Journal Pre-proof

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PII: S0377-2217(19)30942-7
DOI: https://doi.org/10.1016/j.ejor.2019.11.031
Reference: EOR 16167

To appear in: *European Journal of Operational Research*

Received date: 27 July 2018
Accepted date: 15 November 2019

Highlights

- A fixed cost allocation model for a two-stage system is built
- It is first work to apply efficiency invariance principle in two-stage system
- Both cooperative and non-cooperative scenarios are investigated
- The former scenarios are based on overall efficiency invariance principle
- The latter scenarios are based on divisional efficiency invariance principle

Fixed cost allocation based on the principle of efficiency invariance in two-stage systems

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Abstract

Fixed cost allocation among groups of entities is a prominent issue in numerous organisations. Addressing this issue has become one of the most important topics of the data envelopment analysis (DEA) methodology. In this study, we propose a fixed cost allocation approach for basic two-stage systems based on the principle of efficiency invariance and then extend it to general two-stage systems. Fixed cost allocation in cooperative and noncooperative scenarios are investigated to develop the related allocation plans for two-stage systems. The model of fixed cost allocation under the overall condition of efficiency invariance is first developed when the two stages have a cooperative relationship. Then, the model of fixed cost allocation under the divisional condition of efficiency invariance wherein the two stages have a noncooperative relationship is studied. Finally, the validation of the proposed approach is demonstrated by a real application of 24 nonlife insurance companies, in which a comparative analysis with other allocation approaches is included.

Keywords: Data envelopment analysis, Fixed cost allocation, Cooperative model, Noncooperative model, Efficiency invariance principle

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1. Introduction

Data envelopment analysis (DEA) is a mathematical programming approach that is used to evaluate the efficiency of groups of decision-making units (DMUs) with multiple inputs and outputs (Charnes, Cooper, and Rhodes, 1978). It has been widely used in many fields, such as education, geoponics, healthcare and transportation (Emrouznejad and Yang, 2018). In recent years, DEA has also been applied to allocate fixed cost, because it can examine the effect of feasible allocation plans by the empirical description of the production possibility set based on the real productions (Beasley, 2003; Cook and Kress, 1999; Li, Yang, Liang, and Hua, 2009). The issue of fixed cost allocation is commonly encountered in real life when some agents (i.e. DMUs) use a common platform. An example provided by Cook and Zhu (2005) is the allocation of a manufacturer's advertising expenses to local retailers. Another example is the allocation of a bank's common television or newspaper advertising expenses to its branches. The crucial issue in fixed cost allocation is how to make an allocation plan for assigning cost to multiple DMUs.

So far, many DEA works have focused on fixed cost allocation issues on the basis of efficiency invariance principle or efficiency maximization principle. It should be noted that the efficiency of a DMU here is relative efficiency by comparing with other DMUs. The principle of efficiency invariance refers to the invariance in the efficiency of DMUs before and after allocation. Cook and Kress (1999) firstly studied the issue of fixed cost allocation by using DEA. Their proposed allocation procedure solved linear programming problems on the basis of efficiency invariance principle and Pareto-minimality principle. In a subsequent work, Jahanshahloo, Lotfi, Shoja, and Sanei (2004) argued that Cook and Kress (1999)'s method exists computational difficulties. Then they presented an approach that does not require the solution of linear programming problems but require some simple mathematical formulae to allocate the cost. Cook and Zhu (2005) pointed that Cook and Kress (1999)'s approach cannot be used directly to determine cost allocation among DMUs although it can examine existing rules for equitable cost allocation. Thus, they modified Cook and Kress (1999)'s approach by providing a practical approach to the problem of cost allocation. Lin (2011a) proved that the method proposed by Cook and Zhu (2005) has no feasible solution under special constraints. To obtain a feasible allocation plan, Lin (2011a) improved Cook and Zhu (2005)'s approach and set output targets in accordance with the amount of fixed cost shared by each DMU. Furthermore, Lin (2011b) proposed a DEA method for allocating cost and distributing common revenue among DMUs, which reflects the relative efficiency and the input-output scales of DMUs. More importantly, Lin and Chen (2016) illustrated that the "Pareto-minimality" introduced by Cook and Kress (1999) is inappropriate and then proposed a method based on super efficiency invariance and practical feasibility. Mostafaee (2013) presented an alternative allocation approach where the efficiency and the return to scale (RTS) of all DMUs remain unchanged after allocation. Lin, Chen, and Li (2016) proposed a method based on the principle of unit invariance to address the uniqueness of allocation plans successfully and allocate positive resources along with a positive target to each DMU. Amirteimoori and Kordrostami (2005) established a method based on a common set of weight and the efficiency invariance principle. However, Jahanshahloo, Sadeghi, and Khodabakhshi (2017) showed that the efficiency invariance principle is not necessarily satisfied when using Amirteimoori and Kordrostami (2005)'s method. Hence, they presented two equitable approaches based on the efficiency invariance principle and a common set of weight principle.

The principle of efficiency maximization indicates that the efficiency of all DMUs will be improved after cost allocation. Beasley (2003) firstly provided a cost allocation method based on this principle. Later, Si, Liang, Jia, Yang, Wu, and Li (2013) extended the work of Beasley (2003). They examined the equity of the proportional sharing method, and investigated the relationship between the extended method of proportional sharing and other DEA-based allocation methods. Li, Yang, Chen, Dai, and Liang (2013) utilised the DEA approach for equitable fixed cost allocation by considering the effect of allocation to each DMU. To identify a fair scheme, they introduced the concept of satisfaction degree and proposed a max-min model to generate a unique allocation plan. Du, Cook, Liang, and Zhu (2014) established a cost allocation method on the basis of the cross-efficiency concept. Khodabakhshi and Aryavash (2014) proposed that the allocation must be directly proportional or inversely proportional to the inputs and the outputs. Lin and Chen (2017) introduced a global modified additive DEA (MAD) model to allocate fixed cost by optimizing the global MAD-efficiency. Li, Zhu, and Liang (2019) suggested that each DMU should propose an allocation plan to punish itself so as to guarantee the acceptability of the allocation plan. Considering the game relations in the allocation process, Li, Li, Emrouznejad, Liang, and Xie (2019) proposed a cooperative game allocation approach for cost allocation. To guarantee the uniqueness of the allocation result, Chu and Jiang (2019) defined the concept of utility of each DMU and obtained the cost allocation result by maximizing the minimum utility.

According to the structure of the studied system, we mainly classify the fixed cost allocation studies into two categories. One category is cost allocation for single-stage systems, including the studies mentioned above. Although most studies on fixed cost allocation of single-stage system satisfy either efficiency maximization or efficiency invariance, several studies solve allocation issues from another way. For example, Yang and Zhang (2015) proposed a modified Shapley value to solve fixed cost allocation in single systems fairly, and they established a new Gini coefficient to evaluate the fairness of the allocation plan. To help managers incorporate different sub-objectives, Pendharkar (2018) utilised a hybrid genetic algorithm and DEA framework to solve the multicriterion issue of fixed cost allocation. Wu, Chu, and Liang (2016) considered that setting efficient targets by traditional allocation approaches is unfair to certain DMUs. Thus, they incorporated DEA and the closest target technique in the allocation problem. Some previous works based on efficiency maximization principle assumed that all DMUs become efficient after fixed cost allocation, such as Li, Yang, Chen, Dai, and Liang (2013). However, Ding, Chen, Wu, and Wei (2018) thought the achievement of a common technological level by all DMUs is impractical and thus presented a new approach that accounts for technological heterogeneity. Most of studies mentioned above regarded the fixed cost as a new input for the DMUs, but Li, Yang, Liang, and Hua (2009), Lin and Chen

(2017) allocated fixed cost as a complement of other cost inputs on the basis of DEA approach, and this method was extended to two-stage systems by Zhu, Zhang, and Wang (2019). Zhang, Wang, Qi, and Wu (2018) combined game theory and DEA approach to solve the problem of transmission cost allocation. Considering competitive and cooperative relationships among DMUs, Li, Zhu, and Liang (2018) integrated cooperative game theory and cross-efficiency method to develop a unique and fair allocation plan. Studies of this category ignored the internal structure of systems by considering them as 'black boxes' (Yu, Chen, and Bo, 2016).

Another category is fixed cost allocation for two-stage systems. Operational systems usually contain multiple stages. For example, banks usually contain deposit and lending processes. The substages of each bank branch use common facilities and thus each substage should be obliged to afford fixed cost allocation. So far, a few works have considered fixed cost allocation among two-stage systems. For instance, Yu, Chen, and Bo (2016) proposed an alternative approach to solve the problem of fixed cost allocation, which is based on two-stage DEA models and cross-efficiency concept. Ding, Zhu, Zhang, and Liang (2019) dealt with the fixed cost allocation problem for a general two-stage network structure, by introducing the concepts of satisfaction degree and fairness degree. Zhu, Zhang, and Wang (2019) treated the fixed cost as an additional input factor shared in two-stage DMUs and proposed three allocation procedures based on different objectives in reality. Chu, Wu, Chu, and Zhang (2019) considered the competition between the two stages of DMUs in fixed cost allocation issues and regarded all the first stages and all the second stages as two unions. Then the allocation plan is calculated by using satisfaction degree bargaining model. Considering internal structure, Li, Zhu, and Chen (2019) used DEA methodology to determine relative efficiency and allocated costs based on both common weights and operation size. To sum up, these works on fixed cost allocation in two-stage systems are based on the principle of efficiency maximization and neglect the possible relationship between the two stages.

In this study, we focus on fixed cost allocation in two-stage systems, wherein potential conflicts between the two stages arise from intermediate measures (Liang, Cook, and Zhu, 2008). In real life, there are many two-stage systems in government or business, and the fixed cost allocation issues commonly encountered when some agents (i.e. DMUs) use a common platform. Besides, the cooperation and competitive relationships between two stages may exist. Thus, in two-stage systems, assigning proper cost to each DMU and its substages considering the relationship between the two stages is a crucial problem. Assuming cooperative (or noncooperative) relationship between the two stages, we build a cooperative (or noncooperative) model for deciding how to allocate fixed cost in an equitable and fair way. The fair way means the relative efficiency of each DMU remains unchanged after allocation (Cook and Kress, 1999). Why do we keep efficiency invariant? We know that all DMUs' relative efficiency depends on existing inputs and outputs measures, which are out of control of other DMUs, hence the fixed cost allocation plan should be implemented based on the current efficiency scores (Li, Song, Dolgui, and Liang, 2017; Cook and Kress, 1999). In addition, from operation's view, the given inputs and outputs adequately explain the production function, so the allocated cost should have no effect on this function (Cook and Kress, 1999). Therefore, the efficiency should be invariable after allocation in a short term.

Compared with the previous works, the proposed approach in this paper has four main differences (some specific comparisons can be seen in Table 1). First, we build a DEA-based approach for fixed cost allocation in two-stage systems, which is based on the principle of efficiency invariance. Second, our approach studies both cooperative and noncooperative relationships between the two stages of systems during fixed cost allocation. The former relationship is investigated from a centralized point, whereas the latter one is studied from a competitive point. Third, our allocation plan is beneficial to the leader stage when the two stages are noncooperative. Furthermore, our approach makes the difference between the maximum allocation value and the minimum allocation value of DMUs smaller than that of other approaches, which is illustrated in our application.

