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Highlights

- A model for leak identification in pipes via the Cauchy problem for the heat equation is researched.
- The model is reformulated to fit the application of a recently proposed regularising method.
- Analyses of the regularizing method is presented.
- The regularizing method is implemented using an open source Finite element code.
- Conclusions and further research in this area are given.

Leak identification in saturated unsteady flow via a Cauchy problem

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Abstract

This work is an initial study of a numerical method for identifying multiple leak zones in saturated unsteady flow. Using the conventional saturated groundwater flow equation, the leak identification problem is modelled as a Cauchy problem for the heat equation and the aim is to find the regions on the boundary of the solution domain where the solution vanishes, since leak zones correspond to null pressure values. This problem is ill-posed and to reconstruct the solution in a stable way, we therefore modify and employ an iterative regularizing method proposed in [13, 14]. In this method, mixed well-posed problems obtained by changing the boundary conditions are solved for the heat operator as well as for its adjoint, to get a sequence of approximations to the original Cauchy problem. The mixed problems are solved using a Finite element method (FEM), and the numerical results indicate that the leak zones can be identified with the proposed method.

Keywords: Cauchy problem; heat equation; iterative regularization method; leak identification; mixed problem.

1. Introduction

Pipeline (or aquifer) network reliability is an important issue and of constant concern in fluid transportation systems. Although pipelines are protected against damage (from external impact, internal over-pressure, corrosion, etc.) pressure surges in the network induce leaks and line breaks. To identify such leaks, we shall investigate solving the following Cauchy problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in} \quad \Omega \times (0,T), \\ u = \varphi & \text{on} \quad \Gamma_0 \times (0,T), \\ \partial_n u = \psi & \text{on} \quad \Gamma_0 \times (0,T), \\ u(x,0) = 0 & \text{for} \quad x \in \Omega. \end{cases} \tag{1}$$

Here, u is related to pressure, Γ_0 is a part of the boundary $\partial\Omega$ of the bounded solution domain Ω. The term $\partial_n u$ denotes the normal derivative of u, with n being the outward unit normal to the boundary.

The aim is to find the solution on the remaining part of the boundary of Ω. A zero pressure value there would indicate the presence of a leak. Problem (1) is a classical Cauchy problem, well-known of being ill-posed thus posing a challenge in finding the missing boundary values in a stable way.

In Section 2, we shall go through some derivations and motivations leading to (1) as well as give a background to the iterative method [13] that we use for the stable determination of the solution.

There have been various attempts for leak identification. Due to the wide variety of pipeline systems, many methods and techniques for leak identification have been proposed in the literature with various applicability and restrictions.

To date most approaches have worked in the stationary (time-independent) setting. For example, in [4], it was proposed to locate leaks by identifying the friction parameter of the fluid together with cross-correlation of the output estimation error. In [5], leak identification in a saturated and homogenous porous medium was considered using a different approach, where the leak is considered as a fault or crack in conjunction with a numerical method introduced in [20]. Crack-based models have in general problems in capturing small localised leaks and have trouble with singularities near the crack tips (end-points).

Instead, the same stationary governing model as in [5] was used in [9] but the leak identification was formulated as a Cauchy problem, and the numerical solution was obtained by minimizing an energy-functional.

Thus, the main goals and motivation for the present study can be formulated as:

Generalize the model in [9] to the time-dependent setting. Moreover, employ and implement the method [13] for the stable numerical solution. Furthermore, conclude whether the numerical results are promising in the sense that they can indicate the presence of one or more leak zones.

The last goal mentioned might sound a bit modest but since the underlying leak identification problem is ill-posed, we do not expect to obtain excellent numerical results; only indication of the presence of a leak is to be aimed for. Then, once an estimate of the leak zone has be found more specialised methods, such as level set methods, can be applied to improve the accuracy. However, such additional refinement is not within the aim of the present study. Also, from a practical point of view, having a method that can indicate that there is a leak zone, high accuracy might not be required since as soon as a leak zone is indicated service patrols can be put on alert to inspect and handle the leak.

Before going into derivation of the model, we conclude this introduction with some practical aspects behind leak identification, and give an outline of this work. There are mainly two categories of methods employed to detect product leaks along a pipeline, externally based (direct) or internally based (inferential). Externally based methods usually detect leaks outside of a pipeline and can be performed by traditional methods such as direct inspection by line patrols, as well as sensing technologies like hydrocarbon sensing via fiber optic or dielectric cables. Internally based methods, commonly known as computational pipeline monitoring (CPM), use instruments to monitor internal pipeline parameters (such as pressure, flow, and temperature), which can be used as data for manual or electronic computation, see further [12]. Also, leak identification is an important matter when the contamination of aquifers by pollutants is concerned. Pipeline systems vary in their physical characteristics and operational functions, thus no external or internal method is universally applicable or possesses all the features and functionality required for perfect leak detection performance.

For the outline of this work, in Section 2, we give some background on the equations and assumptions leading to the model (1). In Section 3, we motivate the choice of regularizing strategy. The chosen method involves the solution to mixed boundary value problems, therefore, in Section 4, we review some properties of weak solutions to the heat equation. In Section 5, we give the iterative method, and in Section 6, three numerical examples are given, both for single and multiple leaks, showing promising results meaning that indication of a leak zone is given by the method and the procedure is stable with respect to noise. Finally, in Section 7, we investigate the sensitivity of the obtained solution with respect to the various parameters involved such as the mesh size and time-step.

2. Model formulation and theory on leak identification

For stationary flow, we shall employ the idea of [9], of formulating the leak identification as a Cauchy problem, but consider the unsteady time-dependent setting. Moreover, we include in the present work to also identify multiple leak zones, without any prior assumptions on the leak position or length.

To formulate the model that we shall use, we start with the conventional groundwater flow equation, see [3], given by

$$S(x,t)\partial_t h = \nabla \cdot (K(x,t)\nabla h) + f(x,t),$$

where h is the piezometric head, K is the conductivity (permeability) tensor, S the storativity and f the strength of any sources or sinks. This is a diffusion-type equation and can be derived by combining the mass conservation (fluid balance) with the conservation of momentum (Darcy's law), see [3] and [18] for a derivation. Further assuming that the permeability tensor can be diagonalized and also that the viscosity of the fluid and the porosity of the medium are both constant, and no sinks or sources are present, we can normalize and work with the standard heat equation. Also, using the relation between head and pressure, see [18], the unknown function can be assumed to be the pressure.

