k. \end{aligned} \quad (7)$$

Proof: This proposition can be proven by adapting the proof of Proposition 2 in [Karny, 2006].

Note that the solution of the optimal randomised controller as specified in this proposition is not restricted by the pdf that characterises the error or the controller. It provides the general solution for the randomised controller without constraints on the required pdfs. However, the evaluation of the analytic solution for this randomized controller is not possible except for the special case of linear and Gaussian pdfs. Therefore, to facilitate the understanding and the analytical solution of the proposed tracking error based FPD, the next section will demonstrate the solution to the probabilistic tracking control for a class of linear stochastic systems with multiplicative noise.

IV. SOLUTION OF THE PROBABILISTIC TRACKING CONTROL FOR LINEAR STOCHASTIC SYSTEMS

The theory developed in the previous section will be applied here to derive the analytic solution of the probabilistic tracking control for linear stochastic systems with multiplicative noise. Stochastic systems with multiplicative noises arise naturally in networked control systems where multiplicative noises are used to model packet loss. Previous works have considered this class of stochastic systems where the multiplicative noise is used to model packet loss [Wei et al. 2013] and time delay [Zhang et al. 2015] that happens during packet transmission in communication networks. This is different to parameters uncertainty [Liu et al. 2010], [Lee et al. 2001], [Xie et al. 1992] where the uncertainty of the parameters are usually grouped with the parameters of the state and can be considered stochastic or deterministic. The development of a robust control solution for these systems has been a long standing and still unsolved problem.

A. Model Description

Consider a stochastic linear discrete time system with multiplicative Gaussian noise described by,

$$x_k = \tilde{A}x_{k-1} + \tilde{B}u_{k-1} + \tilde{D}x_{k-1}v_{k-1}, \quad (8)$$

where $x_k \in \mathfrak{X}^n$ is the system state, and $u_k \in \mathfrak{X}^m$ is the system input as defined before, A, B, D are system matrices with appropriate dimensions, and $v_k \in \mathfrak{X}$ is an independent Gaussian noise with zero mean and covariance Q .

It should be noted that in real world situations the parameters of the stochastic model (8) are not known in general, thus need to be estimated. However, since the current value of the system state is affected by noise, its value cannot be completely specified by the previous control and previous state values. Therefore, the probabilistic description of the stochastic model (8) needs to be estimated on-line using observed data from the stochastic system dynamics to describe the probabilistic evolution of the system state. The on-line estimation process of the stochastic system parameters and consequently the system state distribution will be discussed next.

B. Estimation of the Probabilistic Description of the System Tracking Error

As discussed in the previous section, due to the stochastic nature of the system dynamics, only the probabilistic description of the system state can be specified. This can be obtained by estimating the system parameters of the stochastic equation of the system state given in (8). Therefore, given our prior knowledge of the linear dynamics of the system and the fact that it is driven by multiplicative noise, the required model of the system equation (8) can be assumed to have the following form,

$$x_k = Ax_{k-1} + Bu_{k-1} + Dx_{k-1}v_{k-1}, \quad (9)$$

where A, B and D are the estimates of the matrices \tilde{A}, \tilde{B} and \tilde{D} respectively. Then these parameters can be estimated on-line by updating their values at each time instant, k when a new measurement of the state value becomes available. In particular, rewrite (9) as follows,

$$\begin{aligned} x_k &= \begin{bmatrix} A & B & D \end{bmatrix} \begin{bmatrix} x_{k-1} \\ u_{k-1} \\ x_{k-1}v_{k-1} \end{bmatrix}, \\ &= \vartheta \chi_{k-1}, \end{aligned} \quad (10)$$

where $\vartheta = \begin{bmatrix} A & B & D \end{bmatrix}$, and $\chi_{k-1} = \begin{bmatrix} x_{k-1} & u_{k-1} & x_{k-1}v_{k-1} \end{bmatrix}^T$. Here, χ_{k-1} has dimension $(2n + m) \times 1$, and ϑ has dimension $n \times (2n + m)$, where n and m are the dimensionality of the state vector and control input respectively as stated earlier. Then given a new observation of the system state x_k the parameter vector, ϑ can be estimated. Since the matrix χ_{k-1} is not a square matrix, the estimation of the parameter vector can be achieved by first multiplying both sides of (10) by χ_{k-1}^T and then solving for the parameter vector ϑ ,

$$x_k \chi_{k-1}^\dagger = \vartheta, \quad (11)$$

where χ_{k-1}^\dagger is an $1 \times (2n + m)$ matrix known as the pseudo-inverse of χ_{k-1} and is given by,