Table 1

Comparison among several previous methods and our method

Authors	Allocations principles	Structure	Models	Approaches
Mostafaee (2013)	Efficiency invariance	Single stage	The combined primal-dual form of the one stage DEA	Step 1: Assess the pre-allocation efficiency scores of each DMU and obtain the optimal intensity variables. Step 2: Allocate fixed cost by minimizing the gaps among allocated fixed costs based on the optimal values of intensive variables and the pre-allocation efficiency scores obtained from step 1.
Lin and Chen (2017)	Efficiency maximization	Single stage	One-stage DEA model by optimizing the global MAD-efficiency	Step 1: Evaluating the efficiencies of all DMUs based on MAD model. Step 2: Allocate fixed cost by minimizing the difference between the maximum and minimum portions over the minimum fixed cost possibly paid by DMUs.
Ding, Zhu, Zhang, and Liang (2019)	Efficiency maximization	General two-stage network	Additive two-stage models	Step 1: Calculate the efficiency by additive two-stage models Step 2: Allocate fixed cost by maximizing average satisfaction degree
Khodabakhshi and Aryavash (2014)	Efficiency maximization	Single stage	One-stage DEA model	Step 1: Determine the minimum and maximum and values of efficiency. Step 2: Calculate the share of DMU from step 1.
Li, Li, Emrouznejad, Liang, and Xie (2019)	Efficiency maximization principle	Single stage	A cooperative game DEA model	Step 1: Define the characteristic function Step 2: Allocate the fixed cost based on nucleolus solution.
Yu, Chen, and Bo (2016)	Efficiency maximization principle	Two-stage system	Two-stage cross-efficiency DEA	Step 1: Calculate the initial values of E_j^1 and E_j^2. Step 2: Put the new E_j^1 and E_j^2 into model to obtain next optimal solution until the values cannot further increase. Step 3: Compute optimal value of cost allocation.
Zhu, Zhang, and Wang (2019)	Efficiency maximization principle	Two-stage system	Three procedures based on different objectives.	Step 1: Consider two situations: (1) the lower bound of α and β (shared resource ratio of stage 1) are zero. (2) the lower bound of α and β are positive Step 2: Propose three procedures based on different objectives to obtain allocation plan.
Li, Zhu, and	Efficiency	Two-stage	Minimize the	Step 1: Optimize the allocation plan by

Chen (2019)	maximization principle	system	deviation between the efficient allocations and size allocations	considering the operation size. Step 2: Minimizing the maximal deviation repeatedly
Chu, Wu, Chu, and Zhang (2019)	Efficiency maximization principle	Two-stage system	Satisfaction degree bargaining game model	Step 1: Propose the leader-follower fixed cost allocation approach and the new concept of satisfaction degree. Step 2: Incorporate the Nash bargaining game theory to propose a satisfaction degree bargaining game model. Step 3: Get the allocation plan.
The current paper	Efficiency invariance principle	Basic two-stage and general two-stage system	Additive two-stage models and leader-follower allocation model	Step 1: Using the additive model and leader-follower model to get initial efficiency Step 2: Deduce efficiency invariance condition by models in step 1. Step 3: Allocate fixed cost in cooperative and noncooperative scenarios.

The rest of this paper is organised as follows. The traditional DEA model and the fixed cost allocation approach proposed by Cook and Kress (1999) are briefly introduced in Section 2. The two-stage DEA models for measuring the efficiency of DMUs before and after allocation, and our fixed cost allocation models are presented in Section 3. Section 4 shows the validation of this approach by using a real application of 24 nonlife insurance companies. A comparative analysis with other allocation methods of two-stage systems is also given in this section. Section 5 discusses the generalization of the proposed model to the general two-stage system. Conclusions and directions for future research are given in Section 6.

2. Preliminaries

2.1 Traditional DEA model

Suppose that there are a set of homogeneous DMUs, and each DMU_j $(j = 1, \ldots, n)$ uses I inputs $x_i (i = 1, \ldots, I)$ to produce K outputs $y_k (k = 1, \ldots, K)$. The output-oriented CCR (Charnes, Cooper, and Rhodes, 1978) model is given as follows:

$$Min \quad \sum_{i=1}^{I} v_i x_{id} \tag{1}$$

$$s.t. \quad \sum_{i=1}^{I} v_i x_{ij} - \sum_{k=1}^{K} u_k y_{kj} \geq 0, j = 1, \dots, n$$

$$\sum_{k=1}^{K} u_k y_{kd} = 1$$

$$v_i, u_k \geq 0, i = 1, \dots, I; k = 1, \dots, K.$$

where u_k is the weight appointed to the *k*th output, and v_i is the weight appointed to the *i*th input. u_k, v_i are variables in model (1). The dual of model (1) is as follows:

$$Max \quad \theta_d \tag{2}$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_{jd} x_{ij} \leq x_{id}, i = 1, \dots, I$$

$$\sum_{j=1}^{n} \lambda_{jd} y_{kj} \geq \theta_d y_{kd}, k = 1, \dots, K$$

$$\lambda_{jd} \geq 0, j = 1, \dots, n$$

where θ_d and $\lambda_j \geq 0$ are variables. DMU_d is called an efficient DMU if and only if the optimal objective function value of models (1) and (2) equals one. Otherwise, it is called an efficient DMU.

2.2 Fixed cost allocation method of Cook and Kress (1999)

In this section, we briefly present the allocation method of Cook and Kress (1999), which can guarantee the efficiency of each DMU invariant after allocation.

Before the fixed cost allocation, the efficiency of DMU_d can be evaluated by model (1) or (2). After the allocation, we assume the allocated fixed cost to DMU_j is $r_j, j = 1, \dots n$. Then, similar to the traditional DEA model (1), the efficiency of DMU_d after allocation can be measured by the following model in which the allocated fixed cost is considered as a new input.

$$Min \quad \sum_{i=1}^{I} v_i x_{id} + v_{I+1} r_d \tag{3}$$

$$s.t. \quad \sum_{i=1}^{I} v_i x_{ij} + v_{I+1} r_j - \sum_{k=1}^{K} u_k y_{kj} \geq 0, j = 1, \dots, n.$$

$$\sum_{k=1}^{K} u_k y_{kd} = 1$$

$$v_i, u_k \geq 0, v_{I+1} > 0, i = 1, \dots, I; k = 1, \dots, K.$$

where v_{I+1} is variable which represents the weight of the allocated fixed cost, $r_j, j = 1, \dots n$ are constant. Model (3) is a linear programming model. Considering the fact that a non-allocation plan is feasible through making $v_{I+1} = 0$, we let $v_{I+1} > 0$. Its dual programming model is as follows:

$$Max \quad \theta_d' \tag{4}$$

$$s.t.\quad \sum_{j=1}^{n} \lambda_{jd} x_{ij} \leq x_{id}, i = 1, \dots, I$$

$$\sum_{j=1}^{n} \lambda_{jd}\, r_j \leq r_d$$

$$\theta'_d y_{kd} - \sum_{j=1}^{n} \lambda_{jd} y_{kj} \leq 0, k = 1, \dots, K$$

$$\lambda_{jd} \geq 0, j = 1, \dots, n$$

It is important to know "how to set the allocated fixed cost to each DMU that can guarantee the efficiency invariable before and after allocation?" Cook and Kress (1999) proposed a principle called *efficiency invariance principle* which means the optimal objective function value of model (1) equals to the optimal objective function value of model (3) for DMU_d. To realize the efficiency invariance principle, v_{I+1} in model (3) must be out of the basis in the final simplex tableau (Jahanshahloo, Lotfi, Shoja, and Sanei, 2004). To better explain what is "out of the basis", we give the following definition.

Definition 1. For each DMU_d, v_{I+1} remains out of the basis means that the reduced cost is nonnegative. That is

$$C_{v_{I+1}} - Z_{v_{I+1}} \geq 0 \Rightarrow C_{v_{I+1}} - C_B B^{-1} A \geq 0 \Rightarrow r_d - \sum_{j=1}^{n} \lambda_{jd}^* r_j \geq 0$$

$$\Longrightarrow r_d \geq \sum_{j=1}^{n} \lambda_{jd}^* r_j \tag{5}$$

where $\lambda_{jd}^* (j = 1, \dots, n)$ are the optimal dual variables of model (3) (Cook and Kress, 1999; Jahanshahloo, Lotfi, Shoja, and Sanei, 2004).

To set the value of θ_d and θ'_d equal, the second constraint of model (4) must be redundant. That is, λ_{jd}^* in formula (5) must be the optimal solution of model (2). Therefore, $r_j, j = 1, \dots, n$ should satisfy $r_d \geq \sum_{j=1}^{n} \lambda_{jd}^* r_j$. One limitation of this allocation is that it is not unique. If the overall fixed cost was distributed in its entirety among only inefficient DMUs in any proportion (Cook and Kress, 1999), the DEA efficiency scores would not change, and invariance assumption is satisfied. Hence, another condition, "the Input Pareto-Minimality" should be satisfied as well. The Input Pareto-Minimality is defined as follows:

Definition 2. The Input Pareto-Minimality of fixed cost allocation plan means that no cost can be transferred from one DMU to another without violating the invariance principle (Cook and Kress, 1999).

In order to satisfy efficiency invariance and Input Pareto-Minimality principle, the constraint $r_d = \sum_{j=1}^{n} \lambda_{jd}^{*} r_j$ should be satisfied for all inefficient DMU_d. Denote S_d as the constraints in model (1) corresponding to the efficient reference set for DMU_d, we have $r_d = \sum_{j \in S_d} \lambda_{jd} r_j$, because other dual variables equal to zero according to complementary slackness. It should be noted that, Lin and Chen (2016) subsequently illustrated that this economic explanation "Pareto-Minimality" for the equality constraints is not appropriate and they suggested calling it as "practical feasibility assumption".

3. Proposed allocation model based on efficiency invariance in two-stage systems

Considering the internal structure of each DMU, we propose a fixed cost allocation model for basic two-stage systems based on the efficiency invariance principle.

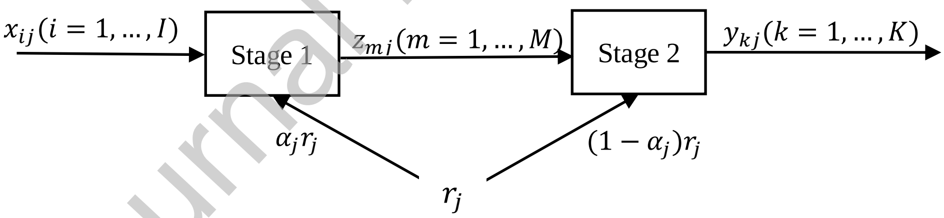

Fig. 1. A basic two-stage system

Fig. 1 depicts a two-stage network structure wherein outputs from the first stage become the inputs of the second stage. Suppose that we have n DMUs, and each $DMU_j (j = 1, \ldots, n)$ uses I nonnegative inputs $x_{ij}(i = 1, \ldots, I)$ to produce M intermediates $z_{mj}(m = 1, \ldots, M)$. These intermediates are viewed as the inputs of the second stage to generate K outputs $y_{kj}(k = 1, \ldots, K)$. In this paper, analogous to Cook and Kress (1999), we regard the allocated fixed cost r_j to each DMU as a new input. Let R denote the overall fixed cost of all DMUs, and r_j denote the cost allocated to DMU_j. The decision maker of each DMU can freely allocate the cost between Stages 1 and 2. $\alpha_j r_j$ is the fixed cost allocated to the first stage of DMU_j, whereas the rest of cost denoted by $(1-\alpha_j)r_j$ is allocated to the second stage of DMU_j (Chen, Du, Sherman, and Zhu, 2010; Yu, Chen, and Bo, 2016; Cook and Hababou, 2001). Chen, Du, Sherman, and Zhu (2010) stated that the value of $\alpha_j(0 \le \alpha_j \le 1)$ is set within a range to balance allocation between the two stages. L and U are the lower bound and upper bound of α_j, respectively. In reality, the value of L and U are determined by decision makers or branch consultant (Cook and Hababou, 2001) based on the actual economic bearing capacity of Stages 1 and 2.