Let $\Omega \subset \mathbb{R}^2$ be the sufficiently smooth flow region (a pipe or aquifer for example) and assume that we can measure data over a time interval $(0,T)$, $T>0$, on a part of the boundary $\Gamma = \partial\Omega$, say on an open arc Γ_0. On the remaining part Γ_1 of the boundary, where $\Gamma_1 = \Gamma \setminus \overline{\Gamma}_0$, there is no possibility to measure data (the pipe can be partly buried for example). Relating the above equation to pressure, see [7], we then have the model (1), where u is related to pressure, n is the outward unit normal to the boundary and (φ,ψ) is a pair related to pressure and flux on $\Gamma_0 \times (0,T)$.

We assume that the data (φ,ψ) are "compatible", that is this pair is indeed the pressure and flux of a function u being a solution to the heat equation. For simplicity, we assume a zero initial condition but the method we propose can easily be adjusted to a more general initial condition. From the assumption that the data are given and compatible, there exists a solution (in the classical sense), uniqueness of a solution is well-established, see [21].

3. Motivation of the regularizing strategy

The model (1) is ill-posed in the sense that small perturbations in the data can cause arbitrarily large errors in the calculated solution. Thus, to reconstruct u and to accurately identify the leak zones, regularizing methods need to be applied.

For the corresponding stationary Cauchy problem, there are numerous papers on solving it via iterative regularizing methods based on the ideas presented in [15]. However, to the authors knowledge, considerable fewer works based on such iterative methods have been presented for time-dependent problems. Some works in this direction are [10, 19, 1, 11, 2], where numerical regularizing methods for the parabolic heat equation based on minimizing energy functionals are developed.

We shall employ the method introduced in [13], which have the capability of handling multiple leak zones since it puts no restriction on how the boundary Γ is partitioned and the boundary parts do not need to be separated. In this method, one solves mixed well-posed problems (obtained by changing the boundary conditions) for the heat operator in (1) as well as for its adjoint, to get a sequence of approximations to the original Cauchy problem.

In the method [13], weights are introduced to handle singularities that can occur in the mixed problems where the boundary condition changes type. Since we are only interested in finding the solution in the interior of Γ_1, we shall consider the limiting case when these weights are all equal to one. The mixed problems needed to be solved in the iterations will be numerically solved using a standard FEM capable of handling mixed boundary value problems, see [6] and [22].

We recall again that one novelty of the undertaken work is to implement the method [13] and to produce numerical results for situations corresponding to two-dimensional spatial domains having multiple leak zones. The discretisation is done via the FEM and we shall investigate dependence of the solution with respect to the mesh size and other relevant parameters. As mentioned above, for Cauchy problems for the heat equation, there are few numerical results presented and mainly for one-dimensional domains [16], for higher dimensions, see [10, 19, 1, 11, 2]. Thus, it is of importance to do further numerical investigations of methods for Cauchy problems for time-dependent problems, making our work timely.

Note that the proposed approach can easily be adjusted to $\mathbb{R}^3$ and can also handle the case when material and flow properties are space and time-dependent. However, our focus in this paper is to consider the simplest case to see how the method performs, and in particular to see if leaks can be determined via (1). It is challenging enough to implement and investigate the procedure for two-dimensional spatial domains, thus higher dimensional solution domains and non-homogeneous flows are deferred to future work.

4. Weak solutions

In the iterative procedure for solving the Cauchy problem (1), we need mixed boundary value problems for the heat equation, and we therefore review some of their properties below and then formulate the procedure in the next section.

The iterative procedure that we shall formulate will involve the following mixed boundary value problems for the heat equation and the backward formally adjoint one, defined respectively by :

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \ \Omega \times (0,T), \\ u = \eta & \text{on } \ \Gamma_1 \times (0,T), \\ \partial_n u = \psi & \text{on } \ \Gamma_0 \times (0,T), \\ u(x,0) = 0 & \text{for } \ x \in \Omega, \end{cases} \tag{2}$$

and

$$\begin{cases} \partial_t v + \Delta v = 0 & \text{in } \ \Omega \times (0,T), \\ v = 0 & \text{on } \ \Gamma_1 \times (0,T), \\ \partial_n v = \xi & \text{on } \ \Gamma_0 \times (0,T), \\ v(x,T) = 0 & \text{for } \ x \in \Omega. \end{cases} \tag{3}$$

We shall employ a standard FEM to solve these two problems numerically, see [6] and [22], and we thus recall the concept of a weak solution. The element u is a weak solution of (2) provided that u satisfies

$$\int_0^T \int_\Omega [-u\partial_t w + (\nabla u \cdot \nabla w)] \, dx \, dt = \int_0^T \int_{\Gamma_0} \psi w \, dS \, dt \tag{4}$$

and $u|_{\Gamma_1 \times (0,T)} = \eta$, $u(x,0) = 0$ for a class of test functions w. Provided data is smooth enough there exists a unique weak solution $u \in L^2((0,T); H^1(\Omega))$ and this can be shown using the standard variational approach, see [17].

We do think of the boundary data as smooth functions although in this formulation one can take the data from trace spaces, for example $\eta(\cdot, t)$ in $H^{1/2}(\Gamma_1)$ for almost every t in the time interval $(0, T)$. Then the restriction of u to the boundary is well-defined using the trace map.

For parabolic equations, there are several other ways of introducing a weak solution, for example, demanding more smoothness with respect to the time variable. The formulation of a weak solution chosen here fits with the theoretical framework from [13] as well as with the numerical method for the numerical implementation (FEM).

Note that a concept of very weak solutions, demanding the solution to be only square integrable, was used in [13]. This was mainly a theoretical trick to be able to prove convergence of the regularizing procedure. In fact, choosing test functions having second order weak derivatives in space as required in [13] and employing integration by parts in space in (4), one can verify that (4) is a weak solution in the sense required in [13]. This was not clearly stated in [13] and is probably a reason why that method has not been tested earlier since most standard FEM packages require a relation of the from (4).

We point out two difficulties. Since the boundary parts where we impose the various boundary conditions are not separated, we can not in general expect square integrable derivatives with respect to space of more than first order, that is $u(\cdot, t) \in H^k(\Omega)$ with $k = 1$ and not higher. Moreover, as is clearly pointed out in Remark 2.7 in [8], if we allow for corners on the solution domain (such as Lipschitz domains corresponding to, for example, a rectangular pipe or aquifer), then in general the normal derivative of u does not exist in L^2 on (all of) the boundary. We do not aim to investigate this in more detail, instead we make the following convention.

Assumption on Γ_1
We assume that the boundary part Γ_1 is C^2-smooth (in particular, it does not contain any corner points).

With this assumption, the normal derivative of the solution exists on Γ_1 and is locally square integrable, see Lemma 6.2 in [13], and this is needed in the next section.

