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Global Optimization for a Developed Price Discrimination Model: A Signomial Geometric Programming Based Approach

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Abstract

This paper presents a price discrimination model for a manufacturer who acts in two different markets. In order to have a fair price discrimination model and compare monopoly and competitive markets, it is assumed that there is no competitor in the first market (monopoly market) and there is a strong competitor in the other market (competitive market). The manufacturer objective is to maximize the total benefit in both markets. The decision variables are selling price, lot size, marketing expenditure, customer service cost, flexibility and reliability of production process, set up costs and quality of products. The proposed model in this paper is a signomial geometric programming problem which is difficult to solve and find the globally optimal solution. So, this signomial model is converted to a posynomial geometric type and using an iterative method, the globally optimal solution is found. To illustrate the capability of the proposed model, a numerical example is solved and the sensitivity analysis is implemented under different conditions.

Keywords: Global optimization, Lot sizing, price discrimination, production planning, signomial geometric programming

1. Introduction

Price discrimination is one of the most important and effective strategies. It can help the manufacturer increase its market share and maximize the profit. This paper considers a manufacturer who produces a single product in two

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markets. The proposed model maximizes the profit of manufacturer in both markets. Since, the functions of demand, unit production, interest and depreciation, and maintenance costs are assumed in a power form, so, the proposed model is a geometric programming (GP) model. GP is an effective method to solve a class of non-linear programming problems. In the previous studies, GP has been used in different fields such as inventory control (Islam, 2008), engineering design (Xu, 2013), project management (Scott and Jefferson, 1995) and etc.

Kochenberger (1971) was the first researcher who used geometric programming method to solve a nonlinear economic order quantity (EOQ) problem. Cheng (1991) solved an EOQ model using the geometric programming. Lee (1993) proposed GP method to maximize the profit of a retailer. His model optimized selling price and order quantity for both no-quality discounts and continuous quality discounts. Drezner et al. (1995) presented an EOQ model with two products when one can be substituted for the other at a given unit cost. They introduced an algorithm to find the optimal solution of proposed non-linear model. Kim and Lee (1998) applied GP for constrained non-linear problems with non-concave objective functions. Chen (2000) proposed a GP profit-maximization inventory model to find out optimal quality level, selling quantity and purchasing product price in the intermediate firms. Abuo-El-Ata et al. (2003) introduced a probabilistic multi item inventory model under two different assumptions and derived the optimal maximum inventory level by using geometric programming technique. Tripathy et al. (2003) considered reliability in an EOQ model with imperfect production process. In addition, they assumed that the unit production cost is directly related to process reliability and inversely related to the demand rate. Teng and Yang (2004) developed the classical EOQ model for not only time-varying demand but also fluctuating unit cost. Also, they considered shortages and partial backlogging in their model and used a simple search algorithm to find the local minimum of the proposed convex function. Sadjadi et al. (2005) proposed an integrated inventory model to find out production lot size, marketing expenditure and selling price of product. They used GP to calculate the optimal solution of the proposed model. Liu (2006) assumed that the demand quantity and unit cost were imprecise and calculated the bounds of profit using duality theorem of GP approach. Mandal and Roy (2006) developed a multi-item finite production lot size model with hybrid numbers for cost parameters. Jung and Klein (2006) established three EOQ models to find out optimal order quantity and price per unit. Safaei et al. (2006) proposed a price discrimination model in order to maximize profit of a firm which produced a single product in two markets. Since, their model was a constrained signomial GP (SGP) with two positive terms in objective function, they used genetic algorithm (GA) to solve it. They find a local optimal solution using GA for the proposed model. Leung (2007) developed an economic production quantity (EPQ) model by considering flexible and imperfect production process. He applied the GP technique to find out optimal solution for set up cost, order quantity and production process reliability. Islam (2008) used a global criteria method and GP technique to solve a multi-objective marketing planning model. Fathian et al. (2009) analyzed the pricing method and service quality for electronic-business companies which sold their products via web supermarkets. Ghazi Nezami et al. (2009) presented a SGP model to determine selling price, marketing expenditure and economic production quantity. They converted the model into a posynomial geometric one and solved it using an iterative algorithm. Sadjadi et al. (2010) proposed a new integrated model to determine products' price, marketing expenditure, lot size, set-up cost, inventory holding cost and production process reliability, simultaneously. They derived the optimal solution using the GP technique. Shen et al. (2011) proposed a deterministic global optimization algorithm for solving a fractional programming problem. Their proposed algorithm reformulated the problem as a monotonic optimization type, and provided a solution which was adequately guaranteed to be close to the actual optimal solution. Koteb and Fergany (2011) proposed a GP multi-item inventory model to find the optimal levels of order quantity, demand rate, and leading time. By considering flexibility and reliability, Sadjadi et al. (2012) suggested a GP model for finding lot size, price and marketing expenditure. Shen and Bai (2013) presented a branch-and-bound algorithm for solving generalized GP problems (GGP) with discrete variables. They indicated their conclusions by reporting the computational results for several examples and small randomly generated problems. Also, in order to globally solve the GGP problem, a branch-reduction-bound algorithm was presented by Shen and Li (2013). They proved the convergence of their algorithm and solved the several numerical examples to demonstrate its feasibility and efficiency. Ghosh and Roy (2013) applied a GP technique to solve a nonlinear goal programming problem. Omrani and Keshavarz (2014) proposed an uncertain EOQ model with interval exponents and coefficients. They maximized the profit and found the lower and upper bounds for the objective function and decision variables. Sadjadi et al. (2015) introduced a cubic production cost function in an inventory GP model. They converted the SGP into a posynomial geometric one and solved it by convex optimization tools. Aliabadi et al. (2017) presented a SGP model of joint partial delayed payments, pricing and marketing strategies in a supply chain with a retailer and multiple customers. They transformed the model to a reversed constraint programming and obtained the optimal solution. Tabatabaei et al. (2017) presented a production lot sizing model where the cost of production was depended on the production size. Their problem formulation was a SGP problem and the GP method has been used to solve their model. Shen et al. (2019) proposed a practicable contraction approach for solving the sum of the generalized polynomial ratios problem with generalized polynomial constraints. In their proposed approach, simple transformation and contraction strategies have been utilized to reduce the original nonconvex problem as a standard GP problem. Finally, they demonstrated he tractability and effectiveness of their approach by solving several numerical examples. In many real world applications, the relevant models are in signomial form with high degree of difficulty. The degree of difficulty (DD) is equal to the number of independent linear equations minus the number of dual variables. These types of problems are belonging to the non-convex class of optimization problems. To solve the non-convex problems, researchers usually use metaheuristic algorithms to find out locally solutions. For instance, Safaei et al. (2006) applied a genetic algorithm to solve a SGP problem.

Recently, researchers found the global optimal solution of SGP models using the iterative algorithms. For instance, Aliabadi et al. (2018) proposed a supply chain EOQ model for single NIDIs (Non-Instantaneous Deteriorating Items) problem that consists of supplier, relater and customers. Their inventory model form was a constrained SGP model that solved by approximation method. Jabbarzadeh et al. (2019) proposed a sustainable EOQ model for multi-items with fuzzy resources and hybrid cost parameters. They formulated the model as a multi-objective SGP problem and used the iterative algorithm to find the optimal solution. Rabbani and Aliabadi (2019) proposed an inventory model by considering delayed payments and shortages. The problem was formulated as a SGP form and solved by iterative algorithm. Three recent studies have used the iterative algorithm presented by Xu (2014).

In this paper, a price discrimination model is considered with a manufacturer who produces a single product in two markets. The proposed model is a SGP form which is difficult to solve and find the globally optimal solution. So, this study also applied the iterative algorithm proposed by Xu (2014) to find out the globally optimal solution. Table 1 provides comprehensive comparisons between the proposed model and method in this paper with recent studies. As it can be seen, only the proposed model in this paper considers the two features of “price discrimination model” and “use of approximation for SGP” at the same time.

The rest of this paper is organized as follows. In the next section, the problem is introduced. In section 3, the mathematical model is presented. Section 4 discusses the iterative algorithm to solve the proposed model. In order to illustrate the capability of the algorithm, several numerical examples are solved in section 5. Also, section 5 presents the sensitivity analysis on the parameters and finally, the conclusion of the paper is summarized in section 6.

Table 1: Comparisons of the model and method in this paper with references

Authors	First Model	Use of Approximation	Final Model	Price Discrimination	Proposed Method
Safaei et al. (2006)	SGP	No	SGP	Yes	Genetic Algorithm
Ghazi Nezami et al. (2009)	SGP	Yes	Posynomial GP	No	Iterative Algorithm
Sadjadi et al. (2015)	SGP	Yes	Posynomial GP	No	Matlab based modeling system (CVX)- Proposed by Boyd and Michael (2009)
Aliabadi et al. (2017)	SGP	No	Reversed Constraint Program	No	Dual GP
Tabatabaei et al. (2017)	SGP	Yes	Posynomial GP	No	Matlab based modeling system (CVX)- Proposed by Grant and Boyd (2014)
Aliabadi et al. (2018)	SGP	Yes	Posynomial GP	No	Iterative Algorithm- Proposed by Xu (2014)
Jabbarzadeh et al. (2019)	SGP	Yes	Posynomial GP	No	Iterative Algorithm- Proposed by Xu (2014)
Rabbani and Aliabadi (2019)	SGP	Yes	Posynomial GP	No	Iterative Algorithm- Proposed by Xu (2014)
This paper	SGP	Yes	Posynomial GP	Yes	Iterative Algorithm- Proposed by Xu (2014)

2. Problem statement

Consider a manufacturer who produces a single product in two markets and uses different production processes in each market. In order to compare two different type of markets (monopoly and competitive markets) we assume that there is no competitor in the first market, so, the manufacturer does not need any marketing plan. In other word, the manufacturer only focuses on pricing strategy to promote the first market and increase his profit. In the second market, there is a strong rival. Hence, there are different income and expenses for the manufacturer in each market. The problem statement is shown in Figure 1.

