EXACT SOLUTIONS OF EINSTEIN'S EQUATIONS
FOR AXISYMMETRIC GRAVITATIONAL FIELDS

by

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SUMMARY

In this thesis Einstein's equations for vacuum axi-symmetric, stationary, gravitational fields are considered. Five analytic solutions of these equations are presented. In each case the analytic solutions are generated by a non-linear ordinary differential equation of the second order.

Some particular integrals of these generating differential equations are given, resulting in some known and unknown metrics. The known metrics are, the Kerr and the Tomimatsu-Sato class. In the derivation these known metrics are shown to have a common origin. It is further shown that they result from a parameter in the generating differential equation assuming certain eigenvalues.

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CHAPTER ONE

INTRODUCTION

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INTRODUCTION

In this thesis we shall only be concerned with obtaining analytic solutions of Einstein's equations of gravitation for vacuum axisymmetrical gravitational fields, which are stationary. The cosmical constant is taken to be zero.

In Chapter 2, Einstein's equations, appropriate to the case under consideration, are written out in a straightforward manner, no use being made of either, the methods of differential forms, or the complex symbolism of Ernst. The remainder of Chapter 2 is devoted to showing that the field equations will be satisfied, provided the metric coefficients are determined by one, or the other, of two sets of equivalent equations. These sets of equations are called set A and set B.

Chapter 3 gives an account of our abortive attempts to find new solutions of the field equations, working with set A.

In Chapter 4, set B is considered and it is shown that it is possible, by choosing a certain 2-dimensional harmonic function Δ appropriately, to obtain five analytic solutions of the field equations. These are generated in each case by an ordinary differential equation of the second order.

In Chapters 5, 6, 7, 8 and 9 the generating differential equation is derived corresponding to each of these five possible forms for Δ. In each case the generating differential equation turns out to be non-linear, and because of this we are unable to give their general solutions, however, some particular integrals have been obtained resulting in some known and unknown metrics.

From the physical point of view, Chapters 5, 6 and 7 are the most important. In Chapters 5 and 6, we show that the Kerr-Tomimatsu-Sato metrics, are contained in our analytic solutions as particular cases of a more general integral. In our derivation of these metrics it is, however necessary to consider separately the three cases when $a^2 > m^2$, $a^2 < m^2$ and $a^2 = m^2$, a and m being respectively the angular momentum and mass of the bounded source. Chapter 5 gives the results appropriate to $a^2 < m^2$ and Chapter 6 those for $a^2 > m^2$. In Chapter 7, we show that the Kerr metric for $a^2 = m^2$ is a particular case of our more general integral. Since in the limit as $a^2 \rightarrow m^2$ all the Tomimatsu-Sato metrics reduce to the Kerr with $a^2 = m^2$ we can say that these are also included in the results of Chapter 7.

From the forms of the generating differential equations of Chapters 5, 6 and 7 it seems unlikely that these general metrics can be regarded as three special cases of a single metric as is possible with the Kerr metric.

Further, the above mentioned particular integrals contained in the results of Chapters 5 and 6 arise by putting a constant in our generating differential equation equal to $-4n^2$ (n, a non-zero integer), e.g. n = 1 gives the Kerr metric, with n = 2,3,4 we get the Tomimatsu-Sato metrics, corresponding to their classification index δ taking the values 2, 3 and 4. We conjecture, with Tomimatsu and Sato that solutions will exist for all values of n. Although we are unable to prove that this will be the case, at least in our derivation the problem is well defined, viz. we have only to show that a certain ordinary differential equation of the second order subject to certain boundary conditions will only admit a solution which is a rational function of its argument.

The metrics arising from the results of Chapters 8 and 9 are not asymptotically flat and so can only represent fields inside a bounded region of space.

Chapter 10 is a short note on the possibility of our metrics being of the special relativity type in some coordinate system.

Conclusions are summarized in Chapter 11.

CHAPTER TWO

DIFFERENTIAL EQUATIONS FOR STATIONARY AXISYMMETRICAL GRAVITATIONAL FIELDS IN A VACUUM

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We shall follow Papapetrou, Lewis and Van Stockum and write the metric in the form:

$$ds^2 = e^{2\psi}[(dx^1)^2 + (dx^2)^2] + A(dx^3)^2 + 2B\,dx^3dx^4 - C(dx^4)^2 \qquad \dots (2.1)$$

where ψ, A, B and C are functions of x^1 and x^2 only.

So that

$$(g_{ij}) = \begin{bmatrix} e^{2\psi} & 0 & 0 & 0 \\ 0 & e^{2\psi} & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & B & -C \end{bmatrix} \qquad \dots (2.2)$$

and

$$(g^{ij}) = \begin{bmatrix} e^{-2\psi} & 0 & 0 & 0 \\ 0 & e^{-2\psi} & 0 & 0 \\ 0 & 0 & C\Delta^{-2} & B\Delta^{-2} \\ 0 & 0 & B\Delta^{-2} & -A\Delta^{-2} \end{bmatrix} \qquad \dots (2.3)$$

where i,j = 1,2,3,4 and

$$\Delta^2 = AC + B^2 \qquad \ldots (2.4)$$

We note by using (2.4) that our metric (2.1) can be rewritten as:

$$ds^2 = e^{2\psi}[(dx^1)^2 + (dx^2)^2] + C^{-1}\Delta^2(dx^3)^2 - C^{-1}(Bdx^3 - Cdx^4)^2 \qquad \ldots (2.5)$$

Using (2.2) and (2.4) we get

$$g = |g_{ij}| = -e^{4\psi}\Delta^2$$

$$\therefore \qquad \sqrt{-g} = e^{2\psi}\Delta \qquad \ldots (2.6)$$

Notation

Throughout this and the following Chapters we shall use the notation that suffix 1 denotes partial differentiation with respect to x^1; suffix 2 denotes partial differentiation with respect to x^2 and a dash denotes ordinary differentiation of a function of a single variable with respect to its argument.

The three index symbols Γ^i_{jk} , $(i,j,k = 1,2,3,4)$ can now be calculated by substituting from (2.2) and (2.3) in

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{im}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^m}\right)$$

the non-zero Γ^i_{jk}'s being:

$$\Gamma^1_{11} = \psi_1$$
$$\Gamma^2_{11} = -\psi_2$$
$$\Gamma^1_{12} = \psi_2$$
$$\Gamma^2_{12} = \psi_1$$
$$\Gamma^3_{13} = \tfrac{1}{2}\Delta^{-2}(CA_1 + BB_1)$$
$$\Gamma^4_{13} = \tfrac{1}{2}\Delta^{-2}(BA_1 - AB_1)$$
$$\Gamma^3_{14} = \tfrac{1}{2}\Delta^{-2}(CB_1 - BC_1)$$
$$\Gamma^4_{14} = \tfrac{1}{2}\Delta^{-2}(BB_1 + AC_1)$$
$$\Gamma^1_{22} = -\psi_1$$
$$\Gamma^2_{22} = \psi_2$$
$$\Gamma^3_{23} = \tfrac{1}{2}\Delta^{-2}(CA_2 + BB_2)$$
$$\Gamma^4_{23} = \tfrac{1}{2}\Delta^{-2}(BA_2 - AB_2)$$
$$\Gamma^1_{33} = -\tfrac{1}{2}e^{-2\psi}A_1$$
$$\Gamma^3_{24} = \tfrac{1}{2}\Delta^{-2}(CB_2 - BC_2)$$
$$\Gamma^4_{24} = \tfrac{1}{2}\Delta^{-2}(BB_2 + AC_2)$$
$$\Gamma^2_{33} = -\tfrac{1}{2}e^{-2\psi}A_2$$
$$\Gamma^1_{34} = -\tfrac{1}{2}e^{-2\psi}B_1$$
$$\Gamma^2_{34} = -\tfrac{1}{2}e^{-2\psi}B_2$$
$$\Gamma^1_{44} = \tfrac{1}{2}e^{-2\psi}C_1$$
$$\Gamma^2_{44} = \tfrac{1}{2}e^{-2\psi}C_2$$

If we now substitute from the above in the Ricci tensor,

$$R_{jk} = \Gamma^i_{rk}\Gamma^r_{ij} - \Gamma^r_{jk}\frac{\partial}{\partial x^r}(\ln\sqrt{-g}) + \frac{\partial^2}{\partial x^j \partial x^k}(\ln\sqrt{-g}) - \frac{\partial \Gamma^i_{jk}}{\partial x^i}$$

we get,

$$R_{13} = R_{14} = R_{23} = R_{24} \equiv 0$$

$$R_{11} = \psi_{11}+\psi_{22}+\Delta^{-1}(\psi_2\Delta_2-\psi_1\Delta_1)+\Delta^{-1}\Delta_{11}-\tfrac{1}{2}\Delta^{-2}(A_1C_1+B_1^2) \qquad \ldots (2.7)$$

$$R_{22} = \psi_{11}+\psi_{22}+\Delta^{-1}(\psi_1\Delta_1-\psi_2\Delta_2)+\Delta^{-1}\Delta_{22}-\tfrac{1}{2}\Delta^{-2}(A_2C_2+B_2^2) \qquad \ldots (2.8)$$

$$R_{12} = -\Delta^{-1}(\psi_1\Delta_2+\psi_2\Delta_1)+\Delta^{-1}\Delta_{12}-\Delta^{-2}(A_1C_2+A_2C_1+2B_1B_2)/4 \qquad \ldots (2.9)$$

$$R_{33} = \tfrac{1}{2}e^{-2\psi}\{A_{11}+A_{22}-\Delta^{-1}(A_1\Delta_1+A_2\Delta_2)+ A\Delta^{-2}(A_1C_1+B_1^2+A_2C_2+B_2^2)\} \qquad \ldots (2.10)$$

$$R_{44} = -\tfrac{1}{2}e^{-2\psi}\{C_{11}+C_{22}-\Delta^{-1}(C_1\Delta_1+C_2\Delta_2)+ C\Delta^{-2}(A_1C_1+B_1^2+A_2C_2+B_2^2)\} \qquad \ldots (2.11)$$

$$R_{34} = \tfrac{1}{2}e^{-2\psi}\{B_{11}+B_{22}-\Delta^{-1}(B_1\Delta_1+B_2\Delta_2)+ B\Delta^{-2}(A_1C_1+B_1^2+A_2C_2+B_2^2)\} \qquad \ldots (2.12)$$

In our case the Einstein equations are $R_{ij} = 0$. We shall replace the equations $R_{33} = 0$, $R_{34} = 0$ and $R_{44} = 0$, by the set :

$$CR_{33} - AR_{44} + 2BR_{34} = 0 \qquad \ldots (2.13)$$

$$BR_{44} + CR_{34} = 0 \qquad \ldots (2.14)$$

$$CR_{33} + AR_{44} = 0 \qquad \ldots (2.15)$$

which are linearly independent provided $C \neq 0$; $\Delta \neq 0$. This we shall assume to be the case throughout. In fact we shall assume $\Delta \neq$ constant.

Substituting from (2.10), (2.11) and (2.12) in (2.13) and using (2.4) we get

$$\Delta_{11} + \Delta_{22} = 0 \qquad \ldots (2.16)$$

Substituting from (2.11) and (2.12) in equation (2.14) we get

$$(\Delta^{-1}C^2D_1)_1 + (\Delta^{-1}C^2D_2)_2 = 0 \qquad \ldots (2.17)$$

where

$$D = BC^{-1} \qquad \ldots (2.18)$$

Now equation (2.17) is the condition for the existence of a function $F(x^1, x^2)$, such that:

$$D_1 = -\Delta C^{-2}F_2 \qquad \ldots (2.19)$$

$$D_2 = \Delta C^{-2}F_1 \qquad \ldots (2.20)$$

The condition $D_{12} = D_{21}$ then leads to the equation

$$\tau = F_{11}+F_{22}+\Delta^{-1}(\Delta_1 F_1+\Delta_2 F_2)-2C^{-1}(C_1 F_1+C_2 F_2) = 0 \quad \dots (2.21)$$

Substituting from equations (2.10) and (2.11) in equation (2.15) and eliminating A and B using equations (2.4) and (2.18), we get after substituting for D_1 and D_2 from equations (2.19) and (2.20):

$$\sigma = C_{11}+C_{22}+\Delta^{-1}(C_1\Delta_1+C_2\Delta_2)-C^{-1}(C_1^2+C_2^2-F_1^2-F_2^2) = 0 \quad \dots (2.22)$$

The remaining equations to be satisfied are $R_{11} = 0$, $R_{22} = 0$ and $R_{12} = 0$, the first two we replace by

$$R_{11} - R_{22} = 0 \quad \dots (2.23)$$

$$R_{11} + R_{22} = 0 \quad \dots (2.24)$$

Substituting from equations (2.7) and (2.8) in equations (2.23) and (2.24) gives:

$$\psi_2\Delta_2-\psi_1\Delta_1 = \Delta_{22}+\tfrac{1}{4}\Delta^{-1}(A_1C_1+B_1^2-A_2C_2-B_2^2) \quad \dots (2.25)$$

$$\psi_{11}+\psi_{22} = \tfrac{1}{4}\Delta^{-2}(A_1C_1+B_1^2+A_2C_2+B_2^2) \quad \dots (2.26)$$

where we have used (2.16).

From (2.9) $R_{12} = 0$ implies

$$\psi_1\Delta_2+\psi_2\Delta_1 = \Delta_{12}-\tfrac{1}{4}\Delta^{-1}(A_1C_2+A_2C_1+2B_1B_2) \qquad \ldots (2.27)$$

Solving equations (2.25) and (2.27) for ψ_1 and ψ_2 we get

$$\psi_2(\Delta_1^2+\Delta_2^2) = \Delta_2\Delta_{22}+\Delta_1\Delta_{12}+(4\Delta)^{-1}[\Delta_2(A_1C_1+B_1^2-A_2C_2-B_2^2) - \Delta_1(A_1C_2+A_2C_1+2B_1B_2)] \qquad \ldots (2.28)$$

$$\psi_1(\Delta_1^2+\Delta_2^2) = \Delta_2\Delta_{12}-\Delta_1\Delta_{22}-(4\Delta)^{-1}[\Delta_2(A_1C_2+A_2C_1+2B_1B_2) + \Delta_1(A_1C_1+B_1^2-A_2C_2-B_2^2)] \qquad \ldots (2.29)$$

Substituting for A and B from equations (2.4) and (2.18), in equations (2.28) and (2.29), and using (2.19) and (2.20) we get

$$\psi_2(\Delta_1^2+\Delta_2^2) = \Delta_2\Delta_{22}+\Delta_1\Delta_{12}-\tfrac{1}{2}C^{-1}C_2(\Delta_1^2+\Delta_2^2) + \tfrac{1}{4}C^{-2}[\Delta_2(C_2^2+F_2^2-C_1^2-F_1^2)+2\Delta_1(C_1C_2+F_1F_2)]\Delta \qquad \ldots (2.30)$$

$$\psi_1(\Delta_1^2+\Delta_2^2) = \Delta_2\Delta_{12}-\Delta_1\Delta_{22}-\tfrac{1}{2}C^{-1}C_1(\Delta_1^2+\Delta_2^2) - \tfrac{1}{4}C^{-2}[\Delta_1(C_2^2+F_2^2-C_1^2-F_1^2)-2\Delta_2(C_1C_2+F_1F_2)]\Delta \qquad \ldots (2.31)$$

We now introduce a new function $S(x^1, x^2)$, defined by

$$e^{2\psi} = C^{-1}e^{S}(\Delta_1^2 + \Delta_2^2) \qquad \ldots (2.32)$$

In terms of S equations (2.30) and (2.31) read

$$2(\Delta_1^2+\Delta_2^2)S_1 = \Delta C^{-2}\{\Delta_1(C_1^2+F_1^2-C_2^2-F_2^2)+2\Delta_2(C_1C_2+F_1F_2)\} \qquad \ldots (2.33)$$

$$2(\Delta_1^2+\Delta_2^2)S_2 = -\Delta C^{-2}\{\Delta_2(C_1^2+F_1^2-C_2^2-F_2^2)-2\Delta_1(C_1C_2+F_1F_2)\} \qquad \ldots (2.34)$$

where we have used equation (2.16). Also equation (2.26) becomes in terms of S,

$$S_{11} + S_{22} = -\tfrac{1}{2}C^{-2}(C_1^2+C_2^2+F_1^2+F_2^2) \qquad \ldots (2.35)$$

where we have used equations (2.16) and (2.22).

We next calculate $\phi = S_{12}-S_{21}$, and $S_{11}+S_{22}$, using only equations (2.16), (2.33) and (2.34). We get

$$\phi = \frac{\Delta C^{-2}}{\Delta_1^2+\Delta_2^2}\{\sigma(C_1\Delta_2-C_2\Delta_1)-\tau(F_1\Delta_2-F_2\Delta_1)\} \qquad \ldots (2.36)$$

$= 0$, by equations (2.21), and (2.22) and hence the consistency condition $S_{12} = S_{21}$ is satisfied.

$$S_{11} + S_{22} = -\tfrac{1}{2}C^{-2}(C_1^2+C_2^2+F_1^2+F_2^2)$$
$$+ \frac{\Delta C^{-2}}{\Delta_1^2+\Delta_2^2}\{\sigma(C_1\Delta_1+C_2\Delta_2) + \tau(F_1\Delta_1+F_2\Delta_2)\}$$
$$\dots (2.37)$$

which is the same as (2.35), since $\sigma = \tau = 0$. Hence, equation (2.35) is derivable from equations (2.16), (2.21), (2.22), (2.33) and (2.34), and so may be disregarded.

