Nonlinear σ model for disordered superconductors

I. V. Yurkevich and Igor V. Lerner
School of Physics and Astronomy, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom
(Received 4 July 2000; published 24 January 2001)

We suggest a variant of the nonlinear σ model for the description of disordered superconductors. The main distinction from existing models lies in the fact that the saddle point equation is solved nonperturbatively in the superconducting pairing field. It allows one to use the model both in the vicinity of the metal-superconductor transition and well below its critical temperature with full account for the self-consistency conditions. We show that the model reproduces a set of known results in different limiting cases, and apply it for a self-consistent description of the proximity effect at the superconductor-metal interface.

DOI: 10.1103/PhysRevB.63.064522 PACS number(s): 74.20.−z, 74.50.+r

I. INTRODUCTION

Since a seminal paper by Wegner,[1] a field-theoretic approach to disordered systems based on the nonlinear σ model (NLσM) became one of the most powerful tools in describing localization effects and mesoscopic fluctuations. The main advantage of this approach lies in formulating the theory in terms of low-lying excitations (diffusion modes), which greatly simplifies perturbative and renormalization group calculations and, on the other hand, allows a nonperturbative treatment.

Such an approach has been successfully extended to the description of disordered superconductors.[2–4] It was based on the fermionic representation[5] of Wegner's NLσM extended to include the electron-electron interaction.[6] The starting point in these works[2–4] was a microscopic model of interacting electrons in a random potential. The effective NLσM includes an extra bosonic field describing the superconducting order parameter Δ. Then the lowest-order expansion in Δ is used. This makes such an approach a good working tool in the vicinity of the superconducting transition where all the interaction channels can be easily included which makes it very useful in describing different aspects of the metal-superconducting transitions.

An alternative approach to the NLσM for dirty superconductors[7–10] starts from the Bogoliubov–de Gennes equations (or, equivalently, Gorkov's equations) without imposing a self-consistency condition on the superconducting order parameter Δ which is considered as given. Then the initial many-body problem turns into a single-particle one which makes applicable powerful techniques based on the supersymmetric NLσM.[11] Such a supersymmetric approach has been recently developed in Ref. 10 and applied to the description of nonperturbative aspects of the proximity effect in superconducting–normal-metal structures. In this approach Δ was taken into account just by the boundary conditions (Andreev reflection) for the normal region. A natural disadvantage of this (and any supersymmetric) approach is that no interaction can be included beyond the mean-field approximation; thus it is impossible to describe an effect on the superconducting order parameter of disorder in the normal metal (or even inside the superconducting region).

A NLσM developed in this paper starts from a microscopic model of electrons in a random potential with BCS attraction, and the order parameter Δ is treated as a dynamical field, similar to the earlier developed microscopic approach.[2–4] We are using the standard fermionic replica approach[5] in temperature techniques.[6] For a long time, it was widely believed that such an approach cannot be used for nonperturbative analysis. However, it was recently shown[12,13] that this is not the case, since the well-known exact nonperturbative result was reproduced from the fermionic replica NLσM, as well as more recently[14] with the Keldysh technique.

In the initial approach[6] to interactions within the NLσM, a saddle-point approximation was identical to that of the noninteracting problem. This scheme was recently greatly improved[15] by choosing (within the Keldysh technique) the saddle point, taking account of the interaction which considerably simplified any further analysis. Such an analysis has been directly extended to dirty superconductors in Ref. 16. We consider a model where, for simplicity, the Coulomb repulsion is not included. A distinctive feature of our approach is a change of the saddle point (and of a subsequent initial approximation) in the presence of the superconducting order parameter. This is similar but not identical to the choice suggested in Ref. 15 (when applied to the Coulomb interaction, it would lead to a different variant of the NLσM). The NLσM (Ref. 15) is optimized to maximally simplify the lowest perturbational order while by sacrificing this we arrive at quite a general formulation of the model with different specific approximations being made for different applications.

As usual, we restrict our consideration to the limit of dirty superconductors when $\Delta \ll 1/\tau_{\rm el} \ll \varepsilon_F$ (or, equivalently, $v_F\tau_{\rm el} \ll \xi$ where $\tau_{\rm el}$ is the elastic mean free time, and ξ is the correlation length is dirty superconductors). After describing in detail an alternative saddle-point approximation, we show how the model reproduces a set of known results in different limiting cases, and apply it for a self-consistent description of the proximity effect at the superconductor-metal interface.

II. BASIC MODEL

We consider the standard BCS Hamiltonian in the presence of a random potential $u(\mathbf{r})$. For completeness, we start by outlining the standard procedure[2] of a field-theoretic representation in the temperature technique for this Hamiltonian. The corresponding action has the form

$$S=S_0+S_i, \tag{1a}$$

$$S_0=\int dx\, \psi_s^*(x)\left[\frac{\partial}{\partial\tau}+\hat{\xi}+u(\mathbf{r})\right]\psi_s(x), \tag{1b}$$

$$S_i=\lambda_0\int dx\, \psi_\uparrow^*(x)\psi_\downarrow^*(x)\psi_\downarrow(x)\psi_\uparrow(x). \tag{1c}$$

Here $\psi_s(x)$ is a Grassmannian field[17,5] antiperiodic in imaginary time τ with period $1/T$, $x\equiv(\mathbf{r},\tau)$, $s=(\uparrow,\downarrow)$ is the spin index, λ_0 is the BCS coupling constant, and from now on we set $\hbar=1$.

The random potential $u(\mathbf{r})$ is supposed to be Gaussian with zero mean and the standard pair correlator,

$$\langle u(\mathbf{r})u(\mathbf{r}')\rangle=\frac{1}{2\pi\nu\tau_{\rm el}}\delta(\mathbf{r}-\mathbf{r}'), \tag{2}$$

with ν being the density of states and $\tau_{\rm el}$ the elastic mean free time. The operator $\hat{\xi}$ in Eq. (1b) is defined as

$$\hat{\xi}=\frac{1}{2m}\left(-i\nabla-\frac{e}{c}\mathbf{A}\right)^2-\mu,$$

where $\mathbf{A}$ is a vector potential of an external magnetic field.

