An iterative regularizing method for an incomplete boundary data problem for the biharmonic equation

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An incomplete boundary data problem for the biharmonic equation is considered, where the displacement is known throughout the boundary of the solution domain whilst the normal derivative and bending moment are specified on only a portion of the boundary. For this inverse ill-posed problem an iterative regularizing method is proposed for the stable data reconstruction on the underspecified boundary part. Convergence is proven by showing that the method can be written as a Landweber-type procedure for an operator formulation of the incomplete data problem. This reformulation renders a stopping rule, the discrepancy principle, for terminating the iterations in the case of noisy data. Uniqueness of a solution to the considered problem is also shown.

1 Introduction

For some higher-order partial differential equations and systems, data on all of the boundary of the solution domain is not enough to give a well-posed problem. A typical situation in applications is when some physical condition is known to hold throughout the boundary of a body but additional data needed can only be measured on a portion due to, for example, a hostile environment. Examples in thermoelasticity where such incomplete data render ill-posed problems are given in [49, 60].

We shall study a problem with incomplete boundary data for the biharmonic equation. Let $D \subset \mathbb{R}^m$, $m \geq 2$, be a doubly-connected domain lying between the two boundary surfaces Γ_1 and Γ_2, with the boundary of D being $\Gamma = \Gamma_1 \cup \Gamma_2$. Each boundary surface is simple (no self-intersections) closed (the surface has no boundary and is connected) and is sufficiently smooth, and Γ_1 lies in the bounded interior of Γ_2. When $m = 2$, the region D is the domain contained between two simple closed non-intersecting curves. For ease of presentation, we restrict the considerations to two-dimensions, $m = 2$. The results and method presented can, with minor changes, be carried over to $m > 2$.

Let u be a solution to the biharmonic equation

$$\Delta^2 u = 0 \quad \text{in } D \tag{1}$$

and suppose additionally that u satisfies the following conditions on the boundary,

$$u = f \quad \text{on } \Gamma, \qquad Nu = g \quad \text{on } \Gamma_2 \qquad \text{and} \qquad Mu = h \quad \text{on } \Gamma_2. \tag{2}$$

The operators are given by ($m = 2$)

$$\begin{aligned} Nu &= \frac{\partial u}{\partial n}, \\ Mu &= \nu \Delta u + (1-\nu)\left(u_{x_1x_1} n_1^2 + 2u_{x_1x_2} n_1 n_2 + u_{x_2x_2} n_2^2\right), \\ Vu &= -\frac{\partial}{\partial n}\Delta u + (1-\nu)\frac{\partial}{\partial s}\left((u_{x_1x_1} - u_{x_2x_2}) n_1 n_2 - u_{x_1x_2}(n_1^2 - n_2^2)\right), \end{aligned} \tag{3}$$

with M in plate theory representing the normal bending moment, V the shear force (both divided by the flexural bending stiffness), $\partial/\partial s$ and $\partial/\partial n$ mean differentiation with respect to the arc length s and the outward unit normal $n = (n_1, n_2)$,

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respectively, and Δu the Laplace operator. We assume, as is typical in applications, that $0 < \nu < 1$ (ν is known as the Poisson ratio). The operator V will be used in a Green's formula later on.

The biharmonic equation has classical applications in modelling bending in plate theory with u being the elastic displacement (sometimes the deflection, the normal component of the displacement) see for example [63, Chapt. 5] and [76, Chapt. 8], and in Stokes flow [51, Chapt. 2.3] (u is the stream function) but it can also model other phenomena such as in radar science [6] (u is the height or topography function). We refer hereafter to u as the displacement.

For problem (1)–(2), we propose an iterative method, which at each iteration step solves well-posed mixed boundary value problems for the biharmonic equation. In 1989, Kozlov and Maz'ya proposed the alternating iterative method [47] for solving some inverse ill-posed problems notably the Cauchy problem for self-adjoint strongly elliptic operators, where data is given on a part of the boundary. For models not being self-adjoint, other iterative methods have been developed, early works are [10,27], and for equations not being strongly elliptic an extension of the alternating method is given in [11]. From a physical point of view, the alternating method is natural in that it updates traces on the boundary with corresponding traces from the previous iteration step. In [9], an iterative method was proposed (also mentioned in [61, 79]) having the similar feature as the alternating method but being of Landweber-type. The method we propose builds on [9].

