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Vacuum Spacetimes with Constant Weyl Eigenvalues

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Abstract. Einstein spacetimes (that is vacuum spacetimes possibly with a non-zero cosmological constant Λ) with constant non-zero Weyl eigenvalues are considered. For type Petrov II & D this assumption allows one to prove that the non-repeated eigenvalue necessarily has the value $2\Lambda/3$ and it turns out that the only possible spacetimes are some Kundt-waves considered by Lewandowski which are type II and a Robinson-Bertotti solution of type D.

For Petrov type I the only solution turns out to be a homogeneous pure vacuum solution found long ago by Petrov using group theoretic methods. These results can be summarised by the statement that the only vacuum spacetimes with constant Weyl eigenvalues are either homogeneous or are Kundt spacetimes. This result is similar to that of Coley et al. who proved their result for **general** spacetimes under the assumption that all scalar invariants constructed from the curvature tensor **and all its derivatives** were constant.

1. Introduction

In this paper special classes of solutions of Einstein's vacuum field equations in general relativity, possibly with a non-zero cosmological constant Λ,

$$R_{ab} = \Lambda g_{ab} \tag{1}$$

are considered. The eigenvalues λ and eigenbivectors V^{ab} of the Weyl tensor satisfy the equation

$$C^{ab}{}_{cd}V^{cd} = 2\lambda V^{ab} \tag{2}$$

where the factor two, which arises due to the contraction over a pair of antisymmetric indices, is included for later convenience. Owing to the duality conditions satisfied by the Weyl tensor there are essentially only three roots of the characteristic equation and these sum to zero due to the trace-free nature of the Weyl tensor (see, for example, chapter 4 of Stephani et al. [1]).

In this paper the class of fields considered are Petrov types I, II & D fields where all Weyl eigenvalues are constant; the conformally flat fields are of constant curvature and need not be considered further whereas for Petrov types III and N fields the assumption of the constancy of the Weyl eigenvalues do not restrict these fields as all the eigenvalues are necessarily zero. For Petrov type I fields it may be assumed that the Weyl eigenvalues are non-zero since Brans [2] showed that pure vacuum Petrov type I fields with a zero Weyl eigenvalue do not exist. His proof generalises trivially to the case when Λ is non-zero.

There are a number of motivations for investigating fields of this sort. Firstly, most known vacuum solutions of the field equations are either algebraically special or admit a group of isometries and so it would be interesting to enlarge the class of known vacuum solutions of Petrov type I. Making simplifying assumptions regarding the scalar curvature invariants seems to be a promising approach which has hitherto been largely neglected. Secondly, in recent years there has been considerable interest in general spacetimes in which all the scalar invariants constructed from the curvature tensor and its covariant derivatives are constant. For example Coley and his collaborators [3] showed that such spacetimes were either homogeneous or are a special class of Kundt-waves [4]. In this paper it is shown that a similar result holds for four dimensional Einstein spaces under the weaker assumption that the scalar curvature invariants constructed from the curvature tensor **alone** are constant and non-zero. It would be interesting to see how far this result could be extended to non-vacuum spacetimes.

To conclude this section some useful results on the Petrov classification are briefly reviewed. The conditions on the Weyl eigenvalues are expressed in the notation of the Newman-Penrose [5] formalism which will be used in the analysis later in the paper (see chapters 3 & 4 of Stephani et al. [1] for more details). For Petrov types I & D there is a complex null tetrad: $\ell^a = (u^a + e_3^a)/\sqrt{2}$, $n^a = (u^a - e_3^a)/\sqrt{2}$, $m^a = (e_1^a + \imath e_2^a)/\sqrt{2}$ in which the NP Weyl tensor components satisfy

$$\Psi_0 = \Psi_4 = (\lambda_2 - \lambda_1)/2 \qquad \Psi_1 = \Psi_3 = 0 \qquad \Psi_2 = -\lambda_3/2, \tag{3}$$

where, the λ_A's are the Weyl eigenvalues and $(e_1^a, e_2^a, e_3^a, u^a)$ is an orthonormal Weyl principal tetrad. For type D we have in addition $\Psi_4 = \Psi_0 = 0$ as $\lambda_1 = \lambda_2$. For Petrov type II the NP Weyl tensor components satisfy

$$\Psi_0 = \Psi_1 = \Psi_3 = 0 \qquad \Psi_4 = -2 \qquad \Psi_2 = -\lambda_3/2. \tag{4}$$

Thus the constancy of the Weyl eigenvalues may be restated as (where in all cases $\Psi_1 = \Psi_3 = 0$):

Type I: Ψ_2 and Ψ_4 are both non-zero constants;
Type D: Ψ_2 is a non-zero constant and $\Psi_0 = \Psi_4 = 0$;
Type II: Ψ_2 is a non-zero constant, $\Psi_0 = 0$ and $\Psi_4 = -2$.