5. An iterative procedure for the Cauchy problem (1)

To handle the difficulty concerning the smoothness of the solution of (2) and (3) near corner points, certain weights were introduced in the regularizing procedure presented in [13]. However, since we are only interested in detecting leaks in the interior of the boundary Γ_1, and this part is assumed to be smooth, we shall consider the limiting case in [13], when the weights are all equal to one.

Note that we are working with a classical and standard weak solution and the numerical routine chosen for the numerical solution of the mixed problems are capable of handling polygonal domains. There could still potentially be an advantage to use weights in particular if high accuracy is needed near corner points, but investigating this is deferred to a future work.

Thus, having the weights all being put equal to one, the iterative procedure for the stable solution of the leak identification problem (1) is the following:

(i) Choose an arbitrary smooth and square integrable initial function η_0 and put $k = 0$.

(ii) Solve problem (2) with the Dirichlet boundary condition changed to $u = \eta_k$ on $\Gamma_1 \times (0, T)$ to obtain the solution u_k in $L^2((0, T); H^1(\Omega))$ in the weak sense (4).

(iii) Then solve the problem (3) with $\partial_n v = u_k - \varphi$ on $\Gamma_0 \times (0, T)$ and denote the solution by v_k belonging to $L^2((0, T); H^1(\Omega))$.

(iv) Let $\eta_{k+1} = \eta_k + \gamma \partial_n v_k|_{\Gamma_1 \times (0,T)}$, put $k = k + 1$, and repeat the steps (ii)–(iv).

In the case of no noise in the data, the procedure continues until the desired level of accuracy has been achieved.

Note that, as pointed out in the previous section, using the standard trace map, the restriction of the weak solution of (2) to Γ_1 is well-defined. Moreover, due to the smoothness assumption on Γ_1 stated in the previous section and since v_k is zero on Γ_1, using local regularity results for parabolic equations it follows that the normal derivative of the solution of (3) on $\Gamma_1' \times (0, T)$ exists and is sufficiently smooth, where $\overline{\Gamma_1'} \subset \Gamma_1$. Thus, the various restrictions to the boundary in the procedure are well-defined at least locally.

For the convergence of the obtained sequence, from Theorem 7.1. in [13], we have:

Theorem:
Let u be the solution to (1) and let u_k be the k-th approximation in the above procedure. Let Γ_1' be an arc of the boundary Γ_1 with $\overline{\Gamma_1'} \subset \Gamma_1$. Then

$$\|u - u_k\|_{L^2(\Gamma_1' \times (0,T))} \to 0, \tag{5}$$

for any (sufficiently smooth) data element η_0*.*

The proof of Theorem 7.1. in [13] is based on a reformulation of the procedure in terms of iterations on the boundary; the procedure is shown to be equivalent to a Landweber-Fridman type regularization. Moreover, using local estimates for parabolic equations, one can show convergence of derivatives of u as well on Γ_1. The discrepancy principle can be applied as a stopping rule in the case of noisy data.

For the numerical implementation given in the next section of the mixed problems (2) and (3), we employ a standard FEM that can handle possible singularities near the end points of Γ_1.

6. Numerical examples

In this section, we illustrate the numerical results obtained using the procedure described in the previous section. To test the efficiency of the proposed numerical method, we solve the Cauchy problem (1) in a 2-dimensional setting in space. The corresponding direct problems (2) and (3) are solved using a FEM obtained from the `Freefem++` package. This FEM is based on time-marching, and we use $T = 40$ and step size $\delta t = 1$ with the regularizing parameter $\gamma = 0.1$. The stopping criteria in time is:

$$\|\eta_k - \eta_{k-1}\|_{L^2(\Omega)} \leq \varepsilon,$$

where ε is a constant that refers to the accepted error tolerance. The initial guess to start the procedure is chosen as $\eta_0 = 0$, which clearly satisfies the smoothness assumptions of Section 2.

Example 1: (General Cauchy heat conduction problem) We first solve (numerically) the Cauchy problem associated with the heat transient equation for an annular domain Ω, with inner and outer radius $r_1 = 1$ and $r_2 = 2$ respectively, where the boundary part Γ_0 is the outer circle and Γ_1 the inner one, both circles being centred at the origin.

The synthetic data are generated on a different mesh to avoid what is known as the inverse crime, by solving the following forward problem:

$$\begin{cases} \partial_t u_0 - \Delta u_0 = 0 & \text{in} \quad \Omega \times (0,T), \\ \partial_n u_0 = t & \text{on} \quad \Gamma_0 \times (0,T), \\ u_0 = \mathrm{Re}(-t/(z-a)) & \text{on} \quad \Gamma_1 \times (0,T), \\ u_0(x,0) = 0 & \text{for} \quad x \in \Omega, \end{cases}$$

where $z = x + iy$ and a is a source point incorporated to generate a singular type behaviour expected near the leak zone (corresponding to a pressure drop or rise); when the source point approaches the boundary Γ_1 the restriction of the solution to Γ_1 has a more narrow peak, see Figures 1 and 3.

Taking the restriction of this solution and its normal derivative (which both are well-defined, see further Section 2) to $\Gamma_0 \times (0, T)$, we obtain the (synthetic) Cauchy data φ and ψ in (1).

In Table 1, we present numerical results for different values of the parameter a for the reconstruction of u on the boundary Γ_1 for time $t = 40$. The source point is placed at three different locations on the negative real axis. The number of nodes stated is the total number used on all of the boundary.

As can be seen from this table, the number of iterations needed are rather large but do decay for values of the parameter a closer to 0. This is expected since with the source point located further away from the boundary the less singular the behaviour of the solution is. Also, by varying the regularization parameter γ in the procedure one can possibly decrease the number of iterations further.

a	nr. of nodes	nr.elements	ε	nr. iterations
-0.8 (Figs. 1–2)	613	1076	10^{-3}	30910
-0.9 (Fig. 2)	613	1076	10^{-3}	51745
-0.5 (Figs. 3–4)	613	1076	10^{-3}	8148

Table 1: Parameters and results for Example 1

In Figures 1 and 3, the reconstructions of the Dirichlet data in Example 1 on Γ_1 are presented for $a = -0.8$ and $a = -0.5$, respectively, for time $t = 5$, 15 and 40. Keeping in mind one of the aims of this work, namely to have a numerical method that can indicate the presence of a leak which would correspond to the peak in these figures, we can conclude that this should be possible judging from these figures; the numerically calculated solution has a peak at the right place although the hight is not perfectly captured. Accuracy is improved when a is further from the boundary ($a = -0.5$), and as mentioned above, this is expected since with the source point located further away from the boundary the less singular the behaviour of the solution is.