SUMMARY

From the above argument we see that the functions Δ, F, C and S, are determined by the following, consistent set of equations,

$$\Delta_{11} + \Delta_{22} = 0 \qquad \dots (2.38)$$

$$\sigma = C_{11}+C_{22}+\Delta^{-1}(C_1\Delta_1+C_2\Delta_2)-C^{-1}(C_1^2+C_2^2-F_1^2-F_2^2) = 0$$
$$\dots (2.39)$$

$$\tau = F_{11}+F_{22}+\Delta^{-1}(F_1\Delta_1+F_2\Delta_2)-2C^{-1}(C_1F_1+C_2F_2) = 0$$
$$\dots (2.40)$$

$$2(\Delta_1^2+\Delta_2^2)S_1 = \Delta C^{-2}\{\Delta_1(C_1^2+F_1^2-C_2^2-F_2^2)+2\Delta_2(C_1C_2+F_1F_2)\}$$
$$\dots (2.41)$$

$$2(\Delta_1^2+\Delta_2^2)S_2 = -\Delta C^{-2}\{\Delta_2(C_1^2+F_1^2-C_2^2-F_2^2)-2\Delta_1(C_1C_2+F_1F_2)$$
$$\dots (2.42)$$

This set of equations we shall call set A

By considering equations (2.16), (2.33), (2.34) and (2.35), it will be observed that equations (2.39) and (2.40) are deducible from them, provided that the determinant,

$$\begin{vmatrix} C_1\Delta_2-\Delta_1 C_2 & F_2\Delta_1-F_1\Delta_2 \\ C_1\Delta_1+C_2\Delta_2 & F_1\Delta_1+F_2\Delta_2 \end{vmatrix}$$

is non-vanishing. This we express in the form of a theorem.

THEOREM

Suppose that functions Δ, F, C and S, can be found which satisfy the following set of equations

$$\Delta_{11} + \Delta_{22} = 0 \qquad \ldots (2.43)$$

$$2(\Delta_1^2+\Delta_2^2)S_1 = \Delta C^{-2}\{\Delta_1(C_1^2+F_1^2-C_2^2-F_2^2)+2\Delta_2(C_1C_2+F_1F_2)\} \qquad \ldots (2.44)$$

$$2(\Delta_1^2+\Delta_2^2)S_2 = -\Delta C^{-2}\{\Delta_2(C_1^2+F_1^2-C_2^2-F_2^2)-2\Delta_1(C_1C_2+F_1F_2)\} \qquad \ldots (2.45)$$

$$S_{11}+S_{22} = -\tfrac{1}{2}C^{-2}(C_1^2+C_2^2+F_1^2+F_2^2) \qquad \ldots (2.46)$$

and in addition the determinant

$$\Gamma = \begin{vmatrix} C_1\Delta_2-\Delta_1 C_2 & F_2\Delta_1-F_1\Delta_2 \\ C_1\Delta_1+C_2\Delta_2 & F_1\Delta_1+F_2\Delta_2 \end{vmatrix} \qquad \ldots (2.47)$$

is non-vanishing. Then Δ, F, C and S, also satisfy the equations of set A.

PROOF

From equations (2.43), (2.44) and (2.45) the condition, $S_{12} = S_{21}$, leads to the equation,

$$\sigma(C_1\Delta_2 - C_2\Delta_1) + \tau(F_2\Delta_1 - F_1\Delta_2) = 0 \qquad \ldots (2.48)$$

If we calculate $S_{11} + S_{22}$, from equations (2.44) and (2.45) and substitute the result in equation (2.46), we get, after using equation (2.43)

$$\sigma(C_1\Delta_1 + C_2\Delta_2) + \tau(F_1\Delta_1 + F_2\Delta_2) = 0 \qquad \ldots (2.49)$$

Now since $\Gamma \neq 0$, equations (2.48) and (2.49) give $\sigma = \tau = 0$.

We shall call the set of equations (2.43) to (2.46) inclusive, set B. The sets A and B being completely equivalent when $\Gamma \neq 0$.

CHAPTER THREE

INVESTIGATION OF THE EQUATIONS OF SET A

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We shall take the solution of equation (2.38) to be $\Delta = x^1$. That this can always be done, without loss of generality can be seen from the following argument.

Suppose we were to choose $\Delta = u(x^1,x^2) \neq x^1$, $u(x^1,x^2)$ satisfying equation (2.38). Then the coordinate transformation $x^i \to \bar{x}^i$ $(i = 1,2,3,4)$, defined by

$$\bar{x}^1 = u(x^1,x^2), \quad \bar{x}^2 = v(x^1,x^2), \quad \bar{x}^3 = x^3, \quad \bar{x}^4 = x^4$$

$v(x^1,x^2)$ being fixed by the equations, $v_2 = u_1$, $v_1 = -u_2$, would not change the form of our metric, and would make $\Delta = \bar{x}^1$ in the $\bar{x}$-frame.

With $\Delta = x^1$ and using the notation, $x^1 = \rho$, $x^2 = z$, the equations of set A become:

$$\nabla^2 C - C^{-1}\{(\nabla C)^2 - (\nabla F)^2\} = 0 \qquad \ldots (3.1)$$

$$\nabla^2 F - 2C^{-1}\nabla C.\nabla F = 0 \qquad \ldots (3.2)$$

$$S\rho = \tfrac{1}{2}\rho C^{-2}(C_\rho^2 + F_\rho^2 - C_z^2 - F_z^2) \qquad \ldots (3.3)$$

$$Sz = \rho C^{-2}(C\rho Cz + F\rho Fz) \qquad \ldots (3.4)$$

where for any functions $f(\rho,z)$ and $g(\rho.z)$ we define:

$$\nabla^2 f = f_{\rho\rho} + f_{zz} + \rho^{-1} f_{\rho}$$

$$\nabla f . \nabla g = f_{\rho} g_{\rho} + f_z g_z$$

and $(\nabla f)^2 = \nabla f . \nabla f$.

Of the set of equations (3.1), (3.2), (3.3), (3.4), equations (3.1) and (3.2) are the most difficult to solve, and as there are no known techniques for handling them, all we can do is to proceed by trial and error.

The case when F is a function of C has been discussed by Papapetrou (1), and the case when D is a function of $\underline{C}_{\rho}$, by Lewis (2) and van Stockum (3).

As regards trying to find the general solution of equations (3.1) and (3.2), the only possibility seems to be that of trying to find functions $\xi(F,C)$, $\eta(F,C)$ such that, if we take two arbitrary, but independent solutions $u(\rho,z)$, $v(\rho,z)$ of the equation

$$\nabla^2 f = 0 \qquad \ldots (3.5)$$

and then by putting $u = \xi(F,C)$, $v = \eta(F,C)$, substituting these respectively in the left hand side of equation (3.5), we can deduce equations (3.1) and (3.2) as a consequence.

The following considerations show that this is not possible, and no such functions exist. For, in terms of F and C, the equations $\nabla^2 u = 0$ and $\nabla^2 v = 0$ give, respectively,

$$\xi_F\nabla^2 F+\xi_C\nabla^2 C+\xi_{FF}(\nabla F)^2+\xi_{CC}(\nabla C)^2+2\xi_{FC}\nabla C.\nabla F = 0 \quad \ldots (3.6)$$

$$\eta_F\nabla^2 F+\eta_C\nabla^2 C+\eta_{FF}(\nabla F)^2+\eta_{CC}(\nabla C)^2+2\eta_{FC}\nabla C.\nabla F = 0 \quad \ldots (3.7)$$

where a lettered suffix on ξ (or η) denotes partial differentiation of ξ (or η) with respect to that letter.

Now if we are going to deduce equations (3.1) and (3.2), from equations (3.6) and (3.7), it is clear that none of the quantities ξ_C, ξ_F, η_C, η_F can be zero.

If we now eliminate $\nabla^2 C$ from between equations (3.6) and (3.7) we get,

$$(\xi_F\eta_C-\xi_C\eta_F)\nabla^2 F + (\eta_C\xi_{FF}-\xi_C\eta_{FF})(\nabla F)^2 + (\eta_C\xi_{CC}-\xi_C\eta_{CC})(\nabla C)^2 + 2(\eta_C\xi_{FC}-\xi_C\eta_{FC})\nabla C.\nabla F = 0 \quad \ldots (3.8)$$

Eliminating $\nabla^2 F$ from between (3.6) and (3.7) gives,

$$(\xi_C\eta_F-\xi_F\eta_C)\nabla^2 C + (\eta_F\xi_{FF}-\xi_F\eta_{FF})(\nabla F)^2 + (\eta_F\xi_{CC}-\xi_F\eta_{CC})(\nabla C)^2 + 2(\eta_F\xi_{FC}-\xi_F\eta_{FC})\nabla C.\nabla F = 0 \quad \ldots (3.9)$$

Since u and v are assumed to be independent we have $\xi_F\eta_C - \xi_C\eta_F \neq 0$.

If (3.8) and (3.9) are to be identical to equations (3.1) and (3.2) we shall require,

$$\eta_C\xi_{CC} - \xi_C\eta_{CC} = 0 \qquad \ldots (3.10)$$

$$\eta_C\xi_{FF} - \xi_C\eta_{FF} = 0 \qquad \ldots (3.11)$$

$$\eta_C\xi_{FC} - \xi_C\eta_{CF} = -C^{-1}(\xi_F\eta_C - \xi_C\eta_F) \qquad \ldots (3.12)$$

$$\eta_F\xi_{CC} - \xi_F\eta_{CC} = -C^{-1}(\xi_C\eta_F - \eta_C\xi_F) \qquad \ldots (3.13)$$

$$\eta_F\xi_{FC} - \eta_{FC}\xi_F = 0 \qquad \ldots (3.14)$$

$$\eta_F\xi_{FF} - \xi_F\eta_{FF} = C^{-1}(\xi_C\eta_F - \eta_C\xi_F) \qquad \ldots (3.15)$$

Solving equations (3.10), (3.11) and (3.12) we get,

$$\xi = \alpha + \beta C^{-1}|f'|^{-\frac{1}{2}} \ , \quad \eta = \gamma + \beta C^{-1} f|f'|^{-\frac{1}{2}} \ ,$$

where α, β, γ are constants, and f is an arbitrary function F. Also since $\xi_C \neq 0$, $\eta_C \neq 0$ we have $\beta \neq 0$. Substituting these values of ξ and η in equation (3.13) we get $\beta = 0$, which is a contradiction.

It would appear that to search for the general solutions of equations (3.1) and (3.2), is perhaps far too ambitious, and that we may have more success if we look for a method of deriving particular integrals of equations (3.1) and (3.2). With this aim in view, we take a known solution, say the Kerr solution, and try to force equations (3.1) and (3.2) to give it in the simplest possible form. For the Kerr solution, C and F have the forms,

$$C = 1 - 2mr(r^2+a^2\cos^2\theta)^{-1} \qquad \dots (3.16)$$

$$F = -2ma\cos\theta(r^2+a^2\cos^2\theta)^{-1} \qquad \dots (3.17)$$

where m and a are constants, and r,θ are functions of ρ,z defined by

$$\rho = \sin\theta(r^2-2mr+a^2)^{\frac{1}{2}} \qquad \dots (3.18)$$

$$z = \cos\theta(r-m) \qquad \dots (3.19)$$

The first thing we note is that if we calculate C-1+iF — ($i = \sqrt{-1}$) we get,

$$C - 1 + iF = -2m(r-ia\cos\theta)^{-1} \qquad \dots (3.20)$$

Now the function in the bracket on the right hand side of (3.20) is quite simple, it is after all just a function of r plus a function of θ. This suggests that equations (3.1) and (3.2) be combined into one equation for the complex function X = C+iF. We get

$$\nabla^2 X = \frac{2}{X + \overline{X}}(\nabla X)^2 \qquad \dots (3.21)$$

where $\overline{X}$ = C-iF.

Then from equation (3.20) we get,

$$X = 1 + 2m(ia\cos\theta - r)^{-1} \qquad \ldots (3.22)$$

Although X is a fairly simple function of r, and θ, r and θ are rather complicated functions of ρ and z (see equations (3.18) and (3.19)). To overcome this we must change the independent variables ρ and z. We cannot use r and θ, defined by equations (3.18) and (3.19), since a and m would appear in the resulting equation. Instead we choose,

$$\rho = K\sinh x\sin y \qquad \ldots (3.23)$$

$$z = K\cosh x\cos y \qquad \ldots (3.24)$$

where x and y are related to r and θ by $r-m = K\cosh x$, $\theta = y$ and $K = \sqrt{m^2-a^2}$ (assuming $m^2 > a^2$).

In terms of x and y equation (3.21) reads

$$\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \coth x\,\frac{\partial X}{\partial x} + \cot y\,\frac{\partial X}{\partial y}$$

$$= \frac{2}{X + \bar{X}}\left\{\left(\frac{\partial X}{\partial y}\right)^2 + \left(\frac{\partial X}{\partial y}\right)^2\right\} \qquad \ldots (3.25)$$

and in terms of x and y equation (3.22) reads

$$X = \frac{\frac{ia}{m}\cos y - \frac{K}{m}\cosh x + 1}{\frac{ia}{m}\cos y - \frac{K}{m}\cosh x - 1} \qquad \ldots (3.26)$$

The form of (3.26) suggests that we change the dependent variable X to Z by putting

$$X = \frac{Z + 1}{Z - 1} \qquad \text{... (3.27)}$$

and in terms of Z equation (3.25) reads

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + \coth x \frac{\partial Z}{\partial x} + \cot y \frac{\partial Z}{\partial y}$$

$$= \frac{2\bar{Z}}{Z\bar{Z}-1} \left\{ \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 \right\} \qquad \text{... (3.28)}$$

where $\bar{Z}$ is the complex conjugate of Z.

The solution of equation (3.28) which generates the Kerr metric is then

$$Z = \cos\lambda \cosh x + i \sin\lambda \cos y \qquad \text{... (3.29)}$$

where $\sin\lambda = a/m$.

Inspection of equation (3.29) suggests a further change of independent variables to

$$x' = \cosh x \qquad \text{... (3.30)}$$

$$y' = \cos y \qquad \text{... (3.31)}$$

Using equations (3.30) and (3.31), equation (3.28) becomes

$$\frac{\partial}{\partial x'}\left\{\left(x'^2-1\right)\frac{\partial Z}{\partial x'}\right\}+\frac{\partial}{\partial y'}\left\{\left(1-y'^2\right)\frac{\partial Z}{\partial y'}\right\}$$

$$=\frac{2\bar{Z}}{Z\bar{Z}-1}\left\{\left(x'^2-1\right)\left(\frac{\partial Z}{\partial x'}\right)^2+\left(1-y'^2\right)\left(\frac{\partial Z}{\partial y'}\right)^2\right\}$$

... (3.32)

and from equation (3.29) we know that $Z = x'\cos\lambda + iy'\sin\lambda$ is a solution of equation (3.32).

To sum up then, in order to get the Kerr solution in its simplest form, we first combine our equations (3.1) and (3.2) using the complex function X (= C+iF) to give equation (3.21), we then change both the dependent variable (X) and independent variables (ρ, z) using the equations

$$X = \frac{Z+1}{Z-1}$$

$$\rho = \alpha(x^2-1)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}$$

$$z = \alpha xy$$

where α is any real constant. This then leads us to Ernst's equation

$$\frac{\partial}{\partial x}\left\{\left(x^2-1\right)\frac{\partial Z}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\left(1-y^2\right)\frac{\partial Z}{\partial y}\right\}$$

$$=\frac{2\bar{Z}}{Z\bar{Z}-1}\left\{\left(x^2-1\right)\left(\frac{\partial Z}{\partial x}\right)^2+\left(1-y^2\right)\left(\frac{\partial Z}{\partial y}\right)^2\right\}$$

... (3.33)

The Kerr solution is then given by $Z = x\cos\lambda + iy\sin\lambda$, where λ is any real constant.

It is now clear that even if we try to find a method of deriving particular integrals of equations (3.1) and (3.2) we are led in the direction of increasing complexity. Equation (3.33) tells us very little, other than if Z is a solution so are Z^{-1} and $Ze^{i\alpha}$, where α is any real constant.

CHAPTER FOUR

THE POSSIBILITY OF FINDING PARTICULAR INTEGRALS OF THE FIELD EQUATIONS $R_{ij} = 0$, USING THE EQUATIONS OF SET B

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THE POSSIBILITY OF FINDING PARTICULAR INTEGRALS OF THE FIELD EQUATIONS $R_{ij} = 0$, USING THE EQUATIONS OF SET B

In the previous Chapter it was shown that equations (2.39) and (2.40), of set A, are very difficult to satisfy. One way of getting round this difficulty would be to look for solutions of set B, which satisfy the condition $\Gamma \neq 0$. For then, by our theorem, equations (2.39) and (2.40) would be satisfied automatically.