Averaging over u with the help of the standard replica trick gives the quartic in ψ term in the action. Using the Hubbard-Stratonovich transformation, one decouples both this term and the BCS term, Eq. (1c), the former with the help of a matrix field $\hat{\sigma}=\hat{\sigma}(\mathbf{r};\tau,\tau')$ and the latter with the help of a pairing field $\Delta=\Delta(\mathbf{r};\tau)$, which will eventually play the role of the order parameter. This results in the following effective action:

$$\mathcal{S}[\hat{\sigma},\Delta,\Psi]=\frac{\pi\nu}{8\tau_{\rm el}}\mathrm{Tr}\,\hat{\sigma}^2+\frac{1}{\lambda_0}\int dx|\Delta(x)|^2+\int dx\,\overline{\Psi}(x)\times\left[-\hat{\tau}_3^{\rm tr}\frac{\partial}{\partial\tau}-\hat{\xi}+\frac{i}{2\tau_{\rm el}}\hat{\sigma}+i\hat{\Delta}\right]\Psi(x). \tag{3}$$

Here the replicated Grassmannian fields are

$$\overline{\Psi}\equiv(\mathcal{C}\Psi)^T=\frac{1}{\sqrt{2}}(\psi_{si}^*,-\psi_{si}),\quad \Psi^T=\frac{1}{\sqrt{2}}(\psi_{si},\psi_{si}^*),$$

where $i=1,\ldots,N$ are the replica indices ($N=0$ in the final results). The standard doubling of these fields ($\psi\to\Psi$) is convenient to separate diffuson and Cooperon channels for electrons propagating in the random potential; $\mathcal{C}$ is the charge conjugating matrix defined by the above equation. The matrix fields $\hat{\sigma}$ and $\hat{\Delta}$ are defined in the space spanned by $\Psi\otimes\overline{\Psi}$ which is convenient to think of as a direct product of the $N\times N$ replica sector, 2×2 spin sector, and 2×2 "time-reversal" sector. The field $\hat{\sigma}$ is defined by its symmetries,

$$\hat{\sigma}^\dagger=\hat{\sigma},\quad \hat{\sigma}=\mathcal{C}\hat{\sigma}^T\mathcal{C}^{-1}, \tag{4}$$

and Tr in Eq. (3) refers to a summation over all the matrix indices, an integration over $\mathbf{r}$, and a double integration over τ (as $\hat{\sigma}$ is not diagonal in τ).

The field $\hat{\Delta}$ is an Hermitian and self-charge-conjugate matrix field, which is diagonal in the replica indices and coordinates $\mathbf{r}$ and τ, and has the following structure in the spin and time-reversal space:

$$\hat{\Delta}=-(\Delta'\hat{\tau}_2^{\rm tr}+\Delta''\hat{\tau}_1^{\rm tr})\otimes\hat{\tau}_2^{\rm sp}, \tag{5}$$

where Δ' and Δ'' are real and imaginary parts of the (scalar) pairing field Δ; $\hat{\tau}_i^{\rm tr}$ and $\hat{\tau}_i^{\rm sp}$ are Pauli matrices ($i=0,1,2,3$ with $\hat{\tau}_0=1$) that span the time-reversal and spin sectors, respectively.

The integral over electron degrees of freedom is performed in a usual way, so that one reduces the effective action (in the Matsubara-frequency representation) to the following form:

$$\mathcal{S}=\frac{\pi\nu}{8\tau_{\rm el}}\mathrm{Tr}\,\sigma^2+\frac{1}{T\lambda_0}\sum_\omega\int d\mathbf{r}|\Delta_\omega(\mathbf{r})|^2-\frac{1}{2}\mathrm{Tr}\ln\left[-\hat{\xi}+\frac{i}{2\tau_{\rm el}}\sigma+i(\hat{\epsilon}+\hat{\Delta})\right]. \tag{6}$$

Here $\hat{\epsilon}=\mathrm{diag}\,\epsilon_n$, while $\epsilon_n=\pi(2n+1)T$ is the fermionic frequency and $\omega=\epsilon-\epsilon'$ is the bosonic one. Since Δ is diagonal in the imaginary time τ, it is a matrix field in the Matsubara frequencies.

The action (6) is a standard starting point for a further field-theoretic analysis. To construct a working model, one needs to expand in some way the Tr ln term in Eq. (6). Our goal here is to derive a field-theoretic model which is fully self-consistent in terms of the superconducting order parameter Δ and does not use a small-Δ expansion. We restrict our considerations to the limit of dirty superconductors when $\Delta\ll1/\tau_{\rm el}\ll\varepsilon_F$. Otherwise, we do not impose any limitations on Δ, and will derive the model applicable both in the vicinity of the transition and deeply in the superconducting regime.

III. SADDLE POINT

Our starting point is to construct a saddle-point approximation to the action (6) in the presence of the field $\hat{\Delta}$. As usual, we vary the action with respect to the field σ which gives

$$\sigma(\mathbf{r})=\left\langle\mathbf{r}\left|\left[-\hat{\xi}+\frac{i}{2\tau_{\rm el}}\sigma+i(\hat{\epsilon}+\hat{\Delta})\right]^{-1}\right|\mathbf{r}\right\rangle. \tag{7}$$

