A complete ranking of decision making units with interval data

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Abstract: Interval Data Envelopment Analysis (Interval DEA) deals with the problem of efficiency assessment when the inputs and/or outputs of Decision Making Units (DMUs) are given as interval data. This paper focuses on the problem of ranking DMUs with interval data. First, we define extreme efficient units, super efficiency score, the best and the worst efficiency (inefficiency) frontiers in the interval DEA context. Then, we propose a novel method based on the lower and upper super efficiency scores of a unit under evaluation and the distance of that unit to four developed frontiers. Our method ranks all efficient and inefficient units which is one of the main advantages of it. Our method uses several essential criteria simultaneously to rank units with interval data. These criteria increase the discrimination power of our proposed method. Potential application of this method is illustrated with a dataset consisting of 30 branches of the social security insurance organization in Tehran.

Keywords: Data Envelopment Analysis, Interval DEA, Decision Making Unit, Ranking.

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Gholam Reza Jahanshahloo was a professor in Applied Mathematics Group in Kharazmi University of Iran. His research interests included Operations Research, performance management and Data Envelopment Analysis. He had supervised many M.Sc. and Ph.D. students in these areas and had published many papers in international journals. Unfortunately, he passed away in 2017.
1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric methodology for assessing the relative efficiency of Decision Making Units (DMUs) with multiple inputs and multiple outputs (Charnes et al. (1978), Banker et al. (1984), Färe et al. (1985), Zhu (2002), Cooper et al. (2006)). It assigns an efficiency measure between 0 and 1 to each unit. The larger the efficiency score, the better performance the unit under evaluation has. A DMU is efficient if its efficiency score is equal to 1, otherwise it is inefficient. The original DEA models consider the situation that all inputs and outputs have certain values. However, this assumption can be violated due to the existence of uncertainty in data. The problem of the evaluation of units with imprecise data has attracted attentions of several scholars. For example, Cooper et al. (1999) developed Imprecise Data Envelopment Analysis (IDEA) method. Their method can be applied in the situation where there exist both imprecisely and exactly-known data in which the IDEA models are transformed into linear programming problems. Kim et al. (1999) proposed a procedure to incorporate partial data into DEA. Their original model was a complicated non-linear model that was transformed into a linear programming problem by applying a linear scale transformation and the variable change technique.

Lee et al. (2002) proposed methods to determine the inefficiency of units such as slacks, returns to scale and so on in IDEA. These information helps the Decision Maker (DM) to improve the efficiency of units. Despotits and Smirlis (2002) proposed an approach to define the upper and lower bounds for the efficiency score of units with imprecise data. Their idea shows that the units with imprecise data do not have constant efficiency scores and their efficiency scores depend on the choice of data. Therefore, one of the attractive issues in IDEA is to determine the upper and lower bounds for units (See Cooper et (2001), Entani et al. (2002), Zhu (2003), Jahanshahloo et al. (2004), Wang et al. (2005), Amirteimoori and Kordostami (2005), Smirlis (2006), Park (2007), Toloo et al. (2008), Park (2010), Kao and Liu (2011), Esmaeili (2012), Hatami Marbini et al. (2014), Sun et al. (2014), and Khalili Damghani et al. (2015) for more studies about IDEA models). Kordostami and Jahani Sayyad Noveiri (2014) proposed a method to estimate the optimistic and pessimistic efficiency scores of units with fuzzy data and then integrated them into a geometric average efficiency.

Emrouznejad and Yang (2016) proposed a performance index based on efficient and anti-efficient frontiers in DEA models without explicit inputs (DEA-WEI) and developed the corresponding performance index in quadratic DEA-WEI models. Piri et al. (2016) proposed a method to evaluate the efficiency scores of DMUs with interval data in which the lower and upper bounds of intervals can take both negative and positive values. Amirteimoori et al. (2017) suggested an approach to integrate the optimistic and pessimistic perspectives to obtain the interval efficiency scores of units with interval data. Azizi et al. (2017) proposed a method to obtain the upper and lower bounds for the
efficiency scores of units with imprecise data when some input and/or output can be
specified as intervals, and some of them can be given as exact values and the other can be
determined as ordinal preferences information. Jiang et al. (2018) developed a DEA model
to measure the scale efficiency of DMUs with imprecise data and analyzed the sensitivity
and stability of their model for scale efficiency. Toloo et al. (2018) developed a
methodology to handle uncertain inputs, outputs and dual role factors and proposed models
to obtain the interval efficiency scores of units based on the optimistic and pessimistic
viewpoints and then suggested an integrated model to identify a unique status of each
imprecise dual factors.

There are another issues that examine uncertainty. In practice, temporal representation
is an attractive problem in a wide range of fields, such as computer science, philosophy,
psychology and so on. For instance, information system deals with the problem of outdated
data. In order to consider questions such as ‘which employees worked for us last year and
made over 15000$’ we need to represent temporal information. See F.Allen (1983) for
more studies about temporal representation. S Ganapathy et al. (2013) combined temporal
features with the fuzzy min-max neural network that is based on a classifier to select the
effective decision in medical diagnosis. See Laxman and sastry (2006), Zhang et al. (2009),
wai and Lee (2008), Simson (1992)) for more studies about temporal data mining,
classification, fuzzy min-max, neural network.

The traditional DEA models cannot discriminate among the efficient units because they
get identical efficiency scores equal to one. In this regard, several ranking approaches have
been developed in the DEA literature. For a review on ranking methods in DEA see Adler
et al. (2002). One of the attractive topics in IDEA is to rank units. Jahanshahloo et al.
(2006) extended TOPSIS method in Interval DEA. Wang et al. (2005) considered the
efficiency assessment of units in the presence of interval and/or fuzzy data. They proposed
two linear CCR models to obtain the interval efficiency of DMUs with interval data and
then applied the interval efficiencies of all units by a minimax regret-based approach to
rank units. Wu et al. (2013) proposed a two-phases approach in which the first phase
obtains the interval cross-efficiency score of DMUs with interval data and the second phase
ranks units by applying an improved TOPSIS technique. Khodabakhshi and Aryavash
(2015) developed a method for ranking units with stochastic data. Rafiee Sani and
Alorezaee (2017) developed some fuzzy versions of trade-off DEA models by applying
some ranking methods based on the comparison of $\alpha$ -cuts. Shavazipour et al. (2017)
proposed an approach to rank extreme efficient units with fuzzy data. Their model is based
on the Tchebycheff norm. Ebrahimi (2019) considered DEA with stochastic data and
applied the expected efficiency of units to present a method for ranking DMUs.

Given the importance of ranking the units in DEA, in particular in Interval DEA, we
focus on the ranking of DMUs with interval data and propose a novel approach to rank
units. In this study, one of our main motivations is to extend some concepts from the
traditional DEA into interval DEA and the other is to develop a powerful method to
discriminate and rank DMUs with interval data, hence, we propose an approach using
several essential criteria simultaneously to rank units with interval data. These criteria
increase the discrimination power of our method.