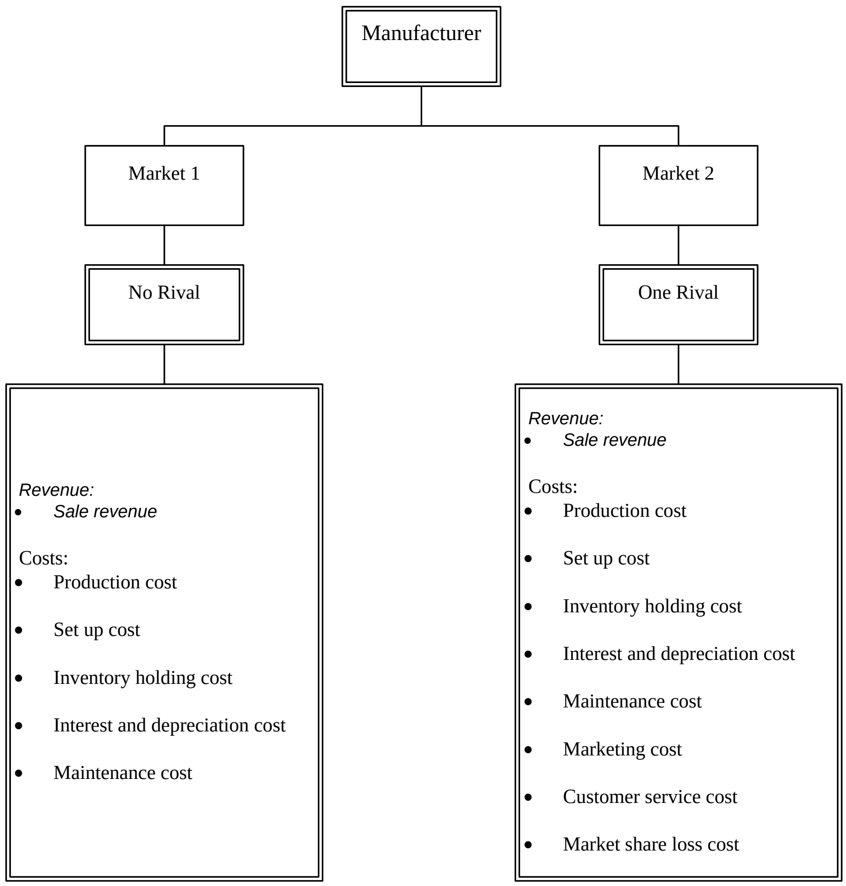

Figure 1: Manufacture's position in two markets with their revenue and costs

The manufacturer wants to find out optimal lot sizes and prices of the products. In addition, manufacturer determines the production quality level, marketing and customer service expenditures. Also, he installs flexible production process (such as flexible machinery) on the production line to reduce the set up time and cost in each production period. Installing a modern and flexible production process need capital investments which, consequently, results in higher interest and depreciation costs (Cheng, 1989; Leung, 2007). Likewise, in order to further increase reliability, manufacturer has to increase the capital investments on the factors such as production technology, monitoring devices and training of personnel. The high reliability has some advantages such as reduction in maintenance costs and rates of defective and scarp. In contrast, the costs of production, interest and depreciation will increase. Hence, the manufacturer should take suitable decisions about level of flexibility and reliability for its production processes.

The nomenclature and formulation which used in this paper are as follows:

Nomenclature:

D_t	Total demand of market *t* which is covered by the manufacturer (*t*=1, 2)	b_t	Required resource to produce each item in market *t* (*t*=1, 2) (e.g. machine hour)
C_t	Unit production cost in market *t* (*t*=1, 2)	R_t	Total available resources in market *t* (*t*=1, 2)
i_t	Rate of inventory holding cost in market *t* (*t*=1, 2) (per unit per unit time)	$N_t(r_t)$	Maintenance cost per production cycle in market *t* (*t*=1, 2)
P_t	Total demand of market *t* (*t*=1, 2)	$E_t(a_t, r_t)$	Interest and depreciation cost per production cycle in market *t* (*t*=1, 2)
ρ_t	Percent of total demand of market *t* (*t*=1, 2)which is covered by the manufacturer	H_t	Inventory holding cost in market *t* (*t*=1, 2)
w_t	Required space for storage each item in market *t* (*t*=1, 2)	ME_2	Marketing cost in market two
W_t	Total available space for storage manufactured items in market *t* (*t*=1, 2)	SV_2	Customer service cost in market two
B_M	Total available budget for the marketing in market two	B_S	Total available budget for the customer service in market two
ME_2	Marketing cost in market two	SV_2	Customer service cost in market two
τ	coefficient of share loss cost in market two	L_2	Share loss cost in market two
p'	Selling price of rival's product in market two		

Decision variables:

p_t	Unit price of the product in market *t* (*t*=1, 2)	a_t	Set-up cost in market *t* (*t*=1, 2)
Q_t	Economic production quantity (EPQ) in market *t* (*t*=1, 2)	M_i	Amount of investments in marketing method i= 1, …, m, per unit time in market two
r_t	Level of the production process reliability in market *t* (*t*=1, 2) (percent of non-defective products in a lot)	S_j	Amount of investments in customer service strategy j= 1, …,s, per unit time in market two
q	level of the product's quality in market two from the viewpoint of customers		

Formulation:

$$D_1(p_1) = k_1 p_1^{-\alpha_1} \tag{1}$$

$$D_2(p_2, q, M_i, S_j) = k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} \mathbf{M}_i^{\beta_i} \prod_{j=1}^{s} \mathbf{S}_j^{\sigma_j} \tag{2}$$

$$C_1(Q_1, r_1) = u_1 Q_1^{-\theta_1} r_1^{\delta_1} \tag{3}$$

$$C_2(Q_2, q, r_2) = u_2 Q_2^{-\theta_2} q^{\varphi} r_2^{\delta_2} \tag{4}$$

$$H_t(h_t, r_t, Q_t) = \frac{1}{2} i_t C_t r_t Q_t T_t, \quad t = 1, 2 \tag{5}$$

$$E_t(a_t, r_t) = d_t a_t^{-\mu_t} r_t^{\nu_t}, \quad t = 1, 2 \tag{6}$$

$$N_t(r_t) = n_t r_t^{-\eta_t}, \quad t = 1, 2 \tag{7}$$

$$ME_2(M_i) = \sum_{i=1}^{m} M_i \tag{8}$$

$$SV_2(S_j) = \sum_{j=1}^{s} S_j \tag{9}$$

$$L_2(p_2, p') = k_2 . \tau . \frac{p_2}{p_2 + p'} \tag{10}$$

Equation (1) is widely used by researchers (Lee, 1993; Lee et al, 1996; Kim and Lee, 1998; Liu, 2006). In equation (1), demand per unit time in market one is considered as a decreasing power function of price per unit. Here, α_1 is price elasticity to demand in the first market ($\alpha_1 > 1$). k_1 is scaling constant ($k_1 > 0$) and represents the other relevant factors. In fact, it is a candidate for all other variables which removed from the model. The sign of coefficient α_1 reflects the negative slope of demand. In equation (2), demand per unit time in market two is a decreasing power function of price per unit and increasing power function of all three production quality, marketing and customer service expenditures. The coefficient α_2 is the price elasticity to demand ($\alpha_2 > 1$), k_2 is scaling constant ($k_2 > 0$) and γ is the quality elasticity of demand. Here, $M = (M_1, M_2, ..., M_m)$ is a vector to represent marketing expenditure per unit in different marketing channels and β_i is the elasticity of demand with respect to expenditures in marketing channel i ($0 < \beta_i < 1$). Also, $S = (S_1, S_2, ..., S_s)$ is a vector to represent customer service expenditure per unit in various scenario and σ_j is the elasticity of demand with respect to expenditures in customer service strategy j ($0 < \sigma_j < 1$). Sadjadi et al. (2012) indicated that demand is inversely related to the price of product and directly related to the both marketing expenditure and quality of product. Also, Sadjadi et al. (2015) identified that demand is inversely related to the price of product and is directly related to the both marketing and customer service expenditures. This study combines these strategies and constructs the demand function in market two as shown in equation (2).