$$\chi_{k-1}^\dagger = \chi_{k-1}^T (\chi_{k-1} \chi_{k-1}^T)^{-1}. \quad (12)$$

Remark 1: As can be seen from Equation (12), the pseudo inverse matrix does have the property that $\chi_{k-1}\chi_{k-1}^\dagger = I$ where I is the identity matrix. However, note that $\chi_{k-1}\chi_{k-1}^\dagger \neq I$ in general. If the matrix $\chi_{k-1}\chi_{k-1}^T$ is singular then Equation (11) does not have a unique solution. In this case, if the pseudo inverse is defined as,

$$\chi_{k-1}^\dagger = \lim_{\iota \rightarrow 0} \chi_{k-1}^T (\chi_{k-1}\chi_{k-1}^T + \iota I)^{-1}, \quad (13)$$

then, the limit can be shown to be always exist and that the limiting value guarantees the optimal solution of Equation (11).

Following the estimation of these parameters, the conditional distribution of the system state is shown to be Gaussian described by,

$$s(x_k|x_{k-1}, u_{k-1}) \sim N(Ax_{k-1} + Bu_{k-1}, Dx_{k-1}Qx_{k-1}^T D^T), \quad (14)$$

where $Ax_{k-1} + Bu_{k-1}$ is the mean of the state calculated using the estimated parameters A and B , and $Dx_{k-1}Qx_{k-1}^T D^T$ is the covariance of the state calculated using the estimated parameter D .

For the objective of deriving a randomised controller that will achieve a narrow tracking error distribution centred around zero, thus guaranteeing an accurate tracking of the system state to the desired value, the tracking error distribution needs to be specified. This can be obtained from the definition of the tracking error given in (2).

The dynamical description of the tracking error can then be obtained by substituting (9) into (2), which yields,

$$\begin{aligned} e_k &= Ax_{k-1} + Bu_{k-1} + Dx_{k-1}v_{k-1} - x_r, \\ &= Ae_{k-1} + Bu_{k-1} + Dx_{k-1}v_{k-1} + Fx_r, \end{aligned} \quad (15)$$

where we have introduced the definition $F = A - I$. From Equations (3), (14), and (15), the distribution of the tracking error is Gaussian with mean μ_k and covariance Σ_k specified as follows,

$$s(e_k|u_{k-1}, e_{k-1}) \sim N(\mu_k, \Sigma_k) \quad (16)$$

where

$$\mu_k = Ae_{k-1} + Bu_{k-1} + Fx_r \quad (17)$$

$$\begin{aligned} \Sigma_k &= \text{cov}(e_k|u_{k-1}, e_{k-1}) \\ &= E\{(e_k - \mu_k)(e_k - \mu_k)^T\} \\ &= Dx_{k-1}Qx_{k-1}^T D^T. \end{aligned} \quad (18)$$

C. Randomised Control Solution

In this section the generalised fully probabilistic control solution of the tracking problem for the stochastic linear system with multiplicative noise defined in (8) is derived. As discussed in earlier sections, the pdf of the system tracking error is assumed to be unknown, thus estimated on-line as explained in Section IV-B. The purpose of the designed controller here is to make the pdf of the tracking error $s(e_k|u_{k-1}, e_{k-1})$ follow a predefined ideal pdf $s^I(e_k|u_{k-1}, e_{k-1})$ and bring the tracking error to zero. Thus,

the ideal distribution of the system tracking error described by (16) is specified as,

$$s^I(e_k|u_{k-1}, e_{k-1}) \sim N(0, \Sigma_2), \quad (19)$$

where Σ_2 specifies the allowed fluctuations of the tracking error around its zero mean value. In addition, the ideal distribution of the sought randomised controller, $c(u_{k-1}|e_{k-1})$ is taken to be Gaussian with the following form,

$$c^I(u_{k-1}|e_{k-1}) \sim N(\mu_u, \Gamma), \quad (20)$$

where Γ is the covariance matrix of the ideal distribution of the control input and μ_u is the mean of the ideal distribution of the control input. To achieve the objective that the optimised randomised controller brings the tracking error between the system state and its desired value to zero, the mean value of the ideal distribution of the controller, μ_u is calculated from (15) to be,

$$\begin{aligned} \lim_{k \rightarrow \infty} [E\{e_k\}] &= \lim_{k \rightarrow \infty} [E\{Ae_{k-1}\} + E\{Bu_{k-1}\} + E\{Dx_{k-1}v_{k-1}\} \\ &\quad + Fx_r], \\ 0 &= 0 + \lim_{k \rightarrow \infty} [E\{Bu_{k-1}\} + Fx_r], \\ \lim_{k \rightarrow \infty} [E\{u_{k-1}\}] &= \mu_u = -(B^T B)^{-1} B^T Fx_r. \end{aligned} \quad (21)$$

Given the pdf of the tracking error defined in (16) and the ideal pdfs of the tracking error and controller defined in (19) and (20) respectively, the performance index for the class of linear stochastic systems defined in (9) can then be shown to be given by the following theorem.