For the reader not familiar with ill-posed problems, we point out that perfect agreement is seldom found for such models. By careful selection of the regularizing parameter (γ together with the iteration index in this case)

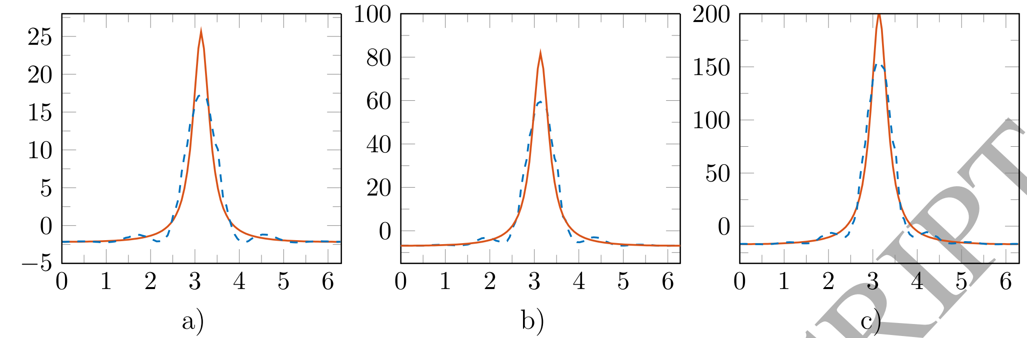

Figure 1: Reconstruction (– – –) of Dirichlet data (—) in Example 1 for $a = -0.8$ with a) $t = 5$ b) $t = 15$ and c) $t = T = 40$

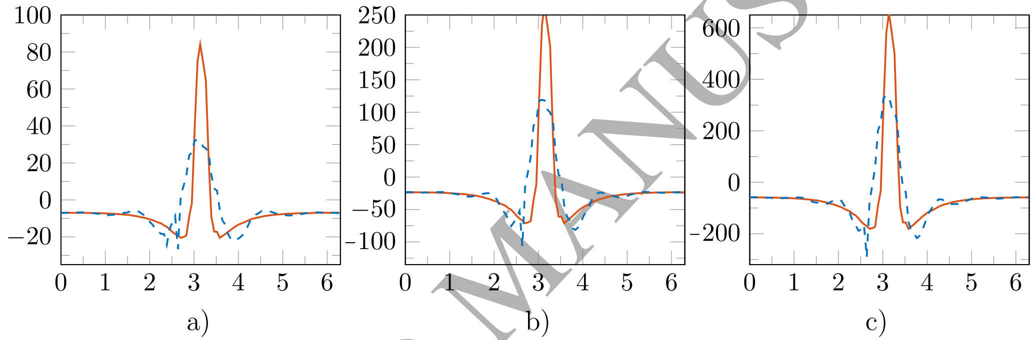

Figure 2: Reconstruction (– – –) of Neumann data (—) in Example 1 for $a = -0.8$ with a) $t = 5$ b) $t = 15$ and c) $t = T = 40$

it it usually possible to improve the results. However, we are only after indications and not the highest accuracy.

In Figures 2 and 4, the reconstructions of the Neumann data for Example 1 on Γ_1 are presented for $a = -0.8$ and $a = -0.5$, respectively, for $t = 5, 15$ and 40. As expected, the reconstructions are less accurate than for the Dirichlet data and in particular where the derivative varies rapidly. This is a known phenomena, since differentiation in itself is a numerically unstable process. Noteworthy here, albeit not having high accuracy, is that the numerical approximation follows the behaviour of the exact solution. Also here, accuracy is improved with a further from the boundary ($a = -0.5$).

Encouraged by these reconstructions, we then consider two examples simulating the identification of leak zones.

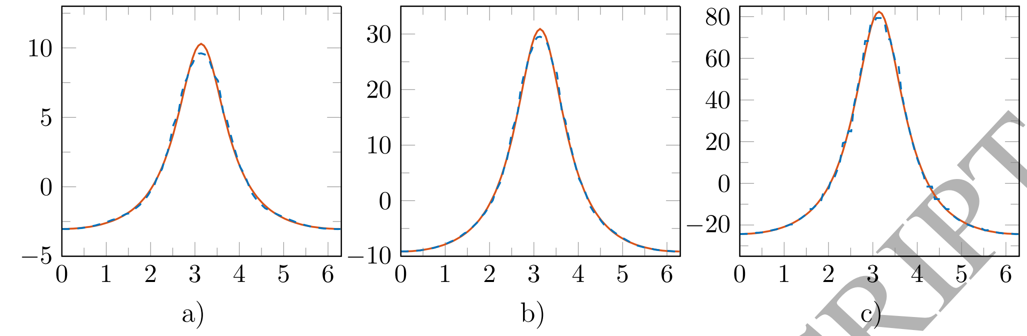

Figure 3: Reconstruction (– – –) of Dirichlet data (—) in Example 1 for $a = -0.5$ with a) $t = 5$ b) $t = 15$ and c) $t = T = 40$

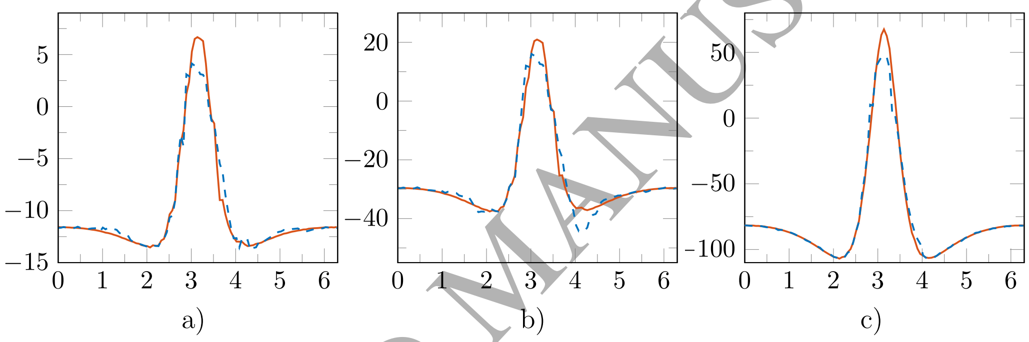

Figure 4: Reconstruction (– – –) of Neumann data (—) in Example 1 for $a = -0.5$ with a) $t = 5$ b) $t = 15$ and c) $t = T = 40$

Example 2. (Application: Single leak identification) In this example, we apply the present iterative method to a problem in hydrogeology. We consider an underground aquifer flowed by a liquid saturating a porous media. The aim is to identify leaks on an inaccessible part of the boundary by exploiting overspecified measurements on the remaining part as described in Section 1. The leak is characterised by null pressure values, and we thus have to locate parts where $u = 0$. We assume that the domain is a rectangle, see Figure 5, and the leak is located in-between $x = 0.5$ and $x = 0.6$ for $y = 0.5$ (with (x, y) being standard coordinates for a point in $\mathbb{R}^2$).

