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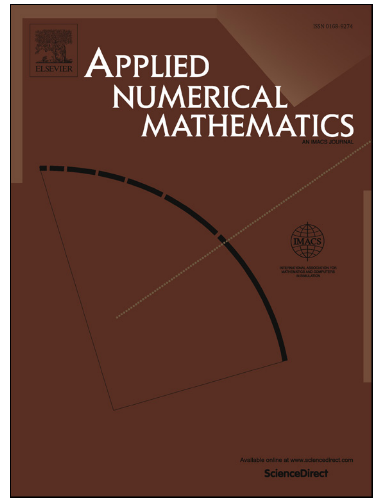

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A boundary integral equation method for numerical solution of parabolic and hyperbolic Cauchy problems

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Abstract

We present a unified boundary integral approach for the stable numerical solution of the ill-posed Cauchy problem for the heat and wave equation. The method is based on a transformation in time (semi-discretisation) using either the method of Rothe or the Laguerre transform, to generate a Cauchy problem for a sequence of inhomogenous elliptic equations; the total entity of sequences is termed an elliptic system. For this stationary system, following a recent integral approach for the Cauchy problem for the Laplace equation, the solution is represented as a sequence of single-layer potentials invoking what is known as a fundamental sequence of the elliptic system thereby avoiding the use of volume potentials and domain discretisation. Matching the given data, a system of boundary integral equations is obtained for finding a sequence of layer densities. Full discretisation is obtained via a Nyström method together with the use of Tikhonov regularization for the obtained linear systems. Numerical results are included both for the heat and wave equation confirming the practical usefulness, in terms of accuracy and resourceful use of computational effort, of the proposed approach.

Keywords: Cauchy parabolic and hyperbolic problems, System of elliptic equations, Single layer potentials, Boundary integral equations, Nyström method, Tikhonov regularization.
2010 MSC: 35R25, 35J05, 65R20

1. Introduction

In [10], an overview is given of a single-layer approach for the stable numerical solution to the classical Cauchy problem for the Laplace equation (for both two and three dimensional regions), that is for the reconstruction of a harmonic function from knowledge of its function value and normal derivative on a part of the boundary of a domain. Using the representation of the solution as a layer potential, by matching the given Cauchy data, a system of boundary integral equations is obtained for the determination of the layer density distribution over the boundary. Since only boundary integrals are involved, the dimension is reduced compared to domain discretisation methods; this becomes particularly advantageous when simulating complex phenomena for the Laplace operator, a recent example is in the field of chemistry in designing new materials at nano-scale [18].

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The single-layer approach surveyed in [10], and which follows naturally from ideas in [29] and [6], is flexible and can be applied for various solution domains, both in two and three dimensions, including unbounded regions, as well as to situations where cracks are present. For the discretization, a Nyström type method is used for planar domains and in 3-dimensions a method in the same spirit is applied based on [40] (does not involve discretisation of the boundary surface) together with Tikhonov regularization for the obtained linear system.

The Cauchy problem (ill-posed) can also be formulated for the time-dependent heat equation, and it also makes sense for the wave equation. By the Cauchy problem for time-dependent models, we mean that data in the form of function values and the normal derivative is given on a lateral part of the cylindrical space-time domain. It should not be confused with the classical well-posed Cauchy initial value problems, where a set of data is given at the bottom of the cylindrical domain. We ask the reader to bear in mind that we always mean the lateral Cauchy problem but will rarely write out the word lateral.

Since the (lateral) Cauchy problem makes sense for the heat and wave equation, it is then natural to extend the boundary integral equation approach outlined in [10] to cover these equations, and this extension is the investigation of the present paper.

The generalization we present has the clear advantage that it will work with virtually no change for both the heat and wave equation. To achieve this, we shall make use of the works [14, 15]. There, the method of Rothe respectively the Laguerre transform is employed for time-dependent boundary value problems to obtain a sequence of inhomogeneous elliptic equations. Instead of applying the traditional and computationally expensive domain discretisation approach of using a volume potential to solve these inhomogeneous boundary value problems, in [14, 15] fundamental sequences are derived such that the solution of the inhomogeneous sequence of problems can be represented in terms of boundary integrals, thus avoiding domain discretisation. We point out that we use the term system and sequence interchangeably for the total entity of inhomogeneous elliptic problems.

In the unified approach we present, our time-dependent Cauchy problems are reduced, using a suitable discretisation in time (either via the Rothe or Laguerre transform), to obtain a Cauchy problem for an elliptic system. Using the fundamental sequence for this system given in [14] or [15] (depending on the employed transformation in time), we derive a single-layer approach for this stationary Cauchy problem leading to a system of boundary integral equations to determine a sequence of layer densities. Invoking Tikhonov regularization for solving this system, assuming the data to be square integrable, a stable solution is obtained.

Utilizing the semi-discretisation in time, we can then generate a numerical approximation to the original time-dependent Cauchy problem for both the heat and wave equation. Numerical experiments included confirm the stability and feasibility of this strategy.

It is important to remark here that the use of our semi-discretization approaches in time for the heat and wave problems lead to Cauchy problems for the similar elliptic systems. By similar is understood that the difference is only in the choice of (constant) coefficients in the obtained elliptic equations. If instead time-dependent integral equations are applied as a mean of solving the above Cauchy problems, then these integrals are principally different for the heat and wave equation, and thus have to be treated differently in terms of discretisation and implementation. For the use of heat potentials for the parabolic Cauchy problem we refer to [20].

Before we give the outline of this work, we mention that there is an extensive literature on regularizing methods for the Cauchy problem and we do not attempt to give a general list of references. We only state a few related integral approaches. Methods for Cauchy problems based on the boundary element method (BEM) have been developed (mainly for bounded planar regions), see for example, [31, 32] (there are plenty more works by these authors, also now in the direction of meshless methods [25]). Boundary integrals of the kind considered in the present work can be used to solve direct mixed problems and such problems can be used iteratively to obtain a solution to the Cauchy problem, see [8] and [22], and references therein. An analysis of a layer approach for the Cauchy problem for the Helmholtz equation was recently presented in [5]. The reader can get a glimpse of other works and results by consulting, for example, [23, 24, 7, 4, 9, 13].

We point out that the considered integral equation approach can be applied in the case of a homogeneous linear differential equation (or a system) of parabolic or hyperbolic type with constant coefficients and for inhomogeneous materials. In the latter case, we need to know the corresponding Green's matrix, see [12]. There are also iterative methods for non-linear inverse problems related to the reconstruction of a portion of the boundary of the solution domain, which lead to the solution of a set of direct time-dependent problems at each iteration step. The suggested integral equation approach is then suitable to apply to such a problem also [16].

For the outline, in Section 2, we show how to discretize our Cauchy problem with respect to time using either the method of Rothe or the Laguerre transform. In Section 3, we present the layer approach for the stationary elliptic system obtained after semi-discretisation with respect to time. The system obtained by matching the layer ansatz against the given Cauchy data is analyzed in the space of square integrable functions, see Theorem 3.3. Full discretization involving the Nyström method and Tikhonov regularization for bounded planar domains is outlined in Section 5. In the final section, Section 6, numerical simulations are given both for the heat and wave equation.

2. Semi-discretization with respect to time

We assume that we have a two-dimensional physical body modelled as a doubly connected domain D in $\mathbb{R}^2$. This domain is such that it has two simple closed boundary curves Γ_1 and Γ_2, with Γ_1 lying in the interior of Γ_2. From a theoretical point of view, there is no hindrance in considering 3-dimensional domains. However, to work out the details and handling the various singularities in the kernels of the integral equations we derive, we focus on the planar case. To extend it to higher dimensions one can build on what is presented here together with results from [11].

