A complete ranking of decision making units with interval data

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Abstract: Interval Data Envelopment Analysis (Interval DEA) deals with the problem of efficiency assessment when the inputs and/or outputs of Decision Making Units (DMUs) are given as interval data. This paper focuses on the problem of ranking DMUs with interval data. First, we define extreme efficient units, super efficiency score, the best and the worst efficiency (inefficiency) frontiers in the interval DEA context. Then, we propose a novel method based on the lower and upper super efficiency scores of a unit under evaluation and the distance of that unit to four developed frontiers. Our method ranks all efficient and inefficient units which is one of the main advantages of it. Our method uses several essential criteria simultaneously to rank units with interval data. These criteria increase the discrimination power of our proposed method. Potential application of this method is illustrated with a dataset consisting of 30 branches of the social security insurance organization in Tehran.

Keywords: Date Envelopment Analysis, Interval DEA, Decision Making Unit, Ranking.

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1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric methodology for assessing the relative efficiency of Decision Making Units (DMUs) with multiple inputs and multiple outputs (Charnes et al. (1978), Banker et al. (1984), Färe et al. (1985), Zhu (2002), Cooper et al. (2006)). It assigns an efficiency measure between 0 and 1 to each unit. The larger the efficiency score, the better performance the unit under evaluation has. A DMU is efficient if its efficiency score is equal to 1, otherwise it is inefficient. The original DEA models consider the situation that all inputs and outputs have certain values. However, this assumption can be violated due to the existence of uncertainty in data. The problem of the evaluation of units with imprecise data has attracted attentions of several scholars. For example, Cooper et al. (1999) developed Imprecise Data Envelopment Analysis (IDEA) method. Their method can be applied in the situation where there exist both imprecisely and exactly-known data in which the IDEA models are transformed into linear programming problems. Kim et al. (1999) proposed a procedure to incorporate partial data into DEA. Their original model was a complicated non-linear model that was transformed into a linear programming problem by applying a linear scale transformation and the variable change technique.

Lee et al. (2002) proposed methods to determine the inefficiency of units such as slacks, returns to scale and so on in IDEA. These information helps the Decision Maker (DM) to improve the efficiency of units. Despotits and Smirlis (2002) proposed an approach to define the upper and lower bounds for the efficiency score of units with imprecise data. Their idea shows that the units with imprecise data do not have constant efficiency scores and their efficiency scores depend on the choice of data. Therefore, one of the attractive issues in IDEA is to determine the upper and lower bounds for units (See Cooper et (2001), Entani et al. (2002), Zhu (2003), Jahanshahloo et al. (2004), Wang et al. (2005), Amirteimoori and Kordrostami (2005), Smirlis (2006), Park (2007), Toloo et al. (2008), Park (2010), Kao and Liu (2011), Esmaeili (2012), Hatami Marbini et al. (2014), Sun et al. (2014), and Khalili Damghani et al. (2015) for more studies about IDEA models). Kordrostami and Jahani Sayyad Noveiri (2014) proposed a method to estimate the optimistic and pessimistic efficiency scores of units with fuzzy data and then integrated them into a geometric average efficiency.

Emrouznejad and Yang (2016) proposed a performance index based on efficient and anti-efficient frontiers in DEA models without explicit inputs (DEA-WEI) and developed the corresponding performance index in quadratic DEA-WEI models. Piri et al. (2016) proposed a method to evaluate the efficiency scores of DMUs with interval data in which the lower and upper bounds of intervals can take both negative and positive values. Amirteimoori et al. (2017) suggested an approach to integrate the optimistic and pessimistic perspectives to obtain the interval efficiency scores of units with interval data. Azizi et al. (2017) proposed a method to obtain the upper and lower bounds for the efficiency scores of units with imprecise data when some input and/or output can be specified as intervals, and some of them can be given as exact values and the other can be determined as ordinal preferences information. Jiang et al. (2018) developed a DEA model to measure the scale efficiency of DMUs with imprecise data and analyzed the sensitivity and stability of their model for scale efficiency. Toloo et al. (2018) developed a methodology to handle uncertain inputs, outputs and dual role factors and proposed models to obtain the interval efficiency scores of units based on the optimistic and pessimistic viewpoints and then suggested an integrated model to identify a unique status of each imprecise dual factors.

There are another issues that examine uncertainty. In practice, temporal representation is an attractive problem in a wide range of fields, such as computer science, philosophy, psychology and so on. For instance, information system deals with the problem of outdated data. In order to consider questions such as ‘which employees worked for us last year and made over 15000$’ we need to represent temporal information. See F.Allen (1983) for more studies about temporal representation. S Ganapathy et al. (2013) combined temporal features with the fuzzy min-max neural network that is based on a classifier to select the effective decision in medical diagnosis. See Laxman and sastry (2006), Zhang et al. (2009), wai and Lee (2008), Simson (1992)) for more studies about temporal data mining, classification, fuzzy min-max, neural network.

The traditional DEA models cannot discriminate among the efficient units because they get identical efficiency scores equal to one. In this regard, several ranking approaches have been developed in the DEA literature. For a review on ranking methods in DEA see Adler et al. (2002). One of the attractive topics in IDEA is to rank units. Jahanshahloo et al. (2006) extended TOPSIS method in Interval DEA. Wang et al. (2005) considered the efficiency assessment of units in the presence of interval and/or fuzzy data. They proposed two linear CCR models to obtain the interval efficiency of DMUs with interval data and then applied the interval efficiencies of all units by a minimax regret-based approach to rank units. Wu et al. (2013) proposed a two-phases approach in which the first phase obtains the interval cross-efficiency score of DMUs with interval data and the second phase ranks units by applying an improved TOPSIS technique. Khodabakhshi and Aryavash (2015) developed a method for ranking units with stochastic data. Rafiee Sani and Alirezaee (2017) developed some fuzzy versions of trade-off DEA models by applying some ranking methods based on the comparison of $\alpha -$cuts. Shavazipour et al. (2017) proposed an approach to rank extreme efficient units with fuzzy data. Their model is based on the Tchebycheff norm. Ebrahimi (2019) considered DEA with stochastic data and applied the expected efficiency of units to present a method for ranking DMUs.

Given the importance of ranking the units in DEA, in particular in Interval DEA, we focus on the ranking of DMUs with interval data and propose a novel approach to rank units. In this study, one of our main motivations is to extend some concepts from the traditional DEA into interval DEA and the other is to develop a powerful method to discriminate and rank DMUs with interval data, hence, we propose an approach using several essential criteria simultaneously to rank units with interval data. These criteria increase the discrimination power of our method.

The contribution of this study is to develop a powerful method for ranking DMUs with interval data as our proposed approach has all desirable features expected for ranking methods. First, we suggest two linear programming models to compute the lower and upper super efficiency scores of a unit following the method of Anderson and Peterson (1993). As in the traditional super efficiency score, it is desirable to obtain the large lower and upper super efficiency scores in interval DEA. Then, we define the terms extreme efficient units and non-dominated DMUs in Interval DEA for the first time. Secondly, we introduce two efficiency frontiers, namely the best and the worst efficiency frontiers, and two inefficiency frontiers, called the best and the worst inefficiency frontiers and after that we formulate four linear programming problems to measure the distance of each unit from these frontiers. It is clear that, units closer to efficiency frontiers and more far from the inefficiency frontiers have better performance. Hence, by using these distances and the defined lower and upper super efficiency scores of units, we assign a vector with four components to each unit. Finally, we sort these vectors by a lexicographic order. In our method, as we expect, the rank order of extreme efficient units is better than other DMUs. Also, if a unit dominates another, it gets a higher rank than the dominated unit.

The rest of this paper is organized as follows: section 2 reviews the interval DEA preliminaries. In section 3, we present a complete ranking method for DMUs with interval data. Two numerical examples are provided in section 4. Section 5 concludes the paper.

2 Preliminaries and basic definitions

Consider a system of n DMUs, denoted by $DMU_j, j = 1, \dots, n,$ where each unit consumes m different inputs to generate s different outputs. The i^{th} input and r^{th} output for DMU_j are denoted by x_{ij} and y_{rj}, respectively, for $i = 1, \dots, m$ and $r = 1, \dots, s$. Also, suppose that input and output values are not deterministic for all units and $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$, where the lower and upper bounds are positive and finite values. Assume that DMU_o is the unit under evaluation.

Wang et al. (2005) considered the following production possibility set (PPS) in Interval DEA:

$$T = \{(x, y) |\, x \geq \sum_{j=1}^{n} \lambda_j x_j^L, y \leq \sum_{j=1}^{n} \lambda_j y_j^U, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\}$$

They formulated two linear programming models (1a) and (1b) to measure the lower and upper bounds for the efficiency score of DMU_o as reported in Table 1.

Table 1. The lower and upper efficiency score.

