DOCTOR OF PHILOSOPHY

Bifurcation study for a vertical channel with constant flux and large aspect ration

Richard L. Jones

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Thesis Summary

Using suitable coupled Navier-Stokes Equations for an incompressible Newtonian fluid, we investigate the linear and non-linear steady state solutions for both a homogeneously and a laterally heated fluid, with finite Prandtl Number ($Pr = 7$) in the vertical orientation of the channel. Both models are studied within the Large Aspect Ratio narrow-gap and under constant flux conditions with the channel closed.We use direct numerics to identify the linear stability criterion in parametric terms as a function of Grashof Number (Gr) and streamwise infinitesimal perturbation wavenumber (making use of the generalised Squire's Theorem). We find higher harmonic solutions at lower wavenumbers with a resonance of 1:3 exist, for both of the heating models considered. We proceed to identify 2D secondary steady state solutions, which bifurcate from the laminar state. Our studies show that 2D solutions are found not to exist in certain regions of the pure manifold, where we find that 1:3 resonant mode 2D solutions exist, for low wavenumber perturbations. For the homogeneously heated fluid , we notice a jump phenomenon existing between the pure and resonant mode secondary solutions for very specific wavenumbers. We attempt to verify whether mixed mode solutions are present for this model by considering the laterally heated model with the same geometry. We find mixed mode solutions for the laterally heated model showing that a bridge exists between the pure and 1:3 resonant mode 2D solutions, of which some are stationary and some travelling. Further, we show that for the homogeneously heated fluid that the 2D solutions bifurcate in hopf bifurcations and there exists a manifold where the 2D solutions are stable to Eckhaus criterion, within this manifold we proceed to identify 3D tertiary solutions and find that the stability for said 3D bifurcations is not phase locked to the 2D state. For the homogeneously heated model we identify a closed loop within the neutral stability curve for higher perturbation wavenumubers and analyse the nature of the multiple 2D bifurcations around this loop for identical wavenumber and find that a temperture inversion occurs within this loop. We conclude that for a homogeneously heated fluid it is possible to have abrupt transitions between the pure and resonant 2D solutions, and that for the laterally heated model there exist a transient bifurcation via mixed mode solutions.

Keywords: Floquet, Stability, Chebyshev, Hopf, Eckhaus, Resonance

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To Chloe, Lucy and Olivia,

my best friends.

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Finally, thank you Chloe, my wife, for standing by me and always being there for me.

1 Introduction

CONTENTS

1.1 Nomenclature

α	streamwise perturbations
β	spanwise perturbations
g	acceleration of gravity
ψ	toroidal part of velocity field
q	volume strength of the heat source
ϕ	poloidal part of velocity field
$\mathbf{u}$	velocity field
T	temperature
π	pressure term
ν	kinematic viscosity
κ	thermal diffusivity
ρ	density
γ	coefficient of thermal expansion
t	time
R	Reynolds Number
Pr	Prandtl number
Gr	Grashof number

1.2 Rational

"Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations." [$http://www.claymath.org/millenium/Navier-Stokes\_Equations$]

The extract above comes from the Clay Institute, which set out five Millenium Problems each with a 1,000,000 dollar prize. This study hopes to add some further insight into the problem of solving elliptical partial differential equations and illustrate some interesting solutions to the equations. As yet there is no analytical method to solve these equations as they have an infinite number of degrees of freedom [29] p.73. We use numerical methods to analyse the quantitive and qualitative nature of the solutions.

Ever since the time of the ancient greeks scientists and philosophers have debated and investigated the nature of fluids and fluid flow, we all recall Archimides' bouyancy principle from our school physics lessons. Bernoulli in the 18th century produced a plethora of work on fluid dynamics which is studied by many undergraduate students. Hand-in-hand with any investigations into the nature of fluid flow is the development of the necessary mathematics. After the discoveries by Newton and Leibnitz the door was opened to extend The Calculus into many physical phenomena. In the early 19th century Navier developed the early theories for fluid motion, which were incomplete but nevertheless an important advancement in understanding for the time. Sometime later Stokes perfected the work by Navier and the well known Navier-Stokes equation was born. During this time Poincare was working on the foundations of Chaos Theory, whose theories we use to this day. Major advancements in knowledge and understanding the complex nature of fluids came with the onset of the industrial revolution in the Britain. As mankind began to use steam and hydraulic systems the importance of understanding the mechanics of fluids became ever more important, especially the nature of turbulence. In the modern age of speed there is still a great deal of research required to understand the mechanisms of turbulence and how systems become turbulent. Formula 1 racing cars and modern aircraft are constantly changing geometries to both enhance and reduce drag. Understanding and being able to predict weather patterns has become increasingly critical in light of current research into global warming. Many around the globe have been effected by the effects of volcanic activity and the movement of tectonic plates caused by the mechanisms of internal magma heating and flow. We are surrounded by natural and man-made boundaries that effect the fluid flows in their direct vicinity. Being able to predict seemingly random events has become one of the major modern world physical and mathematical challenges.

Of the scientists working in the field of thermodynamics around the turn of the 19th century, five stand out; Reynolds, Rayleigh, Kelvin, Prandtl and Helmholz. Reynolds [42] important work in 1883 looked at the nature of turbulence for fluid flow in pipes and came to the conclusion that the link between the transition of a fluid to a turbulent state was a stability problem, a fundamental paradigm still used today. He also demonstrated that open channels show abrupt transitions to turbulence termed "Bypass Transitions".

Rayleigh's work developed the fundamental means for stability analysis by normal modes [16], which we also employ to this day. Kelvin and Helmholz developed the ideas of vortex flow and we use theorems today to decompose a solenoidal velocity field into in-phase and quadrature components. Prandtl's paper in 1905 describes boundary layer separation for low viscosity fluids.

One of the major advancements in the later half of the C20th has been the study of Chaos and non-linearity and the idea that seemingly random behaviour can be deterministic. Chaotic [29] p.13, behaviour can be time evolving and/or spatially evolving, the later being of particular interest in fluid flows. Hilborne [29] uses the Lorentz Model (a simplified Rayleigh-Benard) to describe the transitions or bifurcations [p.11] that a fluid can undergo when heated from below (the Rayleigh-Benard model), which show that a closed channel exhibits specific steps or transitions on the journey to turbulence. He gives an overview of how a fluid changes from a laminar (steady time independent) state to a convective state, where 2D convection rolls are able to circulate in differing directions dependent on a change in control parameter. He describes how the convection rolls can oscillate between directions periodically and aperiodically as time evolves. We know also that other spatial bifurcations can take place leading to cell structures forming in the fluid layer [13], dependant on how rapidly the fluid is heated, in terms of supercritical and subcritical bifurcations [16]. It is known that under certain conditions both spatial and temporal symmetries can be broken [29] p.28. Recent work on pattern competition [11] for a horizontal channel, homogeneously heated, with one boundary adiabatic and the other conducting attempts to predict where blow-holes will occur on the surface using CFD techniques. Imagine being able to predict exactly where the blow holes will occur when heating a pan of rice. This leads on to bubble chamber modelling which has safety applications within the nuclear industry. A more recent study of Rayleigh-Benard convection with a free boundary looks at a large gap model for atmospheric modeling with direct-numerical methods, which moves atmospheric modeling towards better weather prediction [46].

The process by which fluid flows undergo transition from smooth time-independant laminar flow to chaotic turbulent flow is a very complex and popular topic of research, however, the mechanisms of the early stages of turbulent birth are still not fully understood and are currently the focus of much research.

Finding 3D or higher order solutions is of particular importance because we begin to approach fully turbulent flow. Prandtl's paper of 1905 begins to explore the mechanisms of boundary layer separation which are explained well in [1]. Robinson [44] extends earlier ideas on the nature of the 3D structure present in a low Reynolds number turbulent boundary layer in terms of hairpin vortices. Recent research by Itano and Generalis has calculated some higher 3D stable solutions for plane couette flow in a vertical slot [32] using the same numerical methods as applied in this work. Thus providing numerical proof that earlier hypotheses regarding the hairpin structures were indeed correct. Recent research by Waleffe [49] into coherent 3D structures within the turbulent regime for Couette and Pouseiulle flow by direct numerical methods shows the existence of a Generic Self-Sustaining Process within shear flow that gives another insight into the transition to turbulent flow. Understanding the nature of boundary layer separation is of fundamental importance due to it's effects. We many wish to reduce drag created on airplane wings, racing cars or on the hulls of ships. We may wish to increase drag by designing small imperfections into the surface layer, i.e. a golf ball.