2.1. Lateral Cauchy heat problem and its time-discretisation

We first formulate the ill-posed Cauchy problem for the heat equation. The physical body is then conducting and the temperature together with flux measurements are taken on the external boundary curve (given data functions f_2 respectively g_2). As a mathematical model, the bounded function u is a solution to

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial u}{\partial t} = \Delta u & \text{in } D \times (0,T), \\ u = f_2 & \text{on } \Gamma_2 \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = g_2 & \text{on } \Gamma_2 \times (0,T), \\ u(\cdot,0) = 0 & \text{in } D, \end{cases} \tag{1}$$

where the functions f_2 and g_2 are given and sufficiently smooth, ν is the unit outward normal to the boundary, and the constant thermal diffusivity $c > 0$ and final time $T > 0$ are also given. We assume that data are compatible such that there exists a solution; the solution is unique according to [39] but does not in general depend continuously on the data. Note that for uniqueness, the initial condition is irrelevant, see for example [23, Theorem 3.3.10]. Hence, of particular interest is to find the Cauchy data on the boundary part Γ_1.

We then outline two strategies for reducing the above transient Cauchy problem to a sequence of stationary ones.

A) *Rothe's method.* The time derivative in the heat equation (1) is discretized by a finite difference approximation. Thus, on the equidistant mesh

$$\{t_p = (p+1)h_t,\ \ p = -1, \dots, N_t - 1,\ \ h_t = T/N_t,\ \ N_t \in \mathbb{N}\}$$

we approximate the solution u to (1) by the sequence $u_p \approx u(\cdot, t_p)$, $p = 0, \dots, N-1$; the elements of this sequence satisfy the equations

$$\Delta u_p - \gamma^2 u_p = -\gamma^2 u_{p-1}, \quad u_{-1} = 0, \quad \gamma^2 = \frac{1}{ch_t}$$

with boundary conditions

$$u_p = f_{2,p}, \quad \frac{\partial u_p}{\partial \nu} = g_{2,p}, \quad \text{on } \Gamma_2,$$

where $f_{2,p} = f_2(\cdot, t_p)$ and $g_{2,p} = g_2(\cdot, t_p)$. Considering u_{p-1} as known, then the above equation for u_p is a Cauchy problem for a Helmholtz type operator; this is also an ill-posed problem, see, for example, [4]

It is possible to use higher order approximations for the time-derivative. To obtain a scheme using a second order approximation in time, we assume that u is two-times continuously differentiable with respect to the time variable. Then the following differences can be derived,

$$\frac{\partial u}{\partial t}(x, t_p) = \frac{u(x,t_p) - u(x,t_{p-1})}{h_t} + \frac{h_t}{2} u_{tt}(x, t_p - \theta_p h_t), \quad \theta_p \in [0,1]\,,$$
$$\frac{\partial u}{\partial t}(x, t_{p-1}) = \frac{u(x,t_p) - u(x,t_{p-1})}{h_t} - \frac{h_t}{2} u_{tt}(x, t_p - \tilde{\theta}_p h_t), \quad \tilde{\theta}_p \in [0,1]\,.$$

Substituting these relationships in (1) and adding the obtained equations, we get

$$\frac{2}{ch_t} u_p - \frac{2}{ch_t} u_{p-1} - \Delta u_p - \Delta u_{p-1} + O(h_t^2) = 0.$$

From this, taking into account that $u_{-1} = 0$ and by using induction, we then obtain the approximation

$$\Delta u_p - \gamma^2 u_p = \sum_{m=0}^{p-1} \beta_{p-m} u_m, \quad p = 0, \dots, N_t - 1, \quad \gamma^2 = \frac{2}{ch_t}, \quad \beta_i = (-1)^i \frac{4}{ch_t}, \tag{2}$$

for $i = 1, \dots, N_t - 1$.

We point out that the method of Rothe [38] is nowadays a standard approach for reducing well-posed time-dependent initial-boundary value problems. General accounts on this method are, for example, [30, Chapt. 3, Sect. 16] and [26, 37], where in particular convergence results can be found in various function spaces.

B) *Laguerre transform.* In this approach, we search for the solution u of (1) as the (scaled) Fourier expansion with respect to the Laguerre polynomials, that is an expansion of the form

$$u(x,t) = \kappa \sum_{p=0}^{\infty} \tilde{u}_p(x) L_p(\kappa t), \tag{3}$$

where

$$\tilde{u}_p(x) := \int_0^\infty e^{-\kappa t} L_p(\kappa t) u(x,t)\, dt, \quad p = 0, 1, 2, \ldots. \tag{4}$$

Here, L_p is the Laguerre polynomial of order p, and $\kappa > 0$.

For the Fourier–Laguerre coefficients $\tilde{u}_p(x)$, by using the recurrence relations for the Laguerre polynomials, it can be shown (see [14]) that these satisfy the following sequence of Cauchy problems,

$$\Delta \tilde{u}_p - \beta \tilde{u}_p = \beta \sum_{m=0}^{p-1} \tilde{u}_m \quad \text{in } D \tag{5}$$

with boundary conditions

$$\tilde{u}_p = \tilde{f}_{2,p} \quad \text{on } \Gamma_2 \quad \text{and} \quad \frac{\partial u_p}{\partial \nu} = \tilde{g}_{2,p} \quad \text{on } \Gamma_2. \tag{6}$$

Here,

$$\tilde{f}_{2,p}(x) := \int_0^\infty e^{-\kappa t} L_p(\kappa t) f_2(x,t)\, dt, \quad p = 0, 1, 2, \ldots$$

and

$$\tilde{g}_{2,p}(x) := \int_0^\infty e^{-\kappa t} L_p(\kappa t) g_2(x,t)\, dt, \quad p = 0, 1, 2, \ldots$$

are the Fourier–Laguerre coefficients of the given boundary values and $\beta = \kappa/c$. The numerical approximation to the solution of the Cauchy problem (1) is a partial sum of the series (4). Therefore, we have $p = 0, \ldots, N$, with $N \in \mathbb{N}$, in (5)–(6).

2.2. Lateral Cauchy problem for the wave equation and time-discretisation

The ill-posed Cauchy problem for the wave equation (with speed of sound $a > 0$) is to search for a solution u such that

$$\begin{cases} \dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial t^2} = \Delta u & \text{in } D \times (0,T), \\ u = f_2 & \text{on } \Gamma_2 \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = g_2 & \text{on } \Gamma_2 \times (0,T), \\ \dfrac{\partial u}{\partial t}(\cdot,0) = u(\cdot,0) = 0 & \text{in } D. \end{cases} \tag{7}$$

This type of Cauchy problem is considerable less studied than the corresponding parabolic one. Suitable Sobolev spaces for this Cauchy problem is given in [3, Sect. 3.1]. Uniqueness of a solution (in case it exists) holds, see [2, p. 888]. It is an ill-posed problem, that is stability of a solution can not be guaranteed. An example highlighting the instability is presented in [27, Sect. 1]. In terms of applications, reconstructions from lateral Cauchy data for the wave equation occur, for example, in the imaging method of thermoacoustic tomography, see [17].

As for the parabolic problem, both the method of Rothe and the Laguerre transform can be applied to reduce the problem to a sequence of stationary ones. We briefly give some details.