The synthetic data are again generated on a finer mesh to avoid the inverse crime by solving the following forward problem:

$$\begin{cases} \partial_t u_0 - \Delta u_0 = 0 & \text{in} \quad \Omega \times (0,T), \\ \partial_n u_0 = 60 & \text{on} \quad \Gamma_0 \times (0,T), \\ u_0 = 10 & \text{on} \quad \Gamma_1 \times (0,T), \\ u_0 = 0 & \text{on} \quad \Gamma_2 \times (0,T), \\ u_0(x,0) = 0 & \text{for} \quad x \in \Omega. \end{cases}$$

The parameters are kept the same as in the previous example.

In Figure 6, we present the comparison between the exact and the reconstructed solution on Γ_1 at time $t = 10$, both for the Dirichlet and Neumann data. From the behaviour of the approximation to the Dirichlet data seen in Figure 6a), we believe that there is merit in the proposed procedure. There is a dip in pressure at the correct location. However, the hight of the dip is not perfectly reconstructed. This is not to be expected due to the ill-posedness, and the fact that we have not elaborated in finding the best possible initial guess to start the procedure and not chosen the mesh in an optimal way. Thus, it should be possible with further processing, for example, using a better initial guess and possibly also invoking level set techniques, to further improve the accuracy and better reconstruct the dip where the pressure goes to zero indicating the presence of a leak.

Further evidence of the feasibility of the proposed method is seen from the reconstruction of the Neumann data in Figure 6b). As mentioned above, this is usually difficult to accurately obtain for an ill-posed problem; note here that the synthetic solution tested against (obtained via the chosen FEM) is not fully accurate either showing that finding the normal derivative for a well-posed problem is challenging not to mention for an ill-posed problem. Therefore, it is pleasing to see that the proposed method also for the Neumann data function tries to locate the change near the leak zone. Even without post-processing looking at the result in Figure 6, one could get a rough idea of the presence of a leak zone and its location.

Example 3. (Application: Multiple leak zone identification) We consider a case where two leaks are located on the boundary Γ_1. The synthetic

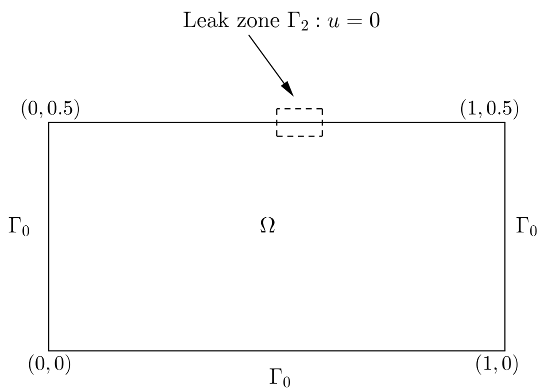

Figure 5: Domain in Example 2

data are generated by solving the forward problem of Figure 7, that is

$$\begin{cases} \partial_t u_0 - \Delta u_0 = 0 & \text{in} \quad \Omega \times (0,T), \\ u_0 = -60 & \text{on} \quad \Gamma_0^{(1)} \times (0,T), \\ \partial_n u_0 = 0 & \text{on} \quad \Gamma_0^{(2)} \times (0,T), \\ u_0 = -30 & \text{on} \quad \Gamma_0^{(3)} \times (0,T), \\ u_0 = 10 & \text{on} \quad \Gamma_1^{(1)} \times (0,T), \\ u_0 = 0 & \text{on} \quad (\Gamma_1^{(2)} \cup \Gamma_1^{(3)}) \times (0,T), \\ u_0(x,0) = 0 & \text{for} \quad x \in \Omega. \end{cases}$$

Two different cases are investigated. First, in Figures 8 and 9, the reconstructions are given on Γ_1 for two different time points ($t = 10$ and $t = 20$) for the case when the two leaks are of identical size (in space), and the leaks are located in-between $x = 0.25$ and $x = 0.3$ and $x = 0.55$ and $x = 0.6$, respectively. In Figures 10 and 11, the similar reconstructions are shown when the leak zones have different size and are located between $x = 0.28$ and $x = 0.3$ and $x = 0.55$ and $x = 0.6$, respectively.

From the Figures 8a) and 9a), we see that we have a similar behaviour as for the Dirichlet data reconstruction of Figure 6a). Thus, that the numerical solution of Figure 6a) was dipping near the leak zone was probably not a coincidence; we see this behaviour also in Figures 8a) and 9a), where the numerical approximation dips at both the leak zones. Moreover, the normal

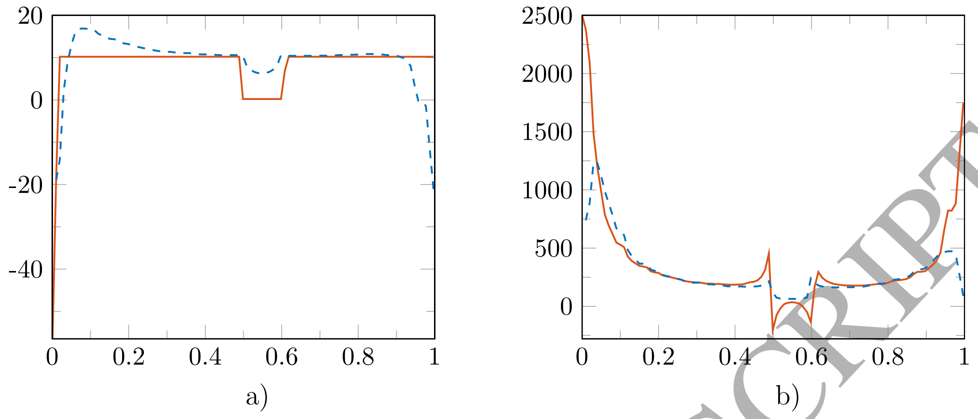

Figure 6: Reconstruction (– – –) and exact (—) solution on Γ_1 with $t = 10$ for a) Dirichlet data and b) Neumann data, in Example 2

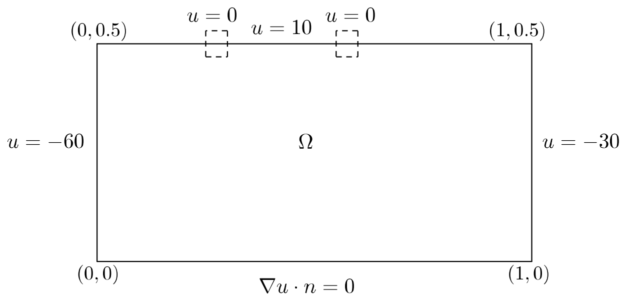

Figure 7: Domain in Example 3

derivative tries to follow the exact solution as well. It is known that the numerical approximations of time-dependent Cauchy problems deteriorates in time, although in this example this effect is not much highlighted.

