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Universal Envelopes of Discontinuous Functions

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Doctor of Philosophy

ASTON UNIVERSITY
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Aston University

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Thesis summary

This thesis is a contribution to computable analysis in the tradition of Grzegorczyk, Lacombe, and Weihrauch. The main theorem of computable analysis asserts that any computable function is continuous. The solution operators for many interesting problems encountered in practice turn out to be discontinuous, however. It hence is a natural question how much partial information may be obtained on the solutions of a problem with discontinuous solution operator in a continuous or computable way. We formalise this idea by introducing the notion of continuous envelopes of discontinuous functions. The envelopes of a given function can be partially ordered in a natural way according to the amount of information they encode. We show that for any function between computably admissible represented spaces this partial order has a greatest element, which we call the universal envelope. We develop some basic techniques for the calculation of a suitable representation of the universal envelope in practice. We apply the ideas we have developed to the problem of locating the fixed point set of a continuous self-map of the unit ball in finite-dimensional Euclidean space, and the problem of locating the fixed point set of a nonexpansive self-map of the unit ball in infinite-dimensional separable real Hilbert space.

Keywords: Computable Analysis, Set-valued function, Hyperspace Topology, QCB-space, Complete Lattice

* * *

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I am very grateful to Matthew de Brecht for pointing out to me that the class of represented spaces X with $\eta_X^{\mathscr{O}^2} : X \to \mathscr{O}^2(\mathrm{X})$ split mono can be characterised as the class of injective objects in QCB_0 relative to Σ-split embeddings. I thank my supervisor, Michal Konečný, for helpful discussions throughout my studies and useful comments on earlier versions of this thesis. This work has further benefited from comments by and discussions with Ulrich Berger, Vasco Brattka, Pieter Collins, Matthew de Brecht, Martín Escardó, Achim Jung, Arno Pauly, Matthias Schröder, Florian Steinberg, Thomas Streicher, Paul Taylor, and Martin Ziegler.

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Chapter 1

Introduction

Computable analysis is an extension of the classical theory of computing from discrete to continuous data, such as real numbers or real functions. The definition of computable real number already appears as one of the central notions in Turing's seminal paper [102, 103] which introduces the "machines" now named for him. The notion of computable real function was introduced independently by Grzegorczyk [50, 48, 49] and Lacombe [66]. Based on these ideas Kreitz and Weihrauch [65, 105, 106] developed a general theory of computation on second-countable T_0 spaces. This theory was further extended by Schröder [87, 88] to the Cartesian closed category of T_0 quotients of countably based spaces, which constitute, in a certain sense, the largest class of topological spaces which can be endowed with a reasonable computability structure [88, Theorem 13]. Related but non-equivalent models of computation include Banach-Mazur computability [4, 71] and Markov computability [70]. A comprehensive account of the history of the field is given in [2]. Computable analysis is closely related to constructive analysis [6, 7] on the one hand and to rigorous numerical computation [101, 74] on the other.

An algorithm on continuous data, as defined within computable analysis, is ultimately a computable transformation of integer sequences and as such can in principle be directly implemented on a digital computer. There exist a number of libraries for practical numerical computation based on the ideas of computable analysis [76, 75, 5, 63, 67]. This is in stark contrast to more idealised models of continuous computation such as the Real-RAM [91] or BSS-machine [9, 8] whose algorithms cannot be directly implemented on a physical machine. Attempts to implement such algorithms on a digital computer are notorious for their erratic behaviour.

Thus, computable analysis offers itself as a rigorous mathematical foundation for numerical analysis. One of the main basic results of the field is that any computable function which operates on continuous data has to be continuous with respect to a suitable topology [106, 87, 88]. In the language of numerical analysis this says that only well-posed problems can be solved algorithmically. This precludes the computation of operations on real numbers such as equality tests and comparison which are considered basic operations in the aforementioned more idealised computational models, but lead to unpredictable behaviour when implemented. On the other hand one is immediately confronted with the issue that the solution operators for a great many problems of practical interest exhibit discontinuities in general. A - naturally very incomplete - list of examples might include the problems of solving nonlinear equations [95, 79, 3], global optimisation [95], solving ordinary [82] or partial [83] differential equations, finding solutions to linear equations or finding Eigenbases for singular matrices [113], finding the spectrum of a linear operator [51], or safety verification for hybrid systems [28].

Hence, if one hopes to find an algorithmic solution to any such problem the first step has to be to find a well-posed reformulation of the problem. In view of the ubiquity of ill-posed computational problems and the fundamental significance of finding a suitable well-posed reformulation for them it makes sense to ask if there is a systematic way of assigning to each discontinuous function a continuous one which - in a certain sense - reflects the properties of the original function as closely as possible.

The aim of this thesis is to develop a systematic approach to the study of continuous reformulations of discontinuous problems and of the amount of information such reformulations contain. In order to have a notion of computability available we will work in the category of *computably admissible represented spaces* [87, 81], which we prefer to call *computable T_0 spaces*, as computable admissibility can be viewed as an effective version of T_0 separation, cf. the discussion in [81, Section 9].

Let $f\colon X \to Y$ be a potentially discontinuous function between computable T_0 spaces. We propose to define a *reformulation* of f as a continuous function which encodes partial information on f. This idea can be formalised as follows: Embed Y into a complete lattice L via a map $\xi_L\colon Y \to L$ and say that a function $F\colon X \to L$ encodes partial information on f if $F(x) \leq \xi_L \circ f(x)$ for all $x \in X$.

We effectivise the classical notion of complete lattice as follows: A *computable complete lattice* is a computable T_0 space L which admits uniformly computable compact meets and overt joins with respect to its specialisation order. In other words, a computable complete lattice is a computable T_0 space which is simultaneously a $\mathscr{K}$-algebra and a $\mathscr{V}$-algebra with computable structure maps, where $\mathscr{K}$ is the upper powerspace monad and $\mathscr{V}$ is the lower powerspace monad. Every suitably represented ω-continuous lattice is a computable complete lattice. Computable complete lattices turn out to have excellent closure properties: They are closed under finite products and retracts, and form an exponential ideal in the category of represented spaces.

This leads us to the following preliminary definition: An *envelope* of a function $f\colon X \to Y$ consists of a computable complete lattice L together with a computable map $\xi_L\colon Y \to L$ called the *inclusion map* and a continuous map $F\colon X \to L$ satisfying $F(x) \leq \xi_L \circ f(x)$ for all $x \in X$ in the specialisation order of L. We usually just write $F\colon X \to L$ for the envelope, letting the rest of the data be implicit. See the end of this chapter for examples.

An envelope $F\colon X \to L$ induces a continuous function

$$\mathfrak{F}\colon X \to \mathscr{P}(\mathscr{O}(Y)),\ \mathfrak{F}(x) = \uparrow\left\{\xi_L^{-1}(U) \in \mathscr{O}(Y) \mid U \in \mathscr{O}(L), F(x) \in U\right\}$$

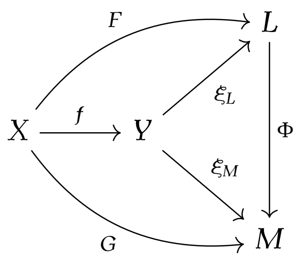
where $\mathscr{P}(\mathscr{O}(Y))$ is the algebraic lattice of all sets of open subsets of Y, endowed with its Scott topology. This function satisfies $\mathfrak{F}(x) \subseteq \{U \in \mathscr{O}(Y) \mid f(x) \in U\}$ for all $x \in X$. In this sense F can be viewed as an effective encoding of partial topological information on f.

For each fixed inclusion map $\xi_L\colon Y \to L$ with values in a computable complete lattice L the set of all continuous functions $F\colon X \to L$ with $F(x) \leq \xi_L \circ f(x)$ for all $x \in X$ has a greatest element in the pointwise order induced by the specialisation order on L. This result relies on L being a complete lattice. We call this F the *principal L-envelope*. Similar results are quite well-known in domain theory, cf. e.g. [40] or [36, Lemma 3.5, Theorem 3.6].

Envelopes can be ordered in a natural way according to the amount of information they contain: If $F\colon X \to L$ and $G\colon X \to M$ are envelopes of f with inclusion maps $\xi_L\colon Y \to L$ and $\xi_M\colon Y \to M$ we say that F *tightens* G if there exists a continuous map $\Phi\colon L \to M$ with

1. $\Phi \circ \xi_L \leq \xi_M$.

2. $\Phi \circ F \geq G$.



The first condition guarantees in particular that $\Phi \circ F \colon X \to M$ is an envelope of f with inclusion map ξ_M.

Note that if F tightens G, then the encoded maps $\mathfrak{F} \colon X \to \mathscr{P}(\mathscr{O}(Y))$ and $\mathfrak{G} \colon X \to \mathscr{P}(\mathscr{O}(Y))$ satisfy the relation $\mathfrak{F}(x) \supseteq \mathfrak{G}(x)$ for all $x \in X$. The function $\Phi \colon L \to M$ in the tightening relation can be viewed as an effective witness for this relation. In particular, if F and G are equivalent with respect to the tightening order, then they encode the same function of type $X \to \mathscr{P}(\mathscr{O}(Y))$. In this sense equivalent envelopes can be viewed as equivalent encodings of the same object.

In order to ensure that the tightening order is well-behaved, we have to put further constraints on the class of lattices we admit as co-domains of envelopes. Without further assumptions it could happen that $F \colon X \to L$ fails to tighten $G \colon X \to M$ not because G encodes information on f that is not contained in F, but simply because there do not exist sufficiently many continuous maps of type $L \to M$. This naturally leads to the requirement that the lattices we allow as co-domains be *injective* in an appropriate sense. We call a computably complete lattice L *computably injective* if it is an injective object in the category of computable T_0 spaces relative to the class of computable Σ-split embeddings. The notion of Σ-split subspace was motivated and extensively studied by Taylor [96].

A computable map $e \colon A \to B$ between computable T_0 spaces is called a computable Σ-split embedding if the map $\mathscr{O}^e \colon \mathscr{O}(B) \to \mathscr{O}(A)$ has a computable section $s \colon \mathscr{O}(A) \to \mathscr{O}(B)$. We show that injective objects of this kind can be characterised as those computable T_0 spaces X where the natural embedding $\nu_X \colon X \to \mathscr{O}^2(X)$ has a computable left inverse (Proposition 3.18). It follows that any space which is injective in this sense is automatically a computable complete lattice, as the class of computable complete lattices is closed under retracts. The class of computably injective lattices is again closed under finite products and retracts, and forms an exponential ideal in the category of represented spaces.

An envelope $F \colon X \to L$ of f which tightens every envelope $G \colon X \to M$ of

f will be called *universal*. In this case F can be viewed as a best continuous approximation to f in the sense that it encodes the largest possible amount of partial information on f. Of course, this best continuous approximation is only unique up to equivalence with respect to the tightening preorder, but recall that equivalent envelopes can be viewed as equivalent encodings of the same function of type $X \to \mathscr{P}(\mathscr{O}(Y))$.

To show that the universal envelope of a function $f\colon X \to Y$ really contains all information that is "continuously obtainable" from f we introduce the following concept: A *continuous probe* for f is a pair of continuous functions

$$\begin{cases} \alpha\colon \widetilde{X} \to X \\ \beta\colon \widetilde{X} \times Y \to Z \end{cases}$$

such that for all $x \in \widetilde{X}$ the point $(x, \alpha(x)) \in \widetilde{X} \times X$ is a point of continuity for the function $\psi(x_0, x_1) = \beta(x_0, f(x_1))$. If α and β are computable functions we call (α, β) a *computable probe* for f. A probe can be viewed as an algorithm (relative to some oracle) which uses f as a subroutine in a continuous way to compute the function $\beta(x, f \circ \alpha(x))$. It is essentially a special kind of Weihrauch reduction [16] of f to a continuous function.

Let $F\colon X \to L$ be a universal envelope of f. We show in Theorem 4.34 that any probe (α, β) where $\beta\colon \widetilde{X} \times Y \to M$ takes values in a continuous lattice M factors through F in the sense that there exists a continuous map $\widetilde{\beta}\colon \widetilde{X} \times L \to M$ with

1. $\widetilde{\beta}(x, \xi_L(y)) \leq \beta(x, y)$ for all $y \in Y$ and all $x \in \widetilde{X}$.

2. $\widetilde{\beta}(x, F(x)) = \beta(x, f \circ \alpha(x))$ for all $x \in X$.

As any computably countably based space embeds into the continuous lattice $\Sigma^{\mathbb{N}}$, this result applies to a fairly wide range of probes.

Informally speaking, any sufficiently well-behaved algorithm which uses f as a subroutine in such a way that the end-result of the entire computation depends continuously on the input data can - in a sense - use the universal envelope as a subroutine instead. Conversely, a good description of the universal envelope yields a good description of the probes of f.

This result immediately leads to the question how the extension $\widetilde{\beta}$ can be obtained and whether it is computable whenever β is computable. We show that if $F\colon X \to L$ is an envelope whose inclusion map $\xi_L\colon Y \to L$ is a Σ-split embedding such that the map $\mathscr{O}^{\xi_L}\colon \mathscr{O}(L) \to \mathscr{O}(Y)$ has a computable section $s\colon \mathscr{O}(Y) \to \mathscr{O}(L)$

which satisfies $F(x) \in s(U)$ for all $x \in X$ and all $U \in \mathscr{O}(Y)$ such that x is contained in the interior of $f^{-1}(U)$, then a certain extension $\widetilde{\beta}\colon \widetilde{X} \times L \to Y$ can be computed uniformly in β. We call an envelope F with this property *uniformly Σ-complete*. We show that if the inclusion map of F is a proper embedding in the sense of Hofmann and Lawson [54], then F is universal if and only if it is uniformly Σ-complete (Theorem 4.18).

We show that any function $f\colon X \to Y$ between computable T_0 spaces has a universal envelope (Theorem 4.8). The proof is constructive in the sense that it yields a concrete representative of the universal envelope, but this representative is not very illuminating. For instance, it only yields a rather tautological description of the encoded function $\mathfrak{f}\colon X \to \mathscr{P}(\mathscr{O}(Y))$ and a similarly tautological description of the probes of f.

The situation becomes simpler for a certain class of problems if one is willing to settle for a slightly smaller class of probes. Let $f\colon X \to Y$ be a function which sends a computable T_0 space to a computably countably based computable Hausdorff space Y. If there exists a continuous function $B\colon X \to \mathscr{K}(Y)$ with $f(x) \subseteq B(x)$ for all $x \in X$ then we can find an envelope of the form $F\colon X \to \mathscr{K}_\perp(Y)$ with inclusion map $\kappa_\perp\colon Y \to \mathscr{K}_\perp(Y)$, $y \mapsto \uparrow y$, such that any computable probe with values in $\mathscr{K}([0,1]^{\mathbb{N}})$ factors computably through F (see Theorem 4.28 for details). Note that any probe with values in a computable metric space Z can be made into a probe with values in $\mathscr{K}([0,1]^{\mathbb{N}})$ by choosing a computable embedding $Z \to [0,1]^{\mathbb{N}}$.

We also develop some basic techniques for finding a good description of the universal envelope of more general problems. We introduce the notion of *retracts* (Definition 4.40), a notion of reducibility between functions that allows us to derive a description of the universal envelope of one function from a description of the universal envelope of another. The dense subset lemma (Lemma 4.43) allows us to reduce the problem of showing universality of a given envelope to the problem of showing universality for a restriction to a dense subset.

In Chapter 5 we illustrate and motivate the ideas we have introduced by applying them to two non-trivial computational problems in fixed point theory: the problem of locating the fixed point set of a continuous self-map of the unit ball in finite-dimensional Euclidean space, and the problem of locating the fixed point set of a nonexpansive self-map of the unit ball in infinite-dimensional separable real Hilbert space.

We show that the greatest amount of positive information that can be obtained on the fixed point set of a given continuous self-map of the unit ball in finite-dimensional Euclidean space is the information that is contained in the Brouwer mapping degree. The corresponding universal envelope is computable and uniformly Σ-complete.

We show that given a nonexpansive self-map of the unit ball in separable real Hilbert space one can compute arbitrarily good upper bounds on its fixed point set in the upper Vietoris topology induced by the weak topology, and this is the best one can do. The corresponding universal envelope is computable but not uniformly Σ-complete. In fact there exist computable probes for this function which do not factor computably through the universal envelope. We can however show that any continuous probe which is computable with respect to the standard representation for the weak topology computably factors through the universal envelope.

Example 1. To illustrate the basic concepts presented so far, let us consider a very simple example. Consider the Heaviside function

$$H\colon \mathbb{R} \to \{0,1\},\ H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Embed $\{0,1\}$ into the lattice $L = \{\bot, 0, 1, \top\}$ in the obvious way. Then the best continuous approximation of $\xi_L \circ H$ is given by

$$\widetilde{H}\colon \mathbb{R} \to L,\ \widetilde{H}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \bot & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Let $G\colon \mathbb{R} \to M$ be an arbitrary envelope of H with inclusion map $\xi_M\colon \{0,1\} \to M$. Then $G(0) \leq \xi_M(0) \wedge \xi_M(1)$. It follows that the envelope $\widetilde{H}$ tightens G via the map $\Phi\colon L \to M$ which sends $0 \in L$ to $\xi_M(0) \in M$, $1 \in L$ to $\xi_M(1) \in M$, $\bot \in L$ to $\xi_M(0) \wedge \xi_M(1) \in M$, and $\top \in L$ to $\top \in M$. Hence $\widetilde{H}$ is a universal envelope.

Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions with $f(0) = g(0)$. Then the function

$$\beta\colon \mathbb{R} \times \{0,1\} \to \mathbb{R},\ \beta(x,i) = \begin{cases} f(x) & \text{if } i = 0, \\ g(x) & \text{if } i = 1, \end{cases}$$

is a probe for H (with $\alpha\colon \mathbb{R} \to \mathbb{R}$ being the identity). It extends to the computable function

$$\widetilde{\beta}\colon \mathbb{R} \times L \to \mathscr{K}_\perp(\mathbb{R}),\ \widetilde{\beta}(x,\ell) = \begin{cases} \emptyset & \text{if } \ell = \top, \\ \{f(x)\} & \text{if } \ell = 0, \\ \{g(x)\} & \text{if } \ell = 1, \\ \{f(x), g(x)\} & \text{if } \ell = \perp, \end{cases}$$

with $\widetilde{\beta}(x, \xi_L(i)) = \beta(x,i)$ and $\widetilde{\beta}(x, \widetilde{H}(x)) = \beta(x, H(x))$.

Example 2. To provide a more substantial example, consider the problem of finding positive information on the set of zeroes of a continuous real function $h\colon \mathbb{R} \to \mathbb{R}$. This problem is formally captured by the function

$$\texttt{zeroes}_<\colon C(\mathbb{R}) \to \mathscr{V}(\mathbb{R}),\ h \mapsto \{x \in \mathbb{R} \mid h(x) = 0\}.$$

Here, $\mathscr{V}(\mathbb{R})$ denotes the lower powerspace of $\mathbb{R}$, see Definition 2.19.

Computing the zero set of a given function as an element of $\mathscr{V}(\mathbb{R})$ amounts to verifying for a given open set $U \in \mathscr{O}(\mathbb{R})$ if there exists a zero in U. This suggests to consider the lattice $L = \Sigma^{(\mathbb{Q},\mathbb{Q})}$, where $(\mathbb{Q},\mathbb{Q})$ denotes the discrete space of open intervals with rational endpoints, with inclusion map

$$\xi_L\colon \mathscr{V}(\mathbb{R}) \to \Sigma^{(\mathbb{Q},\mathbb{Q})},\ \xi_L(A) = \lambda(a,b).\begin{cases} \top & \text{if } (a,b) \cap A \neq \emptyset, \\ \perp & \text{otherwise.} \end{cases}$$

Thus, suppose we are given an open interval (a,b) with rational endpoints. If $h(a) \cdot h(b) < 0$ then the function h has a zero in (a,b) by the intermediate value theorem.

The function $h(x) = x^2$ has a unique zero at $x = 0$, but there exist arbitrarily small perturbations of h without any zeroes.

The function $h(x) = \max(x-1, \min(x+1, 0))$ has as its zero set the interval $[-1,1]$. We can certify the existence of a zero in each open interval $(a,b) \supseteq [-1,1]$ by observing that h changes its sign, but if (a,b) is an interval with $(a,b) \subseteq [-1,1]$ then there exist arbitrarily small perturbations of h without any zeroes in (a,b).

These examples suggest that the best we can do is to observe the occurrence of a sign-change. This leads us to consider the envelope

$$F\colon C(\mathbb{R}) \to L,\ F(h) = \lambda(a,b).\begin{cases} \top & \text{if } h(a) \cdot h(b) < 0, \\ \perp & \text{otherwise.} \end{cases}$$

Note that this is not the principal L-envelope, which - by an elementary argument - is given by

$$G\colon C(\mathbb{R}) \to L,\ G(h) = \lambda(a,b).\begin{cases} \top & \text{if } \exists (a',b') \subseteq (a,b).\,(h(a') \cdot h(b') < 0), \\ \perp & \text{otherwise.} \end{cases}$$

Nevertheless, F tightens G via the map

$$\Phi\colon L \to L,\ \Phi(x) = \lambda(a,b).\sup\left\{x(a',b') \mid (a',b') \subseteq (a,b)\right\}.$$

In fact, F is uniformly Σ-complete and hence the universal envelope of $\texttt{zeroes}_<$, but the proof is not entirely straightforward. It essentially follows from Corollary 5.5.

Chapter 2

Background

In this chapter we will mainly collect some basic definitions and "folklore" results, mostly without proofs, not all of which are easy to find in one place. Schröder's PhD thesis [87] is a very comprehensive source which includes most of the material covered here, but can be difficult to navigate. Our point of view closely follows that of Pauly's recent survey [81], which is close in spirit to Escardó's synthetic topology [42] and, to some extent, to Taylor's Abstract Stone Duality [97]. We will however (have to) put a greater emphasis on the connections to classical topology. A very readable account of the topological aspects of **QCB**-spaces is given in [38]. We also require some basic results and definition from the theory of continuous lattices which we briefly recall here. A standard reference is [45].

The final section of this chapter contains some original results on the commutativity of the powerspace monads on computably countably based spaces. These results are computable analogues of topological results that were recently obtained by de Brecht and Kawai [33].

Notational and terminological conventions

We denote the natural numbers by $\mathbb{N} = \{0, 1, \dots\}$, the rational numbers by $\mathbb{Q}$, and the real numbers by $\mathbb{R}$.

If $A \subseteq X$ is a subset of a topological space X, we write A° for its interior and $\mathrm{cl}\ A$ for its closure. We call a topological space *compact* if every open cover has a finite subcover. Thus, we do not require compact spaces to be Hausdorff. A space is *locally compact* if every point has a compact neighbourhood basis.

Let X be a partially ordered set. For a subset $A \subseteq X$ of X we denote by $\downarrow A = \{x \in X \mid \exists y \in A.\, x \leq y\}$ and $\uparrow A = \{x \in X \mid \exists y \in A.\, x \geq y\}$ the downwards and upwards closure respectively. For a point $x \in X$ we let $\downarrow x = \big\downarrow\{x\}$ and $\uparrow x = \big\uparrow\{x\}$.

Joins in a partially order set are denoted by $\vee$ or sup and meets are denoted by $\wedge$ or inf. The greatest element of a partial order is its *top* $\top$, the smallest element its *bottom* $\bot$.

We write $f\colon \subseteq X \to Y$ to indicate that f is a partial function sending X to Y. In this case we write dom f for its domain.

Let $T\colon C \to D$ be a functor between categories C and D. If X is an object of C we write $T(X)$ for the object of D that T assigns to X. We write T^n for the n^{th} iterate of T. If $f\colon X \to Y$ is a morphism in C and T is covariant we write $T_f\colon T(X) \to T(Y)$ for the induced map. We also write $f_*\colon T(X) \to T(Y)$ instead of T_f if T is clear from the context. Similarly, if T is a contravariant functor, we write $T^f\colon T(Y) \to T(X)$ or simply $f^*\colon T(Y) \to T(X)$.

2.1 Computing on Baire space

On the most basic level, computable analysis is about the study of algorithms on integer sequences. Officially, our underlying computational model is the Turing machine. As always, the exact details of this computational model do not matter and there is no benefit in defining the model more formally.

Baire space, $\mathbb{N}^{\mathbb{N}}$, is the space of natural number sequences with the product topology. This topology is induced by the metric

$$d(p,q) = 2^{-\inf\{n \in \mathbb{N} \mid p(n) \neq q(n)\}}.$$

In this formula we use the convention $\inf \emptyset = \infty$ and $2^{-\infty} = 0$.

The familiar notion of computability of natural number functions $\phi\colon \mathbb{N} \to \mathbb{N}$ can be generalised to functions $\Phi\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ on natural number sequences in a straightforward manner, by feeding the input sequence as an oracle to the algorithm.

Definition 2.1. Let $f\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a partial function on Baire space. Let $M^?$ be an oracle Turing machine. We say that $M^?$ *computes* f if for all $p \in \text{dom}(f)$ and all $n \in \mathbb{N}$, given oracle access to p and n as its input, the machine $M^?$ halts and outputs the number $f(p)(n)$. We say that f is *computable* if there exists some machine which computes f.

Definition 2.2. Let $\Omega\colon \mathbb{N} \to \mathbb{N}$ be some function. Let $f\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a partial function on Baire space. Let $M^{?,?}$ be an oracle Turing machine with two oracle tapes. We say that $M^{?,?}$ *computes* f *relative to* Ω if for all $p \in \mathrm{dom}(f)$ and all $n \in \mathbb{N}$, given oracle access to Ω and p, and n as its input, the machine $M^{?,?}$ halts and outputs the number $f(p)(n)$. We say that f is *computable relative to* Ω if there exists some machine which computes f relative to Ω.

We write $M^{\Omega,p}(n) = k$ if $M^{?,?}$ halts and outputs k on input n and with oracles Ω and p. Thus, $M^{?,?}$ computes f relative to Ω if and only if $M^{\Omega,p}(n) = f(p)(n)$ for all $p \in \mathrm{dom}(f)$ and all $n \in \mathbb{N}$.

The most fundamental basic observation of computable analysis is that relative computability is the same as continuity.

In order to show this, we need to introduce some notation. Let $\mathbb{N}^*$ denote the set of all finite integer sequences. Let $\sqsubseteq$ denote the prefix-relation on $\mathbb{N}^*$. Extend this relation to $\mathbb{N}^* \times \mathbb{N}^{\mathbb{N}}$ in the obvious manner. For a point $p \in \mathbb{N}^{\mathbb{N}}$ and a number $n \in \mathbb{N}$ write $p|_{\leq n}$ for the finite sequence $\langle p(0), \dots, p(n)\rangle$. If $(u_n)_n$ is a sequence of finite sequences $u_n \in \mathbb{N}^*$ and $p \in \mathbb{N}^{\mathbb{N}}$ is an integer sequence, write $u_n \to p$ if $u_n \sqsubseteq p$ for all $n \in \mathbb{N}$ and if for all $l \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that u_n has length at least l for all $n \geq m$.

Theorem 2.3. *Let $f\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a partial function. Then f is continuous if and only if it is computable relative to some oracle.*

Proof. Assume that f is computable relative to some oracle Ω. Fix a machine M which computes f relative to Ω. Let $p \in \mathrm{dom}(f)$. Let $n \in \mathbb{N}$. By definition the machine M halts with oracles Ω and p for each number input $k \in \{0, \dots, n\}$. Let $s_k \in \mathbb{N}$ denote the number of steps that M takes for each input k and let $s = \max_{k \leq n} s_k$. Then for all $k \leq n$, the queries that M makes to p on input k are at most of size s. It follows that if $q \in \mathbb{N}^{\mathbb{N}}$ satisfies $q(i) = p(i)$ for all $i \leq s$ then $M^{\Omega,q}(k) = M^{\Omega,p}(k)$ for all $k \leq n$. As the machine computes f we have shown that if $q \in \mathrm{dom}(f)$ satisfies $d(p,q) < 2^{-s}$ then $d(f(p), f(q)) < 2^{-n}$. Hence f is continuous at p.

Conversely, assume that f is continuous. We construct an oracle Ω relative to which f becomes computable. For each $u \in \mathbb{N}^*$ consider the set

$$A_u = \{v \in \mathbb{N}^* \mid \forall p \in \mathrm{dom}(f). (u \sqsubseteq p \to v \sqsubseteq f(p))\}.$$

This set is clearly directed (with respect to the prefix-ordering) and non-empty, so that we can define the function

$$\Omega\colon \mathbb{N}^* \to \mathbb{N}^*,\ \Omega(u) = \sup A_u.$$

Note that Ω is monotone by definition. We claim that for all $p \in \operatorname{dom}(f)$ we have $\Omega(p|_{\leq k}) \to f(p)$ as $k \to \infty$. Let $n \in \mathbb{N}$. As f is continuous there exists $m \in \mathbb{N}$ such that $d(f(p), f(q)) < 2^{-n}$ for all q with $d(p, q) < 2^{-m}$. But by definition of Ω this means that $\Omega(p|_{\leq m})$ is a prefix of $f(p)$ of length n. The claim follows. The function f can be computed relative to the oracle Ω as follows: given a point $p \in \operatorname{dom}(f)$ and $n \in \mathbb{N}$, evaluate $\Omega(p|_{\leq k})$ for $k = 1, 2, \ldots$ until the result has length $\geq n$. Output the n^{th} symbol of the result. As $\Omega(p|_{\leq k}) \to f(p)$ it follows that the algorithm halts and outputs the correct result. □

2.2 Computing on represented spaces

Computability on continuous structures, such as the real numbers, is introduced by means of *representations*.

Definition 2.4. A *represented space* is a set X together with a partial surjection $\delta_X\colon \subseteq \mathbb{N}^{\mathbb{N}} \to X$ called the *representation* of X.

Definition 2.5. Let $f\colon X \to Y$ be a function between represented spaces. A *realiser* for f is a partial function $F\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $\operatorname{dom} F \supseteq \operatorname{dom} \delta_X$ such that we have

$$\delta_Y \circ F(p) \in f \circ \delta_X(p)$$

for all $p \in \operatorname{dom} \delta_X$.

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Definition 2.6. Let $f\colon X \to Y$ be a function between represented spaces. Then f is called *computable* if it has a computable realiser and *relatively computable* if it has a continuous realiser.

It is often convenient to express the assertion that a given function $f\colon X \to Y$ is computable (relative to some oracle) by saying that $f(x)$ is uniformly computable in x (relative to some oracle). For instance we could say that the sum $x + y$ of two real numbers x and y is uniformly computable in x and y.

We obtain two categories of represented spaces: The category **QCB** where the morphisms are the (total and single-valued) relatively computable functions and the category QCB where the morphisms are the (total and single-valued) computable functions. In the sequel we will only consider the category QCB and call this "the category of represented spaces". All the results we present relativise to an arbitrary oracle and thus yield analogous results in the category **QCB**.

Isomorphisms in our category are defined as usual:

Definition 2.7. Let X and Y be represented spaces. A *computable isomorphism* is a computable map $f\colon X \to Y$ with a computable inverse map $g\colon Y \to X$. A *computable embedding* is a computable map $f\colon X \to Y$ which is a computable isomorphism onto its range.

Any represented space X can be made into a topological space by endowing it with the final topology of the representation δ_X. We call this *the standard topology on* X or just *the topology on* X. Note that this topology is necessarily sequential, as the topology on $\mathbb{N}^{\mathbb{N}}$ is sequential.

Sequential topologies play an important role in the theory of represented spaces, so let us recall some basic definitions. A subset U of a topological space is called *sequentially open* if for any convergent sequence $(x_n)_n$ in X whose limit is in U there exists an index $m \in \mathbb{N}$ such that for all $n \geq m$ we have $x_n \in U$. Complements of sequentially open sets are called *sequentially closed*. A set A is sequentially closed if and only if the limit of every convergent sequence in A belongs to A. Any open set is sequentially open, but not necessarily vice versa. A topology is called *sequential* if all its sequentially open sets are open. The collection of all sequentially open sets of a topology τ forms a sequential topology, called the *sequentialisation* of τ.

Let X be a represented space. We call a topology τ on the set X *compatible with the topology on* X if its sequentialisation coincides with the standard topology on X. In this case we also say that the topology on X is compatible with the topology τ.

Proposition 2.8. *Let X be a represented space. Let $A \subseteq X$ be an arbitrary subspace, represented by the co-restriction of δ_X to A. Then the topology on A is the sequentialisation of the relative topology induced by X. In general the topology on A can be strictly finer than the relative topology.*

Proof. That the topology is the sequentialisation of the relative topology follows from Section 4.1 together with Theorem 7 in [88]. See also [87, Section 4.1.5]. An example where the topology is finer than the relative topology is given in [43, Example 1.8]. □

Proposition 2.9. *Let X and Y be represented spaces. Then the set-theoretic product $X \times Y$ admits a representation making it into the product in the category of represented spaces. The topology on $X \times Y$ is compatible with the product topology. In general the topology on $X \times Y$ can be strictly finer than the product topology.*

Proof. For the compatibility result, see the proof of the more general Proposition 2.10 below. For an example where the topology is finer than the product topology, see [43, Example 1.11]. □

Proposition 2.10. *Let $(X_n)_n$ be a sequence of represented spaces. Then the set-theoretic product $\prod_{n\in\mathbb{N}} X_n$ admits a representation making it into the product in the category of represented spaces. The topology on $\prod_{n\in\mathbb{N}} X_n$ is compatible with the product topology. In general the topology on $\prod_{n\in\mathbb{N}} X_n$ can be strictly finer than the product topology.*

Proof. For the compatibility result, see Section 4.3 and Theorem 7 in [88]. See also [87, Section 4.1.4]. That the topology can be strictly finer than the product follows from Proposition 2.9 above. □

Theorem 2.11. *Let X and Y be represented spaces. Then the set Y^X of all relatively computable functions from X to Y admits a representation making it into the exponential in the category of represented spaces. The topology on Y^X is compatible with the compact-open topology. In general the topology on Y^X can be strictly finer than the compact-open topology.*

Proof. Combine Theorem 7 and Section 4.4 in [88]. See also [87, Section 4.2].

For an example where the topology is strictly finer than the compact-open topology, choose $X = \mathbb{Q}$ with the euclidean topology and $Y = \Sigma$ (see Definition 2.12 below). Then Y^X carries the Scott topology by Theorem 2.16 below. But the Scott topology is strictly finer than the compact-open topology as was shown in [30]. See also Definition 2.51 and the paragraph following it for a discussion of this. □

The composition of two functions $f \in Y^X$ and $g \in Z^Y$ is uniformly computable in f and g, see e.g. [81, Proposition 3].

Theorem 2.11 is the basis for certain hyperspace constructions which play a fundamental role throughout this thesis.

Definition 2.12. Sierpinski space Σ is the represented space with underlying set $\{\bot, \top\}$ and representation

$$\delta_\Sigma \colon \mathbb{N}^\mathbb{N} \to \Sigma, \; \delta_\Sigma(p) = \begin{cases} \bot & \text{if } p(n) = 0 \text{ for all } n \in \mathbb{N}, \\ \top & \text{otherwise.} \end{cases}$$

Proposition 2.13. *Let X be a represented space. Then a subset $U \subseteq X$ is open in the topology of X if and only if its characteristic function*

$$\chi_U \colon X \to \Sigma, \; (\chi_U(x) = \top \;\Leftrightarrow\; x \in U)$$

is relatively computable.

Proof. Assume that $\chi_U \colon X \to \Sigma$ is relatively computable. Then, by definition, the function $\chi_U \circ \delta_X \colon \operatorname{dom}(\delta_X) \to \Sigma$ is relatively computable. It follows from Theorem 2.3 that $\chi_U \circ \delta_X$ is continuous. It follows that the set $(\chi_U \circ \delta_X)^{-1}(\top)$ is open. But, $(\chi_U \circ \delta_X)^{-1}(\top) = \delta_X^{-1}(U)$. It follows that U is open in the final topology induced by δ_X.

Assume that $U \subseteq X$ is open. As δ_X is continuous, there exists an open set $V \in \mathcal{O}(\mathbb{N}^\mathbb{N})$ with $\delta_X^{-1}(U) = V \cap \operatorname{dom} \delta_X$. It hence suffices to show that the characteristic function of V is relatively computable. Choose a computable bijection $\pi \colon \mathbb{N} \to \mathbb{N}^*$ and let U_n be the open set of all $p \in \mathbb{N}^\mathbb{N}$ having $\pi(n)$ as a prefix. Then $(U_n)_n$ forms a basis of the topology of $\mathbb{N}^\mathbb{N}$, so that $V = \bigcup_{k \in \mathbb{N}} U_{n_k}$ for some sequence $(n_k)_k$. Now, observe that V is computable relative to the sequence $(n_k)_k$. $\square$

Definition 2.14. Let X be a represented space. The space $\mathcal{O}(X)$ of *opens of* X is the exponential Σ^X.

Computable points of $\mathcal{O}(X)$ are called *semi-decidable sets*. Similarly one obtains the space $\mathcal{A}(X)$ of *closed subsets* of X by identifying a closed set $A \subseteq X$ with its complement as an element of $\mathcal{O}(X)$.

Proposition 2.15. *Let X and Y be represented spaces. Let $f \colon X \to Y$ be a relatively computable function. Then f is continuous with respect to the topologies on X and Y. The converse need not hold true.*

Proof. If $f \colon X \to Y$ is relatively computable, then the map

$$f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X),\ U \mapsto f^{-1}(U)$$

is well-defined and relatively computable, as composition of continuous functions is uniformly computable in the functions, and $f^{-1}(U)$ is the composition of the characteristic function of U with f. It follows that f is continuous.

