Accepted Manuscript

Uniqueness and counterexamples in some inverse source problems

M. Slodička, B. Tomas Johansson

PII: S0893-9659(16)30045-3
DOI: http://dx.doi.org/10.1016/j.aml.2016.02.001
Reference: AML 4942

To appear in: *Applied Mathematics Letters*

Received date: 19 November 2015
Revised date: 1 February 2016
Accepted date: 2 February 2016

Available online at www.sciencedirect.com
ScienceDirect

Applied Mathematics Letters 00 (2016) 1–6

Uniqueness and counterexamples in some inverse source problems☆

M. Slodička

Department of Mathematical Analysis, research group of Numerical Analysis and Mathematical Modeling (NaM^2), Ghent University, Galglaan 2 - S22, Gent 9000, Belgium

B. Tomas Johansson

Mathematics, EAS, Aston University, B4 7ET Birmingham, UK

Abstract

Uniqueness of a solution is investigated for some inverse source problems arising in linear parabolic equations. We prove new uniqueness results formulated in Theorems 3.1 and 3.2. We also show optimality of the conditions under which uniqueness holds by explicitly constructing counterexamples, that is by constructing more than one solution in the case when the conditions for uniqueness are violated.

Keywords: heat equation, inverse source problem
2010 MSC: 65M32, 65M20

1. Introduction

Mathematical models related to inverse source problems (ISP:s) for parabolic equations arise in various applications such as in the location of a pollutant source in groundwater flow and in the design and control of various heat processes. Partly due to its importance in applications there has been a recent surge on research for various types of ISP:s for parabolic problems, see for example references in [1, 2].

A classical uniqueness result, [3], asserts that for a heat (or diffusion) process with coefficients solely depending on the space variables, a time-independent heat source can be uniquely reconstructed from final time data. It is straightforward to see that there cannot be any uniqueness result for this type of inverse problem in the class of general heat sources that depends on both space and time. It is, however, not immediately clear in terms of uniqueness of a spacewise dependent heat source from final time data in the case when the heat or diffusion process has time-dependent coefficients. That particular case is researched in [2].

In the present work, we shall investigate the given conditions for uniqueness derived in [2], and construct examples showing that when these conditions are violated there is more than one solution. Moreover, we generalize the results

☆The research was supported by the IAP P7/02-project of the Belgian Science Policy.
Email address: marian.slodicka@ugent.be (M. Slodička)
URL: http://cage.ugent.be/~ms (M. Slodička)

in [2], holding for spacewise dependent heat sources, to a wider class of sources being the product of a known time-dependent part and an unknown (to be reconstructed) spacewise dependent part. Sources of the latter form have been used, for example, in [4].

2. Optimality of the conditions for uniqueness derived in reference [2]

Consider a bounded domain Ω with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. To set out to investigate conditions for uniqueness, we start by being more precise and recall that the following inverse source problem (ISP) is studied in [2]: Find a solution pair $\{u(x,t), f(x)\}$ such that

$$\begin{cases} \partial_t u + \nabla \cdot (-\boldsymbol{A}(x,t)\nabla u) + c(x,t)u = f(x) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

given the final time data

$$u(x,T) = \psi_T(x), \qquad \text{for } x \in \Omega \tag{2}$$

where $T > 0$. It is assumed that

$$\boldsymbol{A}(x,t) = \left(a_{i,j}(x,t)\right)_{i,j=1,\ldots,n}, \qquad \boldsymbol{A} = \boldsymbol{A}^t,$$

together with the requirement

$$\boldsymbol{\xi}^t \cdot \boldsymbol{A}\boldsymbol{\xi} = \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq C|\boldsymbol{\xi}|^2 \tag{3}$$

for every x, t and any $\boldsymbol{\xi}^t = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, where $n \geq 1$.