Theorem 1: Using the pdf description of the tracking error dynamics specified by (16), the ideal distribution of the tracking error dynamics given by (19) and the ideal distribution of the controller given by (20) in (6) and (7) gives the following performance index,

$$-\ln(\gamma(e_k)) = 0.5(e_k^T S_k e_k + P_k e_k + w_k), \quad (22)$$

where,

$$S_{k-1} = -A^T M_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k^T A + M_2 + A^T M_k A, \quad (23)$$

$$\begin{aligned} P_{k-1} &= 2x_r^T (M_2 + F^T M_k A) + P_k A \\ &\quad + 2\mu_u^T \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k^T A \\ &\quad - 2x_r^T F^T M_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k^T A \\ &\quad - P_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k^T A, \end{aligned} \quad (24)$$

$$\begin{aligned} w_{k-1} &= x_r^T (M_2 + F^T M_k F - F^T M_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k^T F) \\ &\quad \times x_r + w_k + P_k F x_r + \mu_u^T (\Gamma^{-1} - \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} \Gamma^{-T}) \\ &\quad \times \mu_u - 0.25 P_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T P_k^T \\ &\quad + 2\mu_u^T \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k^T F x_r \\ &\quad - P_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k^T F x_r \\ &\quad + \mu_u^T \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} B^T P_k^T - \ln(\pi) \\ &\quad + \ln((\Gamma^{-1} + B^T M_k B)^{-1}), \end{aligned} \quad (25)$$

and where,

$$\begin{aligned} M_k &= \Sigma_2^{-1} + S_k, \\ M_2 &= D^T S_k Q D. \end{aligned} \quad (26)$$

Proof: The claimed quadratic form of the optimal performance function specified in (22) can be verified subsequently by backward induction. The proof starts by evaluating γ in (7), repeated here,

$$\begin{aligned} \gamma(e_{k-1}) &= \int c^J(u_{k-1} | e_{k-1}) \exp[-\beta_1(u_{k-1}, e_{k-1}) \\ &\quad - \beta_2(u_{k-1}, e_{k-1})] du_{k-1}. \end{aligned} \quad (27)$$

This evaluation requires the evaluation of β_1 and β_2 . Starting with β_1 ,

$$\begin{aligned} \beta_1(u_{k-1}, e_{k-1}) &= \int s(e_k | u_{k-1}, e_{k-1}) \ln \frac{s(e_k | u_{k-1}, e_{k-1})}{s^J(e_k | u_{k-1}, e_{k-1})} de_k \\ &= \int N(\mu_k, \Sigma_k) \left(-0.5 \ln(|\Sigma_k| |\Sigma_2|^{-1}) - 0.5(e_k - \mu_k)^T (\Sigma_k)^{-1} \right. \\ &\quad \left. \times (e_k - \mu_k) + 0.5 e_k^T (\Sigma_2)^{-1} e_k \right) de_k. \end{aligned} \quad (28)$$

To solve (28), the following rule from [Golub and Meurant, 2009] is required,

$$\ln(\det(A_1)) = \text{tr}(\ln(A_1)), \quad (29)$$

where A_1 should be a positive definite matrix. Since $(|\Sigma_k| |\Sigma_2|^{-1})$ is positive definite, the $\ln(|\Sigma_k| |\Sigma_2|^{-1})$ term in (28) can be rewritten as,

$$\ln(|\Sigma_k| |\Sigma_2|^{-1}) = \ln(|\Sigma_k \Sigma_2^{-1}|) = \text{tr}(\ln(\Sigma_k \Sigma_2^{-1})). \quad (30)$$

Assumption 1: Since the objective of the sought randomised optimal controller is to make the distribution of the tracking error of the system dynamics as close as possible to the specified ideal distribution, it is expected that at steady state the covariance of the tracking error dynamics will become close to the covariance of the specified ideal distribution. This means that,

$$\|\Sigma_k \Sigma_2^{-1} - I\| < 1. \quad (31)$$

Remark 2: Please note that the covariance of the noise, Q affecting the system will not be too large in real world systems. This in turns means that $\Sigma_k = D x_{k-1} Q x_{k-1}^T D^T$ will not be too large as well. Therefore the above assumption is a valid assumption. This will be proven numerically in the numerical results section, Section V.