The contribution of this study is to develop a powerful method for ranking DMUs with
interval data as our proposed approach has all desirable features expected for ranking
methods. First, we suggest two linear programming models to compute the lower and upper
super efficiency scores of a unit following the method of Anderson and Peterson (1993).
As in the traditional super efficiency score, it is desirable to obtain the large lower and
upper super efficiency scores in interval DEA. Then, we define the terms extreme efficient
units and non-dominated DMUs in Interval DEA for the first time. Secondly, we introduce
two efficiency frontiers, namely the best and the worst efficiency frontiers, and two
inefficiency frontiers, called the best and the worst inefficiency frontiers and after that we
formulate four linear programming problems to measure the distance of each unit from
these frontiers. It is clear that, units closer to efficiency frontiers and more far from the
inefficiency frontiers have better performance. Hence, by using these distances and the
defined lower and upper super efficiency scores of units, we assign a vector with four
components to each unit. Finally, we sort these vectors by a lexicographic order. In our
method, as we expect, the rank order of extreme efficient units is better than other DMUs. Also, if a unit dominates another, it gets a higher rank than the dominated unit.

The rest of this paper is organized as follows: section 2 reviews the interval DEA preliminaries. In section 3, we present a complete ranking method for DMUs with interval data. Two numerical examples are provided in section 4. Section 5 concludes the paper.

2 Preliminaries and basic definitions

Consider a system of n DMUs, denoted by \( DMU_j, j = 1, ..., n \), where each unit consumes \( m \) different inputs to generate \( s \) different outputs. The \( i^{th} \) input and \( r^{th} \) output for \( DMU_j \) are denoted by \( x_{ij} \) and \( y_{rj} \), respectively, for \( i = 1, ..., m \) and \( r = 1, ..., s \). Also, suppose that input and output values are not deterministic for all units and \( x_{ij} \in \{ x_{ij}^l, x_{ij}^u \} \) and \( y_{rj} \in \{ y_{rj}^l, y_{rj}^u \} \), where the lower and upper bounds are positive and finite values. Assume that \( DMU_o \) is the unit under evaluation.

Wang et al. (2005) considered the following production possibility set (PPS) in Interval DEA:

\[
T = \{ (x, y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_{ij}^l, y \leq \sum_{j=1}^{n} \lambda_j y_{rj}^l, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n \}
\]

They formulated two linear programming models (1a) and (1b) to measure the lower and upper bounds for the efficiency score of \( DMU_o \), as reported in Table 1.

<table>
<thead>
<tr>
<th>The lower efficiency score</th>
<th>The upper efficiency score</th>
</tr>
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<tbody>
<tr>
<td>( E_{oo}^L = \max \sum_{r=1}^{s} \mu_r y_{r0}^l + u_0 )</td>
<td>( E_{oo}^U = \max \sum_{r=1}^{s} \mu_r y_{r0}^u + u_0 )</td>
</tr>
<tr>
<td>( \text{s.t. } \sum_{i=1}^{m} w_i x_{i0}^l = 1, )</td>
<td>( \text{s.t. } \sum_{i=1}^{m} w_i x_{i0}^l = 1, )</td>
</tr>
<tr>
<td>( \sum_{r=1}^{m} \mu_r y_{rj}^l - \sum_{i=1}^{m} w_i x_{ij}^l + u_0 \leq 0, \forall j, )</td>
<td>( \sum_{r=1}^{m} \mu_r y_{rj}^u - \sum_{i=1}^{m} w_i x_{ij}^l + u_0 \leq 0, \forall j, )</td>
</tr>
<tr>
<td>( w_i, \mu_r \geq \varepsilon, \forall i, r. )</td>
<td>( w_i, \mu_r \geq \varepsilon, \forall i, r. )</td>
</tr>
</tbody>
</table>

where \( \varepsilon > 0 \) is Non-Archimedean.

The optimal value of models (1a) and (1b) were called the lower and the upper efficiency score of \( DMU_o \) by Wang et al. (2005), respectively. It is clear that \( E_{oo}^L \leq 1, E_{oo}^U \leq 1 \) and \( E_{oo}^L \leq E_{oo}^U \).

The efficient and inefficient units in Interval DEA were defined as follows by Wang et al. (2005):

**Definition 1.** The unit \( DMU_o = (x_o, y_o) \) is efficient, if \( E_{oo}^U = 1 \). Otherwise, it is inefficient.

In the next section, we propose an original approach to rank DMUs with interval data.
3 A complete ranking of decision making units with interval data

In this section, we present a new method for ranking units in the presence of interval data based on some concepts that are defined in the following subsections. Subsection 3.1 extends the concept of super efficiency to Interval DEA and formulates two linear programming models to obtain the lower and the upper super efficiency scores of DMUs. Based on the obtained lower and upper super efficiency scores, we classify the set of all units into three subsets and then we present the definition of extreme efficient units in the context of Interval DEA. Subsection 3.2 defines the best-case and worst-case convex hulls and obtains two efficiency and two inefficiency frontiers. Then, four linear programming problems are formulated to measure the distance of each unit from these frontiers. It is clear that, units closer to efficiency frontiers and more far from the inefficiency frontiers are preferred. Finally, subsection 3.3 suggests an approach to rank units based on assigning a vector with four components to each unit and lexicographic order.

3.1 Extending the super efficiency concept to Interval DEA

Anderson and Peterson (1993) introduced the super efficiency concept in traditional DEA. Their approach removes a DMU from the set of the observed units and constructs the new PPS by the remaining units and reformulates models (1a) and (1b) by using the new PPS constructed by the remaining units to determine the lower and upper super efficiency scores of units with interval data. The models are reported in Table 2.

<table>
<thead>
<tr>
<th>The lower super efficiency score</th>
<th>The upper super efficiency score</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^L_0 = \max \sum_{r=1}^{s} \mu_r y^L_{r0} + u_0$</td>
<td>$E^U_0 = \max \sum_{r=1}^{s} \mu_r y^U_{r0} + u_0$</td>
</tr>
<tr>
<td>s.t. $m$ $\sum_{i=1}^{m} w_ix^L_{i0} = 1,$</td>
<td>s.t. $m$ $\sum_{i=1}^{m} w_ix^U_{i0} = 1,$</td>
</tr>
<tr>
<td>$\sum_{i=1}^{m} \mu_r y^L_{rj} - \sum_{i=1}^{m} w_ix^L_{ij} + u_0 \leq 0, j \neq o,$</td>
<td>$\sum_{i=1}^{m} \mu_r y^U_{rj} - \sum_{i=1}^{m} w_ix^U_{ij} + u_0 \leq 0, j \neq o,$</td>
</tr>
<tr>
<td>$w_i, \mu_r \geq \varepsilon, \forall i, r.$</td>
<td>$w_i, \mu_r \geq \varepsilon, \forall i, r.$</td>
</tr>
</tbody>
</table>

It is clear that $E^L_0 \leq E^U_0$. On the other hand, $E^L_0 \leq E^U_0$ because any optimal solution of model (1a) is a feasible solution for model (2a). Also, any optimal solution of model (1b) is a feasible solution of model (2b), therefore, $E^U_0 \leq E^U_0$. Regarding the obtained lower and upper super efficiency scores, all units can be classified into the following three subsets:

- $E^{++} = \{ j \in \{1, \ldots, n\} | E^L_j > 1 \}$
- $E^+ = \{ j \in \{1, \ldots, n\} | E^L_j \leq 1, E^U_j > 1 \}$
- $E^- = \{ j \in \{1, \ldots, n\} | E^L_j < 1, E^U_j \leq 1 \}$

$E^{++}$ includes all decision making units that their lower super efficiency score and as a result, their upper super efficiency score are greater than one. $E^+$ includes all DMUs that their lower super efficiency score is less than or equal to one and their upper super
efficiency score is greater than one. $E^-$ includes all units that their lower super efficiency score is less than one and their upper super efficiency score is less than or equal to one.