For an example of a continuous function that is not relatively computable, let $q\colon \mathbb{N} \to \mathbb{Q}$ be a standard enumeration of the rational numbers. Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ denote the space of all sequences $(n_k)_k$ where the sequence $(q(n_k))_k$ converges to a real number. Let R be the represented space with underlying set $\mathbb{R}$ and representation $\delta_R\colon D \to R$, where δ_R maps a sequence $(n_k)_k$ to the limit of the sequence $(q(n_k))_k$. It easily follows from Proposition 2.13 that the final topology of δ_R is the indiscrete topology. Consequently, every function $f\colon \mathbb{N}^{\mathbb{N}} \to R$ is continuous. The cardinality of the continuous functions of type $\mathbb{N}^{\mathbb{N}} \to R$ is hence strictly larger than that of the continuum. But since there are only countably many Turing machines and only continuum-cardinality many oracles, there are only continuum-cardinality many relatively computable functions. □

Theorem 2.16. *Let X be a represented space. Then the topology on $\mathcal{O}(X)$ is the Scott topology.*

Proof. Combine [38, Corollary 5.16] and [38, Theorem 7 (iii)]. □

For a more thorough discussion of the following definitions see [81].

Definition 2.17. Let X be a represented space. A subset $K \subseteq X$ is called *compact* if the set

$$\{U \in \mathcal{O}(X) \mid K \subseteq U\}$$

is an open subset of $\mathcal{O}(X)$. The space $\mathcal{K}(X)$ of *compacts of X*, also called the *upper powerspace of X*, is obtained by identifying each such set with the corresponding element of $\mathcal{O}(\mathcal{O}(X))$.

A computable point of $\mathcal{K}(X)$ is also called a *computably compact set*. A represented space X is called *computably compact* if X is a computable point in $\mathcal{K}(X)$. In other words, X is computably compact if and only if it is semi-decidable for a given open set $U \in \mathcal{O}(X)$ if U is equal to X.

Proposition 2.18. *Let X be a represented space. Then $K \subseteq X$ is compact in the sense of definition 2.17 if and only if it is a saturated compact subset of the topological space X. The topology on $\mathcal{K}(X)$ is the sequentialisation of the* *upper Vietoris topology, i.e., the topology which is generated by all sets of the form* $\{K \in \mathscr{K}(X) \mid K \subseteq U\}$*, where* $U \in \mathscr{O}(X)$.

Proof. See [87, Proposition 4.4.9 (1)], where this is called the "miss"-topology. □

Definition 2.19. The space $\mathscr{V}(X)$ of *overts of* X, also called the *lower power-space of* X, is the space of closed subsets of X, made into a represented space by identifying each closed set $A \subseteq X$ with the set

$$\{U \in \mathscr{O}(X) \mid A \cap U \neq \emptyset\} \in \mathscr{O}(\mathscr{O}(X)).$$

A computable element of $\mathscr{V}(X)$ is also called a *computably overt set.*

Proposition 2.20. *Let* X *be a represented space. Then the topology on* $\mathscr{V}(X)$ *is the sequentialisation of the lower Vietoris topology, i.e., the topology generated by all sets of the form* $\{A \in \mathscr{V}(X) \mid A \cap U \neq \emptyset\}$*, where* $U \in \mathscr{O}(X)$.

Proof. See [87, Proposition 4.4.5], where this is called the "lower Fell topology".
□

Definition 2.21. Let X be a represented space. The space $\mathscr{F}(X)$ of *located* subsets of X is the space of closed subsets of X, made into a represented space by identifying each closed set $A \subseteq X$ with the point $(A, A) \in \mathscr{A}(X) \times \mathscr{V}(X)$.

Certain separation axioms for topological spaces have computable counterparts. A space is Hausdorff if and only if the diagonal $\Delta_X = \{(x, x) | x \in X\}$ is a closed subset of the space $X \times X$. This suggests the following definition:

Definition 2.22. A represented space is called *computably Hausdorff* or *a computable Hausdorff space* if the diagonal $\Delta_X \subseteq X \times X$ is a computable point of the space $\mathscr{A}(X \times X)$. Equivalently, a space is computably Hausdorff if and only if inequality of points is semi-decidable.

It should be noted that, since products in the category of qcb-spaces are in general different from topological products (cf. Proposition 2.9), a computably Hausdorff space need not necessarily be a Hausdorff topological space. For a concrete example, see for instance [44, Example 6.2]. The property that the diagonal Δ_X is a closed subset of the product $X \times X$ in the category of sequential spaces is sometimes called *sequential Hausdorffness*. The above example shows that it is strictly weaker than topological Hausdorffness. Of course, the two notions coincide for countably based spaces.

A space is T_0 if a point is uniquely determined by its filter of open neighbourhoods. A space is computably T_0 if a point can be computably recovered from its filter of open neighbourhoods:

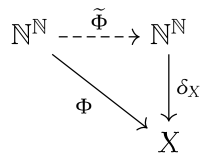
Definition 2.23. A represented space is called *computably* T_0 or *a computable* T_0 *space* if the map

$$\nu_X : X \to \mathcal{O}^2(X),\ \nu_X(x) = \{U \in \mathcal{O}(X) \mid x \in U\}$$

is a computable embedding.

Historically, the systematic study of represented spaces was initiated in order to understand what it means for a representation to capture a given topology on a set in the best way possible. This leads to the notion of *admissible representation*:

Definition 2.24 (Schröder, [88, 87])**.** Let X be a set. Let τ be a topology on X. Let $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ be a representation. We say that δ_X is *admissible* for the topology τ_X if it is continuous and every partial function $\Phi :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ which is continuous with respect to the usual topology on $\mathbb{N}^{\mathbb{N}}$ and the topology τ factors through δ_X:



We call a represented space X *admissibly represented* if the representation is admissible for the topology on X. In general, a representation can be admissible for many different topologies. These can be characterised in terms of the sequentialisation:

Proposition 2.25. *Let X be an admissibly represented space. Then the representation is admissible for a topology τ on X if and only if the sequentialisation of τ coincides with the standard topology on X.*

Proof. See for instance Lemma 8 in [88]. □

Theorem 2.26 (Schröder, [88, Theorem 4])**.** *Let X be a represented space and Y be an admissibly represented space. Then a function $f : X \to Y$ is continuous with respect to the standard topologies on X and Y if and only if it is relatively computable.*

Theorem 2.27. *Let X be a represented space. Then X is relatively computably T_0 if and only if it is admissibly represented.*

Proof (Sketch). Assume that X is admissibly represented. Then by [88, Theorem 13] X is a T_0 space. It follows that the map $\nu_X : X \to \mathcal{O}^2(X)$ is injective. It hence has a partial inverse $\nu_X^{-1} : \nu_X(X) \to X$. The pre-image of an open set $U \in \mathcal{O}(X)$ under ν_X^{-1} is evidently given by $\{\mathcal{U} \in \mathcal{O}^2(X) \mid U \in \mathcal{U}\} \cap \nu_X(X)$. As this set is clearly open in the relative topology on $\nu_X(X)$, it follows from Proposition 2.8 that ν_X^{-1} is continuous. By Theorem 2.26, the map ν_X^{-1} is relatively computable.

For the other direction we need two facts: Firstly, that Σ is admissibly represented, which follows from Proposition 2.13. Secondly, that if Y is admissibly represented, and X is an arbitrary represented space, then Y^X is admissibly represented [87, Section 4.2].. Now, assume that X is relatively computably T_0. Consider a partial continuous function $\Phi : \subseteq \mathbb{N}^{\mathbb{N}} \to X$. Then the function $\nu_X \circ \Phi : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathcal{O}^2(X)$ is continuous, and hence relatively computable thanks to Theorem 2.26. By assumption, the function $\nu_X^{-1} : \nu_X(X) \to X$ is relatively computable. As the composition of two relatively computable functions is relatively computable, the function $\Phi = \nu_X^{-1} \circ \nu_X \circ \Phi$ is relatively computable as well. It hence has a continuous realiser $\widetilde{\Phi} : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, which makes the diagram in Definition 2.24 commute. □

Corollary 2.28. *Let X be a represented space and Y be a computable T_0 space. Then a function $f : X \to Y$ is continuous with respect to the standard topologies on X and Y if and only if it is relatively computable.*

Any T_0 space X can be made into a partially ordered set by endowing it with its *specialisation order*. A point $x_0 \in X$ is below a point $x_1 \in X$ in the specialisation order, in symbols $x_0 \leq x_1$, if every open set of X that contains x_0 also contains x_1. It is useful to note that compatible topologies induce the same specialisation order:

Proposition 2.29. *Let X be a computable T_0 space. Let $(U_n)_n$ be a sequence of open sets that generate a compatible topology on X. Then $x \leq y$ with respect to the specialisation order on X if and only if $x \in U_n$ implies $y \in U_n$ for all $n \in \mathbb{N}$. In particular, compatible topologies induce the same specialisation order.*

Proof. If $x \leq y$ in the specialisation order on X then $x \in U_n$ implies $y \in U_n$ for all $n \in \mathbb{N}$ by definition.

Conversely, assume that $x \in U_n$ implies $y \in U_n$ for all $n \in \mathbb{N}$. Then the constant sequence $(y)_n$ converges to x with respect to the topology generated by the U_n's. As the topology is assumed to be compatible with the topology on X, the constant sequence $(y)_n$ converges to x in the topology on X. Hence $y \geq x$. □

2.3 Computable monads and algebras

Some of the concepts that play a role in this thesis are most naturally phrased in the language of computable monads. These are simply monads on the category of represented spaces satisfying the following local computability condition:

Definition 2.30. Let E be a covariant endofunctor on the category of represented spaces. We say that E is *locally computable* if for all represented spaces X and Y the map

$$Y^X \to E(Y)^{E(X)},\ f \mapsto E_f$$

is computable. Locally computable *contravariant* functors are defined analogously.

Definition 2.30 is not entirely satisfactory, as the computability of E is not uniform on the Hom-sets. It seems very difficult, however, to give such a definition, as the objects of our category form a proper class, for which we do not have a notion of "computability structure" available. This is why we use the term "locally computable", to emphasize that the algorithm is allowed to depend on the Hom-set. It is worth pointing out that all endofunctors we consider in this thesis use "essentially the same algorithm" on each Hom-set.

The composition of two locally computable endofunctors is locally computable. The powerspace construction $\mathscr{O}$ defines a contravariant locally computable endofunctor. The powerspace constructions $\mathscr{V}$ and $\mathscr{K}$ define covariant locally computable endofunctors.

Recall that a *monad* on a category C is an endofunctor $T\colon C \to C$ together with two natural transformations: The *unit* $\eta^T\colon \mathrm{id}_C \to T$ and the *multiplication* $\mu^T\colon T^2 \to T$ such that for each object X in C the following diagrams commute:

$$\begin{array}{ccc} T^3(X) & \xrightarrow{T\mu_X} & T^2(X) \\ {\scriptstyle \mu_{T(X)}}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ T^2(X) & \xrightarrow[\mu_X]{} & T(X) \end{array} \qquad \begin{array}{ccc} T(X) & \xrightarrow{\eta_{T(X)}} & T^2(X) \\ {\scriptstyle T(\eta_X)}\downarrow & \searrow\!\!\searrow & \downarrow{\scriptstyle \mu_X} \\ T^2(X) & \xrightarrow[\mu_X]{} & T(X) \end{array}$$

See Chapter VI of [69] for an introduction to monads.

Definition 2.31. Let M be a monad on the category of represented spaces with unit η_M and multiplication μ_M. We say that M is a *computable monad* if M is a locally computable endofunctor and η^M_X and $\mu^M_X\colon M^2(X) \to M(X)$ are computable maps for each represented space X.

Note that the condition that η^M_X and μ^M_X be computable morphisms is redundant as it is already contained in the condition that these maps be morphisms in the category of represented spaces. We have mentioned it only for emphasis.

The composition of two computable monads is a computable monad. Both $\mathscr{V}$ and $\mathscr{K}$ are computable monads in the category of computable represented spaces and in the category of computable T_0 spaces.

The unit of $\mathscr{K}$ is given by

$$\kappa_X\colon X \to \mathscr{K}(X),\ \kappa(x) = \uparrow x\ .$$

The multiplication is given by

$$\bigcup\colon \mathscr{K}(\mathscr{K}(X)) \to \mathscr{K}(X),\ I \mapsto \bigcup_{K\in I} K.$$

The unit of $\mathscr{V}$ is given by

$$\vartheta_X\colon X \to \mathscr{V}(X),\ \vartheta(x) = \downarrow x\ .$$

The multiplication is given by

$$\mathrm{cl}\bigcup\colon \mathscr{V}(\mathscr{V}(X)) \to \mathscr{V}(X),\ I \mapsto \mathrm{cl}(\bigcup_{A\in I} A).$$

Another important example of a locally computable endofunctor which is a computable monad on both the represented spaces and the computable T_0 spaces is the functor $\mathscr{O}^2$, with unit

$$\nu_X\colon X \to \mathscr{O}^2(X),\ \nu_X(x) = \{U \in \mathscr{O}(X) \mid x \in U\}$$

and multiplication

$$\mu^{\mathscr{O}^2}_X\colon \mathscr{O}^4(X) \to \mathscr{O}^2(X),\ \mu^{\mathscr{O}^2}_X(\mathscr{U}) = \left\{U \in \mathscr{O}(X) \mid \nu_{\mathscr{O}(X)}(U) \in \mathscr{U}\right\}.$$

As a final example of a computable monad, consider the adjunction of a bottom element to a represented space:

Definition 2.32. Let X be a represented space. The space $X_\bot$ has underlying set $X \cup \{\bot\}$, where $\bot$ is some point not contained in X, and the following representation $\delta_{X_\bot} \colon \subseteq \mathbb{N}^\mathbb{N} \to X_\bot$:

The domain of $\delta_{X_\bot} \colon \subseteq \mathbb{N}^\mathbb{N} \to X_\bot$ consists of the constant zero sequence in $\mathbb{N}^\mathbb{N}$ together with those sequences $(x_n)_n$ where there exists an $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \leq N$ and the sequence $(x_{n+N+1} - 1)_{n \in \mathbb{N}}$ is in the domain of δ_X.

The constant zero sequence is the unique name of $\bot$. Every other sequence defines a δ_X-name of some element $x \in X$, and $\delta_{X_\bot}$ maps each such sequence to the corresponding x.

We obtain a locally computable endofunctor which sends a space X to $X_\bot$ and a map $f \colon X \to Y$ to the map

$$f_\bot \colon X_\bot \to Y_\bot,\ f_\bot(x) = \begin{cases} \bot & \text{if } x = \bot, \\ f(x) & \text{otherwise.} \end{cases}$$

For every X we have a natural open embedding $X \to X_\bot$ and this defines the unit of a computable monad M.

Recall that an *algebra* of a monad M with multiplication μ and unit η is an object X together with a map $h \colon M(X) \to X$ called the *structure map* such that the following diagrams commute:

$$\begin{array}{ccc} M^2(X) & \xrightarrow{h_*} & M(X) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle h} \\ M(X) & \xrightarrow[h]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & M(X) \\ & {\scriptstyle \mathrm{id}_X}\searrow & \downarrow{\scriptstyle h} \\ & & X \end{array}$$

We have the following remarkable observation:

Proposition 2.33. *Let X be a represented space. Then the space $\mathscr{O}(X)$ is both a $\mathscr{V}$-algebra and a $\mathscr{K}$-algebra. The structure maps are given by*

$$\bigcup \colon \mathscr{V}(\mathscr{O}(X)) \to \mathscr{O}(X),\ A \mapsto \bigcup_{U \in A} U$$

and

$$\bigcap \colon \mathscr{K}(\mathscr{O}(X)) \to \mathscr{O}(X),\ K \mapsto \bigcap_{U \in K} U$$

respectively.

Proof. We have

$$x \in \bigcup_{U \in A} U \Leftrightarrow \exists U \in A.(x \in U).$$

It follows that $\bigcup$ is computable by the definition of $\mathscr{V}(\mathscr{O}(X))$.

Similarly, we have

$$x \in \bigcap_{U \in K} U \Leftrightarrow \forall U \in K.(x \in U).$$

It follows that $\bigcap$ is computable by the definition of $\mathscr{K}(\mathscr{O}(X))$.

To verify that $\bigcup$ and $\bigcap$ are the structure maps of $\mathscr{O}(X)$ as a $\mathscr{V}$- and $\mathscr{K}$-algebra is a routine calculation. □

Concisely put, Proposition 2.33 says that the lattice $\mathscr{O}(X)$ admits uniformly computable overt joins and compact meets.

2.4 Computably countably based spaces

In certain situations, especially in Section 2.6, it will be necessary to restrict our attention to computable T_0 spaces which are countably based with a computable basis. These can be defined as follows:

Definition 2.34. A *computably countably based space* is a represented space X which computably embeds into the space $\mathscr{O}(\mathbb{N})$.

Any computably countably based space is automatically a computable T_0 space, as it is computably isomorphic to a subspace of $\mathscr{O}(\mathbb{N})$. Any embedding $i\colon X \to \mathscr{O}(\mathbb{N})$ gives rise to the countable basis $(i^{-1}(B_n))_n$ of the topology of X, where $B_n = \{U \in \mathscr{O}(\mathbb{N}) \mid n \in U\}$. Conversely, if $(U_n)_n$ is a countable basis for the topology of X, we have an embedding

$$j\colon X \to \mathscr{O}(\mathbb{N}), j(x) = \{n \in \mathbb{N} \mid x \in U_n\}.$$

If the map j is a computable embedding, we call $(U_n)_n$ a *computable basis* for X.

Note that the basis $(i^{-1}(B_n))_n$ is a computable basis for every computable embedding $i\colon X \to \mathscr{O}(\mathbb{N})$. If $j\colon X \to \mathscr{O}(\mathbb{N})$ is an embedding induced by the basis $(U_n)_n$ then $j^{-1}(B_n) = U_n$. Hence, we have a bijection between computable bases of X and computable embeddings $X \to \mathscr{O}(\mathbb{N})$. In particular a represented space is computably countably based if and only if it has a computable basis.

Note that any computable basis for X is computable as a sequence in $\mathscr{O}(X)$.

Proposition 2.35. *Let X be a countably based T_0-space. Then X admits an admissible representation which makes it into a computably countably based space.*

Proof. Choose a countable basis $(U_n)_n$ and represent $x \in X$ by the representation where a name of $x \in X$ is any sequence $(n_i)_i$ such that for all $n \in \mathbb{N}$ we have $x \in U_n$ if and only if there exists $i \in \mathbb{N}$ with $n_i = n$. For details, see e.g. Section 2.2 in [88] and references therein. □

Computably countably based spaces are essentially the same as Weihrauch's *effective topological spaces* [106] and as the *effectively traceable spaces* introduced by Brattka and Pauly [23].

Proposition 2.36. *The following are equivalent for a computable T_0 space X:*

1. *The space X admits a computably open representation, that is, its representation is computably equivalent to a representation $\delta_X\colon \subseteq \mathbb{N}^{\mathbb{N}} \to X$ such that the function*

$$(\delta_X)_*\colon \mathscr{O}(\mathbb{N}^{\mathbb{N}}) \to \mathscr{O}(X)$$

   which sends an open subset of Baire space to an open subset of X is well-defined and computable.

2. *The space X admits a computably fibre-overt representation, i.e., its representation is computably equivalent to a representation $\delta_X\colon \subseteq \mathbb{N}^{\mathbb{N}} \to X$ such that the function*

$$\delta_X^{-1}\colon X \to \mathscr{V}(\mathbb{N}^{\mathbb{N}}),\ x \mapsto \operatorname{cl} \delta_X^{-1}(\{x\}).$$

   is computable.

Any computably countably based space satisfies both conditions.

Proof. We have $\operatorname{cl} \delta_X^{-1}(\{x\}) \cap U \neq \emptyset$ if and only if $\delta_X^{-1}(\{x\}) \cap U \neq \emptyset$ if and only if $x \in \delta_X(U)$. This establishes the equivalence between the two items.

Assume that the topology of X has a computable countable basis. Then the representation is equivalent to the representation where $p \in \mathbb{N}^{\mathbb{N}}$ is a name of $x \in X$ if and only if the sequence $(U_{p(n)})_n$ contains all basic open sets which contain x (permitting repetition). This representation is clearly computably open. □

For a computably countably based space X the powerspaces $\mathscr{O}(X)$, $\mathscr{K}(X)$, and $\mathscr{V}(X)$ can be represented using sequences of (intersections and unions of) basic open sets. These more concrete representations are useful for many constructions. Analogous representations are introduced in [106] for subsets of euclidean space and in [24] for subsets of general computable metric spaces.

Definition 2.37. Let X be a computably countably based space with a computable basis $(U_n)_n$.

1. The basis $(U_n)_n$ is *computably closed under finite intersections* if there exists a computable function $\mathsf{cap}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $U_{\mathsf{cap}(n,m)} = U_n \cap U_m$ for all $n, m \in \mathbb{N}$.

2. The basis $(U_n)_n$ is *computably closed under finite unions* if there exists a computable function $\mathsf{cup}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $U_{\mathsf{cup}(n,m)} = U_n \cup U_m$ for all $n, m \in \mathbb{N}$.

In the following we will prove certain results that rely on the existence of a computable basis which is computably closed under finite unions and intersections. This is a very mild assumption, as any computable basis can be extended to a basis which is computably closed under finite unions and intersections:

Proposition 2.38. *Let X be a computably countably based space. Let $(U_n)_n$ be a computable basis for X. Then there exist a computable basis $(V_n)_n$ for X which is computably closed under finite unions and finite intersections, and a computable map $j\colon \mathbb{N} \to \mathbb{N}$ with $V_{j(n)} = U_n$ for all $n \in \mathbb{N}$.*

Proof. Let $\langle\cdot\rangle\colon \mathbb{N}^* \to \mathbb{N}$ be a computable bijection with computable inverse.
Let

$$V_{\langle\langle n_0^0,\dots,n_{s_0}^0\rangle,\dots,\langle n_0^t,\dots,n_{s_t}^t\rangle\rangle} = \bigcup_{i=0}^{t}\bigcap_{j=0}^{s_i} U_{n_j^i}.$$

Then $(V_n)_n$ is clearly closed under finite unions and intersections. We can put $j(n) = \langle\langle n\rangle\rangle$.

It remains to show that $(V_n)_n$ is a computable basis. Let

$$i\colon X \to \mathcal{O}(\mathbb{N}),\ i(x) = \{n \in \mathbb{N} \mid x \in V_n\}.$$

Then i is a computable map, as finite intersections and unions in $\mathcal{O}(X)$ are computable by Proposition 2.33, so that the predicate $x \in V_n$ is uniformly semi-decidable in x and n. As $(U_n)_n$ is a computable basis of X which can be effectively recovered from $(V_n)_n$ using the map j, it follows that i is injective and its partial inverse is computable. □

Proposition 2.39. *Let X be a computably countably based space with a computable countable basis $(U_n)_n$ which is computably closed under finite intersections. Then the standard representation of $\mathcal{O}(X)$ is computably equivalent to the representation where $p \in \mathbb{N}^{\mathbb{N}}$ is a name for $U \in \mathcal{O}(X)$ if and only if*

$$U = \bigcup_{n \in \mathbb{N}} U_{p(n)}.$$

Proof. We can assume that X is represented by the representation where $p \in \mathbb{N}^{\mathbb{N}}$ represents $x \in X$ if and only if the sequence $(U_{p(n)})_n$ contains all basic open sets which contain x. Given $p \in \mathbb{N}^{\mathbb{N}}$ we can effectively compute $\bigcup_{n \in \mathbb{N}} U_{p(n)}$ as an element of $\mathscr{O}(X)$. Conversely, a name of an open set $U \in \mathscr{O}(X)$ is a name of a function $u \colon \mathbb{N}^{\mathbb{N}} \to \Sigma$ with $u(p) = u(q)$ whenever p and q represent the same point. Note that we can assume u to be total as any partial computable map from $\mathbb{N}^{\mathbb{N}}$ to Σ extends uniformly computably to a total one. There exists a function $v \colon \mathbb{N}^* \to \Sigma$ which can be effectively computed from u such that for all $a, b \in \mathbb{N}^*$, with a being a prefix of b, we have $v(a) \leq v(b)$, and such that for all convergent sequences $(s_n)_n$ in $\mathbb{N}^*$ we have

$$\lim v(s_n) = u(\lim s_n).$$

This function is essentially a Kleene-Kreisel associate of u. From this we can compute the sequence of all finite strings $s \in \mathbb{N}^*$ which are mapped by v to $\top$. Each such finite string represents a finite intersection of basic open subsets of U. As the basis is assumed to be computably closed under finite intersections, we can effectively compute the index of this intersection from the string. If $x \in U$, then any sufficiently long prefix of any name of x is eventually mapped to $\top$ by v. The claim follows. □

Proposition 2.40. *Let X be a computably countably based space with a computable countable basis $(U_n)_n$ which is computably closed under finite intersections and unions. Then the standard representation of $\mathscr{K}(X)$ is computably equivalent to the representation where $p \in \mathbb{N}^{\mathbb{N}}$ is a name for $K \in \mathscr{K}(X)$ if and only if the sequence $(U_{p(n)})_n$ contains all basic open sets which contain K.*

In particular, the space $\mathscr{K}(X)$ is computably countably based, a computable basis being given by the sequence $([U_n])_n$, where

$$[U_n] = \{K \in \mathscr{K}(X) \mid K \subseteq U_n\}.$$

Proof. Given a standard name of a compact set $K \in \mathscr{K}(X)$ we can clearly compute a sequence of basic open sets which contains all basic open sets which contain K.

Conversely, assume that we are given a sequence $(W_n)_n$ containing all basic open sets which contain K. Computing a name of K in the standard representation amounts to providing an algorithm which takes as input an open set $U \in \mathscr{O}(X)$ and halts if and only if $K \subseteq U$. Represent $\mathscr{O}(X)$ using the representation from Proposition 2.39.

Given $U \in \mathcal{O}(X)$ as a list of basic open sets $(U_{p(n)})_n$ with $U = \bigcup_{n\in\mathbb{N}} U_{p(n)}$, compute the sequence $V_n = \bigcup_{k\le n} U_{p(k)}$ of finite unions of the $U_{p(n)}$'s. If V_n appears somewhere in the sequence $(W_n)_n$ then halt.

As K is compact, K is contained in U if and only if it is contained in V_n for some $n \in \mathbb{N}$. If K is contained in V_n then V_n is contained in the sequence $(W_n)_n$. This shows that the algorithm halts if and only if $K \subseteq U$. □

Proposition 2.41. *Let X be a computably countably based space with a computable countable basis $(U_n)_n$ which is computably closed under finite intersections. Then the standard representation of $\mathcal{V}(X)$ is computably equivalent to the representation where $p \in \mathbb{N}^{\mathbb{N}}$ is a name for $A \in \mathcal{V}(X)$ if and only if the sequence $(U_{p(n)})_n$ contains all basic open sets which intersect A.*

Proof. Given a standard name of $A \in \mathcal{V}(X)$ we can clearly compute a list of all basic open sets which intersect A.

Suppose we are given a sequence $(W_n)_n$ of all basic open sets which intersect A. Computing a name of A in the standard representation amounts to providing an algorithm which takes as input an open set $U \in \mathcal{O}(X)$ and halts if and only if $A \cap U \neq \emptyset$. Represent $\mathcal{O}(X)$ using the representation from Proposition 2.39.

Given a list of basic open sets $(U_{p(n)})_{n\in\mathbb{N}}$ with $U = \bigcup_{n\in\mathbb{N}} U_{p(n)}$, halt if and only if there exists $n \in \mathbb{N}$ such that $U_{p(n)}$ is contained in the list $(W_n)_n$.

The set U intersects A if and only if one of the $U_{p(n)}$'s intersects A if and only if $U_{p(n)}$ is contained in the list $(W_n)_n$. This shows that the algorithm halts if and only if $U \cap A \neq \emptyset$. □

2.5 Continuous lattices

In Chapter 3 we will introduce *computable Σ-split injective lattices* which play a central role throughout this thesis. They can be viewed as natural generalisations of *continuous lattices*. It hence makes sense to recall some of the most basic facts about the latter. Almost everything we present here can be found in the standard reference [45]. Most of the concepts we discuss here make sense for general directed complete partial orders (dcpo's). For our purpose it is sufficient to consider complete lattices, and we will specialise all definitions accordingly.

Definition 2.42. A *complete lattice* is a partially ordered set L in which every subset $A \subseteq L$ has a greatest lower bound $\inf A$ and a least upper bound $\sup A$.

The greatest lower bound of a set A is also referred to as its *meet* and alternatively denoted by $\bigwedge_{x\in A} x$. Similarly, the least upper bound of A is called its *join* and denoted by $\bigvee_{x\in A} x$.

Binary meets and joins are denoted by $x \wedge y$ and $x \vee y$ respectively.

Definition 2.43 ([45, Definition I-1.1]). Let L be a complete lattice. Let $x, y \in L$. Then x is said to be *way below* y, in symbols, $x \ll y$ if for all subsets $A \subseteq L$ with $y \leq \sup A$ there exists a finite subset $A' \subseteq A$ with $x \leq \sup A'$. An element $x \in L$ which is way below itself is called *compact*.

Definition 2.44. Let L be a complete lattice. The lattice L is called *algebraic* (see [45, Definition I-4.2]) if every $x \in L$ is the supremum of the compact points below it. In symbols, if

$$x = \sup \{y \in L \mid y \ll y \leq x\}$$

for all $x \in X$.

The lattice L is called *continuous* (see [45, Definition I-1.6]) if every $x \in L$ is the supremum of the points way below it, in symbols, if

$$x = \sup \{y \in L \mid y \ll x\}.$$

Any algebraic lattice is continuous but not vice versa. The way-below relation of a continuous lattice satisfies the following *interpolation property*:

Theorem 2.45 ([45, Theorem I-1.9]). *Let L be a continuous lattice. Let $x, y \in L$ with $x \ll y$. Then there exists a point $z \in L$ with $x \ll z \ll y$.*

Any complete lattice L can be made into a topological space by endowing it with its Scott topology. Recall that a subset U of a lattice X is Scott open if it is upwards closed and for all directed sets $D \subseteq X$ whose supremum is in U we have $D \cap U \neq \emptyset$ [45, Definition II-1.3]. If L is a continuous lattice then its lattice structure can be completely recovered from this topology: the specialisation order induced by the Scott topology is the same as the original order on L [45, Theorem II-3.8].

With respect to its Scott topology, any continuous lattice is a locally compact sober space [45, Corollary II-1.13]. The continuous lattices with their Scott topology are precisely the injective objects in the category of T_0 spaces relative to the class of topological embeddings [45, Theorem II-3.8]. Hence, the second-countable continuous lattices are precisely the retracts of $\mathscr{O}(\mathbb{N})$.

The notion of continuous lattice can be effectivised using the concept of lattice bases (cf. [45, Chapter III-4, p. 243]).

Definition 2.46. Let L be a lattice. A *basis* for L is a subset $B \subseteq L$ such that for all $x \in L$ we have

$$x = \sup \{y \in B \mid y \ll x\}.$$

A lattice is called *countably based* if it has a countable basis.

A lattice has a basis if and only if it is continuous. If L is a (continuous) lattice with a basis B then the sets $(\twoheaduparrow x)_{x \in B}$ where

$$\twoheaduparrow x = \{y \in L \mid y \gg x\}$$

form a basis for the Scott topology (see the proof of Theorem III-4.5 in [45]). If L is algebraic then a basis is given by the compact elements of L and in this case the sets $(\uparrow x)_{x \ll x}$ form a basis for the Scott topology.

In particular, any countably based continuous lattice is a countably based topological space. Conversely, if a continuous lattice is a countably based space with respect to its Scott topology, then it has a countable basis in the sense of Definition 2.46 (see again the proof of Theorem III-4.5 in [45]).

Any countably based continuous lattice can be endowed with a computability structure which makes it into a computable T_0 space. The following definition is essentially a special case of [93, Definition 3.1].

Definition 2.47. A *computable continuous lattice* is a continuous lattice L with a countable basis $B = (x_n)_n$, which has the additional property that any two elements in B have an upper bound in B, such that the relation

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid x_m \ll x_n\}$$

is a computably enumerable subset of $\mathbb{N} \times \mathbb{N}$.

The *standard representation* of L (with respect to the basis B) is obtained by identifying L with a subspace of $\mathscr{O}(\mathbb{N})$ under the embedding

$$L \to \mathscr{O}(\mathbb{N}),\ x \mapsto \{n \in \mathbb{N} \mid x \gg x_n\}.$$

The standard representation of a computably countably based lattice L makes it into a computably countably based computable T_0 space: As the sets of the form $\twoheaduparrow x_n$ with $x_n \in B$ form a basis for the Scott topology, the standard representation of a computable countably based lattice is a special case of the standard representation of an effective topological space in the sense of Weihrauch [106], and any such space is a computably countably based computable T_0 space.

Another way of effectivising the notion of continuous lattice is to start with the observation that in the category of topological spaces, the countably based continuous lattices are precisely the retracts of $\mathscr{O}(\mathbb{N})$. The same is true in the category of qcb_0-spaces, as was pointed out to me by Thomas Streicher and Matthias Schröder. It was shown in [38, Corollary 6.11] that every core-compact $\mathbf{QCB}_0$-space is already countably based. As every continuous lattice in $\mathbf{QCB}_0$ is core-compact, it follows that the continuous lattices in $\mathbf{QCB}_0$ are precisely the retracts of $\mathscr{O}(\mathbb{N})$.

From this perspective it seems natural to define computable continuous lattices as the computable retracts of $\mathscr{O}(\mathbb{N})$. Let us recall the definition of computable retract first:

Definition 2.48. Let A and B be computable T_0 spaces. Then A is said to be a *computable retract* of B if there exists a computable map $s\colon A \to B$ with a computable left inverse $r\colon B \to A$, *i.e.*, the maps s and r satisfy $r \circ s = \mathrm{id}_A$.

It follows from a characterisation of effectively given domains due to Smyth [93, Theorem 3.4] that the computable continuous lattices are indeed exactly the computable retracts of $\mathscr{O}(\mathbb{N})$. We recall the proof in our special case here:

Proposition 2.49. *Let X be a represented space. Then X is computably isomorphic to a computable continuous lattice if and only if X is a computable retract of $\mathscr{O}(\mathbb{N})$.*

Proof. Let X be a computable continuous based lattice with its standard representation with respect to a chosen basis $(x_n)_n$. Then X is identified with a subspace of $\mathscr{O}(\mathbb{N})$, so we have a computable embedding $s\colon X \to \mathscr{O}(\mathbb{N})$. Using that the way-below relation on basis elements is semi-decidable, define a computable map $r\colon \mathscr{O}(\mathbb{N}) \to \mathscr{O}(\mathbb{N})$ as follows:

$$r(U) = \{m \in \mathbb{N} \mid \exists n \in U. x_m \ll x_n\}.$$

We claim that r is a retraction onto $s(X)$. Let $U \in \mathscr{O}(\mathbb{N})$. We claim that

$$m \in r(U) \;\Leftrightarrow\; x_m \ll \sup\{x_n \mid n \in U\}.$$

It then follows that $r(U) = s(\sup\{x_m \mid m \in U\})$. If $m \in r(U)$ then by definition there exists $n \in U$ with $x_m \ll x_n$, so that $x_m \ll \sup\{x_n \mid n \in U\}$. Conversely, if $x_m \ll \sup\{x_n \mid n \in U\}$ then by the interpolation property of the way-below relation there exists $x \in X$ with $x_m \ll x \ll \sup\{x_n \mid n \in U\}$. It follows from the definition of the way-below relation that $x \leq x_1 \vee \cdots \vee x_N$ for some $N \in \mathbb{N}$. As $x_m \ll x$ we obtain $x_m \ll x_1 \vee \cdots \vee x_N$. As any two elements of the basis are assumed to have an upper bound in the basis, it follows that there exists a basis element x_n with $x_m \ll x_1 \vee \cdots \vee x_N \leq x_n$. Hence $m \in r(U)$. Thus, r takes values in $s(X)$. If $U = s(x)$ then $r(U) = U$, for if $m \in \mathbb{N}$ such that there exists $n \in U$

with $x_m \ll x_n$ then $x_m \ll x_n \ll x$ so that $m \in U$ by definition of s. Hence r is a retraction onto $s(X)$.

Conversely, assume that X is a computable retract of $\mathcal{O}(\mathbb{N})$. Then there exists a computable map $s\colon X \to \mathcal{O}(\mathbb{N})$ with a computable left inverse $r\colon \mathcal{O}(\mathbb{N}) \to X$.

We will construct a computable continuous lattice L which is isomorphic to X. Let $(S_n)_n$ be a computable enumeration of all finite subsets of $\mathbb{N}$. Define a relation $\ll_X$ on $\mathbb{N}$ as follows:

$$m \ll_X n \;\Leftrightarrow\; S_m \subseteq s \circ r(S_n).$$

Note that this relation is semi-decidable.

Let $L \subseteq \mathcal{O}(\mathbb{N})$ be the space of all $\ll_X$-ideals, viewed as a subspace of $\mathcal{O}(\mathbb{N})$. More explicitly, an open set $U \in \mathcal{O}(\mathbb{N})$ is an element of L if and only if it satisfies

1. If $n \in U$ and $m \ll_X n$ then $m \in U$.
2. If $n, m \in U$ then there exists $k \in U$ with $n \ll_X k$ and $m \ll_X k$.

Note that L is a continuous lattice, as it is a retract of $\mathcal{O}(\mathbb{N})$. A retraction is given by the map

$$R\colon \mathcal{O}(\mathbb{N}) \to L,\; R(U) = \left\{ n \in \mathbb{N} \mid S_n \subseteq s \circ r\Big(\bigcup_{m \in U} S_m\Big) \right\}.$$

By definition of L we have for all $U \in L$ the equation:

$$U = \bigcup_{n \in U} \{m \in \mathbb{N} \mid m \ll n\}.$$

Therefore we can characterise the way-below relation in L as follows:

$$x \ll y \;\Leftrightarrow\; \exists n \in y. x \subseteq \{m \in \mathbb{N} \mid m \ll_X n\}.$$

It follows that the elements

$$x_n = \{m \in \mathbb{N} \mid m \ll_X n\}$$

form a basis of L, and $x \gg x_n$ if and only if $n \in x$. Hence, L is a computable continuous lattice with the standard representation induced by the basis $(x_n)_n$.