For each DMU_d under the assumption of constant returns to scale (CRS), the efficiency value of Stages 1 and 2 considering the fixed cost can be obtained by the following formulae.

$$\theta_j^1 = \frac{\sum_{m=1}^{M} w_m z_{mj}}{\sum_{i=1}^{I} v_i x_{ij} + v_{I+1}\alpha_j r_j} \tag{6}$$

$$\theta_j^2 = \frac{\sum_{k=1}^{K} u_k y_{kj}}{\sum_{m=1}^{M} w_m z_{mj} + w_{M+1}(1-\alpha_j) r_j} \tag{7}$$

3.1 Cooperative models for fixed cost allocation in two-stage systems

In many cases, Stages 1 and 2 must work together to optimize overall performance. An alternative method to measure the efficiency of two-stage systems is to consider them from a centralized perspective and determine the set of the optimal weights of z_m. Chen, Du, Sherman, and Zhu (2010) set the overall efficiency by the weighted average of the two stages' efficiency scores as follows:

$$\theta = p_1\theta_j^1 + p_2\theta_j^2 \tag{8}$$

$$p_1 + p_2 = 1$$

where p_1 and p_2 represent the relative importance of efficiency in the first and second stages, respectively. One way to identify the relative importance of each stage is the proportion of the overall inputs devoted to Stages 1 and 2. Considering the fixed cost, the concrete calculation formulae are as follows:

$$p_1 = \frac{\sum_{i=1}^{I} v_i x_{ij} + v_{I+1}\alpha_j r_j}{\sum_{i=1}^{I} v_i x_{ij} + v_{I+1}\alpha_j r_j + \sum_{m=1}^{M} w_m z_{mj} + w_{M+1}(1-\alpha_j) r_j} \text{ and}$$

$$p_2 = \frac{\sum_{m=1}^{M} w_m z_{mj} + w_{M+1}(1-\alpha_j) r_j}{\sum_{i=1}^{I} v_i x_{ij} + v_{I+1}\alpha_j r_j + \sum_{m=1}^{M} w_m z_{mj} + w_{M+1}(1-\alpha_j) r_j}.$$

The denominator represents the total inputs of DMU_j, whereas $\sum_{i=1}^{I} v_i x_{ij} + v_{I+1}\alpha_j r_j$ represents the inputs of Stage 1, and $\sum_{m=1}^{M} w_m z_{mj} + w_{M+1}(1-\alpha_j) r_j$ represents the inputs to Stage 2 (Chen, Du, Sherman, and Zhu, 2010).

Thus, considering the fixed cost, the overall performance of two-stage DMU can be evaluated by solving the following model. The efficiency score of the two substages is simultaneously calculated by the optimal weights of model (9),

$$Max \quad p_1 \frac{\sum_{m=1}^{M} w_m z_{md}}{\sum_{i=1}^{I} v_i x_{id} + v_{I+1}\alpha_d r_d} + p_2 \frac{\sum_{k=1}^{K} u_k y_{kd}}{\sum_{m=1}^{M} w_m z_{md} + w_{M+1}(1-\alpha_d) r_d} = E_d \qquad (9)$$

$$s.t. \quad \frac{\sum_{m=1}^{M} w_m z_{mj}}{\sum_{i=1}^{I} v_i x_{ij} + v_{I+1}\alpha_j r_j} \leq 1, j = 1, \ldots, n \qquad (9.1)$$

$$\frac{\sum_{k=1}^{K} u_k y_{kj}}{\sum_{m=1}^{M} w_m z_{mj} + w_{M+1}(1-\alpha_j) r_j} \leq 1, j = 1, \ldots, n \qquad (9.2)$$

$$v_i, u_k, w_m \geq 0, v_{I+1}, w_{M+1} > 0 \qquad (9.3)$$

$$i = 1, \ldots, I; k = 1, \ldots, K; m = 1, \ldots, M$$

where p_1, p_2 represent the relative weights of efficiency in the first and second stages respectively. Constraints (9.1) and (9.2) show that the efficiency scores of Stages 1 and 2 must not exceed 1. Putting p_1, p_2 into model (9), and then it can be transformed into the following model:

$$\text{Max} \quad \frac{\sum_{m=1}^{M} w_m z_{md} + \sum_{k=1}^{K} u_k y_{kd}}{\sum_{i=1}^{I} v_i x_{id} + v_{I+1}\alpha_d r_d + \sum_{m=1}^{M} w_m z_{md} + w_{M+1}(1-\alpha_d) r_d} = E_d \qquad (10)$$

$s.t.$ (9.1) - (9.3)

Similar to Chen, Du, Sherman, and Zhu (2010) and Yu, Chen, and Bo (2016), we also assume that $v_{I+1} = w_{M+1} = \omega$ for all $DMU_j, j = 1, \ldots, n$ in model (10), because the allocated cost $\alpha_j r_j$ and $(1-\alpha_j) r_j$ are the same type of inputs. Then, by applying Charnes–Cooper transformation (Charnes and Cooper, 1962), let $\tau = \frac{1}{\sum_{i=1}^{I} v_i x_{id} + \sum_{m=1}^{M} w_m z_{md} + \omega r_d}, v_i' = \tau v_i, w_m' = \tau w_m, u_k' = \tau u_k, \omega' = \tau\omega$, then model (10) is transformed into the following model,

$$Max \quad w_m' z_{md} + u_k' y_{kd} \qquad (11)$$

$$s.t. \quad \sum_{m=1}^{M} w_m' z_{mj} - \sum_{i=1}^{I} v_i' x_{ij} - \omega' \alpha_j r_j \leq 0$$

$$\sum_{k=1}^{K} u_k' y_{kj} - \sum_{m=1}^{M} w_m' z_{mj} - \omega'(1-\alpha_j) r_j \leq 0$$

$$\sum_{i=1}^{I} v_i' \, x_{ij} + \sum_{m=1}^{M} w_m' \, z_{md} + \omega' r_d = 1$$

$$u_k', v_i', w_m' \geq 0, \omega' > 0$$

$$k = 1, \dots, K; m = 1, \dots, M; i = 1, \dots, I; j = 1, \dots, n$$

Model (11) is an input-oriented DEA model, similarly, we can obtained the following output-oriented DEA model:

$$\text{LP1 } Min \quad \sum_{i=1}^{I} v_i' \, x_{id} + \sum_{m=1}^{M} w_m' \, z_{md} + \omega' r_d = \frac{1}{e_d^{co}} \tag{12}$$

$$s.t. \quad \sum_{m=1}^{M} w_m' z_{mj} - \sum_{i=1}^{I} v_i' \, x_{ij} - \omega' \alpha_j r_j \leq 0$$

$$\sum_{k=1}^{K} u_k' \, y_{kj} - \sum_{m=1}^{M} w_m' \, z_{mj} - \omega' (1 - \alpha_j) r_j \leq 0$$

$$\sum_{m=1}^{M} w_m' z_{md} + \sum_{k=1}^{K} u_k' y_{kd} = 1$$

$$u_k', v_i', w_m' \geq 0, \omega' > 0$$

$$k = 1, \dots, K; m = 1, \dots, M; i = 1, \dots, I; j = 1, \dots, n$$

The optimal overall efficiency after allocation is denoted by e_j^{co*}, which is calculated by model LP1. Besides, the optimal efficiency of each stage after allocation is obtained by the optimal solutions $(v_i'^*, w_m'^*, u_k'^*, \omega'^*)$. The two divisional efficiency after allocation are denoted by e_j^{co1*}, e_j^{co2*}.

Before fixed cost allocation, let r_j equal zero for all $j = 1, \dots, n$ in model (12), then we can obtain the initial overall efficiency and the initial divisional efficiency by solving model LP2. The initial overall efficiency and divisional efficiency are explained by Definition 3.

$$\text{LP2 } Min \quad \sum_{i=1}^{I} v_i' \, x_{id} + \sum_{m=1}^{M} w_m' z_{md} = \frac{1}{E_d^{co}} \tag{13}$$

$$s.t. \quad \sum_{m=1}^{M} w_m' z_{md} - \sum_{i=1}^{I} v_i' x_{ij} \leq 0$$

$$\sum_{k=1}^{K} u_k' y_{kj} - \sum_{m=1}^{M} w_m' z_{mj} \leq 0$$

$$\sum_{m=1}^{M} w_m' z_{md} + \sum_{k=1}^{K} u_k' \, y_{kd} = 1$$

$$u_k', v_i', w_m' \geq 0$$

$$k = 1, \dots, K; m = 1, \dots, M; i = 1, \dots, I; j = 1, \dots, n$$

Definition 3. The initial efficiency is defined as the efficiency before allocation. The initial overall efficiency (E_j^{co}, E_j^{no}) is the efficiency of two-stage DMU before allocation. The initial divisional efficiency is the efficiencies of two substages before allocation, and we denote the initial divisional efficiencies of two substages as E_j^{co1}, E_j^{co2} and E_j^{no1}, E_j^{no2} in two scenarios, respectively.

How to guarantee the efficiency before and after allocation invariant? Cook and Kress (1999) stated that a necessary condition for equitable allocation is that no DMU can use new input to improve its relative efficiency. Comparing LP1 and LP2, the overall efficiency invariance requires to maintain the variables ω' of LP1 out of the basis. However, achieving the allocation plan through models LP1 and LP2 is impossible. We can attempt to impose such a condition by duality theory. The respective dual models of LP1 and LP2 are

DP1 $Max \quad \theta_d$ (14)

$s.t. \quad \sum_{j=1}^{n}(\lambda_{jd}^2 - \lambda_{jd}^1) z_{mj} \leq (1-\theta_d) z_{md}, m = 1, \dots, M$ (14.1)

$\sum_{j=1}^{n} \lambda_{jd}^1 x_{ij} \leq x_{id}, i = 1, \dots, I$ (14.2)

$\sum_{j=1}^{n} \lambda_{jd}^2 y_{kj} \geq \theta_d y_{kd}, k = 1, \dots, K$ (14.3)

$\sum_{j=1}^{n} \left[\lambda_{jd}^1 \alpha_j r_j + \lambda_{jd}^2 (1-\alpha_j) r_j\right] \leq r_d$ (14.4)

$\lambda_{jd}^1, \lambda_{jd}^2 \geq 0, j = 1, \dots, n$ (14.5)

DP2 $Max \quad \theta_d$ (15)

$s.t.$ (14.1) - (14.3)

(14.5)

