Travelling waves in a Complex Ginzburg-Landau equation with time-delay feedback

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One of the simplest ways to create nonlinear oscillations is the Hopf bifurcation. The spatiotemporal dynamics observed in an extended medium with diffusion (e.g., a chemical reaction) undergoing this bifurcation is governed by the complex Ginzburg-Landau equation, one of the best-studied generic models for pattern formation, where besides uniform oscillations, spiral waves, coherent structures and turbulence are found. The presence of time delay terms in this equation changes the pattern formation scenario, and different kind of travelling waves have been reported. In particular, we study the complex Ginzburg-Landau equation that contains local and global time-delay feedback terms. We focus our attention on plane wave solutions in this model. The first novel result is the derivation of the plane wave solution in the presence of time-delay feedback with global and local contributions. The second and more important result of this study consists of a linear stability analysis of plane waves in that model. Evaluation of the eigenvalue equation does not show stabilisation of plane waves for the parameters studied. We discuss these results and compare to results of other models.

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Contents

1. INTRODUCTION

This thesis focuses on travelling wave patterns in a generic oscillatory nonlinear medium with an extra element of time delay added. We begin our study by defining nonlinear dynamics and examining an oscillatory system. From this we can study the special case of Hopf bifurcation, one of the simplest ways to create nonlinear oscillations. Pattern formation in oscillatory systems are patterns such as concentric wave patterns or spiral waves, and they are the immediate consequences of the nonlinear dynamics of the system. Spirals are observed in bacterial colonies, chemical reactions, all of this can be described by the complex Ginzburg-Landau equation, which is the basis of the model of this study. However we are not interested in spirals, we are interested in a simpler case, one-dimensional travelling waves. Travelling waves represent an important solution in nonlinear systems and are a paradigmatic and well known example of pattern formation. In this thesis, we will study the existence and stability of a specific travelling wave solution in the complex Ginzburg-Landau equation subjected to time-delay feedback. This chapter focuses on the basic concepts underlying the dynamics of the system we will be studying.

Nonlinear dynamics

We start by defining a dynamical system. There are two main types of dynamical systems: differential equations and iterated maps [1]. We will be focusing on differential equations as they are widely used in the field of science and engineering [2]. Dynamical systems can also be seen as functions that describe the state of a system as a function of time and that satisfy the equations of motion of the system [3]. It could be a system of first order ordinary differential equations or partial differential equations that aims to model a real system. This real system may be of physical, chemical, biological or other nature. The state of a dynamical system is described by a set of variables and these span a phase space, a space in which all possible states of a system are represented. A dynamical system involves two parts, a state vector $x \in R^n$ which defines exactly the state of some real or hypothetical system and a function $f : R^n \to R^n$, which describes how the system evolves over time.

We restrict ourselves to evolution equations. This means we have one or more equations in the form of a first-order differential equation in time with a nonlinear function on the right-hand side. The equation is in the form $\dot{u}_i = f(\{u_i\})$, where the dot denotes the time derivative. This equation describes the evolution of a system depending on a continuous time variable t. The basic solution of this evolution equation are fixed points. Fixed point solutions describe stationary solutions, for which $\dot{u}_i = 0$, for all i. However stationary solutions are not the only type of solutions we have in evolution equations. Indeed there exist a class of solutions that we are particularly interested in, which are oscillatory solutions.

Oscillatory systems

The simplest oscillatory system is one that shows a simple periodic behaviour: $u_i(t) = u_i(t+T)$ for all i, where T is the period and oscillations per time is given by $f = \frac{1}{T}$, called the frequency of the motion: $T = \frac{2\pi}{\omega}$ and $f = \frac{\omega}{2\pi}$.

If oscillations appear in a dissipative system, the closed trajectories are called limit cycles. How do we know if a system is dissipative? This depends on the way the system interacts with its environment. Dissipativity is closely related to the notion of energy and is characterised by the property at any time t, the amount of energy which the system can conceivably supply to its environment can not exceed the amount of energy that has been supplied to it. As time evolves a dissipative system absorbs a fraction of its supplied energy. The energy may be transformed into various forms such as heat, increase of entropy, mass, electromagnetic radiation or other kinds of energy loss [4].

To grasp the understanding of an oscillatory system, it is crucial for us to familiarise ourselves with limit cycles. A limit cycle is an isolated trajectory in phase space. Isolated meaning that adjacent trajectories are not closed but they either spiral toward or away from the limit cycle as time approaches infinity. In the case where all adjacent trajectories approach the limit cycle, we define this limit cycle as stable. Otherwise the limit cycle is unstable, or in exceptional cases, half stable [5]. A few prominent examples of stable limit cycles are: the beating of a heart, the periodic firing of a pacemaker neuron and hormone secretion. These systems oscillate even in the absence of any external periodic forces and we say they exhibit self-sustained oscillations [2]. A typical oscillatory chemical system is the Belousov-Zhabotinsky reaction which we will discuss later in further detail.

Oscillatory systems differ from excitable systems as they consist of a stable limit cycle in phase space whereas excitable systems consist of a stable fixed point. Naturally, excitable systems are in a stable state but applying a super-threshold perturbation leads to a large excursion in phase space.

The oscillatory systems we will study are the systems of complex spatial patterns in media that naturally oscillate in time. In this section we will consider a prominent example of such a system, chemical oscillatory systems. The occurrence of oscillations in a chemical system was initially published in 1828 by a German physicist G.T. Fechner. Later in 1899, a German chemist W. Ostwald reported a periodic increase and decrease in the rate of chromium dissolution in acid. In both cases, chemical oscillations were observed through an inhomogeneous reaction, which are reactions in which the reactants and products are in the different physical state (solid, liquid, gas) [6]. Prior to about the 1920, most scientists assumed that oscillations in homogeneous reactions (reactions in which the reactants and products are in the same physical states) were not possible. This assumption was due to the lack of understanding of how oscillations occur. Initially it was believed that these oscillations violate the second law of thermodynamics.

After many decades, finally the scientific community was convinced that these oscillations are possible due to the work done on the Belousouv-Zhabotinsky reaction in the 1950s by the Russian chemist B. Belousov. The reaction was originally developed to model the functional complexity of the Krebs cycle [6]. The Krebs cycle (also known as the citric acid cycle) plays a vital role in metabolism. It is the final pathway where the oxidative metabolism of carbohydrates, amino acids and fatty acids converge their carbon structure and convert to CO_2 [7].

Initially the reaction mixture studied by Belousov contained bromate, citric acid and ceric ions (Ce^{+4}). Belousov observed a periodic change as the solution changed from yellow to clear and then back to yellow. The periodic change was an indication of oxidation, as Ce^{+3} was oxidised losing an electron to become Ce^{+4}. However Belousov's work was not accepted by the scientific community. Later in 1961, a graduate student A. Zhabotinsky took this study further. However he replaced citric acid with malonic acid, which is now the typical BZ reagent [6].

Typically, oscillations are described in terms of their amplitude ρ and frequency Ω. If both of them are constant in time, a solution can be cast into the form $A = \rho \cdot e^{-i\Omega t}$, with t being time, and A a complex variable (also called 'complex amplitude'), as in the complex Ginzburg-Landau equation introduced below.

Waves and pattern formation in oscillatory systems

What do we mean by patterns? Patterns are ubiquitous, whether it may be the hexagonal patterns on a giraffe's skin, spiral chaos in heart muscle fibrillation or even spiral turbulence in chemical systems. Therefore, pattern formation only occurs in a spatially-extended medium. Patterns also play a fundamental role in biology for example embryology is a part of biology which is concerned with the formation and development of the embryo and this development is a sequential process. In embryology the development of pattern and form are found in morphogenesis [8].

Early works on pattern formation were motivated by convection [9], which was studied extensively due to its natural occurrence in the environment. We experience convection in our daily lives, for e.g. heating in a room. Convection is the overturning of a fluid that is heated from below, resulting in the fluid expanding and becoming less dense. This cycle is repetitive as the fluid rises away from the heat source but then cools down, becoming denser and so falls back under the influence of gravity. The continuous rise and fall of the fluid forms spatial patterns. These patterns can be seen as either stripes or convection rolls [9]. It is important to note that the pattern we discuss here relies on continuous energy and/or mass flow, the medium is therefore far from thermal equilibrium. Most of these concepts have been explored in the Rayleigh-Bénard convection, a paradigmatic lab experiment which describes the convection between two horizontal plates [9]. Rayleigh-Bénard convection was based on convection experiments carried out by Henri Bénard which were later analysed and published by Lord Rayleigh. Rayleigh-Bénard convection describes the buoyancy driven flow of a fluid heated from below and cooled from above. This situation is a classical problem in fluid dynamics and has played a vital role in the development of stability theory, pattern formation and in the study of spatial-temporal chaos [10].

Stable patterns correspond to asymptotic solutions (as $t \rightarrow \infty$) of the underlying equations. Patterns may be regular (e.g. spirals or hexagons), or irregular, when regular pattern become unstable. Now we will introduce space, which means the evolution equation now has terms that also depend on space. Examples of such equations are reaction-diffusion equations for chemical systems and the Navier-Stokes equation for fluid systems.

A reaction-diffusion model describes the distribution of one or more substances in space. The distribution fluctuates under the influence of two processes. Firstly local chemical reactions which form substances and secondly due to diffusion [11], a mechanism by which particles in a fluid are transported from an area of higher concentration to an area of lower concentration [9]. This causes substances to spread over a surface in space. A reaction-diffusion equation is a combination of Fick's Law of Diffusion and the chemical reaction rate law [12]. Therefore it consists of a diffusion and a reaction term and is given in the form:

$$\frac{dc}{dt} = D\nabla^2 c + R(c,m). \tag{1}$$

Adolf Fick (1829-1901), a German physiologist proposed the phenomenological law for diffusion. The law stated that the flux j is proportional to the concentration gradient and is given by the following:

$$j = -D\frac{\partial c}{\partial x}, \tag{2}$$

where D is the diffusion coefficient and c the concentration. The negative sign is an indication of the fact that the diffusion is from an area with high density to another of low density [13, 14]. Fick's law of diffusion resembles the Fourier law of heat conduction. Due to the conservation of mass, $\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x} j$ and Eq. (2), we obtain the one-dimensional diffusion equation as follows:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}. \tag{3}$$

The term $R(c,m)$ corresponds to the chemical reaction rate law and mathematically expresses the relationship between rates and concentrations for a chemical reaction [15].