Next theorem provides a sufficient condition for efficiency of a unit with interval data.

**Theorem 1.** The unit $DMU_o = (x_o, y_o)$ is efficient, if $o \in E^+$ or $o \in E^{++}$. 

**Proof:** If $o \in E^+$ or $o \in E^{++}$ then $E^U_o > 1$. We claim that, if $o \in E^+$ or $o \in E^{++}$ then $E^U_o = 1$. By contradiction, let $E^U_o < 1$. Assume that $(\hat{\mu}_o, \hat{\omega}_o, \hat{\alpha}_o)$ is an optimal solution of model (1b), therefore, we have:

$$E^U_o = \sum_{r=1}^{s} \hat{\mu}_{ro} y_{ro} + \hat{\alpha}_o < 1.$$  

Also, suppose that $(\mu^*_o, \omega^*_o, u^*_o)$ is an optimal solution of model (2b), we have:

$$\sum_{r=1}^{s} \mu^*_r y_{ro} - \sum_{i=1}^{m} w^*_i x_{io} + u^*_o \leq 0, \quad j = 1, ..., n, j \neq o$$

$$\sum_{i=1}^{m} w^*_i x_{io}^l = 1,$$

$$E^U_o = \sum_{r=1}^{s} \mu^*_r y_{ro} + u^*_o > 1.$$  

Regarding that $u^*_o$ is a free variable in model (2b), we define:

$$\bar{\mu}_{ro} = \mu_{ro}, \quad r = 1, ..., s,$$

$$\bar{\omega}_io = \omega^*_i, \quad i = 1, ..., m,$$

$$\bar{u}_o = u^*_o - (E^U_o - 1) < u^*_o.$$  

Therefore, $(\bar{\mu}, \bar{\omega}, \bar{u}_o)$ is a feasible solution of model (1b), because:

$$\sum_{r=1}^{s} \bar{\mu}_{ro} y_{ro} - \sum_{i=1}^{m} \bar{\omega}_io x_{io}^u + \bar{u}_o < \sum_{r=1}^{s} \mu^*_r y_{ro} - \sum_{i=1}^{m} w^*_i x_{io} + u^*_o \leq 0, \quad j \neq o,$$

$$\sum_{r=1}^{s} \mu^*_r y_{ro} - \sum_{i=1}^{m} \bar{\omega}_io x_{io}^l + \bar{u}_o = \sum_{r=1}^{s} \mu^*_r y_{ro} - \sum_{i=1}^{m} w^*_i x_{io}^l + u^*_o - (E^U_o - 1) =$$

$$E^U_o - 1 - (E^U_o - 1) = 0,$$

$$\sum_{i=1}^{m} \bar{\omega}_io x_{io}^l = \sum_{i=1}^{m} w^*_i x_{io}^l = 1.$$  

The value of the objective function of model (1b) for this feasible solution is:

$$\sum_{r=1}^{s} \bar{\mu}_{ro} y_{ro} + \bar{u}_o = \sum_{r=1}^{s} \mu^*_r y_{ro} + u^*_o - (E^U_o - 1) = E^U_o - (E^U_o - 1) = 1.$$

which is a contradiction with the optimality of $(\bar{\mu}_o, \bar{\omega}_o, \bar{\alpha}_o)$ for model (1b), thus, $E^U_o = 1$ and $DMU_o$ is efficient according to Definition 1.

In traditional DEA, $DMU_o$ is an extreme efficient unit if its super efficiency score is greater than one. We define the extreme efficient unit in Interval DEA following the definition of extreme efficient unit in traditional DEA.
Definition 2. The unit $DMU_o = (x_o, y_o)$ is an extreme efficient unit if at least one of its lower super efficiency score or its upper super efficiency score be greater than one. In the other word, $DMU_o$ is an extreme efficient unit in Interval DEA if $o \in E^+$ or $o \in E^{++}$.

In the following example, we determine the extreme efficient units in a numerical example with five units in the presence of interval data.

Example 1. Consider five decision making units with interval data. Each DMU consumes one input to produce one output. The second and third columns of Table 5 reports the data and Figure 1 shows the PPS. Columns 4, 5 and 6 of Table 5 show the lower efficiency score, the upper efficiency score and the status of efficiency of units, respectively. As we see, the upper efficiency score of units A, B and C are equal to 1 and hence they are efficient. The upper efficiency score of unit D and E are less than 1 and hence they are inefficient. We solve models (2a) and (2b) to obtain the lower super efficiency and upper super efficiency scores of DMUs and then classify all units into $E^+$, $E^+$ and $E^-$. The results are summarized in columns 7, 8 and 9 of Table 5. The last column of Table 5 shows that each unit is extreme efficient or not. Note that $A, B \in E^+$, according to Definition 2, they are extreme efficient units while C is an efficient unit belongs to $E^-$, units D and E are inefficient and belong to $E^-$. Therefore, according to Definition 2, units C, D and E are not extreme efficient units.

3.2 Efficient and inefficient frontiers

This section considers two convex hulls of DMUs, namely the best-case ($L^{BC}$) and worst-case ($L^{WC}$) convex hulls. $L^{BC}$ is made by the points that represent the best mode of units, similarly, $L^{WC}$ is made by the points that represent the worst mode of units.

\[ L^{BC} = \{(x, y) | x = \sum_{j=1}^{n} \lambda_j x_j^I, y = \sum_{j=1}^{n} \lambda_j y_j^I, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n\} \]  
\[ L^{WC} = \{(x, y) | x = \sum_{j=1}^{n} \lambda_j x_j^U, y = \sum_{j=1}^{n} \lambda_j y_j^U, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n\} \]

Figure 2 shows $L^{BC}$ and $L^{WC}$ for five DMUs reported in Example 1. Thick lines specify $L^{BC}$ and dashed lines specify $L^{WC}$. Shadowed region is the subscription of $L^{BC}$ and $L^{WC}$.

In the following, we define two efficiency and two inefficiency frontiers by considering the frontiers of $L^{BC}$ and $L^{WC}$.

We consider two sets $T_1^{BC}$ and $T_2^{BC}$ constructed by the frontier of $L^{BC}$ as follows:

\[ T_1^{BC} = \{(x, y) | x \geq \sum_{j=1}^{n} \lambda_j x_j^I, y \leq \sum_{j=1}^{n} \lambda_j y_j^I, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n\} \]  
\[ T_2^{BC} = \{(x, y) | x \leq \sum_{j=1}^{n} \lambda_j x_j^I, y \geq \sum_{j=1}^{n} \lambda_j y_j^I, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, ..., n\} \]

And then, we define the best efficiency and the best inefficiency frontiers, namely $\partial T_1^{BC}$ and $\partial T_2^{BC}$, as the frontiers of $T_1^{BC}$ and $T_2^{BC}$, respectively. Note that $\partial T_1^{BC}$ and $\partial T_2^{BC}$ are named as the best efficiency and the best inefficiency frontiers because they are made by the best mode of all units.