To satisfy the principle of efficiency invariance, the optimal value of DP1 and DP2 must be equal. Thus, constraint (14.4) in DP1 must be redundant. The optimal solution of DP1 must also be the optimal solution of DP2. Denote $\lambda_{jd}^{1*}, \lambda_{jd}^{2*}$ as the optimal variables of model DP2 when DMU_d is evaluated, then we have $\sum_{j=1}^{n}[\lambda_{jd}^{1*}\alpha_j r_j + \lambda_{jd}^{2*}(1-\alpha_j) r_j] \leq r_d$. Evidently, this formula cannot determine equitable allocation sufficiently, and then the input practical feasibility principle is necessary. Therefore, the efficiency invariance condition becomes $r_d = \sum_{j=1}^{n}[\lambda_{jd}^{1*}\alpha_j r_j + \lambda_{jd}^{2*}(1-\alpha_j) r_j]$. We obtain the allocation plan by solving the following model,

$$Min \quad \left|\sum_{j=1}^{n} \alpha_j r_j - \sum_{j=1}^{n}(r_j - \alpha_j r_j)\right| = obj1 \tag{16}$$

$$s.t. \quad \sum_{j=1}^{n}[\lambda_{jd}^{1*}\alpha_j r_j + \lambda_{jd}^{2*}(1-\alpha_j)r_j] = r_d, d = 1,\dots,n$$

$$\sum_{j=1}^{n} r_j = R$$

$$L_j \le \alpha_j \le U_j$$

where $\lambda_{jd}^{1*}, \lambda_{jd}^{2*} (d = 1,\dots,n)$ are obtained by solving model DP2 n times. Model (16) is a nonlinear programming model because of $\sum_{j=1}^{n} \lambda_{jd}^{1*}\alpha_j r_j$ and $\sum_{j=1}^{n} \lambda_{jd}^{2*}\alpha_j r_j$ in certain constraints. In accordance with Chen, Du, Sherman, and Zhu (2010), we let $b_j = \alpha_j r_j (j = 1,\dots,n)$, model (16) can be converted into a linear programming model.

$$Min \quad \left|\sum_{j=1}^{n} b_j - \sum_{j=1}^{n}(r_j - b_j)\right| = obj1 \tag{17}$$

$$s.t. \quad \sum_{j=1}^{n}[\lambda_{jd}^{1*} b_j + \lambda_{jd}^{2*}(r_j - b_j)] = r_d, d = 1,\dots,n$$

$$\sum_{j=1}^{n} r_j = R$$

$$L_j r_j - b_j \le 0$$

$$-U_j r_j + b_j \le 0$$

In cooperative scenario, decision makers should select one reasonable objective, which is similar to the approach of Cook and Zhu (2005). To minimize the deviation between Stage 1 and Stage 2, we let *obj*1 equal $Min \mid \sum_{j=1}^{n} b_j - \sum_{j=1}^{n}(r_j - b_j) \mid$. The linear programming can be solved one time with an objective function *obj1* to avoid the uniqueness. If there are still several optimal solutions, we can set a second objective function, for example, $Min \sum_{j=1}^{n}\left|r_j - \bar{r}\right|$, $Min \frac{1}{n}\sum_{j=1}^{n}\left(r_j - \bar{r}\right)^2$, $\bar{r} = R/n$.

3.2 Noncooperative models for fixed cost allocation in two-stage systems

The models presented in Section 3.1 are based on the cooperative relationship, wherein the two stages work together to obtain the optimal system result. In this section, we consider the noncooperative relationship between the two stages when the fixed cost is allocated to each DMU. Given such a competitive condition, we introduce the Leader-Follower concept to explore the divisional efficiency invariance condition. Li, Chen, Liang, and Xie (2012) used the noncooperative approach developed by Liang, Cook, and Zhu (2008) to analyse this extended two-stage structure, they assumed one stage as the leader and another stage as the follower. The leader stage is more important than the follower stage. This leader–follower methodology is introduced from the Stackelberg game. For example, the existence of a manufacturer and retailer wherein the manufacturer plays a crucial part is called *leader-follower competition* or *Stackelberg competition*.

Before fixed cost allocation, we should illustrate the noncooperative model in detail. Above all, the leader stage should be identified. Although Li, Chen, Cook, Zhang, and Zhu (2018) proposed an approach to uncover the leader stage of two-stage system, they assumed the leader stage of each DMU is unknown beforehand. However, here we assume the leader stage is known and identified based on expert opinions before allocation. In this paper, we assume that the leader is the first stage, and the follower is the second stage. Thus, the efficiency of Stage 2 is obtained when the leader's efficiency remains fixed. Certainly, Stage 2 can also be the leader if it is more important than Stage 1 in certain companies.

The initial efficiency of the first stages of DMUs is obtained by solving model (1), here we rename model (1) as LP3. Let w_m^* and v_i^* be the optimal set of weights, and E_d^{no1} is the initial efficiency of Stage 1 for DMU_d. The intermediate product z_{mj} is the only connection between Stages 1 and 2. Therefore, E_d^{no1} should be introduced in the next model when the efficiency of the second stage is calculated. The optimal efficiency score of Stage 2 (follower) is calculated by maintaining Stage 1's (leader) efficiency score. The model is as follows:

$$\text{LP4} \quad Min \quad \sum_{m=1}^{M} w_m z_{md} = \frac{1}{E_d^{no2}} \tag{18}$$

$$s.t. \quad \sum_{k=1}^{K} u_k y_{kj} - \sum_{m=1}^{M} w_m z_{mj} \leq 0\,, j = 1, \ldots, n$$

$$\sum_{m=1}^{M} w_m z_{mj} - \sum_{i=1}^{I} v_i x_{ij} \leq 0\,, j = 1, \ldots, n$$

$$\sum_{m=1}^{M} w_m z_{md} - E_d^{no1*} \sum_{i=1}^{I} v_i x_{id} = 0$$

$\sum_{k=1}^{K} u_k y_{kd} = 1$
$v_i, w_m, u_k \geq 0, m = 1, \dots, M; i = 1, \dots, I; k = 1, \dots, K$

LP3 and LP4 represent the output oriented CCR (Charnes and Cooper, 1962) model. Once we obtain the efficiency of the first stage, the second stage can only consider variables w_m that maintain $E_d^{no1} = E_d^{no1*}$ (Liang, Cook, and Zhu, 2008). Therefore, we consider $\sum_{m=1}^{M} w_m z_{mj}$ as the single input to model LP3 that maintains the optimal objective value E_d^{no1*}(Liang, Cook, and Zhu, 2008).

In noncooperative scenario, we denote the overall optimal efficiency by $E_d^{no1*} * E_d^{no2*}$. Using the same method as that used by Cook and Kress (1999), we set the condition of the efficiency invariance of Stage 1 as $\sum_{j=1}^{n} \lambda_{jd}^{*} \alpha_j r_j = \alpha_d r_d$, where λ_{jd}^{*} is the optimal variables of DP3 when DMU_d is evaluated.

DP3 $Max\ \theta_d' = \frac{1}{E_d^{no1*}}$ (19)

$s.t.\quad \sum_{j=1}^{n} \lambda_{jd} z_{ij} \geq \theta_d' z_{md}$

$\sum_{j=1}^{n} \lambda_{jd} x_{ij} \leq x_{id}$

$\lambda_{jd}, \theta_d' \geq 0, j = 1, \dots, n; m = 1, \dots, M$

Similarly, the condition of efficiency invariance for the second stage is as follows: $\sum_{j=1}^{n} [\varphi_{jd}^{*} \left(1 - \alpha_j\right) r_j + \beta_{jd}^{*} \alpha_j r_j] - (1 - \alpha_d) r_d = E_d^{no1*} \eta_{1d}^{*} \alpha_d r_d$, φ_{jd}^{*}, β_{jd}^{*}, η_{1d}^{*} are the optimal variables of DP4.

DP4 $Max\ \eta_{2d} = \frac{1}{E_d^{no2*}}$ (20)

$s.t.\quad \sum_{j=1}^{n} -\varphi_{jd} y_{kj} + \eta_{2d} y_{kd} \leq 0, k = 1, \dots, K$

$\sum_{j=1}^{n} \left(\left(\varphi_{jd} - \beta_{jd} \right) z_{mj} \right) + \eta_{1d} z_{md} \leq z_{md}, m = 1, \dots, M$

$\sum_{j=1}^{n} \beta_{jd} x_{ij} - E_d^{no1*} \eta_{1d} x_{id} \leq 0, i = 1, \dots, I$

$\varphi_{jd}, \beta_{jd} \geq 0$

Li, Yang, Chen, Dai, and Liang (2013) argued that each DMU is selfish to pay the minimum allocated cost. In the same way, leader stage (Stage 1) prefers to afford less cost than follower stage (Stage 2), so we set the objective function as $obj2 =$

$Min \sum_{j=1}^{n} \alpha_j r_j$. Obviously, this objective function is non-linear, so we let $b_j = \alpha_j r_j, j = 1, \ldots, n$, and then the allocation plan in the noncooperative scenario can be get by model (21). $\varphi_{jd}^*, \beta_{jd}^*, \eta_{1d}^*, \eta_{2d}^*$ are obtained by solving model DP4 n times.

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{n} b_j = obj2 \qquad (21)\\
s.t. \quad & \sum_{j=1}^{n} \varphi_{jd}^* r_j - r_d + \sum_{j=1}^{n} \left(\beta_{jd}^* - \varphi_{jd}^*\right) b_j - E_d^{no1*} \eta_{1d}^* b_d + b_d = 0\\
& \sum_{j=1}^{n} \lambda_{jd}^* b_j = b_d, d = 1, \ldots, n\\
& \sum_{j=1}^{n} r_j = R\\
& L_j r_j - b_j \leq 0\\
& -U_j r_j + b_j \leq 0
\end{aligned}
$$

Finally, decision makers should select one reasonable objective function *obj2* based on actual situations. If there are several optimal solutions, we can set a second objective function, for example, $Min \sum_{j=1}^{n} |r_j - \bar{R}|$, $Min \frac{1}{n} \sum_{j=1}^{n} \left(r_j - \bar{r}\right)^2$, $\bar{r} = r/n$.

4. Application to insurance companies

Table 2 displays a dataset to illustrate our developed approach. The dataset consists of the operation data of 24 nonlife insurance companies in Taiwan, which was firstly shown in Kao and Hwang (2008). Nonlife insurance industries provide services to clients to obtain profit. The whole process of nonlife insurance service can be divided into two stages—premium acquisition and profit generation—to generate profit for insurance companies. The first stage is characterised by insurance services and the reception of direct written premiums from clients or reinsurance premiums. In the second stage, the acquired premiums are invested in a portfolio to generate profit and marketable securities, as well as real estate and mortgage loans. The system uses operation (X_1) and insurance expenses (X_2) to generate directly written (Z_1) and reinsurance premiums (Z_2). Then, such premiums are utilised to obtain underwriting (Y_1) and investment profits (Y_2).

Suppose that the companies intend to build a common platform for information sharing by spending one million NT$. Problems may arise that how to allocate the fixed cost allocation between two stages among different DMUs. Following the framework proposed in this study, we first calculate the efficiency score of Stages 1 and 2 before fixed cost allocation through Chen's additive model (Chen, Du, Sherman, and Zhu, 2010) and Liang's (Liang, Cook, and Zhu, 2008) noncooperative model. After introducing fixed cost as a new input to each stage, we calculate the optimal dual solutions of models LP1, LP2, LP3 and LP4. Finally, we obtain the optimal results on the basis of efficiency invariance and practical feasibility principles.