Ill-posed data completion problems have been studied earlier for the biharmonic equation, see [54, 55, 59, 85] (there are further works in this direction by those authors), but either data is then only given on a part of the boundary or the model is written as a product of two problems for the Laplace equation. The alternating method for (1)–(2) is studied in [58]. For further ill-posed problems for the biharmonic equation, see [77], [36], [21], [82] and the monograph [7]. That the problem we study is ill-posed will follow from a reformulation of it as an operator equation with a compact operator. For a concrete example of the instability of a data completion problem for the biharmonic equation, see [83].

The novelty of the present work is then to extend the method [9], originally given for second-order Cauchy problems, to an incomplete data problem (1)–(2) for a higher order elliptic equation, and to show convergence of this method.

Reconstruction or completion of data for the Laplace equation and other second-order elliptic equations from what is known as Cauchy data (function value and the normal derivative specified) on a part of the boundary is a classical problem. It is near impossible to give adequate overview and references to it. Even narrowing down to iterative methods would be too lengthy. To at least give some references to methods for Cauchy problems both direct and iterative together with applications, see the introduction in [19] and for properties of Cauchy problems [38, Chapt. 3], [4, 18]. For some works on the alternating method, we point to the introduction in [8, 13]. Data completion for higher-order equations is considerably less studied.

For the outline of the present work, in Section 2, we give the iterative method and briefly recall in Proposition 2.1 the well-posedness of the problems involved with a solution being understood in the standard weak sense. In the paragraph following that result is a discussion on the admissible range of the Poisson ratio ν, with references to materials having negative ratio ν. In Section 3 convergence of the procedure is given, see Theorem 3.3. Problem (1)–(2) is formulated as an operator equation on the boundary and the procedure is shown to be a Landweber iteration for this equation. Properties of the operator in this formulation are given such as compactness and injectivity. The compactness reflects the ill-posedness of the problem, the result is stated in Proposition 3.1. That the operator is injective hinge on the uniqueness of a solution to (1)–(2), and this uniqueness is shown as a separate result, see Lemma 3.2. Extensions to simply connected domains and domains with corner singularities and to more general fourth order operators are discussed in Section 4, as well as a possible way to numerically implement the given scheme using integral equations.

2 An iterative method for (1)–(2)

The iterative method for the stable solution to (1)–(2) runs follows:

- Choose an arbitrary initial approximation ζ_0 on the boundary part Γ_1.
- The first approximation u_0 of the solution u is obtained by solving (1) supplied with the boundary conditions

$$u_0 = f \quad \text{on } \Gamma, \qquad Mu_0 = \zeta_0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu_0 = g \quad \text{on } \Gamma_2.$$

- Next, v_0 is constructed by solving (1) with the boundary conditions changed to

$$v_0 = 0 \quad \text{on } \Gamma, \qquad Nv_0 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mv_0 = h - Mu_0 \quad \text{on } \Gamma_2.$$

- Given that u_{k-1} and v_{k-1} are known, the approximation u_k is determined from (1) with

$$u_k = f \quad \text{on } \Gamma, \qquad Mu_k = \zeta_k \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu_k = g \quad \text{on } \Gamma_2.$$

Here,

$$\zeta_k = \zeta_{k-1} + \gamma M v_{k-1}|_{\Gamma_1},$$

where $\gamma > 0$ is a relaxation parameter.

- Then v_k is determined from (1) with boundary conditions

$$v_k = 0 \quad \text{on } \Gamma, \qquad N v_k = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad M v_k = h - M u_k \quad \text{on } \Gamma_2.$$

The iterations continues with the last two steps until a suitable stopping rule has been satisfied. We make precise such a criteria in the next section.

Note that at each iteration, data is updated with data of the same kind from the previous step, for example, the bending moment ζ_{k-1} is updated with a bending moment obtained by solving the problem for v_{k-1}. This is natural from a physical point of view but is not always achieved by other methods, and is thus an additional benefit of the given scheme.

In plate theory, the two conditions imposed on each boundary part in the iterations correspond to a clamped edge (displacement and normal derivative given) or a simply supported edge (displacement and bending moment specified), see further [67, 81] for discussion on boundary conditions for the biharmonic equation and their applications.