The lower efficiency score	The upper efficiency score
$E_{oo}^L = \max \sum_{r=1}^{s} \mu_r y_{ro}^L + u_0$ s.t. $\sum_{i=1}^{m} w_i x_{io}^U = 1, \quad (1a)$ $\sum_{r=1}^{s} \mu_r y_{rj}^U - \sum_{i=1}^{m} w_i x_{ij}^L + u_0 \leq 0, \quad \forall j,$ $w_i, \mu_r \geq \varepsilon, \quad \forall i, r.$	$E_{oo}^U = \max \sum_{r=1}^{s} \mu_r y_{ro}^U + u_0$ s.t. $\sum_{i=1}^{m} w_i x_{io}^L = 1, \quad (1b)$ $\sum_{r=1}^{s} \mu_r y_{rj}^U - \sum_{i=1}^{m} w_i x_{ij}^L + u_0 \leq 0, \quad \forall j,$ $w_i, \mu_r \geq \varepsilon, \quad \forall i, r.$

where $\varepsilon > 0$ is Non-Archimedean.

The optimal value of models (1a) and (1b) were called the lower and the upper efficiency score of DMU_o by Wang et al. (2005), respectively. It is clear that $E_{oo}^L \leq 1$, $E_{oo}^U \leq 1$ and $E_{oo}^L \leq E_{oo}^U$.

The efficient and inefficient units in Interval DEA were defined as follows by Wang et al. (2005):

Definition 1. The unit $DMU_o = (x_o, y_o)$ is efficient, if $E_{oo}^U = 1$. Otherwise, it is inefficient.

In the next section, we propose an original approach to rank DMUs with interval data.

3 A complete ranking of decision making units with interval data

In this section, we present a new method for ranking units in the presence of interval data based on some concepts that is defined in the following subsections. Subsection 3.1 extends the concept of super efficiency to Interval DEA and formulates two linear programming models to obtain the lower and the upper super efficiency scores of DMUs. Based on the obtained lower and upper super efficiency scores, we classify the set of all units into three subsets and then we present the definition of extreme efficient units in the context of Interval DEA. Subsection 3.2 defines the best-case and worst-case convex hulls and obtains two efficiency and two inefficiency frontiers. Then, four linear programming problems are formulated to measure the distance of each unit from these frontiers. It is clear that, units closer to efficiency frontiers and more far from the inefficiency frontiers are preferred. Finally, subsection 3.3 suggests an approach to rank units based on assigning a vector with four components to each unit and lexicographic order.

3.1 Extending the super efficiency concept to Interval DEA

Anderson and Peterson (1993) introduced the super efficiency concept in traditional DEA. Their approach removes a DMU from the set of the observed units and constructs the new PPS by the remaining units and then formulates a linear programming problem to extract the super efficiency score of that unit and uses the super efficiency scores of units to rank them. See Adler et al. (2002) for more details. In this section, we extend the super efficiency concept to Interval DEA. Based on the idea of Anderson and Peterson (1993), we remove the unit under evaluation from the set of the observed units with interval data and reformulate models (1a) and (1b) by using the new PPS constructed by the remaining units to determine the lower and upper super efficiency scores of units with interval data. The models are reported in Table 2.

Table 2. The lower and upper super efficiency scores of units with interval data.

The lower super efficiency score	The upper super efficiency score
$E_o^L = \max \sum_{r=1}^{s} \mu_r y_{ro}^L + u_0$ s.t. $\sum_{i=1}^{m} w_i x_{io}^U = 1, \qquad (2a)$ $\sum_{r=1}^{s} \mu_r y_{rj}^U - \sum_{i=1}^{m} w_i x_{ij}^L + u_0 \le 0, \quad j \ne o,$ $w_i, \mu_r \ge \varepsilon, \qquad \forall i, r.$	$E_o^U = \max \sum_{r=1}^{s} \mu_r y_{ro}^U + u_0$ s.t. $\sum_{i=1}^{m} w_i x_{io}^L = 1, \qquad (2b)$ $\sum_{r=1}^{s} \mu_r y_{rj}^U - \sum_{i=1}^{m} w_i x_{ij}^L + u_0 \le 0, \quad j \ne o,$ $w_i, \mu_r \ge \varepsilon, \qquad \forall i, r.$

It is clear that $E_o^L \le E_o^U$. On the other hand, $E_{oo}^L \le E_o^L$ because any optimal solution of model (1a) is a feasible solution for model (2a). Also, any optimal solution of model (1b) is a feasible solution of model (2b), therefore, $E_{oo}^U \le E_o^U$. Regarding the obtained lower and upper super efficiency scores, all units can be classified into the following three subsets:

$$E^{++} = \{ j \in \{1, \dots, n\} \mid E_j^L > 1\} \qquad (3)$$
$$E^{+} = \{ j \in \{1, \dots, n\} \mid E_j^L \le 1, E_j^U > 1\} \qquad (4)$$
$$E^{-} = \{ j \in \{1, \dots, n\} \mid E_j^L < 1, E_j^U \le 1\} \qquad (5)$$

E^{++} includes all decision making units that their lower super efficiency score and as a result, their upper super efficiency score are greater than one. E^{+} includes all DMUs that their lower super efficiency score is less than or equal to one and their upper super efficiency score is greater than one. E^- includes all units that their lower super efficiency score is less than one and their upper super efficiency score is less than or equal to one.

Next theorem provides a sufficient condition for efficiency of a unit with interval data.

Theorem 1. *The unit $DMU_o = (x_o, y_o)$ is efficient, if $o \in E^+$ or $o \in E^{++}$.*

Proof : If $o \in E^+$ or $o \in E^{++}$ then $E_o^U > 1$. We claim that, if $o \in E^+$ or $o \in E^{++}$ then $E_{oo}^U = 1$. By contradiction, let $E_{oo}^U < 1$. Assume that $(\hat{\mu}_o, \widehat{w}_o, \hat{u}_0)$ is an optimal solution of model (1b), therefore, we have:

$$E_{oo}^U = \sum_{r=1}^{s} \hat{\mu}_{ro} y_{ro}^U + \hat{u}_0 < 1.$$

Also, suppose that (μ_o^*, w_o^*, u_0^*) is an optimal solution of model (2b). we have:

$$\begin{aligned}
&\sum_{r=1}^{s} \mu_{ro}^* y_{rj}^U - \sum_{i=1}^{m} w_{io}^* x_{ij}^L + u_0^* \le 0, \quad j = 1, \dots, n, j \neq o \\
&\sum_{i=1}^{m} w_{io}^* x_{io}^L = 1, \\
&E_o^U = \sum_{r=1}^{s} \mu_{ro}^* y_{ro}^U + u_0^* > 1.
\end{aligned}$$

Regarding that u_0 is a free variable in model (2b), we define:

$$\begin{aligned}
&\bar{\mu}_{ro} = \mu_{ro}^*, && r = 1, \dots, s, \\
&\overline{w}_{io} = w_{io}^*, && i = 1, \dots, m, \\
&\bar{u}_0 = u_0^* - (E_o^U - 1) < u_0^*.
\end{aligned}$$

Therefore, $(\bar{\mu}, \overline{w}, \bar{u}_0)$ is a feasible solution of model (1b), because:

$$\begin{aligned}
&\sum_{r=1}^{s} \bar{\mu}_{ro} y_{rj}^U - \sum_{i=1}^{m} \overline{w}_{io} x_{ij}^L + \bar{u}_0 < \sum_{r=1}^{s} \mu_{ro}^* y_{rj}^U - \sum_{i=1}^{m} w_{io}^* x_{ij}^L + u_0^* \le 0, \qquad j \neq o, \\
&\sum_{r=1}^{s} \bar{\mu}_{ro} y_{ro}^U - \sum_{i=1}^{m} \overline{w}_{io} x_{io}^L + \bar{u}_0 = \sum_{r=1}^{s} \mu_{ro}^* y_{ro}^U - \sum_{i=1}^{m} w_{io}^* x_{io}^L + u_0^* - (E_o^U - 1) = \\
&E_o^U - 1 - (E_o^U - 1) = 0, \\
&\sum_{i=1}^{m} \overline{w}_{io} x_{io}^L = \sum_{i=1}^{m} w_{io}^* x_{io}^L = 1.
\end{aligned}$$

The value of the objective function of model (1b) for this feasible solution is:

$$\sum_{r=1}^{s} \bar{\mu}_{ro} y_{ro}^U + \bar{u}_0 = \sum_{r=1}^{s} \mu_{ro}^* y_{ro}^U + u_0^* - (E_o^U - 1) = E_o^U - (E_o^U - 1) = 1.$$

which is a contradiction with the optimality of $(\hat{\mu}_o, \widehat{w}_o, \hat{u}_0)$ for model (1b). thus, $E_{oo}^U = 1$ and DMU_o is efficient according to Definition 1. □

In traditional DEA, DMU_O is an extreme efficient unit if its super efficiency score is greater than one. We define the extreme efficient unit in Interval DEA following the definition of extreme efficient unit in traditional DEA.