For the method of Rothe applied to (7), we use the following finite difference approximation (the mesh points t_p are as in the previous section)

$$\frac{\partial^2 u}{\partial t^2}\,(x, t_p) \approx \frac{u(x,t_p) - 2u(x,t_{p-1}) + u(x,t_{p-2})}{h_t^2}, \quad p = 0, \ldots, N_t - 2.$$

This leads then to the following sequence of elliptic equations to be solved for $u_p(x) \approx u(x, t_p)$,

$$\Delta u_p - \gamma^2 u_p = \alpha_0 u_{p-1} + \alpha_1 u_{p-2} \quad \text{in } D$$

with the constants $\gamma^2 = \frac{1}{a^2 h_t^2}$, $\alpha_0 = -\frac{2}{a^2 h_t^2}$ and $\alpha_1 = \frac{1}{a^2 h_t^2}$.

If we instead apply the above Laguerre transform to the wave equation in (7), we get a sequence for the Fourier-Laguerre coefficients,

$$\Delta \tilde{u}_p - \gamma^2 \tilde{u}_p = \sum_{m=0}^{p-1} \beta_{n-m} \tilde{u}_m \quad \text{in } D$$

with $\beta_p = \frac{\kappa^2}{a^2}(p+1)$, $p = 0, 1, 2, \ldots$, and $\gamma^2 = \beta_0$.

2.3. An elliptic system for the lateral Cauchy heat and wave problems

Interestingly, the various semi-discretisation approaches described in the previous two sections for lateral Cauchy problems for the heat and wave equation, all lead to a stationary problem that can be written into the following form,

$$\Delta u_p - \gamma^2 u_p = \sum_{m=0}^{p-1} \beta_{p-m} u_m \quad \text{in } D, \tag{8}$$

$$u_p = f_{2,p} \quad \text{on } \Gamma_2, \qquad \frac{\partial u_p}{\partial \nu} = g_{2,p} \quad \text{on } \Gamma_2, \tag{9}$$

with given functions $f_{2,p}$ and $g_{2,p}$, $p = 0, \ldots, N$, $N \in \mathbb{N}$, $\gamma^2 = \beta_0$ and the explicit expressions for the (known) constants β_i depend on the type of employed semi-discretization (Rothe or Laguerre) together with the type of underlying partial differential equation (heat or wave equation).

The Cauchy problem (7) can also be considered in a weak formulation. Then the Laguerre transform again leads to a sequence of stationary problems in the corresponding Sobolev spaces (see [35]). This can potentially be advantageous for more complicated discontinuous material parameters and solutions. Since we consider a classical solution, the finite difference method is a suitable classical way for the semi-discretization.

3. Boundary integral equation method for the system (8)–(9)

The structure of the Cauchy problem (8)–(9) is different compared with the standard Laplace equation, however, as we shall show it is still possible to do as in [10], that is to reduce the Cauchy problem to boundary integral equations. The standard way of handling a non-homogeneous equation like (8) is to use a volume potential (leading to computationally expensive domain discretisation). To avoid this and only use boundary integrals, the existence of an explicit fundamental solution is needed. In the present situation of a system of equations, we have to clarify what is meant by a fundamental solution and we then undertake some work to explicitly construct it.

3.1. A fundamental sequence to (8)

We start with the definition and then go ahead and construct the corresponding fundamental solution (a similar definition was employed in [14, 15], where well-posed boundary values problems for equations of the form (8) are studied).

Definition 3.1. *The sequence of functions* $\{\Phi_p\}_{p=0}^N$ *is denoted a fundamental sequence for the equations* (8) *provided that*

$$\Delta_x \Phi_p(x,y) - \gamma^2 \Phi_p(x,y) - \sum_{m=0}^{p-1} \beta_{p-m} \Phi_m(x,y) = \delta(x-y), \tag{10}$$

where δ *is the Dirac delta function.*

This definition makes sense in that writing (8) as a system for a vectorial solution, the reader can check that the above definition is then precisely the standard definition of a fundamental solution for systems.

For the existence of a fundamental sequence, we can rely on abstract existence results of a fundamental solution to (strongly) elliptic systems [33, pp. 197–200]. However, we shall in fact explicitly give a fundamental sequence.

To give such a sequence, we need the modified Bessel functions

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{z}{2}\right)^{2k}, \quad \text{and} \quad I_1(z) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1}, \tag{11}$$

of order zero and one, respectively, and also the modified Hankel functions

$$\begin{aligned} K_0(z) &= -\left(\ln\frac{z}{2} + C\right) I_0(z) + \sum_{k=1}^{\infty} \frac{\psi(k)}{(k!)^2} \left(\frac{z}{2}\right)^{2k}, \\ K_1(z) &= \frac{1}{z} + \left(\ln\frac{z}{2} + C\right) I_1(z) - \frac{1}{2} \sum_{k=0}^{\infty} \frac{\psi(k+1)+\psi(k)}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1} \end{aligned} \tag{12}$$

of order zero and one, respectively. Here, $\psi(0) = 0$,

$$\psi(k) = \sum_{m=1}^{k} \frac{1}{m}, \quad k = 1, 2, \ldots,$$

and $C = 0.57721\ldots$ denotes the Euler constant. The functions I_ℓ and K_ℓ are (independent) solutions to the modified Bessel differential equation with parameter ℓ, for $\ell = 0, 1$; more on these functions can be found in [1, Chapter 9.6]) (other names attached to the modified Hankel function are Bessel function of the third kind, Basset's function and Macdonald's function).

The fundamental solution to the modified Helmholtz equation (8) with zero right-hand side is given by $\frac{1}{2\pi} K_0(\gamma|x-y|)$, see [36, Chapter 7.3.2]. It seems natural then to build on this function in order to generate a fundamental sequence. We need the polynomials v_p and w_p defined by

$$v_p(r) = \sum_{m=0}^{\left[\frac{p}{2}\right]} a_{p,2m} r^{2m} \quad \text{and} \quad w_p(r) = \sum_{m=0}^{\left[\frac{p-1}{2}\right]} a_{p,2m+1} r^{2m+1}, \tag{13}$$

respectively, for $p = 0, 1, \ldots, N$, with the convention that $w_0 = 0$, and $[q]$ is the largest integer not greater than q. The coefficients in (13) are generated by putting $a_{p,0} = 1$ for $p = 0, \ldots, N$, and then using the two recurrence relations,

$$a_{p,p} = -\frac{1}{2\gamma p} \beta_1 a_{p-1,p-1} \tag{14}$$

and

$$a_{p,k} = \frac{1}{2\gamma k}\left\{4\left[\frac{k+1}{2}\right]^2 a_{p,k+1} - \sum_{m=k-1}^{p-1} \beta_{p-m} a_{m,k-1}\right\}, \quad k = p-1, \ldots, 1, \tag{15}$$

for $p = 1, \ldots, N$.

Straightforward calculations show that these polynomials satisfy the following two sequences of (coupled) ordinary differential equations

$$\begin{aligned} v_p''(r) + \frac{1}{r}\, v_p' - 2\gamma w_p' &= \sum_{m=0}^{p-1} \beta_{p-m} v_m, \\ -2\gamma v_p' + w_n''(r) - \frac{1}{r}\, w_p' + \frac{1}{r^2}\, w_p &= \sum_{m=0}^{p-1} \beta_{p-m} w_m \end{aligned} \tag{16}$$

for $p = 0, \ldots, N$.

We then have,

Theorem 3.2. *The sequence of functions* $\{\Phi_p\}_{p=0}^N$ *with*

$$\Phi_p(x,y) = K_0(\gamma|x-y|)v_p(|x-y|) + K_1(\gamma|x-y|)w_p(|x-y|) \tag{17}$$

for $p = 0, \ldots, N$*, where* K_0 *and* K_1 *are the modified Hankel functions of order zero and one, see* (12)*, and* v_p *and* w_p *are the polynomials given by* (13)*, constitute a fundamental sequence of* (8) *in the sense of* Definition 3.1.