Similarly, we consider two sets $T_1^{WC}$ and $T_2^{WC}$ made by the frontier of $L^{WC}$ as follows:
\[
T_{1}^{WC} = \{ (x, y) | x \geq \sum_{j=1}^{n} \lambda_{j} x_{j}^{U}, y \leq \sum_{j=1}^{n} \lambda_{j} y_{j}^{L}, \sum_{j=1}^{n} \lambda_{j} = 1, \lambda_{j} \geq 0, j = 1, ..., n \} \quad (10)
\]

\[
T_{2}^{WC} = \{ (x, y) | x \leq \sum_{j=1}^{n} \lambda_{j} x_{j}^{U}, y \geq \sum_{j=1}^{n} \lambda_{j} y_{j}^{L}, \sum_{j=1}^{n} \lambda_{j} = 1, \lambda_{j} \geq 0, j = 1, ..., n \} \quad (11)
\]

And then, we define the worst efficiency and the worst inefficiency frontiers, namely \( \partial T_{1}^{WC} \) and \( \partial T_{2}^{WC} \), as the frontiers of \( T_{1}^{WC} \) and \( T_{2}^{WC} \), respectively. Note that, \( \partial T_{1}^{WC} \) and \( \partial T_{2}^{WC} \) are named as the worst efficiency and the worst inefficiency frontiers because they are made by the worst mode of all units.

Figure 3 illustrates the efficiency and inefficiency frontiers for decision making units reported in Example 1. Top thick lines show the best efficiency frontiers, lower thick lines show the worst efficiency frontier, top dashed lines show the best inefficiency frontier and lower dashed lines show the worst inefficiency frontier.

After defining the frontiers \( \partial T_{1}^{BC} \), \( \partial T_{2}^{BC} \), \( \partial T_{1}^{WC} \) and \( \partial T_{2}^{WC} \), one of the attractive issues is to measure the distance of each unit from them. Therefore, we formulate models (12a) and (12b), reported in Table 3, to determine the minimum distance of each DMU from the best and the worst efficiency frontiers.

Table 3. The distance of DMU\(_{a}\) from the efficiency frontiers.

<table>
<thead>
<tr>
<th>The distance from the best efficiency frontier</th>
<th>The distance from the worst efficiency frontier</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ Z_{a} = \max \sum_{i=1}^{m} w_{i}^{-} s_{i}^{-} + \sum_{r=1}^{s} w_{r}^{+} s_{r}^{+} ]</td>
<td></td>
</tr>
<tr>
<td>s.t. [ \sum_{j=1}^{n} \lambda_{j} x_{j}^{U} + s_{i}^{-} = x_{i}^{L}, \forall i, ] [ \sum_{j=1}^{n} \lambda_{j} y_{j}^{L} - s_{r}^{+} = y_{r}^{R}, \forall r, ] [ \sum_{j=1}^{n} \lambda_{j} = 1, ] [ \lambda_{j} \geq 0, \forall j, ] [ s_{r}^{+} \geq 0, \forall r, ] [ s_{i}^{-} \geq 0, \forall i. ]</td>
<td></td>
</tr>
<tr>
<td>[ Z_{a} = \max \sum_{i=1}^{m} w_{i}^{-} s_{i}^{-} + \sum_{r=1}^{s} w_{r}^{+} s_{r}^{+} ]</td>
<td></td>
</tr>
<tr>
<td>s.t. [ \sum_{j=1}^{n} \lambda_{j} x_{j}^{U} + s_{i}^{-} = x_{i}^{L}, \forall i, ] [ \sum_{j=1}^{n} \lambda_{j} y_{j}^{L} - s_{r}^{+} = y_{r}^{R}, \forall r, ] [ \sum_{j=1}^{n} \lambda_{j} = 1, ] [ \lambda_{j} \geq 0, \forall j, ] [ s_{r}^{+} \geq 0, \forall r, ] [ s_{i}^{-} \geq 0, \forall i. ]</td>
<td></td>
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</table>

The first and second constraints of model (12a) is made by adding the slacks \( s_{i}^{-} \) and \( s_{r}^{+} \) for all \( i = 1, ..., m \) and \( r = 1, ..., s \), to the inequalities \( x_{i} \geq \sum_{j=1}^{n} \lambda_{j} x_{j}^{U} \) and \( y_{r} \leq \sum_{j=1}^{n} \lambda_{j} y_{j}^{L} \) in \( T_{1}^{BC} \). Regarding that, \( s_{i}^{-} (i = 1, ..., m) \) represents the distance of \( x_{i}^{L} (i = 1, ..., m) \) from the input of a point on the best efficient frontier and \( s_{r}^{+} (r = 1, ..., s) \) represents the distance of \( y_{r}^{R} (r = 1, ..., s) \) from the output of a point on the best efficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of DMU\(_{a}\) from the best efficient frontier.

The first and second constraints of model (12b) is made by adding the slacks \( s_{i}^{-} \) and \( s_{r}^{+} \) for all \( i = 1, ..., m \) and \( r = 1, ..., s \), to the inequalities \( x_{i} \geq \sum_{j=1}^{n} \lambda_{j} x_{j}^{U} \) and \( y_{r} \leq \sum_{j=1}^{n} \lambda_{j} y_{j}^{L} \) in \( T_{1}^{WC} \). Regarding that, \( s_{i}^{-} (i = 1, ..., m) \) represents the distance of \( x_{i}^{L} (i = 1, ..., m) \) from the input of a point on the worst efficient frontier and \( s_{r}^{+} (r = 1, ..., s) \) represents the distance of \( y_{r}^{R} (r = 1, ..., s) \) from the output of a point on the worst efficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of DMU\(_{a}\) from the worst efficient frontier.
In models (12a) and (12b), $w^{-}$ and $w^{+}$ are given weight vectors by decision maker (DM).

Similarly, models (13a) and (13b), reported in Table 4, determine the minimum distance of each unit from the best and the worst inefficiency frontiers.

Table 4. The distance of $DMU_{o}$ from the efficiency frontiers.

