# Memory effects in a non-equilibrium growth model

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We study *memory effects* in a kinetic roughening model. For $d = 1$, a new dynamic scaling is uncovered in the memory dominated phases; Kardar-Parisi-Zhang (KPZ) [1] scaling being restored in the absence of noise. $d_c = 2$ represents the critical dimension where memory is shown to smoothen the roughening front ($\alpha \leq 0$). Studies on a discrete atomistic model in the same universality class re-confirm the analytical results in the large time limit, while a new scaling behavior shows up for $t < \tau$, $\tau$ being the memory characteristic of the atomistic model. Results can be generalized for other non-conservative systems.

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## INTRODUCTION

Memory effects in intrinsically diffusive dynamics have been the subject of much speculation and excitement in recent years [2–12]. This has been partly due to the fact that along with dimensionality and associated symmetries [9–12, 18, 19], memory, or equivalently time delay, has now been shown to have a highly non-trivial effect both in experimental [5–8] as well as in theoretical situations [9–13]. Whether it be the *spatial correlations* [6, 8, 10, 11] or otherwise the *temporal correlations* [14–17], long-ranged (LR) spatio-temporal correlations are expected to contribute to the universality class of a non-equilibrium system [19] and affect experimental measurements.

Theoretical modeling of interface dynamics incorporating non-local interactions was initiated in a seminal work by Mukherji-Bhattacharjee [12]. The central premise was a generalization of the KPZ non-linearity in [1] to its non-local equivalent through a site:site coupling of gradients, thereby connecting lateral motions between far-off sites (identical in spirit to the non-local Liouville equation studied in [18]). Later attempts [11, 13, 18, 20–22] generally retained the origin of the non-linear LR term while incorporating modifications as demanded by the experimental eg. sputtering [10] or physical eg. correlated noise spectrum [11, 13, 18, 20, 21] situations. What all these studies have done is to confirm once and for all that LR spatial interactions do change the roughness exponent which could now be favorably compared to experimental observations [23]. In all such attempts, though, the attention was restricted to the *spatial correlations* only without attaching much importance whatsoever on the LR nature of temporal fluctuations. As recent biologically motivated studies [16, 17] show, this could be a major loophole in the analysis since temporal correlations of fluctuating fronts could have a marked impact on the spatio-temporal probability distribution of the dynamical process. This article is intended to plug this gap by studying the role of memory related temporal fluctuations in stochastic growth models.

The preliminary question that we address here is the following: what happens if a perturbation at some arbitrary time $t_0$ at site $\vec{x}$ affects the dynamics of the same site $\vec{x}$ at some later time $t$? To address this problem, we start with a *dynamic renormalization group* (DRG) study of a continuum model that incorporates the effects of memory. This analytical prescription is complemented by a study of two independent discrete models: one using a Langevin simulation of the continuum model and the other from a discrete atomistic model with appropriate growth rules. As is later shown, the discrete model belongs to the same universality class as the continuum model in the equilibrium limit ($t \to$ large), whereas for times $t < \tau$, $\tau$ being the delay time associated with the atomistic model (details to follow), an entirely different scaling regime exists that is independent of the specific value of $\tau$.

The article is organized as follows. In the first section, we define the continuum model used to study a memory-dependent dynamical equation of motion. In the next section, a detailed derivation of the method and eventual results of the renormalization group (RG) solution of the equation of motion follows. The section immediately following the RG analysis is a complementary theoretical derivation of the RG scaling results (comparison is restricted to qualitative levels) using the self-consistent mode coupling (SCMC) method. The following section deals with numerical studies of the discretized continuum equation and an independent atomistic simulation incorporating an externally impressed time delay.

## CONTINUUM MODEL IN THE PRESENCE OF A MEMORY TERM

We define our model using the prescription of [11, 12]. However, instead of a site:site interaction at the same time instant $t$, in this model, the fluctuation at a site $\vec{x}$ at time $t$ interacts with the fluctuation of the same site at some other time $t + t'$ ($t' \neq 0$). The consequent equation of motion reads

$$\frac{\partial h}{\partial t}(\vec{x},t) = \nu\nabla^2 h(\vec{x},t) + \frac{1}{2}\int_0^t dt' v(t')\vec{\nabla}h(\vec{x},t+t') \cdot \vec{\nabla}h(\vec{x},t-t') + \eta(\vec{x},t) \quad (1)$$