As $1/\tau_{\rm el}$ is much greater than both temperature T and the order parameter Δ, the matrix $\hat{\epsilon}+\hat{\Delta}$ plays the role of a symmetry breaking field. We look for a solution in a way similar to that in the metallic phase where such a role is played by the matrix $\hat{\epsilon}$ alone. In the metallic phase, the saddle-point equation with $\epsilon\neq0$ has a unique solution $\hat{\sigma}=\Lambda$, where Λ is diagonal in ϵ and unit in the replica and spin sectors:

$$\Lambda=\mathrm{diag}\{\mathrm{sgn}\,\epsilon\}. \tag{8}$$

For $\epsilon=0$ a degenerate solution to the saddle-point equation is given by any matrix $\hat{\sigma}$ of the symmetry (4) obeying the condition $\sigma^2=1$. Such a matrix can be represented as $\hat{\sigma}=U^\dagger\Lambda U$, with U belonging to an appropriate symmetry group.[18]

Similarly, a solution to Eq. (7) in the presence of $\hat{\epsilon}+\hat{\Delta}$ is given by

$$\hat{\sigma}_{s.p.}=V_\Delta^\dagger\Lambda V_\Delta, \tag{9}$$

where V_Δ is the matrix that simultaneously diagonalizes both $\hat{\sigma}$ and $\hat{\epsilon}+\hat{\Delta}$. This means that it should be found together with the yet unknown eigenvalues $\lambda=\mathrm{diag}\,\lambda_\epsilon$ from

$$\hat{\epsilon}+\hat{\Delta}=V_\Delta^\dagger\lambda V_\Delta. \tag{10}$$

Naturally, one expects V_Δ to become a unit matrix above the superconducting transition temperature T_c.

Assuming that both fields $\Delta(\boldsymbol{r})$ and $\sigma(\boldsymbol{r})$ are smooth functions of $\boldsymbol{r}$ and looking for a spatially independent solution to Eq. (7) (i.e., ignoring at this stage the fact that $\hat{\xi}$ and V_Δ do not commute), one substitutes expressions (9) and (10) into Eq. (7), thus reducing it to

$$\sigma=\left\langle \mathbf{r}\left|\left[-\hat{\xi}+\frac{i}{2\tau_{\mathrm{el}}}\Lambda+i\lambda\right]^{-1}\right|\mathbf{r}\right\rangle. \tag{11}$$

The scale of λ is defined by $\epsilon\sim T$ and Δ which are both $\ll 1/\tau_{\mathrm{el}}$ in a dirty superconductor. Thus it is easy to verify that the saddle point is given by Eq. (9) with the eigenvalues Λ, Eq. (8), being not affected by the presence of superconductivity. Let us stress that this saddle point is obtained by a nonperturbative in Δ rotation (9) of the metallic saddle point Λ. This should lead to an effective functional valid anywhere in the superconducting phase rather than only in the vicinity of T_c.

Such an effective functional which includes fluctuations around the saddle point is obtained in the standard way. First, one constructs a saddle-point manifold of matrices σ obeying the saddle-point equation at $\lambda=0$, and then one expands the Tr ln term in Eq. (6) in both the symmetry breaking term λ and gradients of the fields V. The saddle-point manifold is convenient to represent as follows:

$$\sigma=V_\Delta^\dagger Q V_\Delta, \quad Q=U^\dagger\Lambda U, \tag{12}$$

where Q represents the saddle-point manifold in the metallic phase and σ is obtained from Q by the same rotation (9) as $\sigma_{s.p.}$ is obtained from the metallic saddle point Λ. Therefore, Q is defined, as in the metallic phase, on the coset space $S(2N)/S(N)\otimes S(N)$ where, depending on the symmetry, S represents the unitary, orthogonal, or symplectic group. Before describing the expansion, let us stress that one could expand the Tr ln term without making the rotation (12), i.e., in powers of $\nabla\sigma$ and of $\epsilon+\Delta$. Although this would be formally an expansion within the same manifold, performing first the rotation (12) simplifies enormously all the subsequent considerations and leads to a new variant of the nonlinear σ model.

After substituting Eq. (12) into Eq. (6), one obtains the following representation for the Tr ln term:

$$\delta\mathcal{S}=-\tfrac{1}{2}\mathrm{Tr}\ln\{\hat{G}_0^{-1}+V_\Delta[\hat{\xi},V_\Delta^\dagger]-i(U\lambda U^\dagger)\},$$

where

$$\hat{G}_0\equiv\left(\hat{\xi}-\frac{i}{2\tau_{\mathrm{el}}}\Lambda\right)^{-1}.$$

The expansion to the lowest powers of gradients and λ is easily performed and results after some straightforward calculations in the following action:

$$\mathcal{S}=\frac{1}{T\lambda_0}\sum_\omega\int d\mathbf{r}|\Delta_\omega|^2+\frac{\pi\nu}{2}\mathrm{Tr}\left[\frac{D}{4}(\partial Q)^2-\lambda Q\right], \tag{13}$$

where Tr refers to a summation over all the matrix indices and Matsubara frequencies, as well as to an integration over $\boldsymbol{r}$. The long derivative in Eq. (13) is defined as

$$\partial Q\equiv\nabla Q+\left[\mathbf{A}_\Delta-\frac{ie}{c}\mathbf{A}\,\hat{\tau}_3^{\mathrm{tr}},Q\right]\equiv\partial_0 Q+[\mathbf{A}_\Delta,Q], \tag{14}$$

where the matrix $\mathbf{A}_\Delta$ is given by

$$\mathbf{A}_\Delta=V_\Delta\partial_0 V_\Delta^\dagger, \tag{15}$$

and ∂_0 is the long derivative (14) in the absence of the pairing field Δ. Both V_Δ and λ should be found from the diagonalization of $\epsilon+\Delta$, Eq. (10). Although such a diagonalization cannot be done in general, it will be straightforward in many important limiting cases. For $\Delta=0$, the field $\mathbf{A}_\Delta$ vanishes, $\partial\to\partial_0$ and $\lambda\to\epsilon$, so that the functional (13) goes over to that of the standard nonlinear σ model for noninteracting electrons.