In equation (3), unit production cost in market one is considered as a decreasing power function of lot size and increasing power function of process reliability. Here, u_1 is scaling constant ($u_1 > 0$), θ_1 is lot size elasticity to unit production cost ($0 < \theta_1 < 1$) and δ_1 is reliability elasticity of unit production cost ($\delta_1 > 1$). In equation (4), the unit production cost in market two is a function that is inversely related to the lot size and directly related to the both product's quality and process reliability. The parameters u_2, θ_2 and δ_2 in market two are the similar to the u_1, θ_1 and δ_1 in market one ($u_2 > 0$, $0 < \theta_2 < 1$, $\delta_2 > 1$). φ is the product's quality elasticity of unit production cost in market two ($\varphi \geq 1$). Similar structures for the unit production cost in both markets have been introduced by other researchers. For example, Cheng (1991) assumed that the unit production cost is a decreasing power function of demand rate and increasing power function of process reliability. Jung and Klein (2001) considered that the unit production cost is inversely related to the demand per unit time. Jung and Klein (2006) proposed different functions for unit production cost in related to the demand and lot size production. Koteb and Fergany (2011) considered that unit production cost is a decreasing power function of demand rate. Sadjadi et al. (2012) assumed that the unit production cost is inversely related to the lot size and directly related to the product quality and process reliability. Ghosh and Roy (2013) assumed that the unit production cost is a decreasing power function of lot size. However, the inverse relationship between unit production cost and production lot size is the behind of scale economies concept in production. The justification of directly relationship between the unit production cost and product's quality is the competition in market two. So, in order to increase customer satisfaction, the manufacturer needs to purchase higher quality raw materials, train the firm's personnel in better condition, control strictly the production process and etc. These activities increase the unit production cost. On the other hand, in order to achieve higher degree in production reliability in both markets, higher investment on corresponding factors such as production machinery and training of firm's personnel, among other factors, should be done. Therefore, in this paper, it is assumed that the unit cost production, in both markets, is directly related to the process reliability. The equation (5) introduces the inventory holding costs in markets one and two, respectively (see Appendix A).

The equation (6) introduces the interest and depreciation costs in markets one and two, respectively. These cost functions have inverse relation with the set up cost and direct relation with the process reliability in each market. Both relations are in power form. In equation (6), d_1 and d_2 are scaling constants ($d_1, d_2 > 0$), μ_1 and μ_2 are set up cost elasticities to the interest and depreciation cost, ν_1 and ν_2 are process reliability elasticities to the interest and depreciation cost in each market. In order to decrease set up cost, more flexible production machinery must be used which it needs more capital investments. It means that interest and depreciation costs are increased, so, there is an inverse relation between set up cost and interest and depreciation cost. On the other hand, the directly relation between process reliability and interest and depreciation cost causes higher production process reliability. But, it needs to more investments on corresponding factors such as production facility, personnel training, strictly control and etc. It means that interest and depreciation costs are increased (this function is proposed by Van Beek and Van Putten (1987) and Leung (2007)).

The equation (7) introduces maintenance costs in each market as a decreasing power function of production process reliability. The proposed cost functions have an inverse relation with their corresponding process reliability. Here, n_1 and n_2 are scaling constant ($n_1, n_2 > 0$) and η_1 and η_2 are production reliability elasticities to maintenance costs ($0 < \eta_1 < 1, 0 < \eta_2 < 1$). The Eq. (7) shows that by increasing the process reliability, the failure-prone of the machinery is reduced which means decreasing of maintenance cost. The relationship between process reliability and maintenance cost was considered by Sadjadi et al. (2012). Equations (8) and (9) show the total marketing and customer service costs in the production horizon in market two. Equation (10) represents the cost of losing market share in market two.

Due to the rival in market two, the unit selling price of manufacturer's product has great influence on his market penetration which is symmetric in his selling price. Thus, the rival's market share is equal to $\frac{p_2}{p_2 + p'}$. By multiplying the rival's market share in the τ and k_2, the cost of losing market share in market two for the manufacturer can be yielded. However, marketing department should acquire information about the selling price of rival by continuous study about him and consequently adjusts own selling price with it. The equation (10) is similar to function proposed by Ghazi Nezami et al. (2009).

Five assumptions are applied in this paper. First; replenishment is instantaneous, in other word, the production rate is infinite. Second; no excess stock held in warehouse. That means, safety stock is zero (SS= 0). Third; shortage is not allowed, thus, there is not back order or lost sale. Fourth; although the second market for the manufacturer is a competitive type and usually the number of rivals in the real world competitive markets are greater than one, but it is assumed that the manufacturer is faced with a substantial rival in this market. Fifth; all lots (batches) are 100% inspected and all defective parts are discarded. The last assumption may be applied in some industries such as pharmaceutical companies. Some previous studies have different approaches about defective items. Jaber (2006) reworked defective items and converted them as good as new condition. Khan et al. (2011) sold defective items to a secondary market at a discounted price. However, according to the state of a specific industry, different decisions about defective parts should be made.

3. Mathematical model

In this section, total profit for the manufacturer is maximized in both markets, simultaneously. Total profit of the manufacturer in per cycle is as follows:

Total profit in per cycle= total revenues in both markets in per cycle - total costs in both markets in per cycle

Total profit = sale revenue (1) - production cost (1) - set up cost (1) - inventory holding cost (1) - interest/depreciation cost (1) - maintenance cost (1) + sale revenue (2) - production cost (2) - set up cost (2) - inventory holding cost (2) - interest/depreciation cost (2) - maintenance cost (2) - marketing cost (2) - customer service cost (2) - market share loss cost (2).

The number in parentheses indicates the markets (1) and (2).

$$Total\ profit\ in\ per\, cycle\ = \left[\sum_{t=1}^{2} \left(SR_t - PC_t - a_t - H_t - E_t - N_t \right) - ME_2 - SV_2 - L_2 \right] \tag{11}$$

Then, by multiplying profit in per cycle in production cycle numbers, total profit of the manufacturer can be calculated. The total profit based on decision variables is as follows:

$$\begin{aligned} Total\ \ profit\ &= \sum_{t=1}^{2} \left(p_t r_t Q_t - C_t Q_t - a_t - \frac{1}{2} i_t C_t r_t Q_t T_t - d_t a_t^{-\mu_t} r_t^{\nu_t} - n_t r_t^{-\eta_t} \right) \frac{D_t}{r_t Q_t} \\ &- \left(T_2 (\sum_{i=1}^{m} M_i) + T_2 (\sum_{j=1}^{s} S_j) + T_2 k_2 \tau \frac{p_2}{p_2 + p'} \right) \frac{D_2}{r_2 Q_2} \end{aligned} \tag{12}$$

Note that $T_t = \dfrac{r_t Q_t}{D_t}$, is the cycle length in market t (t=1,2). In addition, marketing cost, customer service cost and market share loss cost per cycle is the annual marketing cost, customer service cost and market share loss cost multiply by cycle length in market 2 $\left(T_2\right)$, respectively. To maximize the objective function (12), some constraints should be carried by the manufacturer such as limitations on production capacity and storage space in each market, also, available budget for marketing and service customer in second market. In addition, setting a target values on the specific percent of the total demand in the markets (P_1, P_2) is a very common practice for businesses which is added as the constraints. Furthermore, some constraints should be considered by the manufacturer that will be discussed subsequently. Therefore, the proposed model for maximizing total profit of manufacturer is as follow:

Model (1):

$$\begin{aligned} Max \quad \pi_{manufacturer} &= \sum_{t=1}^{2} p_t D_t - C_t D_t r_t^{-1} - a_t D_t r_t^{-1} Q_t^{-1} - \frac{1}{2} i_t C_t r_t Q_t - d_t a_t^{-\mu_t} r_t^{\nu_t - 1} D_t Q_t^{-1} \\ &- n_t r_t^{-\eta_t - 1} D_t Q_t^{-1} - \sum_{i=1}^{m} M_i - \sum_{j=1}^{s} S_j - k_2 \tau p_2 l^{-1} \end{aligned} \tag{13}$$

$$D_t \geq \rho_t P_t, \quad t = 1,2 \tag{14}$$

$$\sum_{i=1}^{m} M_i \leq B_M \tag{15}$$

$$\sum_{j=1}^{s} S_j \leq B_S \tag{16}$$

$$b_t Q_t \leq R_t, \quad t=1,2 \tag{17}$$

$$w_t(r_t Q_t) \leq W_t, \quad \mathrm{t}=1,2 \tag{18}$$

$$l \leq p_2 + p' \tag{19}$$

$$p_1^{\min} \leq p_1 \tag{20}$$

$$p_2^{\min} \leq p_2 \leq p_2^{\max} = (1+\psi)p' \tag{21}$$

$$q \leq 1 \tag{22}$$

$$r_t^{\min} \leq r_t \leq r_t^{\max}, \quad t=1,2 \tag{23}$$

$$p_t, Q_t, r_t, a_t, q, l, M_i, S_j > 0, \; t=1,2; \; i=1,\ldots,\mathrm{m}; \; \mathrm{j}=1,\ldots,\mathrm{s} \tag{24}$$

The objective function (13) maximizes the total annual profit of the manufacturer in both markets. The variable l which used in the last term of objective function is an estimate of the market share loss cost denominator. Constraint (14) ensures that the total demand in both markets should not be less than the minimum market share targeted in both markets. Constraints (15) and (16) take into account the constraint the total budget available for marketing and customer service for all marketing channels and all service customer strategies in market two. Constraint (17) shows the constraint on production capacity of the manufacturer in market one and market two, respectively. Constraint (18) considers the storage space for non-defective products in each market.