Comparing the above scheme with the alternating method, it is similar in the sense that the type of boundary condition alternates on the two boundary parts during the iterations but the alternating method only updates data on the inner boundary Γ_1, see further [9] for explanation of similarities and differences between the methods [47] and [9] in the case of the Cauchy problem for second-order elliptic equations.

2.1 Well-posedness of the direct problems in the procedure

We briefly recall the well-posedness of the problems used in the iterative algorithm. Introduce the bilinear form $a(\cdot,\cdot)$ by

$$a(u,v) = \int_D \left[\nu \Delta u \Delta v + (1-v) \sum_{i,j=1}^{2} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \right] dx. \tag{4}$$

For $0 < \nu < 1$, if $u \in H^2(D)$ is zero on Γ clearly

$$a(u,u) \geq C \|u\|_{H^2(D)},$$

with $H^2(D)$ the Sobolev space of functions having weak and square integrable derivatives up to order two.

One can show, see [12, p. 235], [29, Chapt. IV:2.3.2] or [72, Chapt. 23, p. 270], the following Green type formula

$$a(u,v) = \int_D (\Delta^2 u) v + \int_\Gamma M u N v \, ds + \int_\Gamma (V u) v \, ds, \tag{5}$$

with $v \in H^2(D)$ and the operators in the integrals over the boundary given by (3). The right-hand side in (5) is interpreted in the dual sense meaning that $Mu \in H^{-1/2}(\Gamma)$ and $Vu \in H^{-3/2}(\Gamma)$. Formula (5) follows formally by integrating by parts twice in (4).

It is then a standard routine to set up a weak formulation of each of the problems in the iterative procedure and verify that they are well-posed; details can be found, for example, in [32, Lect. 13], [22, Chapt. 1.2], [72, Chapt. 23] and [81]. We collect the well-posedness in the following.

Proposition 2.1

(i) *Let $f \in H^{3/2}(\Gamma)$, $\zeta \in H^{-1/2}(\Gamma_1)$ and $g \in H^{1/2}(\Gamma_2)$. Then the problem* (1) *supplied with*

$$u = f \quad \text{on } \Gamma, \qquad M u = \zeta \quad \text{on } \Gamma_1 \qquad \text{and} \qquad N u = g \quad \text{on } \Gamma_2$$

has a unique solution depending continuously on the data.

(ii) *Let $f \in H^{3/2}(\Gamma)$ and $\xi \in H^{-1/2}(\Gamma_2)$. Then the problem* (1) *supplied with*

$$v = f \quad \text{on } \Gamma, \qquad N v = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad M v = \xi \quad \text{on } \Gamma_2$$

has a unique solution depending continuously on the data.

Thus, both problems involved in the iterative procedure are well-posed. Note here that we assume that $0 < \nu < 1$. At the end-points it is clear that for $\nu = 0$, uniqueness still holds. For a discussion on uniqueness of a solution when $\nu = 1$, see [17, Theorem 2.2]. It is known that there is a Gårding inequality for the bilinear form $a(\cdot,\cdot)$ in (4) when $-3 < \nu < 1$, see [2], thus well-posedness can be investigated also for negative values of ν although not pursued here (there are materials, termed auxetics and used for example in sport shoes [73] and ski safety devices [5], having a negative Poisson ratio [52]). Well-posedness of various boundary value problems for the biharmonic equation in Lipschitz domains can be found in [65]. For existence and uniqueness of a classical solution to boundary value problems for the biharmonic equation, see, for example, [37, Theorem 2.4.1]. We also note that the various restrictions of solutions to the respective boundary parts are well-defined.

3 Convergence of the proposed procedure

Define an operator $K : H^{-1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_2)$ mapping between the standard Sobolev dual trace spaces $H^{-1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_2)$, by

$$K\zeta = Mu|_{\Gamma_2} \quad \text{for} \quad \zeta \in H^{-1/2}(\Gamma_1), \tag{6}$$

and u satisfies (1) with boundary conditions

$$u = 0 \quad \text{on } \Gamma, \qquad Mu = \zeta \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu = 0 \quad \text{on } \Gamma_2,$$

with the operators N and M given in (3).