The σ model defined by Eqs. (13)–(15) is fully self-consistent, and the value of the superconducting order parameter can be found from it for any temperature and geometry (i.e., with a proper account of the proximity effects, where applicable). The self-consistency condition would easily follow from variation of the action (13) with respect to Δ and finding the optimal configuration for the fields. However, it is convenient to impose the self-consistency requirement only at the very end of the calculations. Any physical observable is then to be found by calculating an appropriate functional average with the functional (13)–(15).

We proceed with illustrating how the model reproduces basic fundamental results for dirty superconductors, then demonstrate how to include consistently weak localization corrections in the vicinity of the superconducting transition in the presence of a magnetic field, and finally show how to take into account the self-consistency of the order parameter in the description of the proximity effect in the *SNS* geometry.

IV. SIMPLEST APPROXIMATION

We show that the basic results for dirty superconductors can be reproduced in the simplest approximation: (i) we neglect all nonzero Matsubara harmonics of the pairing field, i.e., substitute $\hat{\Delta}_0\delta_{\epsilon,-\epsilon'}$ for $\hat{\Delta}_{\epsilon\epsilon'}$; (ii) we neglect disorder-induced fluctuations near the saddle point, i.e., substitute the saddle-point value $Q=\Lambda_\epsilon$. In this case, the matrix $\hat{\epsilon}+\Delta$ reduces to direct product over all integer n of $(\hat{\epsilon}_n+\hat{\Delta}_0)\otimes(\hat{\epsilon}_n-\hat{\Delta}_0)$ where

$$\hat{\epsilon}_n+\hat{\Delta}_0\equiv\begin{pmatrix}\epsilon_n & \Delta_0\\ \Delta_0^* & -\epsilon_n\end{pmatrix}. \tag{16}$$

Here $\Delta_0=|\Delta|e^{i\chi}$ is a two-component field which, naturally, plays the role of the order parameter (we omit the index 0 in $|\Delta|$). Now it is easy to find explicitly the eigenvalues λ and the diagonalizing matrix V_Δ in Eq. (10):

$$\lambda_\epsilon=\sqrt{\epsilon_n^2+|\Delta|^2}\,\mathrm{sgn}\,\epsilon_n,\quad \cos\theta_\epsilon\equiv\epsilon_n/\lambda_\epsilon,$$

$$V_{n\Delta}(\mathbf{r})=\cos\frac{\theta_\epsilon}{2}+\hat{\delta}\,\mathrm{sgn}\,\epsilon_n\sin\frac{\theta_\epsilon}{2}, \tag{17}$$

where $\hat{\delta}\equiv(\hat{\Delta}_0/|\Delta|)\delta_{\epsilon,-\epsilon'}$ is the 4×4 matrix which depends only on the phase χ of the field Δ_0 and repeats the matrix structure of $\hat{\Delta}_0$, Eq. (5), and the full matrix V_Δ is the direct product of all $V_{n\Delta}$.

On utilizing the assumption (ii) above, i.e., $Q=\Lambda$, and substituting the parametrization (17) into Eq. (13), we arrive at the action $\mathcal{S}\equiv\int d^dr\,\mathcal{L}$ with

$$\mathcal{L}=\frac{|\Delta|^2}{\lambda_0T}-2\pi\nu\sum_\epsilon\sqrt{\epsilon^2+|\Delta|^2}+\delta\mathcal{L},$$

$$\delta\mathcal{L}\equiv\frac{\pi\nu D}{2}\sum_\epsilon\left[(\nabla\theta_\epsilon)^2+\sin^2\theta_\epsilon\left(\nabla\chi-\frac{2e}{c}\mathbf{A}\right)^2\right]. \tag{18}$$

Using the parametrization (17) one can easily sum over ϵ to get

$$\delta\mathcal{L}=\frac{\pi\nu D}{8T}\left\{C_1(\nabla|\Delta|)^2+C_2\left(\nabla\chi-\frac{2e}{c}\mathbf{A}\right)^2\right\}, \tag{19}$$

where the stiffness coefficients $C_{1,2}$ are given by

$$C_1=\frac{1}{|\Delta|}\tanh\frac{|\Delta|}{2T}+\frac{1}{2T}\cosh^{-2}\frac{|\Delta|}{2T},$$

$$C_2=2|\Delta|\tanh\frac{|\Delta|}{2T}. \tag{20}$$

The functional (19)–(20) coincides with that obtained in Ref. 9. Expanding coefficients $C_{1,2}$ in Δ, one obtains the Ginzburg-Landau functional as that in Ref. 9. However, the simplest approximation used here (and equivalent to those on which earlier considerations[7–9] were based) is not sufficient even in describing the vicinity of the superconducting transition. In general, one must keep all the Matsubara components of the pairing fields. In the following section, we will show how to do this in the vicinity of the transition in the weak disorder limit.

V. GINZBURG-LANDAU FUNCTIONAL

In the vicinity of the superconducting transition one can expand the action (13) in the pairing field. A further simplification is possible in the weak disorder limit $p_Fl\gg1$: one can integrate out the Q field to obtain an effective action for the Δ field only. In the quadratic in the Δ approximation, the kernel of this action will give an effective matrix propagator of the pairing field, with due account for the disorder, which governs properties of a disordered superconducting sample near the transition.