Current challenges faced within the nuclear reprocessing industry [1] require research into how radioactive self-heating materials undergo transition through the various stages up to fully turbulent flow. Understanding the mechanisms and parameters that govern the hydrodynamic stability of internally heated plasmas has major implications for the safety and design within the industry [http://www.irss-usa.org/pages/documents/HLWRevJun00.pdf]. Past studies were initiated by knowledge gained in the nuclear industry regarding safety in water-cooled reactors (Chernobyl) where high Rayleigh numbers combined with low Prandtl numbers were considered [30]. Current research in planetary geophysics [18] involves the existence of internal heat sources and their effect. Recent interest indicating strong evidence for the existence of a strong internal heat source driving the jets seen on the Jovian atmosphere [18] highlight the need for further research regarding the nonlinear stability of internal heat models in various states including rotation. The idea of a strong internal heat source on Jupiter was originally put forward by Busse [9]. Even under conditions governed by the Proudman-Taylor theorem [1] p. 280, the incidence of the convection column with latitude leads to the onset of instability thus giving the horizontal zonal flows characteristic of Jupiter. Questions regarding the existence of an internal heat source via radioactive heating in the model of the Earth's mantle are posed [15, 4], in [4] the question of heat dissipation and effects of magma flow on crustal heating is investi-

[1] BNF Sellafield HAL and HAST Management - Though personal time spent at NNL Warrington.

gated with an internal heat source. There exists a link between the angle of inclination of a vertical channel and the onset of instability raised in more recent work by Nagata and Generalis [22], which may have implications from the design of nuclear reactors to domestic tumble dryers. Much attention has been given to flows between differentially heated vertical parallel plates (LHF). Batchelor's [5] work into the rate and nature of heat transfer between vertical channels in a rectangular closed section identifies differing flow regimes for different ratios of rectangular section. Batchelor's work was driven by experimentation into cavity insulations for building design and for research into double glazing. Gershuni, Zhukhovitsky and Tarunin [24] later amended Batchelor's findings by identifying the nature of the convection motion within the heat transfer regime. We extend the considerable work already available by Gershuni and Zhukhovitsky [27] ch.X, which also considered a LAR channel with closed ends, constant flux and differing Prandt Number for the linear case only. More recent studies by Liakopolous et. al. [26] on lateral heating and cooling semiconductors show that the convection rolls persist at very high Grashof numbers. Liakopoulos et al. use spectral methods to identify weakly non-linear solutions by reducing the governing set of PDEs to a lower order approximation. This allows for the implementation of more complex geometries as in [26] and to more complex 3D simulations [28].

This work investigates the nature of the transition to turbulence for both a homogeneously heated fluid (HHF) and a laterally heated fluid (LHF) in a vertical channel but with no applied pressure gradient. We choose a closed channel with a large aspect ratio (LAR) because this geometry favours the upper harmonics. In high AR tubes the upper harmonics have greater amplitudes (relative to the fundamental) than do lower AR tubes. Whether harmonics are existent is just a question of their amplitudes. Brass instruments have high aspect ratio and hence sound "brassy" whereas wood instruments have a low aspect ratio so sound "woody". There is an excellent website **[www.navaching.com/shaku/]** that relates some of the mathematics used in this study to the design of Shakuhachi flutes used by Japanese zen buddhist monks. Having a closed channel also enables the maintenance of constant flux.

This study confirms the linear results from previous related studies [25, 27] and extends the current research in view of important advancements in the understanding of higher dimensional solutions within the highly non-linear region within turbulence [32]. The work presented here extends the work initiated by Gershuni and Zhukovitsky [25] as well as complementing and extending the more recent work by Generalis and Nagata [39] who did not have constant flux, instead constant pressure. We confirm that higher order harmonic resonant linear and non-linear solutions exist and extend the work by Fujimura [20] and Knobloch et al. [33], albeit for a different geometry. We perform a similar analysis for our model in extending the work by Gershuni [25, 27], Generalis [39] and Nagata [22] where a viscous incompressible fluid is bounded between two vertical parallel plates of infinite extent maintained at a constant temperature, $T = T_0$. Much work has already been completed by [39] with $pr \rightarrow 0$ for an open channel without constant flux, and Generalis [22] for an internally heated channel with inclination and the application of a pressure gradient with no inflection points in the basic flow profile, we introduce a basic flow profile with two inflection points where we expect the onset of instability as rolls as dictated by Rayleigh's Inflection Point Theorem [1]. This work extends and compliments these works by investigating the cause and effects of resonant solutions on the linear and secondary equilibrium states of the fluid for $pr = 7$ as well as identifying subsequent 3D solutions. This study begins by setting out the required mathematical model for HHF , then linear stability analysis is carried out. Where possible calculation of single mode 2D secondary equilibrium solutions bifurcating from the laminar state are found. Where not possible investigation of whether higher harmonic resonance modes exist is investigated. The works by Knobloch [33] and Fujimura [36] are used as paradigms in the preparation of this study, both of which analysed higher order harmonic resonant solutions for a horizontal model heated from below. Once identified we find the 2D secondary solutions bifurcating from the resonant laminar state. We proceed to analyse the stability of the 2D states outlining the Eckhaus stability criteria and the Hopf stability boundaries. Once we have identified where the 2D solution becomes unstable we proceed to find a steady state 3D tertiary solution and investigate its stability.

Due to the findings for HHF further investigation is required and thus LHF is investigated. Once benchmarking the calculations against known results for Rayleigh-Benard convection [13] was achieved the study proceeds to incline the channel to the vertical orientation and further benchmark against the work by Chait and Korpela [12] and Nagata and Busse [38] who took a vertical channel laterally heated with Prandtl Numbers 0.71, 1000 and 0 respectively. It was necessary to replicate resonant results for the Rayleigh-Benard model produced by [36] where as well as resonant solutions, mixed mode secondary solutions were found also. Based on reliable information from the Rayleigh-

Benard calculations this knowledge was applied to the LHF model. Understanding more about the nonlinear dynamics for a vertical channel acts as an important basis for advancement in the understanding of the dynamics governed by the aforementioned applications. The author has decided to put the mathematics into appendices at the end of the study due to the fact that the mathematics is somewhat bulky and would distract the reader from the main points discussed. The appendices are referred to in the main body of the text as necessary. The outcomes of this study are then put forward in conclusions and discussions, offering a brief outline of future intended research based on these findings. This study is the first to investigate resonant linear and non-linear solutions for an even flow profile with constant flux as well as being the first to find 3D solutions for the homogeneously heated fluid. This work is also the first to numerically calculate resonant secondary solutions for both HHF and LHF.

2 Programming

CONTENTS

2.1 Overview

A major part of the study involves the use of various computer platforms. For the linear analysis, a desktop PC running Windows XP using Compaq Visual Fortran V6.0 with the necessary IMF libraries is sufficient. However, for the non-linear analysis HPC was required due to memory limitations, namely a Cray XD1 running a modified Cray specific build of Linux 9. Programs have been developed for parallel processor implementation and also investigation is underway into the possible use of GPU hardware running the MAGMA[1] library.

2.1.1 High Performance Computing

In order to solve both the matrix algebra problem and the eigenvalue problem a Cray XD1 cluster was used. The memory requirements meant using distributed memory applications. Because the number of terms in the approximation determines the size of the matrix to be solved, efficient and fast solutions of large matrix problems is essential to obtain correct results, at present we can work with matrices up to 10000 square. The programs are written in Fortran 77 and use the set of Chebyshev approximating polynomials which transform the problem into matrix equations. The matrix equations are of two types; 1. The Steady State problem which searches for steady state solutions using a Newton-Raphson iteration. 2. The Dynamic problem which utilises the generalised eigenvalue problem to examine the stabilty of the steady state solutions. The software installed on the Cray is SuSi Linux 9 using the gnu64 complier and the programs are written with fortran 77. The main solver libraries are ScaLAPACK [2], a distributed memory version of LAPACK the Linear Algebra Solver Package and ParPACK the multiprocessor version. The following libraries are also used; acml (v3.5), blacs and BLAS (Basic Linear Algebra Subroutines, level 3 - matrix to matrix operations) and PBLAS for the parallel version. In addition mpich is adopted for the message passing protocol. Developmental programs written for multiprocessor implementation are being investigated and tested at present but were not used for the results obtained in this study. The software implementation allows the matrix and associated processes to be sub-divided across n x n processors. Further optimisation of the system was achieved by only assigning memory to the non-zero elements

[1] http://icl.cs.utk.edu/magma/

[2] The referenced manuals [6, 35], are published online and can be viewed in their entirety at www.netlib.org

of any block.

2.1.2 The Steady State Problem

The program solves the set of equations for $f(x) = 0$ by the Newton method where f is a vector of n functions of the n variables. The solutions of interest lie on surfaces in the space formed by the parameters. Lines within these surfaces are found by obtained solutions for a series of nearby parameters values. An initial vector x_0 obtained by finding a "seed", i.e. a solution that is found by hand using the symmetries to find a converged solution. One forces the code to converge on a solution by "perturbing" it by a small amount in the necessary coefficient table as dictated by the symmetries until the solution converges. Once a converged solution is found this is used to find further solutions with differing Grashof and perturbation wavenumber. This is used to solve the matrix equation

$$f + G\delta x = 0 \tag{2.1}$$

$$G_{ij} = \frac{df_i}{dx_j} \tag{2.2}$$

to obtain corrections δx to the solution vector $x = x + \delta x$. There are two stages to each iteration; setting up the Jacobian G and the vector f from a given x and solving the matrix equations for the corrections. The ScaLAPACK [6] library was used with the DGESV subroutine. Where the solution space is folded is located at an infinite gradient the Newton method may cause problems, this is overcome in the program by reducing the number of Newton iterations significantly and applying suitable small increments in Grashof Number.