We also define $G : H^{3/2}(\Gamma) \times H^{1/2}(\Gamma_2) \to H^{-1/2}(\Gamma_2)$, where

$$G(f,g) = Mw|_{\Gamma_2} \quad \text{for} \quad (f,g) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma_2), \tag{7}$$

and w satisfies (1) with

$$w = f \quad \text{on } \Gamma, \qquad Mw = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nw = g \quad \text{on } \Gamma_2.$$

Both the operators K and G are well-defined and bounded, due to the well-posedness result of Proposition 2.1.

Finding a solution to the Cauchy problem (1)–(2) is then equivalent to solve for an element $\zeta \in H^{-1/2}(\Gamma_1)$ with

$$K\zeta = h - G(f,g), \tag{8}$$

where K and G are defined by (6) and (7), respectively. The operator K is compact; we outline a proof of this. We will need regularity results holding for certain elliptic equations thus we first verify that the biharmonic equation with the given boundary condition fulfils the requirements.

The biharmonic operator is elliptic and since it trivially has real coefficients it is by consequence strongly elliptic. This in turn implies that it is also properly elliptic in the sense of [3]. For the biharmonic operator, this can of course be seen directly since the requested polynomial to be investigated for being properly elliptic is the square of the corresponding one for the Laplace equation, which is properly elliptic, thus having the same roots but with multiplicity two. Hence, that polynomial for the biharmonic operator will have two roots with positive imaginary part (counting multiplicity) as requested for being properly elliptic for a fourth order equation.

The element u in the definition of the operator K, see (6), has boundary conditions satisfying the complementing condition of [3] (more on this condition can be found, for example, in [64, Chapt. VII:52]). In fact, the leading part (highest order derivatives) of each condition imposed on Γ_2 are the same as for what is known as the Navier boundary conditions for the biharmonic equation, and those latter conditions satisfy the complementing condition as shown in [34, Chapt. 2.3].

With the complementing condition satisfied and the biharmonic equation being properly elliptic, local regularity results for such elliptic equations [3] imply that $u \in H^4(D_0)$, which in turn, by the Sobolev embedding theorem [1, Theorem 5.4], gives $u \in C^2(\overline{D}_0)$ for $D_0 \subset D$ being a domain with $\mathrm{dist}(D_0, \Gamma_1) > 0$ (in fact, u will be differentiable to any order in D).

Therefore, the restriction of Mu to Γ_2 has more smoothness than what is required to be an element of $H^{-1/2}(\Gamma_2)$. Thus, the operator K can be decomposed as a bounded mapping first into $H^s(\Gamma_2)$, with some $s > -1/2$, and then into $H^{-1/2}(\Gamma_2)$. Since the embedding of $H^s(\Gamma_2)$ into $H^{-1/2}(\Gamma_2)$ is compact, see [37, Theorem 4.2.2], it follows that K is compact. We have then shown,

Proposition 3.1 *The operator K defined in* (6) *is compact.*

A compact linear operator on an infinite dimensional Hilbert space does not have a bounded inverse. Since (8) is a reformulation of (1)–(2), this latter problem is indeed ill-posed.

We then turn to the kernel of K and show that it consists of the zero element only. For this, we need the following result.

Lemma 3.2 *There can exist at most one solution to the incomplete boundary data problem* (1)–(2).

Proof. By linearity, it is enough to consider the case when u is a solution to (1)–(2) with homogeneous data $f = g = h = 0$. We first show that $\partial^\alpha u = 0$, for $|\alpha| \leq 2$, on Γ_2. With this information, we shall then extend u across Γ_2 and exploit that Δu is a harmonic function vanishing on Γ_2.

Let $x_0 = \left(x_1^{(0)}, x_2^{(0)}\right)$ be a point on Γ_2. We can assume, without loss of generality, that x_2 is the direction of the inward normal at x_0. We can also assume existence of a neighbourhood $\mathcal{U}$ of x_0 with $D \cap \mathcal{U}$ being given by the intersection of $\mathcal{U}$ and the graph domain $x_2 > \alpha(x_1)$, where α is at least twice continuously differentiable and $\alpha'\left(x_1^{(0)}\right) = 0$.

One of the boundary conditions on Γ_2 requires the function u to be zero there. Another boundary condition forces the normal derivative of u to vanish on Γ_2. That condition immediately gives $u_{x_2}(x_0) = 0$, since x_2 is the direction of the inward normal at x_0. Differentiating the relation $u(x_1, \alpha(x_1)) = 0$ and then using $u_{x_2}(x_0) = 0$, give $u_{x_1}(x_0) = 0$. Hence, the derivatives of u up to order one vanish at x_0.