Based on Assumption 1 and lemma 2.6 from [Hall, 2015], (30) can be approximated as follows,

$$\text{tr}(\ln(\Sigma_k \Sigma_2^{-1})) \approx \text{tr}(\Sigma_k \Sigma_2^{-1} - I) \approx \text{tr}(\Sigma_k \Sigma_2^{-1}) - n, \quad (32)$$

where n is the dimension of e_k .

Using (32) in (28) and expanding the terms of (28) we get,

$$\begin{aligned} \beta_1(u_{k-1}, e_{k-1}) &= \int N(\mu_k, \Sigma_k) \left(-0.5 \text{tr}(\Sigma_k \Sigma_2^{-1}) + 0.5n \right. \\ &\quad \left. + 0.5 e_k^T (\Sigma_2^{-1} - \Sigma_k^{-1}) e_k - 0.5 \mu_k^T \Sigma_k^{-1} \mu_k + e_k^T \Sigma_k^{-1} \mu_k \right) de_k, \\ &= 0.5 \mu_k^T \Sigma_k^{-1} \mu_k - 0.5 \text{tr}(\Sigma_k \Sigma_2^{-1}) + 0.5n + 0.5 \int N(\mu_k, \Sigma_k) e_k^T \\ &\quad \times (\Sigma_2^{-1} - \Sigma_k^{-1}) e_k de_k. \end{aligned} \quad (33)$$

The last part in (33), $0.5 \int N(\mu_k, \Sigma_k) e_k^T (\Sigma_2^{-1} - \Sigma_k^{-1}) e_k de_k$ can be evaluated as follows,

$$\begin{aligned} &0.5 \int N(\mu_k, \Sigma_k) e_k^T (\Sigma_2^{-1} - \Sigma_k^{-1}) e_k de_k \\ &= 0.5 (\text{tr}[\Sigma_2^{-1} \Sigma_k] - n) + 0.5 \mu_k^T (\Sigma_2^{-1} - \Sigma_k^{-1}) \mu_k. \end{aligned} \quad (34)$$

Substitute (34) back into (33), we obtain,

$$\beta_1(u_{k-1}, e_{k-1}) = 0.5 \mu_k^T \Sigma_2^{-1} \mu_k. \quad (35)$$

Similarly, $\beta_2(u_{k-1}, e_{k-1})$ can be evaluated as follows,

$$\begin{aligned} \beta_2(u_{k-1}, e_{k-1}) &= - \int s(e_k | u_{k-1}, e_{k-1}) \ln(\gamma(e_k)) de_k, \\ &= \int N(\mu_k, \Sigma_k) [0.5 (e_k^T S_k e_k + P_k e_k + w_k)] de_k, \\ &= 0.5 \mu_k^T S_k \mu_k + 0.5 w_k + 0.5 x_r^T M_2 x_r \\ &\quad + 0.5 e_{k-1}^T M_2 e_{k-1} + x_r^T M_2 e_{k-1} + 0.5 P_k \mu_k, \end{aligned} \quad (36)$$

where we have used,

$$\begin{aligned} \text{tr}(S_k \Sigma_k) &= x_{k-1}^T M_2 x_{k-1}, \\ &= e_{k-1}^T M_2 e_{k-1} + x_r^T M_2 x_r + 2 x_r^T M_2 e_{k-1}, \end{aligned} \quad (37)$$

with $M_2 = D^T S_k Q D$. Thereupon, substituting (35) and (36) in (27) and collecting the terms that multiply the control input, u_{k-1} together yields,

$$\begin{aligned} \gamma(e_{k-1}) &= \int c^J(u_{k-1} | e_{k-1}) \exp[-\beta_1(u_{k-1}, e_{k-1}) \\ &\quad - \beta_2(u_{k-1}, e_{k-1})] du_{k-1}, \\ &= (2\pi|\Gamma|)^{-\frac{1}{2}} \exp \left[-0.5 \left\{ e_{k-1}^T [A^T M_k A + M_2] e_{k-1} \right. \right. \\ &\quad \left. \left. + w_k + 2 x_r^T (F^T M_k A + M_2) e_{k-1} + \mu_u^T \Gamma^{-1} \mu_u \right. \right. \\ &\quad \left. \left. + P_k A e_{k-1} + P_k F x_r + x_r^T (F^T M_k F + M_2) x_r \right\} \right] \\ &\quad \times \int \exp \left[-0.5 \left\{ u_{k-1}^T (B^T M_k B + \Gamma^{-1}) u_{k-1} + 2 u_{k-1}^T (-\Gamma^{-1} \mu_u \right. \right. \\ &\quad \left. \left. + B^T M_k A e_{k-1} + B^T M_k F x_r + 0.5 B^T P_k^T) \right\} \right] du_{k-1}. \end{aligned} \quad (38)$$