Therefore, a reaction-diffusion equation is typically given in the following form: $\frac{dc}{dt} = D\nabla^2 c + R(c,m)$, where $c(x,t)$ is a state variable which describes the density/concentration of a substance. ∇^2 denotes the Laplacian operator, therefore the linear term of the right hand side is proportional to D (the diffusion coefficient) describing the diffusion of the substance. R is the reaction rate, which depends on local chemical concentrations and a set of parameters m. Reaction-diffusion models are formulated to describe the spatiotemporal dynamics (i.e. of patterns) of many systems. Its applications in various fields from chemical and biological phenomena to medicine (physiology, diseases, etc.), genetics, physics, social science, finance, economics, weather prediction, astrophysics, and so on.

Before we elaborate any further, it is important for us to introduce the concept of bifurcation theory. Bifurcation is a complex field of study and is currently heavily researched. Bifurcation theory is based on how a slight variation in a parameter may lead to significant impact on the solution. To be able to clearly grasp the concept of bifurcation theory we must be able to identify fixed points (or equilibrium points) of a differential equation. $\dot{u} = f(u,m)$ is the root of the equation $f(u,m) = 0$, where m is a parameter. So why is this useful to us? The solution of the fixed point depends on the system parameters and if we change one (or more) parameter, the solution changes. While usually changes are only quantitative, sometimes they may be qualitative. Then we say that a bifurcation (or instability) has happened. Often a new solution is found. Here we will be studying the case of supercritical Hopf bifurcation. The normal form of the supercritical Hopf bifurcation is given by the following system [2]:

$$\dot{r} = \mu r - r^3 \tag{4}$$

$$\dot{\theta} = \omega + br^2. \tag{5}$$

At the supercritical Hopf bifurcation at $\mu = 0$, a limit cycle with infinitely small amplitude and finite frequency is born. Let us consider when $r \to 0$, then consequently $\dot{r} \to 0$ and $\dot{\theta} = \omega$. From this we observe that the onset of oscillations is characterized by vanishing amplitude and finite frequency ω.

The complex Ginzburg-Landau equation

The complex Ginzburg-Landau equation (CGLE) is one of the most studied nonlinear equations in applied mathematics and in the nonlinear physics community [16]. The CGLE represents a normal form of a distributed dynamical system in the vicinity of a supercritical Hopf bifurcation [2]. Not only this, but it is also one of the most popular models in the theory of pattern formation.

It reads [16] :

$$\frac{\partial A}{\partial t}(\vec{x},t) = (1 - i\omega)A(\vec{x},t) - (1 + i\alpha)\,|A|^2 A(\vec{x},t) + (1 + i\beta)\nabla^2 A(\vec{x},t), \tag{6}$$

and we will explain it in further detail below.

Why is the CGLE important to us? It defines qualitatively, and often quantitatively, a wide range of phenomena including chemical reaction-diffusion systems, nonlinear waves, second-order phase transitions, Rayleigh-Bénard convection and superconductivity. The equation was initially derived as an amplitude modulation equation for modelling the onset of instability in fluid convection situations. The equation itself describes the change of amplitudes of unstable modes for any process demonstrating a Hopf bifurcation, whilst we take into consideration a range of unstable wavenumbers. The equation can be viewed as a general form for nonlinear wave phenomena in spatially extended oscillatory systems. The primary solution of the CGLE is uniform oscillations, corresponding to the limit cycle appearing at the Hopf bifurcation [16].

Travelling waves, plane waves

A wave is a disturbance caused by an excitation or perturbation and therefore consequently travels through a medium from one location to another via local interaction, whilst transferring energy. One particle applies a force on its neighbouring element causing a displacement of the neighbouring element from its state of equilibrium. This type of wave that is seen travelling through a medium is sometimes referred to as a travelling wave [17]. A travelling wave is a function of u in the form $u(x,t) = f(x - vt)$, where v is a real number which defines the propagating speed of the wave. The wave profile of a travelling wave just propagates by rigid translation with velocity v. Therefore if we want the line $y = x$ to move with a speed of 6 in the y direction, we now have the following equation: $y = x - 6t$. To summarise, in order to create a travelling wave we replace the x term of the equation of a standing wave with $x - vt$ or $x + vt$ depending on the direction in which we want it to move. For e.g. the equation $y = 2\sin(x - 6t)$ holds all the characteristics of the oscillation. It moves in the positive x direction with a speed of 6 and amplitude of 2.

Plane waves are a special kind of travelling wave. Plane waves play a fundamental role in the theory of linear wave equation and are given in the general form $u(x,t) = \rho e^{i\phi} e^{i(kx - \omega t)}$, where ρ represents the positive constant amplitude, $\phi \in [0, 2\pi]$ characterises the initial phase, k the wavenumber and ω the angular frequency. Noting that $\frac{k}{2\pi}$ is the number of waves per unit length and $\frac{\omega}{2\pi}$ is the number of wave per unit time, we can rewrite the equation of plane waves in the form $u(x,t) = \rho e^{i\phi} e^{ik(x - \frac{\omega}{k} t)}$. From this form, we are able to observe that plane waves are indeed travelling waves with propagation velocity $\frac{\omega}{k}$, also known as phase velocity [18].

Time-delay feedback

We are all familiar with ordinary differential equations, how they work and the use of ordinary and partial differential equations to model many systems, as these models assist us to grasp a better understanding of complicated phenomena. However these systems are unable to capture the various key dynamics observed in some systems, namely when the evolution of a variable depends not only on its current value (at time t) but also on its value at a past time $t-\tau$, where τ is the time delay. Mathematically, this class of differential equations are called the delay differential equations (DDEs) [19]. These equations have an extra element of time involved and they may appear to look very similar to ordinary differential equations but they consist of many features which makes their analysis more complex. Delay differential equations are used in many fields of study including, mathematical biology, chemistry, mechanical vibrations and many more [19]. Systems consisting of a feedback control may involve time delays. A simple example for a DDE is given in the form $\frac{dy}{dt} = ky(t-\tau)$, $y(t) = 1$ when $-\tau \leq t < 0$, where we observe an extra term τ corresponding to time delay. Note that the bracket indicates dependence of, not multiplication.

This time delay occurs as a finite time is required to sense information and then react to this information. So how does a DDE differ from an ODE? Let us start of by examining an example of an initial value problem, for $\tau = 0$:

$\frac{dy}{dt} = ky(t)$, $y(0) = 1$

In this equation, the past is not involved in this solution. The knowledge of the present ($y(0) = 1$) is sufficient to predict the future at any time t.

In contrast, in the DDE a moment in the past determines the future value of y. The right-hand side depends on y at time $t-\tau$, where τ is the time delay. The initial condition is now replaced by an initial function defined on a finite interval of time. DDEs must not only provide the value of the solution at the initial point but also the history, solution of y at times prior to the initial point [19].

The model considered in this study contains a time-delay feedback. A feedback loop is a common and powerful tool when manipulating a system [20, 21]. The main objective of a feedback loop is to take the system output into consideration, which then enables the system to adjust its performance to meet a desired output response. Previous work has shown that turbulence in oscillatory distributed systems can be controlled by introducing a delayed global feedback (e.g. [22]). Under the influence of a delayed global feedback, elements of a distributed oscillatory system collectively produce a control signal that is applied back to each of them after adding a certain delay.

Motivation

The motivation of this study is deeper than simply being able to identify generic features of a reaction-diffusion system but being able to manipulate them. The model studied describes oscillatory reaction-diffusion systems and the patterns that are formed in them. We want to be able to manipulate reaction-diffusion systems as this enables us to modify existing patterns, create novel patterns and also suppress spatiotemporal chaos. The complex Ginzburg-Landau equation is not only a model equation for pattern formation but also used in a great number of fields such as mathematical biology and chemical reaction-diffusion systems. There is always a modelling process involved between a real life process/system and the CGLE and this reduction process depends on the system that we are studying. The CGLE is a generic model and hence applicable to all processes that fulfil certain criteria, a few of which are oscillatory dynamics, Hopf bifurcation and diffusive coupling.

Oscillatory processes are common in biomedical sciences and appear in a wide range of phenomena. The time period of such processes may last from a few seconds to hours to days and even weeks. There is great number of areas of current research involving biological oscillators, a few examples are the periodic pacemaker in the heart. Breathing is also a prime example of another physiological oscillator, where the period is of the order of a second. Another example is certain neural activity in the brain, where the cycles have very small periods.

Another system that shows oscillatory behaviour is the cardiac system [23]. Although the primary function of the heart, pumping blood throughout the body is mechanical, the muscular pumping contractions are the consequence of electrical activity. Each heartbeat is the result of a wave of electrical activity. There, we can see another feature of spatiotemporal oscillatory (or excitable) behaviour, namely that waves can become unstable and spatiotemporal chaos is observed. In the case of the heart, spatiotemporal chaos can correspond to fibrillation, a condition that can ultimately lead to cardiac arrest. Our approach to control spatiotemporal chaos as observed in the CGLE is to use time-delay feedback.

The model studied in this thesis contains a time delay term therefore we have a delay differential equation. Delay terms arise naturally in a wide range of real life systems, such as in physiology or for population models [24]. For example, in population models one assumes that the birth rate of a particular species is considered to act instantaneously however there may be a period of time delay. This time delay may be due to various reasons such as the time to reach maturity, the finite gestation period and so on. In our case it appears as an intrinsic feedback. The relevance of delay differential models to real life systems also gives rise to further motivation for studying this particular model.

However in this thesis, we do not study any of the above-mentioned systems in particular but focus on a generic equation for oscillatory systems, the CGLE. The aim of this thesis is to provide and study some solution of the CGLE in the regime of spatiotemporal chaos and in the presence of time-delayed feedback, which represents travelling plane waves. The research is motivated by previous work on this model [25–27].