To show that the derivatives of order two are also zero at x_0, we start by differentiating twice the equality $u(x_1, \alpha(x_1)) = 0$. Since both u_{x_1} and u_{x_2} vanish at x_0 along with $\alpha'(x_1^{(0)}) = 0$, the obtained expression at x_0 after the stipulated differentiation is $u_{x_1x_1}(x_0) = 0$. Turning to the condition $Mu = 0$ on Γ_2, using that x_2 is the direction of the inward normal at x_0, which in particular means $n_1 = 0$ and $n_2 = -1$ when $x = x_0$ in the definition (3) of Mu. From this together with $u_{x_1x_1}(x_0) = 0$, we have $u_{x_2x_2}(x_0) = 0$. Finally, the boundary condition $Nu = 0$ on Γ_2 can locally be written

$$u_{x_1}(x_1, \alpha(x_1))\alpha_1'(x_1) - u_{x_2}(x_1, \alpha(x_1)) = 0.$$

Here, the components of the normal has been expressed as $n_1(x_1) = \alpha'(x_1)$ and $n_2(x_1) = -1$, and the normalizing factor for the unit normal n has been divided out. Differentiating the above local equation for $Nu = 0$ with respect to x_1, that is along Γ_2, gives $u_{x_2x_1}(x_0) = 0$. We have then shown that also all derivatives of order two of u vanish at x_0.

Thus, since x_0 was arbitrary on the boundary part Γ_2, all derivatives of u up to and including order two are vanishing on Γ_2.

Since u and its normal derivative vanish on Γ_2, it is possible to locally extend u across Γ_2 to a biharmonic function in D_1 with D_1 containing a part of D and Γ_2; explicit constructions of extensions are given in [69, Sect. 3], [28, Theorem 1] and [80, Theorem 1]. The function Δu is a harmonic in D_1. A harmonic function is real-analytic in the interior of its domain of definition [45, p. 98]. The harmonic function Δu vanish along a curve in the interior D_1 since, as shown, $u_{x_1x_1}$ and $u_{x_2x_2}$ both vanish along Γ_2. The zeros to a non-trivial real-analytic function in the plane are isolated, therefore, Δu is identically zero in D_1. We then again have a harmonic function, this time u, vanishing along a curve in the interior of D_1 ($u = 0$ on Γ_2). Thus, u is zero in D_1. It follows that u is identically zero throughout the domain D, and uniqueness of a solution to (1)–(2) has been established. □

We note that the this type of proof is used in [43, Chapt. 1.4.2], where uniqueness is shown for a Cauchy problem for the second-order elliptic Stokes system. Rather than working with classical solutions to (1), strong solutions in $H^4(D)$ can be assumed since the Sobolev embedding theorem [1, Theorem 5.4] implies that u then has classical derivatives up to order two throughout D, and the above proof can be carried out similarly. We point out that there are strong unique continuation results for the biharmonic equation, see [68, Theorem 4] and [70, Sect. 4]. Strong unique continuation results for higher-order operators (including the biharmonic equation) can be found in [23, 24, 53, 56]; in [53, Sect. 1.2] is an overview of results and history as well as references to some counter examples to strong unique continuation for higher order operators.

Returning to the kernel of K, assume that $K\zeta = 0$. Then, from the definition of the operator K, there is a solution u to (1) having zero data (2). According to Lemma 3.2, there is a unique solution (1)–(2), thus u is identically zero in $\overline{D}$. Hence, $\zeta = 0$ and the kernel of K is trivial.

We then define an auxiliary operator T mapping in the opposite direction compared with the operator K. Let $T : H^{-1/2}(\Gamma_2) \to H^{-1/2}(\Gamma_1)$, where

$$T\psi = Mv|_{\Gamma_1} \quad \text{for} \quad \psi \in H^{-1/2}(\Gamma_2), \tag{9}$$

and v satisfies (1) with boundary conditions

$$v = 0 \quad \text{on } \Gamma, \qquad Nv = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mv = \psi \quad \text{on } \Gamma_2.$$

The operator T is also well-defined and bounded. The kernel of T is trivial, this follows along the similar lines as shown for the operator K.