Since the denominator of market share loss cost ($p_2 + p'$) is estimated by variable l, maximizing objective function leads l to be infinite. In order to deal with this problem, the constraint (19) is added to the model. By considering the objective function and constraint (19), simultaneously, the variable l will be equal to ($p_2 + p'$). Constraint (20) ensures that the price of production in market one cannot be less than a specific amount. The manufacturer plays a monopolist role in the market one and it is more economical for him to increase his price of production as much as possible. In contrast, the market two is a competitive market for manufacturer and it is necessary to consider the lower and upper bounds for price of production. This constraint is indicated in (21). The upper bound form of production price ($(1+\psi)p'$) means that the production price cannot exceed from the specific percent ($(1+\psi)$ %) of rival's price. Constraint (22) shows the upper bounds on the quality level of products according to percent customer's satisfaction in market two. This constraint ensures that the quality level of product cannot exceed one or 100%. The customer's satisfaction can be obtained by customer surveys. Constraint (23) implies the lower and upper bounds on the production process reliabilities in markets one and two, respectively. In order to avoid lengthy stop production processes in two markets, the lower bound is determined. The justification of upper bound is the lack of affordability. Finally, constraint (24) implies the fact that all variables should be positive.

Now, by substituting D_1, D_2, C_1, and C_2 from equations (1)-(4) into the above model, the final proposed model is expressed as follows:

Model (2):

$$\begin{aligned}
Max \quad & \pi(p_1, p_2, Q_1, Q_2, r_1, r_2, a_1, a_2, q, l, M_i, S_j) = \\
& \Big(k_1 p_1^{1-\alpha_1} - u_1 k_1 p_1^{-\alpha_1} Q_1^{-\theta_1} r_1^{\delta_1 - 1} - a_1 k_1 p_1^{-\alpha_1} r_1^{-1} Q_1^{-1} - 0.5 i_1 u_1 Q_1^{1-\theta_1} r_1^{1+\delta_1} \\
& - d_1 a_1^{-\mu_1} r_1^{\nu_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} - n_1 r_1^{-\eta_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \\
& - u_2 k_2 p_2^{-\alpha_2} q^{\gamma+\varphi} Q_2^{-\theta_2} r_2^{\delta_2 - 1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} - a_2 k_2 p_2^{-\alpha_2} q^{\gamma} r_2^{-1} Q_2^{-1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \\
& - 0.5 i_2 u_2 Q_2^{1-\theta_2} q^{\varphi} r_2^{1+\delta_2} - d_2 a_2^{-\mu_2} r_2^{\nu_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \\
& - n_2 r_2^{-\eta_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} - \sum_{i=1}^{m} M_i - \sum_{j=1}^{s} S_j - k_2 \tau p_2 l^{-1} \Big)
\end{aligned} \tag{25}$$

$$\rho_1^{-1} P_1 k_1^{-1} p_1^{\alpha_1} \leq 1 \tag{26}$$

$$\rho_2 P_2 k_2^{-1} p_2^{\alpha_2} q^{-\gamma} \prod_{i=1}^{m} M_i^{-\beta_i} \prod_{j=1}^{s} S_j^{-\sigma_j} \leq 1 \tag{27}$$

$$(\sum_{i=1}^{m} M_i) B_M^{-1} \leq 1 \tag{28}$$

$$(\sum_{j=1}^{s} S_j) B_S^{-1} \le 1 \tag{29}$$

$$b_t Q_t R_t^{-1} \le 1,\ \ t = 1,2 \tag{30}$$

$$w_t (r_t Q_t) W_1^{-1} \le 1,\ \ \mathrm{t} = 1,2 \tag{31}$$

$$\frac{l}{(p_2 + p')} \le 1 \tag{32}$$

$$p_1^{\min} \le p_1 \tag{33}$$

$$p_2^{\min} \le p_2 \le (1+\psi) p' \tag{34}$$

$$q \le 1 \tag{35}$$

$$r_t^{\min} \le r_t \le r_t^{\max},\quad t = 1,2 \tag{36}$$

$$p_t, Q_t, r_t, a_t, q, l, M_i, S_j > 0,\ t = 1,2;\ \ i = 1,...,\mathrm{m};\ \ \mathrm{j} = 1,...,\mathrm{s} \tag{37}$$

4. Solution approach

The proposed model (2) is very similar to geometric programming (GP) problem class. Therefore, it can be transformed to a conventional GP form (≤ 1) by applying some changes as follows (see Appendix B):

Model (3):

$$Max \quad Z = Min \quad Z^{-1} \tag{38}$$

$$s.t: \quad \Big[Z + \big(u_1 k_1 p_1^{-\alpha_1} Q_1^{-\theta_1} r_1^{\delta_1 - 1} + a_1 k_1 p_1^{-\alpha_1} r_1^{-1} Q_1^{-1} + 0.5 i_1 u_1 Q_1^{1-\theta_1} r_1^{1+\delta_1} + d_1 a_1^{-\mu_1} r_1^{\nu_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1}$$

$$+ n_1 r_1^{-\eta_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} + u_2 k_2 p_2^{-\alpha_2} q^{\gamma+\varphi} Q_2^{-\theta_2} r_2^{\delta_2 - 1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}$$

$$+ a_2 k_2 p_2^{-\alpha_2} q^{\gamma} r_2^{-1} Q_2^{-1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + 0.5 i_2 u_2 Q_2^{1-\theta_2} q^{\varphi} r_2^{1+\delta_2} \tag{39}$$

$$+ d_2 a_2^{-\mu_2} r_2^{\nu_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + n_2 r_2^{-\eta_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}$$

$$+ \sum_{i=1}^{m} M_i + \sum_{j=1}^{s} S_j + k_2 \tau p_2 l^{-1} \big)\Big] \Big/ \Big(k_1 p_1^{1-\alpha_1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \Big) \le 1$$

$$\rho_1^{-1} P_1 k_1^{-1} p_1^{\alpha_1} \le 1 \tag{40}$$

$$\rho_2 P_2 k_2^{-1} p_2^{\alpha_2} q^{-\gamma} \prod_{i=1}^{m} M_i^{-\beta_i} \prod_{j=1}^{s} S_j^{-\sigma_j} \le 1 \tag{41}$$

$$(\sum_{i=1}^{m} M_i) B_M^{-1} \le 1 \tag{42}$$

$$(\sum_{j=1}^{s} S_j) B_S^{-1} \le 1 \tag{43}$$

$$b_t Q_t R_t^{-1} \le 1, \quad t = 1, 2 \tag{44}$$

$$w_t (r_t Q_t) W_t^{-1} \le 1, \quad t = 1, 2 \tag{45}$$

$$\frac{l}{(p_2 + p')} \le 1 \tag{46}$$

$$p_t^{\min} p_t^{-1} \le 1, \quad \mathrm{t} = 1, 2 \tag{47}$$

$$p_2 (1+\psi)^{-1} (p')^{-1} \le 1 \tag{48}$$

$$q \le 1 \tag{49}$$

$$r_t^{\min} r_t^{-1} \le 1, \;\; r_t (r_t^{\max})^{-1} \le 1, \quad t = 1, 2 \tag{50}$$

$$Z, p_t, Q_t, r_t, a_t, q, l, M_i, S_j > 0,\ t = 1,2;\ \ i = 1,\ldots,\mathrm{m};\ \ \mathrm{j} = 1,\ldots,\mathrm{s} \tag{51}$$

As shown, the model (3) is a constrained signomial geometric programming problem where the constraint (39) is divided by two posynomial terms. Also, the constraint (46) is formed by dividing a monomial term over a posynomial term. Hence, the constraints (39) and (46) don't allow the above model to be easily converted to a standard posynomial GP problem. To deal with this issue, the denominator of the constraints (39) and (46) are approximated by the monomial functions as follows:

Assume $f(x) = \sum_e V_e(x)$ is a posynomial function which $V_e(x)$s are the monomial terms. It is clear that the inequality (52) can be written based on the relation between geometric-arithmetic means.

$$f(x) \geq \hat{f}(x) = \prod_e \left(\frac{V_e(x)}{\Gamma_e(y)} \right)^{\Gamma_e(y)} \tag{52}$$

$$\Gamma_e(y) = \frac{V_e(y)}{f(y)}, \forall e \tag{53}$$

where y is a fixed positive point and summation of $\Gamma_e(y)$ must be equal to one ($\sum_e \Gamma_e(y) = 1$). Boyd et al. (2007) indicated that $\hat{f}(x)$ is the best local monomial approximation of $f(x)$ near y. So, an inequality constraint $\dfrac{g(x)}{f(x)} \leq 1$ with posynomial numerator and denominator can be approximated by $\dfrac{g(x)}{\hat{f}(x)} \leq 1$. The inequality (52) shows that the amount of $\hat{f}(x)$ is smaller than of $f(x)$, so it is clear that $g(x)/f(x) \leq g(x)/\hat{f}(x) \leq 1$. By applying the mentioned method to the denominator of constraints (39) and (46), the approximations (54) and (55) can be yielded:

$$k_1 p_1^{1-\alpha_1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \approx (k_1 p_1^{1-\alpha_1} / \Gamma_{11})^{\Gamma_{11}} . (k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} / \Gamma_{12})^{\Gamma_{12}} \tag{54}$$

$$p_2 + p' \approx (p_2 / \Gamma_{21})^{\Gamma_{21}} . (p' / \Gamma_{22})^{\Gamma_{22}} \tag{55}$$

where, Γ_{11}, Γ_{12}, Γ_{21} and Γ_{22} are defined as follows:

$$\Gamma_{11} = \left(k_1 p_1^{1-\alpha_1}\right) \Big/ \left(k_1 p_1^{1-\alpha_1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\right) \tag{56}$$

$$\Gamma_{12} = \left(k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\right) \Big/ \left(k_1 p_1^{1-\alpha_1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\right) \tag{57}$$

$$\Gamma_{21} = p_2 \big/ (p_2 + p') \tag{58}$$

$$\Gamma_{22} = p' \big/ (p_2 + p') \tag{59}$$

It is clear that $\Gamma_{11} + \Gamma_{12} = 1$ and $\Gamma_{21} + \Gamma_{22} = 1$.