The integral in (38) can be calculated by completing the square with respect to u_{k-1} . Consequently $\gamma(e_{k-1})$ can be

shown to be given by,

$$\begin{aligned} \gamma(x_{k-1}) = \exp \left[-0.5 \left\{ -\ln(2\pi) - 0.5 \ln(|\Gamma^{-1} + B^T M_k B|^{-1}) \right. \right. \\ + e_{k-1}^T (-A^T M_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k A \\ + M_2 + A^T M_k A) e_{k-1} + \{ 2x_r^T (M_2 + F^T M_k A) \\ + P_k A + 2\mu_u^T \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k A \\ - 2x_r^T F^T M_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k A \\ - P_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k A \} e_{k-1} + w_k \\ + P_k F x_r + x_r^T (M_2 + F^T M_k F \\ - F^T M_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k F) x_r \\ + \mu_u^T (\Gamma^{-1} - \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} \Gamma^{-1}) \mu_u \\ - 0.25 P_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T P_k^T \\ + 2\mu_u^T \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k F x_r \\ - P_k B (\Gamma^{-1} + B^T M_k B)^{-1} B^T M_k F x_r \\ \left. \left. + \mu_u^T \Gamma^{-1} (\Gamma^{-1} + B^T M_k B)^{-1} B^T P_k^T \right\} \right]. \quad (39) \end{aligned}$$

Note that according to Theorem 1, $-\ln(\gamma(e_{k-1})) = 0.5(e_{k-1}^T S_{k-1} e_{k-1} + P_{k-1} e_{k-1} + w_{k-1})$. Thus equating quadratic terms, linear terms, and constant terms in (39) with S_{k-1} , P_{k-1} , and w_{k-1} respectively yields the definitions stated in Equations (23), (24), and (25). This completes the proof. ■

Following the above verification of the quadratic performance index, the next step is to evaluate the parameters of the optimal controller distribution that will make the pdf of the tracking error follow the given ideal pdf. Based on (6) and (39), the randomised optimal controller that minimises the Kulback Leibler divergence objective function is given by the following theorem.

Theorem 2: The optimal randomised controller that minimises the Kulback Leibler divergence objective function subject to the probability density function of the tracking error defined in (16) and the ideal pdfs of the tracking error and controller defined in (19) and (20) respectively is given by,

$$c(u_{k-1}|e_{k-1}) \sim N(v_{k-1}, \Gamma_{k-1}), \quad (40)$$

where,

$$v_{k-1} = -K_{k-1} e_{k-1} - L_{k-1} \quad (41)$$

$$K_{k-1} = \Gamma_{k-1}^{-1} B^T M_k A \quad (42)$$

$$L_{k-1} = \Gamma_{k-1}^{-1} (B^T M_k F x_r + 0.5 B^T P_k^T - \Gamma^{-1} \mu_u) \quad (43)$$

$$\Gamma_{k-1} = (\Gamma^{-1} + B^T M_k B), \quad (44)$$

where v_{k-1} and Γ_{k-1} are the mean and covariance of the optimal randomised controller respectively, K_{k-1} is the controller feedback gain, and L_{k-1} is a linear shift which manifests from the considered tracking control problem.

Proof: Substituting Equations (20), (35), (36), and (39)

in (6) yields,

$$\begin{aligned} c^*(u_{k-1}|e_{k-1}) = (2\pi\Gamma)^{-0.5} \exp\{-0.5\mu_k^T M_k \mu_k \\ - 0.5e_{k-1}^T M_2 e_{k-1} - 0.5x_r^T M_2 x_r - x_r^T M_2 e_{k-1} - 0.5w_k \\ - 0.5P_k \mu_k - 0.5(u_{k-1} - \mu_u)^T \Gamma^{-1} (u_{k-1} - \mu_u) \\ + 0.5e_{k-1}^T S_{k-1} e_{k-1} + 0.5P_{k-1} e_{k-1} + 0.5w_{k-1}\}. \quad (45) \end{aligned}$$