For Petrov type I the choice of a canonical tetrad is not unique; it depends on the choice of numbering of the 3 spacelike Weyl principal vectors e_A^a used to construct the complex null tetrad. As there are six distinct numberings a given Petrov type I field will manifest itself as six different solutions of the Newman-Penrose equations for the spin coefficients and Weyl tensor components Ψ_i. These fall naturally into 3 pairs depending on the choice of the spacelike principal vector e_3^a used to construct the real null vectors ℓ^a and n^a. The two members of each pair have the same value of Ψ_2, but Ψ_4 changes opposite sign under interchange of e_1^a and e_2^a.

The ambiguity in the numbering of the principal vectors also means that type D fields with $\lambda_3 = \lambda_1 = -\lambda_2/2$ or $\lambda_2 = \lambda_3 = -\lambda_1/2$ appear as special cases in the analysis of type I spacetimes and need to be excluded. They are characterised by the conditions $\Psi_4 = \pm 3\Psi_2$ respectively. Similarly the Petrov type I fields where one Weyl eigenvalue is zero appears as a special case in the analysis and can be immediately excluded by Brans' result [2]. The cases with $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 0$ are characterised by the conditions $\Psi_4 = \pm\Psi_2$ and $\Psi_2 = 0$ respectively.

2. Algebraically Special Spacetimes with Constant Weyl Eigenvalues

In this section the type II and type D fields satisfying the assumptions of §1 are investigated. It is convenient to consider these two cases together; the NP Weyl tensor components satisfy $\Psi_0 = \Psi_1 = \Psi_3 = 0$ and $\Psi_2 (\neq 0)$ is a constant. Thus $D\Psi_2$, $\Delta\Psi_2$, $\delta\Psi_2$, $\bar{\delta}\Psi_2$, DR, ΔR, δR and $\bar{\delta}R$ are all zero and for type II $\Psi_4 \neq 0$ whereas for type D, $\Psi_4 = 0$.

In what follows the equation numbers refer to the Newman-Penrose equations in chapter 7 of Stephani et al. [1]. Using the above restrictions on Ψ_i the Bianchi identities (7.32a,b,e,h) become

$$3\kappa\Psi_2 = 0, \qquad 3\sigma\Psi_2 = 0, \qquad 3\rho\Psi_2 = 0, \qquad 3\tau\Psi_2 = 0. \tag{5}$$

Hence $\kappa = \sigma = \rho = \tau = 0$, and so the spacetime belongs to a special subclass of the Kundt spacetimes [4]. The Ricci identity (7.21q) reduces to $\Psi_2 + R/12 = 0$ and thus $\Psi_2 = -\Lambda/3$ or equivalently $\lambda_3 = 2\Lambda/3$. Thus under the assumption of constancy, *the non-repeated eigenvalue of the Weyl tensor of an Einstein spacetime of type D or II necessarily has the value* $2\Lambda/3$.

These Einstein spaces belong to the subclass of the Kundt spacetimes considered by Lewandowski [6]. In terms of a complex coordinate z and real coordinates u & v, the metric becomes

$$ds^2 = 2P^{-2}dzd\bar{z} - 2du(dv + Wdz + \bar{W}d\bar{z} + Hdu) \tag{6}$$

where $P = P(z, \bar{z}, u)$ and $H = H(z, \bar{z}, u, v)$ are real and $W = W(z, \bar{z}, u)$ is complex; W is independent of v as a consequence of $\tau = 0$. The repeated principal null vector ℓ^a is given by ∂_v. As shown by Lewandowski [6], the metric functions P, H and W may be written as

$$P = 1 + \Lambda z\bar{z}/2 \qquad H = -\Lambda v^2/2 + H_0(z, \bar{z}, u), \qquad W = iL_{,z} \tag{7}$$