Finally, by incorporating the equations (54) and (55) instead of constraints denominator (39) and (46), the proposed model can be expressed as follows:

Model (4):

$$Max \quad Z = Min \quad Z^{-1} \tag{60}$$

$$\begin{aligned} s.t: \quad & \Big[Z + \Big(u_1 k_1 p_1^{-\alpha_1} Q_1^{-\theta_1} r_1^{\delta_1 - 1} + a_1 k_1 p_1^{-\alpha_1} r_1^{-1} Q_1^{-1} + 0.5 i_1 u_1 Q_1^{1-\theta_1} r_1^{1+\delta_1} + d_1 a_1^{-\mu_1} r_1^{\nu_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} \\ & + n_1 r_1^{-\eta_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} + u_2 k_2 p_2^{-\alpha_2} q^{\gamma+\varphi} Q_2^{-\theta_2} r_2^{\delta_2 - 1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \\ & + a_2 k_2 p_2^{-\alpha_2} q^{\gamma} r_2^{-1} Q_2^{-1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + 0.5 i_2 u_2 Q_2^{1-\theta_2} q^{\varphi} r_2^{1+\delta_2} \\ & + d_2 a_2^{-\mu_2} r_2^{\nu_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + n_2 r_2^{-\eta_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \\ & + \sum_{i=1}^{m} M_i + \sum_{j=1}^{s} S_j + k_2 \tau p_2 l^{-1} \Big) \Big] \\ & \Big/ \left(k_1 p_1^{1-\alpha_1} / \Gamma_{11}\right)^{\Gamma_{11}} . \left(k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} \Big/ \Gamma_{12}\right)^{\Gamma_{12}} \Bigg] \le 1 \end{aligned} \tag{61}$$

$$\frac{l}{[(p_2 / \Gamma_{21})^{\Gamma_{21}} . (p' / \Gamma_{22})^{\Gamma_{22}}]} \le 1 \tag{62}$$

and constraints (40)-(45) and (47)-(51).

The above proposed model is a standard geometric programming which can be turned into a nonlinear convex problem and solved efficiently (Boyd et al., 2007).

4.1. Iterative algorithm

Here, in order to solve the above model, the algorithm proposed by Xu (2014) is used. The basic steps of this algorithm are as follows:

Step 0:

for each market t (t=1, 2), choose initial feasible values for the variables p_t, p_2, Q_t, Q_2, r_t, r_2, a_t, a_2, q, l, M_i, S_j and Z, $p_t^{(0)}$, $p_2^{(0)}$, $Q_t^{(0)}$, $Q_2^{(0)}$, $r_t^{(0)}$, $r_2^{(0)}$, $a_t^{(0)}$, $a_2^{(0)}$, $q^{(0)}$, $l^{(0)}$, $M_i^{(0)}$, $S_j^{(0)}$ and $Z^{(0)}$ ($\forall i, \forall j, \forall t$), respectively. Give solution accuracy $\varepsilon > 0$ and set iteration counter $r = 0$.

Step 1:

At the rth iteration, evaluate the monomial terms in the denominator of constraints (39) and (46) with given $p_1^{(r-1)}$, $p_2^{(r-1)}$, $q^{(r-1)}$, $M_i^{(r-1)}$ and $S_j^{(r-1)}$. Compute their relevant parameters $\Gamma_e(p_1^{(r-1)}, p_2^{(r-1)}, q^{(r-1)}, M_i^{(r-1)}, S_j^{(r-1)})$ using the equations (56)-(59).

Step 2:

Perform the condensation on the posynomial denominators of constraints (39) and (46) using equations (54) and (55) with parameters $\Gamma_e(p_1^{(r-1)}, p_2^{(r-1)}, q^{(r-1)}, M_i^{(r-1)}, S_j^{(r-1)})$.

Step 3:

Solve the standard GP model (4) to attain $p_t^{(r)}$, $p_2^{(r)}$, $Q_t^{(r)}$, $Q_2^{(r)}$, $r_t^{(r)}$, $r_2^{(r)}$, $a_t^{(r)}$, $a_2^{(r)}$, $q^{(r)}$, $l^{(r)}$, $M_i^{(r)}$, $S_j^{(r)}$, $Z^{(r)}$, ($\forall i, \forall j, \forall t$)

Step 4:

If $\left\|p_t^{(r)} - p_t^{(r-1)}\right\| \le \varepsilon$, $\left\|p_2^{(r)} - p_2^{(r-1)}\right\| \le \varepsilon$, $\left\|Q_t^{(r)} - Q_t^{(r-1)}\right\| \le \varepsilon$, $\left\|Q_2^{(r)} - Q_2^{(r-1)}\right\| \le \varepsilon$, $\left\|r_t^{(r)} - r_t^{(r-1)}\right\| \le \varepsilon$, $\left\|r_2^{(r)} - r_2^{(r-1)}\right\| \le \varepsilon$, $\left\|a_t^{(r)} - a_t^{(r-1)}\right\| \le \varepsilon$, $\left\|a_2^{(r)} - a_2^{(r-1)}\right\| \le \varepsilon$, $\left\|q^{(r)} - q^{(r-1)}\right\| \le \varepsilon$, $\left\|l^{(r)} - l^{(r-1)}\right\| \le \varepsilon$, $\left\|M_i^{(r)} - M_i^{(r-1)}\right\| \le \varepsilon \quad ;\forall i$, and $\left\|S_j^{(r)} - S_j^{(r-1)}\right\| \le \varepsilon \quad ;\forall j, \forall t$, then stop. Else set $r = r+1$ and return to Step1.

5. Numerical example

In this section, a numerical example is solved to illustrate the validity and computational efficiency of the proposed model. Consider a manufacturer who produces and supply a single product in two different markets. The manufacturer wants to determine the lot sizes, price of products, requirements of his production technology (which defined as set up costs and process reliabilities) in both markets. Also, he tends to determine quality level of product and amount of expenditures for marketing and customer service only in market two with being a rival. It is assumed that the rival sales his/her product for the price of 3.5 $ ($p' = 3.5$). Furthermore, assume that there are three methods (channels) for advertising the products: newspaper (channel 1), TV (channel 2) and internet (channel 3). In addition, there are two types (strategy) of customer services: buying advice (strategy 1) and product warranty (strategy 2). For this example, the required parameters and initial solutions are shown in Tables 2 and 3, respectively.

Table 2: Initial parameters in each market

Market	*Parameters*				
Market 1	k_1 =3×10^8	α_1 =2	u_1 =1	θ_1 =0.010	δ_1 =1.65
	d_1 =10	μ_1 =1	v_1 =1	n_1 =145	η_1 =0.50
	P_1=3×10^6	ρ_1 =0.9	i_1=0.10	b_1=4	R_1 =1000
	w_1 =36	W_1 =5000	$r_1^{\min}$ =0.7	$r_1^{\max}$ =0.95	$p_1^{\min}$ =3.5
Market 2	k_2 =2×10^8	α_2 =2.46	u_2 =1.01	θ_2 =0.009	δ_2 =1.98
	d_2 =11	μ_2 =1.1	v_2 =1.1	n_2 =155	η_2 =0.55
	P_2 =2×10^6	ρ_2 =0.4	i_2=0.12	b_2 =4	R_2 =1200
	w_2 =36	W_2 =5000	$r_2^{\min}$ =0.75	$r_2^{\max}$ =0.98	$p_2^{\min}$ =2.5
	ψ =0.1	γ =0.53	β_1 =0.0010	β_2 =0.0020	β_3 =0.0005
	σ_1 =0.005	σ_2 =0.008	τ =0.03	B_M =200000	B_S =200000

Finally, using the MATLAB based solver GGPLAB (Mutapcic et al., 2006) on an ASUS laptop with Intel(R) Core(TM) i5-4200U CPU @ 1.60HZ, RAM: 6.00 GB (5.89 GB usable) and by considering $\varepsilon = 10^{-6}$, after 44 iterations with about 11.95 second CPU time, the algorithm yields p_1^* = \$4.06, Q_1^* = 168, r_1^* = 0.83 and a_1^* = \$2.87 as the optimal solution in market one and p_2^* = \$2.99, Q_2^* = 148, r_2^* = 0.94, a_2^* = \$3.17 and q^* = 0.68 (i.e. 68% of customers in market two are satisfied from the product) as the optimal solution in market two. Also, the optimal value for total marketing expenditures in the first, second and third advertising channels are M_1^* = \$15558, M_2^* = \$31115 and M_3^* = \$7779, respectively. The optimal volume of customers service cost in first and second strategies are S_1^* = \$76923 and S_2^* = \$123077, respectively. Finally, the total profit for the manufacturer in two markets is Z^* = \$49,501,568. The results show that the manufacturer produces more product with higher price in market one than market two. These results are expected, because usually in real world business, the price and quantity of a product in a monopolistic market are greater than a competitive one.