Definition 2. The unit $DMU_o = (x_o, y_o)$ is an extreme efficient unit if at least one of its lower super efficiency score or its upper super efficiency score be greater than one. In the other word, DMU_o is an extreme efficient unit in Interval DEA if $o \in E^{+}$ or $o \in E^{++}$.

In the following example, we determine the extreme efficient units in a numerical example with five units in the presence of interval data.

Example 1. Consider five decision making units with interval data. Each DMU consumes one input to produce one output. The second and third columns of Table 5 reports the data and Figure 1 shows the PPS. Columns 4, 5 and 6 of Table 5 show the lower efficiency score, the upper efficiency score and the status of efficiency of units, respectively. As we see, the upper efficiency score of units A, B and C are equal to 1 and hence they are efficient. The upper efficiency score of unit D and E are less than 1 and hence they are inefficient. We solve models (2a) and (2b) to obtain the lower super efficiency and upper super efficiency scores of DMUs and then classify all units into E^{++}, E^{+} and E^{-}. The results are summarized in columns 7, 8 and 9 of Table 5. The last column of Table 5 shows that each unit is extreme efficient or not. Note that $A, B \in E^{+}$, according to Definition 2, they are extreme efficient units while C is an efficient unit belongs to E^{-}, units D and E are inefficient and belong to E^{-}. Therefore, according to Definition 2, units C, D and E are not extreme efficient units.

3.2 Efficient and inefficient frontiers

This section considers two convex hulls of DMUs, namely the best-case (L^{BC}) and worst-case (L^{WC}) convex hulls. L^{BC} is made by the points that represent the best mode of units, similarly, L^{WC} is made by the points that represent the worst mode of units.

$$L^{BC} = \{(x, y) \mid x = \sum_{j=1}^{n} \lambda_j x_j^L , y = \sum_{j=1}^{n} \lambda_j y_j^U , \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\} \quad (6)$$

$$L^{WC} = \{(x, y) \mid x = \sum_{j=1}^{n} \lambda_j x_j^U , y = \sum_{j=1}^{n} \lambda_j y_j^L , \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\} \quad (7)$$

Figure 2 shows L^{BC} and L^{WC} for five DMUs reported in Example 1. Thick lines specify L^{BC} and dashed lines specify L^{WC}. Shadowed region is the subscription of L^{BC} and L^{WC}.

In the following, we define two efficiency and two inefficiency frontiers by considering the frontiers of L^{BC} and L^{WC}.

We consider two sets T_1^{BC} and T_2^{BC} constructed by the frontier of L^{BC} as follows:

$$T_1^{BC} = \{(x, y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j^L , y \leq \sum_{j=1}^{n} \lambda_j y_j^U , \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\} \quad (8)$$

$$T_2^{BC} = \{(x, y) \mid x \leq \sum_{j=1}^{n} \lambda_j x_j^L , y \geq \sum_{j=1}^{n} \lambda_j y_j^U , \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\} \quad (9)$$

And then, we define the best efficiency and the best inefficiency frontiers, namely ∂T_1^{BC} and ∂T_2^{BC}, as the frontiers of T_1^{BC} and T_2^{BC}, respectively. Note that, ∂T_1^{BC} and ∂T_2^{BC} are named as the best efficiency and the best inefficiency frontiers because they are made by the best mode of all units.

Similarly, we consider two sets T_1^{WC} and T_2^{WC} made by the frontier of L^{WC} as follows:

$$T_1^{WC} = \{(x,y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j^U, y \leq \sum_{j=1}^{n} \lambda_j y_j^L, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\} \quad (10)$$

$$T_2^{WC} = \{(x,y) \mid x \leq \sum_{j=1}^{n} \lambda_j x_j^U, y \geq \sum_{j=1}^{n} \lambda_j y_j^L, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\} \quad (11)$$

And then, we define the worst efficiency and the worst inefficiency frontiers, namely ∂T_1^{WC} and ∂T_2^{WC}, as the frontiers of T_1^{WC} and T_2^{WC}, respectively. Note that, ∂T_1^{WC} and ∂T_2^{WC} are named as the worst efficiency and the worst inefficiency frontiers because they are made by the worst mode of all units.

Figure 3 illustrates the efficiency and inefficiency frontiers for decision making units reported in Example 1. Top thick lines show the best efficiency frontiers, lower thick lines show the worst efficiency frontier, top dashed lines show the best inefficiency frontier and lower dashed lines show the worst inefficiency frontier.

After defining the frontiers $\partial T_1^{BC}, \partial T_2^{BC}, \partial T_1^{WC}$ and ∂T_2^{WC}, one of the attractive issues is to measure the distance of each unit from them. Therefore, we formulate models (12a) and (12b), reported in Table 3, to determine the minimum distance of each DMU from the best and the worst efficiency frontiers.

Table 3. The distance of DMU_o from the efficiency frontiers.

The distance from the best efficiency frontier		The distance from the worst efficiency frontier	
$Z_o^* = \max \sum_{i=1}^{m} w_i^- s_i^- + \sum_{r=1}^{s} w_r^+ s_r^+$		$Z_o^- = \max \sum_{i=1}^{m} w_i^- s_i^- + \sum_{r=1}^{s} w_r^+ s_r^+$	
s.t.	$(12a)$	s.t.	$(12b)$
$\sum_{j=1}^{n} \lambda_j x_{ij}^L + s_i^- = x_{io}^L,$	$\forall i,$	$\sum_{j=1}^{n} \lambda_j x_{ij}^U + s_i^- = x_{io}^U,$	$\forall i,$
$\sum_{j=1}^{n} \lambda_j y_{rj}^U - s_r^+ = y_{ro}^U,$	$\forall r,$	$\sum_{j=1}^{n} \lambda_j y_{rj}^L - s_r^+ = y_{ro}^L,$	$\forall r,$
$\sum_{j=1}^{n} \lambda_j = 1,$		$\sum_{j=1}^{n} \lambda_j = 1,$	
$\lambda_j \geq 0,$	$\forall j,$	$\lambda_j \geq 0,$	$\forall j,$
$s_r^+ \geq 0,$	$\forall r,$	$s_r^+ \geq 0,$	$\forall r,$
$s_i^- \geq 0,$	$\forall i.$	$s_i^- \geq 0,$	$\forall i.$

The first and second constraints of model (12a) is made by adding the slacks s_i^- and s_r^+ for all $i = 1, \dots, m$ and $r = 1, \dots, s$, to the inequalities $x_i \geq \sum_{j=1}^{n} \lambda_j x_{ij}^L, y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}^U$ in T_1^{BC}. Regarding that, s_i^- $(i = 1, \dots, m)$ represents the distance of x_{io}^L $(i = 1, \dots, m)$ from the input of a point on the best efficient frontier and s_r^+ $(r = 1, \dots, s)$ represents the distance of y_{ro}^U $(r = 1, \dots, s)$ from the output of a point on the best efficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of DMU_o from the best efficient frontier.

The first and second constraints of model (12b) is made by adding the slacks s_i^- and s_r^+ for all $i = 1, \dots, m$ and $r = 1, \dots, s$, to the inequalities $x_i \geq \sum_{j=1}^{n} \lambda_j x_{ij}^U, y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}^L$ in T_1^{WC}. Regarding that, s_i^- $(i = 1, \dots, m)$ represents the distance of x_{io}^U $(i = 1, \dots, m)$ from the input of a point on the worst efficient frontier and s_r^+ $(r = 1, \dots, s)$ represents the distance of y_{ro}^L $(r = 1, \dots, s)$ from the output of a point on the worst efficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of DMU_o from the worst efficient frontier.

In models (12a) and (12b), w^- and w^+ are given weight vectors by decision maker (DM).

Similarly, models (13a) and (13b), reported in Table 4, determine the minimum distance of each unit from the best and the worst inefficiency frontiers.

Table 4. The distance of DMU_o from the efficiency frontiers.