In this Chapter this possibility will be investigated, for the special case when C and F satisfy the condition,

$$C_1 C_2 + F_1 F_2 = 0 \qquad \dots (4.1)$$

Using (4.1) the equationsof set B become,

$$\Delta_{11} + \Delta_{22} = 0 \qquad \dots (4.2)$$

$$2(\Delta_1^2 + \Delta_2^2) S_1 = \Delta\Delta_1 C^{-2} (C_1^2 + F_1^2 - C_2^2 - F_2{}^2) \qquad \dots (4.3)$$

$$2(\Delta_1^2 + \Delta_2^2) S_2 = -\Delta\Delta_2 C^{-2} (C_1^2 + F_1^2 - C_2^2 - F_2^2) \qquad \dots (4.4)$$

$$S_{11} + S_{22} = -\tfrac{1}{2} C^{-2} (C_1^2 + C_2^2 + F_1^2 + F_2^2) \qquad \dots (4.5)$$

Equations (4.3) and (4.4) give,

$$\Delta_2 S_1 + \Delta_1 S_2 = 0 \qquad \ldots (4.6)$$

We shall regard equation (4.6) as an equation for S, and since it is of the first order and of the Lagrange type, its general solution can be written as,

$$S = G(s) \qquad \ldots (4.7)$$

where $s(x^1x^2) =$ constant, is the solution of the equation

$$\frac{dx^1}{\Delta_2} = \frac{dx^2}{\Delta_1} \qquad \ldots (4.8)$$

and G(s) is an arbitrary function of s.

From equations (4.3), and (4.5) we get

$$C_1^2 + F_1^2 = P^2 C^2 \qquad \ldots (4.9)$$

$$C_2^2 + F_2^2 = Q^2 C^2 \qquad \ldots (4.10)$$

where

$$P^2 = -(S_{11}+S_{22}) + \frac{S_1}{\Delta\Delta_1}(\Delta_1^2+\Delta_2^2) \qquad \ldots (4.11)$$

$$Q^2 = -(S_{11}+S_{22}) - \frac{S_1}{\Delta\Delta_1}(\Delta_1^2+\Delta_2^2) \qquad \ldots (4.12)$$

and we assume $P \neq 0$, $Q \neq 0$.

Now equations (4.1), (4.9) and (4.10) will be satisfied if we put

$$C = e^{u} \qquad \dots (4.13)$$

$$u_1 = P\cos t \qquad \dots (4.14)$$

$$u_2 = Q\sin t \qquad \dots (4.15)$$

$$F_1 = Pe^{u}\sin t \qquad \dots (4.16)$$

$$F_2 = -Qe^{u}\cos t \qquad \dots (4.17)$$

where $t(\not\equiv 0)$ is a new function of x^1 and x^2.

The conditions $u_{12} = u_{21}$, $F_{12} = F_{21}$, give, using equations (4.14) to (4.17) inclusive,

$$t_1 Q\cos t + t_2 P\sin t = P_2\cos t - Q_1\sin t \qquad \dots (4.18)$$

$$t_1 Q\sin t - t_2 P\cos t = P_2\sin t + Q_1\cos t + PQ \qquad \dots (4.19)$$

Solving equations (4.18) and (4.19) for t_1 and t_2, we get,

$$t_1 = P_2 Q^{-1} + P\sin t \qquad \dots (4.20)$$

$$t_2 = -(Q_1 P^{-1} + Q\cos t) \qquad \dots (4.21)$$

The condition $t_{12} = t_{21}$ gives using equations (4.20) and (4.21)

$$(P_2 Q^{-1})_2 + (Q_1 P^{-1})_1 - PQ = 0 \qquad \dots (4.22)$$

Now, instead of investigating the conditions on F, C and Δ which do not make the determinant Γ vanish, it is simpler to find conditions on P, Q and Δ which make $\sigma \equiv 0$, $\tau \equiv 0$. This we shall now do by direct substitution from equations (4.13) to (4.17) inclusive.

From the definitions of σ and τ (see equations (2.39) and (2.40)) and equation (4.13), we can write

$$\sigma = e^{u}\{u_{11}+u_{22}+\Delta^{-1}(\Delta_1 u_1+\Delta_2 u_2)+e^{-2u}(F_1^2+F_2^2)\} \quad \ldots (4.23)$$

$$\tau = \Delta^{-1}e^{2u}\{(\Delta e^{-2u}F_1)_1 + (\Delta e^{-2u}F_2)_2\} \quad \ldots (4.24)$$

From equations (4.14) and (4.15) we get, by differentiation, and using equations (4.20) and (4.21),

$$u_{11} + P^2\sin^2 t = P_1\cos t - P_2 Q^{-1}P\sin t \quad \ldots (4.25)$$

$$u_{22} + Q^2\cos^2 t = Q_2\sin t - Q\cos t Q_1 P^{-1} \quad \ldots (4.26)$$

Adding equations (4.25) and (4.26), and using equations (4.16) and (4.17), we get

$$u_{11} + u_{22} + e^{-2u}(F_1^2+F_2^2) = (P_1-QQ_1P^{-1})\cos t + (Q_2-P_2Q^{-1}P)\sin t \quad \ldots (4.27)$$

Substituting from equations (4.14), (4.15) and (4.27) in equation (4.23) gives

$$\sigma = e^{u}\{(P_1 - QP^{-1}Q_1 + \Delta_1\Delta^{-1}P)\cos t + \sin t(Q_2 - PQ^{-1}P_2 + \Delta_2\Delta^{-1}Q)\} \quad \ldots (4.28)$$

Substituting from equations (4.16) and (4.17) in equation (4.24), we get

$$\tau = \Delta^{-1}e^{u}\{(\Delta Pt_1 - \Delta_2 Q - \Delta Q_2 + \Delta u_2 Q)\cos t + \sin t(\Delta_1 P + P_1\Delta - \Delta u_1 P + \Delta Q t_2)\} \quad \ldots (4.29)$$

after differentiation. Substituting from equations (4.14), (4.15), (4.20) and (4.21) in equation (4.29), we get,

$$\tau = e^{u}\{\cos t(PQ^{-1}P_2 - \Delta_2\Delta^{-1}Q - Q_2) - \sin t(QP^{-1}Q_1 - \Delta_1\Delta^{-1}P - P_1)\} \quad \ldots (4.30)$$

Hence, from equations (4.28) and (4.30) we see that the necessary and sufficient conditions for $\sigma = 0$, $\tau = 0$, are

$$P_1 + \Delta_1\Delta^{-1}P - QP^{-1}Q_1 = 0 \quad \ldots (4.31)$$

$$Q_2 + \Delta_2\Delta^{-1}Q - PQ^{-1}P_2 = 0 \quad \ldots (4.32)$$

For the remainder of this Chapter the physical significance of our equations will be disregarded, and the variables x^1, x^2 will be permitted to take complex values. All functions of these variables will be assumed to be analytic, so that they can be continued into the complex plane.

We now make a change of variables from x^1 and x^2, to the conjugate variables z and $\bar{z}$, defined by,

$$z = x^1 + ix^2 \qquad \ldots (4.33)$$

$$\bar{z} = x^1 - ix^2 \qquad \ldots (4.34)$$

z, and $\bar{z}$ being independent variables when x^1 and x^2 are complex. From (4.33) and (4.34) we get the operators,

$$\frac{\partial}{\partial x^1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} , \quad \frac{\partial}{\partial x^2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

Since Δ is harmonic (see equation (4.2)), and is real when x^1 and x^2 are real, we can write

$$\Delta = f(z) + \bar{f}(\bar{z}) \qquad \ldots (4.35)$$

where $f(.)$ is analytic and $\bar{f}(\bar{.})$ is the conjugate analytic function.

In terms of z and $\bar{z}$, equation (4.8) reads

$$\frac{\partial \Delta}{\partial \bar{z}} dz + \frac{\partial \Delta}{\partial z} d\bar{z} = 0$$

or using equation (4.35),

$$\bar{f}'\,dz + f'\,d\bar{z} = 0 \qquad \ldots (4.36)$$

where $f' = df(z)/dz$ and $\bar{f}' = d\bar{f}(\bar{z})/d\bar{z}$. The integral of (4.36) can be written as

$$r(z) + \bar{r}(\bar{z}) = \text{constant},$$

where

$$r' = (f')^{-1}. \qquad \ldots (4.37)$$

Hence

$$s = r(z) + \bar{r}(\bar{z}) \qquad \ldots (4.38)$$

Substituting from equations (4.7), (4.35) and (4.38) in equations (4.11) and (4.12) we get

$$P^2 = \lambda G'' + \nu G' \qquad \ldots (4.39)$$

$$Q^2 = \lambda G'' - \nu G' \qquad \ldots (4.40)$$

where

$$\lambda = -4(f'\bar{f}')^{-1} \qquad \ldots (4.41)$$

$$\nu = 4(f + \bar{f})^{-1} \qquad \ldots (4.42)$$

In terms of z and $\bar{z}$ equations (4.22), (4.31) and (4.32) read,

$$2pq\{(p_z - q_z)_z + (p_{\bar z} - q_{\bar z})_{\bar z} - 2(p_z + q_z)_{\bar z} + 2pq\}$$
$$- (p_z - p_{\bar z})\{(pq)_z - (pq)_{\bar z}\}$$
$$+ (q_z + q_{\bar z})\{(pq)_z + (pq)_{\bar z}\} = 0 \quad \ldots (4.43)$$

$$(p-q)_z + (p-q)_{\bar z} + 2p\{(\ell n\Delta)_z + (\ell n\Delta)_{\bar z}\} = 0 \quad \ldots (4.44)$$

$$(p-q)_z - (p-q)_{\bar z} - 2q\{(\ell n\Delta)_z - (\ell n\Delta)_{\bar z}\} = 0 \quad \ldots (4.45)$$

where $p = P^2$, $q = Q^2$ and we are using the notation

$$p_z = \frac{\partial p}{\partial z}, \quad p_{zz} = \frac{\partial^2 p}{\partial z^2},$$

etc.

Now, from equations (4.39) and (4.40),

$$p-q = 2\nu G'$$
$$\therefore \quad (p-q)z = 2\left(\frac{\nu G''}{f'} + \nu_z G'\right) \quad \ldots (4.46)$$

where we have used equations (4.37) and (4.38). Substituting from equations (4.35) and (4.46) in equations (4.44) and (4.45) we get

$$G''\{\nu(f')^{-1} + \nu(\overline{f'})^{-1} + \lambda(f'+\overline{f'})(f+\overline{f})^{-1}\}$$
$$+ G'\{\nu_z+\nu_{\overline{z}} +\nu(f'+\overline{f'})(f+\overline{f})^{-1}\} = 0 \qquad \dots (4.47)$$

$$G''\{\nu(f')^{-1} - \nu(\overline{f'})^{-1} - \lambda(f'-\overline{f'})(f+\overline{f})^{-1}\}$$
$$+ G'\{\nu_z-\nu_{\overline{z}} + \nu(f'-\overline{f'})(f+\overline{f})^{-1}\} = 0 \qquad \dots (4.48)$$

Substituting from equations (4.41) and (4.42) for λ and ν we see that equations (4.47) and (4.48) are satisfied identically.

Hence $\sigma = 0$, $\tau \equiv 0$ and equations (2.39) and (2.40) are reduced to mere identities.

Substituting from equations (4.39) and (4.40) in equation (4.43) we get

$$\gamma_1 G'(G'')^2 + \gamma_2(G'')^3 + \gamma_3 G'''(G'')^2 + \gamma_4(G'')^2 G''''$$
$$+ \gamma_5(G')^3 + \gamma_6 G''(G')^2 + \gamma_7 G'''(G')^2 + \gamma_8 G''(G''')^2$$
$$+ \gamma_9 G' G'' G''' + \gamma_{10}(G')^2 G'''' + \gamma_{11}(G')^2(G'')^2$$
$$+ \gamma_{12}(G')^4 + \gamma_{13}(G'')^4 = 0$$
$$\dots (4.49)$$

where

$$\gamma_1 = \frac{3}{4}\left(\frac{\nu}{\lambda}\right)^2 \left\{\left(f'^2+\bar{f}'^2\right)\left(f+\bar{f}\right)^{-1}-\left(f''+\bar{f}''\right)\right\} \quad \ldots (4.50)$$

$$\gamma_2 = -\left(\frac{\nu}{\lambda}\right)^2 \quad \ldots (4.51)$$

$$\gamma_3 = 0 \quad \ldots (4.52)$$

$$\gamma_4 = \tfrac{1}{2} \quad \ldots (4.53)$$

$$\gamma_5 = \frac{1}{4}\left(\frac{\nu}{\lambda}\right)^4 \left\{f''+\bar{f}''-\left(f+\bar{f}\right)^{-1}\left(f'^2+\bar{f}'^2\right)\right\} \quad \ldots (4.54)$$

$$\gamma_6 = \tfrac{1}{2}\left(\frac{\nu}{\lambda}\right)^2 \left\{\left(\bar{f}'^2 f''+f'^2\bar{f}''\right)\left(f+\bar{f}\right)^{-1} - f''\bar{f}''\right) \quad \ldots (4.55)$$

$$\gamma_7 = \tfrac{1}{2}\left(\frac{\nu}{\lambda}\right)^2 \left\{f''+\bar{f}'' - \left(f'^2+\bar{f}'^2\right)\left(f+\bar{f}\right)^{-1}\right\} \quad \ldots (4.56)$$

$$\gamma_8 = -\tfrac{1}{2} \quad \ldots (4.57)$$

$$\gamma_9 = \left(\frac{\nu}{\lambda}\right)^2 \quad \ldots (4.58)$$

$$\gamma_{10} = -\tfrac{1}{2}\left(\frac{\nu}{\lambda}\right)^2 \quad \ldots (4.59)$$

$$\gamma_{11} = -2\left(\frac{\nu}{\lambda}\right)^2 \quad \ldots (4.60)$$

$$\gamma_{12} = \left(\frac{\nu}{\lambda}\right)^4 \quad \ldots (4.61)$$

$$\gamma_{13} = 1 \quad \ldots (4.62)$$

Now if λ and ν were substituted from equations (4.41) and (4.42) in equations (4.50) to (4.62) inclusive, the γ_i's $(i = 1,2,\ldots,13)$ would become functions of the derivatives of $f(z)$ and $\overline{f}(\overline{z})$, and if these γ_i's were then substituted in equation (4.49) we would have one equation for the two unknowns $f(z)$ and $G(s)$. If we do not assume, as a prior condition, that any one of the derivatives G', G'', G''', G'''' is zero, we can make equation (4.49) into a differential equation for $G(s)$, provided we choose $f(z)$ such that each γ_i is a function of s only.

Before investigating this possibility we note that if $f(z)$ is replaced by $f(z)+ia$, where a is any real constant, Δ is unaltered. Hence, $f(z)$ is arbitrary to the extent of additive constants, which are purely imaginary. Further, in terms of z and $\overline{z}$ equation (4.1) reads,

$$C_z^2 + F_z^2 - C_{\overline{z}}^2 - F_{\overline{z}}^2 = 0 \qquad \dots (4.63)$$

Suppose we now change from the variables z, $\overline{z}$ to the variables ζ, $\overline{\zeta}$ where,

$$\zeta = \omega(z), \qquad \overline{\zeta} = \overline{\omega}(\overline{z})$$

$\omega(z)$ being any analytic function of z, and $\overline{\omega}(z)$ the conjugate complex function. Equation (4.63) becomes

$$\left(\frac{d\omega}{dz}\right)^2\left(C_\zeta^{\,2}+F_\zeta^{\,2}\right) - \left(\frac{d\overline{\omega}}{d\overline{z}}\right)^2\left(C_{\overline{\zeta}}^{\,2}+F_{\overline{\zeta}}^{\,2}\right) = 0 \qquad \dots (4.64)$$

Now if we assume equation (4.63) is form invariant under the above transformation we shall require

$$C_{\zeta}^2 + F_{\zeta}^2 - C_{\bar{\zeta}}^2 - F_{\bar{\zeta}}^2 = 0 \qquad \ldots (4.65)$$

From equations (4.64) and (4.65) we get

$$\left(\frac{d\omega}{dz}\right)^2 = \left(\frac{d\bar{\omega}}{d\bar{z}}\right)^2$$

$$\therefore \qquad \omega = az + b,$$

where a and b are constants, a either real or purely imaginary, b complex.

It follows that equation (4.63) is only form invariant under transformations of the type

$$\left.\begin{aligned} \zeta &= az + b \\ \bar{\zeta} &= \bar{a}\bar{z} + \bar{b} \end{aligned}\right\} \qquad \ldots (4.66)$$

Since equations (4.2), (4.3), (4.4) and (4.5) are also form invariant under the transformation (4.66), it follows that, if

$$\left.\begin{aligned} C &= C(z,\bar{z}) \\ F &= F(z,\bar{z}) \\ \Delta &= \Delta(z,\bar{z}) \\ S &= S(z,\bar{z}) \end{aligned}\right\} \qquad \ldots (4.67)$$

are solutions of equations (4.1) to (4.5) inclusive, so are

$$\left.\begin{aligned} C &= C(\zeta,\bar{\zeta}) \\ F &= F(\zeta,\bar{\zeta}) \\ \Delta &= \Delta(\zeta,\bar{\zeta}) \\ S &= S(\zeta,\bar{\zeta}) \end{aligned}\right\} \quad \ldots (4.68)$$

However, since our metric (2.1) is form invariant under the transformation (4.66), the solutions (4.67) and (4.68) will give rise to the same metric. Hence, we can replace z by az+b, and $\bar{z}$ by $\overline{az+b}$, where appropriate, without affecting the final metric.