Let us show that X is isomorphic to L. Define two maps

$$f\colon X \to L,\; f(x) = \{n \in \mathbb{N} \mid S_n \subseteq s(x)\},$$

and

$$g\colon L \to X,\; g(y) = r\left(\bigcup \{S_n \mid n \in y\}\right).$$

It is obvious that g is well-defined and computable. The map f is clearly computable. To see that it is well-defined, let $x \in X$. If $n \in f(x)$ and $m \ll_X n$, then by definition we have $S_n \subseteq s(x)$ and $S_m \subseteq s \circ r(S_n)$. Applying $s \circ r$ to the first equation, we obtain $s \circ r(S_n) \subseteq s(x)$, and hence $S_m \subseteq s(x)$ by the second equation. This shows that $f(x)$ is downwards closed with respect to $\ll_X$. To see that it is $\ll_X$-directed, let $n, m \in f(x)$. Then $S_n \cup S_m \subseteq s(x)$ by definition. We can write $s(x) = \bigcup_{S_k \subseteq s(x)} S_k$. Applying $s \circ r$ to this equation yields $s(x) = \bigcup_{S_k \subseteq s(x)} s \circ r(S_k)$. As $S_n \cup S_m$ is a compact element of $\mathscr{O}(\mathbb{N})$, it follows that there exists $k \in \mathbb{N}$ with $S_n \cup S_m \subseteq s \circ r(S_k)$ and $S_k \subseteq s(x)$. In other words, there exists $k \in f(x)$ with $n \ll_X k$ and $m \ll_X k$. This shows that $f(x)$ is an $\ll_X$-ideal, so that f is well-defined.

It is easy to see that $g \circ f = \mathrm{id}_X$. Let us now show that $f \circ g = \mathrm{id}_Y$. By definition we have

$$f \circ g(y) = f \circ r\left(\bigcup \{S_n \mid n \in y\}\right).$$

Using that y is a $\ll_X$-ideal, we obtain that the set $\{S_n \mid n \in y\}$ is a directed subset of $\mathscr{O}(\mathbb{N})$. As $f \circ r$ is continuous, it preserves directed suprema, so that

$$f \circ g(y) = \sup \{f \circ r(S_n) \mid n \in y\}.$$

Note that the supremum of a directed family in L is simply given by the union, so that

$$f \circ g(y) = \bigcup \{f \circ r(S_n) \mid n \in y\}.$$

It follows from the definition of f that

$$f \circ g(y) = \{m \in \mathbb{N} \mid \exists n \in y . m \ll_X n\}.$$

Using that y is downwards closed with respect to $\ll_X$, we obtain that $f \circ g(y) \subseteq y$. Using that y is upwards directed with respect to $\ll_X$, we obtain that $f \circ g(y) \supseteq y$. Hence, f and g are inverses of each other, so that X is computably isomorphic to L. □

Together with Proposition 2.33 we obtain the following result on the computability of joins and meets:

Theorem 2.50. *Every computable continuous lattice is at the same time a $\mathscr{V}$-algebra and a $\mathscr{K}$-algebra. The structure maps are given by join and meet respectively.*

Proof. Let L be a computable complete lattice. Then there exists a computable embedding $s\colon L \to \mathscr{O}(\mathbb{N})$ with computable left inverse $r\colon \mathscr{O}(\mathbb{N}) \to L$. By Proposition 2.33 the lattice $\mathscr{O}(\mathbb{N})$ is simultaneously a $\mathscr{V}$-algebra and a $\mathscr{K}$-algebra with $\bigcup$

and $\bigcap$ being the structure maps. It follows that L is a $\mathscr{V}$-algebra whose structure map is given by the composition of the following maps:

$$\mathscr{V}(L) \xrightarrow{\mathscr{V}_s} \mathscr{V}(\mathscr{O}(\mathbb{N})) \xrightarrow{\bigcup} \mathscr{O}(\mathbb{N}) \xrightarrow{r} L$$

Analogously L is a $\mathscr{K}$-algebra with the structure map being given by the composition of the following maps:

$$\mathscr{K}(L) \xrightarrow{\mathscr{K}_s} \mathscr{K}(\mathscr{O}(\mathbb{N})) \xrightarrow{\bigcap} \mathscr{O}(\mathbb{N}) \xrightarrow{r} L$$

□

2.6 Computable commutativity of the powerspace monads

As opposed to the rest of this chapter, this section contains original results. A recent result due to de Brecht and Kawai [33] asserts that the lower and upper powerspace monads $\mathscr{V}$ and $\mathscr{K}$ satisfy the following commutativity relation for countably based consonant spaces X:

$$\mathscr{V}(\mathscr{K}(X)) \simeq \mathscr{K}(\mathscr{V}(X)) \simeq \mathscr{O}(\mathscr{O}(X)).$$

Here the symbol $\simeq$ indicates that two topological spaces are homeomorphic. The second equality holds true even without the assumption of consonance. Locale-theoretic analogues of these results were proved much earlier by Vickers [104]. In this section we prove a computable version of this result, where the homeomorphisms are replaced with computable isomorphisms. One of the results proved here, Proposition 2.56, will be required in two places: Firstly, it is used in Proposition 3.24 to show that the lattice $\mathscr{K}_\perp(X)$ is computably injective for every computably countably based computable Hausdorff space X. Secondly, it is used in Chapter 4 to prove Lemma 4.47 which is used to simplify the calculation of the universal envelope of certain set-valued functions.

We first recall the definition of consonant space [35]:

Definition 2.51. A topological space X is called *consonant* if the Scott topology on $\mathscr{O}(X)$ coincides with the compact-open topology, or equivalently, if for every Scott-open set $\mathscr{U} \subseteq \mathscr{O}(X)$ there exists a family $(K_i)_{i\in I}$ of compact sets such that $U \in \mathscr{U}$ if and only if $U \supseteq K_i$ for some $i \in I$. Sets with this property are also called *compactly generated*.

Most of the usual spaces considered in analysis are consonant. Every Polish space [35, Theorem 4.1] and even every quasi-Polish space [34] is consonant. Consonant spaces turn out to have rather erratic closure properties. For instance there exists a pair of consonant spaces whose product is not consonant [35, Example 7.2]. Open and closed subspaces of consonant spaces are consonant [35, Proposition 4.2], but G_δ-subspaces need not be [35, Proposition 7.3]. A concrete example of a non-consonant space is the space of rational numbers $\mathbb{Q}$ with the subspace topology inherited from the space $\mathbb{R}$ of real numbers [30].

Proposition 2.52. *Let X be a consonant computably countably based T_0 space. Then there exists a computable isomorphism*

$$\alpha\colon \mathcal{O}(\mathcal{K}(X)) \to \mathcal{V}(\mathcal{O}(X))$$

such that for all $K \in \mathcal{K}(X)$ and all $\mathcal{U} \in \mathcal{O}(\mathcal{K}(X))$ we have

$$K \in \mathcal{U} \;\Leftrightarrow\; \exists U \in \alpha(\mathcal{U}).\,(K \subseteq U)\,.$$

Proof. Let $(U_n)_n$ be a computable basis of X which is computably closed under finite intersections and unions.

Suppose we are given an open set $\mathcal{U} \in \mathcal{O}(\mathcal{K}(X))$. Combining Propositions 2.39 and 2.40 we can uniformly compute in $\mathcal{U}$ a sequence $p \in \mathbb{N}^{\mathbb{N}}$ such that for all $K \in \mathcal{K}(X)$ we have

$$K \in \mathcal{U} \;\Leftrightarrow\; \exists n \in \mathbb{N}.\,\big(K \subseteq U_{p(n)}\big)\,. \tag{2.1}$$

Let $\alpha(\mathcal{U})$ be the closure of the sequence $(U_{p(n)})_n$ in $\mathcal{O}(X)$. Then $\alpha(\mathcal{U})$ is uniformly computable in $\mathcal{U}$ as an overt set of opens. We have to show that the value $\alpha(\mathcal{U})$ does not depend on the choice of name of $\mathcal{U}$, that α is a computable isomorphism, and that α has the stated property. As the sequence $(U_{p(n)})_n$ is by definition dense in $\alpha(\mathcal{U})$, it follows immediately from (2.1) that

$$K \in \mathcal{U} \;\Leftrightarrow\; \exists U \in \alpha(\mathcal{U}).\,(K \subseteq U)\,. \tag{2.2}$$

It follows from (2.2) that the computable function

$$\alpha^{-1}\colon \mathcal{V}(\mathcal{O}(X)) \to \mathcal{O}(\mathcal{K}(X)),\; \alpha^{-1}(A) = \{K \in \mathcal{K}(X) \mid \exists U \in A. K \subseteq U\}$$

is the inverse of α. Hence α is a computable isomorphism.

Finally, let us show that α is well-defined, *i.e.*, that the value $\alpha(\mathcal{U})$ does not depend on the choice of name of $\mathcal{U}$. We can already view α as a multi-valued map $\alpha\colon \mathscr{O}(\mathscr{K}(X)) \rightrightarrows \mathscr{V}(\mathscr{K}(X))$ with $\alpha^{-1}\circ\alpha = \mathrm{id}$. It hence suffices to show that the inverse function α^{-1} is injective. As $\alpha^{-1}\circ\alpha(\mathscr{U}) = \mathscr{U}$ it follows that α^{-1} is constant on the values of α. By injectivity α is single-valued. Thus, let $A, B \in \mathscr{V}(\mathscr{O}(X))$. Assume that $\alpha^{-1}(A) = \alpha^{-1}(B)$. Then for a compact set $K \in \mathscr{K}(X)$ there exists $U \in A$ with $K \subseteq U$ if and only if there exists $V \in B$ such that $K \subseteq V$. In other words if we consider sets of the form $[K] = \{U \in \mathscr{O}(X) \mid K \subseteq U\}$ with $K \in \mathscr{K}(X)$ then $[K]$ intersects A if and only if it intersects B. These sets form a basis for the compact-open topology on $\mathscr{O}(X)$. As X is assumed to be consonant the compact-open topology on $\mathscr{O}(X)$ coincides with the Scott topology. Hence A and B intersect the same basic open sets and therefore have to be equal. □

Proposition 2.53. *Let X be a computably countably based consonant T_0 space. Then there exists a computable isomorphism*

$$\iota\colon \mathscr{V}(\mathscr{K}(X)) \to \mathscr{O}^2(X).$$

If $A \in \mathscr{V}(\mathscr{K}(X))$ and $K \in \mathscr{K}(X)$ then

$$K \in A \;\Leftrightarrow\; \forall U \supseteq K.\,(U \in \iota(A))\,.$$

Conversely, if $\mathscr{U} \in \mathscr{O}^2(X)$ and $U \in \mathscr{O}(X)$ then

$$U \in \mathscr{U} \;\Leftrightarrow\; \exists K \in \iota(\mathscr{U}).\,(K \subseteq U)$$

Proof. Let

$$\iota(A) = \{U \in \mathscr{O}(X) \mid \exists K \in A. K \subseteq U\}\,.$$

The function ι is clearly computable. The space $\mathscr{V}(\mathscr{K}(X))$ can be identified with a subspace of $\mathscr{O}^2(\mathscr{K}(X))$ via the embedding

$$u\colon \mathscr{V}(\mathscr{K}(X)) \to \mathscr{O}^2(\mathscr{K}(X)),\; u(A) = \{U \in \mathscr{O}(\mathscr{K}(X)) \mid \exists K \in A. K \in U\}\,.$$

Let $\alpha\colon \mathscr{O}(\mathscr{K}(X)) \to \mathscr{V}(\mathscr{O}(X))$ be the computable isomorphism from Proposition 2.52. Let

$$\widetilde{\mu}\colon \mathscr{O}^2(X) \to \mathscr{O}^2(\mathscr{K}(X)),\; \widetilde{\mu}(\mathscr{U}) = \{U \in \mathscr{O}(\mathscr{K}(X)) \mid \exists V \in \alpha(U). V \in \mathscr{U}\}\,.$$

The function $\widetilde{\mu}$ is computable. We claim that the function $\mu = u^{-1} \circ \widetilde{\mu}$ is well-defined and the inverse of ι. Let $\mathscr{U} \in \mathscr{O}^2(X)$. Using that X is consonant, let $(K_i)_i$ be a generating family of compacts for $\mathscr{U}$. We have:

$$
\begin{aligned}
&U \in \widetilde{\mu}(\mathscr{U}) \\
&\Leftrightarrow \exists V \in \alpha(U).V \in \mathscr{U} \\
&\Leftrightarrow \exists V \in \alpha(U).\exists i.K_i \subseteq V \\
&\Leftrightarrow \exists i.K_i \in U.
\end{aligned}
$$

The last equivalence uses Proposition 2.52. Thus, $\widetilde{\mu}(\mathscr{U}) = u(A)$ where A is the closure of the family $(K_i)_i$ in $\mathscr{K}(X)$. Hence the function $\mu = u^{-1} \circ \widetilde{\mu}$ is well-defined. It is easy to see that $\widetilde{\mu} \circ \iota = u$. Hence $\mu \circ \iota = \mathrm{id}$. Using that $\mu(\mathscr{U})$ is a generating family for $\mathscr{U}$ we obtain that $\iota \circ \mu(\mathscr{U}) = \mathscr{U}$. Hence μ is the inverse of ι. □

A dual result holds true for the lower powerspace. We begin with a technical lemma.

Lemma 2.54. *Let X be a computably countably based space. Let $(U_n)_n$ be a computable basis of X. Given (a name of) an open set of overts $\mathscr{U} \in \mathscr{O}(\mathscr{V}(X))$ we can compute a sequence $(\langle U_0^i, \dots, U_{n_i}^i \rangle)_i$ of finite sequences of basic open sets with*

$$A \in \mathscr{U} \;\Leftrightarrow\; \exists i \in \mathbb{N}.\forall k \leq n_i.\left(A \cap U_k^i \neq \emptyset\right).$$

Proof. It follows from Proposition 2.41 that $\mathscr{V}(X)$ is a computably countably based space and that a computable basis for $\mathscr{V}(X)$ is given by the sets of the form

$$[U_{i_1}, \dots, U_{i_n}] = \{A \in \mathscr{V}(X) \mid \forall k \leq n.A \cap U_{i_k} \neq \emptyset\}.$$

The claim now follows from Proposition 2.38 and Proposition 2.39. □

Proposition 2.55. *Let X be a computably countably based space. Then the map*

$$\beta\colon \mathscr{O}(\mathscr{V}(X)) \to \mathscr{K}(\mathscr{O}(X)),\ \beta(\mathscr{U}) = \{U \in \mathscr{O}(X) \mid \forall A \in \mathscr{U}.A \cap U \neq \emptyset\}.$$

is a computable isomorphism.

Proof. Given a name of $\mathscr{U}$, compute a sequence $(\langle U_0^i, \dots, U_{n_i}^i \rangle)_i$ of finite sequences of basic open sets as in Lemma 2.54. We claim that $\beta(\mathscr{U})$ consists of all open sets which contain an open set of the form

$$\bigcup_{i \in \mathbb{N}} U_{f(i)}^i \tag{2.3}$$

where $f\colon \mathbb{N} \to \mathbb{N}$ is a function with $f(i) \leq n_i$ for all i. It is clear that any such open set is contained in $\beta(\mathscr{U})$. On the other hand, if $U \in \mathscr{O}(X)$ does not contain such a set then there exists $i \in \mathbb{N}$ such that U does not contain any of the sets $U^i_1, \dots, U^i_{n_i}$. Hence there exist points $y_j \in U^i_j$ with $y_j \in U^i_j \setminus U$. Then $\{y_1, \dots, y_{n_i}\}$ is contained in $\mathscr{U}$ but does not intersect U. It follows that $U \notin \beta(\mathscr{U})$. Now let $\mathscr{H} \in \mathscr{O}^2(X)$. Then $\beta(\mathscr{U})$ is contained in $\mathscr{H}$ if and only if $\mathscr{H}$ contains all sets of the form (2.3). We can semi-decide if $\beta(\mathscr{U})$ is contained in $\mathscr{H}$ as follows: for all pairs $(n, s) \in \mathbb{N}^2$ run an algorithm for s steps that checks if $\mathscr{H}$ contains all finite unions of the form

$$\bigcup_{i=0}^{n} U^i_{f(i)} \tag{2.4}$$

where $f\colon \mathbb{N} \to \mathbb{N}$ is a function with $f(i) \leq n_i$ for all i. If a pair (n, s) is found, halt the computation, indicating that $\beta(\mathscr{U})$ is contained in $\mathscr{H}$. Let us show that this algorithm is correct. On the one hand, if the algorithm halts then $\mathscr{H}$ contains all finite unions of the form (2.4) for some n and hence a-fortiori all infinite unions of the form (2.3), as it is upwards closed. On the other hand assume that the algorithm does not halt. Consider the tree consisting of all finite sequences $\langle k_0, \dots, k_n \rangle$ with $k_i \leq n_i$ such that $\mathscr{H}$ does not contain the finite union $\bigcup_{i=0}^{n} U^i_{k_i}$. If this tree is finite then the algorithm eventually halts. It follows from our assumption that this tree must be infinite. Hence the tree has an infinite path. This path can be identified with a function $f\colon \mathbb{N} \to \mathbb{N}$ with $f(i) \leq n_i$ such that $\mathscr{H}$ does not contain any of the finite unions $\bigcup_{i=0}^{n} U^i_{f(i)}$. Assume $\mathscr{H} \supseteq \beta(\mathscr{U})$. Then $\mathscr{H}$ contains the infinite union $\bigcup_{n\in\mathbb{N}} U^i_{f(i)}$. As $\mathscr{H}$ is Scott-open it already contains some finite union $\bigcup_{i=0}^{n} U^i_{f(i)}$. This contradicts the existence of f. Hence $\mathscr{H}$ does not contain $\beta(\mathscr{U})$. It follows that β is well-defined and computable.

Consider the computable map

$$\beta^{-1}\colon \mathscr{K}(\mathscr{O}(Y)) \to \mathscr{O}(\mathscr{V}(Y)),\ \beta^{-1}(K) = \{A \in \mathscr{V}(Y) \mid \forall U \in K. A \cap U \neq \emptyset\}$$

We will show that β^{-1} is the inverse of β. We have

$$\beta \circ \beta^{-1}(K) = \{U \in \mathscr{O}(Y) \mid \forall A \in \mathscr{V}(Y).\, ((\forall V \in K. A \cap V \neq \emptyset) \to A \cap U \neq \emptyset)\}.$$

It is obvious that $\beta \circ \beta^{-1}(K) \supseteq K$. Let $U \in \beta \circ \beta^{-1}(K)$ and assume $U \notin K$. Then, since K is saturated,

$$\forall V \in K. \exists y \in Y. y \in V \setminus U.$$

Let $f\colon K \to Y$ be a Skolem-function for this, *i.e.*, a function satisfying

$$\forall V \in K. f(V) \in V \setminus U.$$

Then the set $A = \mathrm{cl}\,(\bigcup_{V\in K} f(V))$ intersects all $V \in K$ but does not intersect U. This contradicts the assumption that $U \in \beta \circ \beta^{-1}(K)$. Hence $U \in K$. We have

$$\beta^{-1} \circ \beta(\mathscr{U}) = \{A \in \mathscr{V}(Y) \mid \forall U \in \mathscr{O}(Y). ((\forall B \in \mathscr{U}. B \cap U \neq \emptyset) \to A \cap U \neq \emptyset)\}.$$

Again it is clear that $\beta^{-1} \circ \beta(\mathscr{U}) \supseteq \mathscr{U}$. Let $A \in \beta^{-1} \circ \beta(\mathscr{U})$ and assume $A \notin \mathscr{U}$. Then, because $\mathscr{U}$ is upwards closed,

$$\forall B \in \mathscr{U}. \exists U \in \mathscr{O}(Y). B \cap U \neq \emptyset \wedge A \cap U = \emptyset.$$

Let $f\colon \mathscr{U} \to \mathscr{O}(Y)$ be a Skolem-function for this. Then the open set $\bigcup_{B\in\mathscr{U}} f(B)$ intersects all elements of $\mathscr{U}$ but does not intersect A, contradicting the assumption $A \in \beta^{-1} \circ \beta(\mathscr{U})$. Hence $A \in \mathscr{U}$. It follows that β^{-1} is really the inverse of β. □

Proposition 2.56. *Let Y be a computably countably based space. Then the map*

$$\gamma\colon \mathscr{K}(\mathscr{V}(Y)) \to \mathscr{O}^2(Y),\ \gamma(K) = \{U \in \mathscr{O}(Y) \mid \forall A \in K. A \cap U \neq \emptyset\}$$

is a computable isomorphism.

Its inverse is given by:

$$\gamma^{-1}\colon \mathscr{O}^2(Y) \to \mathscr{K}(\mathscr{V}(Y)),\ \gamma^{-1}(\mathscr{U}) = \{A \in \mathscr{V}(Y) \mid \forall U \in \mathscr{U}. A \cap U \neq \emptyset\}.$$

Proof. Clearly the map γ is computable. Consider the map

$$\widetilde{\gamma^{-1}}\colon \mathscr{O}^2(Y) \to \mathscr{O}^2(\mathscr{V}(Y)),\ \widetilde{\gamma^{-1}}(\mathscr{H}) = \{\mathscr{U} \in \mathscr{O}(\mathscr{V}(Y)) \mid \beta(\mathscr{U}) \subseteq \mathscr{H}\}.$$

The map $\widetilde{\gamma^{-1}}$ is clearly well-defined and computable. We claim that it takes values in range of the canonical embedding

$$i\colon \mathscr{K}(\mathscr{V}(Y)) \to \mathscr{O}^2(\mathscr{V}(Y))$$

so that we obtain a computable function $\gamma^{-1} = i \circ \widetilde{\gamma^{-1}}$. More specifically, we claim that

$$\mathscr{U} \in \widetilde{\gamma^{-1}}(\mathscr{H}) \iff \mathscr{U} \supseteq \{A \in \mathscr{V}(Y) \mid \forall U \in \mathscr{H}. A \cap U \neq \emptyset\}.$$

It then follows that $\gamma^{-1}(\mathscr{H}) = \{A \in \mathscr{V}(Y) \mid \forall U \in \mathscr{H}. A \cap U \neq \emptyset\}$ is well-defined and computable.

First assume $\mathscr{U} \in \widetilde{\gamma^{-1}}(\mathscr{H})$. Let $A \in \mathscr{V}(Y)$ be a set with $A \cap U \neq \emptyset$ for all $U \in \mathscr{H}$. Our goal is to show that $A \in \mathscr{U}$. Suppose not. Then, as $\mathscr{U}$ is upwards closed,

$$\forall B \in \mathscr{U}. \exists U \in \mathscr{O}(Y). (B \cap U \neq \emptyset \wedge A \cap U = \emptyset) .$$

Let $f\colon \mathscr{U} \to \mathscr{O}(Y)$ be a Skolem function for this. Then the set $U = \bigcup_{B \in \mathscr{U}} f(B)$ intersects all elements of $\mathscr{U}$ and does not intersect A. By definition of γ^{-1} we have $\beta(\mathscr{U}) \subseteq \mathscr{H}$ from which it follows that $U \in \mathscr{H}$. But then by assumption on A we have $A \cap U \neq \emptyset$. Contradiction. It follows that $A \in \mathscr{U}$. Now assume that $\mathscr{U} \supseteq \{A \in \mathscr{V}(Y) \mid \forall U \in \mathscr{H}. A \cap U \neq \emptyset\}$. Let $U \in \mathscr{O}(Y)$ be an open set which intersects all members of $\mathscr{U}$. Our goal is to show that $U \in \mathscr{H}$. Suppose not. Then, since $\mathscr{H}$ is upwards closed, we have

$$\forall V \in \mathscr{H}. \exists y \in Y. y \in V \setminus U.$$

Let $f\colon \mathscr{H} \to Y$ be a Skolem-function for this. Then the set $A = \mathrm{cl}\left(\bigcup_{V \in \mathscr{H}} f(V)\right)$ intersects all members of $\mathscr{H}$ and does not intersect U. By assumption $A \in \mathscr{U}$ and hence by assumption on U intersects U. Contradiction. It follows that $U \in \mathscr{H}$.

Let us now show that γ^{-1} is really the inverse function of γ. We have

$$\gamma^{-1} \circ \gamma(K) = \{A \in \mathscr{V}(Y) \mid \forall U \in \mathscr{O}(Y). ((\forall B \in K. B \cap U \neq \emptyset) \to A \cap U \neq \emptyset)\} .$$

If $A \in K$ then clearly $A \in \gamma^{-1} \circ \gamma(K)$. If $A \notin K$ then since K is saturated we have

$$\forall B \in K. \exists U \in \mathscr{O}(Y). (B \cap U \neq \emptyset \wedge A \cap U = \emptyset) .$$

Let $f\colon K \to \mathscr{O}(Y)$ be a Skolem-function for this. Then $U = \bigcup_{B \in K} f(B)$ is an open set with $\forall B \in K. B \cap U \neq \emptyset$ and $A \cap U = \emptyset$. It follows that $A \notin \gamma^{-1} \circ \gamma(K)$. Hence $K = \gamma^{-1} \circ \gamma(K)$. We have

$$\gamma \circ \gamma^{-1}(\mathscr{H}) = \{U \in \mathscr{O}(Y) \mid \forall A \in \mathscr{V}(Y). ((\forall V \in \mathscr{H}. A \cap V \neq \emptyset) \to A \cap U \neq \emptyset)\} .$$

Again it is obvious that $\gamma \circ \gamma^{-1}(\mathscr{H}) \supseteq \mathscr{H}$. Let $U \in \mathscr{O}(Y)$ with $U \notin \mathscr{H}$. Then, since $\mathscr{H}$ is upwards closed, we have

$$\forall V \in \mathscr{H}. \exists y \in Y. (y \in V \wedge y \notin U).$$

Let $f\colon \mathscr{H} \to Y$ be a Skolem-function for this. Then $\mathrm{cl}\left(\bigcup_{V \in \mathscr{H}} f(V)\right)$ is an overt set with $\mathrm{cl}\left(\bigcup_{V \in \mathscr{H}} f(V)\right) \cap U = \emptyset$ and $\mathrm{cl}\left(\bigcup_{V \in \mathscr{H}} f(V)\right) \cap V \neq \emptyset$ for all $V \in \mathscr{H}$. It follows that $\gamma \circ \gamma^{-1}(\mathscr{H}) = \mathscr{H}$. □

Chapter 3

Computable complete lattices

Envelopes, the central objects of our investigation, are functions which take values in a certain class of complete lattices, which we call *computable complete lattices*. The object of this chapter is to introduce these lattices and to establish some basic results about them. The definition is a straightforward effectivisation of the classical definition of complete lattice:

Definition 3.1. A *computable complete lattice* is a computable T_0 space L which is simultaneously a $\mathscr{K}$-algebra and a $\mathscr{V}$-algebra in the category of computable T_0 spaces.

It is easy to see that the structure map of a $\mathscr{K}$-algebra has to be the meet with respect to the specialisation order. Dually, the structure map for a $\mathscr{V}$-algebra has to be the join with respect to the specialisation order. Thus, a computable complete lattice is a computable T_0 space L which uniformly computably admits all compact meets and all overt joins with respect to its specialisation order. More explicitly, the maps

$$\inf\colon \mathscr{K}(L) \to L,\ K \mapsto \inf K$$

and

$$\sup\colon \mathscr{V}(L) \to L,\ A \mapsto \sup A,$$

are required to be well-defined and computable.

Note that if L is a computable complete lattice then the points $\bot = \sup \emptyset$ and $\top = \inf \emptyset$ are automatically computable. Although Definition 3.1 only asks that the supremum exist for all closed subsets, the following proposition shows that any computable complete lattice admits arbitrary suprema. A computable complete lattice is hence indeed a complete lattice with respect to its specialisation order.

Proposition 3.2. *Let L be a computable complete lattice. Let $A \subseteq L$ be an arbitrary subset. Then* $\sup A$ *exists and* $\sup A = \sup(\operatorname{cl} A)$.

Proof. As $\operatorname{cl} A \supseteq A$ we have $\sup(\operatorname{cl} A) \geq \ell$ for all $\ell \in A$. Conversely, let $m \in L$ satisfy $m \geq \ell$ for all $\ell \in A$. Let $\ell \in \operatorname{cl} A$. Let $U \in \mathscr{O}(L)$ be an open set that contains ℓ. Then $U \cap A \neq \emptyset$, so that there exists $\ell' \in A \cap U$. As $m \geq \ell'$ by assumption, it follows that $m \in U$. Thus, $m \geq \ell$ for all $\ell \in \operatorname{cl} A$ and hence $m \geq \sup(\operatorname{cl} A)$. It follows that $\sup(\operatorname{cl} A)$ is the supremum of A. □

Theorem 2.50 asserts that every computable continuous lattice is a computable complete lattice. More generally, it follows from the proof of Theorem 2.50 that computable retracts of computable complete lattices are computable complete lattices (see Proposition 3.7 below for a general proof). By Proposition 2.33 the space $\mathscr{O}(X)$ is a computable complete lattice for every represented space X. Hence every computable retract of a space of the form $\mathscr{O}(X)$ is a computable complete lattice. The case of computable continuous lattices follows as the special case where $X = \mathbb{N}$. We immediately obtain examples of computable complete lattices which are not continuous, such as $\mathscr{O}(\mathbb{N}^{\mathbb{N}})$. We can also see immediately that every computable T_0 space embeds naturally into a computable complete lattice, as any computable T_0 space X embeds naturally into the lattice $\mathscr{O}^2(X)$. Again, the space $\mathscr{O}^2(X)$ fails to be a continuous lattice in general.

Another prototypical example of a computable complete lattice is the space $\mathscr{K}_\perp(X)$ of all compact subsets of a computable Hausdorff space X with a bottom element added:

Proposition 3.3. *Let X be a computable Hausdorff space. Then $\mathscr{K}_\perp(X)$ is a computable complete lattice. The specialisation order is given by reverse inclusion. Joins and meets are given by intersection and union respectively.*

Proof. Intersection, union, and subset inclusion on $\mathscr{K}(X)$ extend to $\mathscr{K}_\perp(X)$ in an obvious way. It is easy to see that the specialisation order is given by reverse inclusion. It follows immediately that joins are given by intersection and that meets are given by union. The semi-decidable predicate $K \subseteq U$ with $K \in \mathscr{K}(X)$ and $U \in \mathscr{O}(X)$ extends to a semi-decidable predicate on $\mathscr{K}_\perp(X) \times \mathscr{O}(X)$ by letting $\perp \subseteq U$ be false for all $U \in \mathscr{O}(X)$. Let $I \in \mathscr{K}(\mathscr{K}_\perp(X))$ and let $U \in \mathscr{O}(X)$. We have the equivalence:

$$\bigcup_{K \in I} K \subseteq U \Leftrightarrow \forall K \in I. K \subseteq U.$$

It follows that the map $\inf\colon \mathcal{K}(\mathcal{K}_\perp(X)) \to \mathcal{K}_\perp(X)$ is well-defined and computable. To see that joins are computable, first observe that directed joins are computable, for if $A \in \mathcal{V}(\mathcal{K}_\perp(X))$ is a directed set then

$$\bigcap_{K \in A} K \subseteq U \;\Leftrightarrow\; \exists K \in A.K \subseteq U.$$

Now, using that X is computably Hausdorff, binary joins in $\mathcal{K}_\perp(X)$ are well-defined and computable. It follows that the map

$$\Delta\colon \mathcal{V}(\mathcal{K}_\perp(X)) \to \mathcal{V}(\mathcal{K}_\perp(X)),\; A \mapsto \{K_1 \cap \dots \cap K_m \mid K_i \in A\}$$

which sends a set A to the set of all finite intersections of members of A is computable with

$$\bigcap_{K \in A} A = \bigcap_{K \in \Delta(A)} K.$$

As $\Delta(A)$ is always directed it follows that all overt joins are uniformly computable. □

Note that if X is not a Hausdorff space then $\mathcal{K}_\perp(X)$ still admits all directed joins, but it is not a computable complete lattice in general, as it lacks binary joins. For instance, if $X = \mathbb{N} \cup \{\infty_1\} \cup \{\infty_2\}$ is the space of natural numbers with two distinct points at infinity adjoined, then $\mathbb{N} \cup \{\infty_1\}$ and $\mathbb{N} \cup \{\infty_2\}$ are compact sets in X which do not admit a join in $\mathcal{K}_\perp(\mathbb{N})$, as their intersection is no longer compact.

For every computable Hausdorff space X we have a natural embedding into $\mathcal{K}_\perp(X)$, which is the unit of a monad $\mathcal{K}_\perp$.

The class of computable complete lattices has excellent closure properties: it admits finite products, forms an exponential ideal in the category of computable represented spaces, and is closed under retracts.

Proposition 3.4. *Let L be a computable complete lattice. Let $M \subseteq L$ be a subspace which is closed under compact meets and overt joins. Then M is a computable complete lattice.*

Proposition 3.5. *Let L and M be computable complete lattices. Then the product space $L \times M$ is a computable complete lattice as well. Joins and meets are given component-wise.*

Proof. Let $\pi_L\colon L \times M \to L$ and $\pi_M\colon L \times M \to M$ denote the canonical projections. Let $p, q \in L \times M$ be points. We claim that $p \leq q$ if and only if $\pi_L(p) \leq \pi_L(q)$ and $\pi_M(p) \leq \pi_M(q)$. For the one direction observe that π_M and π_L are monotone since they are continuous. For the other direction recall that the topology on $L \times M$ is the sequentialisation of the product topology and that a basis for the product topology is given by open sets of the form $U \times V$ where $U \in \mathscr{O}(L)$ and $V \in \mathscr{O}(M)$. Thus, assume that $\pi_L(p) \leq \pi_L(q)$ and $\pi_M(p) \leq \pi_M(q)$. Then for each basic open set $U \times V$ with $p \in U \times V$ it follows that $q \in U \times V$ and thus $p \leq q$ by Proposition 2.29.

To show that $L \times M$ admits computable compact meets, let $K \in \mathscr{K}(L \times M)$. Let

$$p = (\inf \pi_L(K), \inf \pi_M(K)) \in L \times M.$$

By assumption p is uniformly computable in K. Our goal is to show $p = \inf K$. If $q \in K$ then by definition $\pi_L(p) \leq \pi_L(q)$ and $\pi_M(p) \leq \pi_M(q)$ and hence $p \leq q$. If $r \in L \times M$ with $r \leq q$ for all $q \in K$ then $\pi_L(r) \leq \pi_L(q)$ and $\pi_M(r) \leq \pi_M(q)$ for all $q \in K$ and thus $\pi_L(r) \leq \pi_L(p)$ and $\pi_M(r) \leq \pi_M(p)$ and hence $r \leq p$.

To show that $L \times M$ admits computable overt joins, let $A \in \mathscr{V}(L \times M)$. Let

$$p = (\sup(\operatorname{cl} \pi_L(A)), \sup(\operatorname{cl} \pi_M(A))) \in L \times M.$$

Then p is uniformly computable in A. By Proposition 3.2 we have

$$p = (\sup \pi_L(A), \sup \pi_M(A)) \in L \times M.$$

It then follows that $p = \sup A$ with the same arguments as for compact infima. □

Our observation that $\mathscr{O}(X) = \Sigma^X$ is a computable complete lattice for every represented space X can be generalised from Σ to arbitrary computable complete L. In other words, computable complete lattices form an exponential ideal in the category of represented spaces.

Proposition 3.6. *Let X be a represented space. Let L be a computable complete lattice. Then the function space L^X is a computable complete lattice. Overt joins and compact meets are given point-wise.*

Proof. Let $\alpha\colon X \to L$ and $\beta\colon X \to L$ be functions. We first show that $\alpha \geq \beta$ if and only if $\alpha(x) \geq \beta(x)$. As function evaluation is continuous and hence monotone, the "only if"-part is clear. For the other direction recall that the topology on L^X is the sequentialisation of the compact-open topology. It follows that $\alpha \geq \beta$ if and only if $\beta \in [K, U]$ implies $\alpha \in [K, U]$ for all sets of the form

$$[K, U] = \{\gamma\colon X \to L \mid \gamma(K) \subseteq U\}$$

where $K \in \mathscr{K}(X)$ and $U \in \mathscr{O}(L)$. Now assume that $\alpha(x) \geq \beta(x)$ for all $x \in X$.

Let $\beta \in [K, U]$. Then for all $x \in K$ we have $\beta(x) \in U$. It follows that $\alpha(x) \in U$ for all $x \in K$. Hence $\alpha \in [K, U]$.

Let us now construct joins and meets in L^X. The computable map

$$\mathrm{eval}\colon L^X \times X \to L$$

admits computable extensions

$$\mathscr{K}_{\mathrm{eval}}\colon \mathscr{K}(L^X) \times X \to \mathscr{K}(L),\ (K, x) \mapsto \{\gamma(x) \mid \gamma \in K\}$$

and

$$\mathscr{V}_{\mathrm{eval}}\colon \mathscr{V}(L^X) \times X \to \mathscr{V}(L),\ (A, x) \mapsto \mathrm{cl}\,\{\gamma(x) \mid \gamma \in A\}.$$

If $K \in \mathscr{K}(L^X)$ is a compact set we obtain

$$\inf\ K = \lambda x.\inf(\mathscr{K}_{\mathrm{eval}}(K, x)).$$

If $A \in \mathscr{V}(L^X)$ is an overt set we obtain

$$\sup\ A = \lambda x.\sup(\mathscr{V}_{\mathrm{eval}}(A, x)).$$

The proof that this really defines supremum and infimum is analogous to Proposition 3.5. □

Proposition 3.6 is an analogue to a well-known result for continuous lattices going back to Isbell [56, 57]. Note that in general it is not true that the pointwise infimum of a family of continuous functions is again continuous. It is hence somewhat remarkable that this is always true for compact families. In the context of continuous lattices this observation goes back to Keimel and Gierz [60].

As we have mentioned already, computable complete lattices are closed under computable retracts. We give a proof for the sake of completeness:

Proposition 3.7. *Let L be a computable complete lattice. Let X be a computable retract of L. Then X is a computable complete lattice.*

Proof. Let $s\colon X \to L$ be a computable map with a computable left inverse $r\colon L \to X$. Consider the computable map $\sigma\colon \mathscr{V}(X) \to X$ which is given by the composition of the following maps:

$$\mathscr{V}(X) \xrightarrow{s_*} \mathscr{V}(L) \xrightarrow{\sup} L \xrightarrow{r} X.$$

We claim that $\sigma(A) = \sup A$ for all $A \in \mathscr{V}(X)$. Let $x \in A$. Then $s(x) \in s_*(A)$ so that $s(x) \leq \sup s_*(A)$. It follows that $x = r \circ s(x) \leq r(\sup s_*(A)) = \sigma(A)$. Hence $\sigma(A)$ is an upper bound for A. Assume that $b \geq x$ for all $x \in A$. Then $s(b) \geq s(x)$ for all $x \in A$ and hence $s(b) \geq \sup s_*(A)$.