<table>
<thead>
<tr>
<th>The distance from the best inefficiency frontier</th>
<th>The distance from the worst inefficiency frontier</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{o}^{*} = \max \sum_{i=1}^{m} w_{i}^{-} s_{i}^{-} + \sum_{r=1}^{s} w_{r}^{+} s_{r}^{+}$</td>
<td>$Z_{o}^{-} = \max \sum_{i=1}^{m} w_{i}^{-} s_{i}^{-} + \sum_{r=1}^{s} w_{r}^{+} s_{r}^{+}$</td>
</tr>
<tr>
<td>s.t. $\sum_{j=1}^{n} \lambda_{j} x_{ij} - s_{i}^{-} = x_{io}^{-}$, $\forall i$,</td>
<td>s.t. $\sum_{j=1}^{n} \lambda_{j} x_{ij} - s_{i}^{-} = x_{io}^{-}$, $\forall i$,</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} \lambda_{j} y_{rj}^{U} + s_{r}^{+} = y_{ro}^{U}$, $\forall r$,</td>
<td>$\sum_{j=1}^{n} \lambda_{j} y_{rj}^{U} + s_{r}^{+} = y_{ro}^{U}$, $\forall r$,</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} \lambda_{j} = 1$, $\forall j$,</td>
<td>$\sum_{j=1}^{n} \lambda_{j} = 1$, $\forall j$,</td>
</tr>
<tr>
<td>$s_{r}^{-} \geq 0$, $\forall r$,</td>
<td>$s_{r}^{-} \geq 0$, $\forall r$,</td>
</tr>
<tr>
<td>$s_{r}^{+} \geq 0$, $\forall i$.</td>
<td>$s_{r}^{+} \geq 0$, $\forall i$.</td>
</tr>
</tbody>
</table>

The first and second constraints of model (13a) is made by adding the slacks $s_{i}^{-}$ and $s_{r}^{+}$ for all $i = 1, ..., m$ and $r = 1, ..., s$, to the inequalities $x_{i} \leq \sum_{j=1}^{n} \lambda_{j} x_{ij}^{L} y_{r} \geq \sum_{j=1}^{n} \lambda_{j} y_{rj}^{U}$ in $T_{2}^{EC}$. Regarding that, $s_{i}^{-}$ ($i = 1, ..., m$) represents the distance of $x_{io}^{L}$ ($i = 1, ..., m$) from the input of a point on the best inefficient frontier and $s_{r}^{+}$ ($r = 1, ..., s$) represents the distance of $y_{ro}^{U}$ ($r = 1, ..., s$) from the output of a point on the best inefficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of $DMU_{o}$ from the best inefficient frontier.

The first and second constraints of model (13b) is made by adding the slacks $s_{i}^{-}$ and $s_{r}^{+}$ for all $i = 1, ..., m$ and $r = 1, ..., s$, to the inequalities $x_{i} \leq \sum_{j=1}^{n} \lambda_{j} x_{ij}^{L} y_{r} \geq \sum_{j=1}^{n} \lambda_{j} y_{rj}^{L}$ in $T_{2}^{WC}$. Regarding that, $s_{i}^{-}$ ($i = 1, ..., m$) represents the distance of $x_{io}^{L}$ ($i = 1, ..., m$) from the input of a point on the worst inefficient frontier and $s_{r}^{+}$ ($r = 1, ..., s$) represents the distance of $y_{ro}^{L}$ ($r = 1, ..., s$) from the output of a point on the worst inefficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of $DMU_{o}$ from the worst inefficient frontier.

In models (13a) and (13b), $w^{-}$ and $w^{+}$ are given weight vectors by DM.

Theorem 2 proves that all above four models are feasible and bounded.

**Theorem 2. Models (12a), (12b), (13a) and (13b) are feasible and bounded.**

**Proof:** Clearly,

$$\lambda_{o} = 1,$$
$$\lambda_{j} = 0, \quad j = 1, ..., n, \quad j \neq o,$$
$$s_{i}^{-} = 0, \quad i = 1, ..., m,$$
$$s_{r}^{+} = 0, \quad r = 1, ..., s.$$

is a feasible solution for model (12a). From the constraints of model (12a) we have:
\[ s_i^- = x_{io}^l - \sum_{j=1}^{n} \lambda_j x_{lj}^l \leq x_{io}^l, \quad i = 1, \ldots, m, \]
\[ s_r^+ = \sum_{j=1}^{n} \lambda_j y_{ij}^u - y_{ro}^u \leq \sum_{j=1}^{n} \lambda_j y_{oj}^u \leq M_r, \quad r = 1, \ldots, s. \]

where \( M_r = \max_{1 \leq j \leq n} y_{oj}^u \). Since \( x_{lj}^l, x_{lo}^l, y_{ij}^u \) and \( y_{oj}^u \) are finite for all \( i, j, r, o \), therefore, model (12a) is bounded. Similarly, models (12b), (13a) and (13b) are also feasible and bounded.

In the following, we suggest a method for ranking DMUs with interval data, applying the lexicographic order defined as follows:

**Definition 3.** (Ehrgott (2005)) Let \( y^1, y^2 \in \mathbb{R}^p (p \geq 2) \) and \( k^* = \min \{ k | y_{1k}^1 \neq y_{2k}^2 \} \). If \( y_{1k}^1 > y_{2k}^2 \) or \( y_{1k}^1 = y_{2k}^2 \), then \( y^1 \succeq_{lex} y^2 \).

### 3.3 Our proposed ranking method for Interval DEA

In this section, we propose a method for ranking DMUs with interval data. This method assigns a 4-vector, namely \( V_o \), to each unit \( DMU_o \), for \( o \in \{ 1, \ldots, n \} \), and then compare these vectors by lexicographic order. In the following, we describe how each component of \( V_o \) is selected. Each ranking method in DEA is expected to have the feature that the rank of an efficient unit should be better than the rank of an inefficient unit. Therefore, we consider the upper efficiency score of DMUs as the first priority. According to Theorem 1, the upper efficiency score for all units in \( E'^+ \), for all DMUs in \( E^+ \) and for some decision making units in \( E^- \) is equal to 1. Therefore, the upper efficiency score alone cannot distinguish among them. On the other hand, we consider another priority as the rank of each unit in \( E'^+ \) should be better than the rank of each DMU in \( E^+ \). Regarding that \( E^u_o \) plays an essential role of creating the distinction between \( E'^+ \) and \( E^+ \), hence, we consider the lower super efficiency score of \( DMU_o \) as another priority with upper efficiency score, simultaneously. Therefore, we define the first component of \( V_o \), \( o \in \{ 1, \ldots, n \} \), as the maximum of the lower super efficiency score and the upper efficiency score of \( DMU_o \). In the other word, the first component of \( V_o \) is defined as \( \max \{ E^u_o, E^o_o \} \). If the unit has a higher \( E^u_o \) and \( E^o_o \), gets a better rank.

After defining the first component of \( V_o \), we describe how to define the second component of \( V_o \). We consider the next priority as the rank of all units in \( E^+ \) should be better than the rank of all units in \( E^- \). Note that, the selection of the first component of \( V_o \) as described guarantees that the rank of each unit belongs to \( E'^+ \) is better than the rank of each DMU in \( E'^+ \), but it cannot guarantee that the rank of each decision making unit in \( E'^+ \) is better than the rank of each DMU in \( E^- \). Regarding that, \( E^o_o \) plays an essential role of creating the distinction between \( E^+ \) and \( E^- \), therefore, we define the second component of \( V_o \) as \( E^o_o \). It should be noted that, maybe there exist units that have the same values for the first and second components of their assigned vector. So, we need to define other components to make more distinction between units. As we know, the units closer to efficiency frontiers and more far from inefficiency frontiers are preferred. Hence, we consider the distance of units from the efficiency and inefficiency frontiers as our other priorities.