2.1.3 The Dynamic Problem

The dynamic problem consists of solving the complex generalised eigenvalue problem described by eqn.(3.24) for given A and B to find out whether any of the eigenvalues have positive real parts, which indicates instability. The size of the task corresponds to the size of the steady solution which is needed to make the calculation, and the solution depends on a small number of parameters. What is needed is to compute the small number of eigenvalues which have the largest real part (the most dangerous ones). In practice, we have adopted a shift operation to find a small number of these eigenvalues of the largest magnitude of a related problem, which finds the eigenvalues of the true problem which are nearest to a trial complex number τ. The eigenvalues found, μ, are related to the true ones, λ, by the relation [35]

$$\mu = \frac{1}{\lambda - \tau}, \tag{2.3}$$

from which the value λ can be found. The modified eigenvalue problem becomes,

$$\mu x = (A - \tau B)^{-1} B x. \tag{2.4}$$

when developing the parallel processor version using ParPACK . Where A and B are singular we introduce a scalar factor to both sides of eqn.(2.4) to solve the modified simple eigenvalue problem

$$(I - K(A + KI - \tau B)^{-1})^{-1}(A + KI - \tau B)^{-1} B x = \mu x \tag{2.5}$$

where eqn.(2.3) applies and putting $K = 0$ deselects the mode. The user can specify one or more sets of parameters and a series of trial points τ in a region of interest in the complex plane, and the program will compute 2, 4 and 8 eigenvalues nearest to that point for each set of parameters, thus reducing the computing time. ScaLAPACK was used to set up the matrices and perform the matrix multiplications (pzgemv subroutine) alongside ParPACK which solved the generalised eigenvalue problem. Extra library functions and subroutines were required to interface ScaLAPACK and ParPACK (pzgemr2d, pzgetrs etc.) as the structure of the memory arrays differed between the packages.

2.1.4 Graphics

The curve plots were visualised using TecPlot, as were the stream and contour plots of the fluid flow. However, a preprocessing program written using Fortran [3] was required that structures the data into an ordered I, J 2D [31] format that TecPlot can interpret. A desktop PC was sufficient for the graphics preprocessing on the 2D models but due to the memory requirements it was found that writing distributed memory routines could be avoided by using an AppleMac desktop due to the way the OS-X manages virtual memory for the 3D models.

[3] Programs supplied by S.C. Generalis - Aston University

3 Vertical Channel Internally Heated

CONTENTS

3.1 Linear Analysis

Firstly consider a homogeneously heated fluid between two finite perfectly conducting isothermal boundaries with LAR, where the temperature is maintained at the same reference temperature (zero for ease of calculations). The viscous fluid is incompressible with constant density, what goes into the fluid particle comes out. Beginning with an overview of the linear stability then moving onto investigate non-linear solutions for a vertical channel, internally heated fluid (e.g. a vacuum flask) with pr = 7. Previous linear results provided by Gershuni and Zhukovistksi [27] for pr = 0 were reproduced in the preparation of this work to act as a control for the results obtained herein. This work also extends the work by Generalis and Nagata [39, 22] where in [39] pr = 0, there is no constant flux and a pressure gradient was present, in order to analyse the effect of purely hydrodynamic transition to instability. In [22] temperature was included and the channel was inclined at varying angles. The basic velocity profile in [22] was analogous to a Pouseille flow profile with no inflection points within the channel. In this section we investigate the laminar stability of the basic flow with dual inflection points as [27] but with a finite Prandtl number of 7, to analyse the effects of both hydrodynamic and thermal mechanisms on the stability of the system. Importantly we maintain a constant flux by keeping a closed channel of infinite extent, large aspect ratio. The model presented here is very simplified. Because of the setup of the model it is possible to look at the centre of the channel and not at the boundaries thus avoiding boundary layer separation [1]. It is worth noting that travelling waves are more prevalent in pipe flows whereas stationary waves are more prevalent in the models considered in this study. As far as the author is aware there exists no experimental research available at present except for the study presented by Wilkie and Fisher [50], in which the boundary conditions were different, which is discussed later in this section when the boundary conditions are defined.

[1] In subsequent sections it is further shown how the model overcomes boundary layer separation.

3.1.1 Problem Modeling

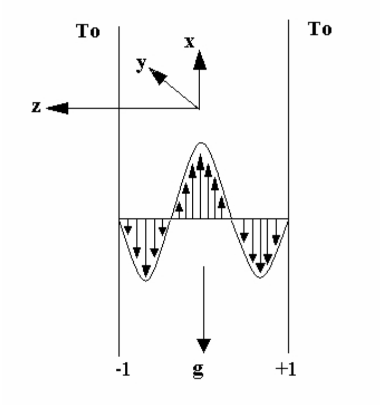


Figure 3.1: *Model Geometry.*

Firstly the coordinate system used is outlined as is the orientation axes in relation to the physical problem, see figure 3.1, with x, y, z as cartesian co-ordinates in the streamwise, spanwise and normal directions with unit vectors $\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$, respectively.

The Boussinesq Approximation [13] is employed, which states that density differences are sufficiently small to be neglected, except where they appear in terms multiplied by g, the acceleration due to gravity for bouyancy driven flows, and also include $\nabla\pi$ [1-3] , the applied dynamic pressure gradient on the fluid particles [2], to obtain the following Navier-Stokes equations for the velocity vector **u** and the temperature variation T:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla\pi + g\gamma T\hat{\mathbf{i}} + \nu\nabla^2\mathbf{u} \tag{3.1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa\nabla^2 T + q \tag{3.2}$$

$$\nabla\cdot\mathbf{u} = 0 \quad , \tag{3.3}$$

$$\int_{-1}^{1}\mathbf{u}dz = 0 \tag{3.4}$$

$$\mathbf{u} = 0\ ,\ T = 0 \quad at \quad z = \pm 1 \tag{3.5}$$

Derivations of Eqns. (3.1) and (3.2) can be readily found in many texts [1] [37] as is an outline of the use of the Boussinesq Approximation [13] and proof of the incompressibility condition (3.3) is dealt with in Appendix C section C.3 using tensors. The incompressibility condition is used as the fluid is Newtonian. Applying that the Div of

[2] refer to the following section on derivation of the perturbation equation for a further handling of the pressure term

the velocity field is zero ensures that this condition is met. [3] Incidentally, this condition is also met automatically when applying the curl and curl curl operators later. The boundary conditions are reflected by eq.(3.5). This model requires isothermal temperature boundaries so set $T_0 = 0$ for the fixed temperature of both the boundaries. For convenience zero is used as this makes computation easier and thus avoiding the introduction of arbitrary constants into any formulations. No-slip conditions at the boundaries for the velocity field $U_0(z)_i = 0$ is adopted. These boundaries are readily modified for diverse models, i.e. an adiabatic boundary where there is no transfer of temperature all along the boundary, so no temperature gradient exists $\frac{\delta T}{\delta z}(z) = 0$ [50] or, heated laterally where there exists a temperature difference across the boundaries which is demonstrated in a further chapter where a vertical channel laterally heated is investigated.

The narrow gap approximation is used which enables the use of cartesian geometry with this model and proceed to non-dimensionalize the system of equations with respect to kinematic viscosity ν for bulk flow because there is a basic flow present, as opposed to thermal diffusivity κ, as thermal diffusivity is greater in air (Pr = 0.71) than for honey with a high Prandtl Number hence for small Pr (7 is acceptable) the equations can be non-dimensionalised with respect to kinematic viscosity. Generalis and Fujimura's work [23] gives a full justification of this theory. It was Kropp and Busse [34] who orginally non-dimensionalised using κ. Based on the previous work with Rayleigh-Benard convection where non-dimensionalisation was carried out with respects to κ because the channel is flat and there is no basic flow, the only force driving the flow is thermal diffusivity. See Appendix A for a full treatment of the non-dimensionalisation. The necessary basic flow and temperature profiles with dual inflection points and even profiles are derived from the linear parts of the N.S. equations, the derivation of the basic flow and temperature profiles using the stated boundary conditions is covered in Appendix B.

The non-dimensionalisation of both the momentum and the temperature equations must be done so that the number of variables are reduced to a few manageable parameters. This is also needed to ignore scaling factors in the model which makes it more applicable. For the non-dimensional description of the problem the following parameters were used:

- d for length,
- $\frac{d^2}{\nu}$ for time,

[3] An excellent resource that explains the basic use of the Div and Curl operators can be found at www.khanacademy.com.