The distance from the best inefficiency frontier	The distance from the worst inefficiency frontier
$W_o^* = \max \sum_{i=1}^{m} w_i^- s_i^- + \sum_{r=1}^{s} w_r^+ s_r^+$	$Z_o^- = \max \sum_{i=1}^{m} w_i^- s_i^- + \sum_{r=1}^{s} w_r^+ s_r^+$
s.t. $\quad (13a)$	s.t. $\quad (13b)$
$\sum_{j=1}^{n} \lambda_j x_{ij}^L - s_i^- = x_{io}^L, \quad \forall i,$	$\sum_{j=1}^{n} \lambda_j x_{ij}^U - s_i^- = x_{io}^U, \quad \forall i,$
$\sum_{j=1}^{n} \lambda_j y_{rj}^U + s_r^+ = y_{ro}^U, \quad \forall r,$	$\sum_{j=1}^{n} \lambda_j y_{rj}^L + s_r^+ = y_{ro}^L, \quad \forall r,$
$\sum_{j=1}^{n} \lambda_j = 1,$	$\sum_{j=1}^{n} \lambda_j = 1,$
$\lambda_j \geq 0, \quad \forall j,$	$\lambda_j \geq 0, \quad \forall j,$
$s_r^+ \geq 0, \quad \forall r,$	$s_r^+ \geq 0, \quad \forall r,$
$s_i^- \geq 0, \quad \forall i.$	$s_i^- \geq 0, \quad \forall i.$

The first and second constraints of model (13a) is made by adding the slacks s_i^- and s_r^+ for all $i = 1, \ldots, m$ and $r = 1, \ldots, s$, to the inequalities $x_i \leq \sum_{j=1}^{n} \lambda_j x_{ij}^L, y_r \geq \sum_{j=1}^{n} \lambda_j y_{rj}^U$ in T_2^{BC}. Regarding that, s_i^- $(i = 1, \ldots, m)$ represents the distance of x_{io}^L $(i = 1, \ldots, m)$ from the input of a point on the best inefficient frontier and s_r^+ $(r = 1, \ldots, s)$ represents the distance of y_{ro}^U $(r = 1, \ldots, s)$ from the output of a point on the best inefficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of DMU_o from the best inefficient frontier.

The first and second constraints of model (13b) is made by adding the slacks s_i^- and s_r^+ for all $i = 1, \ldots, m$ and $r = 1, \ldots, s$, to the inequalities $x_i \leq \sum_{j=1}^{n} \lambda_j x_{ij}^U, y_r \geq \sum_{j=1}^{n} \lambda_j y_{rj}^L$ in T_2^{WC}. Regarding that, s_i^- $(i = 1, \ldots, m)$ represents the distance of x_{io}^U $(i = 1, \ldots, m)$ from the input of a point on the worst inefficient frontier and s_r^+ $(r = 1, \ldots, s)$ represents the distance of y_{ro}^L $(r = 1, \ldots, s)$ from the output of a point on the worst inefficient frontier, hence, we maximize the sum of the slacks of inputs and outputs to determine the distance of DMU_o from the worst inefficient frontier.
In models (13a) and (13b), w^- and w^+ are given weight vectors by DM.

Theorem 2 proves that all above four models are feasible and bounded.

Theorem 2. *Models (12a), (12b), (13a) and (13b) are feasible and bounded.*
Proof: Clearly,

$$\begin{aligned} &\lambda_o = 1, \\ &\lambda_j = 0, \quad j = 1, \ldots, n, \quad j \neq o, \\ &s_i^- = 0, \quad i = 1, \ldots, m, \\ &s_r^+ = 0, \quad r = 1, \ldots, s. \end{aligned}$$

is a feasible solution for model (12a). From the constraints of model (12a) we have:

$$s_i^- = x_{io}^L - \sum_{j=1}^{n} \lambda_j x_{ij}^L \leq x_{io}^L, \qquad i = 1, \dots, m,$$

$$s_r^+ = \sum_{j=1}^{n} \lambda_j y_{rj}^U - y_{ro}^U \leq \sum_{j=1}^{n} \lambda_j y_{rj}^U \leq M_r, \quad r = 1, \dots, s.$$

where $M_r = \max_{1 \leq j \leq n} y_{rj}^U$. Since $x_{ij}^L, x_{ij}^U, y_{rj}^L$ and y_{rj}^U are finite for all i, r, j, therefore, model (12a) is bounded. Similarly, models (12b), (13a) and (13b) are also feasible and bounded.

□

In the following, we suggest a method for ranking DMUs with interval data, applying the lexicographic order defined as follows:

Definition 3. (Ehrgott (2005)) Let $y^1, y^2 \in R^p (p \geq 2)$ and $k^* = \min\{k \mid y_k^1 \neq y_k^2\}$. If $y_{k^*}^1 > y_{k^*}^2$ or $y^1 = y^2$, then $y^1 \geq_{lex} y^2$.

3.3 our proposed ranking method for Interval DEA

In this section, we propose a method for ranking DMUs with interval data. This method assigns a 4-vector, namely V_o, to each unit DMU_o, for $o \in \{1, \dots, n\}$, and then compare these vectors by lexicographic order. In the following, we describe how each component of vector V_o is selected. Each ranking method in DEA is expected to have the feature that the rank of an efficient unit should be better than the rank of an inefficient unit. Therefore, we consider the upper efficiency score of DMUs as the first priority. According to Theorem 1, the upper efficiency score for all units in E^{++}, for all DMUs in E^+ and for some decision making units in E^- is equal to 1. Therefore, the upper efficiency score alone cannot distinguish among them. On the other hand, we consider another priority as the rank of each unit in E^{++} should be better than the rank of each DMU in E^+. Regarding that E_o^L plays an essential role of creating the distinction between E^{++} and E^+, hence, we consider the lower super efficiency score of DMU_o as another priority with upper efficiency score, simultaneously. Therefore, we define the first component of $V_o, o \in \{1, \dots, n\}$, as the maximum of the lower super efficiency score and the upper efficiency score of DMU_o. In the other word, the first component of V_o is defined as $\max\{E_O^L, E_{oo}^U\}$ implying that a unit with a higher E_{oo}^U and E_O^L for $o \in \{1, \dots, n\}$, gets a better rank.

After defining the first component of V_o, we describe how to define the second component of V_o. We consider the next priority as the rank of all units in E^+ should be better than the rank of all units in E^-. Note that, the selection of the first component of V_o as described guarantees that the rank of each unit belongs to E^{++} is better than the rank of each DMU in E^+, but it cannot guarantee that the rank of each decision making unit in E^+ is better than the rank of each DMU in E^-. Regarding that, E_o^U plays an essential role of creating the distinction between E^+ and E^-, therefore, we define the second component of V_o as E_o^U. It should be noted that, maybe there exist units that have the same values for the first and second components of their assigned vector. So, we need to define other components to make more distinction between units. As we know, the units closer to efficiency frontiers and more far from inefficiency frontiers are preferred. Hence, we consider the distance of units from the efficiency and inefficiency frontiers as our other priorities.

Note that, we avoid to define the vector with a lot of components, therefore, we must consider a combination of the distances from the efficiency and inefficiency frontiers as the third and fourth component of V_o, respectively. On the other hand, the components must be selected so the larger value of them indicates the better rank for units. Therefore, we define the third component of V_o as the negative of the average of distances of DMU_o from the best and the worst efficiency frontiers, similarly, the average of distances of DMU_o from the best and the worst inefficiency frontiers is considered as the last component of V_o.

In summary, the preferences in our ranking method to make a powerful distinction between all units are:

1) The maximum value for the lower super efficiency score and the upper efficiency score.
2) The maximum value for the upper super efficiency score.
3) The minimum value for the average of distances of unit from the best and the worst efficiency frontiers.
4) The maximum value for the average of distances of unit from the best and the worst inefficiency frontiers.

Therefore, the assigned 4-vector V_o to DMU_o is $V_o = \left(\max\{E_o^L, E_{oo}^U\}, E_o^U, -\frac{Z_o^*+Z_o^-}{2}, \frac{W_o^*+W_o^-}{2}\right)$. Finally, we rank these vectors according to lexicographic order described in Definition 3.

In the following, we summarize our ranking method as an algorithm for more clarity:

The algorithm of our method

Step 1: Solve models (1b), (2a) and (2b) to obtain the upper efficiency score, the lower super efficiency score and the upper super efficiency score for $DMU_o, o \in \{1, \dots, n\}$.

Step 2: Solve models (12a), (12b), (13a) and (13b) and determine the optimal objective values Z_o^*, Z_o^-, W_o^* and W_o^-, respectively, to measure the distances of DMU_o from the efficiency frontiers and inefficiency frontiers.

Step 3: Define vector $V_o = \left(\max\{E_o^L, E_{oo}^U\}, E_o^U, -\frac{Z_o^*+Z_o^-}{2}, \frac{W_o^*+W_o^-}{2}\right)$ for DMU_o.

Step 4: Compare the vectors $V_j, j \in \{1, \dots, n\}$, by the lexicographic order and obtain a complete ranking of units.

In the following, we present the concept of domination for units with interval data.

Definition 4. Suppose that $o, l \in \{1, \dots, n\}$. If $x_{io}^L \le x_{il}^L, x_{io}^U \le x_{il}^U$ for $i = 1, \dots, m$ and $y_{ro}^L \ge y_{rl}^L, y_{ro}^U \ge y_{rl}^U$ for $r = 1, \dots, s$, then, DMU_o dominates DMU_l.