Note that, we avoid to define the vector with a lot of components, therefore, we must consider a combination of the distances from the efficiency and inefficiency frontiers as the third and fourth component of \( V_o \), respectively. On the other hand, the components must be selected so the larger value of them indicates the better rank for units. Therefore, we define the third component of \( V_o \) as the negative of the average of distances of \( DMU_o \) from the best and the worst efficiency frontiers, similarly, the average of distances of \( DMU_o \) from the best and the worst inefficiency frontiers is considered as the last component of \( V_o \).
In summary, the preferences in our ranking method to make a powerful distinction between all units are:
1) The maximum value for the lower super efficiency score and the upper efficiency score.
2) The maximum value for the upper super efficiency score.
3) The minimum value for the average of distances of unit from the best and the worst efficiency frontiers.
4) The maximum value for the average of distances of unit from the best and the worst inefficiency frontiers.

Therefore, the assigned 4-vector \( V_o \) to \( DMU_o \) is
\[
V_o = \left( \max \{E_{o}^{L}, E_{o}^{U}\}, E_{o}^{U} - \frac{Z_{o}^{e}+Z_{o}^{f}}{2}, \frac{w_{o}^{e}+w_{o}^{f}}{2} \} \right). \]
Finally, we rank these vectors according to lexicographic order described in Definition 3.

In the following, we summarize our ranking method as an algorithm for more clarity:

**The algorithm of our method**

**Step 1:** Solve models (1b), (2a) and (2b) to obtain the upper efficiency score, the lower super efficiency score and the upper super efficiency score for \( DMU_{o}, o \in \{1, \ldots, n\} \).

**Step 2:** Solve models (12a), (12b), (13a) and (13b) and determine the optimal objective values \( Z_{o}^{e}, Z_{o}^{f}, W_{o}^{e} \) and \( W_{o}^{f} \), respectively, to measure the distances of \( DMU_{o} \) from the efficiency frontiers and inefficiency frontiers.

**Step 3:** Define vector \( V_o = \left( \max \{E_{o}^{L}, E_{o}^{U}\}, E_{o}^{U} - \frac{Z_{o}^{e}+Z_{o}^{f}}{2}, \frac{w_{o}^{e}+w_{o}^{f}}{2} \} \) for \( DMU_{o} \).

**Step 4:** Compare the vectors \( V_i, j \in \{1, \ldots, n\} \), by the lexicographic order and obtain a complete ranking of units.

In the following, we present the concept of domination for units with interval data.

**Definition 4.** Suppose that \( o, l \in \{1, \ldots, n\} \). If \( x_{i/o}^{L} \leq x_{i/l}^{L}, x_{i/o}^{U} \leq x_{i/l}^{U} \) for \( i = 1, \ldots, m \) and \( y_{r/o}^{L} \geq y_{r/l}^{L}, y_{r/o}^{U} \geq y_{r/l}^{U} \) for \( r = 1, \ldots, s \), then, \( DMU_{o} \) dominates \( DMU_{l} \).

The next theorem proves that if a unit dominates the other one, then it has the better rank than it.

**Theorem 3.** Let \( DMU_{o} \) dominates \( DMU_{l} \). Then the rank of \( DMU_{o} \) is better than the rank of \( DMU_{l} \) in our method or equivalently \( V_o \succeq_{lex} V_l \).

**Proof:** Suppose that \( DMU_{o} \) dominates \( DMU_{l} \). Therefore, \( x_{i/o}^{L} \leq x_{i/l}^{L}, x_{i/o}^{U} \leq x_{i/l}^{U} \) for \( i = 1, \ldots, m \) and \( y_{r/o}^{L} \geq y_{r/l}^{L}, y_{r/o}^{U} \geq y_{r/l}^{U} \) for \( r = 1, \ldots, s \), and inequality is strict for at least one component. Without loss of generality, we assume that \( x_{k/o}^{L} < x_{k/l}^{L} \). Let \( (\mu_{i}^{o}, w_{i}^{o}, u_{i}^{o}) \) is an optimal solution for model (2a) evaluating \( DMU_{l} \). Hence, we have:

\[
E_{l}^{L} = \sum_{r=1}^{s} \mu_{r}^{o} y_{r/l}^{L} + u_{o}^{o},
\]
\[
\sum_{r=1}^{s} \mu_{r}^{o} y_{r/l}^{U} - \sum_{i=1}^{m} w_{i}^{o} x_{i/l}^{U} + u_{o}^{o} \leq 0, \quad j \neq l,
\]
\[
\sum_{r=1}^{s} \mu_{r}^{o} y_{r/l}^{U} - \sum_{i=1}^{m} w_{i}^{o} x_{i/l}^{U} + u_{o}^{o} \leq \sum_{r=1}^{s} \mu_{r}^{o} y_{r/o}^{U} - \sum_{i=1}^{m} w_{i}^{o} x_{i/o}^{U} + u_{o}^{o} \leq 0,
\]
\[
\sum_{i=1}^{m} w_{i}^{o} x_{i/l}^{U} = 1,
\]
\[
\mu_{r}^{o} w_{i/l}^{o} \geq \varepsilon, \quad \forall i, r.
\]

Since \( x_{i/o}^{L} \leq x_{i/l}^{U} \) for \( i = 1, \ldots, m \), we have:
\[ 1 = \sum_{i=1}^{m} w_{il}x_{il}^{l} \geq \sum_{i=1}^{m} w_{il}x_{il}^{u} = \alpha. \]

It is clear that \( \alpha > 0 \). Now, we prove that \( \left( \frac{1}{\alpha} \mu_{i}^{l}, \frac{1}{\alpha} w_{il}^{l}, \frac{1}{\alpha} u_{il}^{l} \right) \) is a feasible solution for model (2a) evaluating \( DMU_{0} \):

\[ \frac{1}{\alpha} \left( \sum_{r=1}^{s} \mu_{rl}y_{rl}^{l} - \sum_{i=1}^{m} w_{il}x_{il}^{l} + u_{il}^{0} \right) \leq 0, \quad j = 1, ..., n. \]

\[ \frac{1}{\alpha} \left( \sum_{i=1}^{m} w_{il}x_{il}^{u} \right) = 1 \]

\[ \frac{1}{\alpha} (\mu_{rl}) \geq \frac{1}{\alpha} \varepsilon \geq \varepsilon, \quad r = 1, ..., s, \]

\[ \frac{1}{\alpha} (w_{il}) \geq \frac{1}{\alpha} \varepsilon \geq \varepsilon, \quad i = 1, ..., m. \]

Hence, \( E_{0}^{l} \geq E_{l}^{u} \).

Also, suppose that \( (\bar{\mu}_{l}, \bar{w}_{l}, \bar{u}_{l}) \) is an optimal solution for model (1b) evaluating \( DMU_{l} \).