where $\nu$ is the kinematic viscosity and $\eta$ is a white noise defined through the relation $< \eta(\vec{x},t)\eta(\vec{x}',t') = 2D\delta(\vec{x}-\vec{x}')\delta(t-t')$. The memory kernel $v(t)$ is power-law correlated and is defined through the relation $v(t) = \beta_0\delta(t) + \beta_\theta t^{\theta-1}$. For $\beta_\theta = 0$, conventional KPZ dynamics follows while for all other values of $\beta_\theta$ $(0 < \theta < 1)$, the system has a non-negligible memory. As we would shortly find, the infra-red regime $(\omega \to 0)$ is the one of interest and decides the magnitude of temporal correlations in deciding the non-KPZ type universality class of the system, especially for $d = 1$.

## RENORMALIZATION GROUP ANALYSIS INCLUDING 'MEMORY': CRITICAL EXPONENTS

To study this model, we use simple scaling hypothesis followed by momentum renormalization in the mould of [9, 10, 12]. A self-similar scaling of the model gives $\vec{x} \to b\vec{x}$, $t \to b^z t$, $h \to b^\alpha h$ (where $\alpha$ and $z$ are the roughness and dynamic exponents respectively). This yields $\nu \to \nu b^{z-2}$, $\beta_0 \to b^{z+\alpha-2}\,\beta_0$, $\beta_\theta \to b^{\alpha+(\theta+1)z-2}\,\beta_\theta$ and $D \to b^{z-d-2\alpha}D$. For all non-zero values of $\beta_\theta$, the attractor flows over to a non-KPZ fixed point and the Galilean relation $(\alpha + z = 2)$ of a standard KPZ model is also modified, the latter now becoming $\theta$ dependent. The renormalization technique consists in considering a momentum shell between the wave-vectors $\vec{k}$ and $\vec{k}+d\vec{k}$ for the frequency $\omega$ and then integrating out the fast modes between $\Lambda e^{-l} < |\vec{q}| < \Lambda$. At the one-loop level, the flow equations are given by

$$\begin{aligned}
\frac{d\nu}{dl} &= [z-2-K_d\frac{\tilde{v}(2)\tilde{v}(1)D}{\nu^3}\,\frac{d-2}{4d}]\,\nu \\
\frac{d\beta_0}{dl} &= [\alpha+z-2]\,\beta_0 \\
\frac{d\beta_\theta}{dl} &= [\alpha+z(1+\theta)-2]\,\beta_\theta \\
\frac{dD}{dl} &= [z-d-2\alpha]\,D + \frac{D^2K_d}{4\nu^3}\tilde{v}^2(2) \qquad (2)
\end{aligned}$$

where $K_d = S_d/(2\pi)^d$ ($S_d$ is the surface of the d-dimensional hyper-sphere), $\tilde{v}(\omega) = \beta_0 + \beta_\theta\omega^{-\theta}$ and $\Lambda = 1$ without any loss of generality. Due to Galilean invariance, $\beta_0$ is not renormalized at any perturbative order. In terms of the non-dimensional interaction strengths ${U_x}^2 = \frac{D{\beta_x}^2K_d}{\nu^3}$ $(x = 0\ \&\ \theta)$, we get

$$\frac{dU_0}{dl} = (\frac{2-d}{2})U_0 + (\frac{2d-3}{4d}){U_0}^3 + \frac{U_0U_\theta}{8d}[a_0U_0 + a_1U_\theta] \quad (3)$$

In the above, $a_0 = (5d-6)(1+2^{-\theta}) - 2d$ and $a_1 = 2^{-\theta}[(3+2^{-\theta})d - 6]$. This gives

$$\frac{dR}{dl} = -z\theta R, \quad (4)$$

where $R = \frac{U_0}{U_\theta}$. The previous equation suggests that there are no off-axis fixed points in the $U_0$, $U_\theta$ parameter space except at the trivial fixed point $\theta = 0$ (KPZ fixed point). There are only two sets of axial fixed points, the short-ranged one: ${U_0}^{*2} = \frac{2d(d-2)}{2d-3}$, ${U_\theta}^{*2} = 0$ $(\alpha + z = 2$; KPZ fixed point) and the long-ranged non-KPZ fixed point: ${U_0}^{*2} = 0$, ${U_\theta}^{*2} = \frac{4d(d-2-2z\theta)}{a_1}$. It might be noted that our phase diagram has a comparable structure to that of the one in [12]. The quantitative difference, though, lies in the nature of self-consistent evaluation of the dynamic exponent $z$ (or roughness exponent $\alpha$) as a function of the memory parameter $\theta$ and the spatial dimension $d$ of the system.. This is evident from the following expressions that we derive from the flow equations:

$$\begin{aligned}
z &= 2 + \frac{(d-2-2z\theta)(d-2-3z\theta)}{2^\theta a_1} \\
\alpha &= 2 - z(1+\theta) \qquad (5)
\end{aligned}$$

giving

$$6\theta^2z^2 - [5(d-2)\theta + 2^\theta a_1]z + [(d-2)^2 + 2^{\theta+1}a_1] = 0 \quad (6)$$

The equation above can be used to solve for $z$ and $\alpha$. Additionally, we impose the logical constraints that the dynamic exponent $z$ should be real and positive thereby generating the $z$, $\alpha$ vs $\theta$ phase diagrams (Fig. 1). Such outcomes, when contrasted with [28], clearly shows that memory has a non-trivial effect on the non-equilibrium dynamics while when compared with [9], it becomes clear that the contribution from delay is *quantitatively very unlike* that of a spatio-temporal correlation in the noise spectrum. One might, however, argue of a qualitative resemblance between the two [29], in the sense that in either case, the dynamic exponent $z$ always decreases as a function of the external parameter $\theta$ (where $\theta$ is the exponent defining the temporal correlation in [29] while it defines strength of the memory kernel in our case); while the roughness exponent $\alpha$ increases with it. The quantitative difference lies in the convexity property of the $z$ versus $\theta$ plots (or equivalently the concavity of the $\alpha$ vs $\theta$ plot) in each case.

For $d = 1$, we get a series of non-KPZ fixed points, as shown in Fig. 1, the KPZ phase being restored at

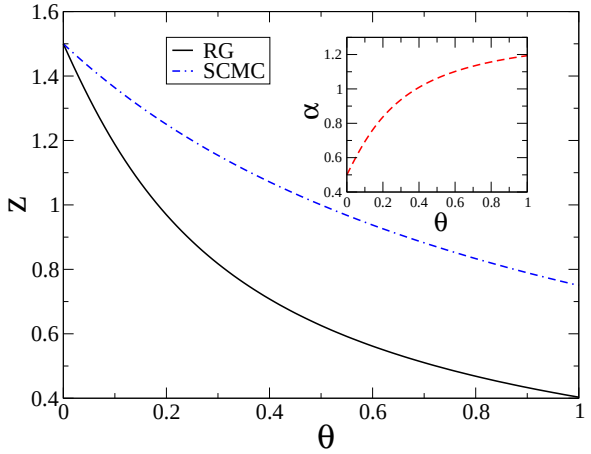


FIG. 1: Phase diagrams for d=1: The outset represents $z$ vs $\theta$; the solid line represents the renormalization group (RG) result calculated using equation (6) while the dotted-dashed line represents the self-consistent mode coupling (SCMC) result calculated from equation (10a). The inset shows the RG result for $\alpha$ plotted against $\theta$.

$\theta = 0$. The outset of this figure represents the $z$ versus $\theta$ results while the inset shows the results for $\alpha$ plotted against $\theta$. It might be noted that for values of $\theta \geq 0.39$ the system enters a *super-rough phase* [25–27] ($\alpha \geq 1$). Although such a behavior generally implies a break-down of the self-affine hypothesis [24], as already shown in numerous theoretical [25] and numerical [27] studies, this is defined as *anomalous scaling*. Studies done on a wide range of kinetic growth processes, including crack propagation, Hele-Shaw flow with quenched disorder, spontaneous imbibition, etc. prove without doubt that such a scaling is very physical and is related to the mean local slope of the interface defined through the non-linear term in the growth process [26]. In the model of our study, we find a consummate proof of this scaling in that the lateral non-linear term, representing the memory contribution, dominates the scaling process as is evident by the existence of such a strongly disordered phase.

The criticality of the system for $d = 2$ is interesting. In the absence of the memory term ($\theta = 0$), the trajectories flow over to an unique Edwards'-Wilkinson [23] fixed point but no stable fixed point exists once the memory is switched on. Clearly $d_c = 2$ represents a *critical dimension* indicating a cross-over from a smooth to a rough phase on either side of $d_c$, a result that is validated also from the self-consistent mode coupling analysis detailed in the following section.

