On the $q = 1/2$ non-extensive maximum entropy distribution

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Abstract

A detailed mathematical analysis on the $q = 1/2$ non-extensive maximum entropy distribution of Tsallis' is undertaken. The analysis is based upon the splitting of such a distribution into two orthogonal components. One of the components corresponds to the minimum norm solution of the problem posed by the fulfillment of the a priori conditions on the given expectation values. The remaining component takes care of the normalization constraint and is the projection of a constant onto the Null space of the "expectation-values-transformation".

1 Introduction

The seminal work of Tsallis [1], which generalizes the concepts of both entropy and expectation values, has rendered a variety of interesting generalized results in connection with multifractals, astrophysics, cosmology, turbulence, thermodynamics, statistical mechanics, etc (see, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). The Tsallis' generalized framework depends upon a real parameter, q, and each $q-$value generates a particular statistics. The limiting case $q \to 1$ yields Shannon's entropy [12] and therefore Jaynes' celebrated results based upon the Maximum Entropy Principle [13, 14].

In the present contribution we will focus attention on the analysis of the $q = 1/2$ case, which arouses special interest because it involves dealing with *linear equations.* This entails making the $q = 1/2$ Tsallis distribution particularly adequate to be used in those situations in which the number M of available expectation values is very large (although, of course, not *so large* as to determine a unique solution). For big M-values, the handling of the nonlinear set of equations arising from considering $q \neq 1/2$ becomes a troublesome task indeed. Typically, such situations take place when the expectation values represent measurements which are obtained as a function of a variable parameter [15, 16]. In addition to the numerical advantage accrued to the linearity of the $q = 1/2$ distribution, one should mention that this distribution has already been shown to be endowed with physical significance by Boghosian [9], being related to the concept of *enstrophy.* Indeed, in a recent publication Boghosian has reported that the density profiles of a pure-electron plasma column during the relaxation to a metaequilibrium state rather that maximize the Boltzmann entropy maximize Tsallis' entropy with $q = 1/2$ [9] The $q = 1/2$ Tsallis distribution appears therefore, as stated, a potentially helpful tool deserving careful analysis from a mathematical point of view. In particular, we wish to shed light on the relation between such a solution and the classical minimum norm one. We will show that the $q = 1/2$ Tsallis distribution can be split into two orthogonal components, each of which has a well defined mathematical meaning. One of the components is the minimum norm solution of the linear problem posed by the fulfillment of the expectation-values constraints. The other component is the projection of a *constant* onto the Null space of the transformation generated by the expectation values. The latter takes care of the normalization constraint.

The paper is organized as follows: In Section 2 Tsallis' approach is briefly summarized, while in Section 3 the case $q = 1/2$ is considered and a detailed mathematical analysis is provided. Some conclusion are drawn in Section 4.

2 The p^q non-extensive maximum entropy distribution

Let us consider a set of N events with probabilities $p_n\,;\, n = 1, \ldots, N$ and let $f_i\,;\, i = 1, \ldots, M$ be a set of M random variables, each of which takes values $f_{i,n}\,;\, n = 1, \ldots, N$. Consider further that, by resort to adequate experimental measurements, one is able to ascertain the expectation values $\overline{f}_i\,;\, i = 1, \ldots, M$ of the corresponding random variables f_i. Tsallis' proposal for determining the probabilities $p_n\,;\, n = 1, \ldots, N$ from the measurements $\overline{f}_i\,;\, i = 1, \ldots, M$ confronts us with a problem that can be rendered in the following terms [4]: for $q \in \mathcal{R}$, and with the set of constraints

$$\sum_{n=1}^{N} p_n^q f_{i,n} = \overline{f}_i \quad ; \quad i = 1, \ldots, M, \tag{1}$$

$$\sum_{n=1}^{N} p_n = 1, \tag{2}$$

maximize the Tsallis entropy S_q

$$S_q = \frac{\sum_{n=1}^{N} p_n^q - \sum_{n=1}^{N} p_n}{1 - q}. \tag{3}$$

The resulting expression for the Tsallis generalized weight p^q [1] adopts the functional form [4]

$$p_n^q = \frac{1}{z_o}[1 - (1-q)\sum_{i=1}^{M} \lambda_i f_{i,n}]^{\frac{q}{1-q}}, \tag{4}$$

where both z_o and the Lagrange Multipliers ($\lambda_i\,;\ i = 1,\ldots,M$) should be determined so as to fulfill constraints (1) and (2). For $q = 1/2$, obviously $\frac{q}{1-q} = 1$, and p_n^q becomes *bilinear* in both $\lambda_i,\ f_{i,n}$.