The proof of this is straightforward; to verify that (10) is satisfied, simply differentiate (17) the necessary number of times remembering that $\frac{1}{2\pi}K_0(\lambda|x-y|)$ is the fundamental solution of the modified Helmholtz equation and making use of the modified Bessel differential equation for the function K_0,

$$z^2 K_0''(z) + z K_0'(z) - z^2 K_0(z) = 0 \tag{18}$$

for $z \neq 0$ together with the relation $K_1 = -K_0'$ and (18), the details are left to the reader.

3.2. *Reduction of* (8)–(9) *to a boundary integral equation*

Following the direct integral approach [10] for the Cauchy problem for the Laplace equation, we search for the solution of the Cauchy problem (8)–(9) in the following potential-layer form

$$u_p(x) = \frac{1}{\pi} \sum_{\ell=1}^{2} \sum_{m=0}^{p} \int_{\Gamma_\ell} q_m^\ell(y) \Phi_{p-m}(x,y)\, ds(y), \quad x \in D \tag{19}$$

with the unknown densities q_m^1 and q_m^2, $m = 0, \ldots, N$, defined on the two (closed) boundary curves Γ_1 and Γ_2, respectively, and Φ_p is given by (17).

The boundary integral operators in (19) have the similar jump properties as the classical single-layer operator for the Laplace equation; this can be verified by noticing from the expansion (12) that the functions in the fundamental sequence each have at most a logarithmic singularity. Therefore, matching (19) against the data (9) and employing the jump properties, we obtain the following system of boundary integral equations

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\sum_{\ell=1}^{2}\int_{\Gamma_\ell} q_p^\ell(y)\Phi_0(x,y)\,ds(y) = F_p(x), \ x\in\Gamma_2, \\[2ex] q_p^2(x) + \dfrac{1}{\pi}\displaystyle\sum_{\ell=1}^{2}\int_{\Gamma_\ell} q_p^\ell(y)\dfrac{\partial\Phi_0(x,y)}{\partial\nu(x)}\,ds(y) = G_p(x), \ x\in\Gamma_2, \end{cases} \tag{20}$$

for $p = 0,\dots,N$, with the right-hand sides

$$F_p(x) = f_{2,p}(x) - \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p-1}\int_{\Gamma_\ell} q_m^\ell(y)\Phi_{p-m}(x,y)\,ds(y)$$

and

$$G_p(x) = g_{2,p}(x) - \sum_{m=0}^{p-1} q_m^2(x) - \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p-1}\int_{\Gamma_\ell} q_m^\ell(y)\frac{\partial\Phi_{p-m}(x,y)}{\partial\nu(x)}\,ds(y).$$

We have then reduced the Cauchy problem (8)–(9) to the system of boundary integral equations (20). As mentioned above in connection with jump properties, kernels appearing in these integral equations can contain logarithmic singularities and we will take this into account when proposing numerical discretisation.

Following the proof of [6, Theorem 4.1], it is possible to verify that the corresponding operator matrix for the system (20), built from the integral operators in (20), has the following properties,

Theorem 3.3. *The operator corresponding to the system* (20) *is injective and has dense range, as a mapping between L^2-spaces on the boundary.*

Therefore, due to this result, Tikhonov regularization can be applied to solve (20) in a stable way.

4. Full discretization of the system of boundary integral equations (20)

We assume that the boundary curves Γ_ℓ, $\ell = 1, 2$, are sufficiently smooth and given by a parametric representation

$$\Gamma_\ell = \{x_\ell(s) = (x_{1\ell}(s), x_{2\ell}(s)),\ s\in[0,2\pi]\}.$$

The system (20) can then be written in parametric form

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\sum_{\ell=1}^{2}\int_0^{2\pi} \psi_p^\ell(\sigma)H_{\ell,2}^0(s,\sigma)\,d\sigma = F_p(s), \ s\in[0,2\pi], \\[2ex] \dfrac{\psi_p^2(s)}{|x_2'(s)|} + \dfrac{1}{2\pi}\displaystyle\sum_{\ell=1}^{2}\int_0^{2\pi} \psi_p^\ell(\sigma)Q_{\ell,2}^0(s,\sigma)\,d\sigma = G_p(s), \ s\in[0,2\pi], \end{cases} \tag{21}$$

for $p = 0,\dots,N$, where for the parametrized densities

$$\varphi_p^\ell(s) = q_p^\ell(x_\ell(s))|x_\ell'(s)|,$$

and where for ease of presentation a new sequence of (unknown) densities has been introduced by

$$\psi_p^\ell(s) = \sum_{m=0}^{p} \varphi_m^\ell(s).$$

The right-hand sides (data functions) in (21) are given by

$$F_p(s) = f_{2,p}(x_2(s)) - \frac{1}{2\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p-1}\int_0^{2\pi} \varphi_m^\ell(\sigma) H_{\ell,2}^{p-m}(s,\sigma)\,d\sigma$$

and

$$G_p(s) = g_{2,p}(x_2(s)) - \frac{1}{2\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p-1}\int_0^{2\pi} \varphi_m^\ell(\sigma) Q_{\ell,2}^{p-m}(s,\sigma)\,d\sigma.$$

The kernels in (21) are given as

$$H_{\ell,k}^0(s,\sigma) = 2\Phi_0(x_k(s), x_\ell(\sigma)), \quad H_{\ell,k}^p(s,\sigma) = 2[\Phi_p(x_k(s), x_\ell(\sigma)) - \Phi_0(x_k(s), x_\ell(\sigma))] \tag{22}$$

and

$$Q_{\ell,k}^0(s,\sigma) = 2\frac{\partial \Phi_0(x,y)}{\partial \nu(x)}\Big|_{x=x_k(s), y=x_\ell(\sigma)},$$

$$Q_{\ell,k}^p(s,\sigma) = 2\frac{\partial [\Phi_p(x,y) - \Phi_0(x,y)]}{\partial \nu(x)}\Big|_{x=x_k(s), y=x_\ell(\sigma)}$$

for $s \neq \sigma$, $\ell, k = 1, 2$, $p = 1, \ldots, N$. The subtraction of the element Φ_0 in the expression for $H_{\ell,k}^p$ respectively $Q_{\ell,k}^p$ is due to the definition of the densities ψ_p. The functions Φ_p are the fundamental sequence given in Theorem 3.2.

4.1. Handling the singularities in the kernels

We write the kernels of the previous section in a such a way that the singularities become explicit making it easer to propose numerical approximations of boundary integrals involving such kernels.

For the singular kernels $H_{\ell,\ell}^p$ in (22), using the explicit expression (17) for the elements in the fundamental sequence, together with the expansions (11)–(12) and the governing relations (13)–(16), careful calculations lead to the following representation,

$$H_{\ell,\ell}^p(s,\sigma) = H_{\ell,\ell}^{p,1}(s,\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) + H_{\ell,\ell}^{p,2}(s,\sigma),$$

where

$$H_{\ell,\ell}^{0,1}(s,\sigma) = -I_0(\gamma|x_\ell(s) - x_\ell(\sigma)|),$$

$$\begin{aligned} H_{\ell,\ell}^{p,1}(s,\sigma) \quad &= \quad -I_0(\gamma|x_\ell(s) - x_\ell(\sigma)|)(v_p(|x_\ell(s) - x_\ell(\sigma)|) - 1) \\ &\quad +I_1(\gamma|x_\ell(s) - x_\ell(\sigma)|)w_p(|x_\ell(s) - x_\ell(\sigma)|) \end{aligned} \tag{23}$$

and

$$H_{\ell,\ell}^{p,2}(s,\sigma) = H_{\ell,\ell}^p(s,\sigma) - H_{\ell,\ell}^{p,1}(s,\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)$$

with diagonal terms

$$H_{\ell,\ell}^{0,2}(s,s) = -2C - 1 - 2\ln\left(\frac{\gamma|x_\ell'(s)|}{2}\right), \quad H_{\ell,\ell}^{p,2}(s,s) = \frac{2a_{p,1}}{\gamma},\ p = 1,2,\ldots,N,$$

with C being the Euler constant defined above and the coefficient $a_{p,1}$ is given by (15).