Uniqueness in the retrieval of the solution and source to the ISP (1)–(2) is proved in [2, Thm. 3.7] under the conditions that

$$\boldsymbol{\xi}^t \cdot \partial_t\left(\boldsymbol{A}^{\frac{1}{2}}\right)\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{\xi} \leq 0, \qquad \forall \xi \in \mathbb{R}^n \qquad \text{and} \qquad \partial_t c(t) \leq 0 \qquad \forall t \in [0,T]. \tag{4}$$

Note that the matrix $\boldsymbol{A}$ obeys (3), and is therefore positive definite in space and time, thus there exists a unique positive definite square root $\boldsymbol{A}^{\frac{1}{2}}$, cf. [5, Chpt. VII, §3]. For scalar coefficients, the conditions (4) translate into the requirements that both $A(t)$ and $c(t)$ decrease in time.

It is natural to ponder on the question whether the requirement to have the coefficients decreasing in time can be relaxed or perhaps dropped altogether and still have uniqueness of a solution to the ISP (1)–(2). Rather surprisingly, the following example shows that without the condition of decay in time we cannot guarantee uniqueness in the above ISP.

Example 1. Consider the following one-dimensional ISP for $x, t \in (0,\pi)$ having zero final time data,

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + c(t)u(x,t) = f(x), \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = 0, \\ u(x,\pi) = 0, \end{cases} \tag{5}$$

with the choice $c(t) = \frac{1-\cos(t)}{\sin(t)} - 1$. Easy calculations show that the coefficient c can be naturally extended to $t = 0$ with $c(0) = -1$, and that $c'(t) = (\cos(t) + 1)^{-1} > 0$. Hence, c is smooth and non-decreasing for t in $(0,\pi)$. Besides the trivial solution $(u, f) = (0,0)$ to (5) we have, as can be readily checked, the following non-trivial one

$$u(x,t) = \sin(x)\sin(t), \qquad f(x) := u_t(x,t) - u_{xx}(x,t) + c(t)u(x,t) = \sin(x).$$

Hence, if the coefficient c is not decreasing in time, the uniqueness of a solution to the ISP can be violated. $\triangle$

The reader might object to the fact that $c(t)$ tends to infinity when t tends to the final time T, and that the integral of $c(t)$ is not finite. However, this is not essential. In fact, it is possible to come up with a more involved example based on results in [6] for a bounded function $c(t)$. We return to this point at the end of the present work. Moreover, one can produce examples in multidimensional solution domains, and a technique for this is outlined in Example 2 below.

3. Time dependent heat sources of the form $f(x)h(t)$

In this section we then extend the proof-technique from [2], which was developed for the case of sources $f(x)$ only depending on the space variables in the ISP (1)–(2), to a wider class of sources being also time-dependent; these sources are of the form of a product of a spacewise dependent source and a time-dependent one that is $f(x,t) = f(x)h(t)$ with the time-dependent part $h(t)$ assumed to be known.

We use a general second-order linear differential operator, involving also a gradient term in the space variables,

$$L(t)u(x,t) = \nabla \cdot (-\boldsymbol{A}(x,t)\nabla u(x,t)) + \boldsymbol{b}^t(x)\nabla u(x,t) + c(x,t)u(x,t). \tag{6}$$

The more general ISP is then to reconstruct $f(x)$ given $h(t)$ such that

$$\begin{cases} \partial_t u(x,t) + L(t)u(x,t) = f(x)h(t) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{7}$$

using the additional final time data (2).

Dividing the governing partial differential equation in (7) by the known (given) function $h(t)$, tacitly assuming that h does not have any zeros, we obtain

$$\partial_t v(x,t) + L(t)v(x,t) + \frac{h'(t)}{h(t)} v(x,t) = \partial_t v(x,t) + \nabla \cdot (-\boldsymbol{A}(x,t)\nabla v(x,t)) + \boldsymbol{b}^t(x)\nabla v(x,t) + \left(c(x,t) + \frac{h'(t)}{h(t)}\right) v(x,t) = f(x)$$

with $v = \frac{u}{h}$ and $\partial_t v = \frac{\partial_t u}{h} - v\frac{h'}{h}$. Thus, after division by h the governing equation can be rewritten in the similar form as the one in (7) with the only change being to replace the coefficient $c(x,t)$ by $c(x,t) + \frac{h'(t)}{h(t)}$ and having a solely spacewise dependent heat source in the right-hand side. Uniqueness for that case is investigated in [2]. Hence, we can show the following result.