It follows that

$$b = r \circ s(b) \geq r(\sup s_*(A)) = \sigma(A).$$

Hence $\sigma(A)$ is the supremum of A.

For compact meets, consider the map $\iota\colon \mathscr{K}(X) \to X$ which is given by the composition of the following maps:

$$\mathscr{K}(X) \xrightarrow{s_*} \mathscr{K}(L) \xrightarrow{\inf} L \xrightarrow{r} X.$$

Then $\iota(K) = \inf K$ by an analogous argument. □

The topology on a computable complete lattice is always weaker than (or equal to) the Scott topology induced by its specialisation order. To prove this we need an auxiliary result:

Proposition 3.8. *Let L be a computable complete lattice. Then the map*

$$\sup\colon L^{\mathbb{N}} \to L,\ (\ell_n)_n \mapsto \sup \{\ell_n \mid n \in \mathbb{N}\}$$

is well-defined and computable.

Proof. The map

$$L^{\mathbb{N}} \mapsto \mathscr{V}(L),\ (\ell_n)_n \mapsto \mathrm{cl}\,\{\ell_n \mid n \in N\}$$

is computable. By Proposition 3.2 the supremum of $\mathrm{cl}\,\{\ell_n \mid n \in N\}$ coincides with the supremum of the sequence $(\ell_n)_n$. As L is a computable complete lattice the supremum of $\mathrm{cl}\,\{\ell_n \mid n \in N\}$ is uniformly computable. The result follows. □

Proposition 3.9. *Let L be a computable complete lattice. Then the topology of L is weaker than (or equal to) the Scott topology induced by the specialisation order on L.*

Proof. Let $D \subseteq L$ be a directed set. We need to show that for all $U \in \mathscr{O}(L)$ we have:

$$\bigvee D \in U \;\Leftrightarrow\; \exists d \in D.\, d \in U.$$

If $\exists d \in D.\, d \in U$ then clearly $\bigvee D \in U$.

To show the other direction, let $d = \bigvee D$. Let $U \in \mathscr{O}(L)$ be an open set with $d \in U$. Fix an algorithm which computes $\sup\colon L^{\mathbb{N}} \to L$. Fix an algorithm which takes as input a name of a point $\ell \in L$ and halts - relative to some oracle - if and only if $\ell \in U$. Choose a dense sequence $(d_n)_n$ in D. Apply the composition of the two algorithms to $(d_n)_n$. As $d \in U$, the composed algorithm will eventually halt.

Upon halting, the algorithm has only read a finite initial segment of the name of $(d_n)_n$. In particular, there exists $N \in \mathbb{N}$ such that the algorithm will halt on input $d_1, d_2, \dots, d_N, d_N, \dots$. It follows that $\sup\{d_1, \dots, d_N\} \in U$. As D is assumed to be directed, we have $\sup\{d_1, \dots, d_N\} \in D$. This proves the claim. □

Corollary 3.10. *Let L be a computable complete lattice. Then every closed ideal in L is principal.*

Proof. Let $I \subseteq L$ be an ideal in L. Then I is downwards closed and directed. Hence its supremum is contained in I by Proposition 3.9. □

If X is a computable Hausdorff space then the predicate $x_0 \neq x_1$ where $x_0, x_1 \in X$ is uniformly semi-decidable in x_0 and x_1. This predicate extends to the semi-decidable predicate $K_0 \cap K_1 = \emptyset$ on $\mathscr{K}_\perp(X)$. Thus, $\mathscr{K}_\perp(X)$ itself behaves somewhat like a Hausdorff space. This motivates the following definition:

Definition 3.11. Let L be a computable complete lattice. Then L is called *computably separated* if the singleton $\{\top\}$ is semi-decidable.

If L is a computably separated computable complete lattice then the relation $\ell_0 \vee \ell_1 = \top$ is uniformly semi-decidable in ℓ_0 and ℓ_1. In particular, the space of maximal elements of $L \setminus \{\top\}$ is a computable Hausdorff space. Note that the space of maximal elements of $L \setminus \{\top\}$ is non-empty, as $\{\top\}$ is open, so that any $x \in L \setminus \{\top\}$ is below a maximal element of $L \setminus \{\top\}$. This uses that $L \setminus \{\top\}$ is Scott-closed, by Proposition 3.9.

Taking the exponential L^X of a computable complete lattice L with a represented space X need not preserve separatedness: The lattice Σ is computably separated but $\Sigma^{\mathbb{N}}$ is not. Observe that for a represented space X the lattice $\mathscr{O}(X)$ is computably separated if and only if the singleton $\{X\} \subseteq \mathscr{O}(X)$ is semi-decidable, *i.e.*, if and only if X is computably compact. This is true for general L:

Proposition 3.12. *Let L be a computably separated computable complete lattice with more than one point. Let X be represented space. Then L^X is computably separated if and only if X is computably compact.*

Proof. The top element of L^X is given by the constant function $(\lambda x.\top)$. Thus the problem of semi-deciding if a given point $\ell \in L^X$ is the top element is equivalent to semi-deciding if $\ell(x) = \top$ for all $x \in X$. On the one hand, if X is computably compact and L is computably separated then this is possible. On the other hand we can embed Σ into L by sending $\top \in \Sigma$ to $\top \in L$ and $\bot \in \Sigma$ to $\bot \in L$. This yields a computable embedding $i\colon \mathcal{O}(X) \to L^X$. Now observe that semi-deciding if $i(U)$ is equal to $\top$ is equivalent to semi-deciding if $U = X$. $\square$

3.1 Injectivity

Recall from the introduction that an envelope $F\colon X \to L$ *tightens* another envelope $G\colon X \to M$ if there exists a continuous map $\Phi\colon L \to M$ which satisfies certain properties. The intended meaning of this relation is that F encodes more information than G. However, without further constraints on the lattice M, the envelope F could fail to tighten G not because G contains information that is not contained in F, but because there aren't sufficiently many continuous maps taking L to M. In order to ensure the existence of sufficiently many continuous maps we require all envelopes to take values in lattices which are *injective* in an appropriate sense.

We recall the definition first:

Definition 3.13. Let $\mathbf{C}$ be a category. Let J be a class of morphisms in $\mathbf{C}$. We say that an object X in $\mathbf{C}$ is *J-injective* if for all morphisms $j\colon A \to B$ in J and all morphisms $f\colon A \to X$ of C there exists a morphism $\bar{f}\colon B \to X$ with $\bar{f} \circ j = f$.

It is well known that in the category of T_0 topological spaces, the injective objects relative to the class of topological embeddings are precisely the continuous lattices with their Scott topology.

As a topological embedding is a continuous map $i\colon X \to Y$ with a continuous partial inverse $i^{-1}\colon i(X) \subseteq Y \to X$, one could naively define a computable embedding to be a computable map $i\colon X \to Y$ with a computable partial inverse $i^{-1}\colon i(X) \subseteq Y \to X$. The problem with this definition is that an embedding in this sense is not a topological embedding, as subspaces in the category of computable T_0 spaces are not the same as subspaces in the category of topological spaces: the topology of a subspace A of a computable T_0 space X is the sequentialisation of the relative topology, which can be strictly finer than the relative topology itself. Consequently, not even Sierpinski space is injective relative to this class of embeddings.

In order to ensure that Σ become injective one should at least ask that an embedding $e\colon X \to Y$ be a map such that the map $\mathcal{O}^e\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ be surjective.

The resulting class of injective object turns out to be too restrictive for our purpose:

Proposition 3.14. *Consider the class of all computable functions of the form* $e\colon A \to B$ *where* A *and* B *are computable* T_0 *spaces, such that the map* $\mathcal{O}^e\colon \mathcal{O}(B) \to \mathcal{O}(A)$ *is surjective. Let* X *be a computably countably based space. If* X *is computably injective relative to this class of functions then* X *is a computable continuous lattice.*

Proof. As X is computably countably based there exists a computable embedding $i\colon X \to \mathcal{O}(\mathbb{N})$. As $\mathcal{O}(\mathbb{N})$ is countably based it is hereditarily sequential. Hence X embeds as a topological subspace of $\mathcal{O}(\mathbb{N})$. In other words, the topology on X is the relative topology induced by $\mathcal{O}(\mathbb{N})$. It follows that the map $\mathcal{O}^i\colon \mathcal{O}^2(\mathbb{N}) \to \mathcal{O}(X)$ is surjective. By injectivity of X it follows that X is a computable retract of $\mathcal{O}(\mathbb{N})$. Hence X is a computable continuous lattice. □

We have observed that every computable T_0 space embeds naturally into the computable complete lattice $\mathcal{O}^2(X)$. This property is lost if we restrict ourselves to lattices which are injective in the above sense. The lattice $\mathcal{O}^2(\mathbb{N}^{\mathbb{N}})$ is countably based but not locally compact, and hence not a continuous lattice. Proposition 3.14 shows that it cannot be injective in the above sense.

It would be interesting to find a complete characterisation of this class of injective spaces. It seems plausible that any such space is already second countable and hence a continuous lattice, but I have been unable to prove this.

In order to get a larger class of injective objects which includes spaces of the form $\mathcal{O}^2(X)$ for every computable T_0 space X we have to further restrict the class of morphisms relative to which we require injectivity. A natural strengthening of the requirement that $\mathcal{O}^e$ be surjective is that the surjectivity of $\mathcal{O}^e\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ be witnessed by a computable single-valued map $s\colon \mathcal{O}(X) \to \mathcal{O}(Y)$:

Definition 3.15. Let A and B be computable T_0 spaces. Then a map $e\colon A \to B$ is called a *computable* Σ*-split embedding* if the map $\mathcal{O}^e\colon \mathcal{O}(B) \to \mathcal{O}(A)$ has a computable section $s\colon \mathcal{O}(A) \to \mathcal{O}(B)$

Proposition 3.16. *Let* X *and* Y *be computable* T_0 *spaces. Let* $e\colon X \to Y$ *be a computable* Σ*-split embedding. Then the partial inverse* $e^{-1}\colon e(X) \subseteq Y \to X$ *is well-defined and computable.*

Proof. Consider the function $e^*\colon \mathscr{O}(Y) \to \mathscr{O}(X)$. Let $s\colon \mathscr{O}(X) \to \mathscr{O}(Y)$ be a computable section for e^*. Then s induces a map $s^*\colon \mathscr{O}^2(Y) \to \mathscr{O}^2(X)$. Define the map

$$e^{-1}\colon e(X) \subseteq Y \to X,\ e^{-1}(y) = \nu_X^{-1} \circ s^* \circ \nu_Y(y).$$

Then e^{-1} is well-defined, computable, and satisfies $e^{-1}(e(x)) = x$. □

Definition 3.17. Let X be a computable T_0 space. We call X *computably Σ-split injective*, or simply *computably injective* for short, if it is an injective object in the category of computable T_0 spaces relative to the class of computable Σ-split embeddings.

Proposition 3.18. *Let X be a computable T_0 space. Then X is computably injective if and only if the natural embedding $\nu_X\colon X \to \mathscr{O}^2(X)$ admits a computable left inverse $\rho_X\colon \mathscr{O}^2(X) \to X$.*

Proof. Assume that $\nu_X\colon X \to \mathscr{O}^2(X)$ has a left inverse $\rho_X\colon \mathscr{O}^2(X) \to X$.

Let $j\colon A \to B$ be a Σ-split embedding. Let $s\colon \mathscr{O}(A) \to \mathscr{O}(B)$ be a section of j^*. Let $f\colon A \to X$ be a map. Let $\bar{s} = s^* \circ \nu_B\colon B \to \mathscr{O}^2(A)$. We claim that $\bar{s} \circ j = \nu_A$. Indeed, we calculate:

$$\bar{s} \circ j(a) = s^* \circ \nu_B \circ j = s^* \circ j^{**} \circ \nu_A = (j^* \circ s)^* \circ \nu_A = \mathrm{id}^*_{\mathscr{O}(A)} \circ \nu_A = \mathrm{id}_{\mathscr{O}^2(A)} \circ \nu_A = \nu_A.$$

Now an extension $\bar{f}\colon B \to X$ is given by the top row of the following diagram:

$$\begin{array}{ccccccc}
B & \xrightarrow{\bar{s}} & \mathscr{O}^2(A) & \xrightarrow{f^{**}} & \mathscr{O}^2(X) & \xrightarrow{\rho_X} & X \\
& {\scriptstyle j}\nwarrow & \uparrow{\scriptstyle \nu_A} & & {\scriptstyle \nu_X}\uparrow & \nearrow{\scriptstyle \mathrm{id}_X} & \\
& & A & \xrightarrow{f} & X & &
\end{array}$$

Conversely, assume that X is computably injective. Consider the embedding $\nu_X\colon X \to \mathscr{O}^2(X)$. Then a section for $\nu_X^*\colon \mathscr{O}^3(X) \to \mathscr{O}(X)$ is given by the map $\nu_{\mathscr{O}(X)}\colon \mathscr{O}(X) \to \mathscr{O}^3(X)$, so that ν_X is a Σ-split embedding.

Indeed, we have:

$$\begin{aligned}
\nu_X^* \circ \nu_{\mathscr{O}(X)}(U) &= \nu_X^*\left(\left\{\mathscr{U} \in \mathscr{O}^2(X) \mid U \in \mathscr{U}\right\}\right) \\
&= \{x \in X \mid U \in \nu_X(x)\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

It follows from the computable injectivity of X that the identity on X extends along ν_X to a map $\rho_X\colon \mathcal{O}^2(X) \to X$. □

Corollary 3.19. *Every computably injective space is a computable complete lattice.*

Proof. By Proposition 3.18 any computably injective space is a computable retract of a computable complete lattice. By Proposition 3.7 the class of computable complete lattices is closed under retracts. □

By virtue of Corollary 3.19 and Proposition 3.18 we can use the terms "computably injective space" and "computably injective lattice" interchangeably. Indeed, by Corollary 3.19 any computably injective space is a computably complete lattice. Conversely, any computably complete lattice L which is an injective object in the category of computably complete lattices with computable maps as morphisms is a retract of $\mathcal{O}^2(L)$ (since $\mathcal{O}^2(L)$ is a computably complete lattice) by the proof of the converse direction of Proposition 3.18. It then follows from the other direction of Proposition 3.18 that L is still computably injective in the larger category of computable T_0 spaces. To emphasise the lattice structure on these spaces we generally prefer the second term.

Proposition 3.20. *Every computable continuous lattice is computably injective.*

Proof. Let L be a computable continuous lattice, effectively given via the basis $(x_n)_n$. Then there exists a computable retraction $r\colon \mathcal{O}(\mathbb{N}) \to L$. Consider the embedding $\nu_L\colon L \to \mathcal{O}^2(L)$. Let

$$f\colon \mathcal{O}^2(L) \to \mathcal{O}(\mathbb{N}),\ f(\mathcal{U}) = \left\{ n \in \mathbb{N} \mid \Uparrow x_n \in \mathcal{U} \right\}.$$

Then $r \circ f$ is a left inverse of ν_L. □

The class of computably injective lattices enjoys the same closure properties as the class of computable complete lattices:

Proposition 3.21. *Let X and Y be computably injective lattices. Then $X \times Y$ is a computably injective lattice.*

Proof. The projections $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ induce maps

$$(\pi_X)_*\colon \mathcal{O}^2(X \times Y) \to \mathcal{O}^2(X)$$

and

$$(\pi_Y)_*\colon \mathcal{O}^2(X \times Y) \to \mathcal{O}^2(Y).$$

As X and Y are computably injective, it follows from Proposition 3.18 that the natural embeddings $\nu_X\colon X \to \mathcal{O}^2(X)$ and $\nu_Y\colon Y \to \mathcal{O}^2(Y)$ have computable left inverses $\rho_X\colon \mathcal{O}^2(X) \to X$ and $\rho_Y\colon \mathcal{O}^2(Y) \to Y$. We then obtain a computable retraction $\mathcal{O}^2(X \times Y) \to X \times Y$ by composing the following maps:

$$\mathcal{O}^2(X \times Y) \xrightarrow{\langle(\pi_X)_*,(\pi_Y)_*\rangle} \mathcal{O}^2(X) \times \mathcal{O}^2(Y) \xrightarrow{\rho_X \times \rho_Y} X \times Y$$

□

Proposition 3.22. *Let L be a computably injective computable complete lattice. Let X be a represented space. Then L^X is again a computably injective computable complete lattice.*

Proof. Consider the map

$$\mathrm{eval}\colon Y^X \times X \to Y.$$

This map extends to a map

$$(\mathrm{eval})^{**}\colon \mathcal{O}^2(Y^X) \times X \to \mathcal{O}^2(Y).$$

Currying yields:

$$\lambda(\mathrm{eval})^{**}\colon \mathcal{O}^2(Y^X) \to \mathcal{O}^2(Y)^X.$$

The retraction $\rho_Y\colon \mathcal{O}^2(Y) \to Y$ induces a map $(\rho_Y)_*\colon \mathcal{O}^2(Y)^X \to Y^X$. We can then define a retraction as the composition of the following maps:

$$\mathcal{O}^2(Y^X) \xrightarrow{\lambda(\mathrm{eval})^{**}} \mathcal{O}^2(Y)^X \xrightarrow{(\rho_Y)_*} Y^X.$$

□

Proposition 3.23. *Every retract of a computably injective lattice is computably injective.*

It follows from Proposition 3.22 that any space of the form $\mathcal{O}(X)$, where X is a represented space, is a computably injective lattice. By Proposition 3.23 any retract of such a space is again a computably injective lattice. On the other hand, any computably injective lattice L is by definition a computable retract of $\mathcal{O}^2(L)$, so that the computably injective lattices are precisely the computable retracts of $\mathcal{O}(X)$ where X is some represented space. In this sense they are natural generalisations of computable continuous lattices, as these are precisely the retracts of $\mathcal{O}(\mathbb{N})$.

In particular, the space $\mathscr{O}^2(X)$ is computably injective for every represented space X, so that any computable T_0 space embeds naturally into a computably injective lattice.

Proposition 2.56 guarantees that the lattice $\mathscr{K}_\perp(X)$ is computably injective for computably countably based computable Hausdorff spaces X:

Proposition 3.24. *Let X be a computably countably based computable Hausdorff space. Then the lattice $\mathscr{K}_\perp(X)$ of compact subsets of X with a bottom element added is computably injective.*

Proof. As X is computably countably based it follows from Proposition 2.40 that the space $\mathscr{K}(X)$ is again computably countably based. It is easy to see that for every computably countably based space Z, the space $Z_\perp$ is again computably countably based. We may hence apply Proposition 2.56 to $\mathscr{K}_\perp(X)$. This yields a well-defined computable isomorphism

$$\begin{aligned}&\gamma^{-1}\colon \mathscr{O}^2(\mathscr{K}_\perp(X)) \to \mathscr{K}(\mathscr{V}(\mathscr{K}_\perp(X))),\\ &\gamma^{-1}(\mathscr{U}) = \{A \in \mathscr{V}(\mathscr{K}_\perp(X)) \mid \forall U \in \mathscr{U}.A \cap U \neq \emptyset\}.\end{aligned}$$

Consider the computable map $\rho_{\mathscr{K}_\perp(X)}\colon \mathscr{O}^2(\mathscr{K}_\perp(X)) \to \mathscr{K}_\perp(X)$ which is defined as follows:

$$\begin{array}{ccc} \mathscr{O}^2(\mathscr{K}_\perp(X)) & \xrightarrow{\rho_{\mathscr{K}_\perp(X)}} & \mathscr{K}_\perp(X) \\ {\scriptstyle\gamma^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle\bigcup} \\ \mathscr{K}(\mathscr{V}(\mathscr{K}_\perp(X))) & \xrightarrow[\mathscr{K}_{\sup}]{} & \mathscr{K}(\mathscr{K}_\perp(X)) \end{array}$$

More explicitly, we have:

$$\rho_{\mathscr{K}_\perp(X)}(\mathscr{U}) = \bigcup\left\{\bigcap_{K\in A} K \mid A \in \mathscr{V}(\mathscr{K}_\perp(X)) \wedge \forall U \in \mathscr{U}.A \cap U \neq \emptyset\right\}.$$

We claim that $\rho_{\mathscr{K}_\perp(X)}$ is a left inverse of $\nu_{\mathscr{K}_\perp(X)}$. Let $K \in \mathscr{K}_\perp(X)$. Then the overt set $\downarrow K \in \mathscr{V}(\mathscr{K}_\perp(X))$ satisfies $\forall U \in \nu_{\mathscr{K}_\perp(X)}.\downarrow K \cap U \neq \emptyset$. It follows from the definition of $\rho_{\mathscr{K}_\perp(X)}$ that $\rho_{\mathscr{K}_\perp(X)} \circ \nu_{\mathscr{K}_\perp(X)}(K) \supseteq K$.

Conversely, let $V \in \mathscr{O}(X)$ be an open set with $K \subseteq V$. Consider the set $[V] = \{H \in \mathscr{K}_\perp(X) \mid H \subseteq V\}$. Then $[V] \in \nu_{\mathscr{K}_\perp(X)}(K)$. Let $A \in \mathscr{V}(\mathscr{K}_\perp(X))$ such that $\forall U \in \nu_{\mathscr{K}_\perp(X)}(K).A \cap U \neq \emptyset$. Then there exists $H \in A$ with $H \in [V]$. Hence $\bigcap_{H\in A} H \subseteq V$. It follows from the definition of $\rho_{\mathscr{K}_\perp(X)}$ that $\rho_{\mathscr{K}_\perp(X)} \circ \nu_{\mathscr{K}_\perp(X)}(K) \subseteq V$. Hence $\rho_{\mathscr{K}_\perp(X)} \circ \nu_{\mathscr{K}_\perp(X)}(K) \subseteq K$.

In total we obtain $\rho_{\mathscr{K}_\perp(X)} \circ \nu_{\mathscr{K}_\perp(X)}(K) = K$ and the claim is shown. □

It follows from the constructive proof of Proposition 3.18 that if $f\colon A \to L$ is a continuous function which takes values in a computably injective lattice L and $e\colon A \to B$ is a computable Σ-split embedding, then the extension $\overline{f} = \rho_L \circ f^{**} \circ s^* \circ \nu_B$ of f to B is uniformly computable in f. Of course, this extension depends on the choice of s. The map $e^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ could have many different sections, each yielding a potentially different extension. On the other hand, the map e^* preserves arbitrary joins and therefore (see *e.g.* [45, Corollary O-3.5]) has an upper adjoint, that is, there exists a - not necessarily continuous - function $s\colon \mathcal{O}(X) \to \mathcal{O}(Y)$ such that for all $U \in \mathcal{O}(Y)$ and all $V \in \mathcal{O}(X)$ we have

$$e^*(U) \subseteq V \;\Leftrightarrow\; U \subseteq s^*(V).$$

See [45, Section O-3] for an introduction to adjunctions. If this upper adjoint is computable it constitutes a canonical choice for the section of e^*. This situation hence deserves special attention.

Finding a good name for maps with this property turns out to be a somewhat non-trivial task. Maps f with the property that the upper adjoint of $\mathcal{O}^f$ is Scott-continuous are called *proper* in [54]. This is justified by the observation that for maps between sober spaces this is equivalent to the classical topological definition of proper map [54, Proposition 3.3]. The requirement of sobriety is not an essential restriction, as sober spaces form a full reflective subcategory of the category of topological spaces. The situation is quite different for $\mathbf{QCB}_0$-spaces as these are not closed under sobrification [47].

Thus, in order to avoid confusion, we choose a different name, which was suggested by Escardó for entirely different reasons [40]:

Definition 3.25. Let X and Y be computable T_0 spaces. A computable map $f\colon X \to Y$ is called *computably finitary* if the upper adjoint of $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is computable.

We will reserve the term "proper map" for a slightly different concept, which is a more direct effectivisation of the classical topological definition:

Definition 3.26. A computable map $f\colon X \to Y$ is called *computably proper* if the map

$$f^{-1}\colon \mathcal{K}(Y) \to \mathcal{K}(X),\; f^{-1}(K) = \{x \in X \mid f(x) \in K\}$$

is well-defined and computable.

Recall from the paragraph after Definition 2.14 that the space $\mathcal{A}(X)$ is the space of closed subsets of X where a closed set $A \subseteq X$ is identified with its complement $A^C \in \mathcal{O}(X)$.

Computably proper maps behave as one would expect (cf. [37, Theorem 6.1]):

Proposition 3.27. *Let $f\colon X \to Y$ be a computable map between computable T_0 spaces. Then the following are equivalent:*

1. *f is computably proper.*
2. *The map*
$$f^{-1}(\uparrow\cdot)\colon Y \to \mathscr{K}(X),\ f^{-1}(\uparrow y) = \{x \in X \mid f(x) \geq y\}$$
is well-defined and computable.
3. *For every computable T_0 space Z, the map*
$$\begin{aligned}&\downarrow(\mathrm{id}_Z \times f[\cdot]) : \mathscr{A}(Z \times X) \to \mathscr{A}(Z \times Y),\\ &\downarrow(\mathrm{id}_Z \times f[A]) = \{(z,y) \in Z \times Y \mid \exists x \in A. f(x) \geq y\}\end{aligned}$$
is well-defined and computable.
4. *The map*
$$\downarrow f[\cdot] : \mathscr{A}(X) \to \mathscr{A}(Y),\ \downarrow f[A] = \{y \in Y \mid \exists x \in A. f(x) \geq y\}$$
is well-defined and computable.

Proof. Clearly, if f is computably proper then the map $f^{-1}(\uparrow\cdot) = f^{-1} \circ \kappa_Y$ is well-defined and computable.

Assume that the map
$$f^{-1}(\uparrow\cdot)\colon Y \to \mathscr{K}(X),\ f^{-1}(\uparrow y) = \{x \in X \mid f(x) \geq y\}$$
is well-defined and computable. Let Z be a computable T_0 space. Let
$$h\colon \mathscr{A}(Z \times X) \to \mathscr{A}(Z \times Y),\ h(A) = \left\{(z,y) \in Z \times Y \mid \{z\} \times f^{-1}(\uparrow y) \cap A \neq \emptyset\right\}.$$
It is easy to see that h is computable and that $h(A) = \downarrow(\mathrm{id}_Z \times f[A])$.

Taking $Z = \{*\}$, we see that the above implies that the map
$$\downarrow f[\cdot] : \mathscr{A}(X) \to \mathscr{A}(Y),\ \downarrow f[A] = \{y \in Y \mid \exists x \in A. f(x) \geq y\}$$
is well-defined and computable.

Assume that the map
$$\downarrow f[\cdot] : \mathscr{A}(X) \to \mathscr{A}(Y),\ \downarrow f[A] = \{y \in Y \mid \exists x \in A. f(x) \geq y\}$$
is well-defined and computable.

Let

$$h\colon \mathscr{K}(Y) \to \mathscr{O}^2(X),\ h(K) = \left\{ U \in \mathscr{O}(X) \mid \downarrow f\left[U^C\right] \cap K = \emptyset \right\}.$$

We claim that $h(K) = i \circ f^{-1}(K)$, where $i\colon \mathscr{K}(X) \to \mathscr{O}^2(X)$ is the natural embedding. We have to show that for every open set $U \in \mathscr{O}(X)$ we have $U \supseteq f^{-1}(K)$ if and only if $\downarrow f\left[U^C\right] \cap K = \emptyset$.

On the one hand, if $\downarrow f\left[U^C\right] \cap K = \emptyset$, then

$$f(U^C \cap f^{-1}(K)) = f(U^C) \cap f(f^{-1}(K)) \subseteq \downarrow f\left[U^C\right] \cap K = \emptyset.$$

It follows that $U^C \cap f^{-1}(K) = \emptyset$.

On the other hand, if $U^C \cap f^{-1}(K) = \emptyset$ then

$$\emptyset = f(U^C \cap f^{-1}(K)) = f(U^C) \cap f(f^{-1}(K)) = f(U^C) \cap K \cap f(X) = f(U^C) \cap K.$$

As K is upwards closed it follows that $\downarrow f\left[U^C\right] \cap K = \emptyset$. □

Any computably proper map is computably finitary and, following the observation by Hofmann and Lawson [54], the two notions agree for maps between sober spaces.

Proposition 3.28. *Let $f\colon X \to Y$ be a computable map. If f is computably proper then the upper adjoint of $\mathscr{O}^f$ is computable. If X is sober then the converse holds true as well.*

Proof. Assume that f is computably proper. Let

$$h\colon \mathscr{O}(X) \to \mathscr{O}(Y),\ h(U) = \left\{ y \in Y \mid f^{-1}(\uparrow y) \subseteq U \right\}.$$

An easy calculation shows that $f^* \circ h \leq \mathrm{id}_{\mathscr{O}(X)}$ and $h \circ f^* \geq \mathrm{id}_{\mathscr{O}(Y)}$. It follows that h is the upper adjoint of f^*.

Now assume that X is sober and that the upper adjoint $s\colon \mathscr{O}(X) \to \mathscr{O}(Y)$ of f^* is computable. Let

$$h\colon \mathscr{K}(Y) \to \mathscr{O}^2(X),\ h(K) = \{U \in \mathscr{O}(X) \mid s(U) \supseteq K\}.$$

Let $i\colon \mathscr{K}(X) \to \mathscr{O}^2(X)$ be the natural embedding. We claim that $h(K) = i \circ f^{-1}(K)$ for all $K \in \mathscr{K}(Y)$. The set $h(K)$ is an open filter of open sets and hence defines a compact set by the Hofmann-Mislove theorem. Its intersection is easily seen to be equal to $f^{-1}(K)$. The claim follows. □

3.2 Right Kan extensions and best continuous approximations

We have seen that extensions of a map $f\colon A \to L$ with values in a computably injective lattice along a map $e\colon A \to B$ are only guaranteed to exist under the rather strong assumption that e be a Σ-split embedding. On the other hand, if L is an arbitrary computable complete lattice and e is an arbitrary continuous map, then there always exists a continuous function $\bar{f}\colon B \to L$ which is, in a certain sense, the closest thing to a continuous extension one can hope to obtain:

Proposition 3.29. *Let L be a computable complete lattice. Let $f\colon A \to L$ and $i\colon A \to B$ be continuous maps. Then the set of all $g\colon B \to L$ with $g \circ i \leq f$ has a greatest element.*

Proof. Let $I = \{g\colon B \to L \mid g \circ i \leq f\}$. This set is nonempty, as it contains the constant function with value $\bot$. As L^B is a complete lattice by Proposition 3.6, the set I has a supremum $\bar{f}\colon B \to L$. Now, I is clearly closed, directed, and downwards closed, and thus a closed ideal in L^B. By Corollary 3.10 it follows that $\bar{f} \in I$. □

Proposition 3.29 essentially goes back to Scott [90]. Escardó [40] observed that the function $\bar{f}$ from Proposition 3.29 is the right Kan extension of the map f along i in the poset-enriched category of computable T_0 spaces.

In general the function

$$R\colon L^A \to L^B,\ f \mapsto \bar{f}$$

which maps a continuous function to its right Kan extension along $i\colon A \to B$ need not be continuous and hence a fortiori not computable. In [40] it is shown that this function is Scott-continuous for continuous lattices L with more than one point if and only if i is a finitary map in the sense that the upper adjoint of $i^*\colon \mathcal{O}(B) \to \mathcal{O}(A)$ is continuous.

Escardó's proof easily generalises to our situation:

Proposition 3.30. *Let $i\colon A \to B$ be a computable map. Let L be a computable complete lattice. Consider the function*

$$R\colon L^A \to L^B,\ f \mapsto \bar{f}$$

which sends a function to its right Kan extension along i. Then the following hold true:

1. *If i is computably proper or L is the one-point lattice then R is computable.*

2. *If L is computably injective and i is computably finitary then R is computable.*

3. *If R is computable then L is the one-point lattice or i is computably finitary.*

In particular, if L is computably injective or A is a sober space then R is computable if and only if L is the one-point lattice or i is computably finitary.

Proof. The function is clearly computable if L is the one-point lattice. Assume that i is computably proper. Then the map

$$i^{-1}(\uparrow\cdot)\colon B \to \mathscr{K}(Y),\ i^{-1}(\uparrow b) = \{a \in A \mid i(a) \geq b\}$$

is well-defined and computable.

Suppose we are given a function $f\colon A \to L$. We can then compute the function

$$\overline{f}(b) = \inf f_*(i^{-1}(\uparrow b))$$

uniformly in f. We claim that $\overline{f}$ is the right Kan extension of f along i. We have

$$\overline{f} \circ i(a) = \inf f_*(i^{-1}(\uparrow i(a))) \leq f(a).$$

If $h\colon B \to L$ satisfies $h \circ i \leq f$ then $h(b) \leq f(a)$ for every a with $i(a) \geq b$ and hence

$$h(b) \leq \inf \{f(a) \mid i(a) \geq b\} = \overline{f}(b).$$

Now assume that L is computably injective and that i is computably finitary. Let $s\colon \mathscr{O}(A) \to \mathscr{O}(B)$ be the upper adjoint of i. By assumption, the map s is computable. Suppose we are given a function $f\colon A \to L$. Compute the extension $\overline{f}\colon B \to L$ of f as in the proof of Proposition 3.18, *i.e.*, let $\overline{f} = \rho_L \circ f^{**} \circ s^* \circ \nu_B$. We claim that $\overline{f}$ is the right Kan extension of f along i. Assume that $h\colon B \to L$ satisfies $h \circ i \leq f$. Then

$$\rho_L \circ (h \circ i)^{**} \circ s^* \circ \nu_B \leq \rho_L \circ f^{**} \circ s^* \circ \nu_B = \overline{f}.$$

Since s is the upper adjoint of i^* we have $s \circ i^* \geq \mathrm{id}_{\mathscr{O}(B)}$. Using this we calculate:

$$\begin{aligned}
\rho_L \circ (h \circ i)^{**} \circ s^* \circ \nu_B &= \rho_L \circ h^{**} \circ i^{**} \circ s^* \circ \nu_B \\
&= \rho_L \circ h^{**} \circ (s \circ i^*)^* \nu_B \\
&\geq \rho_L \circ h^{**} \circ \nu_B \\
&= \rho_L \circ \nu_L \circ h \\
&= h.
\end{aligned}$$

Hence $\overline{f} \geq h$ and the claim follows.

Assume that L contains at least two points. Embed Σ into L by sending $\bot \in \Sigma$ to $\bot \in L$ and $\top \in \Sigma$ to $\top \in L$. Call this embedding $e\colon \Sigma \to L$. This embedding is computably Σ-split: A section of e^* is given by the map $s\colon \mathscr{O}(\Sigma) \to \mathscr{O}(L)$ which sends $\emptyset$ to $\emptyset$, Σ to L, and $\{\top\}$ to some open set which contains $\top \in L$ but does not contain $\bot \in L$. It follows that there exists a computable retraction $L \to \Sigma$.

Therefore the map

$$\Sigma^A \to \Sigma^B$$

which sends a function to its right Kan extension along i is uniformly computable. But this map is just the upper adjoint of i.

Finally, let us prove the claim in the last sentence: If R is computable and L is not the one-point lattice then i is computably finitary by (3).

Conversely, if L is computably injective it follows from (2) that if i is computably finitary then R is computable.

Similarly, if A is a sober space and i is computably finitary then i is even computably proper by Proposition 3.28. It follows from (1) that R is computable.

□

Recall that if $i\colon A \to B$ is a Σ-split embedding and $f\colon A \to L$ is a continuous function with values in a Σ-split injective lattice L then some extension $\overline{f}\colon B \to L$ of f along i can be computed uniformly in f, but this extension depends on the choice of section for i^*. The upper adjoint of i^*, should it be computable, constitutes a canonical choice. This canonical choice of section corresponds to a canonical choice of extension, namely the right Kan extension.

It will be useful to fix a notation for the different kinds of extensions we have introduced so far.

Definition 3.31. Let $i\colon A \to B$ be a computable Σ-split embedding. Let $f\colon A \to L$ be a continuous function with values in a computably injective lattice L.

A section of $\mathscr{O}^i$ will be called a Σ-*section* of i. The upper adjoint of $\mathscr{O}^i$ will be called the *upper* Σ-*adjoint* of i.

Let $s\colon \mathscr{O}(A) \to \mathscr{O}(B)$ be a computable Σ-section of i. Let

$$f/\left(\tfrac{s}{i}\right) = \rho_L \circ f^{**} \circ s^* \circ \nu_B$$

denote the extension of f along i "using" s (cf. the proof of Proposition 3.18).

If $i\colon A \to B$ is a computably finitary map, let

$$f/i = f/\left(\tfrac{s}{i}\right),$$

where $s\colon \mathscr{O}(A) \to \mathscr{O}(B)$ is the upper Σ-adjoint of i.

Proposition 3.29 is a special case of a more general result which is of independent interest and in fact constitutes the starting point for our investigation of continuous envelopes:

Proposition 3.32. *Let $f\colon X \to L$ be a function which takes a computable T_0 space X to a computable complete lattice L. Then the set of all continuous functions $F\colon X \to L$ with $F(x) \le f(x)$ for all $x \in X$ has a greatest element G. If L is a computably continuous lattice then G coincides with f in all points of continuity of f.*

Proof. Let

$$S = \{F\colon X \to L \mid F \text{ is continuous and } \forall x \in X.\,(F(x) \le f(x))\}.$$

Let

$$G(x) = \sup\left(\mathrm{cl}\,\{F(x) \mid F \in S\}\right).$$

If a continuous map $F\colon X \to L$ satisfies $F(x) \le f(x)$ then by construction $F \le G$. We claim that $G \in S$. Since any closed set is computably overt relative to some oracle, so is the set $\mathrm{cl}\,(S)$. Relative to an oracle which makes $\mathrm{cl}\,(S)$ computably overt we can compute $G(x)$ as follows: Compute the range of $\mathrm{cl}\,(S)$ under the function

$$\mathrm{eval}(\cdot, x)\colon L^X \to L.$$

This yields the set

$$\mathrm{cl}\,\{F(x) \mid F \in \mathrm{cl}\,(S)\} \in \mathscr{V}(L).$$

whose supremum is equal to $G(x)$. It follows that G is continuous.