2. LITERATURE REVIEW

In control theory [20], while controlling some solution of the original equation, the feedback term should be equal to zero ($F = 0$) in the moment of control. Consider a feedback F which is non-zero ($F \neq 0$) and while approaching the real solution the feedback signal decays to zero ($F \to 0$), this means we have a non-invasive control. Which means initially, in order to be able to control a system we require the control force to be non-zero but when control is successful, the stabilizing feedback vanishes and all the desirable features of the uncontrolled system are retained [20, 28] . However when F is non-zero we have an invasive system in the moment of control. For many applications, it is desirable to design the feedback such that the magnitude of the control signal decreases as the system approaches the desired state [28].

ETDAS and TDAS: The method of *time-delayed auto synchronization* (TDAS) was first introduced by Pyragas. This method is based on applying a feedback proportional to the deviation of the current system from its state one period in the past [29]. Later, Socolar, Sukow and Gauthier proposed an extension of this method known as *extended time-delay feedback autosynchronization* (ETDAS) [30], which incorporates information about the state of the system at earlier times $t - n\triangle t$ with decreasing weight.

Travelling waves were studied by Bleich and Socolar in 1996 [31]. They researched the control of spatiotemporal dynamics with time-delay feedback (ETDAS method) and suggested a spatially local feedback mechanism for stabilizing periodic orbits in spatially-extended systems. In their work they analysed the CGLE in one dimension to demonstrate how the time-delay feedback plays a vital role in enlarging the stability domain for travelling waves.

Five years later, Socolar collaborated with Harrington to extend on previous work that has demonstrated the possibility of stabilizing plane wave solutions of one-dimensional systems using a spatially local form of time-delayed feedback. Here they showed that the natural extension of this method to two-dimensional systems fails due to the presence of torsion-free unstable perturbations. The linear stability analysis of the CGLE demonstrated that long wavelength, transverse wave instabilities cannot be suppressed by the method of extended time-delay autosynchronization (ETDAS) [32].

In 2002, Beck, Amann, Schöll, Socolar and Just investigated time-delayed feedback control for stabilizing time-periodic spatial patterns in a generic reaction-diffusion system with different coupling schemes, in particular global and local schemes [33].

Experiments by Beta et al. in 2003 were carried out to control chemical turbulence in the catalytic CO oxidation on a Pt(110) single crystal surface [34]. The series of experiments performed, varied both the feedback intensity μ and the delay time τ. The results showed that indeed turbulence can be efficiently suppressed by applying time-delay autosynchronization. A similar experiment to control chemical turbulence by global delayed feedback was previously performed in 2001 by M. Kim et al. which also consisted of varying the feedback intensity and the delay time [35]. The results obtained showed that turbulence could be suppressed and new patterns could be induced by the feedback. Here they used the CGLE with a global phase-shifted feedback to interpret their results. Their experimental and theoretical investigations indicated that global feedback can be used efficiently to control microscopic pattern formation in a surface chemical reaction.

Later in 2004, Beta and Mikhailov also implemented TDAS for controlling spatiotemporal chaos in oscillatory reaction-diffusion systems [36]. Diffusion-induced turbulence in spatially-extended oscillatory media near a supercritical Hopf bifurcation can be controlled by applying global TDAS. The paper focused on the CGLE in the Benjamin-Feir unstable regime and they then analytically investigated the stability of uniform oscillations depending on the feedback parameters. In this paper it is shown that a non-invasive stabilization of uniform oscillations is not possible in this type of system.

In the same year Montgomery and Silber studied the feedback control of travelling wave solutions of the CGLE [37]. They investigated the effectiveness of a noninvasive feedback control scheme in stabilizing travelling wave solutions of the one-dimensional CGLE in the Benjamin-Feir unstable regime through a linear stability analysis. It is important to note that their feedback contains a spatially-shifted component. They derived a sufficient stability criterion which determines whether a travelling wave is stable to all perturbation wavenumbers. Not only this but this criterion also determines an optimal value for the time-delay feedback parameter.

In 2007, Stich, Casal and Díaz investigated the CGLE in the regime of spatiotemporal turbulence and studied numerically how local or a combination of global and local time-delay autosynchronization can be used in order to suppress turbulence by inducing uniform oscillations [25]. Their numerical simulations showed that although a purely local control is unsuitable to produce uniform oscillations, a combination of local and global control can be efficient and also has the ability to create other patterns such as standing waves, amplitude death, or travelling waves.

Also in 2007, Silber and Postlethwaite investigated the spatial and temporal feedback control of travelling wave solutions of the two-dimensional CGLE [38], using the type of feedback that was previously used by Montgomery and Silber in 2004 [37]. As previous research concluded that the Benjamin-Feir unstable travelling waves of the CGLE in two spatial dimensions cannot be stabilized using TDAS [32], they elaborate on how the addition of a spatially-shifted feedback term can be used to stabilize such waves. The main focus is on how spatial terms can be chosen to manipulate the direction of travel of the plane waves.

Kyrychko, Blyuss, Hogan and Schöll studied the effects of a time-delayed feedback control on the appearance and development of spatiotemporal patterns in a reaction-diffusion system [39]. In their study they investigated various types of control schemes and this approach exposed different dynamical regimes, which arise from chaotic state or from travelling waves. In each case of their study the stability boundary was found in the parameter space of the control strength and the time delay, and numerical simulations suggest that diagonal control fails to control the spatiotemporal chaos.

The suppression of spatiotemporal chaos in the CGLE by a combined global and local time-delay feedback was studied further in 2010 by Stich and Beta [26]. Linear stability analysis was performed for two cases. Firstly for uniform oscillations and then secondly, for the fixed point solution that corresponds to amplitude death in the spatially extended systems. Both performed with respect to space-dependent perturbations and then complemented with numerical simulations [26]. This thesis follows a very similar approach and also with a very similar ansatz but with the difference that we are now studying travelling waves. Our model is different as we are using both global and local feedback terms.

Advancing from this study, later in 2013 Stich collaborated with Casal and Beta [27], studying standing waves as solutions of the model studied before [25, 26]. The onset is described as instability of the uniform oscillations with respect to spatially periodic perturbations. The solution of the standing wave pattern was firstly given analytically and then studied through simulations.

Gurevich and Friedrich report on a novel behaviour of solitary localized structures in a real Swift-Hohenberg equation also subjected to a delayed feedback [40]. Their study consisted of a bifurcation analysis of the delayed system and also the derivation of a system of order parameter equations explicitly describing the temporal behaviour of the localized structure in the vicinity of the bifurcation point.

Very recent research by Puzyrev, Yanchuk, Vladimirov and Gurevich focused on the stability of plane wave solutions in complex Ginzburg–Landau equation with delayed feedback [41]. They performed bifurcation analysis of plane wave solutions in a one-dimensional complex cubic-quintic Ginzburg–Landau equation with delayed feedback. As a result of their study they discovered how multi-stability and snaking behaviour of plane waves emerge as time delay is introduced.

3. THE MODEL

3.1. The CGLE

Our model is based on the complex Ginzburg-Landau equation so let us start with some preliminaries. The CGLE is given by [25]:

$$\frac{\partial A}{\partial t}(\vec{x},t) = (1 - i\omega)A(\vec{x},t) - (1+i\alpha)\,|A|^2 A(\vec{x},t) + (1+i\beta)\nabla^2 A(\vec{x},t). \tag{7}$$

The variable A is a complex oscillation amplitude and hence phase space is two-dimensional. It is a function of both space $\vec{x}$ and scaled time t. Space can be 1,2 or 3 dimensional. ω is the linear frequency parameter, α is the nonlinear frequency shift, β is the linear dispersion parameter, and ∇^2 the Laplacian operator. In our context we will reduce the problem to 1 space dimension of finite length/bound L. We have to define boundary conditions for the partial differential equation. They can be:

(a) Fixed: $A(L,t) = a + ib$ where $a, b \in \mathbb{R}$ (and equivalently for $A(0,t)$),

(b) No Flux: $\frac{\partial A}{\partial x}\,|_{x=0,L} = 0$,

(c) Periodic: $A(0,t) = A(L,t)$.

At a fixed boundary condition (also known as the Dirichlet boundary condition), the value of the function on the surface is specified. Typically, the wave decays at that specific boundary. No flux boundary condition (also known as Neumann boundary condition) is a typical boundary condition we experience in chemical reaction-diffusion systems. Here the quantity is not leaving as it is a closed system and the solution of the wave cannot penetrate like for the periodic boundary conditions. In the periodic boundary condition, the value of A at position 0 is the same as A at position L. As a result of this, a travelling wave can disappear at one boundary and reappear at the other. In this way, a travelling wave can be kept in the system infinitely long if the wave is stable.

In our case we are only interested in two types: “No flux” and “periodic” as these relate to travelling waves. Stationary solutions correspond to fixed points and periodic solutions correspond to limit cycles. Regardless of which initial condition we start with, the system dynamics will lead us to the stable solution or one of them if these are multiple stable solutions. The basic stable solution here is the limit cycle, corresponding to uniform oscillations discussed below.

We can split the CGLE into two parts, real and the imaginary part, in the Cartesian form: $A(x,t) = a + ib$.

Real part:

$$\frac{\partial a}{\partial t} = a(1-(a^2+b^2)) + b(\omega + \alpha(a^2+b^2)) + \frac{\partial^2}{\partial x^2}(a - \beta b) \tag{8}$$

Imaginary part:

$$\frac{\partial b}{\partial t} = b(1-(a^2+b^2)) - a(\omega + \alpha(a^2+b^2)) + \frac{\partial^2}{\partial x^2}(b + \beta a) \tag{9}$$

From these equations we see that the CGLE does not exactly correspond to a typical reaction-diffusion equation, since the diffusion of a also influences the dynamics of b (and vice versa), as $\frac{\partial a}{\partial t}$ is proportional to $\frac{\partial^2}{\partial x^2}a - \beta\frac{\partial^2}{\partial x^2}b$. The spatiotemporal dynamics of any reaction-diffusion system close to a Hopf bifurcation is completely described by the CGLE.