Theorem 3.1. *Consider a linear partial differential operator $L(t)$ given by (6) with bounded coefficients (possibly being discontinuous in x) such that $\boldsymbol{A}^t = \boldsymbol{A}$, the condition (3) holds and $\boldsymbol{b} = \boldsymbol{0}$. Let $u_0, \psi_T \in L^2(\Omega)$, $h \in C^2((0,T))$ with h non-zero in the given time-interval. Suppose that*

$$\alpha'(t) \le 0, \text{ where } \alpha(t) := \frac{h'(t)}{h(t)}.$$

Moreover, assume also that

$$\boldsymbol{\xi}^t \cdot \partial_t \left(\boldsymbol{A}^{\frac{1}{2}}\right) \boldsymbol{A}^{\frac{1}{2}} \boldsymbol{\xi} \le 0, \qquad \forall \xi \in \mathbb{R}^n, \qquad \text{and} \qquad \partial_t c(t) \le 0 \qquad \forall t \in [0,T].$$

Then there exists at most one spacewise dependent heat source $f \in L^2(\Omega)$ such that (7) and (2) hold.

Proof. Following on from the discussion preceding the statement of the theorem, the ISP consisting of (7) and (2), can be transformed into the corresponding one studied in [2] for $h = 1$, but in this transformation the coefficient $c(x,t)$ changes into $c(x,t) + \alpha(t)$. The relation $\alpha'(t) \le 0$ then implies

$$\partial_t (c(t) + \alpha(t)) \le 0.$$

Thus, $c(x,t) + \alpha(t)$ satisfies the requirement of [2, Thm. 3.7] and the above result therefore follows. □

We point out that the condition that "h does not have any zeros" is essential for the uniqueness (cf. [1, Example 1], where a similar deduction has been used). We shall further highlight this by giving two examples.

Example 2. Rather than only considering one-dimensional configurations in space, we outline a technique for providing a counterexample to uniqueness of the more general ISP, consisting of (7) and (2), for multi-dimensional domains when $h(t)$ changes sign.

Consider the following multi-dimensional ISP for $(x,t) \in \Omega \times (0,T)$ having final time data being zero,

$$\begin{cases} u_t(x,t) + Au(x,t) &= h(t)f(x) \quad \text{for } (x,t) \in \Omega \times (0,T), \\ u(x,t) &= 0 \quad \text{for } (x,t) \in \Gamma \times (0,T), \\ u(x,0) &= 0 \quad \text{for } x \in \Omega, \\ u(x,T) &= 0 \quad \text{for } x \in \Omega. \end{cases} \tag{8}$$

Take any eigenfunction g of the elliptic operator A, i.e. $Ag = \lambda g$. The space $L^2((0,T))$ is a Hilbert space, therefore there exist elements h such that

$$\int_0^T e^{\lambda s} h(s)\, \mathrm{d}s = 0.$$

An example of such an element is of the form $h(t) = e^{-\lambda t} v(t)$ with $\int_0^T v(s)\, \mathrm{d}s = 0$. Clearly, with such a v (assumed to be non-trivial), the element h changes sign in $[0,T]$. Define

$$\beta(t) := e^{-\lambda t} \int_0^t e^{\lambda s} h(s)\, \mathrm{d}s, \qquad \text{which implies that } \beta(0) = \beta(T) = 0.$$

An easy calculation gives $\beta'(t) + \lambda\beta(t) = h(t)$. Moreover, $u(x,t) := \beta(t)g(x)$ then solves (8) with $f(x) = g(x)$. Thus, besides the trivial solution $(u,f) = (0,0)$ to the ISP (8), we also have the following

$$u(x,t) = \beta(t)g(x), \qquad f(x) = g(x)$$

and there is therefore no uniqueness to (8) when h changes sign in the given time interval. △

To give an explicit form of a solution constructed with the technique outlined in the previous example, we consider a one-dimensional setting in space.