- $\frac{\nu}{d}$ for velocity field($\mathbf{u}$),
- $\frac{1}{d}$ for ∇,
- $\frac{1}{d^2}$ for ∇^2,
- $\frac{qd^2}{2\kappa Gr}$ for temperature,
- $\frac{g\gamma qd^5}{2\nu^2\kappa}$ for Gr, and
- $-\frac{d^3\nabla\pi}{2\nu^2\rho}$ for R.

The Grashof number gives the strength of the internal heat source. The Reynolds number $R = U_{max}d/\nu = -d^3\nabla\pi/2\nu^2\rho$, measures the strength of the applied pressure gradient in the streamwise direction(U_{max} is the maximum laminar velocity), and is the ratio between the inertial forces and the viscous forces, $R = 0$ in the current study. Full non-dimensionalisation of both the momentum and energy equations is given in Appendix A.1 and A.2. After non-dimensionalisation the following Navier-Stokes equations for the velocity vector $\mathbf{u}$ and the temperature variation T are obtained

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = 2R + T\hat{\mathbf{i}} + \nabla^2\mathbf{u} \quad (3.6)$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla\, T = -\frac{1}{Pr}(\nabla^2 T + 2Gr) \quad (3.7)$$

$$\nabla\cdot\mathbf{u} = 0 \quad (3.8)$$

Mass conservation is applied and constant flux is maintained by applying eq.(3.4) which is dealt with in Appendix D, where it is also shown how to derive the integral of the infinite iterative equation in order to apply the mass flux condition into our software model for plotting purposes only.

Taking only the linear parts of the N.S. equations and applying the boundary conditions to the resulting second order differential equation as shown in Appendix B.1 and B.2, results in the basic flow and temperature [27]:

$$U_o(z) = \frac{Gr}{60}(5z^4 - 6z^2 + 1) \quad (3.9)$$

$$T_o(z) = Gr(1 - z^2). \quad (3.10)$$

This basic steady state is derived by assuming that the total mass flux vanishes across any lateral cross- section of the channel. In comparison and for this work, the assumption is made that the remote ends of the channel are closed with LAR and long enough to support the wavelike perturbations applied and therefore the calculations assume the presence of a constant vertical pressure gradient and hence avoiding any boundary layer separation. In the next section derivation of the perturbation equations are made for linear stability analysis employing the Helmholz Decomposition of our solenoidal velocity field [47] , including derivation of the Orr-Somerfeld equation.

3.1.2 Perturbation Equations

To derive the perturbation equations, separate the velocity deviations $\hat{\mathbf{u}}$ from the primary variables, basic flow $\mathbf{u}_0(z)\hat{i}$ and the temperature deviations θ from the basic temperature $T_0(z)$ into average parts (over the x and y coordinates) $\breve{U}(z,t) \equiv \overline{\hat{\mathbf{u}}}$ and $\breve{T}(z,t) \equiv \overline{\theta}$ and a fluctuating part $\breve{\mathbf{u}}$, $\breve{\theta}$ respectively:

$$\hat{\mathbf{u}} = \breve{U}(z,t)\hat{\mathbf{i}} + \breve{\mathbf{u}} \tag{3.11}$$

$$\theta = \breve{T}(z,t) + \breve{\theta}. \tag{3.12}$$

Where the average, indicated by the overbar, is obtained by applying

$$(\alpha\beta/4\pi^2)\int_0^{2\pi/\alpha}\int_0^{2\pi/\beta} dxdy. \tag{3.13}$$

Appendix E gives a full mathematical treatment for the derivation of the mean flow and mean temperature, $\breve{U}(z,t)$ and $\breve{T}(z,t)$ resulting in

$$\partial_z^2\breve{U} + \breve{T} + \partial_z\overline{\Delta_2\phi(\partial_x\partial_z\phi + \partial_y\psi)} = \partial_t\breve{U}, \tag{3.14}$$

$$\partial_z^2\breve{T} + Pr\partial_z\overline{(\Delta_2\phi)\theta} = Pr\partial_t\breve{T}. \tag{3.15}$$

Equations (3.14) and (3.15) comprise of the mean terms and the nonlinear contribution/interaction of the perturbation terms.

It is worth noting that under the laminar regime oscillatory mean flow is not evident and thus it may be ignored at this stage. However, in later calculations it may exist or become time dependent, thus it is included in the necessary calculations and included in the software model. Initially, there is no interest in the time dependence of the mean flow because we do not want to look at fluctuations over the whole surface, only locally. Again, Appendix E discusses this issue further

Further, the soleniodal field $\breve{\mathbf{u}}$ is separated into the poloidal (stream) and toroidal (vorticity) parts ϕ, ψ, by applying the operators $\delta_i = (\nabla \times (\nabla \times \mathbf{k}\cdot))_i$ and $\varepsilon_i = (\nabla \times (\mathbf{k}\cdot))_i$, i.e. $\breve{\mathbf{u}} = \delta\phi + \varepsilon\psi$. By taking the curl and curl curl we also overcome the problem of boundary layer separation, thus avoiding any adverse pressure pressure gradients, i.e. the pressure is scalar. The curl and curl curl is taken with respects to $\mathbf{k}$ across z as the components of the flow normal to the channel boundaries and then normal to the direction of stream flow are required. Decomposition of equation [3.6] is completed by taking the curl and curl curl in z, as it is required to see how the field propogates perpendicular to the stream flow and transverse flow. The non-linear parts of the advection term are ignored at this point. Note that the temperature equation is scalar and is not decomposed. Where, using tensors the curl and curl curl operators are defined as

$$\varepsilon = \varepsilon_{ijk}\lambda_i\delta_j \tag{3.16}$$

$$\delta = \partial_i\partial_z + \lambda_i\Delta \tag{3.17}$$

respectively. See Appendix C.2 for a full derivation of these operators. This method is modeled on the same method first used by Schluter, Lortz and Busse [45], where Rayleigh-Benard convection is considered for both linear and weakly non-linear solutions. In [45] the authors derive the perturbation equations and solve them without the use of computers which seems incredible now. It is shown in Appendix C.3 that employing δ and ε assures that the incompressibility condition is satisfied automatically, thus playing no further part in the calculations. Appendix C.4 outlines the mathematics involved in application of the δ and ε operators on the linear parts of the momentum equation, subsequently the perturbation equations become

$$\begin{aligned}\tfrac{\partial}{\partial t}\nabla^2\Delta_2\phi - \nabla^4\Delta_2\phi + \hat{U}\partial_x\nabla^2\Delta_2\phi - (\partial_z^2\hat{U})(\partial_x\Delta_2\phi)&\\ -\partial x\partial z\theta = \delta\cdot(\mathbf{u}\cdot\nabla\mathbf{u})&\end{aligned} \tag{3.18}$$

$$\begin{aligned}\tfrac{\partial}{\partial_t}\Delta_2\psi - \partial_y\theta - \nabla^2\Delta_2\psi - (\partial_z\hat{U})(\partial_y\Delta_2\phi) + \hat{U}\partial_x\Delta_2\psi&\\ = \varepsilon\cdot(\mathbf{u}\cdot\nabla\mathbf{u})&\end{aligned} \tag{3.19}$$

with eqn. 3.18 being the Orr-Somerfeld Equation, while the temperature equation can be rewritten in the form

$$\tfrac{\partial}{\partial t}\theta + 2Gr(z)\Delta_2\phi + \hat{U}\partial_x\theta - \Delta_2\phi\partial_z\breve{T} - \tfrac{1}{Pr}\nabla^2\theta = (\delta\phi + \varepsilon\psi)\cdot\nabla\theta, \tag{3.20}$$

where the $\breve{}$ has been dropped from the temperature fluctuations and $\Delta_2 \equiv \partial_x^2 + \partial_y^2$ is the planform Laplacian. $\delta\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$ and $\varepsilon\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$ are the nonlinear parts of the Navier-Stokes

Equations and are not decomposed until section 3.2., the non-linear analysis. Eqns.(3.18-3.15) are subject to the homogeneous boundary conditions

$$\phi = \partial\phi/\partial z = \psi = \theta = 0 \qquad at\ z = \pm 1. \tag{3.21}$$

3.1.3 Chebyshev Point Collocation Method

Legendre [43] or Laguerre polynomials [2] could have been used but in this model Chebyshev Polynomials were chosen as they have a faster convergence and simplify the problem with respects to the boundaries used, i.e. ± 1, see [7]. Orzag gives a sound argument for their use in numericaly solving the Orr-Somerfeld equation [41]. Figure 3.2 shows how the flow field is descretised across the channel using coefficients resulting from the normal mode analysis, equation (3.22) the poloidal terms. The channel is also descretised for equation(3.23) but not shown.