3 The $q = 1/2$ case

Before undertaking the analysis of the $p^{\frac{1}{2}}$ distribution, we find it convenient to adopt a vectorial notation. We shall represent a vector, $\mathbf{x}$ say, as $|x\rangle$ and its transpose as $\langle x|$. The *standard basis* $\{|n\rangle; n = 1,\ldots,N\}$ in $\mathcal{R}^N$ is defined as follows: $\langle n|m\rangle = \delta_{n,m}\ ;\ n = 1,\ldots,N\ ;\ m = 1,\ldots,N$, where $\langle .|.\rangle$ stands for *inner product.* Accordingly, the $p^{\frac{1}{2}}$ distribution will be represented as a vector $|p^{\frac{1}{2}}\rangle \in \mathcal{R}^N$, i.e.,

$$|p^{\frac{1}{2}}\rangle = \sum_{n=1}^{N} |n\rangle\langle n|p^{\frac{1}{2}}\rangle = \sum_{n=1}^{N} p_n^{\frac{1}{2}}|n\rangle, \tag{5}$$

and the measurements $\overline{f}_i\,;\ i = 1,\ldots,M$ will be represented as a vector $|\overline{f}\rangle \in \mathcal{R}^M$,i.e.,

$$|\overline{f}\rangle = \sum_{i=1}^{M} |i\rangle\langle i|\overline{f}\rangle = \sum_{i=1}^{M} \overline{f}_i|i\rangle. \tag{6}$$

Furthermore, we define the rank deficient operator $\hat{A} : \mathcal{R}^N \to \mathcal{R}^M$ through the matrix elements $\langle i|\hat{A}|n\rangle = f_{i,n}\,;\ i = 1,\ldots,M\,;\ n = 1,\ldots,N$. Considering that the measurements $\overline{f}_i\,;\ i = 1,\ldots,M$ are linearly independent, $rank(\hat{A}) = M$.

Using this notation, constraints (1) and (2) (for $q = 1/2$) are recast as

$$\hat{A}|p^{\frac{1}{2}}\rangle = |\overline{f}\rangle \tag{7}$$

$$\langle p^{\frac{1}{2}}|p^{\frac{1}{2}}\rangle = 1 \tag{8}$$

with

$$|p^{\frac{1}{2}}\rangle = |z\rangle - \frac{z}{2}\hat{A}^{\dagger}|\lambda\rangle \tag{9}$$

where $\hat{A}^{\dagger}$ stands for the adjoint of $\hat{A}$, $|\lambda\rangle \in \mathcal{R}^M$ is a vector whose components are the Lagrange multipliers λ_i ; $i = 1, \ldots, M$ and $|z\rangle = \sum_{n=1}^{N} \langle n|z\rangle |n\rangle = \sum_{n=1}^{N} z|n\rangle$ is the vectorial representation of the real number $z = z_o^{-1}$.

In order to solve for the Lagrange Multipliers, we introduce (9) into (7). Since we are considering linearly independent measurements, the operator $\hat{A}\hat{A}^{\dagger} : \mathcal{R}^M \rightarrow \mathcal{R}^M$ has an inverse and we obtain

$$|\lambda\rangle = -\frac{2}{z}(\hat{A}\hat{A}^{\dagger})^{-1}|\overline{f}\rangle + \frac{2}{z}(\hat{A}\hat{A}^{\dagger})^{-1}\hat{A}|z\rangle \tag{10}$$

so that $|p^{\frac{1}{2}}\rangle$ becomes

$$|p^{\frac{1}{2}}\rangle = |c\rangle + |\tilde{c}_z\rangle \tag{11}$$

with

$$|c\rangle = \hat{A}^{\dagger}(\hat{A}\hat{A}^{\dagger})^{-1}|\overline{f}\rangle \tag{12}$$

$$|\tilde{c}_z\rangle = |z\rangle - \hat{A}^{\dagger}(\hat{A}\hat{A}^{\dagger})^{-1}\hat{A}|z\rangle. \tag{13}$$

The vector $|\tilde{c}_z\rangle$ depends upon the value of z, which should be determined by the normalization constraint (8). Before fixing such a number, we would like to discuss some general properties of solution (11).