Table 3: Initial solution for starting algorithm

$p_1^{(0)}=15$	$p_2^{(0)}=15$	$Q_1^{(0)}=500$	$Q_2^{(0)}=500$	$r_1^{(0)}=0.80$	$r_2^{(0)}=0.80$	$a_1^{(0)}=20$	$a_2^{(0)}=20$
$q^{(0)}=0.9$	$l^{(0)}=13.5$	$M_1^{(0)}=1500$	$M_2^{(0)}=1500$	$M_3^{(0)}=1500$	$S_1^{(0)}=2000$	$S_2^{(0)}=2000$	$Z^{(0)}=10^6$

One of the difficulties of this problems type is finding the initial solutions for starting the algorithm. The proposed algorithm can obtain the optimal solutions using even an infeasible initial solution in a few runs. For example, consider five different initial solutions for the GP problem. Tables 4 and 5 show that the proposed algorithm can rapidly obtain the global solution with inconsiderable CPU times.

Table 4: Different initial points for starting algorithm

Cases	**Initial point** $(p_1^{(0)}, p_2^{(0)}, Q_1^{(0)}, Q_2^{(0)}, r_1^{(0)}, r_2^{(0)}, a_1^{(0)}, a_2^{(0)}, q^{(0)}, l^{(0)}, M_1^{(0)}, M_2^{(0)}, M_3^{(0)}, S_1^{(0)}, S_2^{(0)}, Z^{(0)})$
A	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
B	(10, 10, 10, 10, 1, 1, 10, 10, 1, 10, 10, 10, 10, 10, 10, 10)
C	(100, 100, 100, 100, 1, 1, 100, 100, 1, 100, 100, 100, 100, 100, 100, 100)
D	(1000, 1000, 1000, 1000, 1, 1, 1000, 1000, 1, 1000, 1000, 1000, 1000, 1000, 1000, 1000)
E	(10000, 10000, 10000, 100000, 1, 1, 1000, 10000, 1, 10000, 10000, 10000, 10000, 10000, 10000, 10000)

Table 5: Comparisons of effects of different initial solutions on the proposed algorithm

Initial solution (case)	CPU time (s)	iterations	Optimal solution $(p_1^*,\ p_2^*, Q_1^*, Q_2^*, r_1^*, r_2^*, a_1^*, a_2^*, q^*, l^*, M_1^*, M_2^*, M_3^*, S_1^*, S_2^*)$
A	11.20	41	(4.06, 2.99, 168, 148, 0.83, 0.94, 2.87, 3.17, 0.68, 6.49, 15558, 31115, 7779, 76923, 123077)
B	11.66	44	(4.06, 2.99, 168, 148, 0.83, 0.94, 2.87, 3.17, 0.68, 6.49, 15558, 31115, 7779, 76923, 123077)
C	12.37	45	(4.06, 2.99, 168, 148, 0.83, 0.94, 2.87, 3.17, 0.68, 6.49, 15558, 31115, 7779, 76923, 123077)
D	12.51	46	(4.06, 2.99, 168, 148, 0.83, 0.94, 2.87, 3.17, 0.68, 6.49, 15558, 31115, 7779, 76923, 123077)
E	12.93	47	(4.06, 2.99, 168, 148, 0.83, 0.94, 2.87, 3.17, 0.68, 6.49, 15558, 31115, 7779, 76923, 123077)
			The optimal value of objective function in each case is equal to 49,501,568

5.1. More numerical examples

In Table 1, a comparison was made between our proposed approach with some recent studies. The results are explained in Table 6.

Table 6: Comparison the results of proposed method with Safaei et al. (2006)

Authors	Method	Optimized solution	Objective value	CPU time (s)	Population Size
Safaei et al. (2006)	Genetic Algorithm	(16.3052, 14.8529, 0.1911)	66,738.05	4.52	20
		(16.2982, 14.8611, 0.1945)	66,738.09	3.79	50
		(16.3208, 14.8488, 0.1928)	66,738.13	7.07	100
		(16.3133, 14.8666, 0.1944)	66,738.13	13.55	150
		(16.3128, 14.8515, 0.1935)	66,738.13	18.80	200
		(16.3106, 14.8592, 0.1936)	66,738.13	24.87	250
	Proposed method	**(16.3157, 14.8580, 0.1938)**	**66,738.13**	**1.74**	**-**

As it can be seen in Table 6, for population size more than 100, the objective value obtained by proposed method is equal to the objective values obtained by Safaei et al. (2006). Our method was faster and uses far less CPU time.

Table 7 presents the comparisons between the proposed method with other researchers for their numerical examples. In this Table, it can be seen that the obtained results by the proposed method are almost similar to others, with much more low iterations and CPU time. There is only a significant difference with Sadjadi et al. (2015). By analyzing their method carefully, we can prove that the obtained solutions reported in their article are not feasible. It is clear that the solutions which are not feasible cannot be optimal. Also, since the proposed method which are used by Aliabadi et al. (2018), Jabbarzadeh et al. (2019) and Rabbani and Aliabadi (2019) is the Xu's method (as shown in Table 1), so the results are the same with this paper but the models are different.

Table 7: Comparisons the results of proposed method with other studies

Authors	Optimized solution	Objective value	CPU time (s)	Iterations
Ghazi Nezami et al. (2009)	(7.1517, 2075.3, 0.0727)	2607.80	-	-
Proposed method	**(7.1517, 2075.3, 0.0727)**	**2607.80**	**0.7041**	**7**
Sadjadi et al. (2015)	(1189.3, 220, 48068, 31874, 29641, 46393, 5.29, 0.86)	11103436	-	-
Proposed method	**(1704.3, 98.8, 37026, 24541, 22819, 35735, 5.46, 0.93)**	**7070402**	**3.33**	**8**
Aliabadi et al. (2017)	(1305, 7.75, 0.0848, 0.337, 0.337, 439.8122, 47.156, 3.835, 0.874)	5757.9254	-	-
Proposed method	**(1305, 7.75, 0.0848, 0.337, 0.337, 439.8122, 47.156, 3.835, 0.874)**	**5757.9254**	-	-
Tabatabaei et al. (2017)	(8.9414, 0.8420, 0.0215)	5.3301	-	-
Proposed method	**(8.9383, 0.8420, 0.0215)**	**5.3301**	**3.4028**	**50**

5.2. Sensitivity analysis

In this section, the sensitivity analysis of optimal solution is discussed to provide some economical and managerial insights.

Market one:

Effects of change in price elasticity to demand on optimal solution

Here, it is investigated the effects of change in the value of α_1 on decision variables. The results are shown in Figures 2a and 2b. As can be seen in Figure 2a, as demand's price elasticity increases, the optimal selling price decreases that is causes to reduction in total profit of manufacturer (Figure 2b). The justification for these inverse relations is that as customers get more sensitive to the purchase price, the manufacturer has to decrease his price of product in order to lose fewer customers.

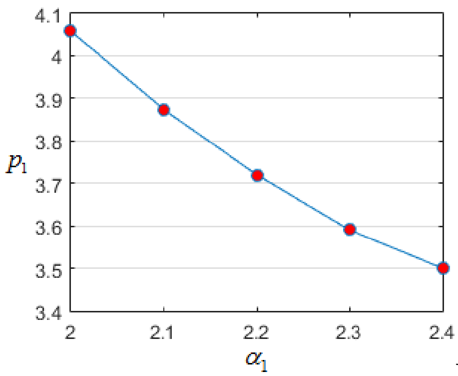

Figure 2a: Effect of change in α_1 on p_1

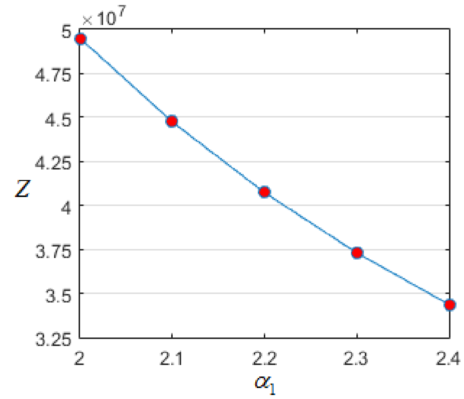

Figure 2b: Effect of change in α_1 on Z

Effects of change in lot size elasticity to unit production cost on optimal solution

As θ_1 has inverse relation with unit production cost, by increasing this elasticity the production cost is decreased and the manufacturer does not have to produces in large lots, so the lot size is decreased. Also, by decreasing the production cost, total profit increases automatically, and it does not require increasing the selling price, therefore the price of production is decreased. The Figures (3a) to (3d) reveal the above explanations.