Table 2

Inputs (*X*), intermediate products (*Z*) and outputs (*Y*) of the 24 nonlife insurance companies in Taiwan

Company	Operations expense (NT\$0)	Insurance expense (NT\$0）	Directly written premium (NT\$0)	Reinsurance premium (NT\$0)	Underwriting profit(NT\$0)	Investment profit (NT\$0)
1. Taiwan Fire	1178744	673512	7451757	856735	984143	681687
2. Chung Kuo	1381822	1352755	10020274	1812894	1228502	834754
3. Tai Ping	1177494	592790	4776548	560244	293613	658428
4.China Mariners	601320	594259	3174851	371863	248709	177331
5. Fubon	6699063	3531614	37392862	1753794	7851229	3925272
6. Zurich	2627707	668363	9747908	952326	1713598	415058
7. Taian	1942833	1443100	10685457	643412	2239593	439039
8. Ming Tai	3789001	1873530	17267266	1134600	3899530	622868
9. Central	1567746	950432	11473162	546337	1043778	264098
10. The First	1303249	1298470	8210389	504528	1697941	554806
11. Kuo Hua	1962448	672414	7222378	643178	1486014	18259
12. Union	2592790	650952	9434406	1118489	1574191	909295
13. Shingkong	2609941	1368802	13921464	811343	3609236	223047
14.SouthChina	1396002	988888	7396396	465509	1401200	332283
15. Cathay Century	2184944	651063	10422297	749893	3355197	555482
16. Allianz President	1211716	415071	5606013	402881	854054	197947
17. Newa	1453797	1085019	7695461	342489	3144484	371984
18. AIU	757515	547997	3631484	995620	692731	163927
19.North America	159422	182338	1141950	483291	519121	46857
20. Federal	145442	53518	316829	131920	355624	26537
21. Royal & Sun Alliance	84171	26224	225888	40542	51950	6491
22. Aisa	15993	10502	52063	14574	82141	4181
23. AXA	54693	28408	245910	49864	0.1	18980
24. Mitsui Sumitomo	163297	235094	476419	644816	142370	16976

4.1 Scenario 1: Cooperative allocation models in two-stage systems

In this *Scenario*, we assume that all all stages of insurance companies cooperate to gain the maximum overall benefit. Thus, we use cooperative models to allocate fixed cost. Our proposed method consists of the following steps:

Step 1: Use model LP2 to calculate the initial efficiency of Stage 1 (E_j^{co1}), Stage 2 (E_j^{co2}) and the initial overall efficiency (E_j^{co}) of DMU_j.

Step 2: Run model DP2 to obtain the optimal values of variables $\lambda_{jd}^{1*}, \lambda_{jd}^{2*}$, and then get the efficiency invariance condition.

Step 3: Obtain the optimal value (b_j^*, α_j^*) of model (17) by setting the objective function as $Min\left|\sum_{j=1}^{n} b_j - \sum_{j=1}^{n}(r_j - b_j)\right|$, α_j is set between 0.3 and 0.7.

Step 4: Introduce the value of α_j, r_j (*j=1, ..., n*) into model LP1 to verify whether efficiency after cost allocation $(e_j^{co1}, e_j^{co2}, e_j^{co})$ is equal to the initial efficiency $(E_j^{co1}, E_j^{co2}, E_j^{co})$. The divisional efficiency after allocation is obtained by using the optimal solutions of model LP1.

Table 3

Fixed cost allocation plan in cooperative scenario

Company	Initial Efficiency			Efficiency after allocation			Allocation plan		
	E_j^{co1}	E_j^{co2}	E_j^{co}	e_j^{co1}	e_j^{co2}	e_j^{co}	$\alpha_j r_j$	$(1-\alpha_j)r_j$	r_j
1. Taiwan Fire	0.9926	0.7045	0.8491	0.9926	0.7045	0.8491	2.5159	1.0783	3.5942
2. Chung Kuo	0.9985	0.6257	0.8122	0.9985	0.6257	0.8122	3.1946	1.3691	4.5637
3. Tai Ping	0.69	1	0.8166	0.69	1	0.8166	2.385	1.0221	3.4071
4. China Mariners	0.7243	0.42	0.5965	0.7243	0.42	0.5965	1.1506	0.4932	1.6438
5. Fubon	0.8307	0.9233	0.8727	0.8307	0.9233	0.8727	14.4907	6.2103	20.701
6. Zurich	0.9606	0.4057	0.6887	0.9669	0.4083	0.6887	3.9185	1.6794	5.5979
7. Taian	0.7521	0.3522	0.5804	0.674	0.4882	0.5804	3.7916	1.6249	5.4165
8. Ming Tai	0.7256	0.378	0.5795	0.712	0.4268	0.5795	6.841	2.9318	9.7728
9. Central	1	0.2233	0.6116	1	0.2681	0.6116	2.6039	1.116	3.7199
10. The First	0.8615	0.5408	0.7131	0.8615	0.5409	0.7131	2.8153	1.2066	4.0219
11. Kuo Hua	0.7292	0.2066	0.5088	0.7626	0.2169	0.5088	2.9536	1.2659	4.2195
12. Union	1	0.7596	0.8798	1	0.7596	0.8798	4.2285	1.8122	6.0407
13. Shingkong	0.8107	0.2431	0.5565	0.7989	0.3009	0.5565	4.6129	1.9769	6.5898
14. South China	0.7246	0.374	0.5773	0.6548	0.4994	0.5773	2.708	1.1605	3.8685

15. Cathay Century	1	0.6138	0.8069	1	0.6361	0.8069	3.7676	1.6146	5.3822
16. Allianz President	0.8856	0.3615	0.6395	0.9284	0.3786	0.6395	1.9797	0.8485	2.8282
17. Newa	0.7232	0.4597	0.6126	0.5719	0.6568	0.6126	3.2305	1.3845	4.615
18. AIU	0.7935	0.3262	0.5868	0.8113	0.3495	0.5868	1.5577	0.6676	2.2253
19. North America	1	0.4112	0.7056	1	0.4112	0.7056	0.4317	0.1851	0.6168
20. Federal	0.9332	0.5857	0.7654	0.8256	0.701	0.7654	0.2383	0.1021	0.3404
21. Royal & Sun Alliance	0.7505	0.2623	0.5412	0.7505	0.2623	0.5412	0.1208	0.0518	0.1726
22. Aisa	0.5895	1	0.7418	0.5895	1	0.7418	0.0416	0.0215	0.0631
23. AXA	0.8426	0.4989	0.6854	0.8426	0.4989	0.6854	0.1027	0.044	0.1467
24.Mitsui Sumitomo	1	0.087	0.5435	1	0.087	0.5435	0.3167	0.1357	0.4524

Table 3 shows the results. The second column in Table 3 presents the efficiency value (obtained with Chen's additive model) of Stage 1 before allocation. The third column refers to the efficiency value of Stage 2, and the fourth column shows the overall efficiency calculated by model LP2 before allocation. The next three columns present efficiency after cost allocation. The last three columns are the fixed cost allocation plans for each substage and DMU. By comparing efficiencies before and after allocation, we find that the overall efficiency scores of all DMUs remain unchanged. This result is consistent with the principle of overall efficiency invariance. However, the divisional efficiency of certain DMUs, such as DMU6, DMU7, DMU8 and DMU11, has changed. Therefore, our allocation model, which is based on the overall condition of efficiency invariance, cannot guarantee that divisional efficiency remains unchanged in the cooperative mode.

The last three columns of Table 3 show that the maximum fixed cost is allocated to DMU5. The intermediate products and inputs of DMU5 far exceed those of others, thus additional input (fixed cost) is required to maintain the overall efficiency invariance. The above phenomenon can also be explained by the size of the input scale. According to the allocation plan to each substage, the allocated proportion to the first stage is approximately equal to 0.7. This value reaches the upper bound of α and satisfies the objective function *obj1*. Table 3 shows that DMU9 and DMU17 have approximately equal overall efficiency, but the inputs of DMU17 are slightly more than that of DMU9, thus DMU17 requires additional fixed cost. Compared with

DMU21, the intermediates of DMU23 are more than DMU21. It is supposed that the second stage of DMU23 affords many fixed cost, while the results show that the second stage of DMU21 (0.0518) bears more cost than DMU23 (0.044). This is due to the value of E_j^{co2} for DMU21 is lower than DMU23. The DMU with low efficiency requires many cost to ensure its efficiency invariant.

4.2 Scenario 2: Noncooperative allocation model in two-stage systems

In the proposed framework, we assume that both stages seek to maximize personal interest. Thus, we use the noncooperative model to allocate fixed cost. The allocation plan is obtained as follows:

Step 1: Use models LP3 and LP4 to calculate the initial efficiency of Stage 1 (E_j^{no1}) and Stage 2 (E_j^{no2}).

Step 2:Run models DP3 and DP4 to obtain the optimal dual variables $\lambda_{jd}^*, \varphi_{jd}^*, \beta_{jd}^*, \eta_{1d}^*, \eta_{2d}^* (d = 1, \dots, n)$ and determine efficiency invariance conditions.

Step 3: Let $L_j = 0.1$ and $U_j = 0.7$, and use model (21) to obtain the fixed cost allocation of each stage.

Step 4: Use the value of α_j, r_j $(j = 1, \dots, n)$ to verify whether efficiency after cost allocation $(e_j^{co1}, e_j^{co2}, e_j^{co})$ is equal to the initial efficiency $(E_j^{co1}, E_j^{co2}, E_j^{co})$.

Table 4 shows the allocation plan for the two-stage network systems and the efficiency of each stage.

Table 4

Fixed cost allocation plan in noncooperative scenario

Company	Initial Efficiency			Efficiency after allocation			Allocation plan		
	E_j^{no1}	E_j^{no2}	E_j^{no}	e_j^{no1}	e_j^{no2}	e_j^{no}	$\alpha_j r_j$	$(1-\alpha_j)r_j$	r_j
1. Taiwan Fire	0.9926	0.7045	0.6993	0.9926	0.7045	0.6993	0.5041	3.5244	4.0285
2. Chung Kuo	0.9985	0.6257	0.6248	0.9985	0.6257	0.6248	0.8366	4.7874	5.624
3. Tai Ping	0.69	1	0.6900	0.69	1	0.6900	0.4422	2.2849	2.7271
4. China Mariners	0.7243	0.42	0.3042	0.7243	0.42	0.3042	0.3201	1.5414	1.8615

5. Fubon	0.8375	0.8057	0.6748	0.8375	0.8057	0.6748	2.6241	15.9881	18.6122
6. Zurich	0.9637	0.401	0.3864	0.9637	0.401	0.3864	0.5003	4.2763	4.7766
7. Taian	0.7521	0.3522	0.2649	0.7521	0.3522	0.2649	0.9068	5.5153	6.4221
8. Ming Tai	0.7256	0.378	0.2743	0.7256	0.378	0.2743	1.3909	7.3714	8.7623
9. Central	1	0.2233	0.2233	1	0.2233	0.2233	0.7088	4.8224	5.5312
10. The First	0.8615	0.5408	0.4659	0.8615	0.5408	0.4659	0.6101	4.1372	4.7473
11. Kuo Hua	0.7405	0.1675	0.1240	0.7405	0.1675	0.1240	0.4931	3.0602	3.5533
12. Union	1	0.7596	0.7596	1	0.7596	0.7596	0.4977	4.4789	4.9766
13. Shingkong	0.8107	0.2431	0.1971	0.8107	0.2431	0.1971	1.0175	5.6744	6.6919
14. South China	0.7246	0.374	0.2710	0.7246	0.374	0.2710	0.6559	3.7778	4.4337
15. Cathay Century	1	0.6138	0.6138	1	0.6138	0.6138	0.4732	4.2584	4.7316
16. Allianz President	0.9072	0.3356	0.3045	0.9072	0.3356	0.3045	0.3038	2.4412	2.745
17. Newa	0.7233	0.4555	0.3295	0.7233	0.4555	0.3295	0.6573	4.1114	4.7687
18. AIU	0.7935	0.3262	0.2588	0.7935	0.3262	0.2588	0.4159	2.0684	2.4843
19. North America	1	0.4112	0.4112	1	0.4112	0.4112	0.14	0.5685	0.7085
20. Federal	0.9332	0.5857	0.5466	0.9332	0.5857	0.5466	0.0411	0.2633	0.3044
21. Royal & Sun Alliance	0.7505	0.2623	0.1969	0.7505	0.2623	0.1969	0.0201	0.1234	0.1435
22. Aisa	0.5895	1	0.5895	0.5895	1	0.5895	0.008	0.0283	0.0363
23. AXA	0.8501	0.4512	0.3836	0.8501	0.4512	0.3836	0.0214	0.1452	0.1666
24. Mitsui Sumitomo	1	0.087	0.0870	1	0.087	0.0870	0.1163	1.0467	1.163

The second to fifth columns in Table 4 present the initial efficiency value (obtained by Liang's (Liang, Cook, and Zhu, 2008) noncooperative model) of two-stage systems, respectively. The next three columns present the efficiency after cost allocation. The last three columns are the fixed cost allocation plan of each stage for DMUs. By comparing efficiencies before and after allocation, we find that the efficiency scores of Stages 1 and 2 remain unchanged, which satisfies the principle of divisional efficiency invariance. According to the allocation plan to each stage, we also know that the cost allocated to leader is less than follower.