where L is a *real* potential satisfying

$$P^2 L_{,z\bar{z}} = -\Lambda L. \tag{8}$$

To complete the identification of the type II & D spacetimes with constant Weyl scalars with those considered by Lewandowski, it is necessary to check that all Lewandowski spacetimes are Petrov type II or D with constant Weyl scalars. Choosing a complex null basis of one-forms:

$$\ell_i dx^i = (H + P^2 W\bar{W})du + dv, \qquad n_i dx^i = du, \qquad m_i dx^i = -P\bar{W}du + P^{-1}dz, \tag{9}$$

a straightforward calculation using the computer algebra system Classi [7] shows that $\Psi_2 = \Lambda/3$ & $\Psi_0 = \Psi_1 = \Psi_3 = 0$. Thus the Weyl tensor is either type II or D and the Weyl eigenvalues are constant. Note that this basis of one-forms is a canonical null tetrad of the Weyl tensor.

The general solution of equation (8) for the potential L is [6]

$$L = \Re(\Lambda P^{-1}\bar{z}f(z, u) - f_{,z}(z, u)) \tag{10}$$

where $f(z, u)$ is an arbitrary function analytic in z. The remaining field equation implies

$$H_{0,z\bar{z}} = \Lambda L_{,z}L_{,\bar{z}} - \Lambda^2 P^{-2} L^2. \tag{11}$$

Given L, this can be integrated to give H_0 up to addition of an arbitrary harmonic function $\Re h_0(z, u)$. It can be seen that the general solution of Petrov type II depends on two arbitrary complex functions $f(z, u)$ and $h_0(z, u)$ analytic in z.

The type D condition $\Psi_4 = 0$ implies, after a somewhat messy calculation (see [8] for more details), that the metric may be written in the form

$$ds^2 = 2(1 + \Lambda z\bar{z}/2)^{-2}dzd\bar{z} - 2dudv - \Lambda v^2 du^2. \tag{12}$$

The metric is decomposable into two 2-spaces of constant curvature and so is a Robinson-Bertotti solution [9, 10]. It is homogenous with a multiply-transitive isometry group of dimension 6.

3. Algebraically General Spacetimes with Constant Weyl Eigenvalues

Now consider Petrov type I fields with constant Weyl eigenvalues. The NP Weyl tensor components satisfy $\Psi_1 = \Psi_3 = 0$ and Ψ_2 & $\Psi_4 (=\Psi_0)$ are non-zero constants. The Bianchi identities (7.32) of [1] become purely algebraic equalities:

$$\begin{aligned}
&(4\alpha - \pi)\Psi_4 + 3\kappa\Psi_2 = 0, &\qquad 3\pi\Psi_2 - \kappa\Psi_4 = 0,\\
&(4\gamma - \mu)\Psi_4 + 3\sigma\Psi_2 = 0, &\qquad 3\mu\Psi_2 - \sigma\Psi_4 = 0,\\
&(4\epsilon - \rho)\Psi_4 + 3\lambda\Psi_2 = 0, &\qquad 3\rho\Psi_2 - \lambda\Psi_4 = 0,\\
&(4\beta - \tau)\Psi_4 + 3\nu\Psi_2 = 0, &\qquad 3\tau\Psi_2 - \nu\Psi_4 = 0.
\end{aligned}$$

Solving these for eight of the spin coefficients in terms κ, σ, ν and λ one obtains:

$$\rho = \psi\lambda, \qquad \tau = \psi\nu, \qquad \mu = \psi\sigma, \qquad \pi = \psi\kappa, \tag{13}$$

$$\alpha = \frac{(\psi^2-1)\kappa}{4\psi}, \qquad \beta = \frac{(\psi^2-1)\nu}{4\psi}, \qquad \epsilon = \frac{(\psi^2-1)\lambda}{4\psi}, \qquad \gamma = \frac{(\psi^2-1)\sigma}{4\psi}. \tag{14}$$

where a subsidiary constant $\psi = \Psi_4/(3\Psi_2)$ has been introduced for later convenience. The following values of ψ are excluded: $\psi \neq 0, \pm 1$ (type D) & $\psi \neq \pm 1/3$ (zero Weyl eigenvalue). These eight spin coefficients may be completely eliminated from the 18 Newman-Penrose Ricci identities (7.21) which thus now only involve derivatives of κ, σ, ν and λ. Here and below equation references of the form (7.xx) refer to Chapter 7 of Stephani et al. [1].