THEOREM

There are just five distinct forms which f(z) can take in order that each γ_i (i = 1,2,...,13) be a function of s only. These (apart from changes in f(z) and z of the form mentioned above) can be written as

(i) $f(z) = z$

(ii) $f(z) = i\beta z^2$

(iii) $f(z) = i\epsilon\cosh z$

(iv) $f(z) = i\epsilon\sinh z$

(v) $f(z) = i\delta e^z$

where β, ϵ and δ are all real constants.

PROOF

(i) If $f(z) = z$ then equations (4.37) and (4.38) give $s = z+\bar{z}$. From equations (4.41) and (4.42) we get

$$\lambda = -4, \qquad \nu = 4(z+\bar{z})^{-1} = 4s^{-1},$$

hence each γ_i $(i = 1,2,\ldots,13)$ is a function of s only.

From now on we shall assume $f'' \neq 0$. In order to obtain the results (ii) to (v) we first note that if a function $\zeta = \zeta(z,\bar{z})$ be expressible as a function of $s = s(z,\bar{z})$ only, it is necessary and sufficient that,

$$\frac{\partial(\zeta,s)}{\partial(z,\bar{z})} = 0$$

i.e. $$s_{\bar{z}}\zeta_z - s_z\zeta_{\bar{z}} = 0$$

or $$f'\zeta_z - \bar{f}'\zeta_{\bar{z}} = 0 \qquad \ldots (4.69)$$

using equations (4.37) and (4.38).

Inspection of equations (4.50) to (4.62) inclusive shows that each γ_i $(i = 1,2,\ldots,13)$, will be a function of s only, if equation (4.69) is satisfied by

(1) $\zeta = \frac{\nu}{\lambda}$

(2) $\zeta = f''+\bar{f}''-(f'^2+\bar{f}'^2)(f+\bar{f})^{-1}$

and (3) $\zeta = (\bar{f}'^2 f''+f'^2\bar{f}'')(f+\bar{f})^{-1}-f''\bar{f}''$.

Substituting

$$\zeta = \frac{\nu}{\lambda} = - f'\bar{f}'(f+\bar{f})^{-1} ,$$

(using equations (4.41) and (4.42)) in equation (4.69), we get

$$(ff'' - f'^2) - (\bar{f}\bar{f}''-\bar{f}'^2) + \bar{f}f'' - f\bar{f}'' = 0 \qquad \ldots (4.70)$$

Now (4.70) is to hold identically in z and $\bar{z}$, so that if we first differentiate partially with respect to z, and then with respect to $\bar{z}$ we get

$$\bar{f}'f''' - f'\bar{f}''' = 0$$

$$\therefore \qquad \frac{f'''}{f'} = \frac{\bar{f}'''}{\bar{f}'} \qquad \ldots (4.71)$$

Since (4.71) is identically true for all z, and $\bar{z}$ we must have

$$f''' = Kf' \qquad \ldots (4.72)$$

where K is any real constant.

There are two cases to consider

Case (1) $K = 0$

The general solution of equation (4.72) is in this case

$$f(z) = Az^2 + Bz + C$$

where A, B and C are complex constants. If we make the transformation $z \to z - \frac{1}{2}BA^{-1}$ we get

$$f(z) = Az^2 + D \qquad \ldots (4.73)$$

where $$D = C - B^2A^{-1}/4.$$

Substituting (4.73) in (4.70) gives

$$A = -\bar{A}, \qquad D = -\bar{D}$$

∴ Since D is imaginary we can write

$$f(z) = i\beta z^2 \qquad \ldots (4.74)$$

β real.

Case (2) $K \neq 0$

Putting $K = a^2$, the general solution of equation (4.72) can be written as

$$f(z) = Ae^{az} + Be^{-az} + C \qquad \ldots (4.75)$$

where A, B and C are complex constants, and we note that a is either real or purely imaginary. Substituting (4.75) in (4.70) we get

$$C + \overline{C} = 0 \qquad \ldots (4.76)$$

$$AB = \overline{AB} \qquad \ldots (4.77)$$

Because of (4.76) there is no loss of generality in putting C = 0. Now it is clear that (4.77) is satisfied if one A, B is zero, in that case

$$f(z) = De^{\pm az} \qquad \ldots (4.78)$$

where D is constant.

The transformation $z \rightarrow \pm\, a^{-1}\{z + i(\pi/2 - \arg D)\}$ then allows us to write (4.78) as

$$f(z) = i\delta e^{z} \qquad \ldots (4.79)$$

where δ is real.

We now assume $A \neq 0$, $B \neq 0$ and write

$$A = \alpha e^{i\theta}, \qquad B = \pm\beta e^{-i\theta},$$

α, β and θ being real constants $\alpha, \beta > 0$, equation (4.77) is then satisfied identically.

Substituting these values of A and B in equation (4.75)

$$f(z) = \alpha e^{az+i\theta} \pm \beta e^{-(az+i\theta)} \qquad \ldots (4.80)$$

The transformation $z \to a^{-1}\{z + \frac{1}{2}\ell n \frac{\beta}{\alpha} + i(\pm \frac{\pi}{2} - \theta)\}$ then allows us to write (4.80) as

$$f(z) = i\epsilon(e^{z} \pm e^{-z}) \qquad \ldots (4.81)$$

Substituting for $f(z)$ from equations (4.74),(4.79) and (4.81) in the other two values of ζ ((2) and (3)), and then substituting these in equation (4.69), we see that it is satisfied identically in each case. Hence the theorem is proven.

The corresponding values of Δ are

(i) $\Delta = x^1$

(ii) $\Delta = ax^1x^2$

(iii) $\Delta = K\sinh x^1 \sin x^2$

(iv) $\Delta = K\cosh x^1 \sin x^2$

(v) $\Delta = \ell e^{x^1} \sin x^2$

where a, K and ℓ are real constants.

In Chapters 5, 6, 7, 8 and 9 the particular integrals corresponding to each of these values of Δ will be worked out, but instead of using the complex coordinates z and $\bar{z}$, we shall return to the real coordinates x^1 and x^2.

CHAPTER FIVE

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = K \sinh x^1 \sin x^2$

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THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = K \sinh x^1 \sin x^2$

It will simplify things a little if we put $x^1 = x$, $x^2 = y$, so that

$$\Delta = K \sinh x \sin y \qquad \dots (5.1)$$

Also suffix 1 will now denote partial differentiation with respect to x, and suffix 2 partial differentiation with respect to y.

The solution of equation (4.8) is in this case

$$\sinh x \operatorname{cosec} y = \text{constant}$$

$$\therefore \qquad s = \sinh x \operatorname{cosec} y \qquad \dots (5.2)$$

From equations (4.7) and (5.2)

$$S_{11} + S_{22} = \{(1+s^2)G'' + 2sG'\}\operatorname{cosec}^2 y \qquad \dots (5.3)$$

and

$$\frac{S_1}{\Delta\Delta_1}\left(\Delta_1{}^2 + \Delta_2{}^2\right) = s^{-1}\left(1+s^2\right)G' \operatorname{cosec}^2 y \qquad \dots (5.4)$$

using equation (5.1). Substituting from (5.3) and (5.4) in equations (4.11) and (4.12) we get

$$P^2 = \operatorname{cosec}^2 y\{-\frac{d}{ds}[(1+s^2)G'] + s^{-1}(1+s^2)G'\} \qquad \dots (5.5)$$

$$Q^2 \doteq -\operatorname{cosec}^2 y\{\frac{d}{ds}[(1+s^2)G'] + s^{-1}(1+s^2)G'\} \qquad \dots (5.6)$$

We now introduce a new function $\theta(s)$ defined by

$$-s\theta = (1+s^2)G' \qquad \dots (5.7)$$

Substituting from equation (5.7) in equations (5.5) and (5.6) we get,

$$P = g(s)\operatorname{cosec} y \qquad \dots (5.8)$$

$$Q = h(s)\operatorname{cosec} y \qquad \dots (5.9)$$

where

$$g^2 = s\frac{d\theta}{ds} \qquad \dots (5.10)$$

and

$$h^2 = s\frac{d\theta}{ds} + 2\theta \qquad \dots (5.11)$$

From equations (5.8) and (5.2), we get

$$P_2 = -(gs)' \operatorname{cosec} y \cot y$$

$$\therefore \qquad P_2 Q^{-1} = -\frac{(gs)'}{h}\cot y$$

$$\therefore \quad (P_2Q^{-1})_2 = \left\{\frac{(sg)'}{h}\right\}' \sinh x \ \mathrm{cosec}\, y \ \cot^2 y + \frac{(gs)'}{h} \ \mathrm{cosec}^2 y$$

$$= \left[s\left\{\frac{(sg)'}{h}\right\}' + \frac{(gs)'}{h}\right]\mathrm{cosec}^2 y - \left\{\frac{(sg)'}{h}\right\}' s$$

(using equation (5.2)). i.e.

$$(P_2Q^{-1})_2 = \frac{d}{ds}\left\{\frac{s}{h}\frac{d}{ds}(sg)\right\}\mathrm{cosec}^2 y - s\,\frac{d}{ds}\left\{\frac{(sg)'}{h}\right\} \qquad \ldots (5.12)$$

Using equations (5.2) and (5.9), we get

$$Q_1 = h' \cosh x \ \mathrm{cosec}^2 y$$

$$\therefore \quad Q_1P^{-1} = \frac{h'}{g} \cosh x \ \mathrm{cosec}\, y$$

$$\therefore \quad (Q_1P^{-1})_1 = \left(\frac{h'}{g}\right)' \mathrm{cosec}^2 y \ \cosh^2 x + \frac{h'}{g} \ \mathrm{cosec}\, y \ \sinh x$$

$$= \left(\frac{h'}{g}\right)' \mathrm{cosec}^2 y + s^2\left(\frac{h'}{g}\right)' + s\left(\frac{h'}{g}\right)$$

(using equation (5.2)). i.e.

$$(Q_1P^{-1})_1 = \frac{d}{ds}\left(g^{-1}\frac{dh}{ds}\right)\mathrm{cosec}^2 y + s\,\frac{d}{ds}\left(sg^{-1}\frac{dh}{ds}\right) \qquad \ldots (5.13)$$

Substituting from equations (5.8), (5.9), (5.12) and (5.13) in equation (4.22) gives

$$\operatorname{cosec}^2 y\left[\frac{d}{ds}\left\{\frac{s}{h}\frac{d}{ds}(sg) + g^{-1}\frac{dh}{ds}\right\} - gh\right]$$

$$- s\frac{d}{ds}\left[h^{-1}\frac{d}{ds}(sg) - sg^{-1}\frac{dh}{ds}\right] = 0$$

$$\ldots (5.14)$$

But

$$h^{-1}\frac{d}{ds}(sg) - sg^{-1}\frac{dh}{ds}$$

$$= \tfrac{1}{2}(sgh)^{-1}\left\{\frac{d}{ds}(s^2 g^2) - s^2\frac{d}{ds}(h^2)\right\}$$

$$= \tfrac{1}{2}(sgh)^{-1}\left\{\frac{d}{ds}\left(s^3\frac{d\theta}{ds}\right) - s^2\frac{d}{ds}\left(s\frac{d\theta}{ds} + 2\theta\right)\right\}$$

using equations (5.10) and (5.11) i.e.

$$h^{-1}\frac{d}{ds}(sg) - sg^{-1}\frac{dh}{ds} \equiv 0$$

and (5.14) reduces to

$$\frac{d}{ds}\left\{sh^{-1}\frac{d}{ds}(sg) + g^{-1}\frac{dh}{ds}\right\} - gh = 0$$

i.e.

$$\frac{d}{ds}\left\{(gh)^{-1}\frac{d}{ds}(h^2 + s^2 g^2)\right\} = 2gh \qquad \ldots (5.15)$$

Multiplying equation (5.15) by $(gh)^{-1}\frac{d}{ds}(h^2+s^2 g^2)$ and integrating gives

$$\left\{\frac{d}{ds}(h^2+s^2g^2)\right\}^2 = 4g^2h^2(h^2+s^2g^2+a) \qquad \ldots (5.16)$$

where a is the constant of integration. Substituting from equations (5.10) and (5.11) in equation (5.16) gives

$$(s^2+1)^2\left\{s\frac{d^2\theta}{ds^2} + 3\frac{d\theta}{ds}\right\}^2 = 4s\frac{d\theta}{ds}\left(s\frac{d\theta}{ds} + 2\theta\right) \ldots$$

$$\ldots \left\{s(1+s^2)\frac{d\theta}{ds} + 2\theta + a\right\} \qquad \ldots (5.17)$$

We can express equation (5.17) in a simpler form if we make the following substitutions

$$s^2 = v \qquad \ldots (5.18)$$

$$\zeta = 2v\theta \qquad \ldots (5.19)$$

Equation (5.17) then reads

$$v^2(1+v)^2\left(\frac{d^2\zeta}{dv^2}\right)^2 = \frac{d\zeta}{dv}\left(v\frac{d\zeta}{dv} - \zeta\right)\left\{(1+v)\frac{d\zeta}{dv} - \zeta + a\right\}$$

$$\ldots (5.20)$$

Solving equation (5.20) for ζ, θ can be found, in terms of s, using equations (5.18) and (5.19), and G(s) can be obtained from equation (5.7).

Although we are unable to give the general solution of equation (5.20), it does seem to possess a remarkable particular integral, this being obtained by putting

$$G(v) = \ell n\{p(v)(1+v)^{-n^2}\} + \text{constant} \qquad \dots (5.21)$$

where $p(v)$ is a polynomial of degree n^2 $(n = 1,2,\dots)$, and by taking the constant a in equation (5.20) to be $-4n^2$. Substituting (5.21) in equation (5.7) gives

$$\theta = 2\{n^2 - (1+v)p' p^{-1}\} \qquad \dots (5.22)$$

where we have used equation (5.18). Substituting from equation (5.22) in equation (5.19) gives

$$\zeta = 4v\{n^2 - (1+v)p' p^{-1}\} \qquad \dots (5.23)$$

Although we are only able to prove that $p(v)$ exists for $n=1$ and $n=2$, there seems no reason why it should not exist for all n. To facilitate the calculations we put

$$w = qp^{-1} \qquad \dots (5.24)$$

$$q = v(1+v)p' \qquad \dots (5.25)$$

then,

$$\zeta = 4(n^2 v - w) \qquad \dots (5.26)$$

(using (5.23)). Substituting from (5.26) in equation (5.20) gives

$$v^2(1+v)^2\left(\frac{d^2w}{dv^2}\right)^2 = 4\left(n^2 - \frac{dw}{dv}\right)\left(v\frac{dw}{dv} - w\right)\left\{(1+v)\frac{dw}{dv} - w\right\} \quad \ldots (5.27)$$

Substituting from equation (5.24) in equation (5.27) gives

$$v^2(1+v)^2\{p(pq''-qp'') - 2p'(pq'-qp')\}^2$$
$$= 4\{n^2p^2 - (pq'-qp')\}\{v(pq'-qp')-pq\}\{(1+v)(pq'-qp')-pq\} \quad \ldots (5.28)$$

Now,

$$pq'-qp' = (1+v)\{v(pp''-p'^2) + p'p\} + vp'p \quad \ldots (5.29)$$

$$v(pq'-qp')-pq = v^2\{(1+v)(pp''-p'^2) + pp'\} \quad \ldots (5.30)$$

$$(1+v)(pq'-qp')-pq = (1+v)^2\{v(pp''-p'^2)+pp'\} \quad \ldots (5.31)$$

where we have used equation (5.25). We also note that

$$pq''-qp'' = \frac{d}{dv}(pq' - qp') \quad \ldots (5.32)$$

Case I

n = 1. The appropriate form for p(v) is

$$p(v) = v + c \qquad \ldots (5.33)$$

where c is an arbitrary constant. Substituting from equation (5.33) in equations (5.29), (5.30) and (5.31) we get

$$pq'-qp' = v^2 + 2cv + c \qquad \ldots (5.34)$$

$$v(pq'-qp')-pq = v^2(c-1) \qquad \ldots (5.35)$$

$$(1+v)(pq'-qp')-pq = c(1+v)^2 \qquad \ldots (5.36)$$

Using (5.32),

$$pq''-qp'' = 2(v+c) \qquad \ldots (5.37)$$

We also note that

$$p^2-(pq'-pq') = c(c-1) \qquad \ldots (5.38)$$

From equations (5.34) to (5.38) inclusive, it follows that equation (5.28) is satisfied by (5.33).

Case II

n = 2. The form for p(v) in this case is

$$p(v) = v^4+4cv^3+6cv^2+4cv+c^2 \qquad \ldots (5.39)$$

where c is a constant. Substituting equation (5.39) in equations (5.29), (5.30) and (5.31) gives

$$pq'-qp' = 4\{v^8+8cv^7+(16c+12c^2)v^6+(20c+36c^2)v^5 + (9c+61c^2)v^4+(12c^3+44c^2)v^3 + (18c^3+10c^2)v^2+8c^3v+c^3\} \quad \ldots (5.40)$$

$$v(pq'-qp')-pq = 4v^2(c-1)\{c(3v+2)-v^3\}^2 \quad \ldots (5.41)$$

$$(1+v)(pq'-qp')-pq = 4c(1+v)^2\{c(1+3v)+v^2(v+3)\}^2 \quad \ldots (5.42)$$

Using equations (5.39) and (5.40) we note that,

$$4p^2-(pq'-p'q) = -4c(1-c)\{c-v^2(2v+3)\}^2 \quad \ldots (5.43)$$

Also using equations (5.39), (5.40) and (5.32)

$$\begin{aligned} p(pq''-qp'')-2p'(pq'-qp') &= 16c(c-1)\{2v^9+9v^8-21cv^6+9v^7-9c(2c+5)v^5 \\ &\quad - 9c(5c+2)v^4-21c^2v^3+9c^3v^2+9c^3v+2c^3\} \\ &= 16c(c-1)\{c-v^2(2v+3)\}\{c(1+3v)+v^2(v+3)\} \\ &\quad \times \{c(3v+2)-v^3\} \end{aligned} \quad \ldots (5.44)$$

From equations (5.41), (5.42), (5.43) and (5.44) it follows that equation (5.28) is satisfied by (5.39) when $n = 2$.