The next theorem proves that if a unit dominates the other one, then it has the better rank than it.

Theorem 3. *Let DMU_o dominates DMU_l. Then the rank of DMU_o is better than the rank of DMU_l in our method or equivalently $V_o \ge_{lex} V_l$.*

Proof: Suppose that DMU_o dominates DMU_l. Therefore, $x_{io}^L \le x_{il}^L, x_{io}^U \le x_{il}^U$ for $i = 1, \dots, m$ and $y_{ro}^L \ge y_{rl}^L, y_{ro}^U \ge y_{rl}^U$ for $r = 1, \dots, s$, and inequality is strict for at least one component. Without loss of generality, we assume that $x_{ko}^L < x_{kl}^L$. Let (μ_l^*, w_l^*, u_0^*) is an optimal solution for model (2a) evaluating DMU_l. Hence, we have:

$$\begin{aligned}
&E_l^L = \sum_{r=1}^{s} \mu_{rl}^* y_{rl}^L + u_0^*, \\
&\sum_{r=1}^{s} \mu_{rl}^* y_{rj}^U - \sum_{i=1}^{m} w_{il}^* x_{ij}^L + u_0^* \le 0, && j \ne l, \\
&\sum_{r=1}^{s} \mu_{rl}^* y_{rl}^U - \sum_{i=1}^{m} w_{il}^* x_{il}^L + u_0^* \le \sum_{r=1}^{s} \mu_{rl}^* y_{ro}^U - \sum_{i=1}^{m} w_{il}^* x_{io}^L + u_0^* \le 0, \\
&\sum_{i=1}^{m} w_{il}^* x_{il}^U = 1, \\
&\mu_{rl}^*, w_{il}^* \ge \varepsilon, && \forall i, r.
\end{aligned}$$

Since $x_{io}^U \le x_{il}^U$ for $i = 1, \dots, m$, we have:

$$1 = \sum_{i=1}^{m} w_{il}^* x_{il}^U \geq \sum_{i=1}^{m} w_{il}^* x_{io}^U = \alpha.$$

It is clear that $\alpha > 0$. Now, we prove that $(\frac{1}{\alpha}\mu_l^*, \frac{1}{\alpha}w_l^*, \frac{1}{\alpha}u_0^*)$ is a feasible solution for model (2a) evaluating DMU_o:

$$\begin{aligned}
&\frac{1}{\alpha}\left(\sum_{r=1}^{s} \mu_{rl}^* y_{rj}^U - \sum_{i=1}^{m} w_{il}^* x_{ij}^L + u_0^*\right) \leq 0, \quad j = 1, \dots, n.\\
&\frac{1}{\alpha}\left(\sum_{i=1}^{m} w_{il}^* x_{io}^U\right) = 1\\
&\frac{1}{\alpha}(\mu_{rl}^*) \geq \frac{1}{\alpha}\varepsilon \geq \varepsilon, \qquad r = 1, \dots, s,\\
&\frac{1}{\alpha}(w_{il}^*) \geq \frac{1}{\alpha}\varepsilon \geq \varepsilon, \qquad i = 1, \dots, m.
\end{aligned}$$

Hence, $E_o^L \geq E_l^L$.

Also, suppose that $(\bar{\mu}_l, \bar{w}_l, \bar{u}_0)$ is an optimal solution for model (1b) evaluating DMU_l. Then, we have:

$$\begin{aligned}
&E_{ll}^U = \sum_{r=1}^{s} \bar{\mu}_{rl} y_{rl}^U + \bar{u}_0,\\
&\sum_{r=1}^{s} \bar{\mu}_{rl} y_{rj}^U - \sum_{i=1}^{m} \bar{w}_{il} x_{ij}^L + \bar{u}_0 \leq 0, \quad j = 1, \dots, n,\\
&\sum_{i=1}^{m} \bar{w}_{il} x_{il}^L = 1,\\
&\bar{\mu}_{rl} \geq \varepsilon, \qquad r = 1, \dots, s,\\
&\bar{w}_{il} \geq \varepsilon, \qquad i = 1, \dots, m.
\end{aligned}$$

Since $x_{io}^L \leq x_{il}^L$ for $i = 1, \dots, m$ and $x_{ko}^L < x_{kl}^L$, we have:

$$1 = \sum_{i=1}^{m} \bar{w}_{il} x_{il}^L > \sum_{i=1}^{m} \bar{w}_{il} x_{io}^L = \beta.$$

It is clear that $\beta > 0$. Now, we prove that $\left(\frac{1}{\beta}\bar{\mu}_l, \frac{1}{\beta}\bar{w}_l, \frac{1}{\beta}\bar{u}_0\right)$ is a feasible solution for model (1b) evaluating DMU_o:

$$\begin{aligned}
&\frac{1}{\beta}\left(\sum_{r=1}^{s} \bar{\mu}_{rl} y_{rj}^U - \sum_{i=1}^{m} \bar{w}_{il} x_{ij}^L + \bar{u}_0\right) \leq 0, \quad j = 1, \dots, n.\\
&\frac{1}{\beta}\left(\sum_{i=1}^{m} \bar{w}_{il} x_{io}^L\right) = 1\\
&\frac{1}{\beta}(\bar{\mu}_{rl}) \geq \frac{1}{\beta}\varepsilon \geq \varepsilon, \qquad r = 1, \dots, s,\\
&\frac{1}{\beta}(\bar{w}_{il}) \geq \frac{1}{\beta}\varepsilon \geq \varepsilon, \qquad i = 1, \dots, m.
\end{aligned}$$

Hence, $E_{oo}^U \geq E_{ll}^U$. This means that $\max\{E_o^L, E_{oo}^U\} \geq \max\{E_l^L, E_{ll}^U\}$.

If $\max\{E_o^L, E_{oo}^U\} > \max\{E_l^L, E_{ll}^U\}$ then regarding the lexicographic order, it is clear that DMU_o has a better rank than DMU_l. Otherwise, we should compare the second components of V_o and V_l. Suppose that $(\hat{\mu}_l, \hat{w}_l, \hat{u}_0)$ is an optimal solution for model (2b) evaluating DMU_l. Hence, with a similar argument, we can prove that $E_o^U \geq E_l^U$. If $E_o^U > E_l^U$ then considering the lexicographic order it is clear that DMU_o obtains a better rank than DMU_l. Otherwise, suppose that $(\lambda^{o*}, s^{-o*}, s^{+o*})$ is an optimal solution for model (12a) evaluating DMU_o. Hence, we have:

$$
\begin{aligned}
& Z_o^* = \sum_{i=1}^{m} w_i^- s_i^{-o*} + \sum_{r=1}^{s} w_r^+ s_r^{+o*} \\
& \sum_{j=1}^{n} \lambda_j^{o*} x_{ij}^L + s_i^{-o*} = x_{io}^L \leq x_{il}^L, \qquad \forall i, i \neq k, \\
& \sum_{j=1}^{n} \lambda_j^{o*} x_{kj}^L + s_k^{-o*} = x_{ko}^L < x_{kl}^L, \\
& \sum_{j=1}^{n} \lambda_j^{o*} y_{rj}^U - s_r^{+o*} = y_{ro}^U \geq y_{rl}^U \qquad \forall r, \\
& \sum_{j=1}^{n} \lambda_j^{o*} = 1, \\
& \lambda_j^{o*} \geq 0, \qquad \forall j, \\
& s_r^{+o*} \geq 0, \qquad \forall r, \\
& s_i^{-o*} \geq 0 \qquad \forall i.
\end{aligned}
$$

Now define:

$$
\begin{aligned}
\tilde{s}_i^- &= x_{il}^L - \sum_{j=1}^{n} \lambda_j^{o*} x_{ij}^L - s_i^{-o*} = x_{il}^L - x_{io}^L \geq 0, \qquad i = 1, \dots, m, i \neq k, \\
\tilde{s}_k^- &= x_{kl}^L - \sum_{j=1}^{n} \lambda_j^{o*} x_{kj}^L - s_k^{-o*} = x_{kl}^L - x_{ko}^L > 0, \\
\tilde{s}_r^+ &= \sum_{j=1}^{n} \lambda_j^{o*} y_{rj}^U - s_r^{+o*} - y_{rl}^U = y_{ro}^U - y_{rl}^U \geq 0, \quad r = 1, \dots, s.
\end{aligned}
$$

Therefore, $(\lambda^{o*}, s^{-o*} + \tilde{s}^-, s^{+o*} + \tilde{s}^+)$ is a feasible solution for model (12a) evaluating DMU_l. Since, $s_k^{-o*} + \tilde{s}_k^- > s_k^{-o*}$ we have:

$$
Z_l^* \geq \sum_{i=1}^{m} w_i^- (s_i^{-o*} + \tilde{s}_i^-) + \sum_{r=1}^{s} w_r^+ (s_r^{+o*} + \tilde{s}_r^+) > \sum_{i=1}^{m} w_i^- s_i^{-o*} + \sum_{r=1}^{s} w_r^+ s_r^{+o*} = Z_o^*.
$$

Similarly, we can prove that $Z_l^- \geq Z_o^-$. Therefore, $\frac{Z_o^* + Z_o^-}{2} < \frac{Z_l^* + Z_l^-}{2}$. Therefore, $V_o \geq_{lex} V_l$. □

Next theorem provides the main property of our ranking method.