Having leak zones of different sizes does not change the results much, the dips are still in the correct regions, see Figures 10–11.

Thus, although the results are not very accurate, for an ill-posed problem we still believe that the behaviour of the approximations compared to the exact solution makes the method merit of further study. In particular, combining it with more post-processing and perhaps a better initial guess can make the leak zones more accurately located.

In total, as mentioned in the previous example, the height of the dips are far from perfect. That the method is capable of at least indicating these dips makes it worthy of future investigations. There are various techniques for further improvements as mentioned at the end of the previous example.

In the next section, we shall investigate the influence of various parameters to illustrate that the obtained results are not overly sensitive to in particular the chosen mesh size and regularizing parameter. A full investigation of the sensitivity with respect to the parameters, including error estimates, is far beyond the scope of the present study.

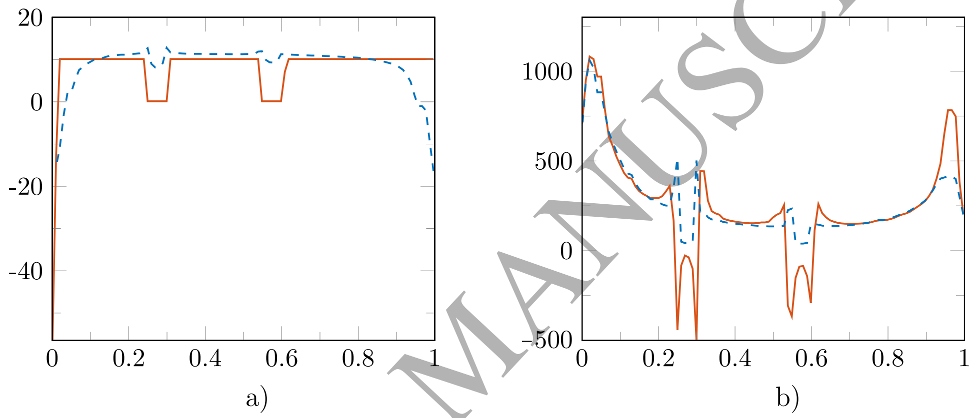

Figure 8: Reconstruction (– – –) and exact (—) solution on Γ_1 with $t = 10$ for a) Dirichlet data and b) Neumann data, in Example 3

7. Variation of the parameters

It is known in general that for an ill-posed problem having a regularizing strategy it should be possible to adjust the mesh size, time-step, stopping index and regularizing parameter such that the obtained numerical approximation will tend to the solution of the Cauchy problem (1). It is a careful interplay between the various parameters. It could still be the case that the numerical solution is very sensitive with respect to one or more of these parameters. To at least indicate the influence of these parameters on the present procedure, we shall produce results when changing one of the parameters keeping the others fixed to thereby see its influence on the obtained numerical reconstruction.

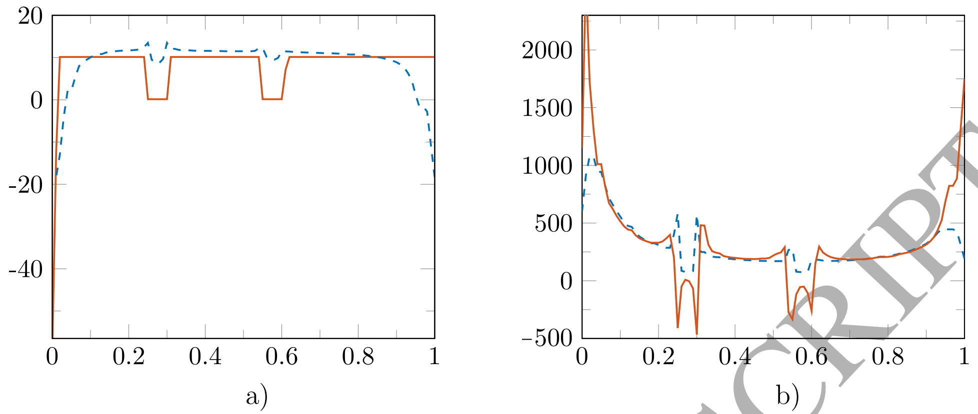

Figure 9: Reconstruction (– – –) and exact (—) solution on Γ_1 with $t = 20$ for a) Dirichlet data and b) Neumann data, in Example 3

7.1. Changing the mesh size

We fix the following parameters: $\delta t = 1$, tolerance $= 10^{-3}$, $\gamma = 0.5$ and $T = 5$. We shall investigate the stability of the solution with respect to the mesh size.

To begin with, changing the mesh size influences the number of iterations needed before reaching the chosen stopping criteria. The dependence of the number of iterations on the mesh size are given in Table 2. From Table 2, it can be seen that increasing the mesh size decreases the number of iterations slightly. More mesh points makes the discretised problem to be closer to the original Cauchy problem, and in particular more numerically unstable, forcing an earlier termination of the iterations.

The accuracy of the reconstructions are somewhat improved as is illustrated in Figures 12–13 for the Dirichlet and Neumann data, respectively. In particular, the improvement of the reconstruction of the Neumann goes up several factors when increasing the mesh size. This is some evidence to our claim in the previous examples that the numerical results can be improved by elaborating on the parameters.

7.2. Variation of the regularizing parameter γ

We consider now the mesh size fixed and composed of 50 nodes on Γ_1 and 100 nodes on Γ_0, and change the regularizing parameter γ.