Let u_k be the iterates obtained from the proposed algorithm. We then have

$$\begin{aligned}\zeta_k &= Mu_{k-1}|_{\Gamma_1} + \gamma\, Mv_{k-1}|_{\Gamma_1} = \zeta_{k-1} + \gamma\, T\,(h - Mu_{k-1}|_{\Gamma_2}) \\ &= \zeta_{k-1} + \gamma\, T\,(h - G(f,g) - K\zeta_{k-1}).\end{aligned} \tag{10}$$

This is the extension of the Landweber method for solving equation (8) given in [79]; in the classical Landweber method the operator T is equal to the adjoint of K, that is $T = K^*$. Given the above properties of K and T with TK being positive, convergence follows from [79, Theorem 2] (the assumption that TK is positive is needed but not explicitly mentioned in that work). We then have to show that TK is indeed positive, or alternatively to show that T is equal to K^*. We find the adjoint of K. Then TK will be a positive self-adjoint operator and it follows from the properties of T and K that TK has a dense range [62, Theorem 2.26].

3.1 The adjoint of the operator K

Let us then show that the adjoint of K is equal to T, with K and T given by (6) and (7), respectively. We shall need a certain inner product in each of the boundary spaces to be used.

Following [47, 48], the required inner product on $H^{-1/2}(\Gamma_1)$ is defined by

$$(\zeta, \xi) = a(u, v), \tag{11}$$

where the bilinear form $a(\cdot,\cdot)$ is given by (4) and u satisfies (1) with boundary conditions

$$u = 0 \quad \text{on } \Gamma, \qquad Mu = \zeta \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu = 0 \quad \text{on } \Gamma_2,$$

and similarly v satisfies (1) with boundary conditions

$$v = 0 \quad \text{on } \Gamma, \qquad Mv = \xi \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nv = 0 \quad \text{on } \Gamma_2.$$

Here, ζ and ξ belong to $H^{-1/2}(\Gamma_1)$, and thus the (weak) solutions belong to $H^2(D)$. The reader can check, using properties of the bilinear form stated above together with Proposition 2.1, that it is a well-defined inner product on $H^{-1/2}(\Gamma_1)$.

Similarly, an inner product on $H^{-1/2}(\Gamma_2)$ is defined by

$$(\psi, \chi) = a(w, z), \tag{12}$$

with $a(\cdot,\cdot)$ from (4), and where w satisfies (1) with boundary conditions

$$w = 0 \quad \text{on } \Gamma, \qquad Nw = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mw = \psi \quad \text{on } \Gamma_2,$$

and similarly z satisfies (1) with boundary conditions

$$z = 0 \quad \text{on } \Gamma, \qquad Nz = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mz = \chi \quad \text{on } \Gamma_2.$$

Here, ψ and χ belong to $H^{-1/2}(\Gamma_2)$, and the (weak) solutions therefore belong to the space $H^2(D)$. The reader can check that also (12) is a well-defined inner product.

From (5), we have the identity

$$a(u, v) = \int_\Gamma MuNv\, ds, \tag{13}$$

valid for $u, v \in H^2(D)$ with u being a weak solution of (1) and $v = 0$ on Γ.

We shall then need a number of auxilary solutions to the biharmonic equation in order to evaluate and manipulate the inner product $(K\zeta, \psi)$ into $(\zeta, T\psi)$, to establish that T is the adjoint of K.

Let u_0 satisfy (1) with boundary conditions

$$u_0 = 0 \quad \text{on } \Gamma, \qquad Mu_0 = \zeta \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu_0 = 0 \quad \text{on } \Gamma_2,$$

and let u_1 satisfy (1) with

$$u_1 = 0 \quad \text{on } \Gamma, \qquad Nu_1 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mu_1 = Mu_0 \quad \text{on } \Gamma_2.$$

Moreover, the element u_2 then satisfies the same type of problem as u_0 but with $Mu_2 = Mu_1$ on Γ_1. The element w_1 satisfies (1) with boundary conditions

$$w_1 = 0 \quad \text{on } \Gamma, \qquad Nw_1 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Mw_1 = \psi \quad \text{on } \Gamma_2.$$

We also need w_2 being a solution to (1) with

$$w_2 = 0 \quad \text{on } \Gamma, \qquad Mw_2 = Mw_1 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nw_2 = 0 \quad \text{on } \Gamma_2.$$