Theorem 4. *The rank of DMUs belonging to* E^{++} *is better than the rank of DMUs in* E^+ *and the rank of DMUs belonging to* E^+ *is better than the units in* E^-.

Proof: Suppose that $o, l, e \in \{1, \dots, n\}$, and let $o \in E^{++}, l \in E^+, e \in E^-$. According to definition of E^{++} and E^+, it is clear that:

$$
E_o^L > 1, E_{oo}^U = 1 \xrightarrow{yields} \max\{E_o^L, E_{oo}^U\} = E_o^L > 1 \quad (14)
$$
$$
E_l^L \leq 1, E_{ll}^U = 1 \xrightarrow{yields} \max\{E_l^L, E_{ll}^U\} = E_{ll}^U = 1 \quad (15)
$$

From (14) and (15), we can conclude that:

$$\max\{E_o^L, E_{oo}^U\} > \max\{E_l^L, E_{ll}^U\}.$$

So, DMU_o has a better rank than DMU_l.

$$E_l^L \le 1, E_{ll}^U = 1 \xrightarrow{yields} \max\{E_l^L, E_{ll}^U\} = E_{ll}^U = 1$$
$$E_e^L < 1, E_{ee}^U \le 1 \xrightarrow{yields} \max\{E_e^L, E_{ee}^U\} \le 1.$$

If $\max\{E_l^L, E_{ll}^U\} > \max\{E_e^L, E_{ee}^U\}$ then $V_l \ge_{lex} V_e$. Otherwise, since $l \in E^+$ and $e \in E^-$ therefore, $E_l^U > 1$ and $E_e^U \le 1$, and we have $E_l^U > E_e^U$. Hence, $V_l \ge_{lex} V_e$. □

In the next section, we provide two numerical example to illustrate our ranking method.

4 Numerical example

Example 2. Consider the data of five DMUs reported in Example 1. As we see in Example 1, Table 5 reports the data units, the lower efficiency score (E_{oo}^L), the upper efficiency score (E_{oo}^U), the lower super efficiency score (E_o^L) and the upper super efficiency score (E_o^U) of DMUs. Now, we apply our ranking method for the data in this example. So, we should solve models (12a), (12b), (13a) and (13b) and obtain the optimal objective values Z_o^*, Z_o^-, W_o^* and W_o^- to measure the distance of each unit from the best efficiency frontier, the worst efficiency frontier, the best inefficiency frontier and the worst inefficiency frontier, respectively. The results are summarized in Table 6.

Now, we should assign a 4-vector $V_o = \left(\max\{E_o^L, E_{oo}^U\}, E_o^U, -\frac{Z_o^* + Z_o^-}{2}, \frac{W_o^* + W_o^-}{2}\right)$, reported in Table 7, to $DMU_o, o \in \{1, \dots, n\}$. Finally, we rank the vectors $V_j, j = 1, \dots, n$, by lexicographic order. The first component of V_A, V_B and V_C are the same and greater than the first component of V_D and V_E, hence, we should compare the second component of V_A, V_B and V_C to determine the rank of A, B and C. As we can see, the second component of V_A, V_B and V_C are 3, 1.17 and 0.83, respectively. Therefore, units A, B and C have the ranks 1, 2 and 3, respectively. Then, we must determine the rank of D and E. The first component of V_D and V_E are 0.20 and 0.25, respectively. Hence, units D and E have the ranks 5 and 4, respectively. The last column of Table 7 reports the obtained rank of units by our ranking method. Our method ranks all efficient and inefficient units. As we can see in Table 1 and Table 7, $A, B \in E^+$ and the rank of them is better than the rank of each unit $C, D, E \in E^-$.

Example 3. In this example, the results of applying our proposed approach to the dataset in Jahanshahloo et al. (2011) are presented. This dataset has 30 decision making units which are branches of Tehran social security insurance organization with three inputs, The number of personal (I_1), the total number of computers (I_2), the area of the branch (I_3) in order to produce four outputs, the total number of insured persons (O_1), the number of insurance policies (O_2), the total number of old age pensioners (O_3) and the received total sum (Income) (O_4). The input /output data are reported in Table 8. We apply our ranking method the dataset in this example. So, we should solve models (1b), (2a) and (2b) to obtain the upper efficiency score (E_{oo}^U), the lower super efficiency score (E_o^L) and the upper super efficiency score (E_o^U) for $DMU_o, o \in \{1, \dots, n\}$ and then, the results are summarized in columns 2, 4 and 5 of Table 9, respectively. In Table 9, column 3 represent the efficiency status of all units according to Definition 1, column 6 shows the category that each unit belongs to it and column 7 specify the extreme efficient units according to Definition 2. Then, we solve models (12a), (12b), (13a) and (13b) and obtain Z_o^*, Z_o^-, W_o^* and W_o^- to measure the distance of each unit from the efficiency frontiers and inefficiency frontiers, the results are reported in columns 8, 9, 10 and 11 of Table 9, respectively.
Then, we assign a 4-vector $V_o = \left(\max\{E_o^L, E_{oo}^U\}, E_o^U, -\frac{Z_o^* + Z_o^-}{2}, \frac{W_o^* + W_o^-}{2}\right)$, reported in the second column of Table 10, to each unit DMU_o. Finally, we rank the assigned vectors to

units by lexicographic order. The obtained rank by our proposed method and the method of Jahanshahloo et al. (2011) are shown in columns 3 and 4 of Table 10. The Spearman's rank order correlation between our proposed method and the method of Jahanshahloo et al. (2011) is 0.76. It can be seen that our method and the method of Jahanshahloo et al. (2011) have a relatively high correlation at least in this instance.

5 Conclusions and further research

In many real world situations, the inputs and/or outputs of decision making units can be given as imprecise data. One of the attractive issues in IDEA is to rank the units. This paper addressed the problem of ranking DMUs with interval data which is a special case of uncertainty in data. The contribution of this study is to develop a powerful method for ranking DMUs with interval data as our proposed approach has all desirable features expected for ranking methods. We extended some concepts in traditional DEA such as super efficiency, extreme efficient unit and dominated units to Interval DEA and then proposed an original approach to rank all units with interval data. Our proposed method was based on four preferences: the maximum value for the lower super efficiency score and the upper efficiency score, the maximum value for the upper super efficiency score, the minimum value for the average of distances of unit from the best and the worst efficiency frontiers and the maximum value for the average of distances of unit from the best and the worst inefficiency frontiers. Then, we assigned a 4-vector to each unit by regarding these preferences. Finally, the rank of DMUs obtained by comparing the assigned vectors with the lexicographic order. Our method ranks all efficient and inefficient units that is one of the main advantages of it. Also it uses several essential criteria simultaneously to rank units with interval data which these criteria increase the discrimination power of our proposed method and this is another advantage of our method. We proved that our proposed method has all desirable features that are expected for a ranking method.

The idea of this paper can be extended for ranking DMUs with interval data by using another method such as TOPSIS instead of lexicography method.

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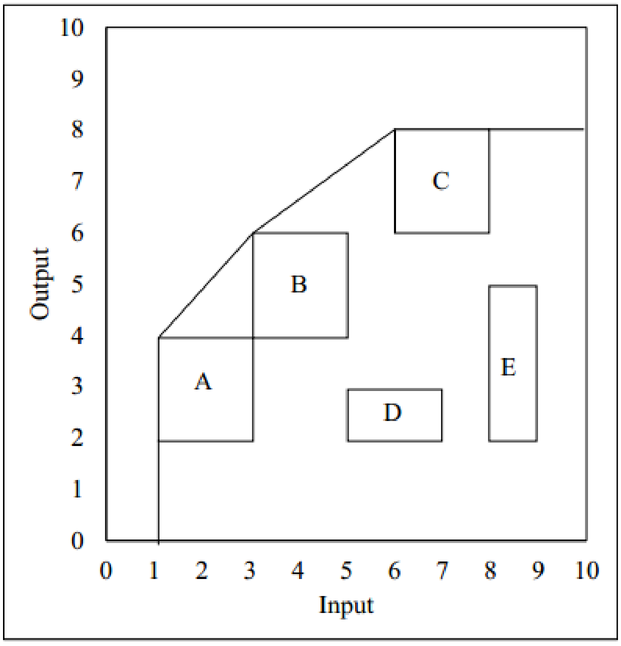

Figure 1. The PPS for five DMUs in Example 1.