Equations (7.21b & h) are equalities for $D\sigma - \delta\kappa$ and result in the purely algebraic identity:

$$24(\kappa\nu - \sigma\lambda)(1-\psi^2) - 12\Psi_2(1-3\psi^2) - R = 0. \tag{15}$$

Similarly (7.21j & q) are both equalities for $\Delta\lambda - \bar{\delta}\nu$ and also results in (15). A second algebraic expression is obtained from (7.21b, ℓ, f & j):

$$6(\kappa\nu - \sigma\lambda)(1-\psi^2)(1-5\psi^2) - \psi^2(18\psi^2\Psi_2 - 42\Psi_2 + R) = 0. \tag{16}$$

From (15) and (16) the following simpler algebraic relations may be deduced:

$$\Psi_2 = \frac{(\kappa\nu - \sigma\lambda)(9\psi^2-1)}{9\psi^2}, \qquad R = 4(\kappa\nu - \sigma\lambda)(3\psi^2+1)^2/3. \tag{17}$$

From these two equations it can be immediately deduced that $\kappa\nu - \sigma\lambda \neq 0$ as $\Psi_2 \neq 0$ and that the pure vacuum case ($R = 0$) is characterised by $\psi = \pm i/\sqrt{3}$. Note that for the pure vacuum case the Weyl eigenvalues are proportional to the three cube roots of -1; for example $\lambda_1 = (-1 + i\sqrt{3})\lambda_3/2$ and $\lambda_2 = (-1 - i\sqrt{3})\lambda_3/2$ for the choice $\psi = +i/\sqrt{3}$.

Equations (7.21a & g) reduce to two linear equations for $D\lambda$ & $\bar{\delta}\kappa$ which may be solved for these two derivatives. Similarly, from the pairs (7.21c & i), (7.21k & m) and (7.21n & p) the derivatives $D\nu, \Delta\kappa, \delta\lambda, \bar{\delta}\sigma, \delta\nu$ and $\Delta\sigma$ are obtained. Thus 8 of the 16 derivatives of κ, σ, ν and λ are now known. In addition to the previous two algebraic equations and the eight equations for single derivatives of κ, σ, ν and λ, five independent Ricci identities involving pairs of derivatives of these spin coefficients remain. A more complete exposition of the calculations here and below may be found in [8].

On applying the commutator $\delta D - D\delta$ to λ and ν, equations for $\kappa\Delta\lambda - \sigma\bar{\delta}\lambda$ & $\kappa\Delta\nu - \sigma\bar{\delta}\nu$ are obtained. Similarly applying the commutator $\bar{\delta}\Delta - \Delta\bar{\delta}$ to σ and κ equations for $\nu D\sigma - \lambda\delta\sigma$ & $\nu D\sigma - \lambda\delta\sigma$ result. The first three of these equations and the five remaining Ricci identities may be solved for the unknown spin coefficient derivatives: $D\kappa, \delta\kappa, D\sigma, \delta\sigma, \Delta\nu, \bar{\delta}\nu, \Delta\lambda$ and $\bar{\delta}\lambda$.

Substituting these in the equation for $\nu D\sigma - \lambda\delta\sigma$ a purely algebraic relation is obtained:

$$\bar{\psi}(1+3\psi^2)(\kappa\nu - \sigma\lambda)^2 = 0. \tag{18}$$

It may be concluded that $\psi = \pm i/\sqrt{3}$ (since $\psi \neq 0$ as type D is excluded and from (17) $\kappa\nu - \sigma\lambda \neq 0$ since $\Psi_2 \neq 0$). Without loss of generality we may choose $\psi = +i/\sqrt{3}$ as the negative sign simply corresponds to interchanging the two Weyl eigenvalues λ_1 and λ_2. Thus, from (17), *the spacetime is a pure vacuum spacetime* ($R = 0$ or equivalently $\Lambda = 0$). To summarise it has been shown that *the only Petrov type I Einstein spacetimes with constant Weyl eigenvalues have* $\Lambda = 0$ *(i.e. are pure vacuum) and the Weyl eigenvalues are proportional to the three cube roots of* -1. Thus, *a fortiori* there are no homogeneous proper Einstein spaces of Petrov type I; this recovers a result of MacCallum & Siklos [11] without using group theoretic methods. In the same paper MacCallum & Siklos showed there are no homogeneous proper Einstein spaces of Petrov type II; a result also confirmed by the analysis in §2 as it is easy to show the Lewandowski metrics are not homogeneous. Note that in the proof of non-existence for Petrov type II the assumption of homogeneity cannot be replaced by the weaker one of constant Weyl eigenvalues.