For the representation of the kernels $Q_{\ell,k}^p$, we introduce the function

$$h_{\ell,k}(s,\sigma) = \frac{(x_{\ell,1}(s) - x_{k,1}(\sigma))x_{\ell,2}'(s) - (x_{2,\ell}(s) - x_{k,2}(\sigma))x_{\ell,1}'(s)}{|x_k(\sigma) - x_\ell(s)|}$$

as well as the polynomials

$$\tilde{v}_p(r) = \gamma \sum_{m=1}^{\left[\frac{p}{2}\right]} a_{p,2m} r^{2m} - 2 \sum_{m=1}^{\left[\frac{p-1}{2}\right]} m a_{p,2m+1} r^{2m}$$

and

$$\tilde{w}_p(r) = \gamma \sum_{m=0}^{\left[\frac{p-1}{2}\right]} a_{p,2m+1} r^{2m+1} - 2 \sum_{m=1}^{\left[\frac{p}{2}\right]} m a_{p,2m} r^{2m-1}.$$

Then the kernels $Q_{\ell,k}^p$ can be written in the form

$$Q_{\ell,k}^0(s,\sigma) = 2\gamma h_{\ell,k}(s,\sigma) K_1(\gamma|x_\ell(s) - x_k(\sigma)|)$$

and

$$\begin{aligned} Q_{\ell,k}^p(s,\sigma) \quad &= \quad 2h_{\ell,k}(s,\sigma)\{K_1(\gamma|x_\ell(s) - x_k(\sigma)|)\tilde{v}_p(|x_\ell(t) - x_k(\sigma)|) \\ &\quad + K_0(\gamma|x_\ell(s) - x_k(\sigma)|)\tilde{w}_n(|x_\ell(s) - x_k(\sigma)|)\} \end{aligned}$$

for $s \neq \sigma$ and $p = 1,2,\ldots,N$.

The kernels $Q_{\ell,\ell}^p$ have logarithmic singularities; similar calculations as when rewriting the singular kernels $H_{\ell,\ell}^p$ give

$$Q_{\ell,\ell}^p(s,\sigma) = Q_{\ell,\ell}^{p,1}(s,\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) + Q_{\ell,\ell}^{p,2}(s,\sigma),$$

where

$$Q_{\ell,\ell}^{0,1}(s,\sigma) = \gamma h_{\ell,\ell}(s,\sigma) I_1(\gamma|x_\ell(s) - x_k(\sigma)|),$$

$$\begin{aligned} Q_{\ell,\ell}^{p,1}(s,\sigma) \quad &= \quad h_{\ell,\ell}(s,\sigma)\{I_1(\gamma|x_\ell(s) - x_k(\sigma)|)\tilde{v}_n(|x_\ell(s) - x_k(\sigma)|) \\ &\quad - I_0(\gamma|x_\ell(s) - x_k(\sigma)|)\tilde{w}_n(|x_\ell(s) - x_k(\sigma)|)\} \end{aligned} \tag{24}$$

and

$$Q_{\ell,\ell}^{p,2}(s,\sigma) = Q_{\ell,\ell}^p(s,\sigma) - Q_{\ell,\ell}^{p,1}(s,\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)$$

with diagonal terms

$$Q_{\ell,\ell}^{0,2}(s,s) = \frac{x_{\ell,2}'(s)x_{\ell,1}''(s) - x_{\ell,1}'(s)x_{\ell,2}''(s)}{|x_\ell'(s)|^2}, \quad Q_{\ell,\ell}^{p,2}(s,s) = 0, \quad p = 1,2,\ldots,N.$$

4.2. Discrete linear equations

The effort in rewriting the kernels now pays off in that we can employ the following standard quadrature rules [28] for numerical discretisation,

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\,d\sigma \approx \sum_{k=0}^{2n-1} f(s_k),$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)d\tau \approx \sum_{k=0}^{2n-1} R_k(s)\,f(s_k),$$

with mesh points

$$s_k = kh, \quad k = 0,\dots,2n-1, \quad h = \pi/n, \tag{25}$$

and the weight functions

$$R_k(s) = -\frac{1}{2n}\left(1 + 2\sum_{m=1}^{n-1}\frac{1}{m}\cos m(s-s_k) - \frac{1}{n}\cos n(s-s_k)\right),$$

in order to approximate the boundary integrals in (21).

Collocating the approximation at the nodal points using the mesh points $\{s_k\}$ lead to the sequence of linear systems

$$\begin{cases} \displaystyle\sum_{j=0}^{2n-1}\left\{\psi^1_{p,j}\frac{1}{2n}H^0_{1,2}(s_i,s_j) + \psi^2_{p,j}\left[R_j(s_i)H^{0,1}_{2,2}(s_i,s_j) + \frac{1}{2n}H^{0,2}_{2,2}(s_i,s_j)\right]\right\} = \tilde{F}_{p,i}, \\ \displaystyle\sum_{j=0}^{2n-1}\left\{\psi_{1,j}\frac{1}{2n}Q^0_{1,2}(s_i,s_j) + \psi^2_{p,j}\left[R_j(s_i)Q^{0,1}_{2,2}(s_i,s_j) + \frac{1}{2n}Q^{0,2}_{2,2}(s_i,s_j)\right]\right\} + \frac{\psi^2_{p,i}}{|x_2'(s_i)|} = \tilde{G}_{p,i} \end{cases} \tag{26}$$

for $i = 0,\dots,2n-1$, with the right-hand sides

$$\begin{aligned} \tilde{F}_{p,i} \;=\; & f_{2,p}(x_2(s_i)) - \sum_{j=0}^{2n-1}\sum_{m=0}^{p-1}\left\{\varphi^1_{m,j}\frac{1}{2n}H^{p-m}_{1,2}(s_i,s_j) + \right. \\ & \left. \varphi^2_{m,j}[R_j(s_i)H^{p-m,1}_{2,2}(s_i,s_j) + \frac{1}{2n}H^{p-m,2}_{2,2}(s_i,s_j)]\right\} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \tilde{G}_{p,i} \;=\; & g_{2,p}(x_2(s_i)) - \sum_{j=0}^{2n-1}\sum_{m=0}^{p-1}\left\{\varphi^1_{m,j}\frac{1}{2n}Q^{p-m}_{1,2}(s_i,s_j)\right. \\ & \left. +\varphi^2_{m,j}[R_j(s_i)Q^{p-m,1}_{2,2}(s_i,s_j) + \frac{1}{2n}Q^{p-m,2}_{2,2}(s_i,s_j)]\right\}, \end{aligned} \tag{28}$$

where $\varphi^\ell_{0,j} = \psi^\ell_{0,j}$ and $\varphi^\ell_{p,j} = \psi^\ell_{p,j} - \psi^\ell_{p-1,j}$, $p = 0,\dots,N$.