The metric corresponding to $n = 1$, is the Kerr metric, and for $n = 2$ the metric is the first member in a series of new metrics obtained by Tomimatsu and Sato (4).

Tomimatsu and Sato obtained their series of solutions, which is characterized by a positive integer δ equal to our n, by working with Ernst's equation (3.33), and by considering possible generalizations of a particular integral of the Weyl class (for the Weyl class $F = 0$ in the notation of Chapter 2).

This is truly a magnificent achievement on the part of these two authors. It should be pointed out, however, that the author of this thesis, discovered equation (5.20) before he was aware of the work of Tomimatsu and Sato.

We shall now show that the Tomimatsu-Sato class is included in our particular integral. According to Tomimatsu and Sato, if we write $Z = \frac{\alpha}{\beta}$, where Z is the dependent variable in equation (3.33), then the members of their class have the following properties,

(i) $\beta \frac{\partial \alpha}{\partial x} - \alpha \frac{\partial \beta}{\partial x}$ is real

(ii) $\beta \frac{\partial \alpha}{\partial y} - \alpha \frac{\partial \beta}{\partial y}$ is purely imaginary

But (i) and (ii) imply that,

$$\frac{\partial Z}{\partial x} = \mu\beta^{-2} \qquad \ldots (5.45)$$

$$\frac{\partial Z}{\partial y} = i\lambda\beta^{-2} \qquad \ldots (5.46)$$

where μ and λ are real functions of x and y.

From (5.45) and (5.46) we get

$$\frac{\partial Z}{\partial x}\frac{\partial \bar{Z}}{\partial y} + \frac{\partial Z}{\partial y}\frac{\partial \bar{Z}}{\partial x} = 0$$

$$\therefore \qquad Z_1\bar{Z}_2 + Z_2\bar{Z}_1 = 0$$

and using equation (3.27) we get

$$X_1\bar{X}_2 + X_2\bar{X}_1 = 0 \qquad \ldots (5.47)$$

and since X = C+iF, we have the condition

$$C_1C_2 + F_1F_2 = 0$$

which is our equation (4.1). Hence, the Tomimatsu-Sato metrics must be included in our particular integral.

Further, inspection of the Tomimatsu-Sato metric for $\delta = 3$, suggests that for n = 3,

$$p(v) = c^3+3c^2(12v^5+30v^4+28v^3+12v^2+3v)$$

$$+ 3c(3v^8+12v^7+28v^6+30v^5+12v^4)+v^9$$

We shall not, however, verify this as we did for the cases when $n = 1$, and $n = 2$, for it is our belief that there is a fundamental principle at work here, which can only be found by further research into the theory of ordinary differential equations of the type (5.20). All that we can say for the present is that when $a = -4n^2$, $(n = 1,2,...)$ it is very probable that equation (5.20) has a particular integral which is a rational function of v, of the type already stated viz. equation (5.23).

If this turns out to be the case it may be of significance for the theory of Rotating Stars. For if we could also prove that in order for the metric generated by the general solution of equation (5.20), to represent a physically real situation e.g. be asymptotically flat at infinite distance, (a necessary condition for this is that ζ be a rational function) and have mass and angular momentum, it is necessary and sufficient for a to equal $-4n^2$, and for ζ to have the form given by equation (5.23), then it may not be too wild to conjecture, that a steady state condition is only possible for a rotating star, when the matter generating the field is in one of a possible number of discrete energy states corresponding to n taking the values 1, 2, 3, ...

As we have said the general solution of equation (5.20) is unknown, it is however possible to give an infinite series expansion for ζ under certain conditions. This is obtained by assuming the following boundary condition on ζ.

At

$$v = 0, \quad \zeta = 0 \quad \text{and} \quad \frac{d\zeta}{dv} = 4ac$$

where c is a constant, and we shall replace a in equation (5.20) by -4a. Then by assuming

$$\zeta = \sum_{n=1}^{\infty} a_n v^n ,$$

and substituting in equation (5.20) we get non-linear difference equations for the a_n's. However, if we first differentiate equation (5.20), and then substitute the infinite series in the resulting equation, we get simpler difference equations for the a_n's (doing it this way avoids having to square the infinite series on the l.h.s. of (5.20)). The result is

$$a_1 = 4ac$$
$$a_2 = 4a^2 c(c-1)$$

$$2\{(n-1)^2(n-2)a_{n-1} + n(n-1)(2n-1)a_n + n^2(n+1)a_{n+1}\}$$
$$= \alpha_n + \beta_n, \qquad n \geq 2,$$

where

$$\alpha_n = \sum_{j=1}^{n} (2n-2j+1)a_{n-j+1}\{a_{j-1}(j-2) + ja_j\}$$

$$\beta_n = \sum_{j=1}^{n} a_{n-j+1}(n-j)\{(j-1)a_{j-1}+ja_j\}$$

where we define $a_o = 4a$.

It should be observed that the above equations uniquely determine the coefficients a_n. Of course, if the above conjecture is correct the infinite series should converge to a rational function when,

$$a = 1^2,\ 2^2,\ \ldots.$$

We shall now prove that the metric corresponding to the solution for n = 1, is in fact the Kerr metric. When n = 1, p(v) = v+c, and from equation (5.23) we get

$$\zeta = \frac{4v(c-1)}{v+c}\ .$$

Equations (5.18) and (5.19) then give

$$\theta = \frac{2(c-1)}{s^2 + c} \qquad \ldots (5.48)$$

Substituting equation (5.48) in equation (5.10) and (5.11) then gives

$$g = \frac{2sp}{p^2 s^2 - q^2} \qquad \ldots (5.49)$$

$$h = \frac{2q}{p^2 s^2 - q^2} \qquad \ldots (5.50)$$

where we have put $c = -\frac{p^2}{q^2}$ where $p^2+q^2 = 1$.

If we put $\ell = \tan\frac{t}{2}$, equations (4.20) and (4.21) become

$$\ell_1 \sin y = -\tfrac{1}{2}(sg)' h^{-1} \cos y(1+\ell^2) + g\ell \qquad \dots (5.51)$$

$$2\ell_2 \sin y = (h-h' g^{-1} \cosh x)\ell^2 - (h+h' g^{-1} \cosh x) \qquad \dots (5.52)$$

where we have used equations (5.8) and (5.9). Substituting equations (5.49) and (5.50) in equations (5.51), (5.52), and noting equation (5.2), we get the solution,

$$\ell = \frac{p\cosh x - 1 - aq\cos y}{a(p\cosh x+1) + q\cos y}$$

where a is any constant. We shall take a = 0, then

$$\ell = \tan\frac{t}{2} = q^{-1}(p\cosh x-1)\sec y \qquad \dots (5.53)$$

$$\therefore \qquad \sin t = \frac{2}{f}\, q\cos y(p\cosh x-1) \qquad \dots (5.54)$$

and

$$\cos t = \frac{1}{f}\{q^2\cos^2 y-(p\cosh x-1)^2\} \qquad \dots (5.55)$$

where

$$f = (p\cosh x-1)^2 + q^2\cos^2 y.$$

Substituting equations (5.8), (5.9), (5.49), (5.50), (5.54) and (5.55) in equations (4.14) and (4.15) we get

$$u_1 = -\frac{2p}{\lambda}\{(p\cosh x-1)^2-q^2\cos^2 y\}\sinh x \qquad \ldots (5.56)$$

$$u_2 = \frac{4q^2}{\lambda}\sin y\cos y(p\cosh x-1) \qquad \ldots (5.57)$$

where

$$\lambda = (p^2\cosh^2 x-1-q^2\cos^2 y)\{(p\cosh x-1)^2+q^2\cos^2 y\}$$

The solution of equations (5.56) and (5.57) is

$$u = \ell n(p^2\cosh^2 x-1+q^2\cos^2 y) - \ell n\{(p\cosh x-1)^2 + q^2\cos^2 y\}$$

Equation (4.13) then gives

$$C = \frac{p^2\cosh^2 x-1+q^2\cos^2 y}{(p\cosh x-1)^2+q^2\cos^2 y} \qquad \ldots (5.58)$$

From equation (5.21)

$$G(s) = \ell n(s^2+c) - \ell n(s^2+1) + \text{constant} \ldots$$

and since we are taking $C = -\frac{q^2}{p^2}$,

$$G(s) = \ell n(p^2 s^2-q^2) - \ell n(s^2+1) + 2\ell n mK^{-1} \qquad \ldots (5.59)$$

where m is a constant.

Substituting from equations (4.7), (5.1), (5.2), (5.58) and (5.59) in equation (2.32) gives

$$e^{2\psi} = m^2\{(p\cosh x-1)^2 + q^2\cos^2 y\} \qquad \ldots (5.60)$$

Substituting (4.16) and (4.17) in equations (2.19) and (2.20) gives

$$D_1 = Q\Delta C^{-1}\cos t \qquad \ldots (5.61)$$

$$D_2 = P\Delta C^{-1}\sin t \qquad \ldots (5.62)$$

where we have used equation (4.13).

Substituting from equations (5.8) and (5.9) in equations (5.61) and (5.62) we get

$$D_1 = h\Delta C^{-1}\operatorname{cosec} y \cos t \qquad \ldots (5.63)$$

$$D_2 = g\Delta C^{-1}\operatorname{cosec} y \sin t \qquad \ldots (5.64)$$

Substituting from equations (5.1), (5.49), (5.50), (5.54), (5.55) in equations (5.63) and (5.64) gives

$$D_1 = -\frac{2qK}{\mu}\sin^2 y \sinh x\{(p\cosh x-1)^2 - q^2\cos^2 y\} \qquad \ldots (5.65)$$

$$D_2 = \frac{4Kpq}{\mu}\sin y\cos y\sinh^2 x(p\cosh x-1) \qquad \ldots (5.66)$$

where

$$\mu = (p^2\cosh^2 x - 1 + q^2\cos^2 y)^2$$

The solution of equations (5.65) and (5.66) is

$$D = -\frac{2K}{q}\cosh x + \nu + \frac{2Kp\sinh^2 x(p\cosh x - 1)}{q(p^2\cosh^2 x - 1 + q^2\cos^2 y)} \qquad \ldots (5.67)$$

where ν is a constant, we choose $\nu = \frac{2K}{pq}$, then equation (5.67) becomes,

$$D = \frac{2Kq\sin^2 y(p\cosh x - 1)}{p(p^2\cosh^2 x - 1 + q^2\cos^2 y)} \qquad \ldots (5.68)$$

From equations (2.18), (5.58) and (5.68) we get

$$B = \frac{2Kq\sin^2 y(p\cosh x - 1)}{p\{(p\cosh x - 1)^2 + q^2\cos^2 y\}} \qquad \ldots (5.69)$$

To get the canonical form of the Kerr metric we put, $K = m$, $q = -\frac{a}{m}$

$$p = -\frac{1}{m}\sqrt{m^2 - a^2} \qquad \ldots (5.70)$$

and make the coordinate transformation,

$$y = \theta \qquad \ldots (5.71)$$

$$r = -m(p\cosh x - 1) \qquad \ldots (5.72)$$

Then equations (5.58), (5.60) and (5.69) become

$$C = 1 - 2mr(r^2 + a^2\cos^2\theta)^{-1} \qquad \ldots (5.73)$$

$$B = \frac{2ma}{p} r\sin^2\theta(r^2 + a^2\cos^2\theta)^{-1} \qquad \ldots (5.74)$$

$$e^{2\psi} = r^2 + a^2\cos^2\theta \qquad \ldots (5.75)$$

Also using equations (5.71) and (5.72) we get

$$dx^2 + dy^2 = \frac{dr^2}{r^2 - 2mr + a^2} + d\theta^2 \qquad \ldots (5.76)$$

and

$$\Delta = p^{-1}\sin\theta\sqrt{r^2 - 2mr + a^2} \qquad \ldots (5.77)$$

Putting $x^3 = p\phi$, $x^4 = t$ and substituting from (5.72) to (5.76) inclusive, in (2.5) we get

$$ds^2 = (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2 + a^2 - 2mr} + d\theta^2\right)$$
$$+ \sin^2\theta\left\{r^2 + a^2 + \frac{2ma^2 r\sin^2\theta}{r^2 + a^2\cos^2\theta}\right\} d\phi^2$$
$$+ \frac{4mar\sin^2\theta}{r^2 + a^2\cos^2\theta}\, d\phi dt - \left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right) dt^2$$

$$\ldots (5.78)$$

which is the canonical form of the Kerr metric. Although, the metric coefficients in (5.78) satisfy Einstein's equations $R_{ij} = 0$ for all values of a and m, our derivation is only valid for $m^2 > a^2$ (see equation (5.70)).

CHAPTER SIX

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = K\cosh x^1 \sin x^2$

CHAPTER SIX

THE PARTICULAR INTEGRAL CORRESPONDING TO Δ = Kcoshx^1sinx^2

As before we put $x^1 = x$, $x^2 = y$, so that

$$\Delta = K\cosh x \sin y \qquad \ldots (6.1)$$

The solution of equation (4.8) is in this case

$$\cosh x \operatorname{cosec} y = \text{constant}$$

$$\therefore \qquad s = \cosh x \operatorname{cosec} y \qquad \ldots (6.2)$$

From equation (4.7), (6.1) and (6.2) we get

$$S_{11}+S_{22} = \operatorname{cosec}^2 y \frac{d}{ds}\{(s^2-1)G'\} \qquad \ldots (6.3)$$

$$\frac{S_1}{\Delta\Delta_1}(\Delta_1{}^2+\Delta_2{}^2) = s^{-1}(s^2-1)G' \operatorname{cosec}^2 y \qquad \ldots (6.4)$$

Substituting from equations (6.3) and (6.4) in equations (4.11) and (4.12) gives,

$$P^2 = \operatorname{cosec}^2 y\left\{-\frac{d}{ds}[(s^2-1)G']+s^{-1}(s^2-1)G'\right\} \qquad \ldots (6.5)$$

$$Q^2 = -\operatorname{cosec}^2 y\left\{\frac{d}{ds}[(s^2-1)G'] + s^{-1}(s^2-1)G'\right\} \qquad \ldots (6.6)$$

If we put

$$-s\theta = (s^2-1)G' \qquad \ldots (6.7)$$

equations (6.5) and (6.6) become

$$P = g(s)\operatorname{cosec} y \qquad \ldots (6.8)$$

$$Q = h(s)\operatorname{cosec} y \qquad \ldots (6.9)$$

where

$$g^2 = s\frac{d\theta}{ds} \qquad \ldots (6.10)$$

$$h^2 = s\frac{d\theta}{ds} + 2\theta \qquad \ldots (6.11)$$

Substituting from equations (6.8) to (6.11) inclusive in equation (4.22) we get,

$$\left(s^2-1\right)^2\left(s\frac{d^2\theta}{ds^2} + 3\frac{d\theta}{ds}\right)^2 = 4s\frac{d\theta}{ds}\left(s\frac{d\theta}{ds} + 2\theta\right) \ldots$$

$$\ldots\left\{s(s^2-1)\frac{d\theta}{ds} - 2\theta - a\right\} \qquad \ldots (6.12)$$

where a is a constant. ((6.12) is obtained in the same way as equation (5.17) was in Chapter 5.)

Putting

$$s^2 = -v \qquad \text{... (6.13)}$$

$$\zeta = 2v\theta \qquad \text{... (6.14)}$$

equation (6.12) reduces to equation (5.20).

Thus we may regard the particular integral of this Chapter as being generated by equation (5.20). In particular the Kerr-Tomimatsu-Sato class for $a^2 > m^2$, is obtained from the results of this Chapter e.g. for the Kerr metric equations (6.10) and (6.11) give

$$g^2 = -\frac{4(c+1)s^2}{(s^2+c)^2}$$

$$h^2 = \frac{4c(c+1)}{(s^2+c)^2}$$

We again put $c = -\frac{q^2}{p^2}$ except that this time we make $q^2-p^2 = 1$, and then when

$$q = -\frac{a}{m}, \qquad p = -\frac{1}{m}\sqrt{a^2 - m^2} \quad .$$

(Compare with equation (5.70)). The remainder of the derivation then being similar to that given in Chapter 5 for the case when $a^2 < m^2$. This point can be best understood by the following argument.