Let $x \in X$ and $U \in \mathscr{O}(L)$ with $G(x) \in U$. As the set S is clearly directed, it follows from Proposition 3.9 that $F(x) \in U$ for some $F \in S$. By definition of S this implies $f(x) \in U$. Hence $G(x) \le f(x)$.

Now let L be computably continuous. Then there exists a computable map $s\colon L \to \mathscr{O}(\mathbb{N})$ with computable left inverse $r\colon \mathscr{O}(\mathbb{N}) \to L$. Consider the best continuous approximation G of the function $s \circ f\colon X \to \mathscr{O}(\mathbb{N})$. Let $x \in X$ be a point of continuity of f. Then x is a point of continuity of $s \circ f$. Let $n \in s \circ f(x)$. Then $s \circ f(x)$ is contained in the open set $[n] = \{U \in \mathscr{O}(\mathbb{N}) \mid n \in U\}$. As $s \circ f$ is continuous in x there exists an open set $W \in \mathscr{O}(X)$ with $x \in W \subseteq (s \circ f)^{-1}(n)$. Put

$$F\colon X \to \mathscr{O}(\mathbb{N}),\ F(z) = \begin{cases} \{n\} & \text{if } z \in W, \\ \emptyset & \text{otherwise.} \end{cases}$$

By construction we have $F(z) \leq s \circ f(z)$ for all $z \in X$ and $F(x) \in [n]$. It follows that $G(x) \in [n]$ or in other words that $n \in G(x)$. As n was an arbitrary element of $s \circ f(x)$ it follows that $G(x) = s \circ f(x)$.

It follows that $r \circ G(x) = r \circ s \circ f(x) = f(x)$ for all points of continuity $x \in X$ of f. As $r \circ G(x) \leq f(x)$ for all $x \in X$ it follows that the best continuous approximation of f coincides with f in all points of continuity. □

Proposition 3.29 follows as a special case from Proposition 3.32 as the map $\bar{f}$ can be defined as the greatest continuous approximation of the map

$$b \mapsto \inf \{f(a) \mid i(a) \geq b\}.$$

This observation is again due to Escardó [40].

Chapter 4

Envelopes

We are now ready to define the main subjects of our investigation: envelopes, the tightening relation, and universality.

We show in Theorem 4.8 that every function $f\colon X \to Y$ between computable T_0 spaces has a universal envelope. While the proof yields a concrete representative of the universal envelope, this representative turns out to be unsatisfactory in many ways.

The next three sections are dedicated to the problem of finding a better description of the universal envelope in certain situations. In Section 4.3 we establish a universality criterion for envelopes with Σ-split inclusion map. This allows us to verify for a candidate envelope if it is universal. In this case we also obtain a good description of how this envelope tightens all other envelopes. A similar description of the tightening relation can be obtained for arbitrary envelopes, as will be discussed in Section 4.4.

Theorem 4.28 shows that for functions with values in a computably countably based computable Hausdorff space which can be "enclosed" by an upper-semi-continuous functions with compact values, we have a good description of an envelope that is universal among the class of envelopes $F\colon X \to L$ where L is a separated computably injective lattice and the inclusion map $\xi_L\colon Y \to L$ sends Y to the maximal elements of L.

In Section 4.6 we make the claim precise that the universal envelope encodes all "continuously obtainable" information on a given function f, by showing that every relativised algorithm which "uses f as a subroutine in a continuous way" factors through the universal envelope.

The last three sections are mainly concerned with further techniques for finding a good description of the universal envelope of a given problem. In Section 4.7 we introduce a notion of reduction between functions that allows us to translate the universal envelope of one function to the universal envelope of another. In Section 4.8 we develop a criterion that allows us to extend a universal envelope on a dense subset to a universal envelope on the whole space. In the final section of this chapter we discuss how to model set-valued valued functions within our framework. We establish a sufficient condition for the universality of envelopes of functions with values in a lower powerspace, similar to the one in Section 4.3. We also show that the universal envelope of an upper semicontinuous function with compact values generically coincides with the function itself.

4.1 Basic definitions and observations

Definition 4.1. Let Y be a computable T_0 space. An *approximation lattice* for Y consists of a computably injective lattice L together with a computable map $\xi_L \colon Y \to L$ called the *inclusion map*.

The approximation lattice L is called *computably separated* if the lattice L is computably separated and the inclusion map sends Y to the maximal elements of $L \setminus \{\top\}$.

Definition 4.2. Let $f \colon X \to Y$ be a function between computable T_0 spaces. An *envelope* of f consists of an approximation lattice L for Y with inclusion map $\xi_L \colon Y \to L$ and a continuous map $F \colon X \to L$ such that $F(x) \leq \xi_L \circ f(x)$ for all $x \in X$. If the map F is computable we call F a *computable envelope*. An envelope is called *computably separated* if L is a computably separated approximation lattice.

As an immediate corollary to Proposition 3.32 we obtain:

Proposition 4.3. *Let $f \colon X \to Y$ be a function between computable T_0 spaces. Let L be an approximation lattice for Y. Then the set of all continuous functions $G \colon X \to L$ with $G(x) \leq \xi_L \circ f(x)$ for all $x \in X$ has a greatest element $F \colon X \to L$. If L is computably continuous then F coincides with $\xi_L \circ f$ in all its points of continuity.*

We call the function F from Proposition 4.3 the *principal L-envelope* of f.

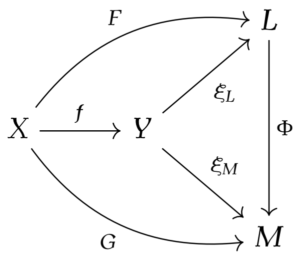
Definition 4.4. Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ and $G\colon X \to M$ be envelopes of f. We say that F *topologically tightens* G if there exists a continuous map $\Phi\colon L \to M$ such that

$$\Phi \circ \xi_L(y) \leq \xi_M(y)$$

for all $y \in Y$ and

$$\Phi \circ F(x) \geq G(x)$$

for all $x \in X$. If the map Φ can be chosen to be computable we say that F *computably tightens* G.



It is easy to see that the tightening relation is reflexive and transitive and hence a preorder on the class of all envelopes.

We will usually drop the adverb "topologically" or "computably" and just say that "F tightens G" if it is either clear from the context or it does not matter up to relativisation which relation we mean. We call two envelopes *equivalent* if they are equivalent with respect to the equivalence relation induced by the tightening order.

Definition 4.5. Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope of f. The envelope F is called *(separated-) universal* if it (is separated and) topologically tightens every continuous (separated) envelope of f.

Example 4.6. Our notion of envelope generalises the following well-known construction in analysis: Let $f\colon X \to \mathbb{R}$ be an arbitrary real-valued function on a topological space X. It is well-known (see e.g. [60]) that there exists a largest lower semicontinuous function $f^-\colon X \to \mathbb{R}$ pointwise below f, and a smallest upper semicontinuous function $f^+\colon X \to \mathbb{R}$ pointwise above f. These are often referred to as the *lower* and *upper semicontinuous envelope of f* respectively. The lower semicontinuous envelope is for instance used in [59] to prove a Hahn-Banach type "sandwich" theorem in semitopological cones.

One can join f^- and f^+ to obtain a best continuous approximation to f with values in the interval domain over $\mathbb{R}$, *i.e.*, the lattice of (possibly degenerate) real intervals, ordered by reverse inclusion. This map agrees with f in all points of continuity (see [39, Theorem 8.8]). This is used in [39] to model Riemann-integrable functions by interval functions, see [39, Theorem 13.9]. A similar idea is used in [41] to embed a function space into a compact function space with larger co-domain.

The above are examples of envelopes in our sense. Assume now that X is a computable T_0 space. Consider the complete lattice $L = [-\infty, +\infty]_{\leq}$ of real numbers with a point at positive and negative infinity added, ordered with the usual ordering. This can be made into a computable T_0 space by endowing it with its Scott topology. In fact, it then becomes a computably injective lattice. A retraction $\rho\colon \mathscr{O}^2(L) \to L$ is given by

$$\rho(\mathscr{U}) = \sup \{x \in [-\infty, +\infty] \mid (x, +\infty] \in \mathscr{U}\}.$$

A function $f\colon X \to \mathbb{R}$ is lower semicontinuous if and only if it is continuous as a function $f\colon X \to L$. The lower semicontinuous envelope of a function $f\colon X \to \mathbb{R}$ is hence the universal L-envelope with inclusion map $\xi_L\colon \mathbb{R} \to L$, $\xi_L(x) = x$. The upper semicontinuous envelope of $f\colon X \to \mathbb{R}$ is the universal envelope in the dual lattice $[-\infty, +\infty]_{\geq}$. The join of these two envelopes is the principal envelope in the interval lattice $I(\mathbb{R})$, which is obtained as the subspace of $\mathscr{K}([-\infty, +\infty])$ consisting of all (potentially degenerate) intervals, together with the empty set.

As one might expect, the principal envelope in the interval lattice can fail to be universal, already for very simple examples. Consider again the Heaviside function (cf. Example 1 in the introduction), this time taking the real numbers as its co-domain:

$$H\colon \mathbb{R} \to \mathbb{R},\ H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if x } \geq 0. \end{cases}$$

Then the principal envelope in the interval lattice is the map:

$$G\colon \mathbb{R} \to I(\mathbb{R}),\ G(x) = \begin{cases} [0,0] & \text{if } x < 0, \\ [0,1] & \text{if } x = 0, \\ [1,1] & \text{if } x > 0. \end{cases}$$

The universal envelope is easily seen to be:

$$F\colon \mathbb{R} \to \mathscr{K}_\perp(\mathbb{R}),\ F(x) = \begin{cases} \{0\} & \text{if } x < 0, \\ \{0,1\} & \text{if } x = 0, \\ \{1\} & \text{if } x > 0. \end{cases}$$

We claim that G does not tighten F. Assume the contrary. Then there exists a continuous map $\Phi\colon I(\mathbb{R}) \to \mathscr{K}(\mathbb{R})$ satisfying:

1. $\Phi([x,x]) \supseteq \{x\}$ for all $x \in \mathbb{R}$.

2. $\Phi \circ G(x) \subseteq F(x)$ for all $x \in \mathbb{R}$.

In particular, we have

$$\{\tfrac{1}{2}\} \subseteq \Phi([\tfrac{1}{2}, \tfrac{1}{2}]) \subseteq \Phi([0,1]) = \Phi \circ G(0) \subseteq F(0) = \{0,1\}.$$

Contradiction.

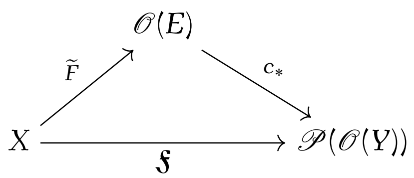
Envelopes can be viewed as encodings of partial topological information on a function. This idea can be made precise as follows:

Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let E be a distributive computably complete lattice and let $c\colon E \to \mathscr{O}(Y)$ be a computable map which preserves overt joins and compact meets. We should think of c as an effective cover of a subspace of $\mathscr{O}(Y)$, *i.e.*, an encoding of observable information of elements of Y. This yields a Scott-continuous map

$$c_*\colon \mathscr{O}(E) \to \mathscr{P}(\mathscr{O}(Y)),\; c_*(U) = \uparrow\{c(e) \mid e \in U\}\,.$$

In other words, open sets of E represent collections of open sets of Y.

Let $F\colon X \to \mathscr{O}(E)$ be a continuous function. Then F and c encode a continuous function $\mathfrak{F}\colon X \to \mathscr{P}(\mathscr{O}(Y))$:



Let

$$j\colon Y \to \mathscr{P}(\mathscr{O}(Y)),\; j(y) = \{U \in \mathscr{O}(Y) \mid y \in U\}\,.$$

Call a pair (F,c) a *co-envelope* of f if $c_* \circ F(x) \subseteq j \circ f(x)$ for all $x \in X$.

In other words, a co-envelope of f is an effective encoding of a function $\mathfrak{F}\colon X \to \mathscr{P}(\mathscr{O}(Y))$ with the property that all elements $U \in \mathfrak{F}(x)$ are "observable properties" of $f(x)$.

If $\mathfrak{F}\colon X \to \mathscr{P}(\mathscr{O}(Y))$ and $\mathfrak{G}\colon X \to \mathscr{P}(\mathscr{O}(Y))$ are co-envelopes given by effective encodings (F, c_F) and (G, c_G) then $\mathfrak{F}$ contains more information than $\mathfrak{G}$ if $\mathfrak{F}(x) \supseteq \mathfrak{G}(x)$ for all $x \in X$. Beyond this, we should ask that this relation be witnessed by a continuous map, so that the information encoded in $\mathfrak{G}$ can be "effectively retrieved" from the information encoded in $\mathfrak{F}$. Thus, we should ask that there be a continuous Skolem-function for the predicate $\forall x \in X.\mathfrak{F}(x) \supseteq \mathfrak{G}(x)$ which can be formally written as

$$\forall x \in X. \forall e \in E_G. \exists e' \in E_F. \left(c_F(e') \subseteq c_G(e) \wedge \left(e \in G(x) \to e' \in F(x)\right)\right).$$

In other words, there should exist a map $t\colon E_G \to E_F$ such that for all $x \in X$ and all $e \in E_G$ we have that $e \in G(x)$ implies $t(e) \in F(x)$ and $c_F \circ t(e) \subseteq c_G(e)$ for all $e \in E_G$. If such a map t exists, we say that $\mathfrak{F}$ *tightens* $\mathfrak{G}$.

In order to make this notion well-behaved we should require that there be sufficiently many continuous maps of type $E_G \to E_F$. This naturally leads to the requirement that E_F and E_G be computably injective lattices.

With this additional assumption the notions of "envelope" and "co-envelope" become - in a sense - dually equivalent:

An envelope $F\colon X \to L$ with inclusion map $\xi_L\colon Y \to L$ can be sent to the co-envelope $F^{**} \circ \nu_X\colon X \to \mathscr{O}^2(L)$ with encoding $\xi_L^*\colon \mathscr{O}(L) \to \mathscr{O}(Y)$. If an envelope $F\colon X \to L$ tightens another envelope $G\colon X \to M$ via a map Φ then the corresponding co-envelope F^{**} tightens the co-envelope G^{**} via the map $\Phi^*\colon \mathscr{O}(M) \to \mathscr{O}(L)$.

Conversely, if $F\colon X \to \mathscr{O}(E)$ is a co-envelope with encoding $c\colon E \to \mathscr{O}(Y)$ then F defines the envelope $F\colon X \to \mathscr{O}(E)$ with inclusion map $c^* \circ \nu_Y\colon Y \to \mathscr{O}(E)$. If the co-envelope $F\colon X \to \mathscr{O}(E)$ tightens the co-envelope $G\colon X \to \mathscr{O}(D)$ via a map $t\colon D \to E$, then the corresponding envelope F tightens the corresponding envelope G via the map $t^*\colon \mathscr{O}(E) \to \mathscr{O}(D)$.

If we start with an envelope $F\colon X \to L$ and apply both functors in succession then we end up with the envelope $F^{**}\colon X \to \mathscr{O}^2(L)$ whose inclusion map is given by $\xi_L^{**} \circ \nu_Y\colon Y \to \mathscr{O}^2(L)$. This envelope is equivalent to F thanks to the injectivity of L. A similar observation applies to co-envelopes.

In this sense we can view envelopes as effective encodings of continuous maps of type $X \to \mathscr{P}(\mathscr{O}(Y))$. Explicitly, an envelope $F\colon X \to L$ with inclusion map $\xi_L\colon Y \to L$ encodes the continuous function

$$\mathfrak{F}\colon X \to \mathscr{P}(\mathscr{O}(Y)),\ \mathfrak{F}(x) = \uparrow\left\{\xi_L^{-1}(U) \in \mathscr{O}(Y) \mid F(x) \in U\right\}$$

Note in particular that equivalent envelopes encode the same function of type $X \to \mathscr{P}(\mathscr{O}(Y))$. Equivalent envelopes can hence be viewed as equivalent encodings of the same object.

Since $\mathscr{P}(\mathscr{O}(Y))$ is an algebraic lattice it follows that $j \circ f$ has a best continuous approximation which is given explicitly by the function

$$\mathfrak{F}(x) = \left\{U \in \mathscr{O}(L) \mid f^{-1}(U) \text{ is a neighbourhood of } x\right\}. \tag{4.1}$$

Indeed, a basis for the Scott topology on $\mathscr{P}(\mathscr{O}(Y))$ is given by the sets of the form $\uparrow U = \{A \in \mathscr{P}(\mathscr{O}(Y)) \mid U \in A\}$ where $U \in \mathscr{O}(Y)$. By definition we have $\mathfrak{F}(x) \in \uparrow U$ if and only if $f^{-1}(U)$ is a neighbourhood of x. It follows that $\mathfrak{F}^{-1}(\uparrow U)$ is an open set for every $U \in \mathscr{O}(Y)$. Hence $\mathfrak{F}$ is continuous.

Let $\mathfrak{G}$ be a continuous approximation of $j \circ f$. Let $U \in \mathfrak{G}(x)$. Then $\mathfrak{G}(x) \in \uparrow U$, so that $\mathfrak{G}^{-1}(\uparrow U)$ is a neighbourhood of x. By assumption we have the inclusion $(j \circ f)^{-1}(\uparrow U) \supseteq \mathfrak{G}^{-1}(\uparrow U)$, so that $f^{-1}(U)$ is a neighbourhood of x. It follows that $\mathfrak{G}(x) \subseteq \mathfrak{F}(x)$.

We will show in Theorem 4.8 that a universal envelope always exists. It follows from Proposition 4.13 that this envelope encodes the function $\mathfrak{F}$ in (4.1). Of course the explicit description of the universal envelope given in (4.1) is somewhat tautological and not very informative. We will dedicate a lot of attention to the problem of finding more interesting descriptions of the universal envelope in concrete situations.

Definition 4.7. Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope of f. Let $x \in X$. A *filter basis for F at x* is a basis of the filter

$$\mathfrak{F}(x) = \Uparrow \left\{ \xi_L^{-1}(U) \mid F(x) \in U \right\} ,$$

i.e., a downwards directed set of open subsets of Y whose upwards closure in $\mathscr{O}(Y)$ is equal to $\mathfrak{F}(x)$.

4.2 Existence of universal envelopes

Theorem 4.8. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Then f has a universal envelope.*

Proof. Let

$$A = \left\{ (U, V) \in \mathscr{O}(X) \times \mathscr{O}(Y) \mid U \subseteq f^{-1}(V) \right\} .$$

Let

$$L = \mathscr{O}(A).$$

By Proposition 3.22 the lattice L is computably injective. Let

$$\xi_L\colon Y \to L,\ \xi_L(y) = \{(U, V) \in A \mid y \in V\} .$$

Then L is an approximation lattice for Y. Let

$$F\colon X \to L,\ F(x) = \{(U, V) \in A \mid x \in U\} .$$

Evidently, F is computable. We have

$$\xi_L \circ f(x) = \{(U, V) \in A \mid f(x) \in V\}.$$

By definition of A if $(U, V) \in A$ satisfies $x \in U$ then $f(x) \in V$ so that

$$F(x) \leq \xi_L \circ f(x).$$

It follows that F is an envelope of f.

We claim that F is universal. Let $G\colon X \to M$ be another envelope of f. Consider the map

$$\widetilde{\Phi}\colon L \to \mathscr{O}^2(M),\ \widetilde{\Phi}(\ell) = \left\{W \in \mathscr{O}(M) \mid (G^{-1}(W), \xi_M^{-1}(W)) \in \ell\right\}.$$

As G is an envelope we have

$$G^{-1}(W) \subseteq f^{-1}(\xi_M^{-1}(W))$$

for all $W \in \mathscr{O}(M)$, so that the map $\widetilde{\Phi}$ is well-defined and computable. As M is computably injective there exists a computable retraction

$$\rho_M\colon \mathscr{O}^2(M) \to M$$

which is a left inverse to the natural embedding

$$\nu_M\colon M \to \mathscr{O}^2(M).$$

Let

$$\Phi\colon L \to M,\ \Phi(\ell) = \rho_M \circ \widetilde{\Phi}(\ell).$$

We have

$$\begin{aligned}
\Phi \circ \xi_L(y) &= \rho_M\left(\left\{W \in \mathscr{O}(M) \mid (G^{-1}(W), \xi_M^{-1}(W)) \in \xi_L(y)\right\}\right)\\
&= \rho_M\left(\{W \in \mathscr{O}(M) \mid \xi_M(y) \in W\}\right)\\
&= \rho_M \circ \nu_M \circ \xi_M(y)\\
&= \xi_M(y).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Phi \circ F(x) &= \rho_M\left(\left\{W \in \mathscr{O}(M) \mid (G^{-1}(W), \xi_M^{-1}(W)) \in F(x)\right\}\right)\\
&= \rho_M\left(\{W \in \mathscr{O}(M) \mid G(x) \in W\}\right)\\
&= \rho_M \circ \nu_M \circ G(x)\\
&= G(x).
\end{aligned}$$

It follows that F tightens G via Φ. As G was chosen arbitrarily it follows that F is universal. □

While the existence of a universal envelope of any given function is an important result in its own right, Theorem 4.8, despite giving an explicit construction, does not establish much beyond the existence itself. For instance, it does not yield a non-tautological description of the filter basis for $\mathfrak{F}$ at any point. Furthermore, a practical implementation of the representative of the universal envelope constructed in Theorem 4.8 would be quite useless. The function $F\colon X \to \mathscr{O}(A)$ is, by means of currying, equivalent to the function $G\colon X \times A \to \Sigma$, where A is the space of all open sets (U, V) with $U \subseteq f^{-1}(V)$. The function G answers the trivial question if a given $x \in X$ is contained in U. The interesting problem is to generate valid inputs for the function G but this is, so to speak, "left to the user".

4.3 Finitary and Σ-split envelopes

In view of the fact that we require all lattices to be injective relative to the class of Σ-split embeddings, it is natural to consider approximation lattices L whose inclusion map $\xi_L\colon Y \to L$ is a Σ-split embedding. In view of the results on the right Kan extension in Section 3.2 it is natural to consider approximation domains whose inclusion map is even finitary.

Definition 4.9. Let Y be a computable T_0 space. An approximation lattice L for Y is called *Σ-split* if ξ_L is a computable Σ-split embedding. Accordingly, an envelope $F\colon X \to L$ with values in a Σ-split approximation domain is called a *Σ-split envelope*.

An approximation lattice L for Y is called *finitary* if its inclusion map is computably finitary, *i.e.*, if the upper adjoint of the map $(\xi_L)^*\colon \mathscr{O}(L) \to \mathscr{O}(Y)$ is computable.

Accordingly, an envelope $F\colon X \to L$ with values in a finitary approximation lattice is called a *finitary envelope*.

For example the approximation lattice $\mathscr{O}^2(Y)$ is Σ-split for each computable T_0 space Y. A computable section is given by the map $\nu_{\mathscr{O}(Y)}\colon \mathscr{O}(Y) \to \mathscr{O}^3(Y)$.

If Y is computably compact computable Hausdorff space then $\mathscr{K}(Y)$ is a finitary approximation lattice for Y. Note that if Y has any finitary approximation lattice then Y is necessarily compact.

Finitary approximation lattices are particularly well-behaved, as any continuous function $f\colon Y \to M$ with values in a computably injective lattice M has a canonical extension $(f/\xi_L)\colon L \to M$ to L along ξ_L which is uniformly computable in f. Finitarity is a strong restriction, however, as the following result shows:

Proposition 4.10. *Let $f\colon X \to Y$ be a function between computable T_0-spaces. Let $F\colon X \to L$ be a finitary envelope of f. Then F is computably tightened by the principal $\mathscr{O}^2(Y)$-envelope of f.*

Proof. By assumption the upper adjoint $s\colon \mathscr{O}(Y) \to \mathscr{O}(L)$ of ξ_L^* is computable. Consider the map

$$G\colon X \to \mathscr{O}^2(Y),\ G(x) = s^* \circ \nu_L \circ F(x).$$

Then

$$\begin{aligned} G(x) &= s^* \circ \nu_L \circ F(x) \\ &\leq s^* \circ \nu_L \circ \xi_L \circ f(x) \\ &= s^* \circ \xi_L^{**} \circ \nu_Y \circ f(x) \\ &= (\xi_L^* \circ s)^* \circ \nu_Y \circ f(x) \\ &\leq \nu_Y \circ f(x). \end{aligned}$$

Hence G is an $\mathscr{O}^2(Y)$-envelope of f.

We claim that G tightens F via the map $\rho_L \circ \xi_L^{**}\colon \mathscr{O}^2(Y) \to L$. We have

$$\rho_L \circ \xi_L^{**} \circ \nu_Y = \rho_L \circ \nu_L \circ \xi_L = \xi_L$$

and

$$\begin{aligned} \rho_L \circ \xi_L^{**} \circ G(x) &= \rho_L \circ \xi_L^{**} \circ s^* \circ \nu_L \circ F(x) \\ &= \rho_L \circ (s \circ \xi_L^*)^* \circ \nu_L \circ F(x) \\ &\geq \rho_L \circ \nu_L \circ F(x) \\ &= F(x). \end{aligned}$$

Hence G computably tightens F. Since G is an $\mathscr{O}^2(Y)$-envelope, the principal $\mathscr{O}^2(Y)$-envelope computably tightens F. □

If $F\colon X \to L$ and $G\colon X \to M$ are envelopes then F tightens G if and only if F tightens G via the map (ξ_M/ξ_L). In the case where F is finitary this map is uniformly computable in ξ_M. This suggests the following generalisation:

Definition 4.11. Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be a Σ-split envelope of f. A Σ-section $s\colon \mathscr{O}(Y) \to \mathscr{O}(L)$ of ξ_L is called *generating* if any envelope $G\colon X \to M$ of f is tightened by F via the extension $\xi_M/(\frac{s}{\xi_L})$.

The existence of generating Σ-sections can be determined with the help of the following notion:

Definition 4.12. Let $f\colon X \to Y$ be a function between computable T_0 spaces. An open set $U \in \mathscr{O}(Y)$ is called a *robust property* of f at x, or by abuse of notation a robust property of $f(x)$, if the preimage $f^{-1}(U)$ is a neighbourhood of x.

If $F\colon X \to L$ is an envelope of f and $U \in \mathscr{O}(Y)$ is a robust property of $f(x)$ we say that F *witnesses* U at x if there exists an open set $V \in \mathscr{O}(L)$ with $F(x) \in V$ and $\xi_L^{-1}(V) \subseteq U$. If F witnesses all robust properties of f we say that F is Σ*-complete*.

Proposition 4.13. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be a universal envelope of f. Then F is Σ-complete.*

Proof. Let $U \in \mathscr{O}(Y)$ be a robust property of $f(x_0)$. By assumption there exists an open set $V \in \mathscr{O}(X)$ with $x_0 \in V \subseteq f^{-1}(U)$. Consider the functions

$$G\colon X \to \Sigma,\ G(x) = \begin{cases} \top & \text{if } x \in V \\ \bot & \text{otherwise} \end{cases}$$

and

$$h\colon Y \to \Sigma,\ h(y) = \begin{cases} \top & \text{if } y \in U \\ \bot & \text{otherwise.} \end{cases}$$

By definition we have $h \circ f(x) \geq G(x)$ for all $x \in X$. It follows that (G, h) is an envelope of f.

As F is assumed to be universal there exists a continuous function $\Phi\colon L \to \Sigma$ with

$$\Phi \circ \xi_L \leq h$$

and

$$\Phi \circ F \geq G.$$

In particular

$$F(x_0) \in \{\ell \in L \mid \Phi(\ell) = \top\}.$$

We have

$$\xi_L^{-1}(\{\ell \in L \mid \Phi(\ell) = \top\}) \subseteq U$$

by assumption. Hence U is witnessed by F. □

Definition 4.14. Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be a Σ-split envelope of f. We say that F is *uniformly* Σ*-complete* if its inclusion map ξ_L^* has a computable section s such that $F(x) \in s(U)$ for all robust properties U of $f(x)$.

Theorem 4.15. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be a Σ-split envelope of f. A Σ-section $s\colon \mathscr{O}(Y) \to \mathscr{O}(L)$ of the inclusion map is generating if and only if $F(x) \in s(U)$ for all robust properties U of $f(x)$. In particular, if F is uniformly Σ-complete then F is universal.*

Proof. Assume that $F(x) \in s(U)$ for all robust properties $U \in \mathscr{O}(Y)$ of $f(x)$. Let $G\colon X \to M$ be an envelope of f. Consider the map

$$\Phi = \xi_M/(\tfrac{s}{\xi_L}) = \rho_M \circ \xi_M^{**} \circ s^* \circ \nu_L \colon L \to M.$$

Then, as in the proof of 3.18, we have $\Phi \circ \xi_L = \xi_M$. As G is an envelope, any property witnessed by G is robust. It follows that

$$\{\xi_M^*(U) \in \mathscr{O}(Y) \mid G(x) \in U\} \subseteq \{U \in \mathscr{O}(Y) \mid F(x) \in s(U)\}.$$

Hence:

$$\begin{aligned}
\Phi \circ F(x) &= \rho_M \circ \xi_M^{**} \circ s^* \circ \nu_L \circ F(x) \\
&= \rho_M \circ \xi_M^{**} \left(\{U \in \mathscr{O}(Y) \mid F(x) \in s(U)\}\right) \\
&\geq \rho_M \circ \xi_M^{**} \left(\{\xi_M^*(U) \in \mathscr{O}(Y) \mid G(x) \in U\}\right) \\
&= \rho_M \circ \nu_M \circ G(x) \\
&= G(x).
\end{aligned}$$

Conversely, assume that the section $s\colon \mathscr{O}(Y) \to \mathscr{O}(L)$ of $\mathscr{O}^{\xi_L}$ satisfies that every envelope $G\colon X \to M$ of f is tightened by F via $\xi_M/(\frac{s}{\xi_L})$. Let U be a robust property of $f(x_0)$. Then there exists an open set $V \in \mathscr{O}(X)$ with $V \subseteq f^{-1}(U)$. As in the proof of 4.13 we obtain an envelope

$$G(x) = \begin{cases} \top & \text{if } x \in V, \\ \bot & \text{otherwise} \end{cases}$$

with inclusion map

$$h(y) = \begin{cases} \top & \text{if } y \in U, \\ \bot & \text{otherwise.} \end{cases}$$

By assumption, F tightens G via the map

$$h/(\tfrac{s}{\xi_L}) = \rho_\Sigma \circ h^{**} \circ s^* \circ \nu_L.$$

We have $G(x_0) = \top$ and thus $h/(\frac{s}{\xi_L}) \circ F(x_0) = \top$. An easy calculation shows

$$\top = h/(\tfrac{s}{\xi_L}) \circ F(x) = \rho_\Sigma \left(\{U \in \mathscr{O}(\Sigma) \mid F(x) \in s \circ h^*(U)\}\right).$$

It follows that $\{\top\} \in \{U \in \mathscr{O}(\Sigma) \mid F(x) \in s \circ h^*(U)\}$ and thus $F(x) \in s(U)$. Hence F is uniformly Σ-complete. $\square$

The existence of uniformly Σ-complete envelopes is quite a special property:

Proposition 4.16. *Let $f: X \to Y$ be a function between computable T_0 spaces. Assume that f has a uniformly Σ-complete envelope. Then the principal $\mathscr{O}^2(Y)$-envelope of f is universal.*

Proof. Let $F: X \to L$ be a uniformly Σ-complete envelope. Let $s: \mathscr{O}(Y) \to \mathscr{O}(L)$ be a generating Σ-section. Consider the map $s^*: \mathscr{O}^2(L) \to \mathscr{O}^2(Y)$. We obtain an envelope

$$G: X \to \mathscr{O}^2(Y),\ G(x) = s^* \circ \nu_L \circ F(x).$$

This is really an envelope, as

$$G(x) \leq s^* \circ \nu_L \circ \xi_L \circ f(x) = s^* \circ \xi_L^{**} \circ \nu_Y \circ f(x) = \nu_Y \circ f(x).$$

If $U \in \mathscr{O}(X)$ is a robust property of $f(x)$ then $F(x) \in s(U)$, and so $U \in G(x)$. It follows that G is uniformly Σ-complete and hence universal. $\square$

Corollary 4.17. *Let $f: X \to Y$ be a function between computable T_0 spaces. Let $F: X \to \mathscr{O}^2(Y)$ be the principal $\mathscr{O}^2(Y)$-envelope of f. If for every robust property $U \in \mathscr{O}(Y)$ of $f(x)$ we have $U \in F(x)$ then F is universal. In this case every envelope $G: X \to M$ of f is tightened by F via the extension $\xi_M / \left(\frac{\nu_{\mathscr{O}(Y)}}{\nu_Y}\right)$.*

For finitary envelopes we can say more:

Theorem 4.18. *Let $f: X \to Y$ be a function between computable T_0 spaces. Let $F: X \to L$ be a finitary envelope of f. Then F is Σ-complete if and only if F is uniformly Σ-complete if and only if F is universal if and only if every envelope $G: X \to M$ is tightened by F via the right Kan extension ξ_M/ξ_L.*

Proof. It follows immediately from the definition of the right Kan extension that if $G: X \to M$ is any envelope with inclusion map $\xi_M: Y \to M$ then G is tightened by F if and only if G is tightened by F via the right Kan extension ξ_M/ξ_L.

Universality implies Σ-completeness by Proposition 4.13 and uniform Σ-completeness implies universality by Theorem 4.15.

It remains to show that Σ-completeness implies uniform Σ-completeness. Thus, assume that F is Σ-complete. Let $U \in \mathscr{O}(Y)$ be a robust property of $f(x)$. Then there exists $V \in \mathscr{O}(L)$ with $\xi_L^*(V) \subseteq U$ and $F(x) \in V$. Let $s: \mathscr{O}(Y) \to \mathscr{O}(L)$ the upper adjoint of ξ_L^*. By assumption s is computable. As s is the upper adjoint, we have $(s \circ \xi_L^*)(U) \supseteq U$ for all $U \in \mathscr{O}(L)$, and hence

$$F(x) \in V \subseteq s \circ \xi_L^*(V) \subseteq s(U).$$

Hence F is uniformly Σ-complete. □

Corollary 4.19. *Let $f\colon X \to Y$ be a function between a computable T_0 space X and a computably compact computable Hausdorff space Y. Let $F\colon X \to \mathscr{K}_\bot(Y)$ be the principal $\mathscr{K}_\bot(Y)$-envelope of f. Then F is universal if and only if for every robust property $U \in \mathscr{O}(Y)$ of $f(x)$ we have $F(x) \subseteq U$. In this case every envelope $G\colon X \to M$ of f is tightened by F via the map $\inf \circ (\mathscr{K}_\bot)_{\xi_M}$.*

Unfortunately there exist functions $f\colon X \to Y$ which do not have a uniformly Σ-complete envelope.

Example 4.20. Let ℓ^2 denote infinite dimensional separable real Hilbert space, made into a computable metric space by endowing it with the metric induced by the ℓ^2-norm and taking as a computable dense sequence the set of all rational sequences with finitely many non-zero entries. Let $(\ell^2)'$ denote the space of continuous linear functionals on ℓ^2, made into a represented space by identification with a subspace of the exponential $\mathbb{R}^{(\ell^2)'}$. See e.g. [25] or [78] for details.

Consider the function $\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}\colon (\ell^2)' \to \ell^2$ which sends a functional $x \in (\ell^2)'$ to the corresponding point $x \in \ell^2$. This problem may seem quite artificial now, but it is closely related to the problem of locating the fixed point set of a nonexpansive self-map of the unit ball in separable real Hilbert space, see Lemma 5.17. By Proposition 4.16 this function has a uniformly Σ-complete envelope if and only if its principal $\mathscr{O}^2(\ell^2)$-envelope is universal.

Let $F\colon (\ell^2)' \to \mathscr{O}^2(\ell^2)$ be the principal $\mathscr{O}^2(\ell^2)$-envelope of $\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}$. We claim that $F(x) = \emptyset$ for all $x \in (\ell^2)'$. It is shown in [25] that $(\ell^2)'$ is computably isomorphic to the partial quotient of $\mathbb{R}^{\mathbb{N}}$ under the map

$$\begin{aligned}
&q\colon \subseteq \mathbb{R}^{\mathbb{N}} \to (\ell^2)',\\
&\mathrm{dom}(q) = \left\{ (x_n)_n \in \mathbb{R}^{\mathbb{N}} \mid x_0 \geq (\sum_{n\geq 1} x_n^2)^{1/2} \right\}\\
&q((x_n)_n) = \lambda y. \sum_{n\geq 1} x_n y_{n-1}.
\end{aligned}$$

It follows that F lifts to a computable map $\widetilde{F}\colon \mathrm{dom}(q) \to \mathscr{O}^2(\ell^2)$ with $\widetilde{F}(z) = F \circ q(z)$ for all $z \in \mathrm{dom}(q)$.

The space ℓ^2 is computably countably based and a computable basis is given by the set of all balls whose radius is a rational number and whose centre is a sequence of rational numbers only finitely many of which are non-zero. Let us refer to balls of this form as "rational balls" for short.

Fix a relativised algorithm computing the map $G\colon \mathrm{dom}(q) \times \mathcal{O}(\ell^2) \to \Sigma$ which is obtained by currying the map $\widetilde{F}\colon \mathrm{dom}(q) \to \mathcal{O}^2(\ell^2)$. Assume that there exists $U \in \mathcal{O}(\ell^2)$ with $U \in F(x)$. Then $x \in U$. In particular U is non-empty. Let $\varepsilon > 0$ be a rational number with $\varepsilon < |x| + 1$.