To integrate over the Q field, one splits the action (13) into $\mathcal{S}\equiv\mathcal{S}_0+\mathcal{S}_\Delta$ where

$$\mathcal{S}_0=-\frac{\pi\nu D}{8}\mathrm{Tr}(\partial_0Q)^2-\frac{\pi\nu}{2}\mathrm{Tr}\,\epsilon Q \tag{21}$$

is the standard nonlinear σ model functional as in the metallic phase. Then one makes a cumulant expansion, i.e., first expands $e^{-(\mathcal{S}_0+\mathcal{S}_\Delta)}$ in powers of $\mathcal{S}_\Delta$, then performs the functional averaging with $e^{-\mathcal{S}_0}$ (denoted below by $\langle\cdots\rangle_Q$), and finally reexponentiates the results. The expansion involves only the first- and second-order cumulants since the higher-order cumulants generate terms of higher order in Δ. Then the only terms which contribute to the action quadratic in Δ are given by

$$\mathcal{S}_{\mathrm{eff}}[\Delta]=\frac{1}{\lambda_0T}\sum_\omega\int d\mathbf{r}|\Delta_\omega|^2-\frac{\pi\nu}{2}\langle\mathrm{Tr}(\lambda-\epsilon)Q\rangle_Q$$
$$-\left\langle\frac{\pi\nu D}{8}\mathrm{Tr}[\mathbf{A}_\Delta,Q]^2+\frac{(\pi\nu D)^2}{8}(\mathrm{Tr}\,Q\partial_0Q\,\mathbf{A}_\Delta)^2\right\rangle_Q. \tag{22}$$

Expanding λ and $\mathbf{A}_\Delta$ to the lowest power in Δ and performing a standard functional averaging, as described in the Appendix, one finds the action quadratic in Δ as follows:

$$\mathcal{S}_{\mathrm{eff}}[\Delta]=\frac{\nu}{T}\sum_\omega\int d\mathbf{r}\,\Delta_\omega^*(\mathbf{r})\langle\mathbf{r}|\hat{\mathcal{K}}_\omega|\mathbf{r}'\rangle\Delta_\omega(\mathbf{r}'), \tag{23}$$

with the operator $\hat{\mathcal{K}}_\omega$ given by

$$\hat{\mathcal{K}}_\omega=\frac{1}{\lambda_0\nu}-2\pi T\sum_{\epsilon(\omega-\epsilon)<0}\left\{\hat{\Pi}_\omega^c+\frac{1}{\pi\nu}\frac{\Pi_{|2\epsilon-\omega|}^d(0)\hat{\mathcal{C}}}{(2\epsilon-\omega)^2}\right\}. \tag{24}$$

Here $\Pi_{|\omega|}^{c,d}(\mathbf{r},\mathbf{r}')=\langle\mathbf{r}|\hat{\Pi}^{c,d}|\mathbf{r}'\rangle$ are the Cooperon and diffuson propagators, respectively, with

$$\hat{\Pi}_{|\omega|}^c=(\hat{\mathcal{C}}+|\omega|)^{-1}, \tag{25}$$

where the operator $\hat{\mathcal{C}}\equiv-D(\nabla-2ie\mathbf{A}/c)^2$ defines the propagation of the Cooperon modes; $\hat{\Pi}^d$ is obtained from $\hat{\Pi}^c$ by putting $\mathbf{A}=\mathbf{0}$.

In the last term in Eq. (24), $\Pi^d_{|\omega|}(0)\equiv\Pi^d_{|\omega|}(\mathbf{r},\mathbf{r})$; this term may be obtained by expanding (in the weak disorder parameter) the Cooperon propagator with the renormalized diffusion coefficient,

$$\hat{\mathcal{C}}\to\left[1-\frac{1}{\pi\nu}\Pi^d_{|\omega|}(0)\right]\hat{\mathcal{C}}.$$

Therefore, this is just a weak localization correction to the free Cooperon propagator $\Pi^c_{|\omega|}(\mathbf{r},\mathbf{r}')$.

The summation over Matsubara frequencies in Eq. (24) is easily performed to yield

$$\hat{\mathcal{K}}_\omega=\ln\frac{T}{T_0}+\psi\left(\frac{1}{2}+\frac{|\omega|-\hat{\mathcal{C}}}{4\pi T}\right)-\psi\left(\frac{1}{2}\right)-\frac{a_\omega\hat{\mathcal{C}}}{4\pi T},\qquad(26)$$

where $T_0\equiv T_{c0}(B=0)$ is the transition temperature of the clean superconductor in the absence of a magnetic field. The weak localization correction is proportional to the coefficient a_ω given by

$$a_\omega(T)=\frac{1}{\pi\nu V}\sum_{\mathbf{q}}\frac{1}{Dq^2}\left\{\psi'\left(\frac{1}{2}+\frac{|\omega|}{4\pi T}\right)-\frac{4\pi T}{Dq^2}\left[\psi\left(\frac{1}{2}+\frac{|\omega|+Dq^2}{4\pi T}\right)-\psi\left(\frac{1}{2}+\frac{|\omega|}{4\pi T}\right)\right]\right\}.$$

For $\omega=0$ the coefficient $a_0\equiv a_{\omega=0}(T)$ can be simplified in the two limits:

$$a_0=\begin{cases}\dfrac{\psi'(1/2)}{\pi\nu L^d}\displaystyle\sum_{L_T^{-1}<q<l^{-1}}\frac{1}{Dq^2}, & L\gg L_T,\\[2ex] -\dfrac{\psi''(1/2)}{8\pi^2\nu L^dT}, & L\ll L_T,\end{cases}\qquad(27)$$

where $L_T\equiv\sqrt{D/T}$ is the thermal smearing length.

The instability of the normal state (i.e., a transition to the superconducting state) occurs when the lowest eigenvalue of the operator $\hat{\mathcal{K}}_\omega$ becomes negative. The eigenfunctions of this operator coincide with the eigenfunctions of the Cooperon operator $\hat{\mathcal{C}}$. The lowest eigenvalue of $\hat{\mathcal{C}}$ is known to be $\mathcal{C}_0=DB/\phi_0$, where ϕ_0 is the flux quanta. This ground-state Cooperon eigenfunction corresponds to the lowest eigenvalue $\mathcal{K}_0$ of the operator $\hat{\mathcal{K}}_\omega$. The condition $\mathcal{K}_0=0$ implicitly defines the line $T_c(B)$ in the (T,B) plane where the transition occurs:

$$\ln\frac{T_c}{T_0}+\psi\left(\frac{1}{2}+\frac{\mathcal{C}_0}{4\pi T_c}\right)-\psi\left(\frac{1}{2}\right)=\frac{a_0\mathcal{C}_0}{4\pi T_c}.\qquad(28)$$

The term on the right-hand side (RHS) of Eq. (28) describes a $1/g$ correction to the main result. This weak localization is linear in the magnetic field B and vanishes as $B\to0$ as expected (Anderson theorem). In a nonzero magnetic field the weak localization correction to the B_c is positive which has a very simple explanation. The superconductivity is destroyed by the magnetic field when the flux over the area with the linear size of the order of the coherence length becomes greater than the flux quanta. The weak localization corrections diminish the diffusion coefficient, which leads to a shrinkage of the coherence length. Therefore, one needs a stronger field to fulfill the condition of coherence destruction. The same reasoning explains the growth of T_c in the fixed magnetic field.