The number of iterations needed are given in Table 3. Rather than producing more figures, we report here without illustrations that the reconstruc-

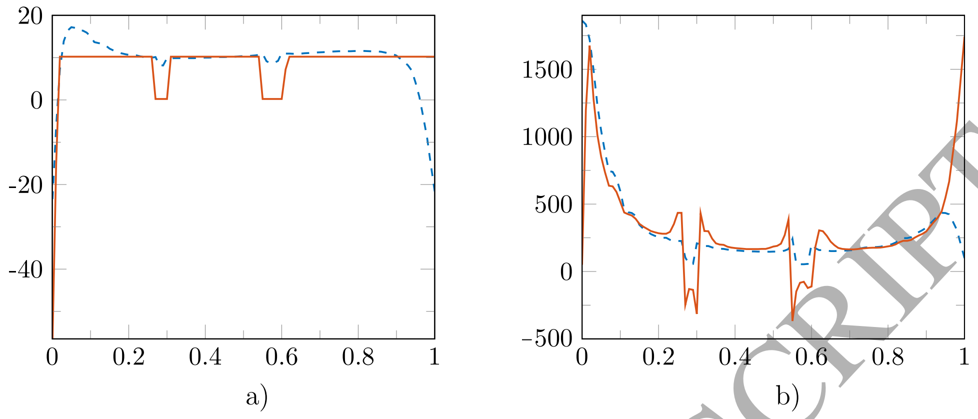

Figure 10: Reconstruction (– – –) and exact (—) solution on Γ_1 with $t = 10$ for a) Dirichlet data and b) Neumann data, in Example 3 with leak zones of different sizes

nr. of nodes on Γ_1	nr.of nodes on Γ_0	total nr. iterations
30	60	1106
50	100	919
200	300	896

Table 2: Number of iterations as a function of the mesh size

tions are sensitive with respect to the regularizing parameter. This is not something specific for our method but a phenomena for regularizing procedures. One could revert to parameter free iterative methods such as conjugate gradient type methods, where only the iteration index need to be chosen. In our case, choosing the parameter γ such that $0.2 \leq \gamma \leq 0.6$ gives stable results improving with smaller values.

A common ad-hoc way to choose the regularizing parameter is to calculate the numerical solution for a range of values of this parameter decreasing to zero. Once unstable (oscillating) numerical results are produced one knows that the regularizing parameter has been chosen too small and no further improvement in terms of accuracy can be achieved.

7.3. Variation of the time step δt

In our final test, we investigate the influence of changing the time step δt. We fix the regularizing parameter $\gamma = 0.5$ and also fix 50 nodes on Γ_1 and 100 nodes on Γ_0.

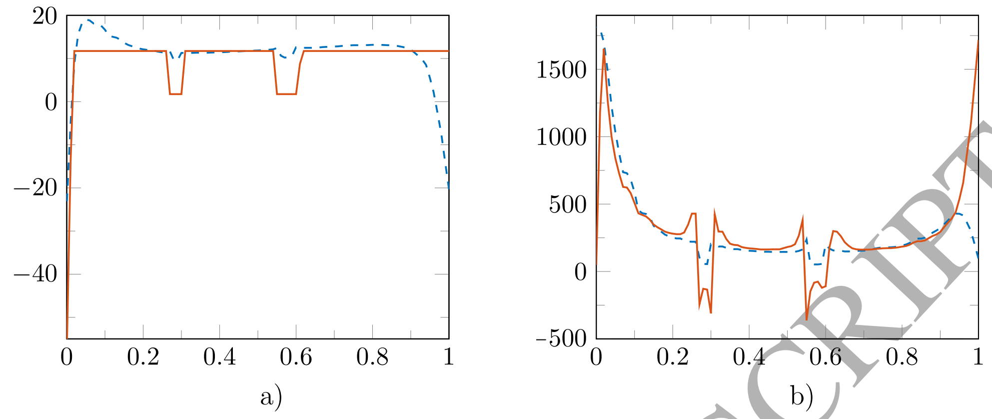

Figure 11: Reconstruction (– – –) and exact (—) solution on Γ_1 with $t = 10$ for a) Dirichlet data and b) Neumann data, in Example 3 with leak zones of different sizes

γ	0.1	0.25	0.5
nr. iterations	228	200	185

Table 3: Number of iterations as a function of the regularizing parameter γ

In Table 4 are given the number of iterations as a function of the time step δt. We notice that there is not much difference in the number of iterations by changing the time step. We report, although not illustrated graphically, that the quality of the reconstructions are similar. This is not surprising since the time-step influences mainly the solution of the direct problems and these are well-posed. Of course, with a too small step size we are closer to the original Cauchy problem but also the FEM chosen has a threshold for the time-step, and with a too small step size numerical inaccuracy will be obtained when solving the direct problems.

We point out that, as a generally known phenomena, the reconstructions of a parabolic Cauchy problem will not be accurate at the final time $t = T$.

δt	0.8	0.9	1	1.1
nr. iterations	217	241	185	202

Table 4: Number of iterations as a function of the time-step δt

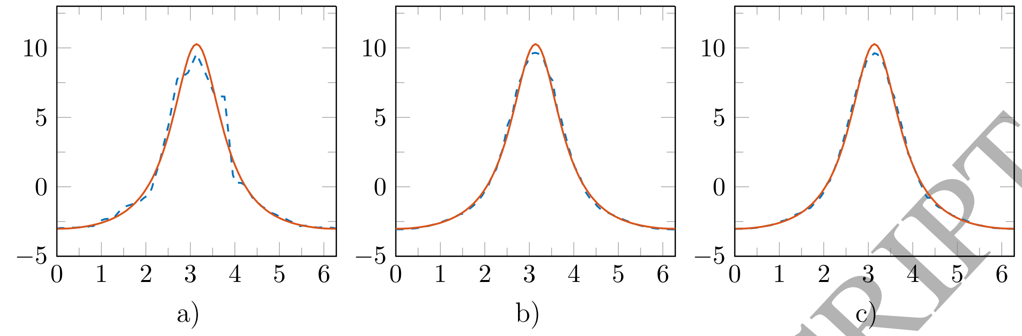

Figure 12: Reconstruction (– – –) of Dirichlet data (—) in Example 1 on Γ_1 with $a = -0.5$ and $t = 5$ for a) 30 nodes on Γ_1 and 60 nodes on Γ_0 b) 50 nodes on Γ_1 and 100 nodes on Γ_0 c) 200 nodes on Γ_1 and 300 nodes on Γ_0