By Proposition 2.39 a name of the set U is given by a sequence $(B_n)_n$ containing all rational balls of radius ε whose centre is contained in U. Consider the sequence $s = (2(|x| + 1), x_0, x_1, \dots) \in \mathrm{dom}(q)$. Feed a name of s and the sequence $(B_n)_n$ as an input to the relativised algorithm which computes the map $G\colon \mathrm{dom}(q) \times \mathcal{O}(\ell^2) \to \Sigma$. Then after finitely many steps the algorithm outputs $\top$. Up until this point the algorithm has only read a finite initial segment $(2(|x|+1), x_0, \dots, x_N)$ of the sequence s and a finite initial segment $B_1, \dots, B_M$ of the sequence $(B_n)_n$. It follows that if $\widetilde{x} \in (\ell^2)'$ with $|\widetilde{x}| \leq 2(|x|+1)$ and $\widetilde{x} \cdot e_n = x \cdot e_n$ for all $n \leq N$ then $\bigcup_{n=1}^{M} B_n \in F(\widetilde{x})$ and hence $\widetilde{x} \in \bigcup_{n=1}^{M} B_n$, as F is assumed to be an envelope of $\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}$. Hence the set of all $\widetilde{x}$'s of this form has a finite cover by balls of radius $|x| + 1$. Contradiction! Hence $F(x) = \emptyset$ for all $x \in (\ell^2)'$.

It follows that F cannot be Σ-complete, as for any $x \in (\ell^2)'$, any open set of the weak topology which contains x is a robust property of $\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}(x)$ and none of these properties are witnessed by F. Hence F cannot be universal.

Example 4.21. Let $f\colon X \to Y$ be an arbitrary function between computable T_0 spaces. Consider the universal envelope of f that was constructed in the proof of Theorem 4.8 The inclusion map $\xi_L\colon Y \to L$ from the proof of Theorem 4.8 is a computable Σ-split embedding. A section for $\mathcal{O}^{\xi_L}$ is given by the map

$$s\colon \mathcal{O}(Y) \to \mathcal{O}^2(A),\ s(V) = \{\mathscr{U} \in \mathcal{O}(A) \mid (\emptyset, V) \in \mathscr{U}\}.$$

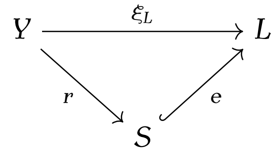
If $G\colon X \to M$ is another envelope, we can extend the inclusion map $\xi_M\colon Y \to M$ to L along ξ_L using this section. The resulting extension $\overline{\xi_M}$ satisfies the equation $\overline{\xi_M} \circ F(x) = \bot$ for all $x \in X$. Thus, this section is far from generating.

4.4 Bases of an envelope

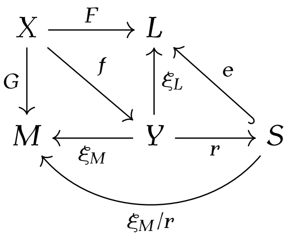
In the case where a given envelope $F\colon X \to L$ is uniformly Σ-complete with generating Σ-section s we have a good description available of how it tightens all other envelopes: If $G\colon X \to M$ is another envelope then F tightens G via the map $\xi_M/(\frac{s}{\xi_L})$, which is uniformly computable in ξ_M.

In order to have a similar description for the tightening relation available in more general cases, we introduce the concept of a *basis* of an envelope. As we will see in the next section, bases play an important role in the description of how the probes of a function factor through the universal envelope.

Definition 4.22. Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope. A computable T_0 space S is called a *basis* for F if the inclusion map $\xi_L\colon Y \to L$ admits a factorisation

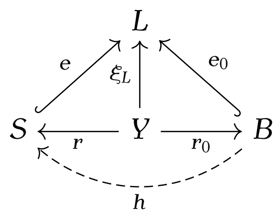


where r is a computable map and e is a computable Σ-split embedding with a computable Σ-section s, such that all envelopes $G\colon X \to L$ which are tightened by F are tightened via the map $(\xi_M/r)/(\frac{s}{e})$.



We call s the *generating* Σ*-section* of the basis.

A basis B of F is called *minimal* if for every basis S of F there exists a computable map $h\colon B \to S$ such that the following diagram commutes:

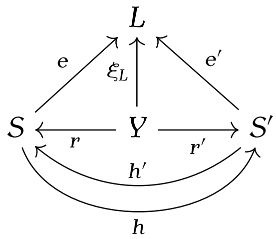


If $F\colon X \to L$ is uniformly Σ-complete and $s\colon \mathscr{O}(Y) \to \mathscr{O}(L)$ is a generating Σ-section for f, then Y is a minimal basis of F with factorisation $\xi_L = \xi_L \circ \mathrm{id}_Y$ and generating Σ-section s.

On the other hand, if $F\colon X \to L$ is any envelope then the lattice L is a basis with factorisation $\xi_L = \mathrm{id}_L \circ \xi_L$ and generating Σ-section $\mathrm{id}_{\mathscr{O}(L)}\colon \mathscr{O}(L) \to \mathscr{O}(L)$.

Proposition 4.23. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope of f. Assume that F has a minimal basis B. Then this minimal basis is unique up to unique isomorphism.*

Proof. Assume that both S and S' are minimal bases of F. Then by assumption there exist computable maps $h\colon S \to S'$ and $h'\colon S \to S'$ such that the following diagram commutes:



We hence have $e' \circ h = e$. It follows that $h = (e')^{-1} \circ e$. Analogously we find that $h' = e^{-1} \circ (e')^{-1}$. Hence h and h' are uniquely determined and inverses of each other. □

We do not require the factorisation of ξ_L through a basis to be an epi-mono factorisation. If it is then the basis is necessarily minimal:

Proposition 4.24. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope of f. Assume that the inclusion map $\xi_L\colon Y \to L$ factors through a basis S as $\xi_L = e \circ r$ where r is surjective. Then S is a minimal basis of F.*

Proof. Let S' be another basis of F with factorisation $\xi_L = i \circ s$. Then, since r is surjective, we have $e(x) = e \circ r(y) = i \circ s(y)$ for some $y \in Y$. It follows that the function $h = i^{-1} \circ e\colon S \to S'$ is well-defined with $h \circ r = s$. □

Example 4.25. Returning to Example 4.20, consider again the function

$$\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}\colon (\ell^2)' \to \ell^2.$$

Consider the envelope

$$F\colon (\ell^2)' \to \mathscr{K}_\bot((\ell^2)'),\ F(x) = \{x\}$$

with inclusion map

$$h\colon \ell^2 \to \mathscr{K}_\bot((\ell^2)'),\ h(x) = \kappa_\bot \circ \mathrm{id}_{\ell^2}^{\mathsf{s}\to\mathsf{w}},$$

where

$$\mathrm{id}_{\ell^2}^{\mathsf{s}\to\mathsf{w}}\colon \ell^2 \to (\ell^2)'$$

is the inverse function of $\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}$.

This envelope is universal, for any envelope $G\colon (\ell^2)' \to M$ is tightened by F via the map $\inf \circ (\mathscr{K}_\perp)_G\colon \mathscr{K}_\perp((\ell^2)') \to M$. By definition the map h factorises as $h = \kappa_\perp \circ \mathrm{id}_{\ell^2}^{\mathsf{s}\to\mathsf{w}}$ where $\kappa_\perp$ has as Σ-section the upper adjoint s of $\mathscr{O}^\kappa$. The envelope F tightens G via the map $G/(\frac{s}{\kappa_\perp})$ and thus in particular via $(\xi_M/\,\mathrm{id}_{\ell^2}^{\mathsf{s}\to\mathsf{w}})/(\frac{s}{\kappa_\perp})$. It follows that $(\ell^2)'$ is a basis for F.

As the map $\mathrm{id}_{\ell^2}^{\mathsf{s}\to\mathsf{w}}$ is surjective it follows that $(\ell^2)'$ is a minimal basis of F, for if ξ_L factors through a basis S as $\xi_L = e \circ r$, then the map $h = e^{-1} \circ \mathrm{id}_{\ell^2}^{\mathsf{s}\to\mathsf{w}}$ is well-defined and computable.

Theorem 4.26. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope of f. Let S be a basis for F with factorisation $\xi_L = e \circ r$ and generating Σ-section s. Then F is universal if and only if*

1. *For every envelope $G\colon X \to M$ of f with inclusion map $\xi_M\colon Y \to M$ the pair $(G, \xi_M/r)$ is an envelope of $r \circ f$.*
2. *The envelope $F\colon X \to L$ of $r \circ f$ with inclusion map $e\colon S \to L$ is uniformly Σ-complete with generating Σ-section s.*

Proof. Assume that F is universal. Let $G\colon X \to M$ be an envelope of f. Then F tightens G via the map $(\xi_M/r)/\left(\frac{s}{e}\right)$. It follows that

$$G \le (\xi_M/r)/(\tfrac{s}{e}) \circ F \le (\xi_M/r)/(\tfrac{s}{e}) \circ e \circ r \circ f = (\xi_M/r) \circ r \circ f.$$

Hence G is an envelope of $r \circ f$ with inclusion map (ξ_M/r). The rest of the proof is analogous to that of Theorem 4.15.

Let us now show the converse direction. If $G\colon X \to M$ is an envelope of f then by assumption G becomes an envelope of $r \circ f$ if the inclusion map is taken to be ξ_M/r. As F, viewed as an envelope of $r \circ f$, is assumed to be uniformly Σ-complete it follows from Theorem 4.15 that F tightens this envelope via some map $\Phi\colon L \to M$ which satisfies $\Phi \circ e \le (\xi_M/r)$ and $\Phi \circ F \ge G$. We have $\Phi \circ e \circ r \le (\xi_M/r) \circ r \le \xi_M$. Thus F tightens G via Φ. □

Theorem 4.27. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be a universal envelope of f. Let S be a basis for F with factorisation $\xi_L = e \circ r$ and generating Σ-section $s\colon \mathscr{O}(S) \to \mathscr{O}(L)$. Let $x \in X$. Then a filter basis for F at x is given by*

$$\left\{ r^{-1}(U) \mid F(x) \in s(U) \right\}.$$

Proof. A filter basis for $\mathfrak{F}(x)$ is given by

$$\{U \in \mathscr{O}(Y) \mid U \text{ is a robust property of } f(x)\}.$$

Let $U \in \mathscr{O}(Y)$ be a robust property of $f(x)$. Let $V \in \mathscr{O}(Y)$ be an open set with $x \in V \subseteq f^{-1}(U)$. Consider the characteristic function $\xi_V \colon X \to \Sigma$ of V and the characteristic function $\xi_U \colon Y \to \Sigma$ of U. Then ξ_V is an envelope of f with inclusion map ξ_U. It follows that F tightens ξ_V via the map $(\xi_U/r)/(\frac{s}{e})$. Let $W \in \mathscr{O}(S)$ denote the open set whose characteristic function is ξ_U/r. The inequality $(\xi_U/r) \circ r \leq \xi_U$ translates to $r^{-1}(W) \subseteq U$. As F tightens ξ_V via $(\xi_U/r)/(\frac{s}{e})$ we obtain

$$\xi_V(x) = \top \leq \rho_\Sigma \circ (h/r)^{**} \circ s^* \circ \nu_L \circ F(x).$$

This leads to $F(x) \in s((h/r)^*(\{\top\})) = s(W)$. As $r^{-1}(W) \subseteq U$ we conclude that

$$\Uparrow\left\{r^{-1}(U) \mid F(x) \in s(U)\right\} \supseteq \{U \in \mathscr{O}(Y) \mid U \text{ is a robust property of } f(x)\}.$$

Hence the result is shown. □

4.5 Separated-universality for compactly majorisable functions

The following theorem gives criterion for when there is a very satisfactory description available for an envelope that is at least optimal amongst the separated ones:

Theorem 4.28. *Let $f \colon X \to Y$ be a function between a computable T_0 space X and a computably countably based computable Hausdorff space Y. Assume there exists a computable map $B \colon X \to \mathscr{K}(Y)$ such that $f(x) \in B(x)$ for all $x \in X$. Then the principal $\mathscr{K}_\perp(Y)$-envelope of f is separated-universal and every separated envelope $G \colon X \to M$ is tightened by F via the map* $\inf \circ (\mathscr{K}_\perp)_{\xi_M}$.

Proof. Let $F \colon X \to \mathscr{K}_\perp(Y)$ be the principal $\mathscr{K}_\perp(Y)$-envelope of f. Let $G \colon X \to M$ be a separated envelope of f. As ξ_M maps Y to the maximal elements of $M \setminus \{\top\}$ we have for all $m \in (M \setminus \{\top\})$:

$$\xi_M(y) \notin \uparrow m \Leftrightarrow \xi_M(y) \vee m = \top.$$

It follows that

$$\xi_M^{-1}(\uparrow m) = \xi_M^{-1}(M \setminus \{m' \in M \mid m' \vee m = \top\}).$$

Hence the set $\xi_M^{-1}(\uparrow m)$ is closed and uniformly computable in m as an element of $\mathscr{A}(Y)$. We can hence define the computable map

$$H\colon X \to \mathscr{K}_\perp(Y),\ H(x) = \xi_M^{-1}(\uparrow G(x)) \cap B(x).$$

We have $G(x) \leq \xi_M \circ f(x)$ for all $x \in X$. Hence $f(x) \in \xi_M^{-1}(\uparrow G(x))$ and so $f(x) \geq H(x)$ for all $x \in X$. Thus H is a $\mathscr{K}_\perp(Y)$-envelope of f and since F is the principal $\mathscr{K}_\perp(Y)$-envelope we have $H \leq F$. By definition this means that

$$B(x) \cap \xi_M^{-1}(\uparrow G(x)) \supseteq F(x).$$

We have

$$\xi_M(B(x) \cap \xi_M^{-1}(\uparrow G(x))) \subseteq \uparrow G(x)\ .$$

Hence

$$\inf \circ (\mathscr{K}_\perp)_{\xi_M} \circ F(x) \geq \inf \left((\mathscr{K}_\perp)_{\xi_M} \left(B(x) \cap \xi_M^{-1}(\uparrow m)\right)\right) \geq G(x).$$

□

Example 4.29. There is no reason for a computably separated-universal envelope to be Σ-complete. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} -x & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

By Theorem 4.28 the envelope $F\colon \mathbb{R} \to \mathscr{K}_\perp(\mathbb{R})$ where

$$F(x) = \begin{cases} \{-x\} & \text{if } x < 0, \\ \{0, 1\} & \text{if } x = 0, \\ \{1\} & \text{if } x > 0 \end{cases}$$

is separated-universal. It is not Σ-complete however, as the set $U = (0, 2) \in \mathscr{O}(\mathbb{R})$ is a robust property of f which isn't witnessed by F. In particular F is not universal. The reason for this is that the filter of robust properties

$$\left\{U \in \mathscr{O}(\mathbb{R}) \mid 1 \in U \wedge \exists n \in \mathbb{N}.(0, \tfrac{1}{n}) \subseteq U\right\}$$

is not Scott-open and hence does not correspond to a compact subset of $\mathbb{R}$.

4.6 Probes

Our next aim is to obtain a better understanding of the amount of information that is encoded in the universal envelope. To this end we introduce *probes* for a function $f\colon X \to Y$ which can be viewed as (relativised) algorithms that use f as a subroutine in such a way that the end result is a continuous function. We show that if F is a universal envelope of f, then a large class of these algorithms can use F as a subroutine instead.

Definition 4.30. Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $U \in \mathscr{O}(X)$ be an open subset of X. A *continuous local probe* for f on U is a continuous map $\varphi\colon Y \to Z$, where Z is a computable T_0 space, such that the function $\varphi \circ f$ is continuous on U. If φ is computable we call φ a *computable local probe.*

Theorem 4.31. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be the universal envelope. Let $\varphi\colon Y \to M$ be a continuous local probe for f on $U \in \mathscr{O}(X)$ with values in a computably injective lattice M. Then there exists a continuous extension*

$$\widetilde{\varphi}\colon L \to M$$

with

$$\widetilde{\varphi} \circ \xi_L(y) \leq \varphi(y)$$

for all $y \in Y$ and

$$\widetilde{\varphi} \circ F(x) = \varphi \circ f(x)$$

for all $x \in U$.

If S is a basis for F with factorisation $\xi_L = e \circ r$ and generating Σ-section s then an extension $\widetilde{\varphi}$ is given by $\widetilde{\varphi} = (\varphi/r)/(\frac{s}{e})$.

Proof. Consider the function

$$G\colon X \to M,\ G(x) = \begin{cases} \varphi \circ f(x) & \text{if } x \in U, \\ \bot & \text{otherwise.} \end{cases}$$

As $\varphi \circ f$ is continuous on U the function G is continuous. It satisfies the inequality $G(x) \leq \varphi \circ f(x)$ for all $x \in X$. It is hence an envelope of f if we consider M to be an approximation domain with inclusion map $\varphi\colon Y \to M$. It follows from the universality of F that F tightens G via a map $\widetilde{\varphi}\colon L \to M$. It is easy to see that the map $\widetilde{\varphi}$ has the desired properties.

If S is a basis as in the statement of the theorem then it follows from the definition of basis that we can choose $\widetilde{\varphi} = (\varphi/r)/(\frac{s}{e})$. □

In particular, if $F\colon X \to L$ is uniformly Σ-complete with generating Σ-section $s\colon \mathscr{O}(Y) \to \mathscr{O}(L)$, then an extension $\widetilde{\varphi}$ as in Theorem 4.31 of a local probe $\varphi\colon Y \to M$ with values in a computably injective lattice is given by the function $\varphi/(\frac{s}{\xi_L})\colon L \to M$. This function is uniformly computable in φ. In particular, any computable local probe has a computable extension in the sense of Theorem 4.31.

More generally, if $F\colon X \to L$ is an envelope with basis S, factorisation given by $\xi_L = e \circ r$, and generating Σ-section s, then any local probe $\varphi\colon Y \to M$ of f extends continuously along r to a local probe $\widetilde{\varphi}\colon S \to M$ of $r \circ f$ which uniformly computably extends to the function $\widetilde{\varphi}/(\frac{s}{e})$. In particular a computable local probe has a computable extension to L as in Theorem 4.31 if and only if it has a computable extension to T along r.

Example 4.32. In general not every computable local probe has a computable extension in the sense of Theorem 4.31. Consider again the function

$$\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}\colon (\ell^2)' \to \ell^2.$$

Recall that the universal envelope is given by

$$\kappa_\perp \circ \mathrm{id}_{(\ell^2)'}\colon (\ell^2)' \to \mathscr{K}_\perp((\ell^2)'),\ x \mapsto \uparrow x\ .$$

Any continuous linear functional $\ell^2 \to \mathbb{R}$ is a local probe for $\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}$. However, there exist continuous linear functionals which are computable as maps of type $\ell^2 \to \mathbb{R}$ but uncomputable as maps $(\ell^2)' \to \mathbb{R}$: It is easy to construct a computable sequence $(x_n)_n \in \mathbb{R}^n$ with a well-defined ℓ^2-norm $\left(\sum_{n\in\mathbb{N}} x_n^2\right)^{1/2}$ which is an uncomputable number (see e.g. [78, Theorem 5.9]). Such a sequence defines a point x in Hilbert space which is computable as an element of $(\ell^2)'$ but not as an element of ℓ^2. The corresponding linear functional $y \mapsto x \cdot y$ is computable as a map $\ell^2 \to \mathbb{R}$ but not as a map $(\ell^2)' \to \mathbb{R}$. This defines a computable local probe for $\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}}$ which does not have a computable extension to $\mathscr{K}_\perp((\ell^2)')$ in the sense of Theorem 4.31.

Definition 4.33. Let $f\colon X \to Y$ be a function between computable T_0 spaces. A *continuous probe* for f consists of two continuous functions $\alpha\colon \widetilde{X} \to X$ and $\beta\colon \widetilde{X} \times Y \to Z$, where $\widetilde{X}$ and Z are computable T_0 spaces, such that for each point $x \in \widetilde{X}$ the point $(x, \alpha(x)) \in \widetilde{X} \times X$ is a point of continuity of the function $\psi(x_0, x_1) = \beta(x_0, f(x_1))$. If α and β are computable we call (α, β) a *computable probe for* f.

Theorem 4.34. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be a universal envelope of f. Let (α, β) be a probe for f where $\beta\colon \widetilde{X} \times Y \to M$ takes values in a computably continuous lattice M. Then β extends to a continuous map*

$$\widetilde{\beta}\colon \widetilde{X} \times L \to M$$

with

$$\widetilde{\beta}(x, \xi_L(y)) \leq \beta(x, y)$$

and

$$\widetilde{\beta}(x, F\circ\alpha(x)) = \beta(x, f\circ\alpha(x)).$$

If S is a basis for F with factorisation $\xi_L = e\circ r$ and generating Σ-section s then an extension $\widetilde{\beta}$ is given by $\widetilde{\beta}(x,\ell) = \big(\lambda t.h(x,t)/(\tfrac{s}{e})\big)(\ell)$ where $h\colon \widetilde{X}\times S\to M$ is the map given by $h(x,t) = (k/r)(t)(x)$ with $k\colon Y\to M^{\widetilde{X}}, k(y) = \lambda x.\beta(x,y)$.

Proof. Consider the map

$$\psi\colon \widetilde{X}\times X\to M,\ \psi(x_0,x_1) = \beta(x_0, f(x_1)).$$

By Proposition 3.32, ψ has a best continuous approximation $\widetilde{\psi}$. As M is assumed to be computably continuous, $\widetilde{\psi}$ coincides with ψ in all its points of continuity. In particular we have for all $x\in\widetilde{X}$:

$$\widetilde{\psi}(x,\alpha(x)) = \beta(x, f(\alpha(x))).$$

Now let

$$G\colon X\to M^{\widetilde{X}},\ G(x)(\widetilde{x}) = \widetilde{\psi}(\widetilde{x},x)$$

and

$$h\colon Y\to M^{\widetilde{X}},\ h(y)(\widetilde{x}) = \beta(\widetilde{x},y).$$

We have

$$G(x)(\widetilde{x}) = \widetilde{\psi}(\widetilde{x},x)\le\psi(\widetilde{x},x) = \beta(\widetilde{x},f(x)) = (h\circ f(x))(\widetilde{x}).$$

Hence G is an envelope of f with inclusion map h. As F is universal, F tightens G via a map $\Phi\colon L\to M^{\widetilde{X}}$. Let $\widetilde{\beta}(x,\ell) = \Phi(\ell)(x)$. Then

$$\widetilde{\beta}(x,\xi_L(y)) = \Phi(\xi_L(y))(x)\le h(y)(x) = \beta(x,y).$$

We further have

$$\widetilde{\beta}(x,F(\alpha(x))) = \Phi(F(\alpha(x)))(x)\ge G(\alpha(x))(x) = \widetilde{\psi}(x,\alpha(x)) = \beta(x,f\circ\alpha(x))$$

and

$$\widetilde{\beta}(x,F(\alpha(x)))\le\widetilde{\beta}(x,\xi_L\circ f\circ\alpha(x))\le\beta(x,f\circ\alpha(x)).$$

The addendum follows from the definition of basis. □

Similarly to the situation with local probes, if $F\colon X\to L$ is uniformly Σ-complete with generating Σ-section $s\colon\mathscr{O}(Y)\to\mathscr{O}(L)$ then for any probe (α,β), with $\beta\colon\widetilde{X}\times Y\to M$ taking values in a computably continuous lattice, a continuous extension $\widetilde{\beta}$ as in Theorem 4.34 is uniformly computable in β. In particular any computable probe has a computable extension.

In general the existence of a computable extension $\widetilde{\beta}$ depends on the existence of a computable map $h\colon X \times S \to M$ to a basis for F with $h(x, r(y)) \leq \beta(x, y)$ for all $x \in \widetilde{X}$ and all $y \in Y$ and $h(x, r \circ f \circ \alpha(x)) = \beta(x, f \circ \alpha(x))$ for all $x \in \widetilde{X}$.

Theorem 4.34 applies "up to embedding" to all probes which take values in a computably countably based space. Let Z be a computably countably based T_0 space. Then there exists a computable embedding $j\colon Z \to \Sigma^{\mathbb{N}}$. Hence any probe (α, β) where $\beta\colon \widetilde{X} \times Y \to Z$ takes values in Z can be made into a probe $(\alpha, j \circ \beta)$ where

$$j \circ \beta\colon \widetilde{X} \times Y \to \Sigma^{\mathbb{N}}$$

takes values in a computably continuous lattice.

Theorem 4.34 essentially characterises the probes for f as those pairs (α, β) where $\beta\colon \widetilde{X} \times L \to M$ satisfies $\beta(x, F \circ \alpha(x)) = \beta(x, f \circ \alpha(x))$. On the one hand any probe "extends" to such a function by Theorem 4.34. On the other hand any such function "restricts" to a probe:

Proposition 4.35. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope of f. Let (α, β) be a pair of continuous functions $\alpha\colon \widetilde{X} \to X$ and $\beta\colon \widetilde{X} \times L \to Z$. Assume that $\beta(x, F \circ \alpha(x)) = \beta(x, \xi_L \circ f \circ \alpha(x))$ for all $x \in \widetilde{X}$. Let*

$$\beta'\colon \widetilde{X} \times Y \to Z,\ \beta'(x, y) = \beta(x, \xi_L(y)).$$

Then (α, β') is a probe for f.

Proof. Let $x \in \widetilde{X}$. Let $\psi(x_0, x_1) = \beta'(x_0, f(x_1))$. Our goal is to show that $(x, \alpha(x))$ is a point of continuity for ψ. Let $((x_0^n, x_1^n))_n$ be a sequence in $\widetilde{X} \times X$ which converges to $(x, \alpha(x))$. As F is continuous, the sequence $\beta(x_0^n, F(x_1^n))$ converges to $\beta(x, F(\alpha(x))) = \beta(x, \xi_L \circ f \circ \alpha(x))$. We have $\beta'(x_0^n, f(x_1^n)) \geq \beta(x_0^n, F(x_1^n))$ so that the sequence $(\beta'(x_0^n, f(x_1^n)))_n$ converges to the same point. □

An analogous result holds true for separated envelopes.

Definition 4.36. Let $f\colon X \to Y$ be a function between computable T_0 spaces. A *separated probe* for f is a probe (α, β) where

$$\beta\colon \widetilde{X} \times Y \to M$$

takes values in a computably separated computable complete lattice M and $\beta(x, y)$ is a maximal element of $M \setminus \{\top\}$ for all $(x, y) \in X \times Y$.

Theorem 4.37. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be a separated-universal envelope of f. Let (α, β) be a separated* *probe for* f *where* $\beta\colon \widetilde{X} \times Y \to M$ *takes values in a computably separated computably continuous lattice. Then* β *extends to a continuous map*

$$\widetilde{\beta}\colon \widetilde{X} \times L \to M$$

with

$$\widetilde{\beta}(x, \xi_L(y)) \leq \beta(x, y)$$

and

$$\widetilde{\beta}(x, F \circ \alpha(x)) = \beta(x, f \circ \alpha(x)).$$

Theorem 4.37 applies "up to embedding" to all probes which take values in a computable metric space. Let Z be a computable metric space. Then there exists a computable embedding $j\colon Z \to [0,1]^{\mathbb{N}}$. Hence any probe $\varphi\colon X \times Y \to Z$ can be made into a separated probe

$$\kappa \circ j \circ \varphi\colon X \times Y \to \mathscr{K}([0,1]^{\mathbb{N}})$$

which takes values in a computably separated computably continuous lattice.

4.7 Retracts

We introduce a notion of reducibility between functions that allows us to reduce the calculation of the universal envelope of one function to the calculation of the universal envelope of another.

Definition 4.38. Let $f\colon X \to Y$ be a function between computable T_0 spaces.

1. A *half-symmetry* of f is a function $\varphi\colon X \to \mathscr{V}(X)$ with $f(\varphi(x)) \subseteq \downarrow f(x)$ for all $x \in X$.

2. Let $x_0, x_1 \in X$. We say that x_0 *reduces to* x_1 *as an instance of* f and write $x_0 \precsim_f x_1$ if there exists a half-symmetry $\varphi\colon X \to \mathscr{V}(X)$ of f with $x_0 \in \varphi(x_1)$. We say that x_0 *is equivalent to* x_1 *as an instance of* f and write $x_0 \sim_f x_1$ if $x_0 \precsim_f x_1$ and $x_0 \succsim_f x_1$.

It follows immediately from the definition that we have the implication:

$$x_0 \precsim_f x_1 \to f(x_0) \leq f(x_1).$$

This implication still holds true if we replace f with a principal envelope:

Proposition 4.39. *Let* $f\colon X \to Y$ *be a function between computable* T_0 *spaces. Let* $F\colon X \to L$ *be a principal envelope of* f*. Then* F *is invariant under the half-symmetries of* f *in the sense that*

$$F(\varphi(x)) \subseteq \downarrow F(x)$$

for all half-symmetries φ of f. In particular we have the implication

$$x_0 \precsim_f x_1 \Rightarrow F(x_0) \leq F(x_1).$$

Proof. Let $\widetilde{x} \in \varphi(x)$. Then since φ is a half-symmetry of f we have $f(\widetilde{x}) \leq f(x)$. Hence

$$F(\widetilde{x}) \leq \xi_L \circ f(\widetilde{x}) \leq \xi_L \circ f(x).$$

It follows that

$$\sup F(\varphi(x)) \leq \xi_L \circ f(x).$$

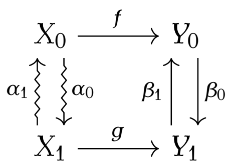
Hence $x \mapsto \sup F(\varphi(x))$ is an envelope of f and since F is the principal L-envelope it follows that $\sup F(\varphi(x)) \leq F(x)$ for all $x \in X$. This shows the first claim.

Now let $x_0, x_1 \in X$ with $x_0 \precsim_f x_1$ and let $\varphi\colon X \to \mathscr{V}(X)$ be a half-symmetry of f with $x_0 \in \varphi(x_1)$. Then

$$F(x_0) \leq \sup F(\varphi(x_1)) \leq F(x_1).$$

□

Definition 4.40. Let $f\colon X_0 \to Y_0$ and $g\colon X_1 \to Y_1$ be functions between computable T_0 spaces. We say that g is a *retract* of f if there exists a diagram



where $\alpha_0\colon X_0 \rightrightarrows X_1$ and $\alpha_1\colon X_1 \rightrightarrows X_0$ are computable multimaps and $\beta_0\colon Y_0 \to Y_1$, and $\beta_1\colon Y_1 \to Y_0$ are computable single-valued maps, such that the following axioms are satisfied:

1. For all $x_1, x_2 \in \alpha_1(x)$ we have $x_1 \sim_f x_2$.
2. For all $x_1, x_2 \in \alpha_0(x)$ we have $x_1 \sim_g x_2$.
3. $f \circ \alpha_1 \leq \beta_1 \circ g$ and $g \circ \alpha_0 \leq \beta_0 \circ f$
4. For all $x_0 \in \alpha_0 \circ \alpha_1(x_1)$ we have $x_0 \succsim_g x_1$.
5. $\beta_0 \circ \beta_1 \leq \mathrm{id}_{Y_1}$.

Note that the first two axioms in Definition 4.40 imply that $f \circ \alpha_1$ and $g \circ \alpha_0$ are single-valued functions. Thus the third axiom is well-typed.

Theorem 4.41. *Let $f\colon X_0 \to Y_0$ and $g\colon X_1 \to Y_1$ be functions between computable T_0 spaces. Assume that g is a retract of f. Let $F\colon X_0 \to L$ be a principal universal envelope of f with inclusion map $\xi_L\colon Y_0 \to L$. Then the envelope $F \circ \alpha_1\colon X_1 \to L$ with inclusion map $\xi_L \circ \beta_1\colon Y_1 \to L$ is a universal envelope of g.*

Furthermore, if T is a basis for F with factorisation $\xi_L = e \circ r$ and generating Σ-section s then T is a basis for $F \circ \alpha_1$ with factorisation $\xi_L \circ \beta_1 = e \circ (r \circ \beta_1)$ and the same generating Σ-section s.

Proof. By Proposition 4.39 and the first assumption of Definition 4.40 the map $F \circ \alpha_1$ is single-valued. Let $x \in X_1$. We have

$$F \circ \alpha_1(x) \leq \xi_L \circ f \circ \alpha_1(x) \leq \xi_L \circ \beta_1 \circ g(x).$$

Hence $F \circ \alpha_1$ is an envelope of g. Let $G\colon X \to M$ be a principal envelope of g. Then by the same argument, $G \circ \alpha_0$ with inclusion map $\xi_M \circ \beta_0$ is an envelope of f. As F is assumed to be universal there exists a map

$$\Phi\colon L \to M$$

with

$$\Phi \circ \xi_L \leq \xi_M \circ \beta_0$$

and

$$\Phi \circ F \geq G \circ \alpha_0.$$

This yields

$$\Phi \circ \xi_L \circ \beta_1 \leq \xi_M \circ \beta_0 \circ \beta_1 \leq \xi_M$$

and

$$\Phi \circ F \circ \alpha_1 \geq G \circ \alpha_0 \circ \alpha_1 \geq G.$$

The last inequality uses Proposition 4.39.

Now assume that T is a basis for F with factorisation $\xi_L = r \circ e$ and generating Σ-section s. It follows from our previous reasoning that F tightens G via the map $(\xi_M \circ \beta_0 / r)/(\frac{s}{e})$. Using that $\beta_0 \circ \beta_1 \leq \xi_M$ we obtain:

$$(\xi_M \circ \beta_0 / r) \circ r \circ \beta_1 \leq \xi_M \circ \beta_0 \circ \beta_1 \leq \xi_M.$$

By definition of $(\xi_M / r \circ \beta_1)$ we conclude that

$$(\xi_M \circ \beta_0 / r) \leq (\xi_M / r \circ \beta_1).$$

In particular

$$(\xi_M / r \circ \beta_1)/(\tfrac{s}{e}) \circ F \geq (\xi_M \circ \beta_0 / r)/(\tfrac{s}{e}) \circ F \geq G.$$

By definition we have

$$(\xi_M/r \circ \beta_1)/(\tfrac{s}{e}) \circ e \circ r \circ \beta_1 \leq \xi_M.$$

Thus $F \circ \alpha_1$ tightens G via the map $(\xi_M/r \circ \beta_1)/(\frac{s}{e})$. It follows that T is a basis of $F \circ \alpha_1$. □

The property of being a retract of a function is quite strong compared with the usual reducibility notions considered in computable analysis. A typical tool for comparing the computational strength of non-computable problems is Weihrauch reduction [16]. Unfortunately a Weihrauch-equivalence between functions does not induce a translation between their universal envelopes:

Example 4.42. Consider the function

$$\texttt{zeroes}\colon C([0,1]) \to \mathscr{F}([0,1]),\ \texttt{zeroes}(f) = \{x \in [0,1] \mid f(x) = 0\}$$

which encodes the computational problem of locating the zero set of a function.

Consider the function

$$\texttt{locate}\colon \mathscr{A}([0,1]) \to \mathscr{F}([0,1]),\ \texttt{locate}(A) = A$$

which encodes the computational problem of making a closed set into a located set.

The two functions are strongly Weihrauch-equivalent due to a well-known construction which goes back to Specker [95].

Given a function $f \in C([0,1])$ we can compute the zero-set of f as an element of $\mathscr{A}(C([0,1]))$ and apply `locate` to obtain $\texttt{zeroes}(f)$. It follows that `zeroes` strongly Weihrauch-reduces to `locate`.

By Proposition 2.39 a set $A \in \mathscr{A}([0,1])$ can be represented by a list of open rational intervals $(I_n)_n$ with $A = (\bigcup_{n\in\mathbb{N}} I_n)^C$. Thus, suppose we are given a name $(I_n)_n$ of $A \in \mathscr{A}([0,1])$. Let $I_n = (a_n, b_n)$. Let $h(x) = \max(0, 1 - |x|)$. We can then compute the function

$$f(x) = \sum_{n\in\mathbb{N}} 2^{-n} h\left(\tfrac{2}{b_n - a_n}\left(x - \tfrac{a_n+b_n}{2}\right)\right).$$

Of course, the function f depends strongly on the name of A. However, we always have $f(x) = 0$ if and only if $x \in A$. It follows that `locate` strongly Weihrauch-reduces to `zeroes`.

The same proof establishes that `locate` is a retract of `zeroes`. However, `zeroes` is not a retract of `locate`. In fact, the amount of information contained in the respective universal envelopes is quite different. It is easy to see that the universal envelope of `locate` is the identity on $\mathscr{A}([0,1])$. This yields the envelope $H\colon C([0,1]) \to \mathscr{A}([0,1])$ of `zeroes` with the inclusion map being the identity $\mathscr{F}([0,1]) \to \mathscr{A}([0,1])$. Let $Z\colon C([0,1]) \to \mathscr{K}(\mathscr{F}([0,1]))$ be the principal $\mathscr{K}(\mathscr{F}([0,1]))$-envelope of `zeroes`. Consider for example the real function $f(x) = 1 - (\frac{3}{2} - 5x)^2$. Then f has two isolated zeroes, namely $\frac{1}{10}$ and $\frac{1}{2}$, and f changes its sign in each of the zeroes. It follows that $Z(f) = \{\{\frac{1}{10}, \frac{1}{2}\}\}$, *i.e.*, Z coincides with `zeroes` in f. Hence Z contains more information on `zeroes` than H.

It is instructive to note that the Weihrauch reduction from `zeroes` to `locate` destroys information on the input f which cannot be recovered continuously from the zero set as an element of $\mathscr{A}([0,1])$. Translating f to its zero set as an element of $\mathscr{A}([0,1])$ and then translating back to a function with the same zero set using the Weihrauch reduction from `locate` to `zeroes` leaves us with a function with two "unstable" zeroes where no sign-change occurs.

4.8 The dense subset lemma

The following lemma allows us to reduce the problem of calculating a universal envelope of a given function $f\colon X \to Y$ to the problem of calculating a universal envelope of a restriction of f to a dense subset.

Lemma 4.43. *Let $f\colon X \to Y$ be a function between computable T_0 spaces. Let $F\colon X \to L$ be an envelope of f. Assume that there exists a dense subset $S \subseteq X$ such that the restriction $F|_S$ is a universal envelope of the restriction $f|_S$. Let T be a basis for $F|_S$ with factorisation $\xi_L = e \circ r$ and generating Σ-section s. Further assume that for all $x \in X$ and all open sets $U \in \mathscr{O}(T)$ we have the implication:*

$$(\exists W \in \nu_X(x).\forall \widetilde{x} \in W \cap S.(F(\widetilde{x}) \in s(U))) \to F(x) \in s(U).$$

Then F is a universal envelope of f with basis T.