The last three columns of Table 4 show that the highest fixed cost is allocated to DMU5 because fixed cost allocation is related to input size. The allocation plan for each substages shows that the allocating proportion to the first stage fluctuates between lower bound and upper bound. This is because the objective of the allocation model is to minimize the sum of allocation to Stage 1 of all DMUs, not minimize the allocation to Stage 1 of each DMU. Table 4 shows that DMU8 and DMU14 have similar divisional efficiency, while DMU8 could afford additional fixed cost allocation because it consumes more expenses than DMU14.

4.3 Comparison

Comparison results of different scenarios

In reality, cooperative and noncooperative relationship among units are general. Different relationships may have different consequences for companies. Hence, explaining the differences and the significances of different relationships for real life environments is necessary.

First, the results show that all the overall efficiency of DMUs with a cooperative relationship are higher than those of DMUs with a noncooperative relationship. This feature verifies the view of win-win cooperation. If substages compete with each other, the overall performance of DMU will decrease. For example, if Stage 1 and Stage 2 are equally important, the decision maker should choose a cooperative working mode to improve company's overall performance. Second, the results show that allocation plan of noncooperative model benefits the leader, because it affords less cost than follower. This illustrates, from a personal point of view, the dominant party gains more in real life. Decision makers can adopt this strategy when a department (Leader stage) has strong profitability. Third, the value of α_j is different. The α_j of each DMU achieves the upper bound U_j in a cooperative relationship, but it is diverse in a noncooperative relationship. The most likely explanation for these results is the different invariance principles. One is based on the principles of overall efficiency invariance, and the other one is based on the principle of divisional efficiency invariance. This phenomenon shows that, in cooperative scenarios, the allocation proportion to substage is related to L_j and U_j, so the decision makers should choose L_j and U_j carefully. In noncooperative scenarios, the leader stage is unwilling to bear many fixed cost, so decision makers should determine the value of L_j by the reality of leader stage.

Comparison with Lin (2011a)

The proposed method is based on efficiency invariance principle, which is defined by Cook and Kress (1999). Some other researchers also extend Cook and Kress (1999)'s method, such as Lin (2011a). However, there are some distinctions.

First, Lin (2011a) proposed methods for allocating fixed cost or resources among DMUs, which ignored the internal structures of DMUs. However, the approach of this paper aims at allocating fixed cost to two-stage network systems, and the relationship between the two stages is considered. Second, the allocation approach of Lin (2011a) is based on efficiency invariance principle, which didn't adopt practical feasibility assumption. Thus, the approach of Lin (2011a) is more flexible when several other constraints are added, and then the allocation plan can be obtained according to the reality of enterprises. However, in this paper, the enterprises' allocation plan can be changed by transforming the objective function of model. Third, Lin (2011a) combined the allocation approach with output targets setting. His approach can solve the problem of how to set targets among all DMUs appropriately.

Comparison with existing methods

To show the usefulness of the proposed approach over the current allocation approaches, we provide a comparative study here. As far as we know, Yu, Chen, and Bo (2016), Zhu, Zhang, and Wang (2019), Chu, Wu, Chu, and Zhang (2019), Li, Zhu, and Chen (2019), Ding, Zhu, Zhang, and Liang (2019) studied fixed cost allocation issues of two-stage network structure. Although Zhu, Zhang, and Wang (2019) studied fixed cost and shared resources allocation in two-stage network system, they put more focus on shared resource allocation rather than fixed cost allocation, thus their study is not comparable to our method. Therefore, here we compare the result with Yu, Chen, and Bo (2016), Chu, Wu, Chu, and Zhang (2019), Li, Zhu, and Chen (2019), Ding, Zhu, Zhang, and Liang (2019), the comparison results are showed in Table 5.

Table 5
Comparison results.

Company	Cooperative model (17)			Noncooperative model (21)			Li, Zhu and Chen (2019)		
	Stage 1	Stage 2	System	Stage 1	Stage 2	System	Stage 1	Stage 2	System
1. Taiwan Fire	2.5159	1.0783	3.5942	0.5041	3.5244	4.0285	1.2163	3.8033	5.0196

2. Chung Kuo	3.1946	1.3691	4.5637	0.8366	4.7874	5.624	1.6337	4.4202	6.0539
3. Tai Ping	2.385	1.0221	3.4071	0.4422	2.2849	2.7271	0.5207	3.523	4.0437
4. China Mariners	1.1506	0.4932	1.6438	0.3201	1.5414	1.8615	0.3798	0.6475	1.0272
5. Fubon	14.491	6.2103	20.701	2.6241	15.9881	18.6122	5.5688	26.1447	31.7135
6. Zurich	3.9185	1.6794	5.5979	0.5003	4.2763	4.7766	1.0983	2.2795	3.3778
7. Taian	3.7916	1.6249	5.4165	0.9068	5.5153	6.4221	1.457	2.8466	4.3037
8. Ming Tai	6.841	2.9318	9.7728	1.3909	7.3714	8.7623	2.1627	4.3533	6.516
9. Central	2.6039	1.116	3.7199	0.7088	4.8224	5.5312	1.9935	0	1.9935
10. The First	2.8153	1.2066	4.0219	0.6101	4.1372	4.7473	1.1608	3.6289	4.7897
11. Kuo Hua	2.9536	1.2659	4.2195	0.4931	3.0602	3.5533	0.7525	0	0.7525
12. Union	4.2285	1.8122	6.0407	0.4977	4.4789	4.9766	1.0442	5.5269	6.5711
13. Shingkong	4.6129	1.9769	6.5898	1.0175	5.6744	6.6919	2.0051	2.1947	4.1998
14. South China	2.708	1.1605	3.8685	0.6559	3.7778	4.4337	0.9864	1.9797	2.966
15.Cathay Century	3.7676	1.6146	5.3822	0.4732	4.2584	4.7316	1.4978	5.0607	6.5585
16. Allianz President	1.9797	0.8485	2.8282	0.3038	2.4412	2.745	0.7685	0.8805	1.649
17. Newa	3.2305	1.3845	4.615	0.6573	4.1114	4.7687	1.0041	4.3006	5.3047
18. AIU	1.5577	0.6676	2.2253	0.4159	2.0684	2.4843	0.4585	0.9545	1.413
19. North America	0.4317	0.1851	0.6168	0.14	0.5685	0.7085	0.1874	0.6297	0.8171
20. Federal	0.2383	0.1021	0.3404	0.0411	0.2633	0.3044	0	0.5245	0.5245
21. Royal & Sun Alliance	0.1208	0.0518	0.1726	0.0201	0.1234	0.1435	0.0113	0.0459	0.0572
22. Aisa	0.0416	0.0215	0.0631	0.008	0.0283	0.0363	0.0032	0.1142	0.1174
23. AXA	0.1027	0.044	0.1467	0.0214	0.1452	0.1666	0.0312	0.0613	0.0925
24. Mitsui Sumitomo	0.3167	0.1357	0.4524	0.1163	1.0467	1.163	0	0.1382	0.1382
SUM	69.9977	30.0026	100	13.7053	86.2947	100	25.9418	74.0584	100

Table 5 (Continued)
Comparison results.

Company	Yu, Chen, and Bo (2016)			Ding, Zhu, Zhang and Liang (2019)			Chu, Wu, Chu, and Zhang (2019)		
	Stage 1	Stage 2	System	Stage 1	Stage 2	System	Stage 1	Stage 2	System
1. Taiwan Fire	1.9945	2.7965	4.7910	1.0368	4.1421	5.1789	0.8367	2.2865	3.1233
2. Chung Kuo	2.9312	2.9464	5.8776	1.228	4.6304	5.8584	1.1251	2.7501	3.8752
3. Tai Ping	1.2725	2.9202	4.1927	0.2945	3.8431	4.1376	0.5363	1.0575	1.5938
4. China Mariners	0.8468	0.3838	1.2306	0.0775	0.682	0.7595	0.3565	0.4402	0.7967
5. Fubon	9.0287	20.0117	29.0404	3.9046	29.3284	33.2330	4.1987	18.5168	22.7155
6. Zurich	2.5250	1.0977	3.6227	0.9937	2.4917	3.4854	1.0946	3.2208	4.3153
7. Taian	2.6280	1.4853	4.1133	0.6955	3.2289	3.9244	1.1998	4.2896	5.4894
8. Ming Tai	4.2745	2.0199	6.2944	1.3076	4.9285	6.2361	1.9389	7.4133	9.3521
9. Central	2.7848	1.1935	3.9783	1.5773	0.0845	1.6618	1.2883	1.3567	2.6449
10. The First	2.0249	2.4530	4.4779	0.4343	4.0724	4.5067	0.9219	3.5628	4.4847

11. Kuo Hua	1.8449	0.7907	2.6355	0.5269	0	0.5269	0.8110	2.4341	3.2451
12. Union	2.5173	4.0280	6.5453	0.9917	6.0322	7.0239	1.0594	3.6730	4.7324
13. Shingkong	3.4179	1.4648	4.8827	1.3882	2.5584	3.9466	1.5632	6.5785	8.1417
14. South China	1.8258	1.0955	2.9212	0.4727	2.2359	2.7086	0.8305	2.6878	3.5183
15.Cathay Century	2.6118	2.9571	5.5689	1.3547	5.6778	7.0325	1.1703	6.9049	8.0751
16.Allianz President	1.4033	0.6014	2.0047	0.6436	0.9902	1.6338	0.6295	1.5083	2.1378
17. Newa	1.8455	2.4714	4.3169	0.3909	4.889	5.2799	0.8641	6.4877	7.3518
18. AIU	1.1841	0.5075	1.6916	0.3601	0.8822	1.2423	0.4078	1.3311	1.7389
19. North America	0.4383	0.1895	0.6278	0.1921	0.6069	0.7990	0.1282	1.0643	1.1925
20. Federal	0.1190	0.2758	0.3948	0	0.5618	0.5618	0.0356	0.7733	0.8089
21. Royal & Sun Alliance	0.0649	0.0278	0.0927	0.006	0.0457	0.0517	0.0254	0.0966	0.1220
22. Aisa	0.0268	0.0626	0.0894	0	0.1258	0.1258	0.0058	0.1781	0.1840
23. AXA	0.0736	0.0411	0.1147	0.0265	0.0592	0.0857	0.0276	0.0000	0.0276
24.Mitsui Sumitomo	0.3465	0.1485	0.4950	0	0	0.0000	0.0535	0.2796	0.3331
SUM	48.0306	51.9697	100	17.9032	82.0971	100	21.1087	78.8916	100

It is obvious that these approaches generate different results due to the research perspectives and efficiency principles. Li, Zhu, and Chen (2019) allocated the fixed cost by considering the operation size and the principle of efficiency maximization. Yu, Chen, and Bo (2016) introduced the concept of cross-efficiency to allocate the fixed cost. Ding, Zhu, Zhang, and Liang (2019) used the concept of satisfaction degree and Chu, Wu, Chu, and Zhang (2019) considered the competition between the DMUs' two stages. These approaches are all based on efficiency maximization principle, while our approach keep the efficiency invariant.