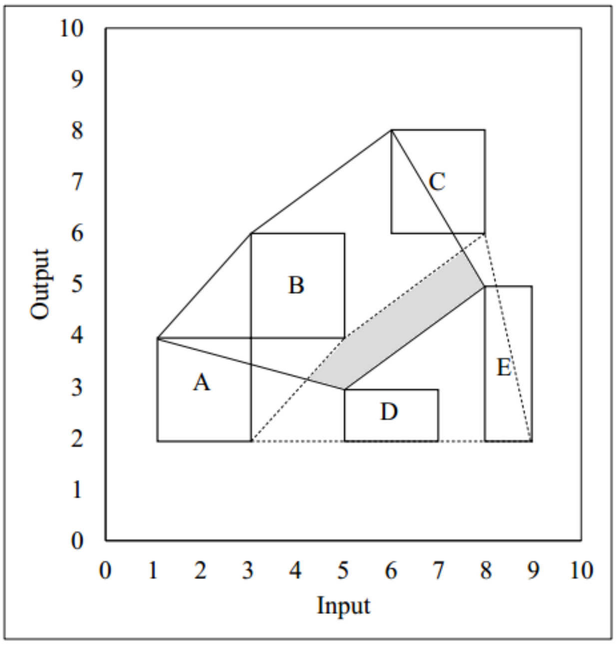

Figure 2. L^{BC} and L^{WC} for units in example 1.

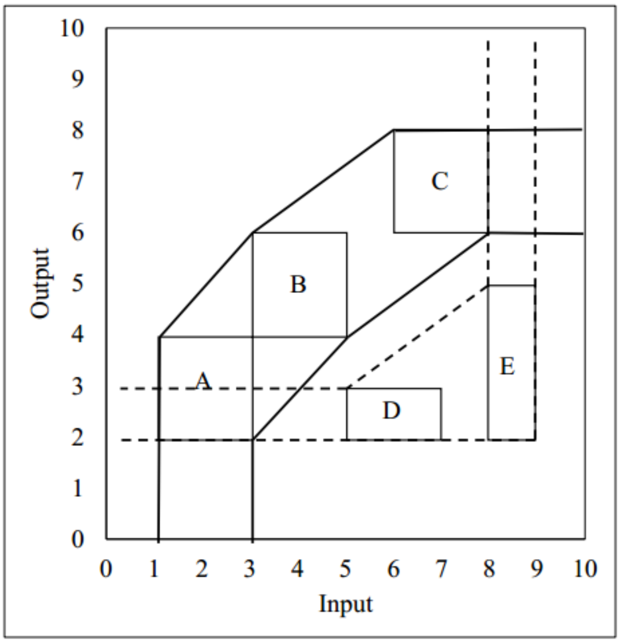

Figure 3. The efficiency and inefficiency frontiers.

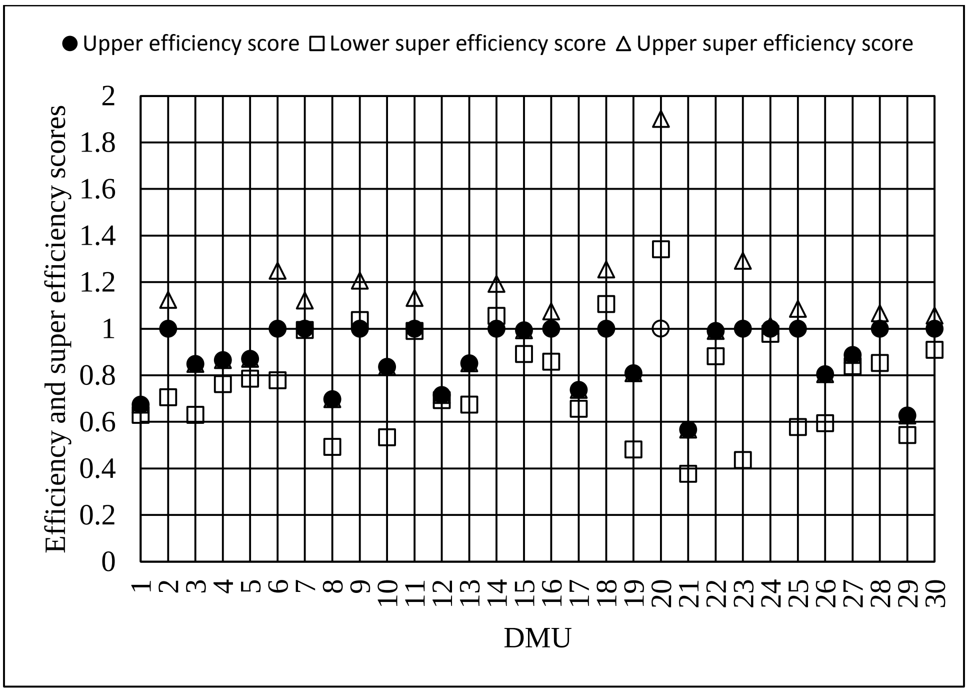

Figure 4. The values of E_{oo}^{U}, E_{o}^{L} and E_{o}^{U} for all units.

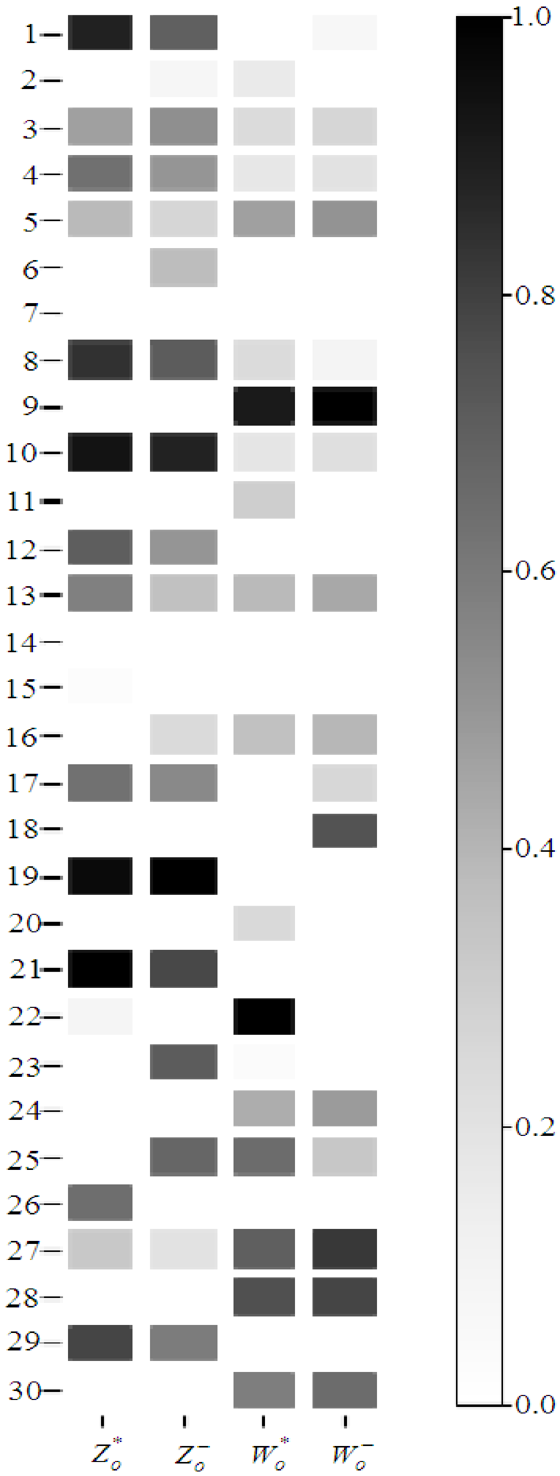

Figure 5. Heatmap graph of the distances of each unit from the frontiers.

Table 5. The data and obtained results for five DMUs in Example 1.

DMU	Input	Output	E_{oo}^L	E_{oo}^U	Efficient	E_o^L	E_o^U	Extreme efficient unit
A	[1, 3]	[2, 4]	0.33	1.00	Yes	1.00	3.00	Yes
B	[3, 5]	[4, 6]	0.20	1.00	Yes	0.20	1.17	Yes
C	[6, 8]	[6, 8]	0.37	1.00	Yes	0.37	1.00	No
D	[5, 7]	[2, 3]	0.14	0.20	No	0.14	0.20	No
E	[8, 9]	[2, 5]	0.11	0.25	No	0.11	0.25	No

Table 6. The distances of each unit from the efficiency and inefficiency frontiers.

DMU	Z_o^*	Z_o^-	W_o^*	W_o^-
A	0.00	0.00	2.75	3.00
B	0.00	0.00	3.00	3.00
C	0.00	0.00	2.00	2.00
D	2.50	2.00	0.00	1.00
E	3.00	3.00	0.00	0.00

Table 7. The obtained rank of units by our proposed method.