Using the description in terms of phase ϕ and real amplitude ρ as introduced by $A = \rho e^{-i\phi}$ the CGLE may also be written in the following polar form:

$$\frac{\partial \rho}{\partial t} = (1-\rho^2)\rho + \nabla^2\rho - \rho(\nabla\phi)^2 + \beta\rho\nabla^2\phi + 2\beta\nabla\phi\nabla\rho, \tag{10}$$

$$\frac{\partial \phi}{\partial t} = \omega + \alpha\rho^2 + (2/\rho)\nabla\rho\nabla\phi + \nabla^2\phi - (\beta/\rho)\nabla^2\rho + \beta(\nabla\phi)^2 \tag{11}$$

This description is useful if we have specific expressions for ρ and ϕ. For instance the solution of uniform oscillation is given by the following:

$\frac{\partial \rho}{\partial t} = 0$ and assume $\phi = \Omega t$, all the spatial gradient terms vanish and we then have: $\frac{\partial \phi}{\partial t} = \Omega$,

$$0 = (1-\rho^2)\rho \tag{12}$$

and

$$\Omega = \omega + \alpha\rho^2 \tag{13}$$

so $\rho = 1$ and $\Omega = \omega + \alpha$, which is the solution of uniform oscillations for the standard CGLE.

For some parameters, the CGLE presents spatiotemporal chaos [16]. This depends on the action of the diffusion and is found for $1 + \alpha\beta < 0$, the so-called Benjamin-Feir-Newell criterion. There has been a lot of effort to modify and control chaos through the application of feedback, see section (**??**) for further details. In our model, we will use a feedback with time delay and hence the equation becomes a delay differential equation.

3.2. The CGLE with time-delay feedback

Let us now introduce our final model, the CGLE for a one-dimensional medium with a combination of local and global time-delayed feedback, which has been introduced in Ref. [25]:

$$\frac{\partial A}{\partial t} = (1 - i\omega)A - (1 + i\alpha)|A|^2 A + (1 + i\beta)\nabla^2 A + F, \tag{14}$$

$$F = \mu e^{i\xi}\{m_l[A(x, t - \tau) - A(x, t)] + m_g[\bar{A}(t - \tau) - \bar{A}(t)]\}, \tag{15}$$

where

$$\bar{A}(t) = \frac{1}{L}\int_0^L A(x, t)dx \tag{16}$$

indicates the spatial average of A over a one-dimensional medium of length L. The feedback strength is described by the parameter μ, and ξ specifies a phase shift between the feedback and the dynamics. The coefficients m_g and m_l denote the global and local feedback contributions, respectively.

After splitting the global and local parts we obtain the following equation:

$$F = F_l + F_g = m_l \mu e^{i\xi}[A(x, t - \tau) - A(x, t)] + m_g \mu e^{i\xi}[\bar{A}(t - \tau) - \bar{A}(t)] \tag{17}$$

The term F_g represents global time-delay feedback, which has been studied extensively [36]. The term F_l is a relatively novel proposal and represents a local feedback [25]. If $A(t) = A(t - \tau)$, the feedback term vanishes.

3.3. Basic solution

Let us now discuss two important solutions of this system: Uniform oscillations and standing waves.

Uniform oscillations are a type of oscillation where the amplitude and frequency are both constant. They represent an elementary solution of spatially extended oscillatory systems. They experience a sinusoidal behaviour in time of the quantity. In our case this sinusoidal quantity is seen in the real or imaginary part, of the complex oscillation amplitude A. The modulus $|A|$ remains constant in time. The amplitude A does not show any space dependence. The solution of feedback induced uniform oscillation is given by:

$$A_0(t) = \rho_0 e^{-i\Omega_0 t} \tag{18}$$

with the amplitude and frequency given by:

$$\rho_0 = \sqrt{1 + \mu(m_g + m_l)\chi_1}, \tag{19}$$

$$\Omega_0 = \omega + \alpha + \mu(m_g + m_l)(\alpha\chi_1 - \chi_2). \tag{20}$$

Where $\chi_{1,2}$ denote effective modulation terms that can be positive or negative. They arise from the feedback and hence depend on ξ and τ :

$$\chi_1 = \cos(\xi + \Omega_0\tau) - \cos\xi \tag{21}$$

$$\chi_2 = \sin(\xi + \Omega_0\tau) - \sin\xi \tag{22}$$

For our model, this solution has been presented in [25]. However, for more details of this derivation please refer to Appendix A.

In general, standing waves (also known as stationary waves) are the result of the overlapping of two waves that consist of the same amplitude and frequency. These waves are travelling in opposite directions but at the same speed. Whilst the waves are moving, at various locations the waves consist of a very large amplitude oscillation while others have zero amplitude and continuous destructive interference. Standing waves refer to a spatial pattern and in one dimension it is a sinusoidal function which is constant in time. The solution of standing waves for a CGLE in the presence of global and local time-delay feedback, is described by the following equation [27]:

$$A_{SW} = e^{-i\Omega_S t}[H_0 + 2B_{k0}\cos(kx)e^{-i\gamma}] \tag{23}$$

The wavenumber k is calculated by the stability analysis of uniform oscillation and H_0, B_{k0}, Ω_s and γ are calculated from the eigenvalue problem of this solution. This solution has been presented in [27]. The standing wave pattern in this case appears as a modulation on top of the uniform oscillation.

3.4. Pattern overview

We have seen that the complex Ginzburg-Landau equation describes the dynamics of a spatially extended system that undergoes a supercritical Hopf bifurcation. The two cases we have here are one with feedback and the other in the absence of feedback. In the absence of feedback ($F = 0$, e.g. for $\mu = 0$), uniform oscillations are unstable and we see spatiotemporal chaos, since we assume $1 + \alpha\beta < 0$. Then, stable uniform oscillations or standing waves can be induced through the feedback.

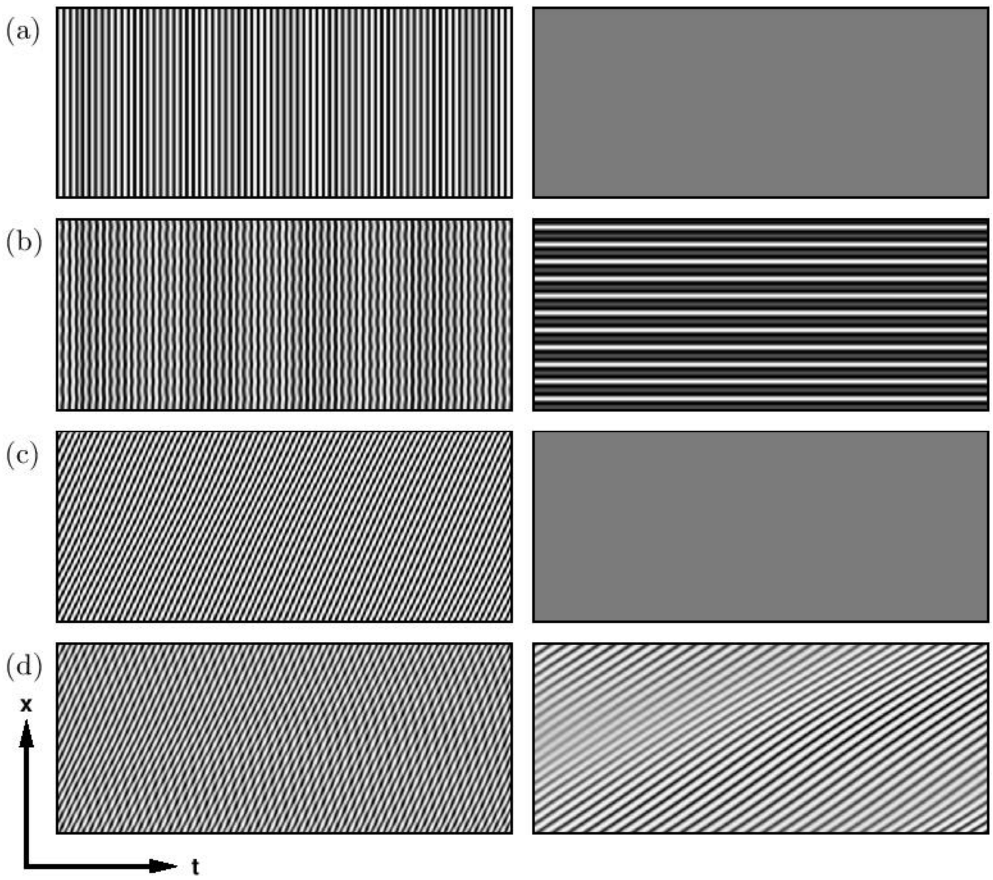

FIG. 1. Space-time plots of grey scale coded the Re A (left panel) and $|A|$ (right panel) for different solutions of our model. (a) uniform oscillations, (b) standing waves, (c) travelling wave with constant amplitude (plane waves), (d) travelling waves with spatially modulated amplitude. System size L=128, displayed time interval t=50. The system parameters are $m_l = 0.7$, $m_g = 0.3$ and $\tau = 0.3$ for (a) and (b). For (c) $m_l = 1.0$, $\mu = 0.6$ and $\tau = 1.05$ and for (d) $m_l = 1.0$, $\mu = 0.9$ and $\tau = 1.05$. The other parameters are $\alpha = -1.4$, $\beta = 2$, $\omega = 2\pi - \alpha$, $\xi = \pi/2$. This figure is based on Fig. 1 and Fig. 3 of [25].

The aim of this project is to study the stabilization of travelling waves through time-delay feedback. In Fig.1 we show an overview of the patterns. In the left panel, we see Re A, which shows the oscillatory nature of the system and in the right panel, $|A|$. In (a), we see uniform oscillations, which have a constant amplitude in space and in time (right panel). In (b), we have standing waves, characterized by a spatially-periodic, temporally-constant pattern for the real amplitude. In (c), we have the type of travelling waves that we will focus on, called plane waves. They are seen travelling in Re A, but with a spatially and temporally constant real amplitude (similar to uniform oscillations). In (d), we have a pattern that is travelling in both Re A and $|A|$. We will not discuss this pattern further. In this project, we want to study the analytical solutions of the plane wave patterns, including their stability.