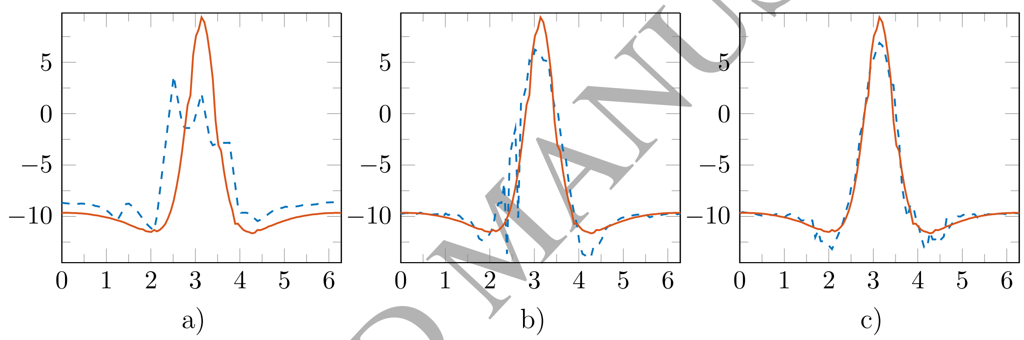

Figure 13: Reconstruction (– – –) of Neumann data (—) in Example 1 on Γ_1 with $a = -0.5$ and $t = 5$ for a) 30 nodes on Γ_1 and 60 nodes on Γ_0 b) 50 nodes on Γ_1 and 100 nodes on Γ_0 c) 200 nodes on Γ_1 and 300 nodes on Γ_0

8. Conclusions

We investigated an iterative numerical method for the stable identification of multiple leak zones in saturated unsteady flow. Using the conventional saturated groundwater flow equation, the leak identification problem was modelled as a Cauchy problem for the (time-dependent) heat equation and leak zones correspond to null pressure values. The iterative method proposed is a limiting case of a method investigated in [13, 14], in the sense that the weights incorporated to improve accuracy of solutions to the heat equation and its adjoint near corner points are put equal to unity, and at each iteration step mixed well-posed boundary value problems for the same governing equation and its adjoint are solved in the classical weak sense.

Few methods prior to the present study looked into the identification of small and localised leaks on the boundary taking into consideration the time-dependency of the model. Most of the earlier studies modelled a leak as a crack but this leads to difficulties when numerically solving the forward problems due to singularities near the crack tips requiring local refinement of the mesh near these in the numerical implementation. Moreover, such crack based leak models have in general problems capturing small localised leaks. In the present study, standard mixed boundary value problems for parabolic heat type operators were used and these are rather straightforward to numerically implement using standard freely available FEM-packages.

Three numerical examples were presented, showing that the proposed approach is capable of indicating the presence of both single and multiple localised leaks. The accuracy of the dips in the numerical solution was not of very high accuracy but since no attempt in optimizing the chosen parameters was done, the accuracy can be further improved. Evidence of this was seen from an included investigation of the influence of the numerical reconstruction on parameters such as the mesh size, regularizing parameter and iteration index. It is important though to obtain Cauchy data on a sufficiently large portion of the boundary of the solution domain otherwise the problems will be too ill-posed. A possible drawback of the given method is the rather large number of iterations needed. However, it is possible to speed up the convergence using conjugate gradient type methods. This together with using weighted spaces to improve accuracy near the corner points, in particular for the normal derivative, are deferred to future investigations.

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References

[1] S. Andrieux and T. N. Baranger; Energy methods for Cauchy problems of evolutions equations. Journal of Physics: Conference Series 135 (2008).

[2] A. Ben Abda, I. Ben Saad, M. Hassine; Recovering boundary data: The Cauchy Stokes system. Applied Mathematical Modelling, 2013 - Elsevier

[3] J. Bear; *Hydraulics of Groundwater*. McGraw-Hill, New York, (1979).

[4] L. Billmann and R. Isermann; Leak detection methods for pipelines. Ninth IFAC World Congress, 1813–1818 (1984).

[5] Z. Belhachmi, A. Karageorghis and A. Taous; Identification and reconstruction of a small leak zone in a pipe by a spectral element method. J. Scientific Computing 27, 111–122 (2006).

[6] J. F. Botha and G. N. Bakkes; Galerkin finite element method and the groundwater flow equation: Convergence of the method. Adv. Water Resources 5, 121–126 (1982).

[7] J. D. Bredehoeft and G. F. Pinder; Mass transport in flowing groundwater. Water resources research, 9, 194–210 (1973).

[8] M. Costabel; Boundary integral operators for the heat equation. Integral Equations Oper. Theory 13, 498–552 (1990).

[9] X. Escriva, T. N. Baranger, N. H. Tlatli; Leak identification in porous media by solving the Cauchy problem. C. R. Mecanique 335, 401–406 (2007).

[10] G. Bastay, V. A. Kozlov and B. O. Turesson; Iterative methods for an inverse heat conduction problem. J. Invese Ill-Posed Problems 9, 375–388 (2001).

[11] D. N. Hào, X. T. Pham, D. Lesnic and B. T. Johansson; A boundary element method for a multi-dimensional inverse heat conduction problem. Int J. Comput. Math. 89, 1540–1554 (2012).

[12] H. Hill; *Technical Review of Leak Detection Technologies*, Volume I, *Crude Oil Transmission Pipelines*. Alaska Department of Environmental Conservation, 1999.

[13] B. T. Johansson; An iterative method for a Cauchy problem for the heat equation. IMA Journal of Applied Mathematics 71, 262–286 (2006).

[14] B. T. Johansson; Determining the temperature from incomplete boundary data. Math. Nachr. 280, 1765–1779 (2007).

[15] V. A. Kozlov and V. G. Maz′ya; On iterative procedures for solving ill-posed boundary value problems that preserve differential equations. Algebra i Analiz 1, 144–170, (1989). English translation: Leningrad Mathematics Journal 1, 1207–1228 (1991).

[16] N.S. Mera, L. Elliott, D.B. Ingham, D. Lesnic; An iterative boundary element method for solving the one-dimensional backward heat conduction problem. International Journal of Heat and Mass Transfer 44,(2001) 1937–1946.

[17] J.-L. Lions, E. Magenes; *Non-homogeneous Boundary Value Problems and Applications*. Vol. I, Springer, Berlin, 1972.

[18] J. W. Mercer and C. R. Faust; Ground-water Modeling: Mathematical models. Ground-Water 18, 212–227 (1980).

[19] H-J Reinhardt , D. N. Hào , J. Frohne J. and F.-T. Suttmeier; Numerical solution of inverse heat conduction problems in two spatial dimensions. J. Inverse Ill-posed Problems. 15 181–198 (2007).

[20] F. Santosa and M. Vogelius; A computational algorithm for determining cracks from electrostatic boundary measurements. Int. J. Eng. Science 29, 917–937 (1991).

[21] J. C. Saut and B. Scheurer; Unique continuation for some evolution equations. J. Differential Equations 66, 118–139 (1987).

[22] V. Thomée; *Galerkin Finite Element Methods for Parabolic Problems*, 2nd Edition. Springer-Verlag, Heidelberg, (2006).