Example 3. Consider the one-dimensional ISP for $x, t \in (0,\pi)$ with zero final time data,

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = h(t)f(x), \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = 0, \\ u(x,\pi) = 0, \end{cases} \tag{9}$$

with $h(t) = \cos(t) + \sin(t)$, which clearly changes its sign in $(0,\pi)$. We have

$$\alpha'(t) := \left(\frac{h'(t)}{h(t)}\right)' = -\frac{2}{(\cos(t) + \sin(t))^2} \le 0.$$

Thus, apart from h having a zero in the given time-interval, the other conditions of Theorem 3.1 are satisfied. Using the technique from the previous example, we find that besides the trivial solution $(u,f) = (0,0)$ to the ISP (9), we also have the following one

$$u(x,t) = \sin(x)\sin(t), \qquad f(x) = \sin(x).$$

△

Now, one can be lead to think that with this class of sources also depending on time, perhaps the condition $\partial_t c(t) \le 0$ in Theorem 3.1 can be removed. However, again via a straightforward example, we show that this condition cannot be removed without violating uniqueness in the ISP consisting of (7) and (2).

Example 4. Consider the following one-dimensional ISP for $x, t \in (0,\pi)$ with zero final time data,

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + 2\dfrac{u(x,t)}{\sin(t)} = h(t)f(x), \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = 0, \\ u(x,\pi) = 0, \end{cases} \tag{10}$$

where $h(t) = 2 + \cos(t) + \sin(t) > 0$ in $[0, \pi]$. One can check that

$$\alpha'(t) := \left(\frac{h'(t)}{h(t)}\right)' = -2\frac{(1 + \cos(t) + \sin(t))}{(2 + \cos(t) + \sin(t))^2} \le 0.$$

We then note that $c(t) := 2\sin^{-1}(t)$ satisfies $c'(t) = -\frac{2\cos(t)}{\sin^2(t)}$, which clearly changes its sign in $(0, \pi)$. Besides the trivial solution $(u, f) = (0, 0)$ to the ISP (10), we also have the following solution pair

$$u(x, t) = \sin(x)\sin(t), \qquad f(x) = \sin(x).$$

△

For the sake of completeness, we state and prove the corresponding generalization of [2, Thm. 3.9] to the case of heat sources $f(x, t) = f(x)h(t)$.

Theorem 3.2. *Consider a linear partial differential operator $L(t)$ given by* (6) *with bounded coefficients* (*possibly discontinuous in x*) *such that $\boldsymbol{A}^t = \boldsymbol{A}$, the condition* (3) *holds, $\nabla \cdot \boldsymbol{b} \le 0$ and $c \ge 0$. Let u_0, $\psi_T \in L^2(\Omega)$, $h \in C^2((0, T))$ with h being non-zero in the given time-interval. Suppose that*

$$\alpha(t) \ge 0,\ \alpha'(t) \ge 0, \alpha''(t) \le 0,\ \textit{where}\ \alpha(t) := \frac{h'(t)}{h(t)}.$$

Moreover, assume that

$$\boldsymbol{\xi}^t \cdot (\partial_t \boldsymbol{A})\, \boldsymbol{\xi} \ge 0, \qquad \textit{and} \qquad \boldsymbol{\xi}^t \cdot \partial_t \left((\partial_t \boldsymbol{A})^{\frac{1}{2}}\right) (\partial_t \boldsymbol{A})^{\frac{1}{2}}\, \boldsymbol{\xi} \le 0, \qquad \forall \xi \in \mathbb{R}^n,$$

$$\partial_t c(t) \ge 0 \qquad \textit{and} \qquad \partial_{tt} c(t) \le 0 \qquad \forall t \in [0, T].$$

Then there exists at most one spacewise dependent heat source $f \in L^2(\Omega)$ such that (7) *and* (2) *hold.*

Proof. Since $\alpha(t) \ge 0$, $\alpha'(t) \ge 0$ and $\alpha''(t) \le 0$, it follows that

$$c(t) + \alpha(t) \ge 0,\ \partial_t (c(t) + \alpha(t)) \ge 0,\ \text{ and }\ \partial_{tt} (c(t) + \alpha(t)) \le 0.$$

Thus, as was pointed out before Theorem 3.1, the structure and requirements of the operator $L(t)$ remains the same when we replace the coefficient $c(x, t)$ by $c(x, t) + \alpha(t)$. Therefore, dividing the governing equation in (7) by $h(t)$ and using the transformations explained in the paragraph before Theorem 3.1, the rest of the proof then follows from [2, Thm. 3.9]. □

One can investigate the necessity of the conditions in Theorem 3.2 and construct similar counterexamples to those above to show that uniqueness of a solution to the ISP consisting of (7) and (2) does not hold when the condition on h or the ones on c are violated, however, this is left to the reader.