4. PLANE WAVES

4.1. The analytical solution

The standard ansatz for a plane wave solution is

$$A_{PW} = \rho e^{-i\Omega t} e^{ikx}. \tag{24}$$

This implies that $|A_{PW}| = \rho$ is constant in space and time as seen in Fig. 1(c). We substitute Eq. (24) into Eq. (14) and obtain:

- $\frac{\partial A}{\partial t} = (-i\Omega)\rho e^{-i\Omega t} e^{ikx} = (-i\Omega)A$
- $|A|^2 = \rho e^{-i\Omega t} e^{ikx} \cdot \rho e^{i\Omega t} e^{-ikx} = \rho^2$
- $\nabla^2 A = (ik)^2 \rho e^{-i\Omega t} e^{ikx} = -k^2 \rho e^{-i\Omega t} e^{ikx} = -k^2 A$
- $A(x,t-\tau) - A(x,t) = \rho e^{-i\Omega(t-\tau)} e^{ikx} - \rho e^{-i\Omega t} e^{ikx} = \rho e^{-i\Omega t} e^{ikx} (e^{i\Omega\tau} - 1) = A(e^{i\Omega\tau} - 1)$

Now for the spatial average Eq. (16), we have:

- $\bar{A}(t) = \frac{1}{L}\left[\frac{1}{ik}\rho e^{-i\Omega t} e^{ikx}\right]_0^L = \frac{1}{L}\left[\frac{1}{ik}\rho e^{-i\Omega t} e^{ikL} - \frac{1}{ik}\rho e^{-i\Omega t}\right] = \frac{1}{ikL}\left[\rho e^{-i\Omega t} e^{ikL} - \rho e^{-i\Omega t}\right]$
- $\bar{A}(t-\tau) = \frac{1}{ikL}\left[\rho e^{-i\Omega(t-\tau)} e^{ikL} - \rho e^{-i\Omega(t-\tau)}\right]$
- $\bar{A}(t-\tau) - \bar{A}(t) = \frac{1}{ikL}\left[\rho e^{-i\Omega(t-\tau)} e^{ikL} - \rho e^{-i\Omega(t-\tau)} - \rho e^{-i\Omega t} e^{ikL} + \rho e^{-i\Omega t}\right] = \frac{1}{ikL}\left[\rho e^{-i\Omega t} e^{ikL} - \rho e^{-i\Omega t}\right]\left[e^{i\Omega\tau} - 1\right] = \bar{A}(e^{i\Omega\tau} - 1)$

Now we calculate $\bar{A}$:

$$\bar{A} = \frac{1}{L}\int_0^L \rho e^{ikx - i\Omega t} dx = \frac{1}{L}\left[\frac{1}{ik}\rho e^{ikx - i\Omega t}\right]_0^L = \frac{1}{L}\left[\frac{1}{ik}\rho e^{ikL - i\Omega t} - \frac{1}{ik}\rho e^{-i\Omega t}\right], \tag{25}$$

and therefore

$$\bar{A} = \frac{1}{ikL}\rho e^{-i\Omega t}(e^{ikL} - 1). \tag{26}$$

Assuming that the system size is much larger than the wavelength, and in particular in the limit $L \to \infty$, these terms vanish and we obtain $\bar{A}(t) = 0$. So we now have:

$$(-i\Omega)A = (1 - i\omega)A - (1 + i\alpha)\rho^2 A + (1 + i\beta)(-k)^2 A + \mu e^{i\xi}\left[m_l A(e^{i\Omega\tau} - 1)\right] \tag{27}$$

dropping A we obtain:

$$-i\Omega = 1 - i\omega - \rho^2 - i\alpha\rho^2 - k^2 - k^2 i\beta + \mu e^{i\xi} m_l (e^{i\Omega\tau} - 1) \tag{28}$$

and then separating the real and imaginary parts, we then have a solution of Eq. (14) with amplitude and frequency given by

$$\rho^2 = 1 - k^2 + \mu m_l \chi_1, \tag{29}$$

$$\Omega = \omega + \alpha\rho^2 + k^2\beta - \mu m_l \chi_2, \tag{30}$$

where

$$\chi_1 = \cos(\xi + \Omega\tau) - \cos\xi, \tag{31}$$

and

$$\chi_2 = \sin(\xi + \Omega\tau) - \sin\xi. \tag{32}$$

Here, $\chi_{1,2}$ denote effective modulation terms that arise from the feedback and hence depend on ξ and τ. For comparison, we give the solution of plane waves in the standard CGLE in Appendix B. If we set $m_l = 0$ in (29) and (30), we recover the solution of Appendix B. The solution does not depend on m_g, this means that the global feedback does not introduce a novel plane wave solution, only the local feedback does.

4.2. Mode separation

In this section, we show how to derive the equations that are needed later in order to perform the linear stability analysis. Since we are interested in the stability of plane waves with respect to perturbation of wavenumber q, we therefore perform a mode separation as we want to separate parts which are unperturbed and then applying perturbations $A_+$and A_-.

We begin with the ansatz :

$$A(x,t) = e^{-i\Omega t} e^{ikx} (\rho + A_+ e^{iqx} + A_- e^{-iqx}) \tag{33}$$

$$= A_{PW} + A_+ e^{-i\Omega t} e^{i(k+q)x} + A_- e^{-i\Omega t} e^{i(k-q)x} \tag{34}$$

Before substituting the ansatz (34) into Eq. (14), we have to determine several terms appearing in Eq. (34), with

$$\begin{aligned}\frac{\partial A}{\partial t} &= (-i\Omega)\rho e^{ikx-i\Omega t} + (-i\Omega)A_+ e^{-i\Omega t} e^{i(k+q)x} + \frac{\partial A_+}{\partial t} e^{-i\Omega t} e^{i(k+q)x} \\ &+ (-i\Omega)A_- e^{-i\Omega t} e^{i(k-q)x} + \frac{\partial A_-}{\partial t} e^{-i\Omega t} e^{i(k-q)x}\end{aligned}$$

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} = (-k)^2 \rho e^{ikx-i\Omega t} - (k+q)^2 A_+ e^{-i\Omega t} e^{i(k+q)x} - (k-q)^2 A_- e^{-i\Omega t} e^{i(k-q)x} \tag{35}$$

Now for $|A|^2 A$, let us start with $|A|^2$

$$A^* = \rho e^{-ikx+i\Omega t} + A_+^* e^{i\Omega t} e^{-i(k+q)x} + A_-^* e^{i\Omega t} e^{-i(k-q)x} \tag{36}$$

$$\begin{aligned}|A|^2 &= A^* A = (\rho e^{-ikx+i\Omega t} + A_+^* e^{i\Omega t} e^{-i(k+q)x} + A_-^* e^{i\Omega t} e^{-i(k-q)x})(\rho e^{ikx-i\Omega t} + A_+ e^{-i\Omega t} e^{i(k+q)x} \\ &+ A_- e^{-i\Omega t} e^{i(k-q)x})\end{aligned}$$

$$= \rho^2 + \rho A_+^* e^{-iqx} + \rho A_-^* e^{iqx} + \rho A_+ e^{iqx} + |A_+|^2 + A_+ A_-^* e^{2iqx} + \rho A_- e^{-iqx} + A_- A_+^* e^{-2iqx} + |A_-|^2 \tag{37}$$

and now for $|A|^2A =$

$$
\begin{aligned}
= \; & \rho^3 e^{ikx-i\Omega t} + \rho^2 A_+^* e^{-i\Omega t} e^{i(k-q)x} + \rho^2 A_-^* e^{-i\Omega t} e^{i(k+q)x} + \rho^2 A_+ e^{-i\Omega t} e^{i(k+q)x} + \rho\, |A_+|^2 e^{ikx-i\Omega t} \\
& + \rho A_+ A_-^* e^{ikx-i\Omega t} e^{2iqx} + \rho^2 A_- e^{-i\Omega t} e^{i(k-q)x} + \rho A_- A_+^* e^{ikx-i\Omega t} e^{-2iqx} + \rho\, |A_-|^2\, e^{-ikx-i\Omega t} \\
& + \rho^2 A_+ e^{-i\Omega t} e^{i(k+q)x} + \rho\, |A_+|^2 e^{ikx-i\Omega t} + \rho A_+ A_-^* e^{ikx-i\Omega t} e^{2iqx} + \rho A_+^2 e^{ikx-i\Omega t} e^{2iqx} \\
& + A_+ |A_+|^2 e^{-i\Omega t} e^{i(k+q)x} + A_+^2 A_-^* e^{-i\Omega t} e^{i(k+q)x} e^{2iqx} + \rho A_+ A_- e^{ikx-i\Omega t} + |A_+|^2 A_- e^{-i\Omega t} e^{i(k-q)x} \\
& + |A_-|^2 A_+ e^{-i\Omega t} e^{i(k+q)x} + \rho^2 A_- e^{-i\Omega t} e^{i(k-q)x} + \rho A_+^* A_- e^{ikx-i\Omega t} e^{-2iqx} + \rho\, |A_-|^2 e^{ikx-i\Omega t} \\
& + \rho A_+ A_- e^{ikx-i\Omega t} + |A_+|^2 A_- e^{-i\Omega t} e^{i(k-q)x} + |A_-|^2 A_+ e^{-i\Omega t} e^{i(k+q)x} + \rho A_-^2 e^{ikx-i\Omega t} e^{-2iqx} \\
& + A_-^2 A_+^* e^{-i\Omega t} e^{i(k-q)x} e^{-2iqx} + |A_-|^2 A_- e^{-i\Omega t} e^{i(k-q)x}
\end{aligned}
$$

$$
\begin{aligned}
& = e^{ikx-i\Omega t} (\rho^3 + \rho\, |A_+|^2 + \rho A_+ A_-^* e^{2iqx} + \rho A_+^* A_- e^{-2iqx} + \rho\, |A_-|^2 + \rho\, |A_+|^2 + \rho A_+ A_-^* e^{2iqx} \\
& + \rho A_+^2 e^{2iqx} + \rho A_+ A_- + \rho A_+^* A_- e^{-2iqx} + \rho\, |A_-|^2 + \rho A_+ A_- + \rho A_-^2 e^{-2iqx}) \\
& + e^{i(k+q)x} e^{-i\Omega t} (\rho^2 A_-^* + \rho^2 A_+ + \rho^2 A_+ + |A_+|^2 A_+ + A_+^2 A_-^* e^{2iqx} + |A_-|^2 A_+ + |A_-|^2 A_+) \\
& + e^{i(k-q)x} e^{-i\Omega t} (\rho^2 A_+^* + \rho^2 A_- + |A_+|^2 A_- + \rho^2 A_- + |A_+|^2 A_- + A_-^2 A_+^* e^{-2iqx} + |A_-|^2 A_-)
\end{aligned}
$$

$$
= e^{ikx-i\Omega t} \varphi_1 + e^{i(k+q)x} e^{-i\Omega t} \varphi_2 + e^{i(k-q)x} e^{-i\Omega t} \varphi_{3,}
$$

where φ_1, φ_2 and φ_3 represent the respective terms in the brackets.