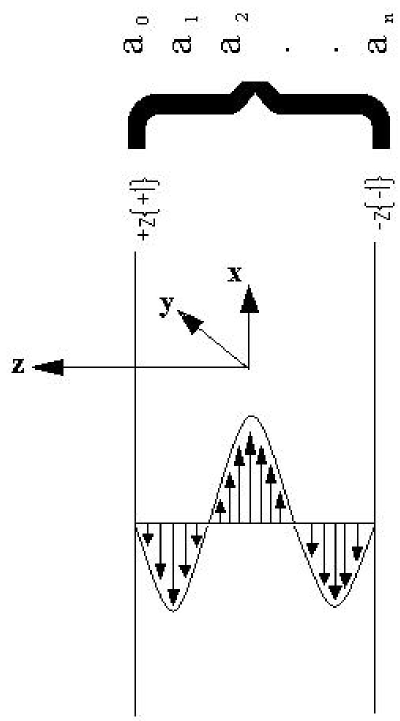


Figure 3.2: *Schematic Representing Chebyshev Descretisation of Poloidal Part Only.*

The 3D problem is reduced to a 2D problem according to Squires Theorem [48] which states that the 2D perturbations are the most dangerous for the linear case and take temperature into account. It is known that the streamwise periodic 2D modes (α) are the most dominant according to [48], hence this study focuses on examining the stability of the basic flow with infinitesimal perturbations of the transverse roll (TW) types ($\partial y = 0; \beta = 0; \alpha \neq 0$), neglecting equation 3.19 and the non-linear parts of eqns. 3.18 and 3.20. It is worth noting that for Rayleigh-Benard convection there is no basic flow and hence no preferred direction of pertubations that are more stable, i.e. both Transverse

Wave (TW) and Longitudinal Wave (LW) perturbations result in the same linear stability characteristics. However as the channel is vertical in this study there exists a basic flow due to gravity and the preferred direction for perturbations is the the streamwise direction. Hence, TWs cause linear instability at lower Grashof than do LWs which cause instability at much higher Grashof, this is illustrated by figure 3.3. For Normal Mode analysis [13] a method of separating variables is employed choosing:

$$\phi = \exp\{i\alpha(x-ct)\}\sum_{l=0}^{L} a_l f_l(z), \tag{3.22}$$

$$\theta = \exp\{i\alpha(x-ct)\}\sum_{l=0}^{L} b_l g_l(z), \tag{3.23}$$

where $f_l(z) = (1-z^2)^2 T_l(z)$ and $g_l(z) = (1-z^2)T_l(z)$ with T_l being the l^{th} order Chebyshev polynomial and $a_0,...,a_l,b_0,...,b_l$ are the unknown complex coefficients. In order that the boundary conditions [eqn. 3.21] are satisfied for ϕ and θ, $(1-z^2)^2$ and $(1-z^2)$ are introduced into the expansions. The Chebyshev collocation point method is used, with the resulting equations defining an algebraic eigenvalue problem

$$A\tilde{\mathbf{x}} = \sigma B\tilde{\mathbf{x}} \tag{3.24}$$

where $\sigma = -i\alpha c$, $\mathbf{x}$=$(a_0,...,a_l,b_0,...,b_l)^T$, and A and B are $2(L+1)$ by $2(L+1)$ complex matrices. The QZ method is utilised along with the IMF visual fortran libraries to solve eqn.(3.24). The real part of σ, σ_r is the rate of decay or amplification of the perturbations and hence used for stability. The imaginary part, $\sigma_i = -\alpha Re[c]$, is the phase velocity $Re[C]$ of the propagating perturbations in the flow.

An important aspect of this work is to define the symmetry groups that define the structure of the surviving harmonics after computation by analysing the nature and structure of the output matrix symmetries by looking at the effect of odd and even values of l and m in the coefficient of phi. The intention is to produce a set of symmetries that enable structures produced by the output files after calculations to be identified. A minimal closed set for a_{lm} is calculated (i.e. do not include trivial elements of the set like $T_0 = 1$ (a_{000}), which correspond to arbitrary shifts of the solution) and this output set will help to identify converged solutions within the output data produced by the computer by their symmetries. Full mathematical treatment and a more in depth explanation of the construction of the symmetry group is given in Appendix F, where in F.1 we construct the linear model symmetry group. Firstly the set of symmetries produced by the linear equations are found. Later, further calculations are made to ascertain which of these symmetries continue to exist once application of the 2D and 3D non-linear part of the decomposed Navier Stokes Equations is applied as this non-linear contribution is the most critical and defines which symmetries survive.

The resulting closed symmetry group emanating from the linear terms of the perturbed Navier-Stokes equation is found to be;

$$
\begin{aligned}
T_l^{+} \sin m^{+}, T_l^{+} \cos m^{+}, \\
T_l^{++} \sin m^{+}, T_l^{++} \cos m^{+},
\end{aligned}
\tag{3.25}
$$

Where + and ++ indicate odd and even functions respectively.

3.1.4 Linear Results

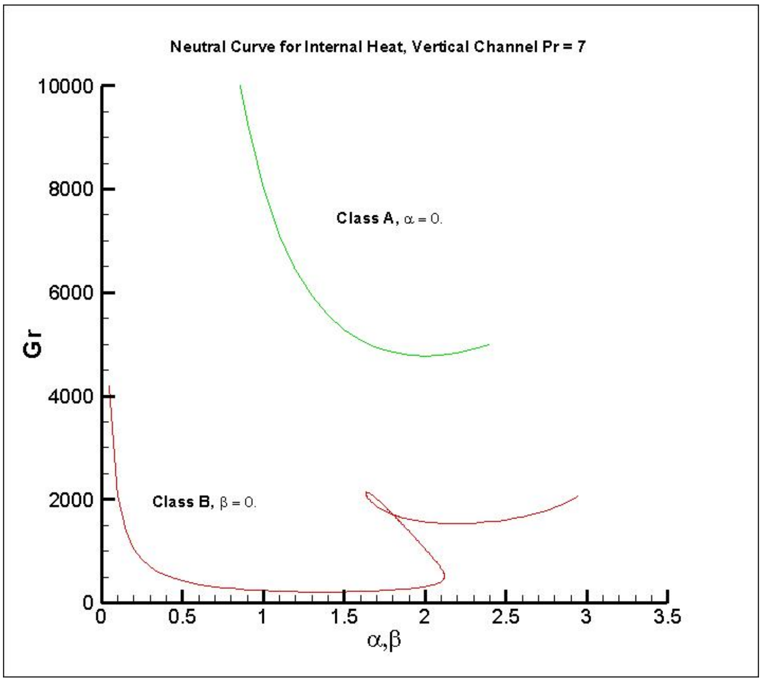


Figure 3.3: *Neutral Stability Curves for both LWs and TWs Vertical Channel.*

The linear stability boundary was calculated using a bisection method that is accurate to 4 decimal places. The region above the locus of the neutral stability curve is linearly unstable. Fig. (3.4) shows the neutral curve obtained with the critical parameters $(\alpha_c, Gr_c) = (1.37, 211.323)$. It can be seen that the thermal effects have a strong influence on the basic state stability with TW perturbations. The neutral stabilty curve for LW perturbations is not shown as it is manifest at much higher Grasshoff numbers and hence, as previously stated, is not required. We see that a closed loop or "island" is present which was caused by thermal convection, this island is as a result of an increase in Prandtl number, from zero, which is to be expected. This island was also identified in [25]. The neutral stability boundary was found that when the order of Chebyshevs was 30 a good convergence was obtained. The eigenvalue associated with the island was identified as lying in the temperature section of the solution matrix. The region above the curve is linearly unstable and below stable, in addition the region enclosed by the island was found to be linearly unstable.

In tracing the island care was taken when using the bisection method as the leading eigenvalues were both positive inside and outside this loop, the second most dangerous eigenvalue was used in tracing this island as it belongs to another mode. The island

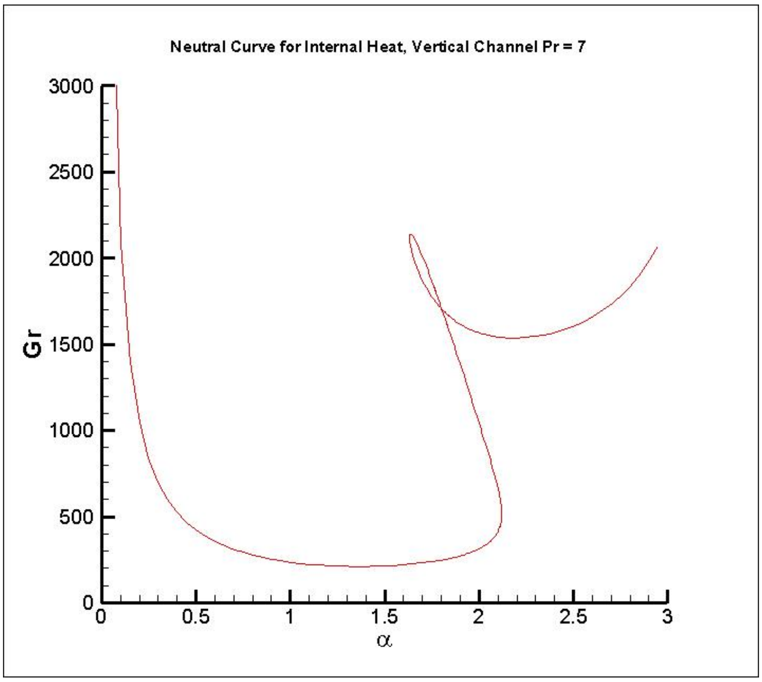


Figure 3.4: *Neutral Stability Curve for TWs.*

boundary was traced using the second eigenvalue as the guide to neutrality.