Note finally that we have calculated the Q averages in Eq. (22) perturbatively, up to the first order in the weak localization correction. It would be straightforward to include the main weak localization corrections in all orders by calculating these averages via the renormalization group. This would lead to renormalizing the diffusion coefficient in the Cooperon propagator (25), thus changing the shape of the $T_c(B)$ curve. However, the value of $T_c(0)$ will again remain unaffected, since the superconducting instability is defined by the appearance of the zero mode in the operator $\hat{\mathcal{K}}$, Eq. (26). This zero mode is homogeneous, and thus does not depend on the value of the diffusion coefficient in the Cooperon propagator.

VI. PROXIMITY EFFECT

A recent supersymmetric version[10] of the NLσM has been specifically formulated for studying the proximity effect in *SNS* junctions. Although this version is very convenient for a nonperturbative analysis, it has the natural disadvantage of the supersymmetric approach: no interaction can be included beyond the mean-field approximation. It means that the superconducting order parameter Δ should be treated as a background field rather than a dynamical one. More specifically, Δ was taken into account[10] just by the boundary conditions (Andreev reflection) at the boundaries of a normal metal, while having been considered as a given field in the superconducting region. This allows for changes in characteristics of the normal metal in the proximity of the superconductor, but not for the possibility of changes in the superconducting order parameter in the proximity of the normal metal.

The action in the normal region (*N*) has the standard form[5,19] while in the superconductor (*S*) we have the NLσM of the form (13). The continuity of the Green function across the N/S boundary requires

$$Q_N|_{N/S}=V_\Delta^\dagger Q_S V_\Delta|_{N/S}.\qquad(29)$$

The *N* region by itself would favor $Q_N=\Lambda$. The proximity leads to a rotation of the matrix Q_N in the *N* region in order to match the structure imposed by the boundary condition (29):

$$Q_N\to V_N^\dagger Q_N V_N,\qquad(30)$$

with the rotation matrix V_N of the same structure as V_Δ in the *S* region so that at the boundary they match each other. Proceeding in the same manner as above we keep only the $\omega=0$ component of the pairing field and neglect the disorder induced fluctuations, i.e., we put $Q_N=Q_S=\Lambda$. Then for the *N* region we have

$$Q \to V_N^\dagger \Lambda\, V_N = \cos\theta_\epsilon + \sin\theta_\epsilon \hat{\delta},$$

$$\hat{\delta}_{\epsilon,\epsilon'} = -\delta_{\epsilon,-\epsilon'}(\cos\chi_\epsilon \hat{\tau}_2^{\mathrm{tr}} + \sin\chi_\epsilon \hat{\tau}_1^{\mathrm{tr}}) \otimes i\sigma_2, \qquad (31)$$

where θ_ϵ and χ_ϵ are now independent variables. In a bulk superconductor, all these parameters were explicit functions of Δ and ϵ, Eq. (17). There is no such a constraint in the normal region. The $(\epsilon,-\epsilon)$ sectors in the normal region are still coupled due to the proximity effect but they may all be different.

In this approximation the action corresponding to the N region decouples into the sum of uncorrelated contributions:

$$\mathcal{S}_N = 2\pi\nu \sum_\epsilon \int d\mathbf{r}\, \mathcal{L}_\epsilon,$$

$$\mathcal{L}_\epsilon = \frac{D}{4}\left[(\nabla\theta_\epsilon)^2 + \sin^2\theta_\epsilon \left(\nabla\chi_\epsilon - \frac{2e}{c}\mathbf{A} \right)^2 \right] - \epsilon\cos\theta_\epsilon. \qquad (32)$$

Now we find the supercurrent $\mathbf{j}_s$ by varying the action (32) with respect to the vector potential $\mathbf{A}$:

$$\mathbf{j}_s = 2e\pi\nu D\, T \sum_\epsilon \left\langle \sin^2\theta_\epsilon \left(\nabla\chi_\epsilon - i\frac{2e}{c}\mathbf{A} \right) \right\rangle_N, \qquad (33)$$

where $\langle \cdots \rangle_N$ stands for functional averaging with the action (32), the functional integration being performed over functions obeying the boundary conditions

$$\chi_\epsilon|_N = \chi|_S, \quad \cos\theta_\epsilon|_N = \frac{\epsilon}{\sqrt{\epsilon^2 + |\Delta|^2}}. \qquad (34)$$

Here $|\Delta|$ and χ are the modulus and phase of the order parameter at the N/S interface.