We point out that the matrix corresponding to (26) is the same for every p but having a recurrence right-hand side containing the solutions of previous systems. Clearly, each of the linear systems has a high-condition number since the Cauchy problem (8)–(9) is ill-posed and Tikhonov regularization therefore have to be incorporated

We end this section by giving formulas for the Cauchy data on the boundary Γ_1. Using (19) we have the following representation of the function value

$$f_{1,p}(x) := u_p(x) = \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p}\int_{\Gamma_\ell} q_m^\ell(y)\Phi_{p-m}(x,y)\,ds(y), \quad x \in \Gamma_1$$

and for the normal derivative

$$g_{1,p}(x) := \frac{\partial u_p}{\partial \nu}(x) = -\sum_{m=0}^{p} q_m^1(x) + \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p}\int_{\Gamma_\ell} q_m^\ell(y)\frac{\partial \Phi_{p-m}(x,y)}{\partial \nu(x)}ds(y), \quad x \in \Gamma_1.$$

The numerical approximation of these expressions $u_{p,n}$ and $\partial u_{p,n}/\partial\nu$ can be obtained using the given quadrature rules via similar calculations as those given above.

Remembering the semi-discretisation in time from Section 2 that gave rise to the elliptic system (8)–(9), the above expressions then generates approximations to the sought time-dependent lateral Cauchy data.

As mentioned, the given approach can also be applied to non-stationary Cauchy problems in 3-dimensional domains, by extending what has been presented together with details from [11].

5. Numerical experiments

We demonstrate the efficiency of our method using examples in two different planar domains. The first domain D_1 is given by the following parametric representation of the boundary curves (see further Fig. 1a)

$$\Gamma_1^1 := \{x_1(s) = (0.6\cos s, 0.4\sin s), s \in [0,2\pi]\}, \quad \Gamma_2^1 := \{x_2(s) = (\cos s, \sin s), s \in [0,2\pi]\}.$$

The second domain D_2 has boundary curves given by (see Fig. 1b)

$$\Gamma_1^2 := \{x_1(s) = (\sqrt{(0.5\cos s)^2 + (0.25\sin s)^2}(\cos s, \sin s), s \in [0,2\pi]\}$$

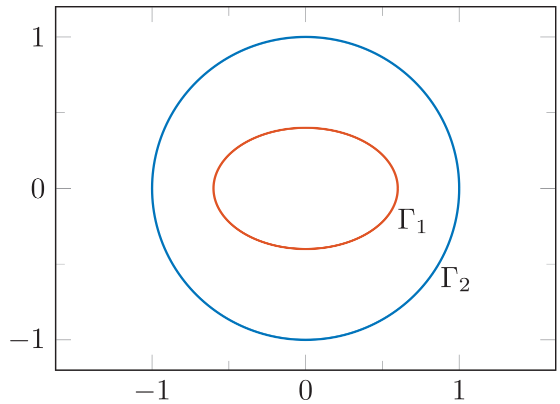
and

$$\Gamma_2^2 := \{x_2(s) = [(\cos s)^{10} + (\sin s)^{10}]^{-0.1}(\cos s, 5\sin s), s \in [0,2\pi]\}.$$

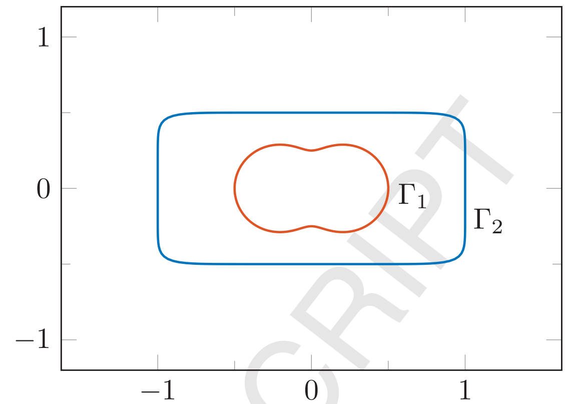
The upper index indicates which domain the boundary curve belongs to, it is not always written out if it is clear from the context which domain that is considered.

We illustrate the robustness of the proposed method for both exact and noisy data. In the case of noisy data, random pointwise errors are added to the normal derivative g_2 on the outer boundary curve with the percentage given in terms of the L^2-norm.

To avoid the "inverse crime", we use a finer mesh in the calculation of the input data. The regularisation parameter α can be chosen by Morozov's discretization principle [34] or by the L-curve criterion [21]. As is common in inverse problems, it is usually more straightforward to run the simulations for a range of values and select by inspection a suitable value for α, this is known as selection by trial and error [28, Chapt. 15.2, p. 271]. In our case, the value for α is then chosen by trial and error; we calculated the numerical solutions for $\alpha = 10^{-\ell}$ with $\ell = 1, \ldots, 15$ and used the value giving the most accurate result.



a) The domain D_1



b) The domain D_2

Figure 1. The two solution domains used in th numerical experiments

Ex. 1 (Sequence of elliptic equations). A key step in our procedure to solve the ill-posed Cauchy problem for both the heat and wave equation, is to be able to solve the corresponding sequence of elliptic Cauchy problems (8)–(9). In Section 3, we outlined a boundary integral equation method for this sequence of problems. In the present example, we therefore test our proposed numerical method for the Cauchy problem for the sequence of elliptic equations (8)–(9).

1.1. The domain chosen is D_1 with boundary curves Γ_1^1 and Γ_2^1 (see Fig. 1a) and the Cauchy data are given as

$$f_{2,p}(x) = \frac{1}{2\pi}\Phi_p(x, x^*), \quad g_{2,p}(x) = \frac{1}{2\pi}\frac{\partial \Phi_p}{\partial \nu}(x, x^*), \quad x \in \Gamma_2^1, \quad p = 0, 1, \dots, N$$

with $x^* = (0, 2)$ and $\{\Phi_p\}_{p=0}^N$ the fundamental sequence of Theorem 3.2. Clearly, the exact solutions of the Cauchy problems (8)–(9) has the form $u_p(x) = \Phi_p(x, x^*)$.

We make use of the following relative L_2-errors on the inner boundary (where no data is specified)

$$e_p^2 = \frac{\displaystyle\int_0^{2\pi} [f_{1,p}(x_1(s)) - u_{p,n}(x_1(s))]^2\, ds}{\displaystyle\int_0^{2\pi} f_{1,p}^2(x_1(s))\, ds}$$

and

$$q_p^2 = \frac{\displaystyle\int_0^{2\pi} [g_{1,p}(x_1(s)) - \frac{\partial u_{p,n}}{\partial \nu}(x_1(s))]^2\, ds}{\displaystyle\int_0^{2\pi} g_{1,p}^2(x_1(s))\, ds}.$$

For the numerical calculation of the integrals in the above two error terms, we use the trapezoidal rule.