Now we calculate $\bar{A}$:

$$= \frac{1}{L}[\int_0^L \rho e^{ikx-i\Omega t}dx + \int_0^L A_+ e^{-i\Omega t} e^{i(k+q)x} dx + \int_0^L A_- e^{-i\Omega t} e^{i(k-q)x} dx]$$

$$= \frac{1}{L}[\frac{1}{ik}\rho e^{ikx-i\Omega t} + \frac{1}{i(k+q)} A_+ e^{-i\Omega t} e^{i(k+q)x} + \frac{1}{i(k-q)} A_- e^{-i\Omega t} e^{i(k-q)x}]_0^L$$

$$= \frac{1}{L}[(\frac{1}{ik}\rho e^{ikL-i\Omega t} + \frac{1}{i(k+q)} A_+ e^{-i\Omega t} e^{i(k+q)L} + \frac{1}{i(k-q)} A_- e^{-i\Omega t} e^{i(k-q)L}) - (\frac{1}{ik}\rho e^{-i\Omega t} + \frac{1}{i(k+q)} A_+ e^{-i\Omega t} + \frac{1}{i(k-q)} A_- e^{-i\Omega t})]$$

$$= \frac{1}{ikL}\rho e^{-i\Omega t}(e^{ikL}-1) + \frac{1}{i(k+q)L} A_+ e^{-i\Omega t}(e^{i(k+q)L}-1) + \frac{1}{i(k-q)L} A_- e^{-i\Omega t}(e^{i(k-q)L}-1) \tag{38}$$

Assuming that the system size is much larger than the wavelength, and in particular in the limit $L \to \infty$, the terms vanish and we obtain $\bar{A}(t) = 0$, and consequently $\bar{A}(t-\tau) = 0$. This means, $F_g = 0$ and we are then just left with the local contribution F_l. Substituting (34) into (15) we have:

$$F_l = \mu m_l e^{i\xi}[A(x,t-\tau) - A(x,t)]$$

$$= \mu m_l e^{i\xi}[(\rho e^{ikx-i\Omega(t-\tau)} + A_+ e^{-i\Omega(t-\tau)} e^{i(k+q)x} + A_- e^{-i\Omega(t-\tau)} e^{i(k-q)x})$$
$$-(\rho e^{ikx-i\Omega t} + A_+ e^{-i\Omega t} e^{i(k+q)x} + A_- e^{-i\Omega t} e^{i(k-q)x})]$$

$$= \mu m_l e^{i\xi}[\rho e^{ikx-i\Omega t}(e^{i\Omega\tau} - 1) + A_+ e^{i(k+q)x} e^{-i\Omega t}(e^{i\Omega\tau} - 1) + A_- e^{-i\Omega t} e^{i(k-q)x}(e^{i\Omega\tau} - 1)]$$

$$= \mu m_l e^{i\xi}[(e^{i\Omega\tau} - 1)(\rho e^{ikx-i\Omega t} + A_+ e^{i(k+q)x} e^{-i\Omega t} + A_- e^{-i\Omega t} e^{i(k-q)x})]$$

$$= \mu m_l e^{i\xi}(e^{i\Omega\tau} - 1)A \tag{39}$$

Substituting all these terms into Eq. (14), we obtain 3 equations:

$$\begin{aligned}(-i\Omega)\rho e^{ikx-i\Omega t} = (1-i\omega)\rho e^{ikx-i\Omega t} - (1+i\alpha)e^{ikx-i\Omega t}\varphi_1 + (1+i\beta)(-k^2)\rho e^{ikx-i\Omega t} \\ + \mu m_l e^{i\xi}\rho e^{ikx-i\Omega t}(e^{i\Omega\tau} - 1)\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{\partial A_+}{\partial t} e^{i(k+q)x} e^{-i\Omega t} &= -(-i\Omega)A_+ e^{i(k+q)x} e^{-i\Omega t} + (1-i\omega)A_+ e^{i(k+q)x} e^{-i\Omega t} - (1-i\alpha)e^{i(k+q)x} e^{-i\Omega t}\varphi_2 \\ &\quad +(1+i\beta)(-(k+q)^2)A_+ e^{i(k+q)x} e^{-i\Omega t} \\ &\quad +\mu m_l e^{i\xi} A_+ e^{i(k+q)x} e^{-i\Omega t}(e^{i\Omega\tau} - 1)\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{\partial A_-}{\partial t} e^{i(k-q)x} e^{-i\Omega t} &= -(-i\Omega)A_- e^{i(k-q)x} e^{-i\Omega t} + (1-i\omega)A_- e^{i(k-q)x} e^{-i\Omega t} - (1-i\alpha)e^{i(k-q)x} e^{-i\Omega t}\varphi_{3,} \\ &\quad +(1+i\beta)(-(k-q)^2)A_- e^{i(k-q)x} e^{-i\Omega t} \\ &\quad +\mu m_l e^{i\xi} A_- e^{i(k-q)x} e^{-i\Omega t}(e^{i\Omega\tau} - 1)\end{aligned} \tag{42}$$

After cancelling terms $\rho e^{ikx-i\Omega t}$, $e^{-i\Omega t}e^{i(k+q)x}$ and $e^{-i\Omega t}e^{i(k-q)x}$we are then left with:

$$(-i\Omega) = (1-i\omega) - (1+i\alpha)\varphi_1 - (1+i\beta)k^2 + \mu m_l e^{i\xi}(e^{i\Omega\tau}-1) \tag{43}$$

$$\begin{aligned}\frac{\partial A_+}{\partial t} &= -(-i\Omega)A_+ + (1-i\omega)A_+ - (1+i\alpha)\varphi_2 - (1+i\beta)(k+q)^2 A_+ \\ &+ \mu m_l e^{i\xi}A_+(e^{i\Omega\tau}-1)\end{aligned}$$

$$\begin{aligned}\frac{\partial A_-}{\partial t} &= -(-i\Omega)A_- + (1-i\omega)A_- - (1+i\alpha)\varphi_3 - (1+i\beta)(k-q)^2 A_- \\ &+ \mu m_l e^{i\xi}A_-(e^{i\Omega\tau}-1)\end{aligned}$$

where

- $\varphi_1 = \rho^2 + 2(|A_+|^2 + |A_-|^2) + 2A_+A_- + e^{2iqx}(2A_+A_-^* + A_+^2) + e^{-2iqx}(2A_+^*A_- + A_-^2)$
- $\varphi_2 = \rho^2(A_-^* + 2A_+) + A_+(|A_+|^2 + 2|A_-|^2) + e^{2iqx}(A_+^2A_-^*)$
- $\varphi_3 = \rho^2(A_+^* + 2A_-) + A_-(2|A_+|^2 + |A_-|^2) + e^{-2iqx}(A_-^2A_+^*)$

We neglect the higher harmonics and keep only the lowest order terms. We now obtain:

$$(-i\Omega) = (1-i\omega) - (1+i\alpha)\rho^2 - (1+i\beta)k^2 + \mu m_l e^{i\xi}(e^{i\Omega\tau}-1) \tag{44}$$

$$\begin{aligned}\frac{\partial A_+}{\partial t} &= (i\Omega)A_+ + (1-i\omega)A_+ - (1+i\alpha)\rho^2(A_-^* + 2A_+) - (1+i\beta)(k+q)^2 A_+ \\ &+ \mu m_l e^{i\xi}A_+(e^{i\Omega\tau}-1)\end{aligned} \tag{45}$$

$$\begin{aligned}\frac{\partial A_-}{\partial t} &= (i\Omega)A_- + (1-i\omega)A_- - (1+i\alpha)\rho^2(A_+^* + 2A_-) - (1+i\beta)(k-q)^2 A_- \\ &+ \mu m_l e^{i\xi}A_-(e^{i\Omega\tau}-1)\end{aligned} \tag{46}$$

Eq. (44) is identical to Eq. (28) and does not give anything new. We are interested in $\frac{\partial A_+}{\partial t}$ and $\frac{\partial A_-}{\partial t}$. However, since A_+ is coupled to A_-^* rather than to A_-, we replace the equation for A_- by an equation for A_-^*.

$$\frac{\partial A_-^*}{\partial t} = (-i\Omega)A_-^* + (1+i\omega)A_-^* - (1-i\alpha)\rho^2(A_+ + 2A_-^*) - (1-i\beta)(k-q)^2 A_-^* \\ + \mu m_l e^{-i\xi} A_-^* (e^{-i\Omega\tau} - 1) \tag{47}$$

Eq. (44) and (28) can be written in the following form:

$$\frac{\partial}{\partial t}\begin{pmatrix} A_+ \\ A_-^* \end{pmatrix} = C \begin{pmatrix} A_+ \\ A_-^* \end{pmatrix} \tag{48}$$

where

$$C = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \tag{49}$$

and

$$\Lambda_{11} = i\Omega + 1 - i\omega - 2\rho^2 - 2i\alpha\rho^2 - (k+q)^2 - i\beta(k+q)^2 + \mu m_l e^{i\xi}(e^{i\Omega\tau} - 1)$$
$$\Lambda_{12} = (-\rho^2 - i\alpha\rho^2)$$
$$\Lambda_{21} = (-\rho^2 + i\alpha\rho^2)$$
$$\Lambda_{22} = -i\Omega + 1 + i\omega - 2\rho^2 + 2i\alpha\rho^2 - (k-q)^2 + i\beta(k-q)^2 + \mu m_l e^{-i\xi}(e^{-i\Omega\tau} - 1)$$

4.3. Linear stability analysis

We now come to the linear stability analysis as such, where we set

$$A_+ = A_+^0 e^{\lambda t} \tag{50}$$

and

$$A_-^* = A_-^{*0} e^{\lambda t}, \tag{51}$$

where A_+^0, A_-^{*0} reflects an initial constant. We multiply by $e^{\lambda t}$ to observe how a small perturbation depending on λ will affect the dynamics of the modes A_+ and A_-^*. If the real part of λ is positive, an initial perturbation grows and if it is negative, an initial perturbation is damped out (decays). Calculating the partial derivative w.r.t. time we obtain:

$$\frac{\partial}{\partial t} A_+ = \lambda A_+^0 e^{\lambda t} = \lambda A_+ \tag{52}$$

$$\frac{\partial}{\partial t} A_-^* = \lambda A_-^{*0} e^{\lambda t} = \lambda A_-^* \tag{53}$$

and therefore we reduce this system to obtain:

$$\lambda \begin{pmatrix} A_+ \\ A_-^* \end{pmatrix} = C \begin{pmatrix} A_+ \\ A_-^* \end{pmatrix} \tag{54}$$

Eq. (54) is a standard linear eigenvalue equation, where λ are the eigenvalues of the matrix C. Alternatively, the perturbation ansatz (50) and (51) could have combined with the mode separation (34), but the derivation would have been more complicated.