The classical trajectory corresponding to the action (32) is nothing but the Usadel equation[20]

$$-\frac{D}{2}\Delta\theta + \frac{D}{4}\sin 2\theta \left(\nabla\chi - \frac{2e}{c}\mathbf{A} \right)^2 + \epsilon\cos\theta = 0,$$

$$\nabla\left[\sin^2\theta \left(\nabla\chi - \frac{2e}{c}\mathbf{A} \right) \right] = 0. \qquad (35)$$

For quasi-one-dimensional (quasi-1D) geometry in the absence of a magnetic field, the Usadel equation (35) can be written as the equation for θ,

$$-\frac{d^2\theta}{dx^2} + \alpha_\epsilon^2 \frac{\cos\theta}{\sin^3\theta} + L_\epsilon^{-2}\sin\theta = 0, \qquad (36)$$

with the self-consistency condition on α_ϵ (see Fig. 1):

$$\chi_N = \alpha_\epsilon \int \frac{dx}{\sin^2\theta_\epsilon}. \qquad (37)$$

Here $\chi_N \equiv \chi_+ - \chi_-$ is the phase difference between two superconducting banks and $L_\epsilon = \sqrt{D/2\epsilon}$ is the coherence length for two particles with the energy difference ϵ propagating in

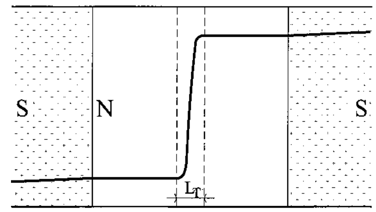

FIG. 1. A spatial dependence of the phase χ_ε across the SNS contact for quasi-1D geometry.

the normal metal. For a long normal bridge between the two superconducting banks, $L \gg L_T \equiv \sqrt{D/2\pi T}$, one may consider separately three regions: those close to the N/S boundaries (with the width of order L_T) and the bulk. Matching the solutions for all the regions, we find the following expression which well approximates the solution for the entire normal region:

$$\theta(x) = 8\tan(\theta_0/4) e^{-L/2L_\epsilon} \sqrt{\cos^2\frac{\chi_N}{2} + \sinh^2\frac{x}{L_\epsilon}},$$

$$\alpha_\epsilon = 32\tan^2(\theta_0/4)\sin\chi_N L_\epsilon^{-1}\exp[-L/L_\epsilon], \qquad (38)$$

where $\theta_0 \equiv \theta_\epsilon|_{N/S}$. In calculating the supercurrent through the normal bridge, one reduces the expression within the angular brackets in Eq. (33) to $\alpha_\epsilon \sin\chi_N$. Then it is enough to keep only the leading term with $\epsilon_0 = \pi T$ because the contributions from all other frequencies are exponentially suppressed as $L_\epsilon < L_T$. Then we obtain the following expression typical for Josephson junctions $j_s = j_c \sin\chi_N$, where j_c is the critical current:

$$j_c = e2^7\pi\nu D\, T\tan^2(\theta_T/4) L_T^{-1}\exp[-L/L_T], \qquad (39)$$

with $\theta_T \equiv \theta_{\epsilon_0}$.

The supercurrent in the superconducting banks is found by varying the action (19) valid in the S region with respect to the vector potential $\mathbf{A}$:

$$\mathbf{j}_s = e\pi\nu D|\Delta|\tanh\frac{|\Delta|}{2T}\frac{\chi_S}{L_S}, \qquad (40)$$

where L_S is the length of the superconductor and χ_S the phase difference between its edges.

It should be stressed that we have varied the action for the entire SNS structure, rather than only for the normal region subject to the boundary conditions at the superconducting banks as in the supersymmetric variant of the NLσM for dirty superconductors.[10] This means that the phase difference across the normal region is not fixed but should be found self-consistently by finding the optimal configuration for the action for the entire SNS structure subject by the matching the fields at the N/S boundaries. This defines the actual phase difference χ_N, Eq. (37), across the normal bridge. Numerically, a similar procedure has been employed in Ref. 21. It is easy to show that the matching condition can be expressed as the continuity of the supercurrents (as varying with respect to the phase difference is equivalent to varying with respect to the vector potential). Thus supercurrent conservation defines the phase difference on the normal bridge,

$$\frac{|\Delta|}{64T}\tanh\frac{|\Delta|}{2T}\chi_S=\frac{L_S}{L_T}e^{-L/L_T}\sin\chi_N\tan^2\frac{\theta_T}{4}, \qquad (41)$$

so that if the width of superconductor banks L_s is sufficiently large, the overall phase drop mainly happens across the banks.

Finally, let us reiterate that the main result of the paper is an alternative variant of the NLσM given by Eqs. (13)–(15). Here we have applied this formalism to a few relatively simple problems mainly to show that it works and has certain advantages over alternative variants of the NLσM. This model has also been applied to a microscopic consideration[22] of the quantum phase slip problem in quasi-1D superconductors[23,24] and to a microscopic derivation of level statistics in nonstandard symmetry classes introduced in Ref. 25. Let us also stress that the method employed in the derivation of Eqs. (13)–(15) can be straightforwardly generalized both to including different types of interactions and to considering the unconventional pairing in dirty superconductors.

ACKNOWLEDGMENTS

We are grateful to R. A. Smith for useful comments. This work has been supported by the Leverhulme Trust under Contract No. F/94/BY.

APPENDIX

To perform the functional averaging in Eq. (22), one should employ some parametrization of the field Q in terms of unconstrained matrices, for example,[11,19]

$$Q=(1-W/2)\Lambda(1+W/2)^{-1},$$

where $W=-W^\dagger$ and $W\Lambda+\Lambda W=0$. The Q integration then reduces to the Gaussian one with weight e^{-S_0} with S_0 obtained from Eq. (21) by expanding Q to second order in W. The Gaussian W integration is carried out with the help of the following contraction rules:

$$\langle \mathrm{Tr}\, MW(\mathbf{r})PW(\mathbf{r}')\rangle = -\frac{2}{\pi\nu}\sum_{\substack{\epsilon\epsilon'<0\\ \alpha,\beta}}[(\hat{\pi}\tau_1)^{\alpha\beta}_{\epsilon\epsilon'}\,\mathrm{tr}\,M^{\alpha\beta}_{\epsilon\epsilon'}\overline{P}^{\beta\alpha}_{\epsilon'\epsilon} + \hat{\pi}^{\alpha\beta}_{\epsilon\epsilon'}\,\mathrm{tr}\,M^{\alpha\alpha}_{\epsilon\epsilon}\,\mathrm{tr}\,P^{\beta\beta}_{\epsilon'\epsilon'}], \qquad (A1)$$