Then, we have:

\[ E_{ll}^{u} = \sum_{r=1}^{s} \bar{\mu}_{rl}y_{rl}^{l} + \bar{u}_{l}. \]

\[ \sum_{r=1}^{s} \bar{\mu}_{rl}y_{rl}^{l} - \sum_{i=1}^{m} \bar{w}_{il}x_{il}^{l} + \bar{u}_{l}^{0} \leq 0, \quad j = 1, ..., n, \]

\[ \sum_{i=1}^{m} \bar{w}_{il}x_{il}^{u} = 1, \]

\[ \bar{\mu}_{rl} \geq \varepsilon, \quad r = 1, ..., s, \]

\[ \bar{w}_{il} \geq \varepsilon, \quad i = 1, ..., m. \]

Since \( x_{il}^{l} \leq x_{il}^{u} \) for \( i = 1, ..., m \) and \( x_{il}^{l} < x_{il}^{l} \), we have:

\[ 1 = \sum_{i=1}^{m} \bar{w}_{il}x_{il}^{l} > \sum_{i=1}^{m} \bar{w}_{il}x_{il}^{u} = \beta. \]

It is clear that \( \beta > 0 \). Now, we prove that \( \left( \frac{1}{\beta} \bar{\mu}_{l}, \frac{1}{\beta} \bar{w}_{l}, \frac{1}{\beta} \bar{u}_{l} \right) \) is a feasible solution for model (1b) evaluating \( DMU_{0} \):

\[ \frac{1}{\beta} \left( \sum_{r=1}^{s} \bar{\mu}_{rl}y_{rl}^{l} - \sum_{i=1}^{m} \bar{w}_{il}x_{il}^{l} + \bar{u}_{il}^{0} \right) \leq 0, \quad j = 1, ..., n. \]

\[ \frac{1}{\beta} \left( \sum_{i=1}^{m} \bar{w}_{il}x_{il}^{u} \right) = 1 \]

\[ \frac{1}{\beta} (\bar{\mu}_{rl}) \geq \frac{1}{\beta} \varepsilon \geq \varepsilon, \quad r = 1, ..., s, \]

\[ \frac{1}{\beta} (\bar{w}_{il}) \geq \frac{1}{\beta} \varepsilon \geq \varepsilon, \quad i = 1, ..., m. \]

Hence, \( E_{0}^{l} \geq E_{l}^{u} \). This means that \( \max\{E_{0}^{l}, E_{0}^{u}\} = \max\{E_{l}^{l}, E_{l}^{u}\} \).

If \( \max\{E_{0}^{l}, E_{0}^{u}\} > \max\{E_{l}^{l}, E_{l}^{u}\} \) then regarding the lexicographic order, it is clear that \( DMU_{0} \) has a better rank than \( DMU_{l} \). Otherwise, we should compare the second components.
of $V_o$ and $V_I$. Suppose that $(\tilde{\mu}_i, \tilde{\omega}_i, \tilde{\alpha}_o)$ is an optimal solution for model (2b) evaluating $DMU_l$. Hence, with a similar argument, we can prove that $E_o^L \geq E_I^L$. If $E_o^U > E_I^U$ then considering the lexicographic order it is clear that $DMU_o$ obtains a better rank than $DMU_I$.

Otherwise, suppose that $(\lambda^{o*}, s^{-o*}, s^{+o*})$ is an optimal solution for model (12a) evaluating $DMU_o$. Hence, we have:

\[
Z_o^* = \sum_{i=1}^{m} w_i^o s_i^{o*} + \sum_{r=1}^{s} w_r^o s_r^{+o*} \leq \sum_{j=1}^{n} \lambda_j^{o*} x_{ij}^L + s_i^{o*} = x_{io}^L \quad \forall i, i \neq k, \\
\sum_{j=1}^{n} \lambda_j^{o*} x_{kj}^L + s_k^{o*} = x_{ko}^L < x_{kb}^L, \\
\sum_{j=1}^{n} \lambda_j^{o*} y_{rj}^L - s_r^{+o*} = y_{ro}^U \geq y_{rt}^U \quad \forall r, \\
\sum_{j=1}^{n} \lambda_j^{o*} = 1, \\
\lambda_j^{j*} \geq 0, \quad \forall j, \\
 s_r^{+o*} \geq 0, \quad \forall r, \\
 s_i^{-o*} \geq 0 \quad \forall i.
\]

Now define:

\[
\tilde{s}^-_i = x_{it}^L - \sum_{j=1}^{n} \lambda_j^{o*} x_{ij}^L - s_i^{o*} = x_{it}^L - x_{ib}^L \geq 0, \quad i = 1, \ldots, m, i \neq k, \\
\tilde{s}^-_k = x_{kt}^L - \sum_{j=1}^{n} \lambda_j^{o*} x_{kj}^L - s_k^{o*} = x_{kt}^L - x_{ko}^L > 0, \\
\tilde{s}^+_r = \sum_{j=1}^{n} \lambda_j^{o*} y_{rj}^L - s_r^{+o*} - y_{ro}^U = y_{ro}^U - y_{rt}^U \geq 0, \quad r = 1, \ldots, s.
\]

Therefore, $(\lambda^{o*}, s^{-o*} + \tilde{s}^-, s^{+o*} + \tilde{s}^+, \tilde{s}^-)$ is a feasible solution for model (12a) evaluating $DMU_l$. Since, $s_k^{o*} + \tilde{s}_k > s_k^{-o*}$ we have:

\[
Z_l^* \geq \sum_{i=1}^{m} w_i^o (s_i^{o*} + \tilde{s}_i^-) + \sum_{r=1}^{s} w_r^o (s_r^{+o*} + \tilde{s}_r^+) > \sum_{i=1}^{m} w_i^o s_i^{o*} + \sum_{r=1}^{s} w_r^o s_r^{+o*} = Z_o^*.
\]

Similarly, we can prove that $Z_I^* \geq Z_o^*$. Therefore, $\frac{Z_o^* + Z_o^*}{2} < \frac{Z_l^* + Z_l^*}{2}$. Therefore, $V_o \geq_{lex} V_I$.

Next theorem provides the main property of our ranking method.

**Theorem 4.** The rank of DMUs belonging to $E^{++}$ is better than the rank of DMUs in $E^+$ and the rank of DMUs belonging to $E^+$ is better than the units in $E^-$.  

**Proof:** Suppose that $a, b, c \in \{1, \ldots, n\}$, and let $a \in E^{++}, b \in E^+, c \in E^-$. According to definition of $E^{++}$ and $E^+$, it is clear that:

\[
E_o^L > 1, \quad E_o^U = 1 \quad \text{yields} \quad \max(E_o^L, E_o^U) = E_o^L > 1 \quad (14) \\
E_I^L \leq 1, \quad E_I^U = 1 \quad \text{yields} \quad \max(E_I^L, E_I^U) = E_I^U = 1 \quad (15)
\]
From (14) and (15), we can conclude that:
\[
\max\{E_{1o}^L, E_{oo}^U\} > \max\{E_{1o}^L, E_{oo}^U\}.
\]

So, DMU₀ has a better rank than DMU₁.

\[
E_{1}^L \leq 1, E_{1}^U = 1 \implies \max\{E_{1}^L, E_{1}^U\} = E_{1}^U = 1
\]

\[
E_{e}^L < 1, E_{ee}^U \leq 1 \implies \max\{E_{e}^L, E_{ee}^U\} \leq 1.
\]

If \(\max\{E_{1}^L, E_{1}^U\} > \max\{E_{e}^L, E_{ee}^U\}\), then \(V_1 \succeq_{lex} V_e\). Otherwise, since \(l \in E^+\) and \(e \in E^-\) therefore, \(E_{1}^U > 1\) and \(E_{e}^U \leq 1\), and we have \(E_{1}^U > E_{e}^U\). Hence, \(V_1 \succeq_{lex} V_e\). \(\square\)

In the next section, we provide two numerical example to illustrate our ranking method.