The obtained results for exact data with Tikhonov regularization parameter $\alpha = 10^{-13}$, and for noisy data with $\alpha = 10^{-2}$, are given in Table 1 for both the Laguerre and Rothe cases (that is for the various choices of parameters in (8)–(9) corresponding to these two cases). We demonstrate the results for various numbers p of Cauchy problems. Here, we used $\kappa = 2$, $\beta_i = \kappa$, $i = 0, \dots, p-1$, for the Laguerre transform

Table 1. Errors for Ex. 1.1

	Laguerre approach				Rothe's approach			
	Exact data		5% noisy		Exact data		5% noisy	
p	e_p	q_p	e_p	q_p	e_p	q_p	e_p	q_p
0	3.37E-4	8.33E-3	6.31E-2	2.76E-1	2.91E-3	3.47E-2	2.11E-1	4.55E-1
5	1.58E-3	1.20E-2	1.07E-1	1.99E-1	2.71E-4	6.60E-3	3.62E-2	1.55E-1
10	9.19E-4	3.02E-2	3.14E-1	1.35E 0	1.45E-4	5.11E-3	2.10E-2	1.23E-1
15	9.21E-4	1.04E-2	5.17E-1	1.50E 0	1.10E-4	4.64E-3	1.60E-2	1.13E-1
20	3.53E-3	1.75E-2	1.36E 0	1.94E 0	8.65E-5	4.42E-3	1.35E-2	1.08E-1

and $h_t = 0.2$, $\gamma^2 = 1/h_t$, $\beta_i = 0$, $i = 1, \dots, p-2$, $\beta_{p-1} = -1/h_t$ for the Rothe method; the discretization parameter is $n = 32$ in (25) for all cases considered.

From Table 1, we see that the error continues to decrease more for the method of Rothe than for the Laguerre transform, before reaching the threshold when the linear systems become too ill-posed for the errors to decrease any further. Once the error starts to increase, it is no sudden increase but rather mild.

Other domains and data can be used and results of the similar kind is generally to be expected.

1.2. To support the claim that other domains and data can be used, we then consider the Cauchy problem for the system (8)–(9) in the domain D_2 (Fig. 1b) instead, with parameters corresponding to the Laguerre transform. The Cauchy data are generated by solving the corresponding direct Dirichlet boundary value problem with boundary functions

$$f_{\ell,p}(x) = \frac{e^2(2 + \kappa p(\kappa(p-1) - 4)}{(\kappa+1)^{p+3}}(x_1 + x_2), \quad x \in \Gamma_\ell^2,\ p = 0, 1, \dots. \tag{29}$$

The corresponding L_2 errors are reflected in Table 2. Here $\kappa = 1$, $\beta_i = \kappa$, $i = 0, \dots, p$ and $n = 32$.

Table 2. Errors for Ex. 1.2: Laguerre approach

	Exact data (α=1E-5)		5% noisy (α=1E-2)	
p	e_p	q_p	e_p	q_p
0	1.11E-3	7.92E-3	3.23E-2	6.96E-2
5	2.09E-3	8.53E-3	5.29E-2	6.26E-2
10	1.12E-3	6.76E-3	2.23E-2	5.51E-2
15	1.91E-3	1.26E-2	1.39E-1	5.29E-1
20	6.65E-3	3.35E-2	2.13E 0	6.51E 0

The behaviour of these results are similar to the ones in Table 1. Again, the errors do not increase dramatically when the number p of the considered Cauchy problem in the sequence (8)–(9) is increased beyond the threshold when the ill-posedness starts to take over the calculations. It is possible also to use the method of Rothe, but not to overload the presentation with data, we left them out.

Ex. 2 (Heat equation). We consider here two Cauchy problems for a heat equation.

2.1. The first one is a test example related to the use of the fundamental solution of heat equation as an exact solution of the corresponding Cauchy problem.

Let

$$u_{ex}(x,t) = \frac{1}{4\pi t} e^{-\frac{|x-x^*|^2}{4t}}, \quad x \in D_1,\ t \in (0,1],\quad x^* = (0,2).$$

It is straightforward to show that in the case of the Laguerre transform there holds

$$\int_0^\infty u_{ex}(x,t)L_k(\kappa t)e^{-\kappa t}dt = \frac{1}{2\pi}\Phi_k(x,x^*), \quad k = 0,1,\dots \tag{30}$$

with $\beta_n = \kappa$.

We solve the parabolic Cauchy problem (1) with $c = 1$ and the data

$$f_2(x,t) = u_{ex}(x,t), \quad g_2(x,t) = \frac{\partial u_{ex}}{\partial \nu}(x,t), \quad x \in \Gamma_2^1,\ t \in (0,1]. \tag{31}$$

We use the following transient L_2-errors

$$e^2 = \frac{\int_0^T \int_0^{2\pi} (f_1(x_1(s),t) - u_{N,n}(x_1(s),t))^2\, ds\, dt}{\int_0^T \int_0^{2\pi} f_1^2(x_1(s),t)\, ds\, dt}$$

and

$$q^2 = \frac{\int_0^T \int_0^{2\pi} (g_1(x_1(s),t) - \frac{\partial u_{N,n}}{\partial \nu}(x_1(s),t))^2\, ds\, dt}{\int_0^T \int_0^{2\pi} g_1^2(x_1(s),t)\, ds\, dt}.$$

The numerical results in the case of the combination of the Laguerre transform and the integral equations approach of Section 3.2, with $n = 64$, $\kappa = 2$, $\alpha = 10^{-9}$ for exact data and $\alpha = 10^{-2}$ for noisy data, are presented in Table 3.

Table 3. Ex. 2.1. Errors: Laguerre approach

	Exact data		5% noisy	
N	e	q	e	q
20	1.81E-2	6.99E-2	5.32E-2	2.41E-1
30	1.20E-2	2.93E-2	5.37E-2	2.48E-1
40	7.60E-3	2.47E-2	5.24E-2	2.61E-1

Table 4 contains errors obtained by instead using the Rothe method and the integral equation approach of Section 3.2. Here, $n = 64$, $\alpha = 10^{-7}$ for exact data and $\alpha = 10^{-3}$ for noisy data. Note here, that it is of crucial importance to balance the relation between time and space discretization parameters.

2.2. In this example, we change the solution domain and input data. To keep the presentation at reasonably length, we do not produce further tables and do not intend to use both approaches. Instead, in this example, we focus on the Laguerre approach and shall generate some figures of the obtained approximations.

The input data for the parabolic Cauchy problem with thermal diffusivity $c = 1$ is generated by solving the Dirichlet initial boundary value problem with boundary functions

$$f_\ell(x,t) = t^2 e^{-t+2}(x_1 + x_2), \quad x \in \Gamma_\ell^2,\ t \geq 0.$$

Table 4. Ex. 2.1. Errors: Rothe approach

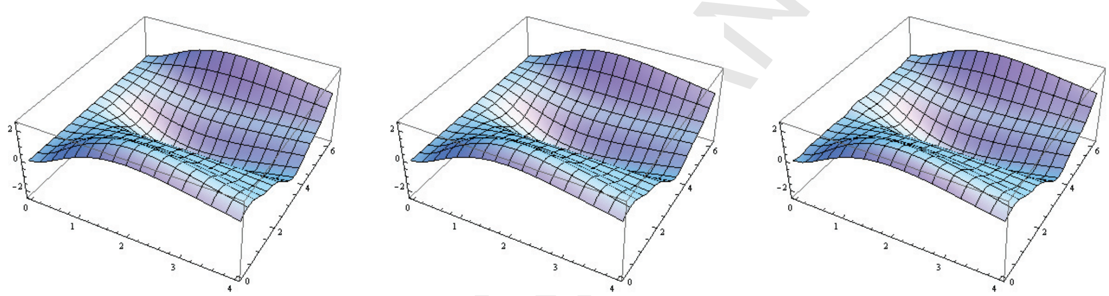
	Exact data		5% noisy	
N	e	q	e	q
5	2.50E-5	2.50E-1	4.97E-2	3.34E-1
10	6.39E-5	1.99E-1	3.64E-2	2.70E-1
15	1.21E-4	1.29E-1	3.45E-2	2.03E-1