4. Comparison of the present conditions for uniqueness with [6, Thm. 2.1]

We point out that Prof. Isakov published an involved uniqueness theorem in [6] for an ISP of the form (7) and (2). In that result, monotonicity of $h(t)$ is assumed, that is $h'(t) > 0$ is required. In [6, Thm. 3.1], a bounded smooth and non-zero function h is constructed with a derivative that changes sign in the given time interval, and with this function a non-trivial source $f(x)$ is constructed to a problem of the form (9). Using the technique preceding Theorem 3.1, with this function h one can then give a non-trivial solution to the problem (5) with a smooth and bounded coefficient $c(t)$ and where $c'(t)$ changes sign. Thus, violating the condition that c shall decrease in time, there is no uniqueness to the ISP (1)–(2) as was already shown in Example 1 but with smooth increasing and unbounded coefficient $c(t)$.

Let us compare assumptions on the data in [6, Thm. 2.1] with those needed in Theorem 3.1. Prof. Isakov requires in [6, Thm. 2.1] that

$$c(t) \ge 0, \qquad \partial_t c \le 0, \qquad h(t) \ge 0, \qquad \partial_t h > 0, \qquad h(T) > 0.$$

Theorem 3.1 does not need $c(t) \geq 0$ and $h(t)$ is not necessarily increasing. Theorem 3.1 needs that

$$\partial_t c \leq 0, \qquad \alpha'(t) = \left(\frac{h'(t)}{h(t)}\right)' = \frac{h''(t)h(t) - (h'(t))^2}{h^2(t)} \leq 0.$$

Thus, Theorem 3.1 is valid also for positive and concave functions $h(t)$, such as

$$h(t) = \sqrt{1-t}, \qquad h'(t) = -\frac{1}{2}\,\frac{1}{\sqrt{1-t}}, \qquad h''(t) = -\frac{1}{4}\,(1-t)^{-3/2} \qquad \text{in } [0, 1-\varepsilon].$$

Moreover, Theorem 3.1 holds true also for non-monotone functions $h(t)$; an explicit example is

$$h(t) = t\sin(t), \qquad h'(t) = \sin(t) + t\cos(t), \qquad \alpha'(t) = \left(\frac{h'(t)}{h(t)}\right)' = -\frac{1}{t^2} - \frac{1}{\sin^2(t)} \leq 0 \qquad \text{in } (0, \pi).$$

Based on these considerations, we may say that Theorem 3.1 is valid also for sources being non-monotone in time, which generalizes results from [6, Thm. 2.1].

References

[1] A. Hasanov, M. Slodička, An analysis of inverse source problems with final time measured output data for the heat conduction equation: A semigroup approach, Applied Mathematics Letters 26 (2) (2013) 207–214.

[2] S. D'haeyer, B. Johansson, M. Slodička, Reconstruction of a spacewise dependent heat source in a time-dependent heat diffusion process, IMA Journal of Applied Mathematics 79 (1) (2014) 33–53.

[3] W. Rundell, Determination of an unknown non-homogeneous term in a linear partial differential equation from overspecified boundary data, Applicable Analysis 10 (1980) 231–242.

[4] A. Erdem, D. Lesnic, A. Hasanov, Identification of a spacewise dependent heat source, Applied Mathematical Modelling 37 (2013) 10231–10244.

[5] L. Ljusternik, W. Sobolev, Elements of spectral analysis, Nauka, Moscow, 1965, Russian.

[6] V. Isakov, Inverse parabolic problems with the final overdetermination, Communications on Pure and Applied Mathematics 44 (2) (1991) 185–209.