## SELF-CONSISTENT MODE COUPLING (SCMC)

In order to cross-check the non-KPZ effects qualitatively, we use self-consistent mode coupling (SCMC) theory as in [11]. The starting point is the Dyson's equation $G^{-1}(\vec{k},\omega) = -i\omega + \nu k^2 + \Sigma(\vec{k},\omega)$ which, together with the self-consistent scheme, gives for the self-energy

$$\Sigma(\vec{k},\omega) = -\int \frac{d^dk}{(2\pi)^d}\frac{d\omega'}{2\pi}\,[\vec{p}.\vec{k}][\vec{p}.(\vec{k}-\vec{p})]\,\mathrm{G}(\vec{k}-\vec{p},\omega-\omega') \times \mathrm{C}(\vec{p},\omega')\,|v(\omega)|^2 \tag{7}$$

The correlation function at an equivalent order is given by

$$\mathrm{C}(\vec{k},\omega) = |\mathrm{G}(\vec{k},\omega)|^2 \int \frac{d^dk}{(2\pi)^d}\frac{d\omega'}{2\pi}\,\vec{p}^2(\vec{k}-\vec{p})^2\,|v(\omega)|^2 \times \mathrm{C}(\vec{k}-\vec{p},\omega-\omega')\,\mathrm{C}(\vec{p},\omega') \tag{8}$$

One can now use the following scaling ansatz for $\Sigma(\vec{k},\omega)$ and $\mathrm{C}(\vec{k},\omega)$

$$\Sigma(\vec{k},\omega) = k^z\, f(\frac{\omega}{|\vec{k}|^z}), \tag{9a}$$

$$\mathrm{C}(\vec{k},\omega) = k^{-(d+2\alpha+z)} g(\frac{\omega}{|\vec{k}|^z}) \tag{9b}$$

In the infra-red limit, we can now combine equations (7, 8, 9a, 9b) and use the self-consistent power counting scheme [11] to get the following values for the exponents

$$z = \frac{2+d}{2(1+\theta)} \tag{10a}$$

$$\alpha = \frac{2-d}{2} \tag{10b}$$

It might be noted that the above analysis validates the more detailed RG results at a qualitative level. The dynamic exponent $z$ as a function of $\theta$ has the same concavity in its structure, although the roughness exponent remains independent of $\theta$ which is a marked difference from the RG result. This should be evident from Fig. 1 where the dynamic exponent $z$, derived independently from dynamic renormalization group and mode coupling theory, are contrasted against each other (outset of the figure). The other agreement is on the value of the critical dimension, defined through the identity $\alpha = 0$, which, as in the RG case, shows $d_c = 2$.

## DISCRETE GROWTH MODEL, INCLUDING 'MEMORY'

To check the strength and consistency of the previous analytical arguments (especially in view of claims to fallibility of DRG arguments in non-local KPZ analyzes

[22]) and possibly more, we now resort to two independent numerical frameworks. The first method involves the simulation of the Langevin equation (1) while the other is our proposal for a new atomistic model that is shown to belong to the same universality class as the continuum model in certain temporal domains. In certain other temporal regimes characterized by an externally impressed delay time $\tau$, such an atomistic model predicts a different scaling behavior that goes beyond the continuum model. For time $t < \tau$, a separate scaling regime exists (Fig. 2) that crosses over to the continuum scaling behavior for $t > \tau$. What this essentially means is that the $t > \tau$ regime signifies the stationarity limit of the discrete model which has a marked difference to the *roughening* observed for smaller values of $t < \tau$.

In the Langevin simulation, we use non-dimensionalized units throughout and a typical Gaussian white noise of low strength. The discretization follows a finite-difference scheme accurate up to order $\mathcal{O}((\triangle x)^2)$. The results, in 1+1 dimensions, generally agrees with the renormalization group results as depicted in Fig. 1 in that the growth exponents $\beta$ (defined later) increase as the values of $\theta$ increase. The results, though, are only a qualitative match with the RG results, the former being results in the long time limit.

A similar scaling follows for the atomistic model (described later) for $t \rightarrow$ large. As already indicated, delay does not change the universality class for the discrete model so long as $t > \tau$.