By definition, $(K\zeta, \psi) = a(u_1, w_1)$, and then using (13), we have

$$(K\zeta, \psi) = \int_{\Gamma_2} Mu_1 Nw_1 \, ds. \tag{14}$$

From this, employing the normal bending moment boundary condition for u_1 on Γ_2 together with (13),

$$\int_{\Gamma_2} Mu_1 Nw_1 \, ds = \int_{\Gamma_2} Mu_0 Nw_1 \, ds = a(u_0, w_1). \tag{15}$$

Combining (14) and (15),

$$(K\zeta, \psi) = a(u_0, w_1). \tag{16}$$

Similarly, by definition, $(\zeta, T\psi) = a(u_0, w_2)$, and using the normal bending moment boundary condition for w_2 on Γ_1 together with (13) and the symmetry of the bilinear form $a(\cdot, \cdot)$,

$$a(u_0, w_2) = \int_{\Gamma_1} Mw_2 Nu_0 \, ds = \int_{\Gamma_1} Mw_1 Nu_0 \, ds = a(u_0, w_1). \tag{17}$$

Thus, from (17)

$$(\zeta, T\psi) = a(u_0, w_1). \tag{18}$$

Comparing (16) and (18), we have

$$(K\zeta, \psi) = (\zeta, T\psi),$$

with $\zeta \in H^{-1/2}(\Gamma_1)$ and $\psi \in H^{-1/2}(\Gamma_2)$ arbitrary. Hence, $T = K^*$.

3.2 Convergence of the iterative procedure

In (10) is shown that the proposed method can be rewritten as a Landweber type iteration for an operator reformulation, (8), of the Cauchy problem (1)–(2). The operators K and T, defined in (6) and (9) respectively, have been shown to satisfy the convergence criteria (K is bounded and injective with T being the adjoint) for the Landweber method, and therefore we have the following convergence of the proposed iterative procedure.

Theorem 3.3 *Let $f \in H^{3/2}(\Gamma_2)$, $g \in H^{1/2}(\Gamma_2)$ and $h \in H^{-1/2}(\Gamma_2)$. Assume that the Cauchy problem* (1)–(2) *has a solution u. Let the regularizing parameter γ satisfy $0 < \gamma < 2/(\|T\|\|K\|)$, and let u_k be the k–th approximation in the given algorithm. Then*

$$\lim_{k\to\infty} \|u - u_k\|_{H^2(D)} = 0$$

for any initial function $\zeta_0 \in H^{-1/2}(\Gamma_1)$.

Convergence of higher derivatives in the interior of D can also be achieved. Indeed, let D' be a domain satisfying $\overline{D'} \subset D$. Using the local regularity results [3], discussed in the two paragraphs preceding Proposition 3.1, imply ($\ell > 0$)

$$\|u_k - u\|_{H^{\ell+2}(D')} \le C\|u_k - u\|_{H^2(D)}.$$

This estimate and Theorem 3.3 show convergence of higher derivatives in D.

The discrepancy principle [66] can be applied as a stopping rule. Assume that we have noisy Cauchy data f^δ, g^δ and h^δ such that

$$\|h - h^\delta\|_{H^{-1/2}(\Gamma_2)} + \|G(f - f^\delta, g - g^\delta)\|_{H^{3/2}(\Gamma)\times H^{1/2}(\Gamma_2)} \le \delta.$$

Let u_k^δ be the iterates with the noisy data. The iterations should be terminated when

$$\|h^\delta - u_k^\delta\|_{H^{-1/2}(\Gamma_2)} \le \tau\delta,$$

where $\tau > 1$.

The regularizing parameter γ can be chosen as $\gamma = 1$. This was shown for the similar procedure given for second-order elliptic equations in [9]. Starting from

$$(TK\zeta, TK\zeta) = a(u_2, u_2)$$

and then following the steps for the similar result in [9], we obtain

$$(TK\zeta, TK\zeta) \le a(u_0, u_0) = \|\zeta\|^2, \tag{19}$$

with $\|\cdot\|$ the norm induced by the inner product (11), and u_0 and u_2 as in the previous section. The norm of the operator TK is therefore by (19) less than or equal to one. The regularizing parameter can thus be chosen as $\gamma = 1$ according to Theorem 3.3.