Suppose that the metric coefficients for the Kerr-Tomimatsu-Sato class are calculated using the equations of

Chapter 5, and we then make the coordinate transformation

$$r - m = \sqrt{m^2 - a^2}\cosh x$$

$$\theta = y$$

on the understanding that

$$m^2 > a^2 \qquad \ldots (6.15)$$

In terms of r and θ the metric coefficients are such that the square root in the term $\sqrt{m^2-a^2}$ is removed. Since the Einstein equations are satisfied identically in r, θ, m and a, the condition (6.15) then becomes irrelevant and we can make the coordinate transformation

$$r - m = \sqrt{a^2 - m^2}\sinh x$$

$$\theta = y$$

thus giving us the particular integral of this Chapter.

We can put this another way, because of the form we have taken for our metric, in which $g_{11} = g_{22}$ the mathematics divides the vacuum gravitational field of rotating bodies of the Kerr-Tomimatsu-Sato type, into three distinct classes viz.

(i) when $m^2 > a^2$ for which the results of Chapter 5 are appropriate,

(ii) when $m^2 < a^2$ for which the results of this Chapter are appropriate,

(iii) when $m^2 = a^2$ for which the results of Chapter 7 are appropriate.

Without knowing the general solutions of the equations of Chapters 5, 6 and 7 we cannot say whether or not the particular integrals will always lead to the same metric in which $g_{11} \neq g_{22}$. If any of the metric coefficients contain a term like v^α, where α is a constant not equal to an integer, the above procedure would not be possible, and we would have distinct solutions.

CHAPTER SEVEN

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = Ke^{x^1} \sin x^2$

CHAPTER SEVEN

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = Ke^{x^1}\sin x^2$

Similarly to Chapter 5 $x^1 = x$, $x^2 = y$.

$$\Delta = Ke^x \sin y \qquad \ldots (7.1)$$

$$s = e^x \operatorname{cosec} y \qquad \ldots (7.2)$$

$$P = g \operatorname{cosec} y \qquad \ldots (7.3)$$

$$Q = h \operatorname{cosec} y \qquad \ldots (7.4)$$

$$g^2 = s\frac{d\theta}{ds} \qquad \ldots (7.5)$$

$$h^2 = s\frac{d\theta}{ds} + 2\theta \qquad \ldots (7.6)$$

$$-\theta = sG' \qquad \ldots (7.7)$$

$$s^2 = v \qquad \ldots (7.8)$$

$$\zeta = 2v\theta \qquad \ldots (7.9)$$

$$\boxed{v^4\left(\frac{d^2\zeta}{dv^2}\right)^2 = \frac{d\zeta}{dv}\left(v\frac{d\zeta}{dv} - \zeta\right)\left(v\frac{d\zeta}{dv} - \zeta + a\right)} \qquad \ldots (7.10)$$

where a is a constant.

Again the general solution of equation (7.10) is unknown, however, it is completely integrable in the in the special case when a = 0, for in that case equation (7.10) reads

$$v^4\left(\frac{d^2\zeta}{dv^2}\right)^2 = \left(\frac{d\zeta}{dv}\right)\left(v\,\frac{d\zeta}{dv} - \zeta\right)^2$$

or

$$\xi^{-\frac{1}{2}}\,\frac{d\xi}{dv} = \pm\,\frac{d}{dv}(v^{-1}\zeta)\,,$$

$$\xi = \frac{d\zeta}{dv}$$

$$\therefore \qquad 4\xi = (v^{-1}\zeta + \alpha)^2\,,$$

where α is the constant of integration

$$\therefore \qquad 4\,\frac{d\zeta}{dv} = (v^{-1}\zeta + \alpha)^2 \qquad \ldots (7.11)$$

Putting

$$\gamma = v^{-1}\zeta + \alpha \qquad \ldots (7.12)$$

equation (7.11) becomes

$$4v\frac{d\gamma}{dv} = (\gamma-2)^2 + 4(\alpha-1) \qquad \ldots (7.13)$$

Three cases can arise,

<u>Case (i) $\alpha - 1 < 0$</u>: and we put $\alpha - 1 = -\beta^2$.

The solution of equation (7.13) is then,

$$\gamma = 2\{c^2(1+\beta) - v^\beta(1-\beta)\}/(c^2-v^\beta)$$

where c is a constant. Using equation (7.12) we get

$$\zeta = v(c^2-v^\beta)^{-1}\{c^2(1+\beta)^2 - v^\beta(1-\beta)^2\} \qquad \dots (7.14)$$

<u>Case (ii) $\alpha - 1 > 0$</u>: and we put $\alpha - 1 = \beta^2$.

The solution of equation (7.13) is

$$\gamma = 2\{1+\beta\tan[\ell n(cv^{\beta/2})]\}$$

$$\therefore \qquad \zeta = v\{1-\beta^2+2\beta\tan[\ell n(cv^{\beta/2})]\} \qquad \dots (7.15)$$

using equation (7.12), where c is a constant.

<u>Case (iii) $\alpha = 1$</u>:

The solution of equation (7.13) is,

$$\gamma = 2 - 4\{\ell n(c^2 v)\}^{-1}$$

where c is a constant. Using equation (7.12) we get

$$\zeta = v\{1 - \frac{4}{\ell n(c^2 v)}\} \qquad \dots (7.16)$$

Case (i)

Although it is possible to calculate the metric coefficients for all values of β, they are rather complicated, except in the case when $\beta = 1$, and it is to this value of β that we shall restrict ourselves.

When $\beta = 1$ equation (7.14) reduces to

$$\zeta = \frac{4vc^2}{c^2 - v} \qquad \ldots (7.17)$$

Substituting from equation (7.17) in (7.9) gives

$$\theta = - \frac{2c^2}{s^2 - c^2} \qquad \ldots (7.18)$$

where we have used equation (7.8). Substituting from equation (7.18) in equations (7.5), (7.6) and (7.7) gives,

$$g = \frac{2cs}{s^2 - c^2} \qquad \ldots (7.19)$$

$$h = \frac{2c^2}{s^2 - c^2} \qquad \ldots (7.20)$$

$$S = G(s) = \ell n(s^2 - c^2) - 2\ell ns + \text{constant} \qquad \ldots (7.21)$$

Substituting from equations (7.19) and (7.20) in equations (7.3), (7.4) and using (7.2) we get

$$P = \frac{2ce^{x}}{e^{2x} - c^2\sin^2 y} \qquad \ldots (7.22)$$

$$Q = \frac{2c^2\sin y}{e^{2x} - c^2\sin^2 y} \qquad \ldots (7.23)$$

Substituting from equations (7.22) and (7.23) in equations (4.20) and (4.21) we get

$$t_1 = \frac{2ce^{x}(\cos y + \sin t)}{e^{2x} - c^2\sin^2 y} \qquad \ldots (7.24)$$

$$t_2 = \frac{2c\sin y(e^{x} - c\cos t)}{e^{2x} - c^2\sin^2 y} \qquad \ldots (7.25)$$

The solution of equations (7.24) and (7.25) is

$$\tan\frac{t}{2} = \frac{\ell e^{x} - c(\ell+\cos y)}{e^{x} + c(1+\ell\cos y)}$$

where ℓ is a constant.

$$\therefore \quad \cos t = \lambda^{-1}\{(1-\ell^2)e^{2x}+2ce^{x}(1+\ell^2+2\ell\cos y)+c^2(1-\ell^2)\sin^2 y\} \qquad \ldots (7.26)$$

$$\sin t = 2\lambda^{-1}\{\ell e^{x}-c(\ell+\cos y)\}\{e^{x}+c(1+\ell\cos y)\} \qquad \ldots (7.27)$$

where

$$\lambda = (1+\ell^2)e^{2x}+2e^x c(1-\ell^2)+c^2\{(1+\ell^2)(1+\cos^2 y)+4\ell\cos y\} \quad \ldots (7.28)$$

Substituting from equations (7.22), (7.23), (7.26), (7.27), and (7.28) in equations (4.14) and (4.15) we get

$$u_1 = 2ce^x\mu^{-1}\{(1-\ell^2)e^{2x}+2ce^x(1+\ell^2+2\ell\cos y)+c^2(1-\ell^2)\sin^2 y\} \quad \ldots (7.29)$$

$$u_2 = 4c^2\sin y\,\mu^{-1}\{\ell e^x-c(\ell+\cos y)\}\{e^x+c(1+\ell\cos y)\} \quad \ldots (7.30)$$

where

$$\mu = (e^{2x}-c^2\sin^2 y)\{(1+\ell^2)e^{2x}+2ce^x(1-\ell^2) + c^2[(1+\ell^2)(1+\cos^2 y)+4\ell\cos y]\} \quad \ldots (7.31)$$

The solution of equations (7.29) and (7.30) is

$$u = -\ell n\{(1+\ell^2)e^{2x}+2ce^x(1-\ell^2)+c^2[(1+\ell^2)(1+\cos^2 y) + 4\ell\cos y]\} + \ell n(e^{2x}-c^2\sin^2 y) + \text{constant} \quad \ldots (7.32)$$

From (4.13) and (7.32) we get

$$C = (1+\ell^2)\nu^{-1}(e^{2x}-c^2\sin^2 y) \qquad \ldots (7.33)$$

where the constant in equation (7.32) has been put equal to $\ell n(1+\ell^2)$ and

$$\nu = (1+\ell^2)e^{2x}+2ce^{x}(1-\ell^2)+c^2\{(1+\ell^2)(1+\cos^2 y) + 4\ell\cos y\} \qquad \ldots (7.34)$$

Substituting from equations (7.1), (7.22), (7.23), (7.26), (7.27) and (7.33) in equations (5.61) and (5.62) we get

$$D_1 = \frac{2c^2}{\epsilon}\sin^2 y e^{x}\{(1-\ell^2)e^{2x}+2ce^{x}(1+\ell^2+2\ell\cos y) + c^2(1-\ell^2)\sin^2 y\} \qquad \ldots (7.35)$$

$$D_2 = \frac{4c}{\epsilon}e^{2x}\sin y\{\ell e^{x}-c(\ell+\cos y)\}\{e^{x}+c(1+\ell\cos y)\} \qquad \ldots (7.36)$$

where we have used (7.34) and

$$\epsilon = (1+\ell^2)(e^{2x}-c^2\sin^2 y)^2 \qquad \ldots (7.37)$$

The solution of equations (7.35) and (7.36) is

$$D = -\frac{2c^2}{\epsilon_1}\{\sin^2 y[e^x(1-\ell^2)+c(1+\ell^2+2\ell\cos y)]$$

$$-(\frac{2\ell}{c}\cos y+b)(e^{2x}-c^2\sin^2 y)\} \qquad \ldots (7.38)$$

where b is a constant, and $\epsilon_1 = (1+\ell^2)^{\frac{1}{2}}\epsilon_{\frac{1}{2}}$.

From equations (2.18), (7.33) and (7.38) we get

$$B = -\frac{2c^2}{\nu}\{\sin^2 y[e^x(1-\ell^2)+c(1+\ell^2+2\ell\cos y)]$$

$$-(\frac{2\ell}{c}\cos y+b)(e^{2x}-c^2\sin^2 y)\} \qquad \ldots (7.39)$$

Finally from equations (2.32), (7.1), (7.21), (7.33) and (7.34) we get

$$e^{2\psi} = \alpha\nu \qquad \ldots (7.40)$$

where α is a constant.

Hence all the metric coefficients are known. If we interpret e^x as a radial spherical polar coordinate we see from equations (7.37) and (7.38) that our metric is only asymptotically flat (i.e. x as $\rightarrow +\infty$, y fixed) when $\ell = b = 0$. In that case by putting $c = m$, $\alpha = 1$, and by making the transformation $r-m = e^x$, $y = \theta$ we see that the metric is the Kerr metric with $a^2 = m^2$ (see equation (5.78)). Thus the Kerr solution for $a^2 = m^2$ is included in equation (7.10) as a special case.

Case (ii)

Similarly to case (i)

$$\theta = \tfrac{1}{2}\{(1-\beta^2) + 2\beta\tan[\ell n(cs^\beta)]\} \quad \ldots (7.41)$$

$$g = \beta\sec\{\ell n(cs^\beta)\} \quad \ldots (7.42)$$

$$h = \beta\tan\{\ell n(cs^\beta)\} + 1 \quad \ldots (7.43)$$

$$G = 2\ell n\{\cos[\ell n(cs^\beta)]\} + \tfrac{1}{2}(\beta^2-1)\ell n s + \text{constant} \quad \ldots (7.44)$$

$$P = \beta\sec\phi\,\text{cosec}\,y \quad \ldots (7.45)$$

$$Q = (1 + \beta\tan\phi)\,\text{cosec}\,y \quad \ldots (7.46)$$

where

$$\phi = \beta x + \ell n\{c\ \text{cosec}^\beta y\} \quad \ldots (7.47)$$

and the equations for t are

$$t_1 = -\beta\sec\phi\,\text{cosec}\,y(\cos y-\sin t) \quad \ldots (7.48)$$

$$-t_2 = \beta\sec\phi + (1+\beta\tan\phi)\,\text{cosec}\,y\cos t \quad \ldots (7.49)$$

Integrating equation (7.48) we get

$$\tan\frac{t}{2} = \{\ell(\sec\phi+\tan\phi) + m\}(\sec\phi+\tan\phi+f)^{-1} \quad \ldots (7.50)$$

where

$$\ell = \sec y(1+\sin y) \quad \ldots (7.51)$$

$$m = \sec y(1-\sin y)f \quad \ldots (7.52)$$

and f is an arbitrary function of y. To find f(y) we put

$$\tan\phi = \frac{a^2-1}{2a} \qquad \sec\phi = \frac{a^2+1}{2a}$$

then using (7.50) in equation (7.49) and equating coefficients of powers of a, we get

$$\beta(1+\ell^2) + \beta\operatorname{cosec}y(1-\ell^2) = 0 \qquad \ldots (7.53)$$

$$-\{2\ell' + \beta\cot y(m-f\ell)\} = \beta(f+m\ell) + \beta\operatorname{cosec}y(f-m\ell) + (1-\ell^2)\operatorname{cosec}y \qquad \ldots (7.54)$$

$$-4m' = \beta(1+\ell^2) + \beta(m^2+f^2) + \beta\operatorname{cosec}y(f^2-m^2) - \beta\operatorname{cosec}y(1-\ell^2) + 4(f-m\ell)\operatorname{cosec}y \qquad \ldots (7.55)$$

$$\beta\cot y(m-f\ell) + 2(fm'-f'm) = -\beta(f+m\ell) + \operatorname{cosec}y\{m^2-f^2+\beta(f-m\ell)\} \qquad \ldots (7.56)$$

$$\beta(m^2+f^2) - \beta\operatorname{cosec}y(-m^2+f^2) = 0 \qquad \ldots (7.57)$$

Substituting from equations (7.51), (7.52) in equations (7.53), (7.54), (7.56) and (7.57) we get mere identities. Substituting from equations (7.51) and (7.52) in equation (7.55) we get

$$f' = -\beta(\sec y + \tan y)(1-\sin y)^{-1} + \sec y f - \beta \sec y f^2$$

... (7.58)

Putting z = siny, equation (7.58) gives

$$(z^2-1)\frac{df}{dz} = \beta(1+z)(1-z)^{-1} - f + \beta f^2$$

... (7.59)

We are unable to give the solution of this Riccati equation. Consequently, we can proceed no further in this case.

Case (iii)

Similarly to case (i)

$$\theta = \tfrac{1}{2}\{1-2[\ell n(cs)]^{-1}\}$$ ... (7.60)

$$g = \{\ell n(cs)\}^{-1}$$ ... (7.61)

$$h = 1 - \{\ell n(cs)\}^{-1}$$ ... (7.62)

$$S = G(s) = \ell n\{\ell n(cs)\} - \tfrac{1}{2}\ell n s + \text{constant}$$

... (7.63)

$$P = \operatorname{cosec} y\{x+\ell n(c \operatorname{cosec} y)\}^{-1}$$ ... (7.64)

$$Q = \operatorname{cosec} y\{x-1+\ell n(c \operatorname{cosec} y)\}\{x+\ell n(c \operatorname{cosec} y)\}^{-1}$$

... (7.65)

and the equations for t are

$$\{x+\ell n(c\ \text{cosec}y)\}t_1 = \text{cot}y(\text{sec}y\sin t-1) \quad \ldots (7.66)$$

$$-\{x+\ell n(c\ \text{cosec}y)\}t_2 = 1 + \text{cot}y\{x-1+\ell n(c\ \text{cosec}y)\}\cos t \quad \ldots (7.67)$$

Integrating equation (7.66) gives

$$\tan\frac{t}{2} = \frac{\ell x + m}{x + p} \quad \ldots (7.68)$$

where

$$\ell = \text{sec}y(1+\sin y) \quad \ldots (7.69)$$

$$m = \text{sec}y\{(1-\sin y)f + (1+\sin y)\ell n(c\ \text{cosec}y)\} \quad \ldots (7.70)$$

$$p = f + \ell n(c\ \text{cosec}y) \quad \ldots (7.71)$$

and f is an arbitrary function y. Using equation (7.68) in equation (7.67) and equating coefficients of powers of x gives

$$-2\ell' = (1-\ell^2)\text{cosec}y \quad \ldots (7.72)$$

$$-2\{\ell'\ell n(c\ \text{cosec}y) + m' + p\ell' - p'\ell)$$
$$= 1 + \ell^2 + \text{cosec}y\{(1-\ell^2)[\ell n(c\ \text{cosec}y) - 1] + 2(p-\ell m)\} \quad \ldots (7.73)$$

$$-2\{pm' - pm' + [\ell n(c\ \text{cosec}y)](m'+p\ell'-p'\ell)\}$$
$$= 2(p+\ell m) + \text{cosec}y\{p^2-m^2+2(p-\ell m)[\ell n(c\ \text{cosec}y)-1]\} \quad \ldots (7.74)$$

$$-2(pm'-pm')\ell n(c\ \text{cosecy}) = \frac{m^2+p^2+\text{cosecy}\{\ell n(c\ \text{cosecy})-1\}}{(p^2-m^2)} \quad \ldots (7.75)$$

Equation (7.75) determines f(y), for if we multiply it by p^{-2}, $p \neq 0$. ($p = 0$ leads to contradictions), we get

$$-2[\ell n(c\ \text{cosecy})]\frac{dw}{dy} = w^2\{1 + \text{cosecy}[1-\ell n(c\ \text{cosecy})]\} - \text{cosecy}\{1 - \ell n(c\ \text{cosecy})\} + 1 \quad \ldots (7.76)$$

where

$$w = mp^{-1} \quad \ldots (7.77)$$

Thus when w is known from equation (7.76), f(y) can be found by substituting from (7.70) and (7.71) in (7.77).