Note that

$$\int_0^\infty f_\ell(x,t)e^{-\kappa t}L_p(\kappa t)dt = f_{\ell,p}(x), \quad p = 0, \ldots, \ \ell = 1, 2,$$

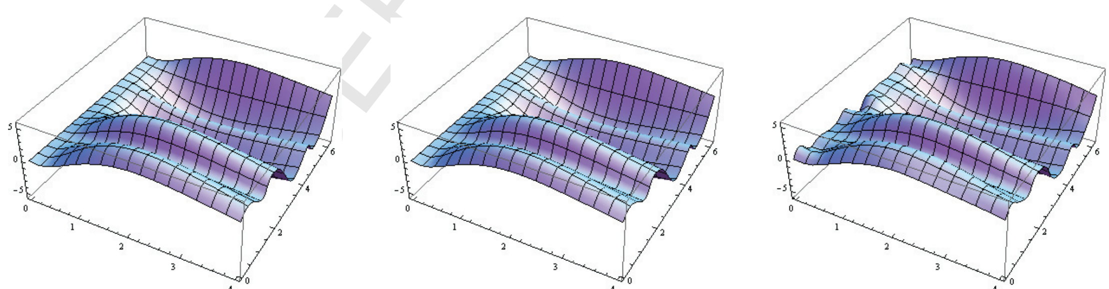
where $f_{\ell,p}$ are defined as in (29).

The results of the numerical experiments using the Laguerre transform and the integral equations approach of Section 3.2, are given graphically in Fig. 2 and Fig. 3 both for exact and noisy data as specified in each figure caption. To generate these figures, we used $\kappa = 2$, $N = 20$, $n = 32$ and $T = 4$.



a) boundary function on Γ_1^2 b) exact data ($\alpha = 1E-5$, $e = 0.0067$) c) 5% noisy ($\alpha = 1E-3$, $e = 0.113$)

Figure 2. Reconstruction of the boundary function on Γ_1^2 (parabolic case)



a) normal derivative on Γ_1^2 b) exact data ($\alpha = 1E-5$, $q = 0.110$) c) 5% noisy ($\alpha = 1E-3$, $q = 1.192$)

Figure 3. Reconstruction of the normal derivative on Γ_1^2 (parabolic case)

It is particularly pleasing to see that also the normal derivative is reconstructed with acceptable accuracy even in the case of noisy data. Differentiation in itself is an ill-posed problem and in general for Cauchy problems to also reconstruct numerically the normal derivative in a stable way is an additional challenge. It is also known [19] that for the ill-posed Cauchy problem for the heat equation, the approximation will in general deteriorate for $t = T$ (causality principle). Tendencies to this deterioration is present in the given figures.

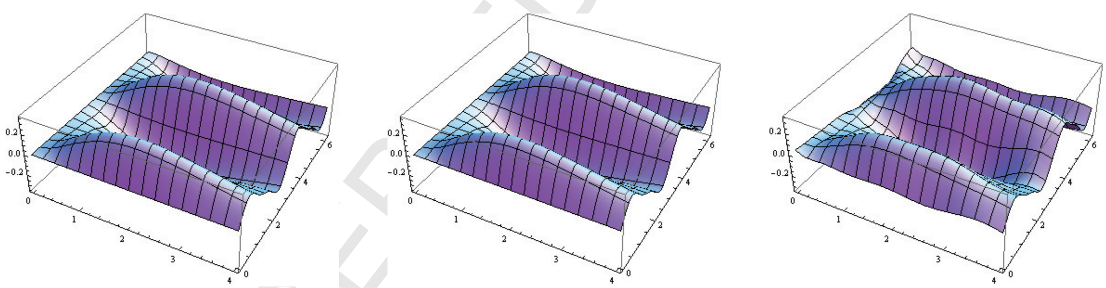
Ex. 3 (Wave equation). The Cauchy problem for the wave equation is, as mentioned above, considerable less studied in particularly from a numerical point of view. This can partly be explained by the more complicated fundamental solution that the wave equation has (involving a Heaviside function) compared with the heat equation, making direct boundary integral approaches complicated. Thus, our aim with this example is to show that one can indeed by-pass this difficulty by the present approach of reducing the wave equation to a sequence of stationary problems. We only produce a set of figures for one set of data to show that the present approach is promising. A fuller investigation for the wave equation is deferred to a future work.

The input data for the Cauchy wave problem (hyperbolic) (7) with wave speed $a = 1$ is generated by solving the Dirichlet initial boundary value problem with boundary functions

$$f_\ell(x,t) = t^2 e^{-t+2} x_1 x_2, \quad x = (x_1, x_2) \in \Gamma_\ell^2, \; t \geq 0.$$

The results of the numerical experiments using the Laguerre transform and the integral equations approach of Section 3.2, are presented in Fig. 4 and Fig. 5. Here, we used $\kappa = 2$, $N = 20$, $n = 32$ and $T = 4$.

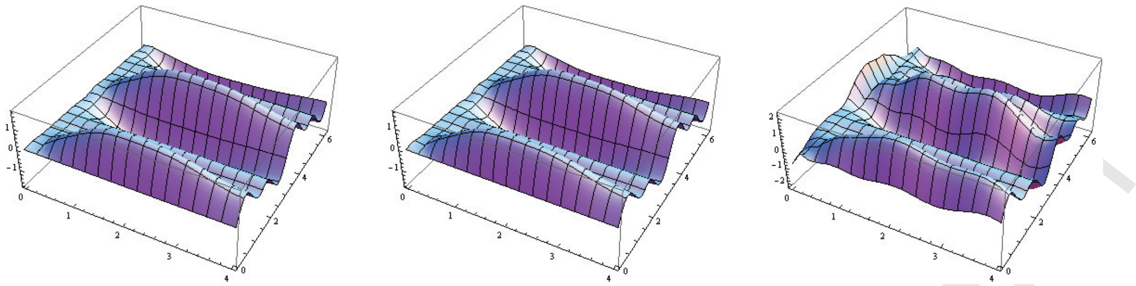
As for the heat equation, as is seen in Fig. 5, the normal derivative can also be reconstructed with some accuracy both with exact and noisy data.



a) boundary function on Γ_1^2 b) exact data ($\alpha = 1E-5$, $e = 0.0068$) c) 5% noisy ($\alpha = 1E-2$, $e = 0.159$)

Figure 4. Reconstruction of the boundary function on Γ_1^2 (hyperbolic case)

To conclude this numerical section, we remark the following. Ideally, one should compare the obtained results with others in the literature. To do this in an objective way tends to be difficult and very much dependent on what is being compared. For the heat equation, we can at least say that the results are comparable to the ones obtained in [**?**]. For the wave equation, we have not been able to locate any numerical results for the two-dimensional case to compare against.



a) normal derivative on Γ_1^2 b) exact data ($\alpha = 1E-5$, $q = 0.104$) c) 5% noisy ($\alpha = 1E-2$, $q = 1.501$)

Figure 5. Reconstruction of the normal derivative on Γ_1^2 (hyperbolic case)

6. Conclusion

A unified boundary integral approach has been developed for the ill-posed Cauchy problem for the heat and wave equation. The transient problems are each reduced, via either the Laguerre transform or the method of Rothe, to the similar sequence of elliptic Cauchy problems. For this sequence, a boundary integral approach, based on the derivation of a fundamental solution, was given and analysed. For the discretisation a Nyström method together with the Tikhonov regularization are employed for the stable numerical approximation. The included numerical experiments showed the feasibility of the approach. In particular, accurate reconstructions of both the solution and its normal derivative on the boundary part where data are initially missing, were obtained with small computational effort both for the heat and wave equation.

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