3.1 Some properties of the $|p^{\frac{1}{2}}\rangle$ distribution

We shall study here the $|p^{\frac{1}{2}}\rangle$ distribution by analyzing its two components $|c\rangle$ and $|c_z\rangle$. The essential tool for the analysis is provided by the spaces $Null(\hat{A})$ and $Null^{\perp}(\hat{A})$ (the orthogonal complement of $Null(\hat{A})$). Let us recall then that $Null(\hat{A})$ is defined as

$$Null(\hat{A}) = \{|b\rangle \in \mathcal{R}^N \, ; \, \hat{A}|b\rangle = 0\} \tag{14}$$

whereas

$$Range(\hat{A}) = \{|f\rangle \in \mathcal{R}^M \,;\, |f\,\rangle = \hat{A}|b\rangle \ for\ \ some\ |b\rangle \in \mathcal{R}^N\}. \tag{15}$$

Proposition 1: The vectors $|c\rangle$ and $|\tilde{c}_z\rangle$ given in (12) and (13) are mutually orthogonal, $|\tilde{c}_z\rangle$ being the orthogonal projection of $|p^{\frac{1}{2}}\rangle$ onto $Null(\hat{A})$, and $|c\rangle$ the orthogonal projection of $|p^{\frac{1}{2}}\rangle$ onto $Null^{\perp}(\hat{A})$.

Proof: Let us give the names $\hat{P}_N$ and $\hat{P}_{N^\perp}$, respectively, to the orthogonal projection operators onto $Null(\hat{A})$ and $Null^{\perp}(\hat{A})$. In order to obtain explicit representations for these projectors, we consider the eigenvectors of the operator $\hat{A}^\dagger A : \mathcal{R}^{\mathcal{N}} \to \mathcal{R}^{\mathcal{N}}$. This is a bounded self adjoint operator which satisfies

$$\hat{A}^\dagger A|\psi_n\rangle = \mu_n|\psi_n\rangle \quad ; \quad \langle\psi_n|\psi_m\rangle = \delta_{n,m} \quad ; \quad n = 1, \ldots, N, \tag{16}$$

with the eigenvalues property: $\mu_1 \geq \mu_2 \ldots \geq \mu_N \geq 0$ [17]. Since $rank(\hat{A}^\dagger A) = M$, we have M nonzero eigenvalues $\mu_n\,;\, n = 1, \ldots, M$ and $(N - M)$ zero eigenvalues $\mu_n\,;\, n = M + 1, \ldots, N$. The vectors $|\phi_n\rangle = \hat{A}|\psi_n\rangle\,;\, n = 1, \ldots, M$ are the eigenvectors of the operator $\hat{A}\hat{A}^\dagger$ with corresponding eigenvalues μ_n, as is easily seen. It is also straightforward to verify that $\langle\phi_n|\phi_m\rangle = \delta_{m,n}\mu_m$. We see then that the vectors $|\psi_n\rangle$ corresponding to a zero eigenvalue give rise to vectors $|\phi_n\rangle$ of zero norm, whereby $|\psi_n\rangle\,;\, n = M + 1, \ldots, N$ span $Null(\hat{A})$. Thus, the explicit representations of $\hat{P}_N$ and $\hat{P}_{N^\perp}$ are as follows:

$$\hat{P}_N = \sum_{n=M+1}^{N} |\psi_n\rangle\langle\psi_n| \tag{17}$$

$$\hat{P}_{N^\perp} = \sum_{n=1}^{M} |\psi_n\rangle\langle\psi_n|. \tag{18}$$

The normalized vectors $|\tilde{\phi}_n\rangle = \frac{|\phi_n\rangle}{\sqrt{\mu_n}} = \frac{\hat{A}|\psi_n\rangle}{\sqrt{\mu_n}}\,;\, n = 1, \ldots, M$ span $Range(\hat{A})$, so that they provide an explicit representation for the orthogonal projection operator onto $Range(\hat{A})$, i.e.,

$$\hat{P}_R = \sum_{n=1}^{M} |\tilde{\phi}_n\rangle\langle\tilde{\phi}_n|. \tag{19}$$