Unfortunately there are no known solutions of equation (7.76). Consequently we can proceed no further in this case.

CHAPTER EIGHT

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = Kx_1$

CHAPTER EIGHT

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = Kx^1$

We put $x^1 = \rho$, $x^2 = z$ and regard ρ and z as cylindrical polar coordinates, i.e each will have the dimensions of length and we can put K = 1 so that

$$\Delta = \rho \qquad \ldots (8.1)$$

and from equations (4.6), (4.7) and (4.8) it follows that

$$S = G(\rho) \qquad \ldots (8.2)$$

Hence

$$S_{11}+S_{22} = G'' \qquad \ldots (8.3)$$

$$\frac{S_1}{\Delta\Delta_1}(\Delta_1{}^2+\Delta_2{}^2) = \rho^{-1}G' \qquad \ldots (8.4)$$

Substituting from equations (8.3) and (8.4) in equations (4.11) and (4.12) we get

$$P^2 = -G'' + \rho^{-1}G' \qquad \ldots (8.5)$$

$$Q^2 = -G'' - \rho^{-1}G' \qquad \ldots (8.6)$$

Putting

$$\rho G' = -\theta \qquad \ldots (8.7)$$

equations (8.5) and (8.6) give

$$P^2 = \rho^{-2}\left(\rho \frac{d\theta}{d\rho} - 2\theta\right) \qquad \ldots (8.8)$$

$$Q^2 = \rho^{-1} \frac{d\theta}{d\rho} \qquad \ldots (8.9)$$

Since P and Q are function of ρ only equation (4.22) reduces to

$$(Q_1 P^{-1})_1 - PQ = 0$$

or

$$(Q_1 P^{-1})_1 = PQ \qquad \ldots (8.10)$$

Multiplying equation (8.10) by $Q_1 P^{-1}$ and integrating gives

$$(Q_1 P^{-1})^2 = Q^2 + 2a$$

where a is the constant of integration. Hence

$$(Q^2)_1^2 = 4P^2 Q^2 (Q^2 + 2a) \qquad \ldots (8.11)$$

Substituting from equations (8.8) and (8.9) in equation (8.11) gives

$$\left\{\frac{d}{d\rho}\left(\rho^{-1} \frac{d\theta}{d\rho}\right)\right\}^2 = 4\rho^{-3} \frac{d\theta}{d\rho}\left(\rho\frac{d\theta}{d\rho} - 2\theta\right)\left(\rho^{-1} \frac{d\theta}{d\rho} + 2a\right)$$

and putting $\rho^2 = v$ we get

$$v^2\left(\frac{d^2\theta}{dv^2}\right)^2 = 2\,\frac{d\theta}{dv}\left(v\,\frac{d\theta}{dv} - \theta\right)\left(\frac{d\theta}{dv} + a\right)$$

... (8.12)

We shall now show that the above integral does not belong, either, to the Papapetrou Class, or to the Lewis-van Stockum Class.

A member of the Papapetrou Class satisfies the condition

$$C_1F_2 - C_2F_1 = 0 \qquad \text{... (8.13)}$$

and if our integral satisfied this condition we would have, substituting from equations (4.13) to (4.17) inclusive, in equation (8.13) $PQ = 0$, which contradicts our hypothesis $P \neq 0$, $Q \neq 0$. If our integral were to be a member of the Lewis-Van Stockum Class it would satisfy the condition

$$x^1(F_2C_2 + F_1C_1) = CF_1 \qquad \text{... (8.14)}$$

which would imply, substituting from equations (4.13) to (4.17) inclusive, in equation (8.14),

$$\rho(P^2 - Q^2)\sin 2t - 2P\sin t = 0$$

i.e. either

$$\text{(i)}\quad t = \frac{n\pi}{2}, \qquad \text{or} \quad \text{(ii)} \quad t = \cos^{-1}\left[\frac{P}{\rho(P^2-Q^2)}\right]$$

would have to be the general solution of equations (4.20) and (4.21). However, inspection of equations (4.20) and (4.21) shows that this is not the case (see below).

Although the general solution of equation (8.12) is not known, it is completely integrable when a = 0. With a = 0, equation (8.12) reads

$$v^2\left(\frac{d^2\theta}{dv^2}\right)^2 = 2\left(\frac{d\theta}{dv}\right)^2\left(v\,\frac{d\theta}{dv} - \theta\right)$$

or

$$\left(\frac{d\zeta}{dv}\right)^2 = 2\zeta\left(\frac{d\theta}{dv}\right)^2 \qquad \ldots (8.15)$$

where

$$\zeta = v\,\frac{d\theta}{dv} - \theta \qquad \ldots (8.16)$$

The solution of equation (8.15) is

$$2\zeta = (\theta + b)^2 \qquad \ldots (8.17)$$

where b is the constant of integration.

Substituting from equation (8.16) in equation (8.17) gives

$$2v\,\frac{d\theta}{dv} = (\theta+b+1)^2 - (2b+1)^2 \qquad \ldots (8.18)$$

Three cases can arise:

Case (i) : 2b+1 = 0

The solution of equation (8.18) is

$$\theta = -\tfrac{1}{2} - \{\ell n(c\sqrt{v})\}^{-1}$$

where c is a constant.

Case (ii) : 2b+1 > 0

If we write $2b+1 = \alpha^2$ the solution of equation (8.18) is

$$\theta = \tfrac{1}{2}\{c^2(\alpha-1)^2 - (\alpha+1)^2 v^{\alpha}\}(v^{\alpha}-c^2)^{-1}$$

where c is a constant.

Case (iii) : 2b+1 < 0

Putting $2b+1 = -\alpha^2$ the solution of equation (8.18) is

$$\theta = \tfrac{1}{2}(\alpha^2-1) + \alpha\tan\{\ell n(cv^{\frac{\alpha}{2}})\}$$

where c is a constant.

We shall now calculate the metric coefficients corresponding to each of these cases.

Case (i)

$$\theta = -\tfrac{1}{2} - \{\ell n(c\rho)\}^{-1} \qquad \ldots (8.19)$$

$(v = \rho^2)$.

Substituting from equation (8.19) in equations (8.7), (8.8) and (8.9) gives

$$G = \ell n\{\beta\rho^{\frac{1}{2}}\ell n(c\rho)\} \qquad \ldots (8.20)$$

where β is a constant.

$$P = \pm\{\rho\ell n(c\rho)\}^{-1}\{1+\ell n(c\rho)\} \qquad \ldots (8.21)$$

$$Q = \pm\{\rho\ell n(c\rho)\}^{-1} \qquad \ldots (8.22)$$

Case (ii)

$$\theta = \tfrac{1}{2}\{c^2(\alpha-1)^2 - (\alpha+1)^2\rho^{2\alpha}\}(\rho^{2\alpha}-c^2)^{-1}$$

$$\ldots (8.23)$$

since $v = \rho^2$.

Substituting from equation (8.23) in equations (8.7), (8.8) and (8.9) gives

$$G = \ell n\{\beta\rho^{\frac{(\alpha-1)^2}{2}}(\rho^{2\alpha}-c^2)\} \qquad \ldots (8.24)$$

where β is a constant.

$$P = \pm\, \rho^{-1}(\rho^{2\alpha} - c^2)^{-1}\{(\alpha+1)\rho^{2\alpha} + (\alpha-1)c^2\} \qquad \ldots (8.25)$$

$$Q = \pm\, 2\alpha c\rho^{\alpha-1}(\rho^{2\alpha}-c^2)^{-1} \qquad \ldots (8.26)$$

Case (iii)

$$\theta = \tfrac{1}{2}(\alpha^2-1) + \alpha\tan\{\ell n(c\rho^{\alpha})\} \qquad \ldots (8.27)$$

using $v = \rho^2$.

Substituting from equation (8.27) in equations (8.7), (8.8) and (8.9) gives

$$G = \tfrac{1}{2}(\alpha^2-1)\ell n\rho - \ell n\{\cos[\ell n(c\rho^{\alpha})]\} + \text{constant} \qquad \ldots (8.28)$$

$$P = \pm\, \rho^{-1}\{1 - \tan[\ell n(c\rho^{\alpha})]\} \qquad \ldots (8.29)$$

$$Q = \pm\, \alpha\rho^{-1}\sec\{\ell n(c\rho^{\alpha})\} \qquad \ldots (8.30)$$

It is possible to express the metric coefficients in terms of P, Q, and G. From equation (8.10) we have $Q_1{}^2 = P^2Q^2$ therefore $Q_1 = \pm\, PQ$, in the event both signs lead to the same result, we shall take the +ve sign, i.e.

$$Q_1 = PQ \qquad \ldots (8.31)$$

Substituting from equation (8.31) in equations (4.20) and

(4.21) gives

$$t_1 = Q_1 Q^{-1} \sin t \qquad \ldots (8.32)$$

$$t_2 = -Q(1+\cos t) \qquad \ldots (8.33)$$

The solution of equations (8.32) and (8.33) is

$$\tan\frac{t}{2} = -\,zQ$$

$$\therefore \qquad \sin t = -\,2zQ(1+z^2Q^2)^{-1} \qquad \ldots (8.34)$$

$$\cos t = (1-z^2Q^2)(1+z^2Q^2)^{-1} \qquad \ldots (8.35)$$

Substituting from equations (8.34) and (8.35) in equations (4.14) and (4.15) gives

$$u_1 = P(1-z^2Q^2)(1+z^2Q^2)^{-1} \qquad \ldots (8.36)$$

$$u_2 = -2zQ^2(1+z^2Q^2)^{-1} \qquad \ldots (8.37)$$

The solution of equations (8.36) and (8.37) is

$$u = \ln(aQ) - \ln(1+z^2Q^2) \qquad \ldots (8.38)$$

where a is a constant.

From equations (4.13) and (8.38) we get

$$C = aQ(1+z^2Q^2)^{-1} \qquad \ldots (8.39)$$

Substituting from equations (4.16), (4.17), (8.1), (8.34), (8.35) and (8.38) in equations (2.19) and (2.20) gives

$$D_1 = \frac{\rho}{a}(1-z^2Q^2) \qquad \ldots (8.40)$$

$$D_2 = -\frac{2}{a}P\rho z \qquad \ldots (8.41)$$

equation (8.41) can be integrated to give

$$D = -\frac{P}{a}\rho z^2 + h(\rho) \qquad \ldots (8.42)$$

where $h(\rho)$ is an arbitrary function of ρ. Substituting from equation (8.42) in equation (8.40) gives

$$\frac{dh}{d\rho} - \frac{z^2}{a}\frac{d}{d\rho}(P\rho) = \frac{\rho}{a} - \frac{z^2}{a}\rho Q^2$$

$$\therefore \quad \frac{dh}{d\rho} = \frac{\rho}{a} - \frac{z^2}{a}\{\rho Q^2 - \frac{d}{d\rho}(P\rho)\}$$

But

$$\rho Q^2 - \frac{d}{d\rho}(P\rho) = \frac{1}{2\rho P}\{2\rho^2 PQ^2 - \frac{d}{d\rho}(P^2\rho^2)\}$$

$$= \frac{1}{2\rho P}\{\rho^2\frac{d}{d\rho}(Q^2) - \frac{d}{d\rho}(P^2\rho^2)\} \text{ (using (8.31))}$$

$$= \frac{1}{2\rho P}\{\rho^2(\rho^{-1}\frac{d^2\theta}{d\rho^2} - \rho^{-2}\frac{d\theta}{d\rho})$$

$$- (\rho\frac{d^2\theta}{d\rho^2} - \frac{d\theta}{d\rho})\} \text{ (using (8.8) and (8.9))}$$

$$\equiv 0$$

$$\therefore \quad h = \frac{1}{2a}(\rho^2 + m) \qquad \ldots (8.43)$$

where m is the constant of integration.

Substituting from equation (8.43) in equation (8.42) gives

$$D = \frac{1}{2a}(\rho^2+m-2P\rho z^2) \qquad \ldots (8.44)$$

From equations (2.18) and (8.44) we get

$$B = Q(\rho^2+m-2P\rho z^2)(1+z^2Q^2)^{-1} \qquad \ldots (8.45)$$

Substituting from equations (8.1), (8.2) and (8.39) in equation (2.32) gives

$$e^{2\psi} = \frac{e^G}{aQ}(1+z^2Q^2) \qquad \ldots (8.46)$$

Hence, all the coefficients are known in terms of P, Q and G. It will be observed that none of these metrics is asymptotically flat.

CHAPTER NINE

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = Kx^1x^2$

CHAPTER NINE

THE PARTICULAR INTEGRAL CORRESPONDING TO $\Delta = Kx^1x^2$

Similarly to Chapter 5 $x = x^1$, $y = x^2$

$$\Delta = Kxy \qquad \text{... (9.1)}$$

$$s = xy^{-1} \qquad \text{... (9.2)}$$

$$-s\theta = (1+s^2)G' \qquad \text{... (9.3)}$$

$$P = gy^{-1} \qquad \text{... (9.4)}$$

$$Q = hy^{-1} \qquad \text{... (9.5)}$$

$$g^2 = s\frac{d\theta}{ds} \qquad \text{... (9.6)}$$

$$h^2 = s\frac{d\theta}{ds} + 2\theta \qquad \text{... (9.7)}$$

and the differential equation for ζ is the same as in Chapter 5, viz. equation (5.20).

In the case when a (see equation (5.20)) < 0, it is possible to express the metric coefficients explicitly in terms of G, g, and h. To see this, we choose s and y as new coordinate variables, using (9.2) we have

$$x = sy \qquad \text{... (9.8)}$$

In terms of s and y equations (4.20) and (4.21) read

$$yty = -\{h^{-1}s(sg)' + g^{-1}h'\} + sgsint - hcot \qquad \text{... (9.9)}$$

$$ts = -h^{-1}(sg)' + gsint \qquad \ldots (9.10)$$

where we have used equations (9.4) and (9.5) and

$$ty = \frac{\partial t}{\partial y} \ , \qquad ts = \frac{\partial t}{\partial s} \ .$$

Theorem

If

$$\alpha = -\frac{1}{2hg}\frac{d}{ds}(h^2 + s^2 g^2) \qquad \ldots (9.11)$$

and

$$\ell = \frac{sg + 2n}{h + \alpha} \qquad \ldots (9.12)$$

where the constant a in equation (5.20) is equal to $-4n^2$. Then the solution of equations (9.9) and (9.10) is given by

$$\tan\frac{t}{2} = -\ell \qquad \ldots (9.13)$$

Proof

Since t is a function of s only $\ell_y = 0$, also

$$\sin t = -\frac{2\ell}{1+\ell^2} \ , \qquad \cos t = \frac{1-\ell^2}{1+\ell^2}$$

∴ equation (9.9) gives

$$\ell^2\left\{\frac{1}{2gh}\frac{d}{ds}(s^2g^2+h^2)-h\right\} + 2sg\ell + h$$
$$+ \frac{1}{2gh}\frac{d}{ds}(s^2g^2+h^2) = 0$$

i.e.

$$\ell^2(\alpha+h) - 2sg\ell + \alpha - h = 0$$

(using (9.11).

i.e.

$$\{\ell-sg(h+\alpha)^{-1}\}^2 + \{\alpha^2-h^2-s^2g^2\}(\alpha+h)^{-2} = 0.$$

But equation (5.16) gives

$$\alpha^2 - (h^2+s^2g^2) = a = -4n^2$$

$$\therefore \quad \{\ell - (sg+2n)(\alpha+h)^{-1}\}\{\ell - (sg-2n)(\alpha+h)^{-1}\} = 0$$

which is satisfied by

$$\ell = (sg+2n)(\alpha+h)^{-1}$$

Therefore equation (9.9) is satisfied identically.