It is possible to verify that a second mode exists by using the results of the neutral stability calculations, by looking at the leading eigenvalues it is evident that more than one solution bifurcates from different parts of the neutral curve. Four points at various positions around the neutral curve (see figure 3.5) were used and the results of the real parts of the leading eigenvalues are shown in table 3.1.

Table 3.1: **Linear Stability Analysis in Loop Manifold**

Ref.	α	Grashof	Matrix Position	Eigenvalue
(i)	1.70	231.515	1	0.2594977342923E-08
(ii)	1.72	1829.678	1	0.9180557725350E+00
	1.72	1829.678	2	-0.2433592690590E-08
(iii)	1.72	1957.917	1	0.17449532217655E+01
	1.72	1957.917	2	0.48440826524054E-08
(iv)	2.00	1565.740	1	0.50096172180642E-09

An observation is that the leading eigenvalues inside the island are considerably larger than those outside because the area inside the island is highly unstable. As is shown in table 3.1, at points (i) and (iv) the expectation is to only have one bifurcating secondary

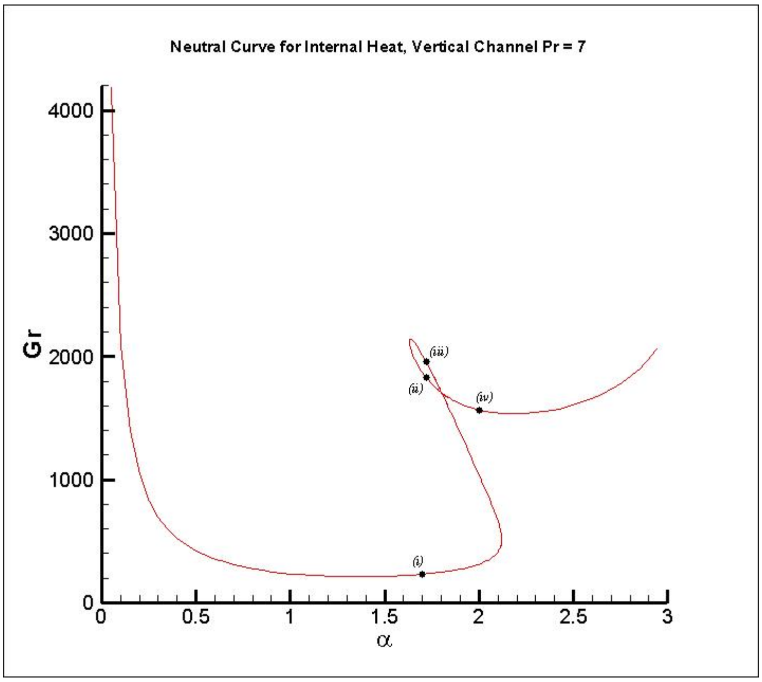


Figure 3.5: *Neutral Stability Curve.*

solution, but at points (ii) and (iii) there are two bifurcating secondary solutions. From the linear analysis there are two distinct neutral curves corresponding to two different modes. One originates from the LHS and low wavenumber perturbations and the other from the RHS with higher wavenumber perturbations. One of the aims of this study is to investigate the nature and interaction of the secondary solutions around the island where the two modes overlap. It is worth noting that the existence of mixed mode solutions needs to be investigated. It is known that mixed mode solutions exist in Rayleigh-Benard convection [36] and are associated with resonant solutions at higher harmonics. Hence, identification of any resonant solutions is now studied.

3.1.5 Resonant 1:3 Analysis

In order to investigate the existence of higher harmonic resonances in the linear regime the classic problem of Rayleigh- Benard convection with well documented, known solutions was reproduced in order to benchmark all the work in this study against the paradigms set out by Chandrasekhar [13], also reproduced were the results for odd and even modes found in [13] p.39 in order to ascertain that the second eigenvalue is required for odd modes and the first for even, this helps us to realise the resonant modes as they are found in the linear regime. Figure 3.6 shows the recreation of the odd and even modes as per

[13].

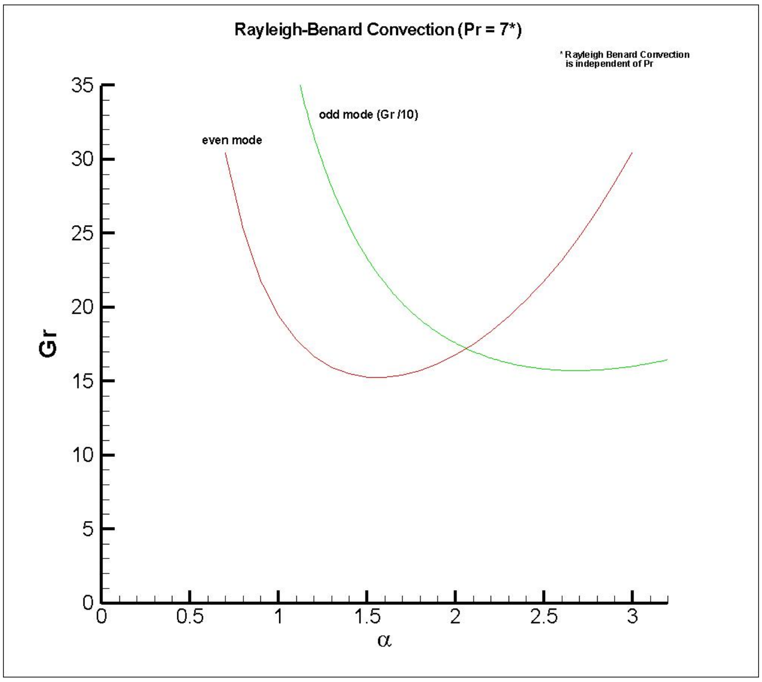


Figure 3.6: *Rayleigh-Benard Neutral Curve with Odd/Even Modes.*

For the linear resonant solutions benchmarking against the work by Fujimura and Mizushima [36], who found a 1:3 resonant solution that gives rise to 1:3 secondary solutions and mixed mode solutions was required. In fact it was this work which outlined the solutions that Busse had earlier overlooked [14]. Fujimura and Mizushima's work had quite an impact because their solutions had additional stability characteristics from those described in Busse's Balloon (stability analysis is undertaken in a later section). It was in [10] where the theoretical work in [14] is shown experimentally, where the mixed mode solutions were also overlooked. Also reproduced were the relevant results found by Knobloch et. al [33] who find 1:2 mode resonant solutions for the Rayleigh-Benard model.

It was subsequently found that a 1:3 mode linear solution exists for our HHF model. We see examples of where resonant solutions may exist in other system configurations [38] for a Pr=0, laterally heated model with a cubic flow profile shows that nonlinear two-dimensional solutions were not obtained for low streamwise wavenumber.

Previous work with Rayleigh-Benard convection on the presence of these resonant solutions are used as a reference for this work [33, 21]. Fujimura's results [21] were also replicated in order to assertain the nature of the resonant solutions found here. During this benchmarking was found a 1:4 resonance within the Rayleigh-Benard model, thus extending the research findings of [21]. As there are identical boundary conditions, the presence of any resonant mode interactions will play a part in the breaking of the mid-plane symmetry [33].

Fujimura [21] introduced $\tilde{z} = 2z$ and expanded ϕ and θ in terms of Chebyshev polynomials $T_n(\tilde{z})$ as

$$\phi = \exp\{i\alpha(x-ct)\}\sum_{n=0}^{N} a_n f_n(\tilde{z}), \tag{3.26}$$

$$\theta = \exp\{i\alpha(x-ct)\}\sum_{n=0}^{N} b_n g_n(\tilde{z}), \tag{3.27}$$

where $f_n(\tilde{z}) = (1-\tilde{z}^2)^2 T_n(\tilde{z})$ and $g_n(\tilde{z}) = (1-\tilde{z}^2)T_n(\tilde{z})$. In this study it is found that it was not necessary to adopt this method directly. We note that the values of Grashof found for the 1:3 resonant neutral stability boundaries are exactly the same as those for the pure mode, both with N=30 but we must take a third of the perturbation wavenumber (α). This fits with the existence of an O(3) symmetry as eluded to by [21] and [33].

Fig. 3.7 shows the interaction between the pure and 1:3 resonant neutral modes. The intersection of the 1:3 resonance mode and the pure mode curves is found at $(\alpha_c = 0.67, Gr_c = 319.3)$. This intersection point is where the search for the first secondary equilibrium solutions begins.