Furthermore, since $|\tilde{\phi}_n\rangle\,;\, n=1,\ldots,M$ are the normalized eigenvectors of the operator $\hat{A}\hat{A}^\dagger$, we have

$$\hat{A}\hat{A}^\dagger = \sum_{n=1}^{M} |\tilde{\phi}_n\rangle \mu_n \langle\tilde{\phi}_n| \tag{20}$$

and

$$(\hat{A}\hat{A}^\dagger)^{-1} = \sum_{n=1}^{M} |\tilde{\phi}_n\rangle \frac{1}{\mu_n} \langle\tilde{\phi}_n| \tag{21}$$

therefore, $\hat{A}\hat{A}^\dagger(\hat{A}\hat{A}^\dagger)^{-1} = (\hat{A}\hat{A}^\dagger)^{-1}\hat{A}\hat{A}^\dagger = \hat{P}_R$.

In addition,

$$\hat{A}^\dagger(\hat{A}\hat{A}^\dagger)^{-1}\hat{A} = \sum_{n=1}^{M} \hat{A}^\dagger|\tilde{\phi}_n\rangle \frac{1}{\mu_n}\langle\tilde{\phi}_n|\hat{A} = \sum_{n=1}^{M} |\psi_n\rangle\langle\psi_n| = \hat{P}_{N^\perp}. \tag{22}$$

We are now in a position to prove that $\hat{P}_{N^\perp}|\tilde{c}_z\rangle = 0$ and $\hat{P}_{N^\perp}|c\rangle = |c\rangle$. Indeed,

$$\hat{P}_{N^\perp}|\tilde{c}_z\rangle = \hat{P}_{N^\perp}|z\rangle - \hat{P}_{N^\perp}\hat{A}^\dagger(\hat{A}\hat{A}^\dagger)^{-1}\hat{A}|z\rangle = \hat{P}_{N^\perp}|z\rangle - \hat{P}_{N^\perp}\hat{P}_{N^\perp}|z\rangle = 0, \tag{23}$$

and

$$\begin{aligned} \hat{P}_{N^\perp}|c\rangle &= \sum_{n=1}^{M} |\psi_n\rangle\langle\psi_n|\hat{A}^\dagger(\hat{A}\hat{A}^\dagger)^{-1}|\overline{f}\rangle = \sum_{n=1}^{M} \hat{A}^\dagger|\tilde{\phi}_n\rangle\frac{1}{\mu_n}\langle\tilde{\phi}_n|\hat{A}\hat{A}^\dagger(\hat{A}\hat{A}^\dagger)^{-1}|\overline{f}\rangle \\ &= \hat{A}^\dagger(\hat{A}^\dagger A)^{-1}\hat{P}_R|\overline{f}\rangle = \hat{A}^\dagger(\hat{A}\hat{A}^\dagger)^{-1}|\overline{f}\rangle = |c\rangle. \end{aligned} \tag{24}$$

Summing up, we have proved that $\hat{P}_{N^\perp}|p^{\frac{1}{2}}\rangle = \hat{P}_{N^\perp}|c\rangle + \hat{P}_{N^\perp}|\tilde{c}_z\rangle = |c\rangle$. On the other hand, since $\hat{P}_{N^\perp}|c\rangle = |c\rangle$, $\hat{P}_N|c\rangle = 0$, and since $\hat{P}_{N^\perp}|\tilde{c}_z\rangle = 0$, $|\tilde{c}_z\rangle = \hat{P}_N|\tilde{c}_z\rangle$. Hence, $\hat{P}_N|p^{\frac{1}{2}}\rangle = \hat{P}_N|c\rangle + \hat{P}_N|\tilde{c}_z\rangle = |\tilde{c}_z\rangle$ $\square$

Corollary 1: The vector $|c\rangle$ given in (12) is the minimum norm solution of equation (7).