As shown above $\Lambda = 0$ and it can be assumed *wlog* that $\psi = +i/\sqrt{3}$. As all 16 derivatives of the spin coefficients κ, σ, ν & λ are now known, the remaining commutators may be applied to these 4 spin coefficients and all the resulting derivative terms eliminated to produce eight purely algebraic compatibility relations.

$$2\kappa\sigma\nu + \sigma^2\lambda + \sigma\bar{\kappa}\bar{\nu} - \sigma\bar{\sigma}\bar{\lambda} + i\sqrt{3}\nu^2\lambda = 0, \qquad \lambda^3 + \sigma\bar{\sigma}\bar{\lambda} - \bar{\kappa}\sigma\bar{\nu} - i\sqrt{3}\kappa^2\lambda = 0, \quad (19a)$$

$$2\kappa\sigma\lambda + \kappa^2\nu + \kappa\bar{\sigma}\bar{\lambda} - \kappa\bar{\kappa}\bar{\nu} + i\sqrt{3}\nu\lambda^2 = 0, \qquad \sigma^3 + \bar{\sigma}\lambda\bar{\lambda} - \bar{\kappa}\bar{\nu}\lambda - i\sqrt{3}\sigma\nu^2 = 0, \quad (19b)$$

$$2\kappa\nu\lambda + \sigma\lambda^2 + \bar{\kappa}\bar{\nu}\lambda - \bar{\sigma}\lambda\bar{\lambda} + i\sqrt{3}\kappa^2\sigma = 0, \qquad \nu^3 + \kappa\bar{\kappa}\bar{\nu} - \kappa\bar{\sigma}\bar{\lambda} - i\sqrt{3}\sigma^2\nu = 0, \quad (19c)$$

$$2\sigma\nu\lambda + \kappa\nu^2 + \bar{\sigma}\nu\bar{\lambda} - \bar{\kappa}\nu\bar{\nu} + i\sqrt{3}\kappa\sigma^2 = 0, \qquad \kappa^3 + \bar{\kappa}\nu\bar{\nu} - \bar{\sigma}\nu\bar{\lambda} - i\sqrt{3}\kappa\lambda^2 = 0. \quad (19d)$$

There are three sets of solutions and, in each of them, all the spin coefficients are constants:

$$\sigma = \lambda = 0, \quad \nu = +\bar{\kappa}, \quad \kappa = \pm k(1 + \epsilon_2 i)/\sqrt{2}, \quad \Psi_2 = 8k^2/3. \quad (20a)$$

$$\lambda = \sigma = \epsilon_2\kappa, \quad \nu = -\kappa, \quad \kappa = \pm k(-\sqrt{3} + i)/2, \quad \Psi_2 = 4k^2(-1 + i\sqrt{3})/3. \quad (20b)$$

$$\lambda = \sigma = i\epsilon_2\kappa, \quad \nu = +\kappa, \quad \kappa = \pm k(1 - i\sqrt{3})/2, \quad \Psi_2 = -4k^2(1 + i\sqrt{3})/3. \quad (20c)$$

Here k is an arbitrary positive constant and $\epsilon_2 = \pm 1$. The remaining spin coefficients are

$$\rho = i\lambda/\sqrt{3}, \qquad \tau = i\nu/\sqrt{3}, \qquad \mu = i\sigma/\sqrt{3}, \qquad \pi = i\kappa/\sqrt{3}, \quad (21)$$

$$\alpha = i\kappa/\sqrt{3}, \qquad \beta = i\nu/\sqrt{3}, \qquad \epsilon = i\lambda/\sqrt{3}, \qquad \gamma = i\sigma/\sqrt{3}. \quad (22)$$

As discussed in §1 the choice of canonical tetrad is not unique; it depends on the ordering of the spacelike eigenvectors used to construct the complex null tetrad. Thus there will be three pairs of distinct solutions of the Newman-Penrose equations for the spin coefficients and Weyl tensor components Ψ_i corresponding to the same spacetime. The choice of $\psi = +i/\sqrt{3}$ effectively singles out one member of each pair and so the appearance of three solutions in (20) is to be expected. The fourfold sign ambiguity in each of these solutions is also to be expected as the vectors of the orthonormal principal tetrad $(e_1^a, e_2^a, e_3^a, u^a)$ are only determined up to signs.