Evaluating the independent of u_{k-1} terms in (45), and completing the square with respect to the control input, u_{k-1} , Equation (45) can be further expressed as,

$$\begin{aligned} c^*(u_{k-1}|e_{k-1}) = (2\pi\Gamma)^{-0.5} \exp \left[-0.5 \left\{ u_{k-1}^T (B^T M_k B \right. \right. \\ + \Gamma^{-1}) u_{k-1} + 2u_{k-1} [B^T M_k F e_{k-1} + B^T M_k F x_r + 0.5 B^T P_k \\ - \Gamma^{-1} \mu_u] + (\Gamma^{-1} \mu_u + B^T M_k F x_r + B^T M_k A e_{k-1} + 0.5 B^T P_k^T)^T \\ \times (B^T M_k B + \Gamma^{-1})^{-1} (\Gamma^{-1} \mu_u + B^T M_k F x_r + B^T M_k A e_{k-1} \\ + 0.5 B^T P_k^T) \left. \right\} \right], \\ = (2\pi\Gamma)^{-0.5} \exp\{-0.5[u_{k-1} - (-B^T M_k A e_{k-1} - B^T M_k F x_r \\ - 0.5 B^T P_k^T + \Gamma^{-1} \mu_u) (\Gamma^{-1} + B^T M_k B)^{-1}]^T (\Gamma^{-1} + B^T M_k B) \\ \times [u_{k-1} - (-B^T M_k A e_{k-1} - B^T M_k F x_r - 0.5 B^T P_k^T + \Gamma^{-1} \mu_u) \\ \times (\Gamma^{-1} + B^T M_k B)^{-1}] (\Gamma^{-1} + B^T M_k B)^{-1} \}. \quad (46) \end{aligned}$$

It can be seen that the mean and covariance of the distribution given in (46) are the mean and covariance of the optimised randomised controller as stated in equations (41) and (44) respectively. This completes the proof. ■

V. NUMERICAL RESULTS

This section will demonstrate the effectiveness of the proposed probabilistic minimisation of the tracking error specified by Theorems 1 and 2 in driving the output of the system dynamics to a predefined desired output value. In particular, the theory developed in Section IV is applied here to a flexible beam system [Flores and Barbieri, 2006] described by the following equation,

$$\dot{x} = Ax + Bu, \quad (47)$$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 38.1425 & 0 & 239.0350 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -47.0569 & 0 & -271.9385 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -6.9241 & 0 & -187.2933 & 0 \end{bmatrix},$$

$$B = [0 \quad 9.4393 \quad 0 \quad -10.7386 \quad 0 \quad -1.7135]^T.$$

Also, $x = [\theta \quad \dot{\theta} \quad q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2]^T$ is the beam system state vector, θ is the angle between the hubs frame and a global (stationary) reference frame, and $q_i, i = 1, 2$ represent the i th flexible mode.

Since the proposed framework is developed for discrete time systems, Equation (47) is discretised using the forward difference method, where the sampling time, h is taken to

be equal to 0.06. In addition, to demonstrate all aspects of the proposed method, a multiplicative noise is added to the original deterministic system equation after it has been discretised. Following the discretisation of Equation (47) and the addition of the multiplicative noise, the following equivalent discrete time description which is also modified by the addition of the multiplicative noise is obtained,

$$x_k = (Ah + I_{n \times n})x_{k-1} + Bhu_{k-1} + Dx_{k-1}v_{k-1}, \quad (48)$$

where, v_k is Gaussian noise with zero mean and variance 0.001, $v_k \sim N(0, 0.001)$, I is the identity matrix, and where,

$$D = 10^{-3} \begin{bmatrix} 9.0 & 6.3 & -0.2 & -5.4 & -21.2 & -5.7 \\ -4.3 & -0.2 & 8.7 & 3.5 & -1.5 & 13.1 \\ 9.0 & 19.1 & -13.7 & -4.4 & 5.2 & 2.5 \\ -22.6 & 21.7 & -10.7 & -6.4 & 3.9 & -8.1 \\ -5.9 & -0.7 & 11.3 & -2.0 & 1.9 & -3.4 \\ 1.7 & 8.4 & 10.3 & -6.4 & -3.8 & 7.0 \end{bmatrix}$$

is randomly generated.

The objective of the sought randomised controller is then specified to be of suppressing the quenching vibration in the beam and stabilising the angle between the hubs frame and a global (stationary) reference frame, θ at the value of 1. Therefore, the reference value that the system state is required to track is taken to be $x_r = [1, 0, 0, 0, 0, 0]^T$. In addition the system state is assumed to start from the following initial state values, $x_0 = [22, 0.3, 1, 0.4, 0.5, 2]^T$.