We solve the eigenvalue problem (54) with the Wolfram Mathematica software [42]. A Mathematica script was generated to determine for a set of parameters, the solution of plane waves (given by ρ and Ω), together with its stability. Note that in order to obtain the stability of a plane wave, we have to fix its wavenumber k. Then, the eigenvalue λ is only a function of the wavenumber of the perturbation q, ($\lambda = \lambda(q)$ is also called the dispersion relation). Since C is a 2×2 matrix, we obtain 2 eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$. The eigenvalues can be complex and can be written as $\lambda^{(i)} = \lambda_1^{(i)} + i\lambda_2^{(i)}$, where the superscript $i = 1, 2$. An example is shown in Fig. 2.

We fix the parameters $m_l = 0.7$, $m_g = 0.3$, $\tau = 0.6$, $\mu = 0.6$, $\alpha = -1.4$, $\beta = 2$, $\omega = 2\pi - \alpha$, $\xi = \frac{\pi}{2}$ and $k = 0.1$ and we obtained $\Omega = 6.59644$ and $\rho^2 = 1.29601$. These values were then substituted into the above matrix (49) to plot both the real eigenvalues $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ on one graph displaying the dispersion relation.

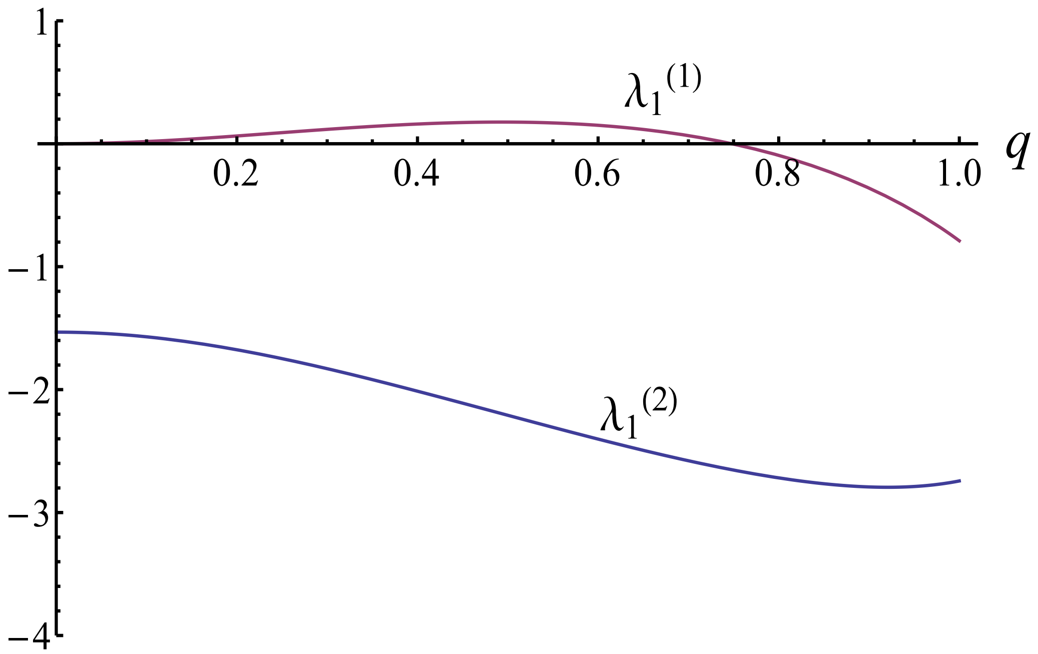

FIG. 2. Dispersion relation for the parameters $\tau = 0.1$, $\mu = 0.9$, $m_l = 0.4$, $m_g = 0.6$, $\alpha = -1.4$, $\beta = 2$, $\omega = 2\pi - \alpha$, $\xi = \frac{\pi}{2}$ and $k = 0.1$ and as a result we obtain $\rho^2 = 0.766205$ and $\Omega = 6.70851$. We show the two real parts $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$. The two curves refer to two different eigenvalue branches. We see here that that the largest real part of the eigenvalues is positive for approximately $q < 0.75$, hence indicating instability in this region.

Another Mathematica [42] script was generated using a loop over all q for 0 to 1 and for all k for 0 to 1. Then we can show for a given combination of μ and τ the largest real part of the eigenvalues for all k and q. This is shown in Fig. 3.

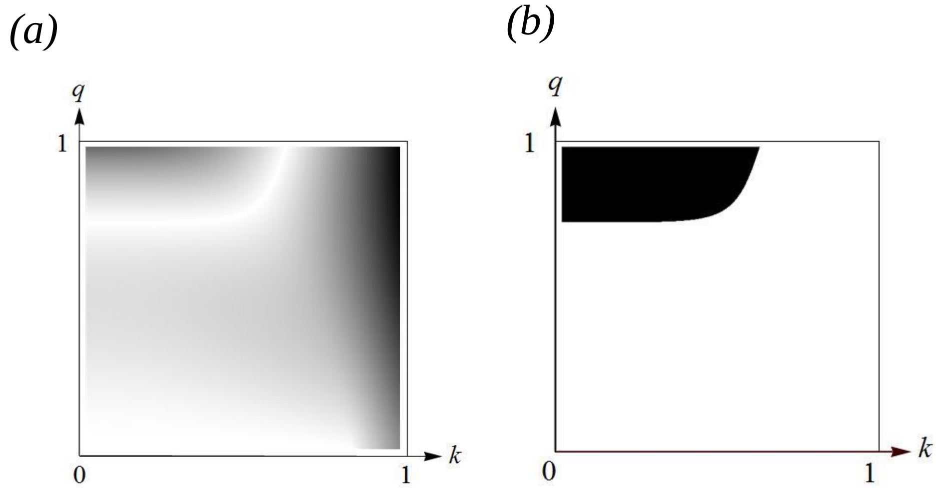

FIG. 3. The stability region for the plane wave solution in the CGLE for $\tau = 0.1$, $\mu = 0.9$, $m_l = 0.4$, $m_g = 0.6$, using the same parameters as in Fig. 2. This figure shows the largest real part of the eigenvalues for all k and q. **(a)** displays the magnitude of the leading eigenvalue, here the white region indicates regions that are close to zero and the darker regions are further away from zero (both negative and positive). **(b)** In the shaded region we experience stability (largest real part of eigenvalue negative) but in order to obtain global stability the solution must be stable with respect to all q. Since the shaded region does not extend for all q there is no global stability for any k.

The figures display the largest real part of the eigenvalues for all k and q. Here we experience stability in the shaded regions, however the solution must be stable with respect to all q to obtain global stability. We do not obtain global stability as the shaded region does not extend for all q. Now we repeat this procedure for different combinations of μ and τ. This is shown in Fig. 4.

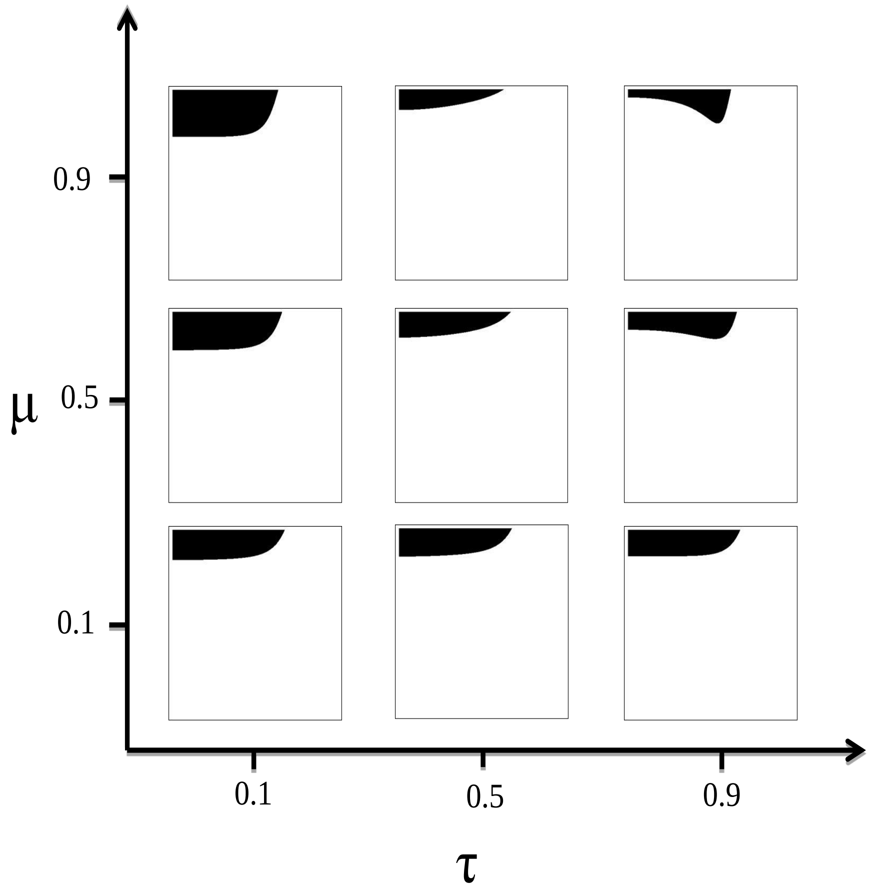

FIG. 4.