4 Extensions and implementation of the procedure

The proposed iterative method for (1)–(2) and accompanying analysis carry over to simply-connected domains. The analysis is slightly more technical since spaces of the form $H_{00}^{k+1/2}(\Gamma_i)$, with $i = 1, 2$ and k a non-negative integer, has to be used for the boundary data along with the corresponding dual spaces, to make the higher-order traces like Mu well-defined on the two boundary parts. Data in such spaces is used in [47, 48] for the alternating method; properties of those spaces can be found for example in [57, Chapt. 1:11.5 and 1:12.1]. If the solution domain is smooth apart from a number of corner points, weighted Sobolev spaces can be included as done for second-order elliptic equations in [44].

The procedure and analysis are also applicable to more general fourth order equations, such as the anisotropic equation for plate bending,

$$\frac{\partial^2}{\partial x_k x_l}\Big(a_{ijkl}(x)\frac{\partial^2 u}{\partial x_i x_j}\Big) = 0 \quad \text{in } D,$$

with $a_{ijkl}(x) = a_{klij}(x) = a_{ijlk}(x) = a_{jilk}(x)$, and with corresponding changes of the boundary operators (3). Important for the analysis of the procedure is that the bilinear form generates an inner product in the respective boundary spaces as in Section 3.1, and that local regularity for elliptic equations can be applied. For properties of a fourth-order equation of the above form, see [14]. Generating analytic solutions and further discussion on boundary conditions for the biharmonic equation is presented in [31].

For the implementation of the proposed procedure, any of the standard numerical methods for partial differential equations can in principle be employed. However, meshless or integral equation based methods have the advantage of reducing the dimension since only data on the boundary is needed. The boundary element method is applied to inverse problems for the biharmonic equation in [54, 85] and meshless methods in [55, 59]. Note that other regularization strategies than iterative can be applied to (8) such as Tikhonov regularization, for its numerical realisation, see [30], [15, Chapt. 2.7] and [35, Chapt. 5].

In [16, 20], it is shown how to obtain numerical approximations for mixed problems for the Laplace equation in two respectively three dimensions using an anzats for the solution as a combination of the classical single- and double-layer potentials with discretisation strategies from [50, 84], and this is used to implement the alternating method. The similar approach and ansatz can be employed for boundary problems for (1). We shall not go into details but recall the expressions for the single- and double-layer potentials for the biharmonic equation.

The fundamental solution for the biharmonic equation in $\mathbb{R}^2$ is

$$G(x, y) = \frac{1}{8\pi}|x-y|^2 \ln|x-y|,$$

satisfying

$$\Delta_x^2 G(x, y) = \delta(x-y), \quad \text{in } \mathbb{R}^2,$$

with δ the Dirac delta function. The elements

$$u(x) = \int_\Gamma \Big(G(x, y)\varphi(y) + (N_y G(x, y))\,\psi(y)\Big)\,ds(y), \quad x \in D, \tag{20}$$

and

$$v(x) = \int_\Gamma \Big(\,(M_y G(x, y))\,\varphi(y) + (V_y G(x, y))\,\psi(y)\Big)\,ds(y), \quad x \in D, \tag{21}$$

are the (interior) single- and double-layer potentials, respectively (expressions are well-defined for all $x \in \mathbb{R}^2 \setminus \Gamma$), with operators as in (3) and sufficiently smooth densities φ and ψ. For properties of the single- and double-layer potentials (20) and (21) and integral equations for the biharmonic equation, see [37, Chapt. 2.4 and Chapt. 10.4.4] and [17, 25, 26, 33, 39–42, 46, 71, 74, 75, 78].

5 Conclusion

A stable iterative procedure has been proposed and investigated for an ill-posed incomplete boundary data problem for the biharmonic equation. The displacement is given on all of the boundary whilst the normal derivative and normal bending moment are only specified on a portion with the remaining part of the boundary being underspecified in terms of data. The method presented builds on [9] and updates data with data of the same type throughout the iterations. Convergence was established by rewriting the inverse ill-posed problem as an operator equation on the boundary, for which the method can be written as a Landweber type procedure for a compact operator. Uniqueness of a solution to the incomplete data problem was established using strong unique continuation results for the biharmonic equation. A way of numerically realising the procedure based on layer-potentials and integral equations is also given.

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