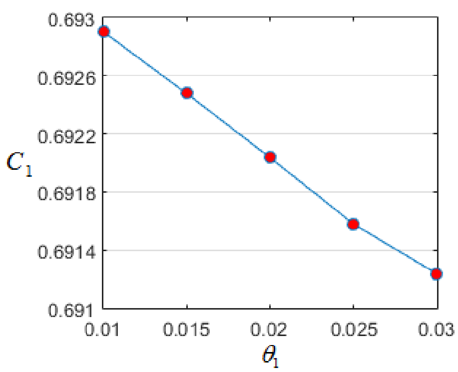

Figure 3a: Effect of change in θ_1 on C_1

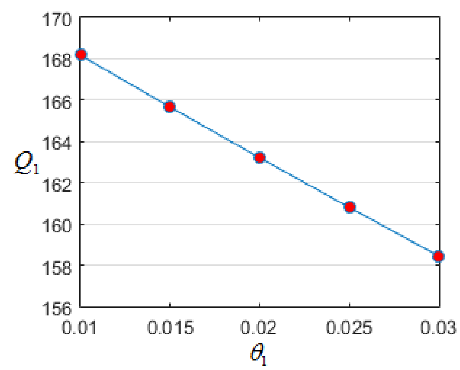

Figure 3b: Effect of change in θ_1 on Q_1

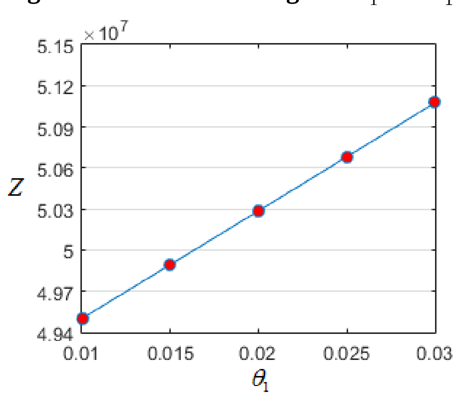

Figure 3c: Effect of change in θ_1 on Z

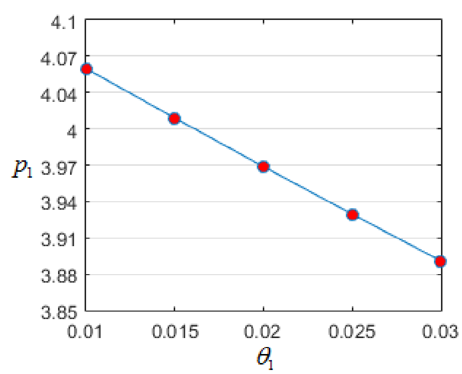

Figure 3d: Effect of change in θ_1 on p_1

Market two:

Effect of change in price elasticity to demand on optimal solution

As the price elasticity to demand increases, the manufacturer has to decrease his selling price in order to maintain favorite market share. Then, by decreasing in p_2, the manufacturer decreases the volume of investments on the marketing expenditures (Figures 4a, 4b and 4c) and volume of investments.

The volume of investments is related to production quality, for instances: purchases lower quality raw materials, trains personnel poorly in quality-related issue, controls easier on production process and etc. Therefore, the numbers of customers who are satisfied from product and, accordingly, the amount of lot size decrease (see Figure 4d). In order to compensate of falling profit, losing customers and decreasing total cost, the manufacturer has to enhance the process reliability by improving the production technology, training personnel, monitoring device, and so on which this actions lead to reduction in maintenance cost. The Figures 4e and 4f show these explanations.

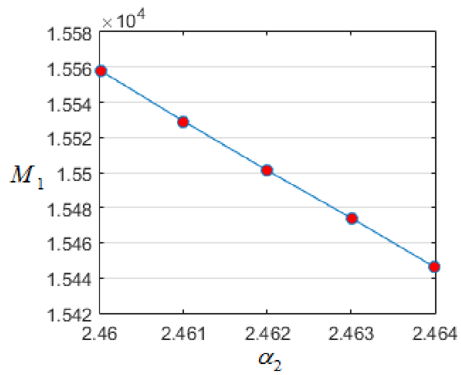

Figure 4a: Effect of change in α_2 on M_1

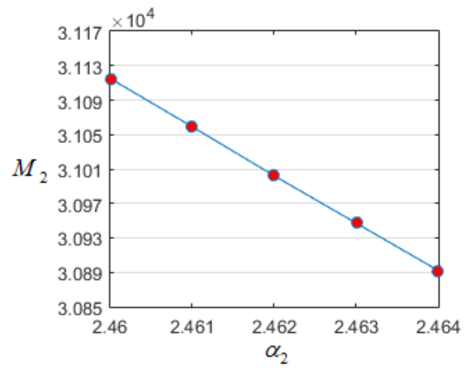

Figure 4b: Effect of change in α_2 on M_2

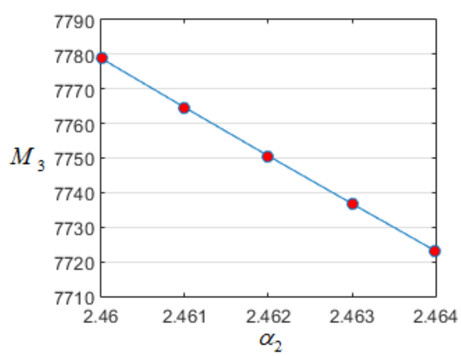

Figure 4c: Effect of change in α_2 on M_3

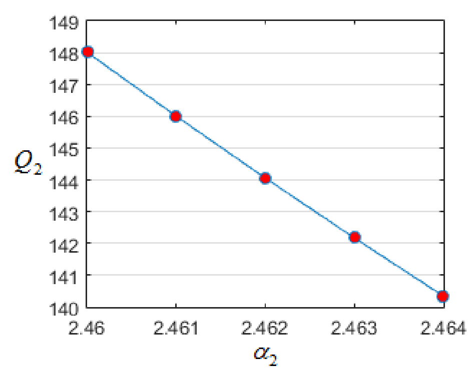

Figure 4d: Effect of change in α_2 on Q_2

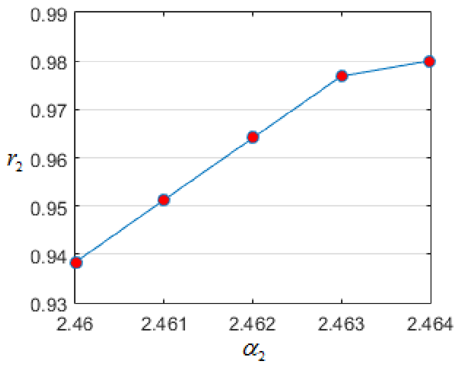

Figure 4e: Effect of change in α_2 on r_2

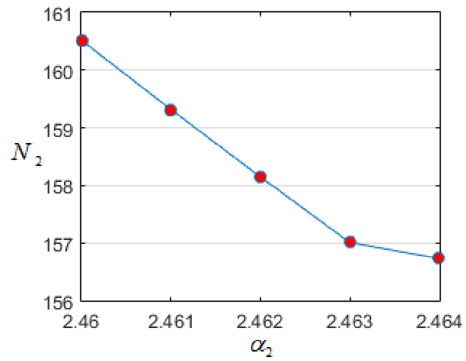

Figure 4f: Effect of change in α_2 on N_2

Effects of change in customer services elasticities to demand on optimal solution

In this sub-section, the values of σ_1 is increased and simultaneously, the value of σ_2 is decreased to survey the changes in decision variables. As shown in Table 8, total profit have inverse relations with the closer amounts of σ_1 and σ_2 and direct relations with the scattered amounts of σ_1 and σ_2. This means as σ_1 and σ_2 get closer to each other, the total profit decrease. For example in the points σ_1 =0.006, σ_2 =0.007 and σ_1 =0.007, σ_2 =0.006, σ_1 and σ_2 are close to each other and total profit is minimum. Also, as the σ_1 and σ_2 get farther from each other, the total profit increase. For example in the point (σ_1 =0.010, σ_2 =0.003) σ_1 and σ_2 are farthest and total profit is maximum. Hence, it is proposed to manufacturer investing in a diverse range of customer service strategies with scatter and incomparable elasticity rather than in service strategies with near and comparable elasticity values, in order to be more useful these strategies.

Table 8: Effect of change in σ_1 and σ_2 ($\sigma_1 + \sigma_2 =$ constant) on total profit

σ_1	0.005	0.006	0.007	0.008	0.009	0.010
σ_2	0.008	0.007	0.006	0.005	0.004	0.003
Z	49501568	49496734	49496733	49501568	49511489	49527089

Effects of change in rival's price on optimal solution

The second market is a competitive and the manufacturer wants to find out the best decisions due to the different performance and strategy of his rival. In this sub section, the rival price is changed and the effect of changes on optimal decision is surveyed. By increasing in the rival's price, a significant number of customers attract to the manufacturer. So, he sells a greater amount of product (see Figure 5a) at a higher price (see Figure 5b) which this cause that the total profit of the manufacturer increases (see Figure 5c). Also, increasing in the numbers of customers is equal to satisfaction from the manufacturer. Therefore, in order to maximize profit, the manufacturer does not need to more invest on modern and capable facilities and machinery, monitoring device, personnel training, and so on which causes the process reliability decrease. The Figure 5d reflects these explanations.