Proof. Let $G\colon X \to M$ be an envelope of f. Then $G|_S$ is an envelope of $f|_S$. As $F|_S$ is universal, the assumptions of Theorem 4.26 are satisfied. It follows that $G \le (\xi_M/r) \circ r \circ f$. Let $x \in X$. Let $U \in \mathscr{O}(M)$ be an open set with $G(x) \in U$. Then $(\xi_M/r)^*(U)$ is a robust property of $r \circ f(\widetilde{x})$ for all $\widetilde{x} \in G^{-1}(U)$. In particular this is true of the points $\widetilde{x} \in G^{-1}(U) \cap S$. We hence have $F(\widetilde{x}) \in s((\xi_M/r)^*(U))$ for all $\widetilde{x} \in G^{-1}(U) \cap S$. By the assumption on F it follows that $F(x) \in s((\xi_M/r)^*(U))$.

The rest of the proof is identical to the proof of Theorem 4.15. We conclude that

$$\{(\xi_M/r)^*(U) \in \mathscr{O}(T) \mid G(x) \in U\} \subseteq \{U \in \mathscr{O}(T) \mid F(x) \in s(U)\}.$$

We then calculate:

$$
\begin{aligned}
(\xi_M/r)/(\tfrac{s}{e}) \circ F(x) &= r_M \circ (\xi_M/r)^{**} \circ s^* \circ \nu_L \circ F(x) \\
&= r_M \circ (\xi_M/r)^{**} \left(\{U \in \mathscr{O}(T) \mid F(x) \in s(U)\}\right) \\
&\geq r_M \circ (\xi_M/r)^{**} \left(\{(\xi_M/r)^*(U) \in \mathscr{O}(T) \mid G(x) \in U\}\right) \\
&= r_M \circ \nu_M \circ G(x) \\
&= G(x).
\end{aligned}
$$

□

4.9 Envelopes of set-valued functions

In analysis one very frequently encounters computational problems which do not admit a unique solution. Such problems can be modelled as multi-valued maps which send a problem instance to the set of all possible solutions. In computable analysis it is customary to understand the computational problem associated with such a map to be the task of finding a particular solution for every given problem instance in a potentially non-extensional way, *i.e.*, given a name of a problem instance to produce a name of a solution. Such semantics are particularly appropriate for showing uncomputability results. For our purpose however it seems most appropriate to model multi-valued functions as set-valued functions which take values in a suitably chosen powerspace. In other words we will study the problem of computing the set of all solutions to a given problem, rather than the problem of obtaining one particular solution. In order to have a representation for the space of sets of solutions available we restrict our attention to set-valued functions with closed values. If $f\colon X \rightrightarrows Y$ is a set-valued function with closed values between computable T_0 spaces the question arises how to represent the space of closed subsets. In principle there are many possible ways of representing closed sets [24] but arguably the three most important represented spaces of closed subsets are:

1. The *upper powerspace* $\mathscr{A}(Y)$.
2. The *lower powerspace* $\mathscr{V}(Y)$.
3. The *joint powerspace* $\mathscr{F}(Y) = \{(A, A) \in \mathscr{A}(Y) \times \mathscr{V}(Y) \mid A \subseteq Y \text{ closed}\}$.

Elements of $\mathscr{A}(Y)$ can be thought of as closed sets encoded with negative information. Elements of $\mathscr{V}(Y)$ can be thought of as closed sets encoded with positive information. Hence elements of $\mathscr{F}(Y)$ encode both negative and positive information on a closed set.

Thus we can associate three natural computational problems with any set-valued function:

1. The *upper formulation* $f_>: X \to \mathscr{A}(Y)$.

2. The *lower formulation* $f_<: X \to \mathscr{V}(Y)$.

3. The *joint formulation* $f: X \to \mathscr{F}(Y)$.

The lower formulation is closely related to the usual non-deterministic semantics for multi-valued functions used in computable analysis, see [19]. Any envelope for the upper or lower formulation is an envelope for the joint formulation. In particular the join of universal envelopes of the upper and lower formulation is an envelope for the joint formulation. Note however that there is no reason to expect that this join be a universal envelope for the joint formulation.

Proposition 4.44. *Let Y be a computable T_0 space. Then $\mathscr{A}(Y)$ is a finitary approximation lattice for $\mathscr{A}(Y)$, the inclusion map being given by the identity.*

Proof. The space $\mathscr{A}(Y)$ is a computable complete lattice as it is just the dual lattice of $\mathscr{O}(Y)$. By the same argument $\mathscr{A}(Y)$ is a computably injective lattice. Thus $\mathscr{A}(Y)$ is an approximation lattice over $\mathscr{A}(Y)$. It is obvious that the identity is a finitary embedding. □

Proposition 4.45. *Let Y be a computable T_0 space. Then $\mathscr{O}^2(Y)$ is an approximation lattice for $\mathscr{V}(Y)$. The inclusion map is given by the canonical embedding*

$$A \mapsto \{U \in \mathscr{O}(Y) \mid A \cap U \neq \emptyset\}.$$

Proposition 4.46. *Let Y be a computably countably based space. Then $\mathscr{O}^2(Y)$ is a finitary approximation lattice for $\mathscr{V}(Y)$.*

Proof. By Proposition 2.56 the map

$$\gamma: \mathscr{K}(\mathscr{V}(Y)) \to \mathscr{O}^2(Y),\ \gamma(K) = \{U \in \mathscr{O}(Y) \mid \forall A \in K. A \cap U \neq \emptyset\}$$

is a computable isomorphism. Its inverse is given by the map

$$\gamma^{-1}: \mathscr{O}^2(Y) \to \mathscr{K}(\mathscr{V}(Y)),\ \gamma^{-1}(\mathscr{U}) = \{A \in \mathscr{V}(Y) \mid \forall U \in \mathscr{U}. A \cap U \neq \emptyset\}.$$

The inclusion map $\xi\colon \mathscr{V}(Y) \to \mathscr{O}^2(Y)$ is given by $\xi(A) = \{U \in \mathscr{O}(Y) \mid A \cap U \neq \emptyset\}$. It follows that $\xi^{-1}(\uparrow\mathscr{U}) = \gamma^{-1}(\mathscr{U})$ so that ξ is computably proper and hence computably finitary. $\square$

It follows from Proposition 4.46 and Theorem 4.18 that if Y is computably countably based then the principal $\mathscr{O}^2(Y)$-envelope F of a function $f\colon X \to \mathscr{V}(Y)$ is universal if and only if for each robust property U of $f(x)$ we have the inclusion $\xi^{-1}(\uparrow F(x)) \subseteq U$.

This criterion still requires us to reason about open sets of the space $\mathscr{V}(Y)$. Luckily, the task can be further simplified:

Lemma 4.47. *Let $f\colon X \to \mathscr{V}(Y)$ be a function between a computable T_0 space X and a computably countably based space Y. Let $F\colon X \to \mathscr{O}^2(Y)$ be an envelope of f with the inclusion map $\xi\colon \mathscr{V}(Y) \to \mathscr{O}^2(Y)$ being the natural embedding. If for all $x \in X$ the set $F(x) \in \mathscr{O}^2(Y)$ contains all $U \in \mathscr{O}(Y)$ such that the set*

$$f^{-1}(U) = \{z \in X \mid f(z) \cap U \neq \emptyset\}$$

is a neighbourhood of x then F is uniformly Σ-complete.

Proof. By Proposition 2.52 we have a computable isomorphism

$$\gamma\colon \mathscr{K}(\mathscr{V}(Y)) \to \mathscr{O}^2(Y).$$

We hence have a map

$$\gamma_*\colon \mathscr{O}(\mathscr{K}(\mathscr{V}(Y))) \to \mathscr{O}^3(Y).$$

Consider the natural embedding

$$\kappa_{\mathscr{V}(Y)}\colon \mathscr{V}(Y) \to \mathscr{K}(\mathscr{V}(Y)).$$

As $\kappa_{\mathscr{V}(Y)}$ is proper, the upper adjoint α of $\kappa^*_{\mathscr{V}(Y)}$ is computable. We can then compute the map

$$\gamma_* \circ \alpha\colon \mathscr{O}(\mathscr{V}(Y)) \to \mathscr{O}^3(Y).$$

This map is a section for ξ^*, where $\xi\colon \mathscr{V}(Y) \to \mathscr{O}^2(Y)$ is the natural embedding. Indeed, we have $\xi = \gamma \circ \kappa_{\mathscr{V}(Y)}$ and hence

$$\xi^* \circ \gamma_* \circ \alpha = (\gamma \circ \kappa_{\mathscr{V}(Y)})^* \circ \gamma_* \circ \alpha = \kappa^*_{\mathscr{V}(Y)} \circ \gamma^* \circ \gamma_* \circ \alpha = \kappa^*_{\mathscr{V}(Y)} \circ \alpha = \mathrm{id}_{\mathscr{O}(\mathscr{V}(Y))}.$$

Our goal is to show that

$$F(x) \in \gamma_* \circ \alpha(\mathscr{U})$$

for every robust property $\mathscr{U}$ of $f(x)$. The claim then follows from Theorem 4.15.

The topology of $\mathscr{V}(Y)$ is generated by sets of the form

$$[U] = \{A \in \mathscr{V}(Y) \mid A \cap U \neq \emptyset\}.$$

By the definition of γ we have $U \in F(x)$ if and only if $\gamma^{-1} \circ F(x) \subseteq [U]$. Let $\mathscr{U} \in \mathscr{O}(\mathscr{V}(Y))$ be a robust property of $f(x)$. Then we can write

$$\mathscr{U} = \bigcup_{i \in I} \left([U_1^i] \cap \cdots \cap [U_{n_i}^i] \right)$$

with $U_j^i \in \mathscr{O}(Y)$. To simplify the notation let us set $U_j^i = U_{n_i}^i$ for $j > n$, so that we can write

$$\mathscr{U} = \bigcup_{i \in I} \bigcap_{j \in \mathbb{N}} [U_j^i].$$

By the axiom of choice we have:

$$\mathscr{U} = \bigcap_{A\colon I \to \mathbb{N}} \bigcup_{i \in I} [U_{A(i)}^i].$$

Note that for all collections $(V_j)_j$ of open subsets of $\mathscr{V}(Y)$ we have:

$$\bigcup_{j \in J} [V_j] = [\bigcup_{j \in J} V_j].$$

It follows that

$$\mathscr{U} = \bigcap_{A\colon I \to \mathbb{N}} [\bigcup_{i \in I} U_{A(i)}^i].$$

Since $\mathscr{U}$ is robust in particular the property $[\bigcup_{i \in I} U_{A(i)}^i]$ is robust for every function $A\colon I \to \mathbb{N}$, so that by the assumption on F we have

$$\bigcup_{i \in I} U_{A(i)}^i \in F(x).$$

Hence, by the definition of γ,

$$\gamma^{-1} \circ F(x) \subseteq [\bigcup_{i \in I} U_{A(i)}^i].$$

And thus

$$\gamma^{-1} \circ F(x) \subseteq \bigcap_{A\colon I \to \mathbb{N}} [\bigcup_{i \in I} U_{A(i)}^i] = \mathscr{U}.$$

It follows that

$$F(x) \in \gamma_* \circ \alpha(\mathscr{U})$$

and the result is shown. □

It is useful to fix a name for the special robust properties that are used in Lemma 4.47:

Definition 4.48. Let $f\colon X \to \mathscr{V}(Y)$ be a function where X and Y are computable T_0 spaces and Y is computably countably based. Let $x \in X$. An open set $U \in \mathscr{O}(Y)$ such that

$$f^{-1}(U) = \{z \in X \mid f(z) \cap U \neq \emptyset\}$$

is a neighbourhood of x is called a *basic robust property* of $f(x)$.

Typically it is easier to calculate a universal envelope for the upper and lower formulation of a set-valued map than to calculate a universal envelope for the joint formulation. In general the join of a universal envelope of the upper formulation and a universal envelope of the lower formulation need not be a universal envelope for the joint formulation. We can however give a sufficient criterion for this, which is stated in Theorem 4.52 below. We need two auxiliary results as a preparation:

Proposition 4.49. *Let Y be a computable T_0 space. If Y is computably compact then $\mathscr{F}(Y)$ is computably compact.*

Definition 4.50. Let X be a computable T_0 space. Then X is called *computably locally compact* if there exists a computable sequence $(\widehat{I}_n)_n$ of compact sets $\widehat{I}_n \in \mathscr{K}(X)$ and a computable sequence $(I_n)_n$ of open sets $I_n \in \mathscr{O}(X)$ such that $I_n \subseteq \widehat{I}_n$ for all $n \in \mathbb{N}$ and $(I_n)_n$ and constitutes a basis for the topology of X. We call any such pair of sequences *computable basis of compact neighbourhoods* for X.

Proposition 4.51. *Let Y be a computably compact and computably locally compact computable T_0 space. Then the maps*

$$\mathscr{A}(Y) \to \mathscr{K}(\mathscr{F}(Y)),\ A \mapsto \{B \in \mathscr{F}(Y) \mid B \subseteq A\}$$

and

$$\mathscr{O}^2(Y) \to \mathscr{K}(\mathscr{F}(Y)),\ \mathscr{U} \mapsto \{B \in \mathscr{F}(Y) \mid \forall U \in \mathscr{U}.(B \cap U \neq \emptyset)\}$$

are well-defined and computable.

Proof. By Proposition 4.49 if Y is computably compact then so is $\mathscr{F}(Y)$. Hence the identity $\mathscr{A}(\mathscr{F}(Y)) \to \mathscr{K}(\mathscr{F}(Y))$ is well-defined and computable. Thus it suffices to compute the maps with co-domain $\mathscr{A}(\mathscr{F}(Y))$. Given $A \in \mathscr{A}(Y)$ and $B \in \mathscr{F}(Y)$ we can verify if $B \nsubseteq A$ by testing if there exists $y \in B$ with $y \notin A$. Computability of the first map follows. Let $(\widehat{I}_n)_n$ be a computable basis of compact neighbourhoods of Y. Given $\mathscr{U} \in \mathscr{O}^2(Y)$ and $B \in \mathscr{F}(Y)$ we can verify if there exists a finite sequence $\langle n_0, \dots, n_k \rangle \in \mathbb{N}^*$ such that $I_{n_0} \cup \dots \cup I_{n_k} \in \mathscr{U}$ but $(\widehat{I}_{n_0} \cup \dots \cup \widehat{I}_{n_k}) \cap B = \emptyset$. As this is the case if and only if there exists $U \in \mathscr{U}$ with $B \cap U = \emptyset$ computability of the second map follows. □

Theorem 4.52. *Let $f\colon X \rightrightarrows Y$ be a multi-valued function between computable T_0 spaces. Let $F_>\colon X \to \mathscr{A}(Y)$ be an $\mathscr{A}(Y)$-envelope of the upper formulation $f_>\colon X \to \mathscr{A}(Y)$. Let $F_<\colon X \to \mathscr{O}^2(Y)$ be an $\mathscr{O}^2(Y)$-envelope of the lower formulation $f_<\colon X \to \mathscr{V}(Y)$. Consider the joint formulation $f\colon X \to \mathscr{F}(Y)$. If every robust property of $f(x)$ contains the set*

$$\{A \in \mathscr{F}(Y) \mid A \subseteq \mathscr{F}_>(x) \wedge \forall U \in F_<(x).(A \cap U \neq \emptyset)\}.$$

then $F_> \times F_<$ is the universal envelope of f. In this case it is uniformly Σ-complete.

Proof. Using Proposition 4.51 we obtain the computable map

$$\mathscr{A}(Y) \times \mathscr{O}^2(Y) \to \mathscr{K}(\mathscr{F}(Y)),\ (A, \mathscr{U}) \mapsto \{C \in \mathscr{F}(Y) \mid C \subseteq A \wedge \forall U \in \mathscr{U}.(C \cap U \neq \emptyset)\}$$

Composition of $F_> \times F_<$ with this map yields a $\mathscr{K}(\mathscr{F}(Y))$-envelope of f. This envelope is uniformly Σ-complete if and only if every robust property of $f(x)$ contains the set

$$\{A \in \mathscr{F}(Y) \mid A \subseteq \mathscr{F}_>(x) \wedge \forall U \in F_<(x).(A \cap U \neq \emptyset)\}.$$

The claim follows. □

A famous result due to Kuratowski asserts that upper and lower semicontinuity coincide generically.

Recall that a multi-valued function $f\colon X \rightrightarrows Y$ is called *lower semicontinuous* if for every open set $U \in \mathscr{O}(Y)$ the *preimage* $f^{-1}(U) = \{x \in X \mid f(x) \cap U \neq \emptyset\}$ is an open subset of X. The function $f\colon X \rightrightarrows Y$ is called *upper semicontinuous* if for every closed set $A \in \mathscr{A}(Y)$ the preimage $f^{-1}(A)$ is a closed subset of X.

A subset of a topological space X is called *comeagre* or a *residual* if it can be expressed as a countable intersection of dense open sets. A *Baire space* is a topological space in which every comeagre set is dense. A property that holds for all points of a residual in a Baire space is also referred to as a *generic property*. The *Baire category theorem* asserts that every complete metric space is a Baire space.

It follows from Kuratowski's result that the principal $\mathscr{O}^2(Y)$-envelope of an upper semicontinuous function f coincides generically with the lower formulation $f_<\colon X \to \mathscr{V}(Y)$. As a preparation we need a result that is of independent interest:

Theorem 4.53. *Let X be a computable T_0 space. Let Y be a complete computable metric space. Let $f\colon X \rightrightarrows Y$ be a multi-valued function. If there exists a continuous function $B\colon X \to \mathcal{K}(Y)$ with $f(x) \subseteq B(x)$ then the principal $\mathcal{O}^2(Y)$-envelope of $f_<\colon X \to \mathcal{V}(Y)$ coincides with $f_<$ in all points of lower semicontinuity.*

Proof. Assume that f is lower semicontinuous in $x \in X$. Let $U \in \mathcal{O}(Y)$ with $f(x) \cap U \neq \emptyset$. Let $y \in f(x) \cap U$. Then there exists an open set V with

$$y \in \mathrm{cl}\,(V) \subseteq U.$$

As f is lower semicontinuous in x the set $f^{-1}(V)$ is a neighbourhood of x. Let $W \in \mathcal{O}(X)$ be an open set with

$$x \in W \subseteq f^{-1}(V).$$

Define the map

$$G\colon X \to \mathcal{O}^2(Y),\; G(z) = \begin{cases} \{U \in \mathcal{O}(Y) \mid U \supseteq \mathrm{cl}\,(V) \cap B(z)\} & \text{if } z \in W, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then G is an $\mathcal{O}^2(Y)$-envelope of f with $G(x) \cap U \neq \emptyset$. It follows that the principal $\mathcal{O}^2(X)$-envelope of f coincides with $f_<$ in x. □

For a proof of the following result see e.g. [1].

Theorem 4.54 (Kuratowski, 1958). *Let $f\colon X \rightrightarrows Y$ be an upper semicontinuous function with values in a complete separable metric space Y. Then the points of lower semicontinuity of f are comeagre.*

As an immediate corollary to the previous two results we obtain:

Theorem 4.55. *Let X be a computable T_0 space. Let Y be a complete computable metric space. If $f\colon X \rightrightarrows Y$ is an upper semicontinuous function with compact values then the principal $\mathcal{O}^2(Y)$-envelope of $f_<\colon X \to \mathcal{V}(Y)$ generically coincides with f, i.e., it coincides with f in a comeagre set.*

Theorem 4.56. *Let X be a computable T_0 space. Let Y be a complete computable metric space. If $f\colon X \rightrightarrows Y$ is an upper semicontinuous function with compact values then the principal $\mathcal{A}(Y) \times \mathcal{O}^2(Y)$-envelope of $f\colon X \to \mathcal{F}(Y)$ generically coincides with f, i.e., it coincides with f in a comeagre set.*

Chapter 5

Calculations

As an application of the theory developed so far, we will calculate the universal envelopes of two non-trivial problems: Locating the fixed point set of a continuous self-map of the unit cube in finite-dimensional euclidean space and locating the fixed point set of a nonexpansive self-map of the unit ball in infinite-dimensional separable real Hilbert space.

It should be emphasized that the problem of "calculating" the universal envelope is a creative process rather than a mechanical one. In each case we will proceed by first guessing the universal envelope and then using the techniques developed in this thesis to verify that it is indeed universal.

The "guesses" are informed by previous computability results. In the case of finding Brouwer fixed points, the Brouwer index yields a sufficient condition for the existence of a fixed point. The index is computable and can be used to compute components of the fixed point set in the upper Vietoris topology [72, 21, 20, 29]. On any open set where the index is zero, the function can be made fixed-point free up to a small perturbation thanks to the Hopf theorem [55]. This suggests that the greatest amount of continuously obtainable information on the fixed point set is encoded in the Brouwer index. We will verify this using Lemma 4.47 in conjunction with the Hopf theorem.

In the case of finding fixed points of nonexpansive maps, it was shown in [78] that the problem finding a fixed point is Weihrauch-equivalent to a "compact choice" operator which sends a compact set in the strong topology to a compact set in the weak topology. We will build on this result, showing that if we restrict to maps with unique fixed points, we obtain a retraction from the problem of computing the identity from the weak topology to the strong topology. As the universal envelope of the latter is easily seen to be the identity of the unit ball with the weak topology, we obtain a universal envelope of a restriction to the problem to a dense subset. The dense subset lemma (Lemma 4.43) then enables us to extend this envelope to the whole space.

5.1 Brouwer fixed points

By the famous Brouwer fixed point theorem any continuous map

$$f\colon [0,1]^n \to [0,1]^n$$

has a fixed point. The problem

$$\mathrm{Fix}\colon C([0,1]^n,[0,1]^n) \rightrightarrows [0,1]^n,\ \mathrm{Fix}(f) = \{x \in [0,1]^n \mid f(x) = x\}$$

of finding fixed points of a given continuous map is well-known to be uncomputable. Its computational content has been extensively studied within computability theory and reverse mathematics [79, 3, 21, 92].

Note that the function

$$\mathrm{Fix}_{>}\colon C([0,1]^n,[0,1]^n) \to \mathscr{A}([0,1]^n)$$

is computable.

The goal of this section is to calculate a universal lower envelope for Fix, *i.e.*, a universal envelope of the function

$$\mathrm{Fix}_{<}\colon C([0,1]^n,[0,1]^n) \to \mathscr{V}([0,1]^n).$$

Our calculation is mainly based on ideas by Collins [29].

The main tool will be the Brouwer mapping degree. Recall that the mapping degree is the unique function

$$\deg\colon C(\mathbb{R}^n,\mathbb{R}^n) \times \mathscr{O}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{Z}$$

with domain

$$\mathrm{dom}(\deg) = \{(f,U,y) \in C(\mathbb{R}^n,\mathbb{R}^n) \times \mathscr{O}(\mathbb{R}^n) \times \mathbb{R}^n \mid U \text{ is bounded and } y \notin f(\partial U)\}$$

which satisfies the following properties:

1. Translation invariance: $\deg(f,U,y) = \deg(f-y,U,0)$.

2. Normalisation: $\deg(\mathrm{id},U,y) = 1$ for all $y \in U$.

3. Additivity: If U_1 and U_2 are open disjoint subsets of U such that we have $y \notin f(\mathrm{cl}\,U \setminus (U_1 \cup U_2))$ then $\deg(f,U,y) = \deg(f,U_1,y) + \deg(f,U_2,y)$.

4. Homotopy invariance: If $H(t, x)$ is a homotopy from f to g satisfying $y \notin H(t, \partial U)$ for all $t \in [0, 1]$ then $\deg(f, U, y) = \deg(g, U, y)$.

The degree yields a sufficient condition for an equation to have a solution. If $\deg(f, U, y)$ is well-defined and non-zero, *i.e.*, if $(f, U, y) \in \mathrm{dom}(\deg)$ and $\deg(f, U, y) \neq 0$ then the equation $f(x) = y$ has at least one solution in U.

Very nice and readable introductions to the mapping degree are given in [73] and [98].

A concrete definition of the degree can be given in terms of singular homology. For our purpose it suffices to establish this for the unit sphere S^n. Let $h\colon S^n \to S^n$ be a self-map of the unit sphere. Then h induces a homomorphism $h_*\colon H_n(S^n) \to H_n(S^n)$, where H_n is the n^{th} singular homology group. As $H_n(S^n) \simeq \mathbb{Z}$ this homomorphism is the action of the multiplication with a number $\alpha \in \mathbb{Z}$. We call α the *mapping degree* of h. See e.g. [52, Chapter 2.2, p. 134ff.] for more details.

This definition relates to our axiomatic definition of degree as follows: Let $f\colon D^n \to \mathbb{R}^n$ be a map on the unit disk D^n. Let $y \in \mathbb{R}^n \setminus f(S^{n-1})$. Then we can define the map

$$h\colon S^{n-1} \to S^{n-1},\ h(x) = \frac{f(x) - y}{|f(x) - y|}.$$

The degree of h is equal to $\deg(f, D^n, y)$.

The *Hopf theorem* asserts that the degree is the only homotopy invariant of self-maps of S^n. This will play an important role in our calculation.

Theorem 5.1 (Hopf, 1927 [55]). *Let $f, g\colon S^n \to S^n$ be self-maps of the n-sphere. Then f and g have the same mapping degree if and only if they are homotopic.*

In particular if $\deg(f, D^n, y) = 0$ then the map $h(x) = \frac{f(x)-y}{|f(x)-y|}$ is homotopic to a constant function. This is the main idea behind the proof of the key lemma 5.8 below.

It can be shown that the degree is computable when the space of open sets is appropriately represented. This was probably first observed by Miller [72] who showed that the degree is computable on rational cubical complexes (see also [21, 20, 22]). The result is based on computational homology [58].

It will be convenient for our purpose to be able to compute the degree on the set of all open sets. Let $\mathscr{U}(\mathbb{R}^n)$ denote the space of open subsets of $\mathbb{R}^n$ which is obtained by identifying an open set U with its two-sided distance function:

$$d_{\text{two-sided}}(\cdot, U)\colon X \to \mathbb{R},\ x \mapsto \begin{cases} d(x, \partial U) & \text{if } x \notin U, \\ -d(x, \partial U) & \text{if } x \in U. \end{cases}$$

Note that the underlying representation is much stronger than the standard representation of open sets. In particular, the space $\mathscr{U}(\mathbb{R}^n)$ is a computable Hausdorff space. Computability of the degree on this class of spaces can be established in a similar way as for cubical complexes using computational homology. For the sake of variety we mention an arguably more elementary proof based on the determinant formula:

Theorem 5.2. *The partial map*

$$\deg\colon \subseteq C(\mathbb{R}^n, \mathbb{R}^n) \times \mathscr{U}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{Z},\ (f, U, y) \mapsto \deg(f, U, y)$$

is computable with semi-decidable domain.

Proof Sketch. The degree $\deg(f, U, y)$ is defined so long as $y \notin f(\partial U)$, and this is uniformly semi-decidable for continuous f and $U \in \mathscr{U}$. To compute $\deg(f, U, y)$, compute a sufficiently good twice differentiable approximation $\widetilde{f}$ to f and a sufficiently good approximation $\widetilde{y}$ to y, which is a regular value of $\widetilde{f}$. It suffices to choose $\widetilde{y}$ with $|y - \widetilde{y}| < d(y, f(\partial U))$ and $\widetilde{f}$ with $\left|f - \widetilde{f}\right| < d(y, f(\partial U))$. The fact that $\widetilde{y}$ can be chosen to be a regular value follows from Sard's theorem. Then $\deg(f, U, y)$ can be computed using the determinant formula:

$$\deg(f, U, y) = \deg(\widetilde{f}, U, \widetilde{y}) = \sum_{x \in \widetilde{f}^{-1}(y)} \operatorname{sgn}\left(\det\left(D\widetilde{f}(x)\right)\right).$$

For more details refer to the construction of the mapping degree in [98, Chapter 16]. □

It will further be convenient to extend the degree to the set of all bounded open sets. From now on we write deg for the map

$$\deg\colon \subseteq C(\mathbb{R}^n) \times \mathscr{U}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{Z}_\perp$$

which extends the previous definition of deg to all triples (f, U, y) where U is a bounded open set and is equal to $\perp$ if and only if $y \in f(\partial U)$. Clearly this map is computable.

Lemma 5.3. *Consider the function*

$$\mathrm{Fix}_<\colon C([0,1]^n, [0,1]^n) \to \mathscr{V}([0,1]^n),\ \mathrm{Fix}_<(f) = \{x \in [0,1]^n \mid f(x) = x\}$$

which sends a continuous self-map of the unit cube to its fixed point set in the lower Vietoris topology. Then $\mathrm{Fix}_<$ *is a computable retract of the function*

$$\widetilde{\mathrm{Fix}}_<\colon C(\mathbb{R}^n, [0,1]^n) \to \mathscr{V}(\mathbb{R}^n),\ \widetilde{\mathrm{Fix}}_<(f) = \{x \in \mathbb{R}^n \mid f(x) = x\}.$$

Proof. Let $s\colon [0,1]^n \to \mathbb{R}^n$ be the subspace inclusion. Choose a computable retraction $r\colon \mathbb{R}^n \to [0,1]^n$. Consider the diagram

$$\begin{array}{ccc} C(\mathbb{R}^n,[0,1]^n) & \xrightarrow{\widetilde{\mathrm{Fix}}_<} & \mathscr{V}(\mathbb{R}^n) \\ \mathrm{ext}\uparrow\;\downarrow\mathrm{res} & & s_*\uparrow\;\downarrow r_* \\ C([0,1]^n,[0,1]^n) & \xrightarrow{\mathrm{Fix}_<} & \mathscr{V}([0,1]^n) \end{array}$$

where

$$\mathrm{res}\colon C(\mathbb{R}^n,[0,1]^n) \to C([0,1]^n,[0,1]^n),\ \mathrm{res}(f) = f|_{[0,1]^n}$$

and

$$\mathrm{ext}\colon C([0,1]^n,[0,1]^n) \to C(\mathbb{R}^n,[0,1]^n),\ \mathrm{ext}(f) = f \circ r.$$

It is easy to see that this defines a retraction from $\widetilde{\mathrm{Fix}}_<$ to $\mathrm{Fix}_<$. □

Theorem 5.4. *Consider the function*

$$\widetilde{\mathrm{Fix}}_<\colon C(\mathbb{R}^n,[0,1]^n) \to \mathscr{V}(\mathbb{R}^n),\ \widetilde{\mathrm{Fix}}_<(f) = \{x \in \mathbb{R}^n \mid f(x) = x\}$$

A universal envelope of $\widetilde{\mathrm{Fix}}_<$ *is given by the map*

$$\widetilde{F}\colon C(\mathbb{R}^n,[0,1]^n) \to \mathscr{O}^2(\mathbb{R}^n),$$
$$\widetilde{F}(f) = \{U \in \mathscr{O}(\mathbb{R}^n) \mid \exists V \subseteq U.\,(\deg(f - \mathrm{id}_{\mathbb{R}^n}, V, 0) \notin \downarrow 0)\}\,.$$

This envelope is uniformly Σ*-complete.*

Corollary 5.5. *Consider the function*

$$\mathrm{Fix}_<\colon C([0,1]^n,[0,1]^n) \to \mathscr{V}([0,1]^n),\ \mathrm{Fix}_<(f) = \{x \in [0,1]^n \mid f(x) = x\}$$

Let $r\colon \mathbb{R}^n \to [0,1]^n$ *be a computable retraction. A universal envelope of* $\mathrm{Fix}_<$ *is given by the map*

$$F\colon C([0,1]^n,[0,1]^n) \to \mathscr{O}^2([0,1]^n),$$
$$F(f) = \{U \in \mathscr{O}([0,1]^n) \mid \exists V \subseteq r^*(U).\,(\deg(f \circ r - \mathrm{id}_{\mathbb{R}^n}, V, 0) \notin \downarrow 0)\}\,.$$

This envelope is uniformly Σ*-complete.*

Remark 5.6. Let $f\colon [0,1]^n \to [0,1]^n$. Let $C \subseteq \mathrm{Fix}(f)$ be a connected component of the fixed point set of f. Call C *robust* if for every open neighbourhood U of C there exists $\varepsilon > 0$ such that every $\widetilde{f}$ with $|f - \widetilde{f}| < \varepsilon$ has a fixed point in U. Then the set of robust components of the fixed point set of f is a generating family (see Definition 2.51) for the set $F(f)$. For a proof idea see [29].

The proof of Theorem 5.4 will be split into several lemmas. We mainly have to show that $\widetilde{F}$ is uniformly Σ-complete. By Lemma 4.47 it suffices to show that $\widetilde{F}$ witnesses all basic robust properties of $\widetilde{\mathrm{Fix}}_<$.

Lemma 5.7. *The map $\widetilde{F}$ is a computable envelope of $\widetilde{\mathrm{Fix}}_<$.*

Proof. That $\widetilde{F}$ is computable follows almost immediately from the computability of the degree. Given an open set $U \in \mathscr{O}(\mathbb{R}^n)$ we can computably enumerate the list of all finite unions of balls with rational centre and radius which are compactly contained in U. As these finite unions of balls form a dense sequence in U we have $U \in \widetilde{F}(h)$ if and only if we can find a finite union of balls $B_1 \cup \cdots \cup B_m$ in this sequence with $\deg(h - \mathrm{id}_{\mathbb{R}^n}, B_1 \cup \cdots \cup B_m, 0) \notin \downarrow 0$.

As $\deg(h - \mathrm{id}_{\mathbb{R}^n}, B_1 \cup \cdots \cup B_m, 0) \notin \downarrow 0$ implies that there is a solution to the equation $h(x) = y$ in U it follows that $\widetilde{F}$ is an envelope. □

The following two lemmas are the core of the proof. They will allow us to characterise the robust properties of $\widetilde{\mathrm{Fix}}_<$. As mentioned earlier, the first of these lemmas is based on the Hopf theorem (Theorem 5.1).

Lemma 5.8. *Let $f \colon \mathbb{R}^n \to [0,1]^n$ be a continuous function. Let $U \in \mathscr{O}(\mathbb{R}^n)$ be a bounded connected open set with $\partial U \cap \mathrm{Fix}(f) = \emptyset$ and $\deg(f - \mathrm{id}_{\mathbb{R}^n}, U, 0) = 0$. Let $|f - \mathrm{id}_{\mathbb{R}^n}| < \varepsilon$ on U. Then there exists a 2ε-perturbation of f which agrees with f on the complement of U and which has no fixed points in U.*

Proof. Let $g(x) = f(x) - x$. We show that there exists a 2ε-perturbation $\widetilde{g}$ of g which agrees with g on the complement of U and has no zeroes in U. Then $\widetilde{g} + x$ is the desired perturbation of f.

Let $\delta > 0$ be a lower bound to $|g(x)|$ on ∂U. Choose a small perturbation g_0 of g with

$$|g_0 - g| < \min\{\delta/4, \varepsilon/2\}$$

on U such that $\deg(g_0, U, 0) = \deg(g, U, 0) = 0$ and such that 0 is a regular value of g_0. Choose $\nu > 0$ so small that $d(x, \partial U) \leq \nu$ implies $|g(x)| > \delta/2$.

Let

$$\alpha \colon \mathbb{R}^n \to \mathbb{R},\ \alpha(x) = \min\{1, \max\{0, 1 + d_{\text{two-sided}}(x, U)/\nu\}\}.$$

Let

$$h_0(x) = \alpha(x) g(x) + (1 - \alpha(x)) g_0(x).$$

Then h_0 is equal to g outside of U and equal to g_0 on the open set

$$\{x \in U \mid d(x, \partial U) > \nu\}.$$

If $x \in U$ with $d(x, \partial U) \leq \nu$ we have:

$$\begin{aligned}
|h_0(x)| &= |\alpha(x)g(x) + (1 - \alpha(x))g_0(x)| \\
&\geq |g_0(x)| - \alpha(x)|g(x) - g_0(x)| \\
&\geq |g(x)| - |g(x) - g_0(x)| - \alpha(x)|g(x) - g_0(x)| \\
&> \delta/2 - \delta/4 - \delta/4 \\
&= 0
\end{aligned}$$

so that h_0 has the same zeroes in U as g_0. As the zero set of g_0 in U is finite we can find a neighbourhood $V \subseteq U$ of the zero set which is homeomorphic to the unit disk D^n. Fix a homeomorphism $\psi \colon D^n \to V$. We have

$$\deg(h_0, V, 0) = \deg(g_0, V, 0) = 0.$$

It follows from the Hopf theorem that there exists a homotopy

$$H_0 \colon [0, 1] \times S^{n-1} \to S^{n-1}$$

with

$$H_0(0, x) = h_0(\psi(x))/|h_0(\psi(x))|$$

and

$$H_0(1, x) = c$$

for some constant $c \in S^{n-1}$. Then the function $|h_0(\psi(x))|H(t, x)$ is a homotopy between $h_0 \circ \psi$ and $|h_0(\psi(x))| \cdot c$ on S^{n-1}. Let

$$\nu = \inf \{|h_0(x)| \mid x \in \partial V\}$$

and

$$\xi = \sup \{|h_0(x)| \mid x \in \partial V\}.$$

Let

$$H_1(t, x) = \max\{\nu, |h_0(\psi(x))| + t(\nu - \xi)\}c.$$

Then H_1 is a homotopy between $|h_0(\psi(x))|c$ and the constant function νc. It follows that there exists a homotopy H between $h_0 \circ \psi$ and νc on S^{n-1}. Let

$$k \colon D^n \to \mathbb{R}^n, \; k(x) = \begin{cases} H(2 - 2|x|, x/|x|) & \text{if } |x| \geq 1/2, \\ \nu c & \text{if } |x| \leq 1/2. \end{cases}$$

Let $h = k \circ \psi^{-1}$. Then $h(x) = h_0 \circ \psi(x)$ on ∂V and h has no zeroes in V. Extend h to $\mathbb{R}^n$ by letting $h(x) = h_0(x)$ outside of V. □

The next lemma is somewhat more elementary but relies on similar ideas:

Lemma 5.9. *Let $f\colon \mathbb{R}^n \to [0,1]^n$ be a continuous function. Let $U \in \mathscr{O}(\mathbb{R}^n)$ be a bounded connected open set with $\partial U \cap \mathrm{Fix}(f) = \emptyset$. Let $x_0 \in U$. Assume that $|f(x) - x| < \varepsilon$ for all $x \in U$. Then there exists a 2ε-perturbation $\widetilde{f}$ of f which agrees with f outside of U such that x_0 is the unique fixed point of $\widetilde{f}$ in U.*

Proof. We use similar arguments as in the first half of the proof of Lemma 5.8.