DMU	V_o	Rank
A	(1.00, 3.00, 0.00, 2.88)	1
B	(1.00, 1.17, 0.00, 3.00)	2
C	(1.00, 0.83, 0.00, 2.00)	3
D	(0.20, 0.20, -2.25, 0.50)	5
E	(0.25, 0.25, -3.00, 0.00)	4

Table 8. The inputs and outputs for 30 branches of the insurance organization.

DMU	I_1^L	I_1^U	I_2^L	I_2^U	I_3^L	I_3^U	O_1^L	O_1^U	O_2^L	O_2^U	O_3^L	O_3^U	O_4^L	O_4^U
1	96	100	86	87	4000	4000	55830	57318	30	45	1307	1350	145	192
2	75	81	88	90	2565	2565	36740	36852	0.001	22	8385	8571	175	486
3	77	80	85	89	1343	1343	38004	38783	11	27	6588	6601	113	276
4	91	94	93	96	1500	1500	35469	36017	10	55	10820	10821	128	316
5	89	92	83	83	1680	1680	52927	54817	9	43	9493	9751	101	263
6	102	105	97	97	3750	3750	70254	78574	7	19	7536	8752	82	615
7	96	100	90	92	3313	3313	32585	37443	47	129	14118	14994	154	392
8	85	90	92	92	1500	1500	42900	47270	11	27	1634	1661	54	220
9	106	112	84	92	1600	1600	85399	87220	43	97	10206	10775	179	289
10	107	111	95	95	1725	1725	46924	47316	9	36	6608	6823	117	342
11	94	101	78	78	1920	1920	36652	44298	81	242	11996	12261	37	286
12	78	79	89	89	4433	4433	39582	39620	11	31	7422	7624	124	184
13	102	102	107	111	2500	2500	56144	58816	30	57	7380	7936	185	430
14	82	88	92	94	2800	2800	87716	90250	28	43	630	660	51	167
15	77	82	92	94	1630	1630	50210	50593	6	16	10247	10256	28	295
16	89	91	85	85	1127	1127	47727	49489	15	30	7302	7542	85	286
17	84	90	104	104	3400	3400	52923	53249	15	28	4740	5058	109	240
18	94	108	91	92	1304	1304	78550	89111	13	25	4745	5151	72	224
19	97	103	95	96	4206	4206	46154	46791	13	21	1611	1636	129	477
20	82	87	100	101	1340	1340	27978	32943	29	325	14473	14820	190	368
21	71	73	88	90	1393	1393	27128	27940	0.001	20	921	973	55	179
22	112	118	120	123	2191	2191	102175	103047	31	49	252	3577	120	320
23	80	86	100	100	2140	2140	31819	35627	12	32	1963	2147	156	522
24	87	93	91	93	1231	1231	51345	55163	35	73	10157	10238	85	205
25	97	103	90	90	1960	1960	72915	74633	40	52	4193	4668	112	427
26	79	83	81	81	3375	3375	42887	44363	11	33	560	628	218	390
27	107	110	101	101	2540	2540	78068	79695	26	46	8963	9338	136	265
28	96	102	87	97	1603	1603	71743	72534	50	92	8762	12569	102	240
29	67	69	81	86	2300	2300	38054	38914	13	33	1405	1477	23	156
30	88	93	90	94	2930	2930	63182	64541	10	32	11143	11609	122	378

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Table 9. The results for 30 branches of the insurance organization.

DMU	E_{oo}^{U}	Efficient	E_o^L	E_o^U	Category	Extreme efficient	Z_o^*	Z_o^-	W_o^*	W_o^-
1	0.674	No	0.629	0.674	E^-	No	0.885	0.694	0.000	0.060
2	1.000	Yes	0.705	1.123	E^+	Yes	0.000	0.073	0.156	0.000
3	0.848	No	0.629	0.848	E^-	No	0.468	0.520	0.235	0.265
4	0.864	No	0.762	0.864	E^-	No	0.633	0.501	0.169	0.200
5	0.870	No	0.785	0.870	E^-	No	0.384	0.263	0.467	0.510
6	1.000	Yes	0.778	1.248	E^+	Yes	0.000	0.373	0.000	0.000
7	1.000	Yes	0.994	1.120	E^+	Yes	0.000	0.000	0.000	0.000
8	0.697	No	0.492	0.697	E^-	No	0.838	0.710	0.238	0.091
9	1.000	Yes	1.036	1.205	E^{++}	Yes	0.000	0.000	0.910	1.000
10	0.836	No	0.533	0.836	E^-	No	0.933	0.881	0.183	0.215
11	1.000	Yes	0.990	1.132	E^+	Yes	0.000	0.000	0.299	0.000
12	0.715	No	0.693	0.715	E^-	No	0.700	0.503	0.000	0.000
13	0.851	No	0.673	0.851	E^-	No	0.577	0.354	0.384	0.440
14	1.000	Yes	1.055	1.192	E^{++}	Yes	0.000	0.000	0.000	0.000
15	0.993	No	0.890	0.993	E^-	No	0.021	0.000	0.000	0.000
16	1.000	Yes	0.857	1.074	E^+	Yes	0.000	0.245	0.355	0.392
17	0.737	No	0.655	0.737	E^-	No	0.629	0.546	0.000	0.261
18	1.000	Yes	1.105	1.254	E^{++}	Yes	0.000	0.000	0.000	0.746
19	0.808	No	0.481	0.808	E^-	No	0.962	1.000	0.000	0.000
20	1.000	Yes	1.340	1.900	E^{++}	Yes	0.000	0.000	0.249	0.000
21	0.566	No	0.376	0.566	E^-	No	1.000	0.774	0.000	0.000
22	0.990	No	0.881	0.990	E^-	No	0.081	0.000	1.000	0.000
23	1.000	Yes	0.435	1.291	E^+	Yes	0.000	0.709	0.031	0.000
24	1.000	Yes	0.977	1.012	E^+	Yes	0.000	0.000	0.423	0.483
25	1.000	Yes	0.577	1.085	E^+	Yes	0.000	0.671	0.652	0.330
26	0.804	No	0.593	0.804	E^-	No	0.641	0.000	0.000	0.000
27	0.887	No	0.839	0.887	E^-	No	0.326	0.203	0.697	0.819
28	1.000	Yes	0.852	1.065	E^+	Yes	0.000	0.000	0.750	0.783
29	0.626	No	0.542	0.626	E^-	No	0.738	0.593	0.000	0.000
30	1.000	Yes	0.908	1.056	E^+	Yes	0.000	0.000	0.584	0.649

Table 10. The rank of units by our method and the method of Jahanshahloo et al. (2011).

DMU	The assigned vector (V_o)	Rank (Our method)	Rank (Jahanshahloo et al. (2011))
1	(0.674, 0.674, -0.790, 0.030)	28	23
2	(1.000, 1.123, -0.036, 0.078)	8	18
3	(0.848, 0.848, -0.494, 0.243)	21	22
4	(0.864, 0.864, -0.567, 0.184)	19	17
5	(0.870, 0.870, -0.324, 0.488)	18	15
6	(1.000, 1.248, -0.187, 0.000)	5	16
7	(1.000, 1.120, 0.000, 0.000)	9	6
8	(0.697, 0.697, -0.774, 0.165)	27	27
9	(1.000, 1.205, 0.000, 0.955)	6	1
10	(0.836, 0.836, -0.907, 0.199)	22	26
11	(1.000, 1.132, 0.000, 0.150)	7	5
12	(0.715, 0.715, -0.601, 0.000)	26	19
13	(0.851, 0.851, -0.456, 0.412)	20	20
14	(1.055, 1.192, 0.000, 0.000)	3	4
15	(0.993, 0.993, -0.011, 0.000)	15	9
16	(1.000, 1.074, -0.122, 0.374)	11	11
17	(0.737, 0.737, -0.588, 0.131)	25	21
18	(1.105, 1.254, 0.000, 0.382)	2	3
19	(0.808, 0.808, -0.981, 0.000)	24	28
20	(1.340, 1.900, 0.000, 0.124)	1	2
21	(0.566, 0.566, -0.887, 0.000)	30	30
22	(0.990, 0.990, -0.041, 0.500)	16	10
23	(1.000, 1.291, -0.355, 0.015)	4	29
24	(1.000, 1.012, 0.000, 0.453)	14	7
25	(1.000, 1.0855, -0.336, 0.491)	10	14
26	(0.804, 0.804, -0.320, 0.000)	23	24
27	(0.887, 0.887, -0.265, 0.758)	17	13
28	(1.000, 1.065, 0.000, 0.766)	12	12
29	(0.626, 0.626, -0.666, 0.000)	29	25
30	(1.000, 1.056, 0.000, 0.617)	13	8