Based on the results of Table 5, we have some findings. First, it is clearly observed that the maximal cost is allocated to the DMU 5, who has large operation size, this suggests that the allocation plan is highly related with operation size in different scenarios. Second, the allocation cost to all stages by Yu, Chen, and Bo (2016) and ours are all positive, while in other methods' results several substages are allocated zero cost because these methods do not have the limits on allocation proportion. Third, we find that the allocation plan of Li, Zhu, and Chen (2019) and Ding, Zhu, Zhang, and Liang (2019) have a larger fluctuation than other approaches. Specifically, the differences between maximum and minimum cost allocated to DMUs by our approach are 20.6379 (cooperative) and 18.5759 (noncooperative), while those of others are bigger than ours. Finally, in noncooperative scenarios, our allocation plan is beneficial to leader stage. It can be seen from the fifth column where the allocation amount of Stages 1 is less than other allocation plan, and the sum of allocation to Stage 1 is the least.

To better compare these allocation schemes, we show the system allocation results in the following bar chart. According to **Fig. 2**, we find that the distribution trend among DMUs is similar, that is to say, the allocation results generated by ours and other approaches are very closely related. However, there are differences in some cases. For example, **Fig. 2** shows that the allocation amount of DMU5 calculated by our approach is less than that of other approaches (Chu, Wu, Chu, and Zhang, 2019; Ding, Zhu, Zhang, and Liang, 2019; Li, Zhu, and Chen, 2019; Yu, Chen, and Bo, 2016), while the allocation amount of DMU4 (or DMU6, DMU11, DMU14, DMU21) calculated by our approach is more than that of other approaches. This is because of the different efficiency principle. Compared with allocation plan based on efficiency maximization principle (Chu, Wu, Chu, and Zhang, 2019; Ding, Zhu, Zhang, and Liang, 2019; Li, Zhu, and Chen, 2019; Yu, Chen, and Bo, 2016), in this research, DMUs with small operation size and low efficiency afford much cost to keep efficiency invariant. Inversely, DMUs with large operation size and high efficiency afford less cost than Chu, Wu, Chu, and Zhang (2019), Ding, Zhu, Zhang, and Liang (2019), Li, Zhu, and Chen (2019), Yu, Chen, and Bo (2016).

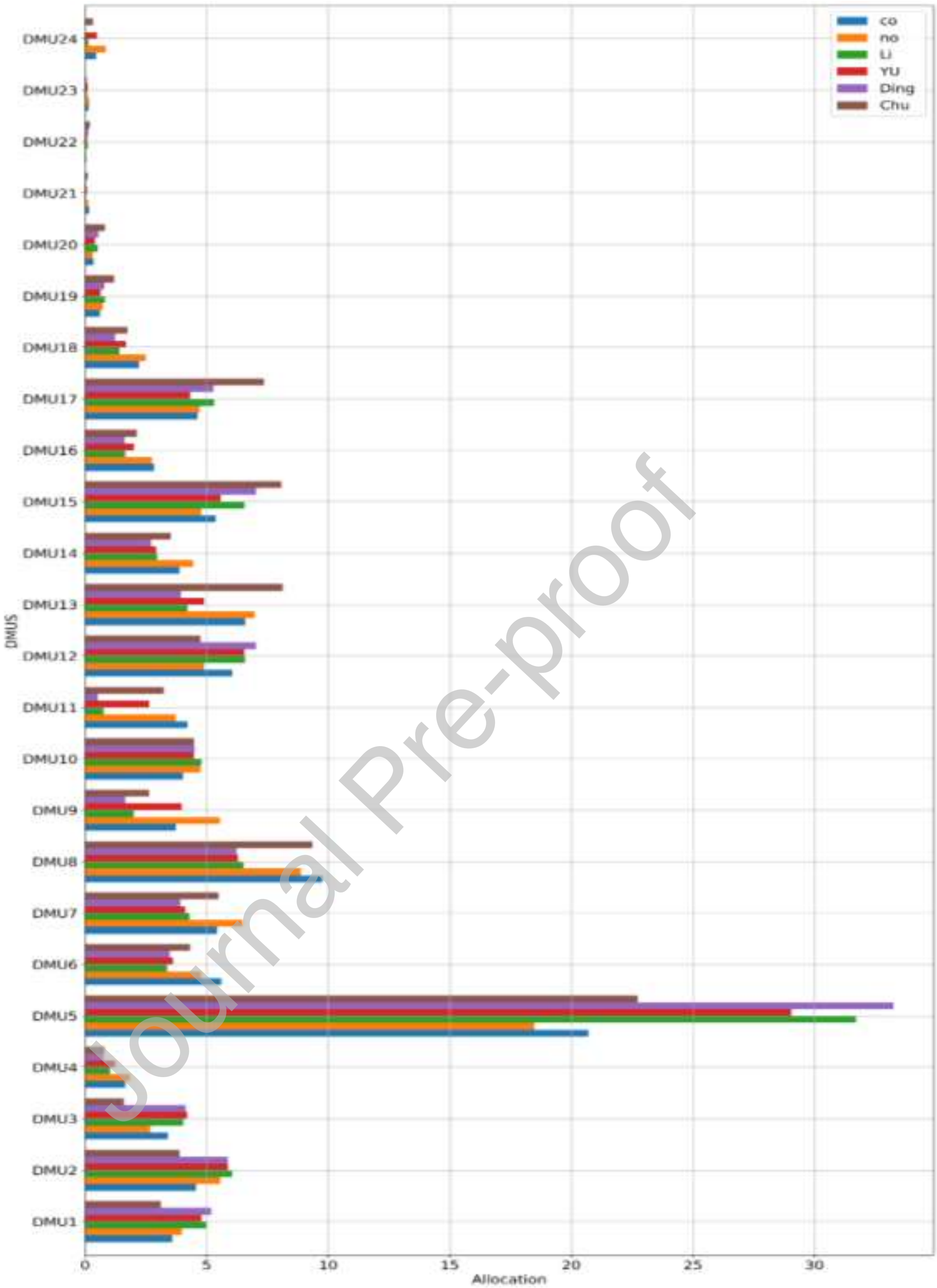

Fig. 2. Comparisons of allocation plan

5. Discussion and generalization

As mentioned before, the proposed approach for fixed cost allocation applies to basic two-stage systems. Here we extend it to general two-stage systems using the same method. As shown in **Fig. 3**, the structure of general two-stage system is characterized by the exogenous output $(y_{\xi j}^1, \xi = 1, \dots, \phi)$ of Stage 1 and exogenous input $(x_{qj}^2, q = 1, \dots, Q)$ of Stage 2. Stage 1 produces exogenous and endogenous outputs. By contrast, Stage 2 uses endogenous outputs from Stage 1 and exogenous inputs to generate the final outputs.

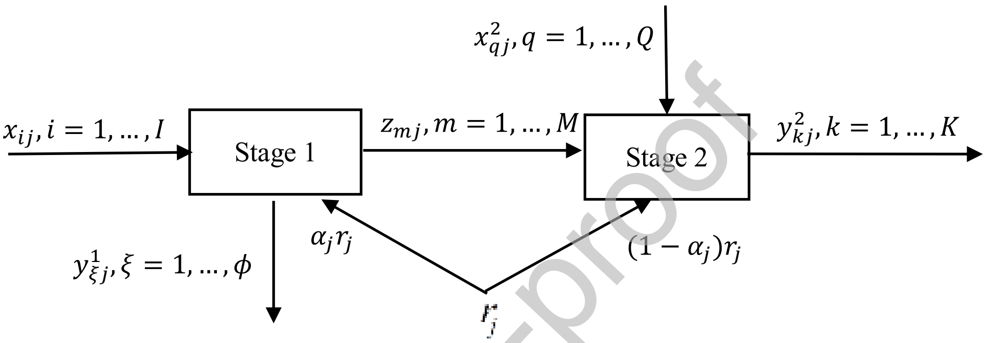

Fig. 3. General two-stage system

For DMUs with general two-stage systems, the related procedures of fixed cost allocation on the basis of efficiency invariance principle in two different scenarios are presented as follows.

Procedure. First, we should explore the efficiency invariance condition in cooperative scenarios. The efficiency of each stage considering fixed cost allocation is calculated by the following formula:

$$e_d^{gco1} = \frac{\sum_{m=1}^{M} w_m z_{md} + \sum_{\xi=1}^{\phi} t_\xi y_{\xi d}^1}{\sum_{i=1}^{I} v_i x_{id}^1 + v_{I+1} \alpha_d r_d}, \quad e_d^{gco2} = \frac{\sum_{k=1}^{K} u_k y_{kd}^2}{\sum_{m=1}^{M} w_m z_{md} + w_{M+1}(1-\alpha_d) r_d + \sum_{q=1}^{Q} h_q x_{qd}^2} \tag{22}$$

By adopting a unified approach similar to Chen, Du, Sherman, and Zhu (2010), we obtain overall efficiency on the basis of the weighted average of the two stages' efficiency scores.

$$e_d^{gco} = p_{1d} e_d^{gco1} + p_{2d} e_d^{gco2} \tag{23}$$

$$= \frac{\sum_{m=1}^{M} w_m z_{md} + \sum_{\xi=1}^{\phi} t_\xi y_{\xi d}^1 + \sum_{k=1}^{K} u_k y_{kd}^2}{\sum_{i=1}^{I} v_i x_{id}^1 + v_{I+1} \alpha_d r_d + \sum_{m=1}^{M} w_m z_{md} + w_{M+1}(1-\alpha_d) r_d + \sum_{q=1}^{Q} h_q x_{qd}^2}$$

Let $v_{I+1} = w_{M+1} = \omega$, then apply Charnes–Cooper transformation and denote

$\pi = 1/(\sum_{i=1}^{I} v_i x_{id}^1 + \sum_{m=1}^{M} w_m z_{md} + \sum_{q=1}^{Q} h_q x_{qj}^2 + \omega r_d)$, $v_i' = \pi v_i$, $w_m' = \pi w_m$, $\omega' = \pi\omega$, $u_k' = \pi u_k$, $t_\xi' = \pi t_\xi$, $h_q' = \pi h_q$, formula (23) become

$$e_d^{gco} = \sum_{m=1}^{M} w_m' z_{md} + \sum_{\xi=1}^{\phi} t_\xi' y_{\xi d}^1 + \sum_{k=1}^{K} u_k' y_{kd}^2 \tag{24}$$

The optimal value of e_d^{gco} can be calculated by the following output-oriented model.