As discussed in Section IV, the parameters of the flexible beam system equation as specified in (48) are assumed to be unknown, therefore are estimated online at each time step. The mean and covariance of the conditional distribution of the beam system dynamics are then specified using the estimated parameters as discussed in Section IV. These estimates of the mean and covariance of the beam system dynamics are then used in Equations (23) and (24) to evaluate the Riccati equation, S_k , as well as P_k which are then both used in (41) to calculate the mean of the control input to be forwarded to the beam. Also, in the simulation experiment, the covariance, Σ_2 of the ideal distribution of the tracking error is taken to be $0.01 * I_{n \times n}$. The covariance, Γ of the ideal distribution of control inputs is taken to be 1. The simulation results are shown in Figure 1 and Figure 2. Figure 1 shows the various states of the flexible beam with their corresponding reference signals. As can be seen from this figure, all the flexible beam system states are accurately tracking their corresponding reference states. This can be confirmed from the magnified figures in Figure 1 which show the steady state values of the beam states. The tracking errors are presented in Figure 2, (a), from which it can also be seen that all the state tracking errors go to zero. These figures on the other hand show large deviation of the beam state values from their corresponding reference values and large tracking errors in the transient period. This is expected as the parameters of the beam equation which are estimated online will not have converged to their true values in this transient period. Once the parameters converge to their true values the beam states show good tracking to

their corresponding reference values. Also, the control input as calculated from (41) is shown in Figure 2, (b). The control input as can be seen from this figure is stable, thus yielding the required results. Finally, the feedback gain as calculated from (42) is shown in Figure 3. This figure shows that all the feedback gains have converged and reached steady state values. To re-emphasise, the numerical results prove the efficacy of the proposed probabilistic tracking control method and show that the mean of tracking error can be minimised to reach zero value.

VI. CONCLUSION

This paper presented a new framework for the design of randomised controllers for complex stochastic and uncertain systems that is based on the minimisation of the Kullback-Leibler divergence of the tracking error of the controlled system. The new proposed framework, considers the design of randomised controllers that take the multiplicative noises that affect the dynamics of the controlled stochastic system into consideration in the optimisation process. The theoretical development of this framework is demonstrated on linear Gaussian stochastic systems that are affected by multiplicative noises. The theoretical findings was then validated on controlling the vibration quenching of flexible beams.

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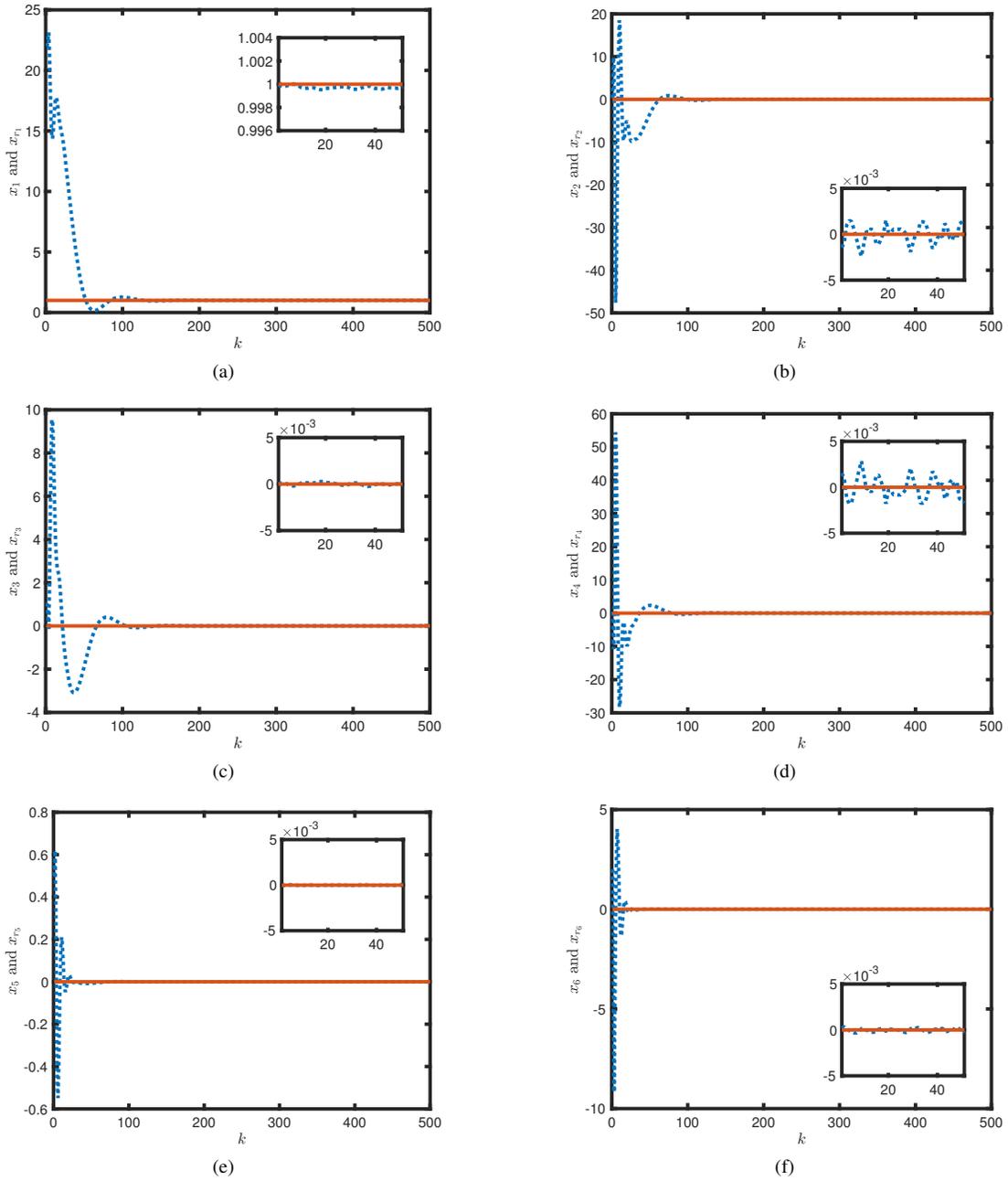