Thus it is expected that the three cases in (20) all correspond to the same underlying spacetime. In fact, the spacetime in question is homogeneous with metric:

$$ds^2 = \frac{1}{4k^2}\left(dx^2 + e^{-4x/\sqrt{3}}dy^2 + e^{2x/\sqrt{3}}\cos(2x)(dz^2 - dt^2) - 2e^{2x/\sqrt{3}}\sin(2x)dzdt\right). \quad (23)$$

This metric is originally due to Petrov [12] who derived it using group theoretic methods. The orthonormal tetrad of Weyl principal vectors corresponding to case (a) in (20) is:

$$u_a dx^a = \tfrac{e^{x/\sqrt{3}}}{2k}\big(\cos x\ dt - \sin x\ dz\big), \qquad e_{3a}dx^a = \frac{e^{x/\sqrt{3}}}{2k}\big(\sin x\ dt + \cos x\ dz\big),$$

$$e_{1a}dx^a = \tfrac{1}{2\sqrt{2}k}\big(dx - e^{-2x/\sqrt{3}}dy\big), \qquad e_{2a}dx^a = \frac{1}{2\sqrt{2}k}\big(dx + e^{-2x/\sqrt{3}}dy\big).$$

A straightforward calculation using the computer algebra system Classi [7] shows that

$$\Psi_0 = \Psi_3 = 0, \quad \Psi_0 = \Psi_4 = 8ik^2/\sqrt{3}, \quad \Psi_2 = 8k^2/3.$$

The metric (23) is the only homogeneous Einstein space of Petrov type I [12]. The analysis above shows that it is *the only Einstein space of Petrov type I with constant Weyl eigenvalues.*

4. Conclusions

Vacuum spacetimes possibly with a non-zero cosmological constant Λ (that is Einstein spacetimes) with constant non-zero Weyl eigenvalues have been considered. For type II & D the non-repeated eigenvalue necessarily has the value $2\Lambda/3$ and so algebraically special pure vacuum spacetimes of this type are ruled out. It is then shown that the only possible spacetimes are some Kundt-waves considered by Lewandowski [6] which are type II and a Robinson-Bertotti solution of type D. The Lewandowski solutions depend on an arbitrary complex function $f(z, u)$ analytic in $z = x + iy$ and an arbitrary real function of x, y & u harmonic in x and y. The solutions are not homogeneous and in general, they admit no isometries. Originally the solutions were found by Lewandowski by considering Kundt solutions with a reduced holonomy group; this paper gives them a new characterisation namely the constancy of their Weyl eigenvalues.

For Petrov type I the only solutions in which all three Weyl eigenvalues are constant must have $\Lambda = 0$ (i.e. must be pure vacuum); there are no proper Einstein spaces satisfying this assumption. Thus *a fortiori* there are no homogenous proper Einstein spaces of Petrov type I. This provides an independent proof of the result of MacCallum & Siklos [11] (see also [1] §12.9) which does not use group theoretic methods. The only metric turns out to be the homogeneous pure vacuum solution found long ago by Petrov [12] using group theoretic methods. It is the only homogenous Einstein spacetime of Petrov type I; this paper shows that it can be characterised uniquely by the weaker assumption of the constancy of the Weyl eigenvalues.

The above results can be summarised in an alternative way by the statement that the only vacuum spacetimes with constant Weyl eigenvalues are either homogeneous or are Kundt spacetimes of the Lewandowski class. This result is similar to that of Coley et al. [3] who proved their result for **general** spacetimes under the assumption that all scalar invariants constructed from the curvature tensor **and all its derivatives** were constant. The result in this paper is restricted to Einstein spaces of Petrov types I, II & D only, but subject to the weaker assumption of the constancy of the scalar invariants constructed from the curvature tensor **alone**.

It is somewhat disappointing that all solutions in this paper had previously been found by other methods, given the generality of initial assumptions (namely the constancy of the Weyl eigenvalues). However, the paper does provide new characterisations of the solutions based on conditions that are weaker than those originally used to derive them. In particular the proof makes no assumptions regarding the isometry or holonomy groups of the spacetimes.

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