$$\begin{aligned}
\text{LPG1}\quad Min\quad & \sum_{i=1}^{I} v_i' x_{id}^1 + \sum_{q=1}^{Q} h_q' x_{id}^2 + \sum_{m=1}^{M} w_m' z_{md} + \omega' r_d = \frac{1}{e_d^{gco}} \\
s.t.\quad & \sum_{m=1}^{M} w_m' z_{mj} + \sum_{\xi=1}^{\phi} t_\xi' y_{\xi j}^1 - \sum_{i=1}^{I} v_i' x_{ij}^1 - \omega' \alpha_j r_j \le 0 \\
& \sum_{k=1}^{K} u_k' y_{kj}^2 - \sum_{m=1}^{M} w_m' z_{mj} - \sum_{q=1}^{Q} h_{qj}' x_{qj}^2 - \omega'\left(1 - \alpha_j r_j\right) \le 0 \\
& \sum_{m=1}^{M} w_m' z_{md} + \sum_{\xi=1}^{\phi} t_\xi' y_{\xi d}^1 + \sum_{k=1}^{K} u_k' y_{kd}^2 = 1 \\
& v_i', h_q', w_m', t_\xi', u_k' \ge 0, \omega' > 0, j = 1, \dots, n \\
& i = 1, \dots, I; m = 1, \dots, M; \xi = 1, \dots, \phi; k = 1, \dots, K; q = 1, \dots, Q
\end{aligned} \tag{25}$$

The dual model of LPG1 is as follows:

$$\begin{aligned}
\text{DPG1}\quad Max\quad & \psi_d \\
s.t.\quad & \sum_{j=1}^{n} \lambda_{jd}^1 x_{ij}^1 \le x_{id}^1 , i = 1, \dots, I \\
& \sum_{j=1}^{n} \lambda_{jd}^2 x_{qj}^2 \le x_{qd}^2 , q = 1, \dots, Q \\
& \sum_{j=1}^{n} [\left(\lambda_{jd}^2 - \lambda_{jd}^1\right) z_{mj}] \le (1 - \psi_d) z_{md} , m = 1, \dots, M \\
& \sum_{j=1}^{n} -\lambda_{jd}^1 y_{\xi j}^1 + \psi_d y_{\xi d}^1 \le 0 , \xi = 1, \dots, \phi \\
& \sum_{j=1}^{n} -\lambda_{jd}^2 y_{kj}^2 + \psi_d y_{kd}^2 \le 0 , k = 1, \dots, K \\
& \sum_{j=1}^{n} (\lambda_{jd}^1 \alpha_j r_j + \lambda_{jd}^2 \left(1 - \alpha_j\right) r_j) \le r_d \\
& \lambda_{jd}^1, \lambda_{jd}^2 \ge 0, j = 1, \dots, n.
\end{aligned} \tag{26}$$

By using the same method as that used by Cook and Kress (1999), the overall condition of efficiency invariance is $\sum_{j=1}^{n} \left[\lambda_{jd}^{1*} \alpha_j r_j + \lambda_{jd}^{2*} \left(1 - \alpha_j\right) r_j\right] = r_d$, $\lambda_{jd}^{1*}, \lambda_{jd}^{2*}$ represents the optimal variable of model (27).

$$\begin{aligned}
Max\quad & \psi_d' \\
s.t.\quad & \sum_{j=1}^{n} \lambda_{jd}^1 x_{ij}^1 \le x_{id}^1 , i = 1, \dots, I
\end{aligned} \tag{27}$$

$$\sum_{j=1}^{n} \lambda_{jd}^{2} x_{qj}^{2} \leq x_{qd}^{2}, q = 1, \dots, Q$$

$$\sum_{j=1}^{n} [(\lambda_{jd}^{2} - \lambda_{jd}^{1}) z_{mj}] \leq (1 - \psi'_d) z_{md}, m = 1, \dots, M$$

$$\sum_{j=1}^{n} -\lambda_{jd}^{1} y_{\xi j}^{1} + \psi_d y_{\xi d}^{1} \leq 0, \xi = 1, \dots, \phi$$

$$\sum_{j=1}^{n} -\lambda_{jd}^{2} y_{kj}^{2} + \psi'_d y_{kd}^{2} \leq 0, k = 1, \dots, K$$

$$\lambda_{jd}^{1}, \lambda_{jd}^{2} \geq 0, j = 1, \dots, n.$$

Second, the condition of efficiency invariance in noncooperative scenarios can be derived by the same way. The following models refer to the divisional efficiency of each stage after allocation.

Leader Stage

$$Min \quad \sum_{i=1}^{I} v_i x_{id}^{1} + \omega \alpha_d r_d = \frac{1}{e_d^{gno1}} \tag{28}$$

$$s.t. \quad \sum_{m=1}^{M} w_m z_{mj} + \sum_{\xi=1}^{\phi} t_\xi y_{\xi j}^{1} - \sum_{i=1}^{I} v_i x_{ij}^{1} - \omega \alpha_j r_j \leq 0$$

$$\sum_{m=1}^{M} w_m z_{md} + \sum_{\xi=1}^{\phi} t_\xi y_{\xi d}^{1} = 1$$

$$j = 1, \dots, n, w_m, v_i, t_\xi \geq 0, \omega > 0$$

$$m = 1, \dots, M; i = 1, \dots, I; \xi = 1, \dots, \phi$$

Follower Stage

$$Min \quad \sum_{m=1}^{M} w_m z_{md} + \sum_{q=1}^{Q} h_q x_{qd}^{2} + \omega (1 - \alpha_d) r_d = \frac{1}{e_d^{gno2}} \tag{29}$$

$$s.t. \quad \sum_{m=1}^{M} w_m z_{mj} + \sum_{\xi=1}^{\phi} t_\xi y_{\xi j}^{1} - \sum_{i=1}^{I} v_i x_{ij}^{1} - \omega \alpha_j r_j \leq 0$$

$$\sum_{k=1}^{K} u_k y_{kj}^{2} - \sum_{m=1}^{M} w_m z_{mj} - \sum_{q=1}^{Q} h_q x_{qj}^{2} - \omega (1 - \alpha_j) r_j \leq 0$$

$$\sum_{\xi=1}^{\phi} t_\xi y_{\xi d}^{1} + \sum_{m=1}^{M} w_m z_{md} - e_d^{gno1*} \left(\sum_{i=1}^{I} v_i x_{id}^{1} + \omega \alpha_d r_d\right) = 0$$

$$\sum_{k=1}^{K} u_k y_{kd}^{2} = 1$$

$$j = 1, \dots, n, v_i, t_\xi, w_m, h_q, u_k \geq 0, \omega > 0$$

$$m = 1, \dots, M; i = 1, \dots, I; \xi = 1, \dots, \phi; k = 1, \dots, K; q = 1, \dots, Q$$

To satisfy the principle of efficiency invariance, the following formula must be established $\sum_{j=1}^{n} \lambda_{jd}^{*} \alpha_j r_j = \alpha_d r_d$, $\sum_{j=1}^{n} [\varphi_{jd}^{*} (1 - \alpha_j) r_j + \beta_{jd}^{*} \alpha_j r_j] - (1 - \alpha_d) r_d -$

$E_d^{gno1*}\eta_{1d}^*\alpha_d r_d = 0$. λ_{jd}^* is the optimal variable of model (30) when DMU_d is evaluated, and $\varphi_{jd}^*, \beta_{jd}^*, \eta_{1d}^*$ are calculated by model (31). E_d^{gno1*}, E_d^{gno2*} represents the efficiency of two substages.

$$
\begin{aligned}
Max \quad & \theta'_d = \frac{1}{E_d^{gno1}} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_{jd} z_{mj} \geq \theta'_d z_{md} \\
& \sum_{j=1}^{n} \lambda_{jd} y_{\xi j}^1 \geq \theta'_d y_{\xi d}^1 \\
& \sum_{j=1}^{n} \lambda_{jd} x_{ij}^1 \leq x_{id}^1 \\
& \lambda_{jd}, \theta'_d \geq 0 \\
& j = 1, \ldots, n; m = 1, \ldots, M;\ \xi = 1, \ldots, \phi
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
Max \quad & \eta_{2d} = \frac{1}{E_d^{gno2}} \\
s.t. \quad & \sum_{j=1}^{n} -\varphi_{jd} y_{kj}^2 + \eta_{2d} y_{kd}^2 \leq 0, k = 1, \ldots, K \\
& \sum_{j=1}^{n} -\beta_{jd} y_{\xi j}^1 + \eta_{1d} y_{\xi d}^1 \leq 0, \xi = 1, \ldots, \phi \\
& \sum_{j=1}^{n} (\varphi_{jd} - \beta_{jd}) z_{mj} + \eta_{1d} z_{md} \leq z_{md}, m = 1, \ldots, M \\
& \sum_{j=1}^{n} \beta_{jd} x_{ij}^1 - E_d^{gno1*} \eta_{1d} x_{id}^1 \leq 0, i = 1, \ldots, I \\
& \sum_{j=1}^{n} \varphi_{jd} x_{qj}^2 \leq x_{qd}^2, q = 1, \ldots, Q \\
& \varphi_{jd}, \beta_{jd} \geq 0, j = 1, \ldots, n
\end{aligned} \tag{31}
$$

The allocation plan can be obtained easily in accordance with the condition of efficiency invariance. When the exogenous inputs and outputs are 0, the general two-stage structure transforms into a basic two-stage system.

6. Conclusions and direction for future studies

Numerous DMUs, such as banks, hospitals, universities or corporations, have two-stage network structures. Previous DEA-based models for fixed cost allocation usually treat DMUs as a "black box". In this work, we aim to open the "black box" of DMUs and investigate their internal structure. Therefore, we extend Cook and Kress

(1999)'s model to two-stage systems by considering the relationship between two subtages of DMUs. We also combine cooperative and noncooperative models with Cook and Kress (1999)'s approach to explore the condition of efficiency invariance. The cooperative model applies to the case when two substages work together to obtain the best overall system performance. By contrast, the noncooperative model is suitable for network systems whose substages compete with each other. This study fills in the gap left by previous studies on the model of fixed cost allocation, which combines the two-stage system with the gaming concept.

We prove that when the two stages work together, the overall efficiency of all DMUs remain unchanged after fixed cost allocation by using our approach. Moreover, when the two stages are noncooperative, we provide an allocation scheme that maintains the divisional efficiencies of two stages of all DMUs. It is found that the allocation results are related to input size and efficiency. If two DMUs' inputs are similar, DMU with low efficiency affords more cost than the other because the DMU with low efficiency requires more cost to ensure its efficiency invariant when the input sizes of two DMUs are the same. Meanwhile, if the efficiency scores of two DMUs are similar, DMU with large input affords more cost than another DMU. These results indicate that our allocation has a direct relationship with efficiencies and the input size of each stage.

This work can be extended to several directions. First, investigating the fixed cost allocation of two-stage systems with cooperative relationships on the basis of the principle of divisional efficiency invariance remains interesting and crucial. Next, our proposed models are based on the principle of efficiency invariance. Therefore, developing an approach that is based on the principle of efficiency maximization to solve this problem is critical. Finally, the leader-follower relationship between two stages are given and identified based on enough priori knowledge in our noncooperative scenario. However, sometimes the leader stages of two-stage systems are unknown in the absence of priori knowledge. Extending our approach to this case is an important and interesting work in the future.

Acknowledgments

The research is supported by National Natural Science Foundation of China (Nos. 71871223, 91846301, 71631008), Innovation - Driven Planning Foundation of Central South University (2019CX041), Major Project for National Natural Science Foundation of China (71790615).

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