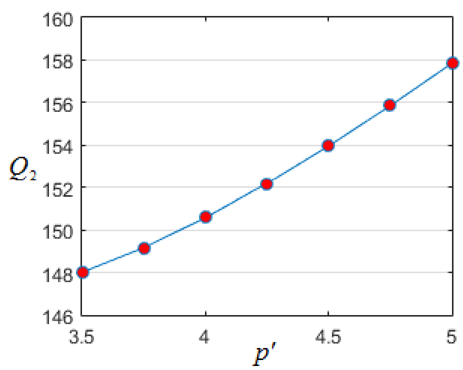

Figure 5a: Effect of change in p' on Q_2

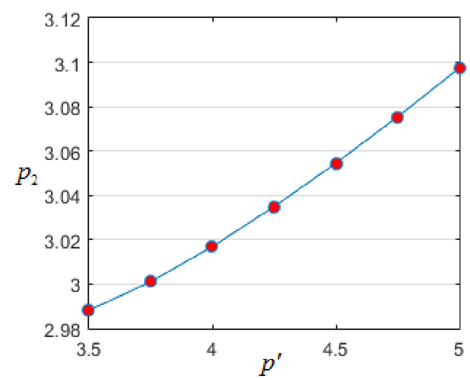

Figure 5b: Effect of change in p' on p_2

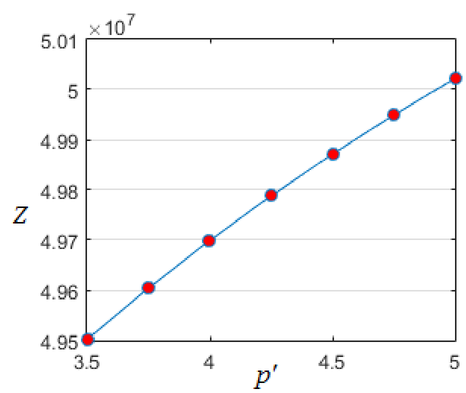

Figure 5c: Effect of change in p' on Z

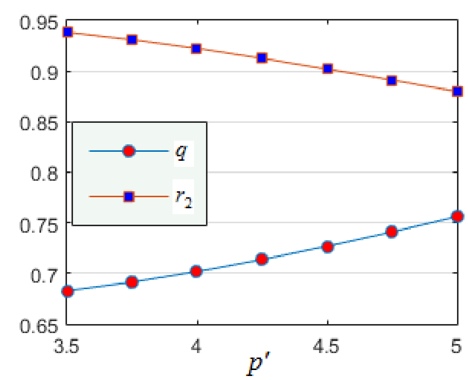

Figure 5d: Effect of change in p' on q and r_2

6. Conclusion

This paper developed a price discrimination model considering a manufacturer who produces and sells a single product in two different markets. Unlike the first market, there was a strong rival in the second market. The manufacturer wanted to maximize sum of total profits in both markets by determining the optimal values of decision variables include selling price and lot size in both markets, and marketing expenditure and customer service cost in market two. Also, the manufacturer aimed to have control over manufacturing requirements in terms of flexibility and reliability of production process. In addition, he tended to obtain the optimal level of his process's reliabilities and set up costs. In order to have a comprehensive model, the quality level of the product was considered as a decision variable, too. The proposed model was a constrained signomial geometric programming which has been converted to a posynomial standard GP form using some of transformations and convexification strategies. Finally, to illustrate the capability of the model, a numerical example was considered and by introducing an iterative algorithm, the global optimum solution was found.

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Appendix A:

Here, the formulation of inventory holding cost is proved.

$$inventory\ holding\ \cos t\ in\, each\, cycle = ic\int_{t=0}^{T} I(t)d(t)$$

$$= ic\int_{t=0}^{T}(rQ - Dt)dt = ic(rQT - \frac{1}{2}DT^{2})$$

where, $I(t)$ is the inventory level in time $t\,(t \in [0,T])$, and $T = \frac{rQ}{D}$, is the cycle length. Now, by substituting $D = \frac{rQ}{T}$ into above equation, will yield:

$$\Rightarrow inventory\ holding\ \cos t\ in\, each\, cycle\ in\ market\ one = \frac{1}{2} i_1 c_1 r_1 Q_1 T_1.$$

$$\Rightarrow inventory\ holding\ \cos t\ in\, each\, cycle\ in\ market\ two = \frac{1}{2} i_2 c_2 r_2 Q_2 T_2.$$

Appendix B:

Here, it is proved that the combination of equations (38) and (39) are equivalent to equation (25). First, a new variable Z is introduced as objective function:

$$\pi(p_1, p_2, Q_1, Q_2, r_1, r_2, a_1, a_2, q, l, M_i, S_j) \equiv Z$$

It is clear that the objective function can be expressed as follows:

$$
\begin{aligned}
&\Rightarrow Max\ Z\\
&s.t: \quad Z \le \Big[k_1 p_1^{1-\alpha_1} - u_1 k_1 p_1^{-\alpha_1} Q_1^{-\theta_1} r_1^{\delta_1 - 1} - a_1 k_1 p_1^{-\alpha_1} r_1^{-1} Q_1^{-1} - 0.5 i_1 u_1 Q_1^{1-\theta_1} r_1^{1+\delta_1}\\
&\quad - d_1 a_1^{-\mu_1} r_1^{\nu_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} - n_1 r_1^{-\eta_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\\
&\quad - u_2 k_2 p_2^{-\alpha_2} q^{\gamma+\varphi} Q_2^{-\theta_2} r_2^{\delta_2 - 1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} - a_2 k_2 p_2^{-\alpha_2} q^{\gamma} r_2^{-1} Q_2^{-1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\\
&\quad - 0.5 i_2 u_2 Q_2^{1-\theta_2} q^{\varphi} r_2^{1+\delta_2} - d_2 a_2^{-\mu_2} r_2^{\nu_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\\
&\quad - n_2 r_2^{-\eta_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} - \sum_{i=1}^{m} M_i - \sum_{j=1}^{s} S_j - k_2 \tau p_2 l^{-1}\Big]
\end{aligned}
$$

Now, after moving the negative terms (terms (2)-(6) and (8)-(15)) from the right to the left hand side we have:

$$
\begin{aligned}
&\Rightarrow Max\ Z\\
&s.t: \quad \Big[Z + u_1 k_1 p_1^{-\alpha_1} Q_1^{-\theta_1} r_1^{\delta_1 - 1} + a_1 k_1 p_1^{-\alpha_1} r_1^{-1} Q_1^{-1} + 0.5 i_1 u_1 Q_1^{1-\theta_1} r_1^{1+\delta_1} + d_1 a_1^{-\mu_1} r_1^{\nu_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1}\\
&\quad + n_1 r_1^{-\eta_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} + u_2 k_2 p_2^{-\alpha_2} q^{\gamma+\varphi} Q_2^{-\theta_2} r_2^{\delta_2 - 1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\\
&\quad + a_2 k_2 p_2^{-\alpha_2} q^{\gamma} r_2^{-1} Q_2^{-1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + 0.5 i_2 u_2 Q_2^{1-\theta_2} q^{\varphi} r_2^{1+\delta_2}\\
&\quad + d_2 a_2^{-\mu_2} r_2^{\nu_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + n_2 r_2^{-\eta_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\\
&\quad + \sum_{i=1}^{m} M_i + \sum_{j=1}^{s} S_j + k_2 \tau p_2 l^{-1}\Big] \le \Big(k_1 p_1^{1-\alpha_1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\Big)
\end{aligned}
$$

Finally, by dividing both sides by the right hand side and rearrangement the terms, we obtain:

$$
\begin{aligned}
&Min\ Z^{-1}\\
&s.t: \quad \Big[Z + \Big(u_1 k_1 p_1^{-\alpha_1} Q_1^{-\theta_1} r_1^{\delta_1 - 1} + a_1 k_1 p_1^{-\alpha_1} r_1^{-1} Q_1^{-1} + 0.5 i_1 u_1 Q_1^{1-\theta_1} r_1^{1+\delta_1} + d_1 a_1^{-\mu_1} r_1^{\nu_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1}\\
&\quad + n_1 r_1^{-\eta_1 - 1} k_1 p_1^{-\alpha_1} Q_1^{-1} + u_2 k_2 p_2^{-\alpha_2} q^{\gamma+\varphi} Q_2^{-\theta_2} r_2^{\delta_2 - 1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\\
&\quad + a_2 k_2 p_2^{-\alpha_2} q^{\gamma} r_2^{-1} Q_2^{-1} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + 0.5 i_2 u_2 Q_2^{1-\theta_2} q^{\varphi} r_2^{1+\delta_2}\\
&\quad + d_2 a_2^{-\mu_2} r_2^{\nu_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j} + n_2 r_2^{-\eta_2 - 1} Q_2^{-1} k_2 p_2^{-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\\
&\quad + \sum_{i=1}^{m} M_i + \sum_{j=1}^{s} S_j + k_2 \tau p_2 l^{-1}\Big)\Big] \Big/ \Big(k_1 p_1^{1-\alpha_1} + k_2 p_2^{1-\alpha_2} q^{\gamma} \prod_{i=1}^{m} M_i^{\beta_i} \prod_{j=1}^{s} S_j^{\sigma_j}\Big) \le 1
\end{aligned}
$$

where $Max\ Z$ and $Min\ Z^{-1}$ are equivalent and the proof is complete.