Let $g(x) = f(x) - x$. It suffices to construct a 2ε-perturbation $\widetilde{g}$ of g which agrees with g on the boundary of U and whose unique zero is x_0.

Let $\delta > 0$ be a lower bound to $|g(x)|$ on ∂U. Choose a small perturbation g_0 of g with

$$|g_0 - g| < \min\{\delta/4, \varepsilon/2\}$$

on U such that 0 is a regular value of g_0. Choose $\nu > 0$ so small that $d(x, \partial U) \leq \nu$ implies $|g(x)| > \delta/2$.

Let

$$\alpha\colon \mathbb{R}^n \to \mathbb{R},\ \alpha(x) = \min\{1, \max\{0, 1 + d_{\text{two-sided}}(x, U)/\nu\}\}.$$

Let

$$h_0(x) = \alpha(x)g(x) + (1 - \alpha(x))g_0(x).$$

Then h_0 is equal to g outside of U and equal to g_0 on the open set

$$\{x \in U \mid d(x, \partial U) > \nu\}.$$

As established in the proof of Lemma 5.8 the function h_0 has the same zeroes in U as g_0. As the zero set of g_0 in U is finite we can find a neighbourhood $V \subseteq U$ of the zero set which is homeomorphic to the unit disk D^n.

Fix a homeomorphism $\psi\colon D^n \to V$ which sends 0 to x_0. Let

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ |x| \cdot g_0 \circ \psi\left(\frac{x}{|x|}\right) & \text{otherwise.} \end{cases}$$

Then $\widetilde{g} = h \circ \psi^{-1}$ is the desired perturbation. □

Hence we arrive at a characterisation of the basic robust properties of $\widetilde{\mathrm{Fix}}_<$:

Lemma 5.10. *Let $U \in \mathscr{O}(\mathbb{R}^n)$ be a basic robust property of $\widetilde{\mathrm{Fix}}_<(h)$. Then $U \in \widetilde{F}(h)$, i.e., U contains an open set V with $\deg(h - \mathrm{id}_{\mathbb{R}^n}, 0, V) \notin \downarrow 0$.*

Proof. As U assumed to be robust there exists $\varepsilon > 0$ such that every map $\widetilde{h}\colon \mathbb{R}^n \to [0,1]^n$ which is ε-close to h has a fixed point in V. Consider the open set

$$W = \{x \in \mathbb{R}^n \mid |h(x) - x| < \varepsilon/4\}.$$

By assumption W is non-empty. It decomposes into finitely many connected components $W_1, \ldots, W_n$ none of which have a fixed point on their boundary. Hence the degrees $\deg(h - \mathrm{id}_{\mathbb{R}^n}, 0, W_i)$ are all well-defined, *i.e.*, different from $\perp$. As U contains a fixed point of h at least one of these components needs to intersect U. If W_i is a component which is not completely contained in U then by Lemma 5.9 there exists a $\varepsilon/2$-perturbation of h which has no fixed point in $W_i \cap U$ and agrees with h outside of W_i. As, by robustness of U, every ε-perturbation of h needs to have a fixed point in U it follows that there exists at least one component which is completely contained in U. If $W_i \subseteq U$ is a component which is contained in U with $\deg(h - \mathrm{id}, 0, W_i) = 0$ then by Lemma 5.8 we can find a $\varepsilon/2$-perturbation of h without fixed points in $W_i \cap U$ which agrees with h outside of W_i. Again, since U is robust, there has to exist a component $W_i \subseteq U$ with $\deg(h - \mathrm{id}, 0, W_i) \neq 0$. □

It follows that $\widetilde{F}_<$ is uniformly Σ-complete and hence universal. Thus Theorem 5.4 is proved.

5.2 Fixed points of nonexpansive mappings

As a second problem we consider the problem of locating the fixed point set of a nonexpansive map on the unit ball in Hilbert space. A map $f\colon X \to Y$ between metric spaces X and Y is called *nonexpansive* if it is Lipschitz-continuous with Lipschitz constant 1, *i.e.*, if

$$\forall x_0, x_1 \in X. \, (d(f(x_0), f(x_1)) \leq d(x_0, x_1)).$$

It was shown independently by Browder, Göhde, and Kirk in 1965 that a nonexpansive self-map of a nonempty, closed, bounded, convex subset of a uniformly convex Banach space has a fixed point. For the sake of simplicity we only consider the special case of the unit ball in separable real Hilbert space ℓ^2 with the usual norm

$$|x| = \left(\sum_{n \in \mathbb{N}} x_n^2\right)^{1/2}.$$

Theorem 5.11 (Browder [27], Göhde [46], Kirk [61], 1965).
Let $f\colon B_{\ell^2} \to B_{\ell^2}$ be a nonexpansive map, i.e.,

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in B_{\ell^2}$. Then f has a fixed point.

The goal of this section is to calculate the universal envelope of the function which assigns to a given nonexpansive map its fixed point set.

Before we state this more formally, let us recall some of the notation from Example 4.20. The separable real Hilbert space ℓ^2 can be made into a computable metric space in the usual way. Its continuous dual $(\ell^2)'$ can be made into a computable T_0 space by identifying it with a subspace of the exponential $\mathbb{R}^{(\ell^2)'}$. Note that the topology on the represented space $(\ell^2)'$ is the sequentialisation of the weak* topology, which does not coincide with the weak* topology itself. As ℓ^2 is self-dual we can interpret $(\ell^2)'$ as ℓ^2 with the (sequentialisation of the) weak topology. With this interpretation in mind we obtain a computable map

$$\mathrm{id}_{\ell^2}^{\mathsf{s}\to\mathsf{w}} \colon \ell^2 \to (\ell^2)',\ x \mapsto x$$

with a discontinuous inverse

$$\mathrm{id}_{\ell^2}^{\mathsf{w}\to\mathsf{s}} \colon (\ell^2)' \to \ell^2,\ x \mapsto x.$$

Finally, let $B_{\ell^2} \subseteq \ell^2$ denote the unit ball in ℓ^2 and let $B_{(\ell^2)'} \subseteq (\ell^2)'$ denote the unit ball in $(\ell^2)'$. Let $\mathscr{N}(B_{\ell^2})$ denote the space of all nonexpansive self-maps of B_{ℓ^2}, made into a computable T_0 space by identifying it with a subspace of $B_{\ell^2}^{B_{\ell^2}}$.

Theorem 5.12. *Consider the function*

$$\mathrm{Fix} \colon \mathscr{N}(B_{\ell^2}) \to \mathscr{F}(B_{\ell^2}),\ f \mapsto \{x \in B_{\ell^2} \mid f(x) = x\}.$$

The universal envelope of Fix *is given by the computable map*

$$F \colon \mathscr{N}(B_{\ell^2}) \to \mathscr{K}(B_{(\ell^2)'}),\ F(h) = \mathrm{Fix}(h)$$

with inclusion map

$$\xi_L \colon \mathscr{F}(B_{\ell^2}) \to \mathscr{K}(B_{(\ell^2)'}),\ \xi_L(A) = \left\{x \in (\ell^2)' \mid x \in A\right\}.$$

Theorem 5.12 relies on the following result which was used in [78] as the key step in the characterisation of the Weihrauch complexity of the Browder-Göhde-Kirk theorem:

Theorem 5.13 ([78, Theorem 5.1])**.** *The function*

$$F \colon \mathscr{N}(B_{\ell^2}) \to \mathscr{K}(B_{(\ell^2)'})$$

is computable, as is its multivalued right inverse

$$F^{-1} \colon \subseteq \mathscr{K}(B_{(\ell^2)'}) \rightrightarrows \mathscr{N}(B_{\ell^2}).$$

We will use Theorem 5.13 to show that the restriction of Fix to those functions which have a unique fixed point is a retract of the identity $B_{(\ell^2)'} \to B_{\ell^2}$. This is the main step in the proof of theorem 5.12. We first need two auxiliary lemmas:

Lemma 5.14. *A sequence $(f_n)_n$ of nonexpansive maps converges in $\mathcal{N}(B_{\ell^2})$ to a map f if and only if $(f_n)_n$ converges to f pointwise.*

Proof. Since evaluation is continuous, convergence in $\mathcal{N}(B_{\ell^2})$ implies pointwise convergence.

For the opposite direction, let $(x_n)_n$ be a computable dense sequence in B_{ℓ^2}. Consider the map

$$i\colon \mathcal{N}(B_{\ell^2}) \to B_{\ell^2}^{\mathbb{N}},\ f \mapsto (f(x_n))_n.$$

We claim that this is an isomorphism onto its image. In order to compute the inverse function we need to compute the map

$$i(\mathcal{N}(B_{\ell^2})) \times B_{\ell^2} \to B_{\ell^2},\ (i(f), x) \mapsto f(x).$$

This is achieved by the following algorithm: given $\varepsilon > 0$, search for a number $n \in \mathbb{N}$ with $|x_n - x| < \varepsilon/2$. Output an approximation of $i(f)(n)$ with error $\varepsilon/2$. Since f is nonexpansive, we have

$$|f(x_n) - f(x)| \leq |x_n - x| < \varepsilon/2$$

and the correctness of the algorithm follows. Now, if $f_n \to f$ pointwise then $i(f_n) \to i(f)$. Since i is an isomorphism it follows that $f_n \to f$ in $\mathcal{N}(B_{\ell^2})$. □

Lemma 5.15 ([78, Lemma 5.4])**.** *Let $f\colon B_{\ell^2} \to \ell^2$ be a nonexpansive map. Assume that* $\mathrm{Fix}(f) \cap B_{\ell^2} \neq \emptyset$*. Let*

$$P_{B_{\ell^2}}\colon \ell^2 \to B_{\ell^2}$$

denote the metric projection onto the unit ball. Then the map

$$P_{B_{\ell^2}} \circ f\colon B_{\ell^2} \to B_{\ell^2}$$

is nonexpansive as well with

$$\mathrm{Fix}(P_{B_{\ell^2}} \circ f) = \mathrm{Fix}(f).$$

We can now prove the announced retraction result. In the following, let $\mathcal{U} \subseteq \mathcal{N}(B_{\ell^2})$ denote the subspace of $\mathcal{N}(B_{\ell^2})$ which consists of all nonexpansive maps with a unique fixed point. The main step is the following lemma:

Lemma 5.16. *Let $f, g \in \mathcal{U}$ be nonexpansive maps with the same unique fixed point $p \in B_{\ell^2}$. Then there exists a half-symmetry*

$$\varphi\colon \mathcal{U} \to \mathcal{V}(\mathcal{U})$$

of $\mathrm{Fix}|_{\mathcal{U}}$ *with $g \in \varphi(f)$.*

Proof. We will construct a map

$$\widetilde{\varphi}\colon B_{(\ell^2)'} \to \mathcal{V}(\mathcal{U})$$

with $\operatorname{Fix}(\widetilde{\varphi}(q)) = \{q\}$ for all $q \in B_{(\ell^2)'}$ and $g \in \widetilde{\varphi}(f)$. As $F\colon \mathcal{U} \to B_{(\ell^2)'}$ is computable by Theorem 5.13 we can then put $\varphi = \widetilde{\varphi} \circ F$ to obtain a half-symmetry with the desired properties. To construct the map $\widetilde{\varphi}$ we use that $B_{(\ell^2)'}$ is computably countably based. It therefore suffices to construct an operation

$$\psi\colon B_{(\ell^2)'} \times \mathbb{N} \rightrightarrows \mathcal{U}$$

which is extensional in its second argument and satisfies $\operatorname{Fix}(\psi(q,n)) = \{q\}$ and $g \in \operatorname{cl}\{\psi(p,n) \mid n \in \mathbb{N}\}$.

For the construction of ψ let us introduce some notation. Let $P_n\colon \ell^2 \to \ell^2$ denote the projection onto the first n coordinates, *i.e.*,

$$P_n\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{n} x_i e_i.$$

Let $S_n\colon \ell^2 \to \ell^2$ denote the right-shift operator

$$S_n\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} x_i e_{i+n}.$$

Let $L_n\colon \ell^2 \to \ell^2$ denote the left-shift operator

$$L_n\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} x_{i+n} e_i.$$

Now, let $q \in B_{(\ell^2)'}$ and $n \in \mathbb{N}$. Let

$$\widetilde{H}_n(x) = P_n \circ g \circ P_n(x) - P_n \circ g \circ P_n(q) + P_n(q).$$

Then $\widetilde{H}_n$ is nonexpansive and $P_n(q)$ is a fixed point of $\widetilde{H}_n$. Let

$$H_n(x) = (1 - 2^{-n})\widetilde{H}_n(x) + 2^{-n}P_n(q).$$

Then $P_n(q)$ is a fixed point of H_n. But H_n is Lipschitz-continuous with Lipschitz constant $(1 - 2^{-n}) < 1$. Hence $P_n(q)$ is the unique fixed point of H_n. Note that H_n is uniformly computable in n and q. Let $A \in \mathcal{N}(B_{\ell^2})$ be some nonexpansive map with unique fixed point $L_n(q)$. Some A with this property is computable from n and q by Theorem 5.13. Let

$$\widetilde{A}_n(x) = (1 - 2^{-n})x + 2^{-n}A(x).$$

Let

$$R_n(x) = S_n \circ \widetilde{A}_n \circ L_n(x).$$

Then R_n is $(1 - 2^{-n})$-Lipschitz and thus has a unique fixed point. This unique fixed point is given by $S_n \circ L_n(q)$. As H_n and R_n are nonexpansive and take values in orthogonal subspaces the map $H_n + R_n$ is nonexpansive as well. As the value of $H_n(x)$ only depends on the first n coordinates of x and the value of $R_n(x)$ only depends on the remaining coordinates, it follows that q is a fixed point of $H_n + R_n$. Conversely, if $H_n(x) + R_n(x) = x$ then it follows that $H_n(P_n(x)) = P_n(x)$, so that $P_n(x) = P_n(q)$ and $R_n(S_n \circ L_n(x)) = S_n \circ L_n(x)$, so that $S_n \circ L_n(x) = S_n \circ L_n(q)$. Hence q is the unique fixed point of $H_n + R_n$. By Lemma 5.15 the map $h_n = P_{B_{\ell^2}}(H_n + R_n)$ is nonexpansive with unique fixed point q. Let the output of ψ on input n and q be h_n. Note that this output depends on the choice of name of q.

It remains to show that if $q = p$ is the unique fixed point of g, then $\psi(q, n)$ converges to g as $n \to \infty$, independent of the choice of name of q. By Lemma 5.14 it suffices to show that $\psi(n, q)$ converges pointwise to g. But this follows easily with standard estimates. □

Lemma 5.17. *Let $\mathscr{U} \subseteq \mathscr{N}(B_{\ell^2})$ denote the set of all nonexpansive maps which have a unique fixed point. Then the restriction $\mathrm{Fix}|_{\mathscr{U}}$ is a retract of the identity $\mathrm{id}_{B_{\ell^2}}^{w\to s} \colon B_{(\ell^2)'} \to B_{\ell^2}$.*

Proof. By Theorem 5.13 the function

$$F^{-1} \colon B_{(\ell^2)'} \rightrightarrows \mathscr{U},\ F^{-1}(x) = \{h \in \mathscr{U} \mid \mathrm{Fix}(h) = \{x\}\}$$

is computable (using that the space of compact singletons of the computable T_0 space $B_{(\ell^2)'}$ is computably isomorphic to the space $B_{(\ell^2)'}$ itself). By Lemma 5.16 all values of F^{-1} are equivalent as instances of Fix. It follows that a retraction is given by the following diagram:

$$\begin{array}{ccc} B_{(\ell^2)'} & \xrightarrow{\mathrm{id}_{B_{\ell^2}}^{w\to s}} & B_{\ell^2} \\ F \uparrow \ \rightsquigarrow F^{-1} & & \Vert \\ \mathscr{U} & \xrightarrow{\mathrm{Fix}} & B_{\ell^2} \end{array}$$

□

We are now ready to prove Theorem 5.12.

Proof of Theorem 5.12. By Lemma 5.17 and Theorem 4.41 the restriction $F|_{\mathscr{U}}$ to the set of all nonexpansive maps with a unique fixed point is the universal envelope of the restriction $\mathrm{Fix}|_{\mathscr{U}}$. As $\mathscr{U}$ is a dense subset of $\mathscr{N}(B_{\ell^2})$ it remains to show that F satisfies the assumptions of Lemma 4.43.

A basis of F is given by $B_{(\ell^2)'}$ with generating Σ-section

$$s \colon \mathscr{O}(B_{(\ell^2)'}) \to \mathscr{O}(\mathscr{K}(B_{(\ell^2)'})),\ s(U) = \left\{K \in \mathscr{K}(B_{(\ell^2)'}) \mid K \subseteq U\right\}.$$

Let $h\colon B_{\ell^2} \to B_{\ell^2}$ be nonexpansive, let $U \in \mathcal{O}(B_{(\ell^2)'})$ with $F(\widetilde{h}) \subseteq U$ for each $\widetilde{h} \in \mathscr{U} \cap W$ where W is some small neighbourhood of h. Choose $\varepsilon > 0$ such that W contains an open ε-ball around h. For each $x \in F(h)$ consider the function

$$\widetilde{h}_x(z) = \varepsilon x + (1 - \varepsilon)h(z).$$

Then $\widetilde{h}_x \in W \cap \mathscr{U}$ with $F(\widetilde{h}_x) = \{x\}$. It follows that $x \in U$. As x was chosen arbitrarily we conclude that $F(h) \subseteq U$ and everything is shown. $\square$

Chapter 6

Open problems and future work

Let us conclude with a list of open problems and directions of future work:

1. The main direction of future work is to apply the theory developed here to further problems. Interesting and relevant discontinuous problems can be found in almost any application domain of continuous computation, such as computational geometry, dynamical systems, hybrid systems, partial differential equations, optimisation, or linear functional analysis.

2. Proposition 3.32 shows that the best continuous approximation of a continuous function with values in a computably continuous lattice L coincides with the function in all points of continuity. The question arises whether the restriction to continuous lattices is necessary or whether this result holds true for all computable complete lattices. A closely related question is whether the continuity of M is required in Theorem 4.34. This theorem would certainly be much more satisfactory without this restriction, for in this case every probe (α, β) where $\beta\colon \widetilde{X} \times Y \to Z$ takes values in an arbitrary computable T_0 space Z factors through the universal envelope up to naturally embedding the co-domain of β into $\mathscr{O}^2(Z)$.

3. The subject of this thesis is the study of best continuous approximations of arbitrary set-theoretic functions. From a purely mathematical point of view it is certainly natural to consider generalisations of this idea, where continuity is replaced with a weaker property, such as measurability or topological reducibility to a given Weihrauch degree.

   While continuous approximations play a special role from a computational point of view due to the strong link between computability and continuity, there is a similar link between, say, Borel measurability and limit computability [11]. Various other models of "hypercomputation", such as computability with finitely many mind-changes [109], probabilistic computability [18], or nondeterministic computability [108, 12] can be captured by appropriate Weihrauch degrees.

Such models of hypercomputation can be viewed as ordinary algorithms which satisfy weaker contracts. For instance, a finitely revising algorithm is allowed to output a certain amount of wrong information, so long as it eventually identifies all wrong information as incorrect and from some point onwards produces only correct information.

In certain situations it may be appropriate, or rather inevitable, to settle for a lower degree of reliability in order to have a greater amount of information on the solution of a problem available.

Independent of any immediate practical considerations, the study of best approximations below certain Weihrauch degrees could prove to be valuable for the study of the computational power of mathematical theorems in the spirit of Weihrauch reducibility and reverse mathematics, since examples such as Example 4.42 seem to suggest that envelopes provide a more fine-grained picture of the finitary computational content of a given problem.

4. One of the aims of computable analysis is to provide a mathematically rigorous language for the specification of algorithms for the processing of continuous data. As such it endeavours to serve as a foundational framework for the theory of numerical computation. Our aim is to extend the scope of this framework by allowing the treatment of discontinuous functions.

   From this point of view the present work constitutes only the very first step in this direction: So far we have developed a theory for reformulating algorithmically unsolvable problems into ones that are algorithmically solvable in principle, but at no point have we attempted to specify any actual algorithms for the solution of the modified problems.

   Thus, the study and implementation of concrete algorithms for the computation of envelopes appears to be another important direction of future research.

Bibliography

[1] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhäuser, 1990.

[2] J. Avigad and V. Brattka. *Computability and analysis: the legacy of Alan Turing*, pages 1 – 47. Lecture Notes in Logic. Cambridge University Press, 2014.

[3] G. Baigger. Die Nichtkonstruktivität des Brouwerschen Fixpunktsatzes. *Archiv für Mathematische Logik und Grundlagenforschung*, 25:183–188, 1985.

[4] S. Banach and S. Mazur. Sur les fonctions calculables. *Ann. Soc. Pol. de Math.*, 16, 1937.

[5] L. Benvenuti, D. Bresolin, A. Casagrande, P. Collins, A. Ferrari, E. Mazzi, A. Sangiovanni-Vincentelli, and T. Villa. Reachability computation for hybrid systems with Ariadne. *IFAC Proceedings Volumes*, 41(2):8960 – 8965, 2008. 17th IFAC World Congress.

[6] E. Bishop. *Foundations of Constructive Analysis*. New York: Academic Press, 1967.

[7] E. Bishop and D. Bridges. *Constructive Analysis*. Springer-Verlag, 1985.

[8] L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation*. Springer, 1998.

[9] L. Blum, M. Shub, and S. Smale. On a theory of computation and complexity over the real numbers: NP- completeness, recursive functions and universal machines. *Bull. Amer. Math. Soc.*, 21(1):1 – 46, 1989.

[10] V. Brattka. Computing Uniform Bounds. *Electronic Notes in Theoretical Computer Science*, 66(1):13 – 24, 2002. CCA 2002, Computability and Complexity in Analysis.

[11] V. Brattka. Effective Borel measurability and reducibility of functions. *Mathematical Logic Quarterly*, 51(1):19–44, 2005.

[12] V. Brattka, M. de Brecht, and A. Pauly. Closed choice and a uniform low basis theorem. *Annals of Pure and Applied Logic*, 163:986–1008, 2012.

[13] V. Brattka and R. Dillhage. Computability of finite-dimensional linear subspaces and best approximation. *Annals of Pure and Applied Logic*, 162:182–193, 2010.

[14] V. Brattka and G. Gherardi. Borel complexity of topological operations on computable metric spaces. In S. B. Cooper, B. Löwe, and A. Sorbi, editors, *Computation and Logic in the Real World*, volume 4497 of *Lecture Notes in Computer Science*, pages 83–97, Berlin, 2007. Springer. Third Conference on Computability in Europe, CiE 2007, Siena, Italy, June 18-23, 2007.

[15] V. Brattka and G. Gherardi. Borel complexity of topological operations on computable metric spaces. *Journal of Logic and Computation*, 19(1):45–76, 2009.

[16] V. Brattka and G. Gherardi. Weihrauch degrees, omniscience principles and weak computability. *J. Symbolic Logic*, 76(1):143–176, 2011.

[17] V. Brattka, G. Gherardi, and R. Hölzl. Las Vegas computability and algorithmic randomness. In E. W. Mayr and N. Ollinger, editors, *32nd International Symposium on Theoretical Aspects of Computer Science (STACS 2015)*, volume 30 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 130–142, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

[18] V. Brattka, G. Gherardi, and R. Hölzl. Probabilistic computability and choice. *Information and Computation*, 242:249–286, 2015.

[19] V. Brattka and P. Hertling. Continuity and computability of relations. Technical report, FernUniversität Hagen, 1994.

[20] V. Brattka, S. Le Roux, J. S. Miller, and A. Pauly. The Brouwer Fixed Point Theorem Revisited. In A. Beckmann, L. Bienvenu, and N. Jonoska, editors, *Pursuit of the Universal: 12th Conference on Computability in Europe*, volume 9709 of *Lecture Notes in Computer Science*, pages 58–67. Springer, Cham, 2016.

[21] V. Brattka, S. Le Roux, and A. Pauly. On the Computational Content of the Brouwer Fixed Point Theorem. In S. B. Cooper, A. Dawar, and B. Löwe, editors, *How the World Computes: Turing Centenary Conference and 8th Conference on Computability in Europe*, volume 7318 of *Lecture Notes in Computer Science*, pages 56–67. Springer, Berlin, Heidelberg, 2012.

[22] V. Brattka, S. Le Roux, and A. Pauly. Connected Choice and the Brouwer Fixed Point Theorem. Preprint. Available at https://arxiv.org/abs/1206.4809v2, 2016.

[23] V. Brattka and A. Pauly. On the algebraic structure of Weihrauch degrees. Preprint. Available at https://arxiv.org/abs/1604.08348v7, 2016.

[24] V. Brattka and G. Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305(1 – 3):43 – 76, 2003.

[25] V. Brattka and M. Schröder. Computing with Sequences, Weak Topologies and the Axiom of Choice. In L. Ong, editor, *Computer Science Logic*, volume 3634 of *Lecture Notes in Computer Science*, pages 462–476. Springer Berlin Heidelberg, 2005.

[26] L. E. J. Brouwer. An intuitionist correction of the fixed point theorem on the sphere. *Proceedings of the Royal Societey. London. Series A.*, 213:1–2, 1952.

[27] F. E. Browder. Nonexpansive nonlinear operators in a Banach space. *Proc. Nat. Acad. Sci. USA*, 54:1041–1044, 1965.

[28] P. Collins. Continuity and computability of reachable sets. *Theoretical Computer Science*, 341(1–3):162–195, 2005.

[29] P. Collins. Computability and Representations of the Zero Set. *Electronic Notes in Theoretical Computer Science*, 221:37–43, 2008.

[30] C. Costantini and S. Watson. On the dissonance of some metrizable spaces. *Topology and its Applications*, 84(1):259 – 268, 1998.

[31] M. de Brecht. Quasi-polish spaces. *Annals of Pure and Applied Logic*, 164(3):356 – 381, 2013.

[32] M. de Brecht. Levels of discontinuity, limit-computability, and jump operators. In V. Brattka, H. Diener, and D. Spreen, editors, *Logic, Computation, Hierarchies*, Ontos Mathematical Logic, pages 93–122. Walter de Gruyter, Boston, 2014.

[33] M. de Brecht and T. Kawai. On the commutativity of the powerspace monads. Preprint. Available at https://arxiv.org/abs/1709.06226v1, September 2017.

[34] M. de Brecht, M. Schröder, and V. Selivanov. Base-complexity classification of qcb_0-spaces. *Computability*, 5(1):75 – 102, 2016.

[35] S. Dolecki, G. H. Greco, and A. Lechicki. When do the Upper Kuratowski Topology (Homeomorphically, Scott Topology) and the Co-compact Topology Coincide? *Transactions of the American Mathematical Society*, 347(8):2869–2884, 1995.

[36] A. Edalat and M. H. Escardó. Integration in Real PCF. *Information and Computation*, 160(1 – 2):128 – 166, 1996.

[37] M. Escardó. Intersections of compactly many open sets are open. available at `https://www.cs.bham.ac.uk/~mhe/papers/compactness-submitted.pdf` - retrieved 13th February 2019, 2009.

[38] M. Escardó, J. Lawson, and A. Simpson. Comparing cartesian closed categories of (core) compactly generated spaces. *Topology and its Applications*, 143(1):105 – 145, 2004.

[39] M. H. Escardó. *PCF extended with real numbers: a domain-theoretic approach to higher-order exact real number computation.* PhD thesis, Imperial College London, 1996.

[40] M. H. Escardó. Properly injective spaces and function spaces. *Topology and its Applications*, 89(1):75 – 120, 1998.

[41] M. H. Escardó. Function-space compactifications of function spaces. *Topology and its Applications*, 120(3):441 – 463, 2002.

[42] M. H. Escardó. Synthetic topology of data types and classical spaces. *Electronic Notes in Theoretical Computer Science*, 87:150pp., 2004.

[43] S. Franklin. Spaces in which sequences suffice. *Fundamenta Mathematicae*, 57(1):107–115, 1965.

[44] S. Franklin. Spaces in which sequences suffice II. *Fundamenta Mathematicae*, 61(1):51–56, 1967.

[45] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. Scott. *Continuous lattices and domains*. Cambridge University Press, 2003.

[46] D. Göhde. Zum Prinzip der kontraktiven Abbildung. *Math. Nachr.*, 30:251–258, 1965.

[47] G. Gruenhage and T. Streicher. Quotients of countably based spaces are not closed under sobrification. *Mathematical. Structures in Comp. Sci.*, 16(2):223–229, 2006.

[48] A. Grzegorczyk. Computable functionals. *Fundamenta Mathematicae*, 42:168–202, 1955.

[49] A. Grzegorczyk. On the definition of computable functionals. *Fundamenta Mathematicae*, 42:232–239, 1955.

[50] A. Grzegorczyk. On the definitions of computable real continuous functions. *Fundamenta Mathematicae*, 44(1):61–71, 1957.

[51] A. C. Hansen. On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators. *Journal of the AMS*, 24:81–124, 2011.

[52] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[53] P. Hertling. A Banach–Mazur computable but not Markov computable function on the computable real numbers. *Annals of Pure and Applied Logic*, 132(2):227 – 246, 2005.

[54] K. H. Hofmann and J. D. Lawson. On the order-theoretical foundation of a theory of quasicompactly generated spaces without separation axiom. *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics*, 36(2):194–212, 1984.

[55] H. Hopf. Abbildungsklassen n-dimensionaler Mannigfaltigkeiten. *Math. Ann.*, 96:209 – 224, 1927.

[56] J. R. Isbell. Function spaces and adjoints. *Mathematica Scandinavica*, 36:317 – 339, Dec. 1975.

[57] J. R. Isbell. Meet-continuous lattices. *Symposia Mathematica*, 16:41 – 54, 1975.

[58] T. Kaczynski, K. Mischaikow, and M. Mrozek. *Computational Homology*. Springer-Verlag New York, 2004.

[59] K. Keimel. Topological cones: functional analysis in a T_0-setting. *Semigroup Forum*, 77:109 – 142, 2008.

[60] K. Keimel and G. Gierz. Halbstetige Funktionen und stetige Verbände. In R.-E. Hoffman, editor, *Continuous Lattices and Related Topics, Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Worksho V), Universität Bremen, Germany*, volume 27 of *Mathematik-Arbeitspapiere*, 1982.

[61] W. A. Kirk. A fixed point theorem for mappings which do not increase distances. *Amer. Math. Monthly*, 72:1004–1006, 1965.

[62] S. C. Kleene and R. E. Vesley. *The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions*. North-Holland Amsterdam, 1965.

[63] M. Konečný. A Haskell library for Approximating Exact Real Numbers (AERN). `https://github.com/michalkonecny/aern2`. retrieved 18th September 2018, 17:30.

[64] V. Kreinovich. *Categories of space-time models*. PhD thesis, Soviet Academy of Sciences, Novosibirsk, 1979. (Russian).

[65] C. Kreitz and K. Weihrauch. Theory of representations. *Theoretical Computer Science*, 38:35 – 53, 1985.

[66] D. Lacombe. Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles iii. *Comptes Rendus Académie des Sciences Paris*, 241:151–153, 1955.

[67] B. Lambov. The basic feasible functionals in computable analysis. *J. Complexity*, 22(6):909–917, 2006.

[68] S. Le Roux and M. Ziegler. Singular coverings and non-uniform notions of closed set computability. *Mathematical Logic Quarterly*, 54(5):545–560, 2008.

[69] S. Mac Lane. *Categories for the Working Mathematician*. Springer New York, second edition, 1998.

[70] A. Markov. On the continuity of constructive functions. *Uspekhi Mat. Nauk*, 9:226 – 230, 1954.

[71] S. Mazur. *Computable Analysis*, volume 33. Razprawy Matematyczne, Warsaw, 1963.

[72] J. Miller. *Π^0_1-classes in computable analysis and topology*. PhD thesis, Cornell University, 2002.

[73] J. Milnor. *Topology from the Differentiable viewpoint*. University Press of Virginia, 1965.

[74] R. E. Moore, R. B. Kearfott, and M. J. Cloud. *Introduction to Interval Analysis*. Society for Industrial and Applied Mathematics, 2009.

[75] N. T. Müller. iRRAM - Exact Arithmetic in C++. `http://irram.uni-trier.de/`, `https://github.com/norbert-mueller/iRRAM`. retrieved 18th September 2018, 17:30.

[76] N. T. Müller. The iRRAM: Exact Arithmetic in C++. In J. Blanck, V. Brattka, and P. Hertling, editors, *Computability and Complexity in Analysis: 4th International Workshop, CCA 2000 Swansea, UK, September 17–19, 2000 Selected Papers*, pages 222–252. Springer Berlin Heidelberg, 2001.

[77] J. Nash. Non-Cooperative Games. *Annals of Mathematics*, 52(2):286–295, 1951.

[78] E. Neumann. Computational Problems in Metric Fixed Point Theory and their Weihrauch Degrees. *Logical Methods in Computer Science*, 11(4), 2015.

[79] V. P. Orevkov. A constructive map of the square onto itself, which moves every constructive point. *Dokl. Akad. Nauk SSSR*, 152:55–58, 1963.

[80] A. Pauly. On the (semi)lattices induced by continuous reducibilities. *Mathematical Logic Quarterly*, 56(5):488–502, 2010.

[81] A. Pauly. On the topological aspects of the theory of represented spaces. *Computability*, 5(2):159–180, 2016.

[82] M. B. Pour-el and I. Richards. A computable ordinary differential equation which possesses no computable solution. *Annals of Mathematical Logic*, 17(1–2):61–90, 1979.

[83] M. B. Pour-el and I. Richards. The wave equation with computable initial data such that its unique solution is not computable. *Advances in Mathematics*, 39(3):215–239, 1981.

[84] M. B. Pour-El and J. I. Richards. *Computability in Analysis and Physics*. Springer, 1989.

[85] H. Scarf. The approximation of fixed points of a continuous mapping. *SIAM J. Appl. Math.*, 15(5):1328–1343, 1967.

[86] A. Schalk. *Algebras for Generalized Power Constructions*. PhD thesis, Technische Hochschule Darmstadt, 1993.

[87] M. Schröder. *Admissible Representations for Continuous Computations*. PhD thesis, FernUniversität Hagen, 2002.

[88] M. Schröder. Extended admissibility. *Theoretical Computer Science*, 284:519–538, 2002.

[89] M. Schröder. A Hofmann-Mislove theorem for Scott open sets. Preprint. Available at `https://arxiv.org/abs/1501.06452v1`, 2015.

[90] D. Scott. Continuous lattices. In F. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer-Verlag, 1972.

[91] M. I. Shamos. *Computational Geometry*. PhD thesis, Yale University, 1978.

[92] S. G. Simpson. *Subsystems of second order arithmetic*, volume 1. Cambridge University Press, 2009.

[93] M. B. Smyth. Effectively given domains. *Theoretical Computer Science*, 5(3):257–274, 1977.

[94] E. Specker. Nicht konstruktiv beweisbare Sätze der Analysis. *J. Symbolic Logic*, 14:145–208, 1949.

[95] E. Specker. Der Satz vom Maximum in der rekursiven Analysis. In A. Heyting, editor, *Constructivity in Mathematics*, pages 254–265. Studies in Logic and The Foundations of Mathematics, 1959.

[96] P. Taylor. Subspaces in Abstract Stone Duality. *Theory and Applications of Categories*, 10(13):300–366, 2002.

[97] P. Taylor. A lambda calculus for real analysis. *Journal of Logic & Analysis*, 2(5):1–115, 2010.

[98] G. Teschl. Topics in real and functional analysis. Available online at `https://www.mat.univie.ac.at/gerald/ftp/book-fa/fa.pdf` - retrieved 1st September 2017, Version: July 27, 2017.

[99] A. S. Troelstra and D. Van Dalen. *Constructivism in Mathematics, Volume 1.* North-Holland, Amsterdam, 1988.

[100] A. S. Troelstra and D. Van Dalen. *Constructivism in Mathematics, Volume 2.* North-Holland, Amsterdam, 1988.

[101] W. Tucker. *Validated Numerics: A Short Introduction to Rigorous Computations.* Princeton University Press, 2011.

[102] A. M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.

[103] A. M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem: A correction. *Proceedings of the London Mathematical Society*, 2(43):544–546, 1937.

[104] S. Vickers. The double powerlocale and exponentiation: A case study in geometric logic. *Theory and Applications of Categories*, 12(13):272 – 422, 2004.

[105] K. Weihrauch. *Computability.* Monographs on Theoretical Computer Science. Springer-Verlag, 1987.

[106] K. Weihrauch. *Computable Analysis.* Springer, 2000.

[107] K. Weihrauch and N. Zhong. Is wave propagation computable or can wave computers beat the turing machine? *Proceedings of the London Mathematical Society*, 85(2):312–332, 2002.

[108] M. Ziegler. Real hypercomputation and continuity. *Theory of Computing Systems*, 41(1):177–206, 2007.

[109] M. Ziegler. Revising type-2 computation and degrees of discontinuity. In D. Cenzer, R. Dillhage, T. Grubba, and K. Weihrauch, editors, *Proceedings of the Third International Conference on Computability and Complexity in Analysis*, volume 167 of *Electronic Notes in Theoretical Computer Science*, pages 255–274, Amsterdam, 2007. Elsevier. CCA 2006, Gainesville, Florida, USA, November 1–5, 2006.

[110] M. Ziegler. Real Computation with Least Discrete Advice: A Complexity Theory of Nonuniform computability. *Annals of Pure and Applied Logic*, 163(8):1108 – 1139, 2012. Continuity, Computability, Constructivity: From Logic to Algorithms.

[111] M. Ziegler. Real computation with least discrete advice: a complexity theory of nonuniform computability with applications to effective linear algebra. *Ann. Pure Appl. Logic*, 163(8):1108–1139, 2012.

[112] M. Ziegler and V. Brattka. Turing computability of (non-)linear optimization. In T. Biedl, editor, *Thirteenth Canadian Conference on Computational Geometry*, pages 181–184. University of Waterloo, 2001. CCCG 2001, August 13–15, 2001.

[113] M. Ziegler and V. Brattka. Computability in linear algebra. *Theoretical Computer Science*, 326(1-3):187–211, 2004.