Same as Fig. 3, but now we try several combinations of τ and μ. Each inset shows in (k,q) space, the region in black where the largest real part of the eigenvalues is negative. We do not find any combination of (τ, μ) for which a plane wave pattern is stable for all q. Other parameters as in Fig. 2.

From our study of the stability of plane waves we have not found stability. Although this may seem like a negative result, this result only indicates we have not found stability for where we have looked. The parameter space is very large with α, β, m_l, m_g, ω and ξ not being varied here. In future work these variables can be used to find a stability of plane waves.

5. DISCUSSION

In this thesis, we have studied the stability of plane waves in a time-delay complex Ginzburg-Landau equation with combined local and global time-delayed feedback. We derived the analytical solution of plane waves of the kind $A_{PW} = \rho e^{ikx-i\Omega t}$, where ρ and Ω are given by Eqs. (29) and (30). Note that the solution only depends on m_l, not on m_g, as a spatially periodic solution does not contribute to the spatial average. We see that here when limit $k \to 0$, we do not recover the solution of uniform oscillations. This is precisely due to the spatial average because if the solution is spatially uniform, the term with m_l is identical to the one with m_g, and both contribute.

We have performed a linear stability analysis of plane waves with respect to perturbations of wavenumber q. As a consequence of this analysis, the eigenvalue equation (54) was obtained. After solving this equation numerically on Mathematica [42] we were able to display the dispersion relation and also determine the curves that limit the stability region of plane waves in (τ , μ)-space; see Fig 4. For the parameters used, no stabilization of plane waves was achieved.

We see that Fig.1(c) seems to contradict our findings. This contradiction may be due to a number of factors. Firstly Fig. 1(c) may be erroneous in the sense that although it is displaying stability, it may eventually become unstable in a later period of time. Secondly, simulations are made for a fixed system size L. This contradicts our theory as in Eq.(39) we make assumptions that the system size is much larger than the wavelength, and in particular in the limit L$\to \infty$ such that the global feedback vanishes. Not only does the global feedback vanishes but we also neglect higher harmonic terms.

Already in 1996 travelling waves were studied by Bleich and Socolar as they researched into the control of spatiotemporal dynamics with time-delay feedback using [31]. However, they used an *Extended TDAS* scheme and focused on the stabilization of plane waves being the solution of the CGLE in the absence of feedback. Similar appoaches were taken by Montgomery and Silber [37] in 2004 to investigate the stabilization of travelling wave solutions of the one-dimensional complex Ginzburg-Landau equation, however they used a spatially shifted scheme and their solution is also different.

In contrast to the above-mentioned studies, our model contains both global and local feedback and the feedback scheme is different. Also, we tried to stabilize a feedback-induced solution, not the plane wave solution of the standard CGLE. Although stabilization was not found, the parameter space is large and can be manipulated for future work (ω, α, β and ξ have not been varied yet). In our study we have only considered linear perturbations, considering nonlinear perturbations would yield another set of equations in the form of ordinary differential equations, which replace the current set of equations. When considering nonlinear perturbations we would have to use methods as used in weakly nonlinear analysis [43].

Our research was motivated by the vast applicability of the CGLE and how the CGLE can be used to describe various phenomena such as chemical oscillations or nonlinear waves. The CGLE resembles a reaction-diffusion system and the motivation to use time-delay feedback. The motivation lies deeper than simply identifying generic features of such a system but being able to manipulate them. Although the CGLE is not directly applicable to any specific chemical or biological system as the ones mentioned in the motivation section without performing a reduction analysis, its behaviour is qualitatively well described whenever the system is close to a Hopf bifurcation.

To conclude, similar work has been done previously but not using the same feedback scheme and not aiming at stabilizing the same solution. The research is interesting to focus on as we now know more about the stability of waves. Although our work may not be immediately applicable to cardiac systems or the Belousov-Zhabotinsky reaction but it may however prove importance for the study of other physical, chemical or biological systems.

6. APPENDIX A

In this appendix, we demonstrate how to derive the solution of feedback-induced uniform oscillations.

$$\frac{\partial A}{\partial t} = (1 - i\omega)A - (1 + i\alpha)|A|^2 A + (1 + i\beta)\nabla^2 A + F, \tag{55}$$

$$F = \mu e^{i\xi}\{m_l[A(x, t - \tau) - A(x, t)] + m_g[\bar{A}(t - \tau) - \bar{A}(t)]\}, \tag{56}$$

where

$$\bar{A}(t) = \frac{1}{L}\int_0^L A(x, t)dx \tag{57}$$

After splitting the global and local parts we obtain the following equation:

$$F = F_l + F_g = m_l \mu e^{i\xi}[A(x, t - \tau) - A(x, t)] + m_g \mu e^{i\xi}[\bar{A}(t - \tau) - \bar{A}(t)] \tag{58}$$

Substituting the ansatz $A_0(t) = \rho_0 e^{-i\Omega_0 t}$ into Eq. (6), we obtain a very long equation. For simplicity we derive each term separately, starting with the differential operator.

- $\frac{\partial A}{\partial t} = (-i\Omega_0)\rho_0 e^{-i\Omega_0 t} = (-i\Omega_0)A$
- $|A|^2 = AA^* = \rho_0 e^{-i\Omega_0 t}\rho_0 e^{i\Omega_0 t} = \rho_0^2$
- $\nabla^2 A = \frac{\partial^2 A}{\partial x^2} = 0$
- $A(x, t - \tau) - A(x, t) = \rho_0 e^{-i\Omega_0(t-\tau)} - \rho_0 e^{-i\Omega_0 t} = \rho_0 e^{-i\Omega_0 t}(e^{i\Omega_0\tau} - 1) = A(e^{i\Omega_0\tau} - 1)$
- $\bar{A}(t) = \frac{1}{L}\int_0^L \rho e^{-i\Omega t}dx = \frac{1}{L}[L\rho e^{-i\Omega t} - 0] = \rho e^{-i\Omega t} = A$ and therefore $\bar{A}(t - \tau) = \rho e^{-i\Omega(t-\tau)}$
- $\bar{A}(t - \tau) - \bar{A}(t) = \rho_0 e^{-i\Omega_0 t}(e^{i\Omega_0\tau} - 1) = A(e^{i\Omega_0\tau} - 1)$

Substituting the obtained terms back into Eq. (6) we now have the following equation:

$$(-i\Omega_0)A = (1 - i\omega)A - (1 + i\alpha)\rho_0^2 A + \mu e^{i\xi}[(m_l + m_g)A(e^{i\Omega_0\tau} - 1)] \tag{59}$$

$$(-i\Omega_0) = (1 - i\omega) - (1 + i\alpha)\rho_0^2 + \mu e^{i\xi}[(m_l + m_g)(e^{i\Omega_0\tau} - 1)] \tag{60}$$

$$-i\Omega_0 = 1 - i\omega - \rho_0^2 - i\alpha\rho_0^2 + \mu(\cos\xi + i\sin\xi)(\cos\Omega_0\tau + i\sin\Omega_0\tau - 1)(m_l + m_g) \tag{61}$$

Now separating real and imaginary terms, we obtain the following two equations, in terms of amplitude ρ:

$$0 = 1 - \rho_0^2 + \mu(m_l + m_g)\chi_1 \tag{62}$$

$$\rho_0 = \sqrt{1 + \mu(m_l + m_g)\chi_1} \tag{63}$$

and frequency Ω_0:

$$-\Omega_0 = -\omega - \alpha\rho_0^2 + \mu(m_l + m_g)\chi_2 = -\omega - \alpha(1 + \mu(m_l + m_g)\chi_1) + \mu(m_l + m_g)\chi_2 \tag{64}$$

$$\Omega_0 = \omega + \alpha + \mu(m_g + m_l)(\alpha\chi_1 - \chi_2) \tag{65}$$

Where $\chi_{1,2}$ denote effective modulation terms that can be either positive or negative. As these terms arise from the feedback therefore $\chi_{1,2}$ depend on ξ and τ.

7. APPENDIX B

The native solution of plane waves reads:

$$A_{PW} = \rho_{PW} e^{ikx} e^{-i\Omega_{PW} t} \tag{66}$$

Substituting (66) into each term of the CGLE [Eq. (6)], we now have (omitting the subscripts):

- $\frac{\partial A}{\partial t} = (-i\Omega)\rho e^{ikx} e^{-i\Omega t} = (-i\Omega)A$
- $|A|^2 = AA^* = \rho e^{ikx-i\Omega t} \cdot \rho e^{-ikx+i\Omega t} = \rho^2$
- $\nabla^2 A = \frac{\partial^2 A}{\partial x^2} = (ik)^2 \rho e^{ikx-i\Omega t} = -k^2 \rho e^{ikx-i\Omega t} = -k^2 A$

Now substituting the evaluated terms into (6), we now have the following:

$$(-i\Omega)A = (1-i\omega)A - (1+i\alpha)\rho^2 A + (1+i\beta)(-k^2)A \tag{67}$$

$$-i\Omega = 1 - i\omega - (1+i\alpha)\rho^2 + (1+i\beta)(-k^2) \tag{68}$$

after separating real and imaginary parts, we then obtain the following:

$$0 = 1 - \rho^2 - k^2, \tag{69}$$

$$\Omega = \omega + \alpha\rho^2 + \beta k^2 \tag{70}$$

or

$$\rho_{PW}^2 = 1 - k^2, \tag{71}$$

$$\Omega_{PW} = \omega + \alpha(1-k^2) + \beta k^2 = \omega + \alpha - \alpha k^2 + \beta k^2 = \omega + \alpha + (\beta - \alpha)k^2. \tag{72}$$

Therefore we now have:

$$A_{PW} = \sqrt{1-k^2}\, e^{ikx - i[\omega + \alpha + (\beta - \alpha)k^2]t} \tag{73}$$

Eq. (73) describes the solution of plane waves within the standard CGLE.

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