Proof: Since $\hat{A}|c'\rangle = 0$ for all $|c'\rangle \in Null(\hat{A})$, the most general solution of equation (7) is amenable to a cast in the fashion $|c\rangle + |c'\rangle$, with $|c\rangle$ given by (12) and $|c'\rangle$ being any vector in $Null(\hat{A})$. Indeed, $\hat{A}(|c\rangle + |c'\rangle) = \hat{A}|c\rangle = \hat{P}_R|\overline{f}\rangle = |\overline{f}\rangle$. It is obvious then that by choosing $|c'\rangle \equiv 0$ the minimum norm solution is obtained $\square$.

Corollary 2: The vector $|\tilde{c}_z\rangle$ given in (13) is the orthogonal projection of a constant z onto

$Null(\hat{A})$.

Proof: According to (22) $|\tilde{c}_z\rangle$ can be expressed as $|\tilde{c}_z\rangle = |z\rangle - \hat{P}_{N^\perp}|z\rangle \equiv \hat{P}_N|z\rangle$ $\Box$.

From the above corollaries we conclude that, among all the vectors $|c'\rangle \in Null(\hat{A})$, the Tsallis $q = 1/2$ approach chooses the one which is just the projection of a constant onto $Null(\hat{A})$. Such a vector plays the role of making sure of increasing the minimum norm so as to give one the possibility of setting equal to unity the norm-value, as required by constraint (8).

Although $|\overline{f}\rangle \in Range(\hat{A})$ by hypothesis, it is appropriate to recall that its components $\langle i|\overline{f}\rangle = \overline{f}_i\,;\; i = 1,\ldots,M$ represent experimental measurements that are always affected by errors. In practice, what is actually available is a vector $|f^o\rangle = |\overline{f}\rangle + |\delta f\rangle$ and situations for which $|f^o\rangle \notin Range(\hat{A})$ may certainly occur. The next proposition deals with this case and shows that $\hat{A}|p^{\frac{1}{2}}\rangle$ renders an approximation to $|f^o\rangle$ which is optimal in a minimum distance sense.

Proposition 2: If $|f^o\rangle \notin Range(\hat{A})$ is the available observation vector, $\hat{A}|p^{\frac{1}{2}}\rangle$ is the unique vector in $Range(\hat{A})$ that minimizes the distance to $|f^o\rangle$.

Proof: The proof stems from the fact that $\hat{A}|p^{\frac{1}{2}}\rangle$ is the orthogonal projection of $|f^o\rangle$ onto $Range(\hat{A})$. Indeed,

$$\hat{A}|p^{\frac{1}{2}}\rangle = \hat{A}|c\rangle = \hat{A}\hat{A}^\dagger(\hat{A}\hat{A}^\dagger)^{-1}|f^o\rangle = \hat{P}_R|f^o\rangle. \tag{25}$$

Accordingly, $|f^o\rangle$ can be written as : $|f^o\rangle = \hat{A}|p^{\frac{1}{2}}\rangle + |\Delta f^o\rangle$ where $|\Delta f^o\rangle \in Range^\perp(\hat{A})$. If we take an arbitrary vector $|g\rangle \in Range(\hat{A})$ and calculate the distance to $|f^o\rangle$ we have: $|||g\rangle - |f^o\rangle||^2 = |||g\rangle - \hat{A}|p^{\frac{1}{2}}\rangle - \Delta f^o\rangle||^2 = |||g\rangle - \hat{A}|p^{\frac{1}{2}}\rangle||^2 + ||\Delta f^o\rangle||^2$. Hence, the distance $|||g\rangle - |f^o\rangle||$ is minimized if $|g\rangle \equiv \hat{A}|p^{\frac{1}{2}}\rangle$ $\Box$

We shall fix now the constant z so as to fulfill the normalization constraint (8) i.e.,

$$\sum_{n=1}^{N} p_n = \langle p^{\frac{1}{2}}|p^{\frac{1}{2}}\rangle = \langle c|c\rangle + \langle\tilde{c}_z|\tilde{c}_z\rangle = \langle c|c\rangle + \langle z|\hat{P}_N|z\rangle = 1. \tag{26}$$