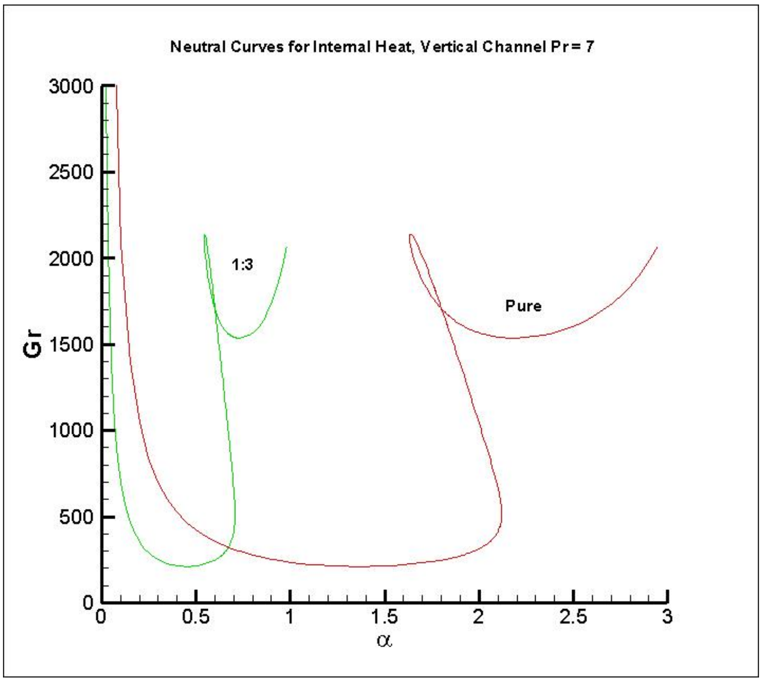


Figure 3.7: *Pure and 1:3 Resonant Mode Neutral Stability Curves.*

3.2 Non-Linear Analysis

3.2.1 Secondary Equilibrium States

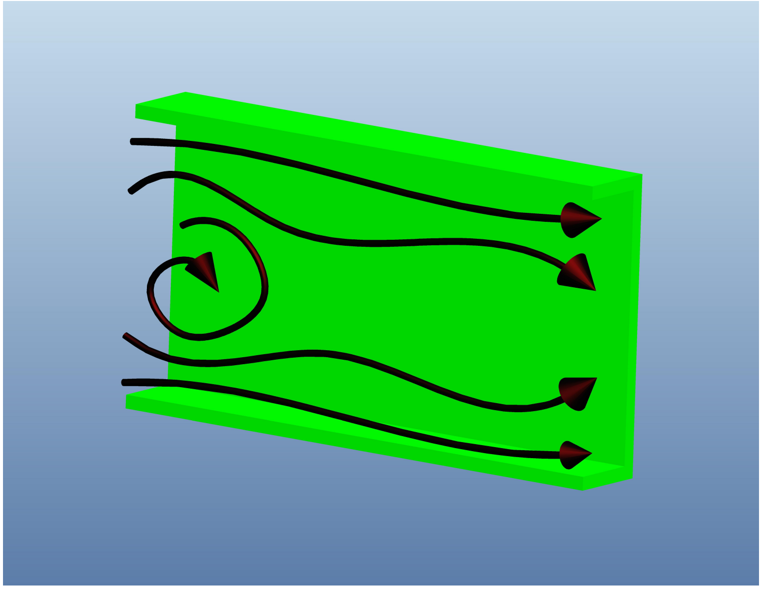


Figure 3.8: *Roll Evolution Schematic.*

In fig 3.8 it is possible to visualise, albeit in a simplistic manner, how secondary 2D non-linear solutions may evolve as we increase the TW perturbation. In this section calculation of the two-dimensional non-linear TW equilibrium solutions that evolve from transverse roll type perturbations on the neutral stability boundaries is outlined. Ignoring the equation for the toroidal part, Eqn.(3.19) and the spanwise dependency ($\partial_y = 0$), but retaining eqn.(3.18), eqn.(3.20) and eqns. (3.14) and (3.15) as well as retaining the original boundary conditions. ϕ and θ are expanded in terms of the set of orthogonal functions

$$\phi = \sum_{m=-\infty,m\neq 0}^{\infty} \sum_{l=0}^{\infty} a_{lm} \exp\{im\alpha(x-ct)\}(1-z^2)^2 T_l(z), \tag{3.28}$$

$$\theta = \sum_{m=-\infty,m\neq 0}^{\infty} \sum_{l=0}^{\infty} b_{lm} \exp\{im\alpha(x-ct)\}(1-z^2) T_l(z), \tag{3.29}$$

while we write the means:

$$\breve{U} = \sum_{l=even,l=0}^{\infty} (1-z^2) T_l(z), \tag{3.30}$$

$$\breve{T} = \sum_{l=even,l=0}^{\infty} (1-z^2) T_l(z), \tag{3.31}$$

Subject to the boundary conditions

$$\phi = \partial\phi/\partial z = \breve{U} = \psi = \breve{T} = \theta = 0 \qquad at\ z = \pm 1. \tag{3.32}$$

For the 2D calculations decomposition of the non-linear terms of the N.S. equations in 3.18 is required by taking the curl curl, $\delta \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$ using eqn 3.17. For the 2D solutions the poloidal part of the ϕ equation 3.18 is only necessary. It is worth noting that the poloidal flow can have many parts hence ϕ is more complex with more components and partial derivatives. The toroidal part of the flow is easier to calculate because it is across the channel ($\pm z$) and there are not so many terms as can be seen in Appendix C.5, where a complete derivation is included. (Decompostion of the toroidal parts is left until required in the section where tertiary (3D) solutions are sought. It is left to the reader to derive, preceeding as follows;

$$\begin{aligned}
&\tilde{\delta} \cdot \left(\tilde{u} \cdot \tilde{\nabla} \tilde{u}\right) \\
&= \tilde{\delta} \cdot \left[\left(\tilde{\delta}\phi + \tilde{\varepsilon}\psi\right) \cdot \tilde{\nabla} \left(\tilde{\delta}\phi + \tilde{\varepsilon}\psi\right)\right] \\
&= \delta_i \left(u_j \nabla_j u_i\right) \\
&= \delta_i \left[\left(\delta_j\phi + \varepsilon_j\psi\right) \nabla_j \left(\delta_i\phi + \varepsilon_i\psi\right)\right] \\
&= \delta_i \left\{\left(\delta_j\phi + \varepsilon_j\psi\right) \partial_j \left[\left(\partial_i\partial_z - \lambda_i\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\psi\right]\right\} \\
&= \delta_i \left\{\left(\delta_j\phi + \varepsilon_j\psi\right) \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\} \\
&= \delta_i \left\{\left[\left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi\right] \quad \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\} \\
&= \left(\partial_i\partial_z - \lambda_i\Delta\right) \left\{\left[\left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi\right] \quad \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}
\end{aligned}$$

Now apply the product rule twice for $\partial_i\partial_z$ and $\lambda_i\Delta$ letting $\Delta = \partial_t\partial_t$ in second expansion. Then expand, simplify, apply the necessary tensor algebra and collect sets of partial derivatives in terms of $\phi - \phi$, $\phi - \psi$ and $\psi - \psi$, as shown in Appendix C. For the 2D symmetry group we only require the partial derivatives of the $\phi - \phi$ that are in terms of x and z in the curl-curl decomposition. Eventually, one should arrive at the following result, where the remaining partial derivatives remaining are;

$$\begin{aligned}
&+ \left(\partial_x\partial_z\phi\right)\left(\partial_x^5\phi\right) + \left(\partial_x\partial_z\phi\right)\left(\partial_x^3\partial_z^2\phi\right) - \left(\partial_x^3\phi\right)\left(\partial_x^3\partial_z\phi\right) - \left(\partial_x^3\phi\right)\left(\partial_x\partial_z^3\phi\right) \\
&- \left(\partial_x^2\phi\right)\left(\partial_x^4\partial_z\phi\right) - \left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right) + \left(\partial_x^2\partial_z\phi\right)\left(\partial_x^4\phi\right) + \left(\partial_x^2\partial z\phi\right)\left(\partial_x^2\partial_z^2\phi\right).
\end{aligned}$$

The next task is to find the associated output symmetry group for the 2D problem. Appendix F.2 outlines the method employed which is similar to that for the linear problem but taking into account the the non-linear terms above. After 2D non-linear analysis it was found that the symmetry set reduced to;

$$T_l^+ \sin m^{++}, T_l^+ \cos m^{++}, T_l^{++} \sin m^+, T_l^{++} \cos m^+. \tag{3.33}$$

That is, the single symmetry involving ϕ is $a_{lm} = 0$ for $l+m$ even. For ϕ and θ we impose the reality condition $a_{-lm} = a^*_{lm}$ and $b_{-lm} = b^*_{lm}$, where * represents the complex conjugate. The outlining mathematics to apply the reality condition can be found at the very end of Appendix F.

The nonlinear secondary equilibrium solutions are found numerically using the Chebyshev collocation point method and the Newton-Raphson iterative method for high enough truncation numbers L and M, see table 3.2.