We now compare such results to the discrete atomistic model as described below. The algorithm for the discrete model originates from a version of the restricted solid-on-solid model by Kim and Kosterlitz [30]. We define the simple-to-follow growth rule in 1d. A particle is dropped at site $i$ only if the nearest neighbor height differences are less than a pre-assigned whole number. This means $h_i \rightarrow h_i + 1$ only if $|h_i - h_{i-1}| < N$ or $|h_{i+1} - h_i| < N$, $N$ (=1 in our simulation) being a whole number that can have values 1, 2, 3, etc. However, such a growth rule defines a simple KPZ universality class. In order to incorporate "delay" or memory in the system, we impose the additional condition: a site $i$ can not accept the next particle until delay time $\tau$ has elapsed and after this time a particle is allowed to stick at the location $i$ only if $|h_{i\pm1} - h_i| < N$. This essentially amounts to throwing away a particle if it falls on a site within time $\tau$ of the site having accepted a previous particle. This is not a problem though, since number conservation of particles is not an intrinsic symmetry of the basic Kardar-Parisi-Zhang model. One can now control the value of this externally impressed delay time $\tau$ and study the second spatio-temporal moments, as shown in Fig. 2.

In Fig 2, we show scaling results for the atomistic simulation data in a log-log plot where the temporal width $w(t) =< [h(x,t'+t) - h(x,t')]^2 >$ is plotted against time $t$ using non-dimensionalized units. The result shows

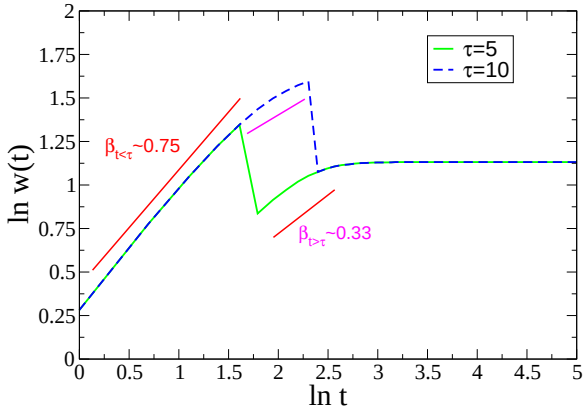


FIG. 2: Atomistic simulation for $t < \tau$ shows a dynamic scaling $\beta \sim 0.75$ that crosses over to the continuum scaling behavior $\beta \sim 0.33$ in the stationarity limit. Results for $\tau = 5$ & 10 confirms that scaling results in both temporal regimes are independent of the actual values of $\tau$.

two different scaling zones in 1+1 dimensions. For $t < \tau$, the growth exponent $\beta \sim 3/4$ while for $t > \tau$, there is a cross-over to the KPZ universality class ($\beta \sim 0.33$). The non-KPZ scaling for $t < \tau$ appears to be a specific property of the discrete model that does not have a continuum analogue.

## CONCLUSIONS

To summarize, we have proposed a model of kinetic roughening that takes into account the presence of memory in the system. Using complementary results from renormalization, self-consistent mode coupling and Langevin simulation, we make an important conclusion: *the nature of perturbation generated by the presence of memory in a non-linear system has a marked quantitative difference to that generated by other external perturbations, including that of a temporally colored noise.* For $d = 1$, the presence of memory contributes to producing a a huge set of non-KPZ type fixed points while for $d = 2$, the only stable phase is the Edwards'-Wilkinson phase. Results from the continuum model are complemented by simulation results from a proposed discrete atomistic model which has an external delay imposed on an otherwise KPZ type growth rule. We find that for systems with large relaxation times ($\tau$ large), the usual KPZ universality class is restored, as is to be expected. However, for small enough delay times ($\tau \sim (\frac{\beta_\theta}{\beta_0})^{\frac{1}{1-\theta}}$), the system shows a unique cross-over to a non-KPZ phase defined by a growth exponent $\sim 3/4$. The memory dominated non-KPZ phase seems to indicate the presence of intermittency in the energy spectrum as could easily be tested from a dimensional analysis.What is most reassuring, though, is the fact that the cross-over value of the delay time $\tau$ does not affect the scaling behavior in

either limit, thereby confirming the universality aspect of this atomistic model. Irrespective of the nature of its non-linearity, we expect other memory-dependent non-equilibrium models to show similar behaviors as long as the dynamics remains non-conservative.

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