Now

$$ts = -\frac{2\ell'}{1+\ell^2}$$

Therefore equation (9.10) gives

$$\ell' = \tfrac{1}{2}h^{-1}(sg)'(1+\ell^2)+g\ell \qquad \ldots (9.14)$$

To prove (9.14) we first note that ℓ satisfies the equation,

$$(\alpha+h)\ell^2+\alpha-h = 2sg\ell \qquad \ldots (9.15)$$

and

$$\alpha' = -hg \qquad \ldots (9.16)$$

(see equation (5.15). Now

$$\ell'(h+\alpha)^2 = (h+\alpha)(sg)'-(sg+n)(h'+\alpha') \quad \text{(using (9.12))}$$
$$= (h+\alpha)(sg)'-(sg+2n)(h'-hg)$$

where we used (9.16). Now $h' = g(hs)^{-1}(sg)'$ (obtained by eliminating θ from between equations (9.6) and (9.7)).

$$\therefore \quad \ell'(h+\alpha)^2 = (h+\alpha)(sg)'-(sg+2n)\left\{\frac{(sg)'}{sh} - (h+\alpha)+\alpha\right\}g$$

$$= (h+\alpha)(sg)'+(sg+2n)g(h+\alpha)$$
$$- (sg+2n)g\left\{\alpha+\frac{(sg)'}{sh}\right\}$$

$$\therefore \qquad \ell' = g\ell + (\alpha+h)^{-2}\left\{(h+\alpha)(sg)' - (sg+2n)\,g\left[\alpha + \frac{(sg)'}{sh}\right]\right\} \qquad \ldots (9.17)$$

But

$$\alpha + \frac{(sg)'}{sh} = -\frac{(1+s^2)}{sh}(sg)' + \frac{(sg)'}{sh} = -\frac{s}{h}(sg)' \qquad \ldots (9.18)$$

where we have used (9.12) and

$$h' = \frac{g}{hs}(sg)'.$$

Substituting (9.18) in (9.17) gives

$$\begin{aligned} \ell' &= g\ell + (\alpha+h)^{-2}\{(h+\alpha)(sg)' + (sg+2n)\,sgh^{-1}(sg)'\} \\ &= g\ell + (\alpha+h)^{-2}(sg)'h^{-1}\{h(h+\alpha) + sg\ell(h+\alpha)\} \qquad \text{(using (9.12))} \\ &= g\ell + \frac{(sg)'}{2}h^{-1}(\alpha+h)^{-1}(2h+2sg\ell) \\ &= g\ell + \frac{(sg)'}{2}h^{-1}(h+\alpha)^{-1}\{2h+(\alpha+h)\ell^2+(\alpha-h)\} \qquad \text{(using (9.15))} \end{aligned}$$

i.e.

$$\ell' = g\ell + \tfrac{1}{2}(sg)'h^{-1}(1+\ell^2)$$

which is equation (9.14). Therefore equation (9.10) is identically satisfied.

Substituting from equations (9.4) and (9.5) in equations (4.14) and (4.15) gives

$$u_1 = gy^{-1}\cos t \qquad \ldots (9.19)$$

$$u_2 = hy^{-1}\sin t \qquad \ldots (9.20)$$

Now

$$u_s = yu_1 = g\cos t \qquad (\text{from } (9.19))$$

i.e
$$u_s = g\frac{(1-\ell^2)}{1+\ell^2} \qquad (\text{from } (9.13))$$

$$\ldots (9.21)$$

$$u_y = su_1 + u_2$$

i.e.
$$yu_y = sg\cos t + h\sin t \quad (\text{using } (9.19) \text{ and } (9.20))$$

i.e.
$$yu_y = (1+\ell^2)^{-1}\{sg(1-\ell^2)-2\ell h\}$$

$$\ldots (9.22)$$

Now using (9.11), (9.12) and (5.16) we can also write ℓ in the form

$$\ell = \frac{\alpha-h}{sg-2n}.$$

$$\therefore \quad \ell(sg-2n) = \alpha-h = \alpha+h-2h$$

$$\therefore \quad \ell^2(sg-2n) = \ell(\alpha+h) - 2h\ell$$

$$= sg+2n - 2h\ell \qquad (\text{using } (9.12))$$

$$\therefore \quad -2n(\ell^2+1) = sg(1-\ell^2) - 2\ell h$$

$$\therefore \quad (\ell^2+1)^{-1}\{sg(1-\ell^2)-2\ell h\} = -2n \qquad \ldots (9.23)$$

Substituting from equation (9.23) in equation (9.22) gives

$$yu_y = -2n \qquad \ldots (9.24)$$

The solution of equations (9.21) and (9.24) is

$$u = \ell n(fy^{-2n}) \qquad \ldots (9.25)$$

where

$$f = (s^2g^2+h^2)^{-\frac{1}{2}}\exp\{-2n\int sg^2(s^2g^2+h^2)^{-1}ds\} \qquad \ldots (9.26)$$

where we have used (9.11), (9.12) and (9.13). From equations (9.25) and (4.13) we get

$$C = fy^{-2n} \qquad \ldots (9.27)$$

Substituting from equations (4.13), (4.16) and (4.17) in equations (2.19) and (2.20) we get

$$D_1 = \Delta QC^{-1}\cos t \qquad \ldots (9.28)$$

$$D_2 = \Delta PC^{-1}\sin t \qquad \ldots (9.29)$$

Substituting from equations (9.1), (9.4), (9.5) in equations (9.28) and (9.29) gives

$$D_1 = \frac{shK}{f}\, y^{2n+1}\cos t$$

$$D_2 = \frac{sgK}{f}y^{2n+1}\sin t$$

$$\therefore \quad D_s = \frac{sgK}{f}y^{2n+2}\sin t \qquad \ldots (9.30)$$

$$D_y = \frac{Ky^{2n+1}}{f}s(sh\cos t+gs\sin t) \qquad \ldots (9.31)$$

The solution of equations (9.30) and (9.31) is

$$D = \frac{Ksy^{2(n+1)}}{2f(n+1)} \{s\alpha(h^2-g^2)-2nhg(1+s^2)\}(s^2g^2+h^2)^{-1} + T$$

(using (9.12) and (9.13). Where T is a constant, we shall take T = 0 so that

$$D = \frac{Ky^{2(n+1)}}{2f(n+1)} s\{s\alpha(h^2-g^2)-2nhg(1+s^2)\}(s^2g^2+h^2)^{-1} \quad \dots (9.32)$$

Equations (9.27), (9.32) and (2.18) give

$$B = \frac{y^2 sK}{2(n+1)} \{s\alpha(h^2-g^2)-2ngh(1+s^2)\}(s^2g^2+h^2)^{-1} \quad \dots (9.33)$$

Substituting from equations (9.1), (9.27) in equation (2.32) we get

$$e^{2\psi} = K^2 e^G y^{2(n+1)}(1+s^2)f \quad \dots (9.34)$$

where we have used equation (9.2).

Also since $x = sy$

$$dx^2+dy^2 = (1+s^2)dy^2 + 2sydsdy+y^2ds^2 \quad \dots (9.35)$$

Substituting from equations (9.1) and (9.27) in equation (2.4) gives

$$A = y^{2n} f^{-1} (K^2 s^2 y^4 - B^2) \qquad \dots (9.36)$$

<u>Final form of the metric when $a = -4n^2$</u>

Substituting from equations (9.27), (9.33), (9.35) and (9.36) in (2.1) we get

$$d\sigma^2 = K^2 e^{G(s)} y^{2(n+1)} f(1+s^2) \{(1+s^2) dy^2 + 2sydsdy + y^2 ds^2\}$$
$$+ y^{2n} f^{-1} (s^2 K^2 y^4 - B^2) d\phi^2 + 2Bd\phi dt - fy^{-2n} dt^2$$
$$\dots (9.37)$$

where we have put $x^3 = \phi$, $x^4 = t$.

Where

$$B = \frac{sy^2 K}{2(n+1)} \{s\alpha(h^2 - g^2) - 2nhg(1+s^2)\} (s^2 g^2 + h^2)^{-1}$$
$$\dots (9.38)$$

$$f = (s^2 g^2 + h^2)^{-\frac{1}{2}} \exp\{-2n \int s g^2 (s^2 g^2 + h^2)^{-1} ds\}$$
$$\dots (9.39)$$

and

$$\alpha = -\frac{1}{2hg} \cdot \frac{d}{ds}(s^2 g^2 + h^2) \qquad \dots (9.40)$$

Since the generating differential equation for this metric is the same as that of Chapter 5, we can use the resultsof that Chapter to express G, B andf in terms of s and y, e.g. when n = 1 G is given by equation (5.59). Substituting from equations (5.49) and (5.50) in equations (9.38), (9.39) and

(9.40) gives

$$f = a(s^2p^2-q^2)(s^4p^2+q^2)^{-1} \qquad \ldots (9.41)$$

when a is a constant.

$$\alpha = 2pq(1+s^2)(p^2s^2-q^2)^{-1} \qquad \ldots (9.42)$$

and

$$B = -s^2pqy^2K(1+s^2)(s^4p^2+q^2)^{-1}.$$

When n = 2, the results are rather complicated e.g. f involves the integral

$$\int sg^2(s^2g^2+h^2)^{-1}ds$$
$$= \frac{1}{2}\int\frac{v(p^2v^3 - 3q^2v - 2q^3)^2 dv}{p^2v^2(p^2v^3-3q^2v-2q^2)^2+q^2(2p^2v^3+3p^2v^2+q^2)^2}$$

where $s^2 = v$, and we are unable to evaluate this integral.

CHAPTER TEN

A CONDITION THAT THE METRIC (2.1) SHALL NOT BE OF THE SPECIAL RELATIVITY TYPE IN SOME COORDINATE SYSTEM

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A CONDITION THAT THE METRIC (2.1) SHALL NOT BE OF THE SPECIAL RELATIVITY TYPE IN SOME COORDINATE SYSTEM

If the metric (2.1) is that of special relativity in some coordinate system, it is necessary that all of the components of the Riemann-Christoffel Curvature Tensor $R_{ijk\ell}$ vanish identically.

Thus if one of these is not zero for the metric (2.1), then it is not special relativity in any coordinate system.

If we calculate R_{1212}, using the results of Chapter 2 we get

$$R_{1212} = -e^{2\psi}\nabla^2\psi \qquad \ldots (10.1)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2}$$

Thus from (10.1) $R_{1212} \neq 0$ if

$$\nabla^2\psi \neq 0 \qquad \ldots (10.2)$$

The result (10.1) is also obvious from the metric (2.1), which if we allow x^1 and x^2 to be complex, and introducing the complex conjugate variables z and $\bar{z}$ defined by

$$z = x^1 + ix^2 , \qquad \bar{z} = x^1 - ix^2 ,$$

the metric (2.1) becomes

$$ds^2 = e^{2\psi} dz d\bar{z} + \text{etc.} \qquad \ldots (10.3)$$

Making the coordinate transformation

$$z = f(w) , \qquad \bar{z} = \bar{f}(\bar{w})$$

where f(w) is analytic and $\bar{f}(w)$ the conjugate analytic function (10.3) becomes

$$ds^2 = e^{2\psi(w,\bar{w})} \frac{df}{dw} \frac{d\bar{f}}{d\bar{w}} dw d\bar{w} + \text{etc.}$$

$$\ldots (10.4)$$

If the metric (10.4) is special relativity in the w-frame we shall require

$$e^{2\psi(w,\bar{w})} f' \bar{f}' = \text{constant}$$

$$\therefore \quad 2\psi(w,\bar{w}) = - \ell n f' - \ell n \bar{f}' + \text{constant}$$

$$\therefore \quad \frac{\partial^2 \psi}{\partial w \partial \bar{w}} = 0$$

$$\therefore \quad \frac{\partial^2 \psi}{\partial z \partial \bar{z}} = 0$$

or

$$\nabla^2 \psi = 0$$

which is the same as equating R_{1212} to zero.

If we calculate $\nabla^2\psi$ for each of the metrics of Chapters 5,7,8 and 9 we see that $\nabla^2\psi \neq 0$. Hence, none of them is trivial.

CHAPTER ELEVEN

CONCLUSION

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CONCLUSION

The results of Chapter 4 show, just as with the other equations of Mathematical Physics, that it is possible even in General Relativity, to have everything depending on the solution of just one ordinary differential equation. The danger here is that an equation like equation (4.1) is not generally coordinate invariant, and so we run the risk of not being able to tell whether or not two different solutions will not after all give rise to the same metric, by making suitable coordinate transformations.

The ordinary differential equation of Chapter 5 seems to indicate that certain non-linear differential equations may have eigenfunction solutions of a particular (in this case rational) form when a parameter assumes certain eigenvalues. From a purely mathematical point of view this is a very interesting possibility. However, further research will be required to decide whether this is true or not. The present theory of non-linear differential equations is mainly concerned with equations which possess non-movable singularities in their solutions. We, however, are interested in equations which have movable singularities, in fact movable poles.

The results of Chapters 5, 6 and 7 show that the very complicated metrics of Kerr and Tomimatsu-Sato have a common origin. In our view the methods we have used give the simplest and most natural way of generalizing the metrics of Schwarzschild and Weyl.

Although we have concerned ourselves only with vacuum solutions, it is clear from the work of Ernst, that all the solutions obtained can be generalized to include certain types of source free electromagnetic fields as well. In this generalization our generating differential equations are unaltered and so we can say, for example, that equations (5.20), (6.12) and (7.10) generate the metrics for a charged rotating body according as $m^2 > a^2 + e^2$, $m^2 < a^2 + e^2$, or $m^2 = a^2 + e^2$ where e is the total charge.

REFERENCES

1. LEVY H — Classification of Stationary Axisymmetric Gravitational Fields.
Nuovo Cimento. Series 10B Vol.56 (1968)

2. LEWIS T — Some Special Solutions of the equations of axially symmetric gravitational fields
Proc.Roy.Soc.London A.136.176 (1932)

3. VANSTOCKUM W J — The gravitational field of a distribution of particles rotating about an axis of symmetry.
Proc.Roy.Irish Acad.135 (1938)

4. TOMIMATSU A and SATO H — New Series of Exact Solutions for Gravitational Fields of Spinning Masses.
Prog.Theor.Phy.Vol.50 No.1 95 (1973)

5. CARTER B — Hamilton-Jacobi and Schrodinger Separable Solutions of Einstein's Equations
Commun.Math.Phys. 10 280 (1968)

6. MORGAN A J A — The reduction by one of the number of independent variables in some systems of partial differential equations
Quart.J.Math. Oxford (1952) pp.250

7. MAJUMDER S D — A Class of Exact Solutions of Einstein's Field Equations.
Phy.Rev. 72 390 (1947)

8. BONNOR W B A Class of Stationary Solutions of the Einstein-Maxwell Equations
Commum.Math.Phys. 34 77 (1973)

9. NEWMAN E and PENROSE R An Approach to Gravitational Radiation by a Method of Spin Coefficients
J.Math.Phys 1 566 (1962)

10. KINNERSLEY W Type D Vacuum Metrics
J.Math.Phys 10 1195 (1969)

11. ERNST F J New Formulation of the Axially Symmetric Gravitational Field Problem
Phy.Rev. 167 1175 (1967)

12. ROBINSON I and SCHILD A Generalization of a Theorem by Goldberg and Sachs.
J.Math.Phys 4 484 (1963)

13. DEBNEY G C; KERR R P and SCHILD A Solutions of the Einstein and Einstein-Maxwell Equations
J.Math.Phys 10 1842 (1969)

14. KERR R P Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics.
Phy.Rev.Lett 11 237 (1963)

15. ROBINSON J; TRAUTMAN A Some Spherical Gravitational Waves in General Relativity.
Proc.Roy.Soc.(London) A.265 463 (1961)

16. KERR R P and DEBNEY G C Einstein Spaces with Symmetrical Groups
J.Math.Phys. 11 2807 (1970)

17. ISRAEL W Differential Forms in General Relativity Communications of the Dublin Institute for Advanced Studies. Series A. No.19 (1970)

18. SYNGE J L Complex 3-Space and the Petrov-Prirani Classification Communications of the Dublin Institute for Advanced Studies. Series A. No.15 (1964)

19. ERNST F J Complex Potential Formulation of the Axially Symmetric Gravitational Field Problem. J.Math.Phys 15 1409 (1974)

20. ONENGUT G and SERDAROGLU M Two-Parameter Static and Five-Parameter Stationary Solutions of the Einstein-Maxwell Equations Nuovo Cimento Series B Vol.27 pp 213 (1975)

21. ROBINSON I: ROBINSON J R and ZUND J D Degenerate Gravitational Fields with Twisting Rays Journal of Mathematics and Mechanics Vol.18 No.9 (1969)

22. PERJES Z Spinor Treatment of Stationary Space-Times J.Math.Phys 11 3383 (1970)

23. CARTER B The Commutation Property of a Stationary, Axisymetric System. Commun.Math.Phys.17 233 (1970)

24. MISRA R M; PANDEY D B; SRIVASTAVA D E and TRIPATHI
New Class of Solutions of the Einstein-Maxwell Fields.
Phy.Rev.D 7 1587 (1973)

25. DAVIES H and CAPLAN T A The Space-Time Inside a Rotating Cylinder
Camb.Phil.Soc. 69 325 (1971)

26. HOFFMAN R B Stationary Axially Symmetric Generalization of the Weyl Solutions in General Relativity
Phy.Rev. 182 1361 (1969)

27. KINNERSLEY W Limits of the Tomimatsu-Sato Gravitational Field
J.Math.Phys 15 2121 (1974)

28. JANIS A I and NEWMAN E T Structure of Gravitational Sources
J.Math.Phys 6 902 (1965)

29. JANIS A I and NEWMAN E T Note on the Kerr Spinning-Particle Metric
J.Math.Phys. 6 915 (1965)

30. GLASS E N Structure of the Tomimatsu-Sato Gravitational Field
Phy.Rev. D.7 3127 (1973)