### 4 Numerical example

**Example 2.** Consider the data of five DMUs reported in Example 1. As we see in Example 1, Table 5 reports the data units, the lower efficiency score \((E_{1o}^L)\), the upper efficiency score \((E_{1o}^U)\), the lower super efficiency score \((E_{oo}^U)\) and the upper super efficiency score \((E_{oo}^U)\) of DMUs. Now, we apply our ranking method for the data in this example. So, we should solve models (12a), (12b), (13a) and (13b) and obtain the optimal objective values \(Z_o, Z_o^*, W_o^*\) and \(W_o^-\) to measure the distance of each unit from the best efficiency frontier, the worst efficiency frontier, the best inefficiency frontier and the worst inefficiency frontier, respectively. The results are summarized in Table 6.

Now, we should assign a 4-vector \(V_o = \left(\max\{E_{1o}^L, E_{oo}^U\}, E_{1o}^U, -\frac{Z_o^2 + Z_o^*}{2}, \frac{W_o^* + W_o^-}{2}\right)\), reported in Table 7, to \(DMU_o, o \in \{1, \ldots, n\}\). Finally, we rank the vectors \(V_1, \ldots, V_n\) by lexicographic order. The first component of \(V_1, V_2\) and \(V_3\) are the same and greater than the first component of \(V_4\) and \(V_5\), hence, we should compare the second component of \(V_4, V_5\), and \(V_6\) to determine the rank of \(A, B\) and \(C\). As we can see, the second component of \(V_1, V_2\) and \(V_3\) are 3, 1.17 and 0.83, respectively. Therefore, units \(A, B\) and \(C\) have the ranks 1, 2 and 3, respectively. Then, we must determine the rank of \(D\) and \(E\). The first component of \(V_D\) and \(V_E\) are 0.20 and 0.25, respectively. Hence, units \(D\) and \(E\) have the ranks 5 and 4, respectively. The last column of Table 7 reports the obtained rank of units by our ranking method. Our method ranks all efficient and inefficient units. As we can see in Table 1 and Table 7, \(A, B \in E^+\) and the rank of them is better than the rank of each unit \(C, D, E \in E^-\).

**Example 3.** In this example, the results of applying our proposed approach to the dataset in Jahanshahloo et al. (2011) are presented. This dataset has 30 decision making units which are branches of Tehran social security insurance organization with three inputs. The number of personal (I₁), the total number of computers (I₂), the area of the branch (I₃) in order to produce four outputs, the total number of insured persons (O₁), the number of insurance policies (O₂), the total number of old age pensioners (O₃) and the received total sum (Income) (O₄). The input/output data are reported in Table 8. We apply our ranking method the dataset in this example. So, we should solve models (1b), (2a) and (2b) to obtain the upper efficiency score \((E_{oo}^U)\), the lower super efficiency score \((E_{oo}^U)\) and the upper super efficiency score \((E_{oo}^U)\) for \(DMU_o, o \in \{1, \ldots, n\}\) and then, the results are summarized in columns 2, 4 and 5 of Table 9, respectively. In Table 9, column 3 represent the efficiency status of all units according to Definition 1, column 6 shows the category that each unit belongs to it and column 7 specify the extreme efficient units according to Definition 2. Then, we solve models (12a), (12b), (13a) and (13b) and obtain \(Z_o^*, Z_o, W_o^*\) and \(W_o^-\) to measure the distance of each unit from the efficiency frontiers and inefficiency frontiers, the results are reported in columns 8, 9, 10 and 11 of Table 9, respectively. Then, we assign a 4-vector \(V_o = \left(\max\{E_{1o}^L, E_{oo}^U\}, E_{1o}^U, -\frac{Z_o^2 + Z_o}{2}, \frac{W_o^* + W_o^-}{2}\right)\), reported in the second column of Table 10, to each unit \(DMU_o\). Finally, we rank the assigned vectors to
The obtained rank by our proposed method and the method of Jahanshahloo et al. (2011) are shown in columns 3 and 4 of Table 10. The Spearman’s rank order correlation between our proposed method and the method of Jahanshahloo et al. (2011) is 0.76. It can be seen that our method and the method of Jahanshahloo et al. (2011) have a relatively high correlation at least in this instance.

5 Conclusions and further research

In many real world situations, the inputs and/or outputs of decision making units can be given as imprecise data. One of the attractive issues in IDEA is to rank the units. This paper addressed the problem of ranking DMUs with interval data which is a special case of uncertainty in data. The contribution of this study is to develop a powerful method for ranking DMUs with interval data as our proposed approach has all desirable features expected for ranking methods. We extended some concepts in traditional DEA such as super efficiency, extreme efficient unit and dominated units to Interval DEA and then proposed an original approach to rank all units with interval data. Our proposed method was based on four preferences: the maximum value for the lower super efficiency score and the upper efficiency score, the minimum value for the average of distances of unit from the best and the worst efficiency frontiers and the maximum value for the average of distances of unit from the best and the worst inefficiency frontiers. Then, we assigned a 4-vector to each unit by regarding these preferences. Finally, the rank of DMUs obtained by comparing the assigned vectors with the lexicographic order. Our method ranks all efficient and inefficient units that is one of the main advantages of it. Also it uses several essential criteria simultaneously to rank units with interval data which these criteria increase the discrimination power of our proposed method and this is another advantage of our method. We proved that our proposed method has all desirable features that are expected for a ranking method.

The idea of this paper can be extended for ranking DMUs with interval data by using another method such as TOPSIS instead of lexicography method.

References


Figure 1. The PPS for five DMUs in Example 1.

Figure 2. $L^B_C$ and $L^W_C$ for units in example 1.
Figure 3. The efficiency and inefficiency frontiers.

Figure 4. The values of $E_o^U$, $E_o^L$ and $E_o^U$ for all units.
Figure 5. Heatmap graph of the distances of each unit from the frontiers.
Table 5. The data and obtained results for five DMUs in Example 1.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Input</th>
<th>Output</th>
<th>$E^L_{o0}$</th>
<th>$E^U_{o0}$</th>
<th>Efficient</th>
<th>$E^L_o$</th>
<th>$E^U_o$</th>
<th>Extreme efficient unit</th>
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<td>3.00</td>
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<tr>
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<td>[4, 6]</td>
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<td>1.17</td>
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<td>[6, 8]</td>
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Table 6. The distances of each unit from the efficiency and inefficiency frontiers.

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Table 7. The obtained rank of units by our proposed method.

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<tr>
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<tr>
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<tr>
<td>E</td>
<td>(0.25, 0.25, -3.00, 0.00)</td>
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Table 8. The inputs and outputs for 30 branches of the insurance organization.

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715
Table 9. The results for 30 branches of the insurance organization.

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<th>$E_{o}^U$</th>
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Table 10. The rank of units by our method and the method of Jahanshahloo et al. (2011).

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<th>Rank (Our method)</th>
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