$$\langle \mathrm{Tr}\, MW(\mathbf{r})\mathrm{Tr}\, PW(\mathbf{r}')\rangle = -\frac{2}{\pi\nu}\sum_{\substack{\epsilon\epsilon'<0\\ \alpha,\beta}}\hat{\pi}^{\alpha\beta}_{\epsilon\epsilon'}\,\mathrm{tr}(M-\overline{M})^{\alpha\beta}_{\epsilon\epsilon'}(P-\overline{P})^{\beta\alpha}_{\epsilon'\epsilon}, \qquad (A2)$$

where the upper indices α,β refer to the time-reversal sector and tr refers only to the matrix indices which are not indicated explicitly. The matrix $\hat{\pi}$ in Eqs. (A1) and (A2) has the following structure in the time-reversal sector:

$$\hat{\pi}_{\epsilon\epsilon'}(\mathbf{r},\mathbf{r}')=\begin{pmatrix}\Pi^d_{|\epsilon-\epsilon'|}(\mathbf{r},\mathbf{r}') & \Pi^c_{|\epsilon-\epsilon'|}(\mathbf{r},\mathbf{r}')\\ \Pi^c_{|\epsilon-\epsilon'|}(\mathbf{r}',\mathbf{r}) & \Pi^d_{|\epsilon-\epsilon'|}(\mathbf{r},\mathbf{r}')\end{pmatrix}, \qquad (A3)$$

where the propagators are solutions to the standard Cooperon and diffuson equations:

$$[-D\nabla^2_{\mathbf{r}}+\omega]\Pi^d_\omega(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'),$$
$$\left[-D\left(\nabla_{\mathbf{r}}-i\frac{2e}{c}\mathbf{A}(\mathbf{r})\right)^2+\omega\right]\Pi^c_\omega(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \qquad (A4)$$

Note that in the absence of a magnetic field these contraction rules go over to those previously derived for orthogonal symmetry.[26]

Next, one expands Q in Eq. (22) up to the fourth power in W and uses the above contraction rules to obtain

$$\langle \mathrm{Tr}(\lambda-\epsilon)Q\rangle_Q=\mathrm{Tr}(\lambda-\epsilon)\Lambda, \qquad (A5)$$

$$\langle \mathrm{Tr}[\mathbf{A}_\Delta,Q]^2\rangle_Q=\mathrm{Tr}[\mathbf{A}_\Delta,\Lambda]^2+\frac{8}{\pi\nu}\sum_{\epsilon(\omega-\epsilon)<0}\frac{\Pi^d_{|2\epsilon-\omega|}(0)}{(2\epsilon-\omega)^2}\times\left|\left(\nabla-\frac{2e}{c}\mathbf{A}\right)\Delta_\omega\right|^2. \qquad (A6)$$

Taking into account that the second term in the brackets in Eq. (22) contributes to the higher-order correction only we arrive at the Ginzburg-Landau functional described in the text.

[1] F. Wegner, Z. Phys. B **35**, 207 (1979).
[2] A.M. Finkelshtein, Pis'ma Zh. Éksp. Teor. Fiz. **45**, 37 (1987) [JETP Lett. **45**, 46 (1987)].
[3] A.M. Finkelstein, Physica B **197**, 636 (1994).
[4] D. Belitz and T.R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).
[5] K.B. Efetov, A.I. Larkin, and D.E. Khmel'nitskii, Zh. Éksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)].
[6] A.M. Finkelshtein, Zh. Éksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)].
[7] R. Oppermann, Physica A **167**, 301 (1990).
[8] V.E. Kravtsov and R. Oppermann, Phys. Rev. B **43**, 10 865 (1991).
[9] V.E. Kravtsov, Z. Phys. B: Condens. Matter **82**, 205 (1991).
[10] A. Altland, B.D. Simons, and D. Taras-Semchuk, Pis'ma Zh. Éksp. Teor. Fiz. **67**, 21 (1998) [JETP Lett. **67**, 22 (1998)]; Adv. Phys. **49**, 321 (2000).
[11] K.B. Efetov, Adv. Phys. **32**, 53 (1983).
[12] A. Kamenev and M. Mezard, Phys. Rev. B **60**, 3944 (1999).

[13] I.V. Yurkevich and I.V. Lerner, Phys. Rev. B **60**, 3955 (1999).
[14] A. Altland and A. Kamenev, cond-mat/0006363 (unpublished).
[15] A. Kamenev and A. Andreev, Phys. Rev. B **60**, 3944 (1999).
[16] M.V. Feigel'man, A.I. Larkin, and M.A. Skvortsov, Phys. Rev. B **61**, 12 361 (2000).
[17] F.A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987).
[18] L.P. Gor'kov, A.I. Larkin, and D.E. Khmel'nitskii, Pis'ma Zh. Éksp. Teor. Fiz. **30**, 248 (1979) [JETP Lett. **30**, 228 (1979)].
[19] B.L. Altshuler, V.E. Kravtsov, and I.V. Lerner, Zh. Éksp. Teor. Fiz. **91**, 2276 (1986) [Sov. Phys. JETP **64**, 1352 (1986)].
[20] K.D. Usadel, Phys. Rev. Lett. **25**, 507 (1970).
[21] R.A. Riedel, L.-F. Chang, and P. Bagwell, Phys. Rev. B **54**, 16 082 (1996).
[22] I.V. Yurkevich and I.V. Lerner (unpublished).
[23] J.S. Langer and V. Ambegaokar, Phys. Rev. **164**, 498 (1967); D.E. McCumber and B.I. Halperin, Phys. Rev. B **1**, 1054 (1970).
[24] J.-M. Duan, Phys. Rev. Lett. **74**, 5128 (1995); A.D. Zaikin, D.S. Golubev, A. van Otterlo, and G.T. Zimanyi, *ibid.* **78**, 1552 (1997).
[25] A. Altland and M.R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[26] I.V. Lerner, Phys. Rev. B **48**, 9462 (1993).