Fig. 1. The state of the simulated flexible beam: (a) state x_1 , dotted line, and reference state x_{r1} , solid line. (b) states x_2 , dotted line, and reference state x_{r2} , solid line. (c) states x_3 , dotted line, and reference state x_{r3} , solid line. (d) states x_4 , dotted line, and reference state x_{r4} , solid line. (e) states x_5 , dotted line, and reference state x_{r5} , solid line. (f) states x_6 , dotted line, and reference state x_{r6} , solid line. Small magnified figures show the steady state values of the beam states.

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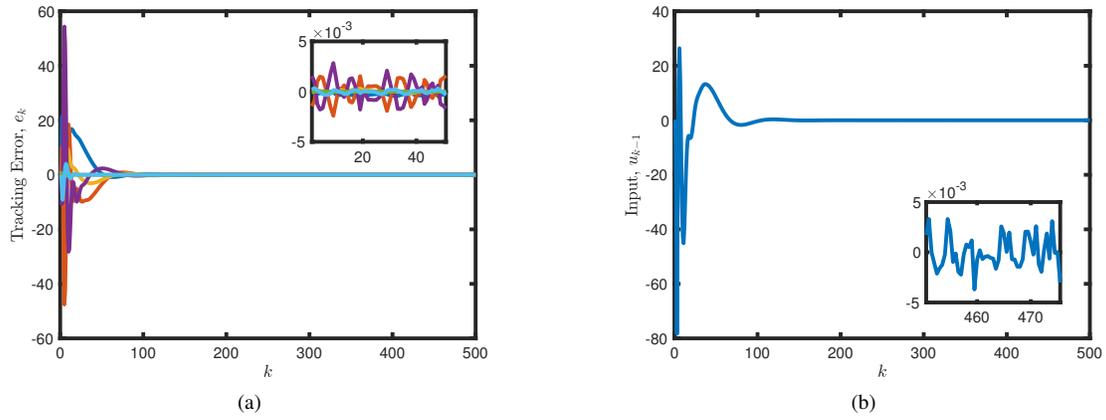


Fig. 2. The tracking error of the states and system input of the simulated flexible beam: (a) tracking error of the flexible beam states. (b) the control input to the flexible beam as calculated from (41).

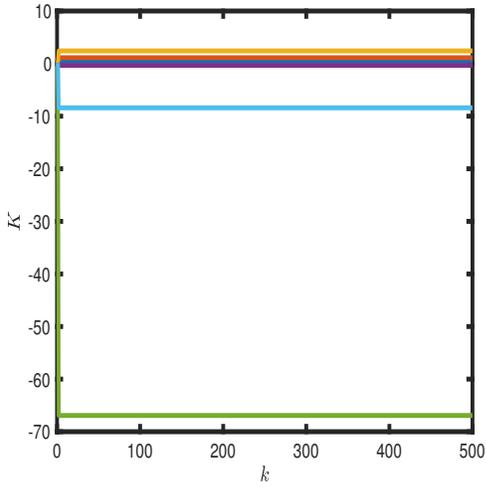


Fig. 3. The feedback gain of the controller as calculated from (42).

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