There exist two values of z satisfying (26), namely:

$$z_{\pm} = \pm\left(\frac{1 - \langle c|c\rangle}{\sum_{j=1}^{N}\sum_{k=1}^{N}\langle j|\hat{P}_N|k\rangle}\right)^{\frac{1}{2}} = \pm\left(\frac{1 - \sum_{n=1}^{M}\frac{|\langle\tilde{\phi}_n|\overline{f}\rangle|^2}{\mu_n}}{N - \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{n=1}^{M}\langle j|\psi_n\rangle\langle\psi_n|k\rangle}\right)^{\frac{1}{2}}. \tag{27}$$

In the next proposition we show that the negative value, z_-, is to be disregarded because it yields a lower entropy than the positive one.

Proposition 3: From the two values z_+ and z_-, satisfying constraint (8), z_+ renders the largest entropy-value.

Proof: Writing explicitly the entropy $S_{\frac{1}{2}}(z_+)$ we have

$$S_{\frac{1}{2}}(z_+) = 2\sum_{n=1}^{N}\langle n|c\rangle + 2\sum_{n=1}^{N}\langle n|\hat{P}_N|z_+\rangle - 2 = 2\sum_{n=1}^{N}\langle n|c\rangle + 2z_+\sum_{n=1}^{N}\sum_{k=1}^{N}\langle n|\hat{P}_N|k\rangle - 2 \tag{28}$$

and, since $z_- = -z_+$,

$$S_{\frac{1}{2}}(z_-) = 2\sum_{n=1}^{N}\langle n|c\rangle + 2\sum_{n=1}^{N}\langle n|\hat{P}_N|z_-\rangle - 2 = 2\sum_{n=1}^{N}\langle n|c\rangle - 2z_+\sum_{n=1}^{N}\sum_{k=1}^{N}\langle n|\hat{P}_N|k\rangle - 2. \tag{29}$$

Notice that $\sum_{n=1}^{N}\sum_{k=1}^{N}\langle n|\hat{P}_N|k\rangle = \langle\tilde{c}_z|\tilde{c}_z\rangle/z^2 \geq 0$, so that on comparing (28) and (29) we gather that $S_{\frac{1}{2}}(z_+) > S_{\frac{1}{2}}(z_-)$ $\Box$

Thus, the distribution $p_j^{\frac{1}{2}} = \langle j|p^{\frac{1}{2}}\rangle$; $j = 1,\ldots,N$ that complies with the constraints (7) and (8) yields the global maximum of the entropy $S_{\frac{1}{2}}$ as given by

$$\begin{aligned} p_j^{\frac{1}{2}} &= \langle j|\hat{A}^{\dagger}(\hat{A}\hat{A}^{\dagger})^{-1}|\overline{f}\rangle + \langle j|\hat{P}_N|z_+\rangle \\ &= \sum_{n=1}^{M}\langle j|\psi_n\rangle\frac{1}{\sqrt{\mu_n}}\langle\tilde{\phi}_n|\overline{f}\rangle + z_+ - z_+\sum_{n=1}^{M}\sum_{k=1}^{N}\langle j|\psi_n\rangle\langle\psi_n|k\rangle \;\; ; \;\; j = 1,\ldots,N \end{aligned} \tag{30}$$

where z_+ is calculated as in (27). It should be stressed, however, that from the above expression the positivity property of the distribution can not be guaranteed.

4 Conclusions

A detailed mathematical analysis, performed on the $q = 1/2$ Tsallis distribution, has been undertaken in the present effort. We have shown that such a distribution is able to be split into two orthogonal components, each endowed with a clear mathematical significance. One of the components corresponds to the minimum norm solution of the (a priori) expectation values equations. The other component allows for the normalization constraint, and is the projection of a constant onto the Null space of the expectation values transformation.

It has been shown that the process of extremizing $S_{\frac{1}{2}}$, restricted by the given constraints, leads to two stationary points. A general expression for the global maximum solution was provided. Furthermore, we have shown that such a solution gives rise to a predictor of the expectation values which minimizes the distance to the given experimental measurements.

We believe the results of our analysis should be of assistance when trying to decide on the use of the $q = 1/2$ Tsallis distribution in a given particular situation.

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