As a measure of the numerical convergence we employ the vector $l_2 - norm$ of the secondary solutions , which is defined by

$$|l_2^a| = \left\{ \sum_{l=0}^{L} \sum_{m=-M, m\neq 0}^{M} a_{lm} a^*_{lm} \right\}^{1/2}, \tag{3.34}$$

for a_{lm} and a similar expression for b_{lm}. Fujimura [21] used another method for plotting the strength of the secondary solution by using $\omega_1 = i\alpha\phi_1$, i.e. the imaginary part of the Fourier expansion, which is the equilibrium amplitude of the particle velocity component at $z = 0$ (midplane) for the primary roll.

Well converged supercritical TW secondary solutions are obtained at L=39 and M=5 for $\alpha > 0.5$ for this the pure mode. See table 3.2, where values for the $l_2 - norm$ for the poloidal component of the velocity field are shown for a sample integer range of L^{th} order Chebyshev and M harmonics for secondary solutions obtained close to the critical Grashoff . The total number of real coefficients in the Fourier expansion of the poloidal part of the velocity fluctuations are given.

The $|l_2|$ is not examined for $Gr > 401$ except for $\alpha \geq 1.6$ where an initial value of $Gr = 3156$ is taken and decremented to avoid the island in the neutral stability curve, also for $\alpha = 1.6$ a value of $Gr = 2191.$ is not exceeded.

Table 3.2: **Values of the l_2^{norm} for various L and M at** $\alpha = 1.37$**, Gr = 220.**

l_2^{norm}	L	M	a_{lm}
0.1978706106333731	3	37	266
0.1978706106333282	3	39	280
0.1978641941579140	4	39	360
0.1978642000637219	5	29	330
0.1978642001234143	5	31	352
0.1978642001187010	5	33	372
0.1978642001180187	5	35	396
0.1978642001183240	5	37	418
0.1978642001182701	5	39	440

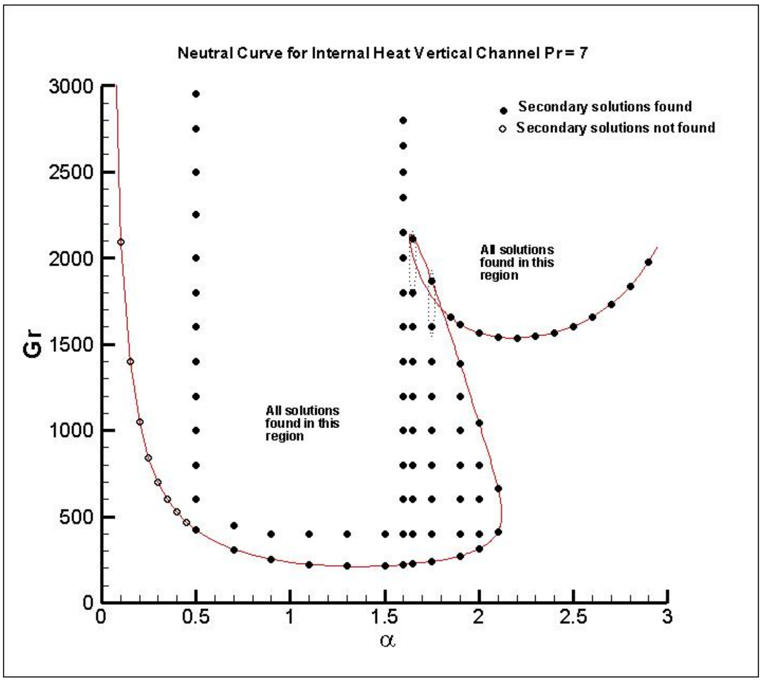


Figure 3.9: *Secondary Solutions.*

Supercritical TW 2D solutions for ϕ were obtained for a variety of parameters. Figure 3.9 shows all the secondary solutions found. For $\alpha < 0.5$ secondary solutions could not be found for this mode. Secondary solutions for $0.5 < \alpha < 1.65$ bifurcate from the neutral curve and the strength of the solution continues to rise as the strength of the internal heat source rises, figure 3.11. It is interesting to note that for $1.65 < \alpha < 1.8$ the solutions bifurcate from the bottom of the neutral curve (Gr < 250) and from the top of the island vertically above for the same wavenumber. This is better visualised in Figure 3.10 where it can be seen that the secondary solution is closed. This indicates that the strength of the secondary solution begins to increase as the strength of the internal heat source increases, however, as the strength of the internal heat source continues to rise the strength of the secondary solution diminishes to zero. Figure 3.9 also shows that for $1.8 < \alpha < 2.1$ and $1750 < Gr < 240$ there are again, secondary solutions that are closed and bifurcate for the same α. Secondary solutions where $\alpha > 1.8$ and $Gr > 1750$ are not closed and do not rejoin the neutral curve and behave as for the solutions where $0.5 < \alpha < 1.65$, that is the strength of the solution continues as the heat strength increases. It can be seen that the profiles of the secondary solutions close to the LHS of the island in figure 3.11(b) (put in here 2D solutions curves), which shows how the solutions behave around the island. It is interesting to note that the solution for $\alpha = 1.6$, just to the left of the island has itself a loop included in the solution. As can be see in figure 3.11(b) the secondary

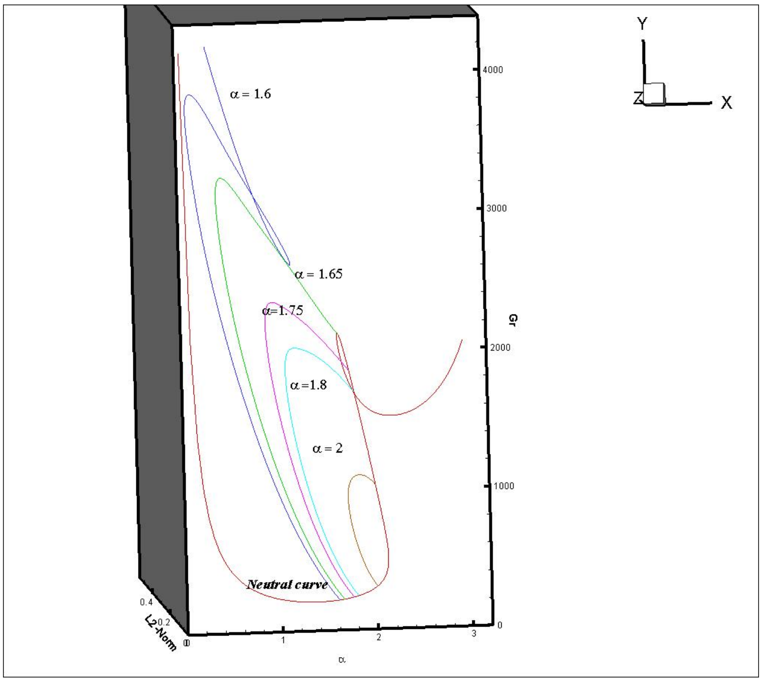


Figure 3.10: *Secondary Solutions in 3D.*

solutions for $\alpha \leq 1.65$, start and finish at the bottom and top of the neutral curve. It is now necessary to investigate whether the solutions from the far RHS of the neutral curve continue across to the left and into the bottom of the island. This is, in fact the case, we are able to trace secondary solutions from the far RHS with large wavenumbers across into the island. For these solutions the L_2^{norm} values continue to rise steadily in the same manner as the solutions where $\alpha > 1.8, Gr > 1500$. Figure 3.12 shows these secondary solutions which bifurcate from the RHS of the neutral curve and continue to bifurcate from the bottom of the island as α is decreased. There exists a situation where

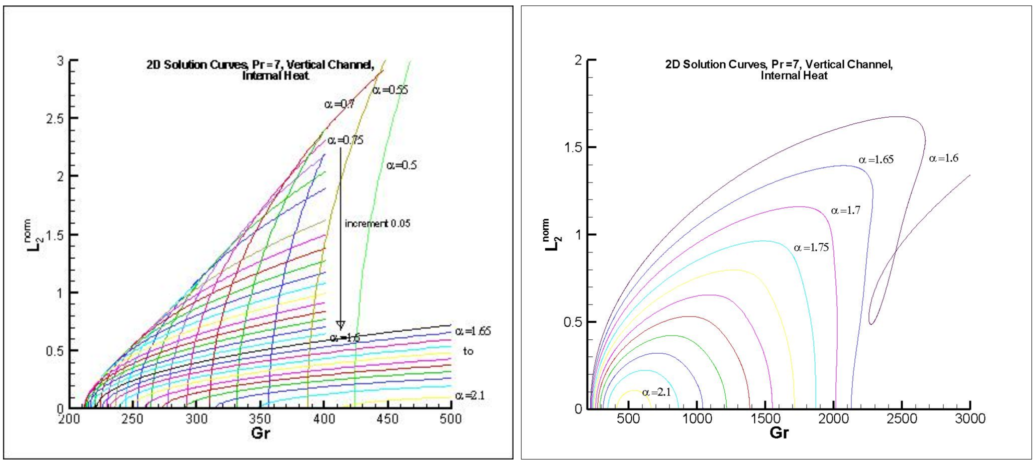


Figure 3.11: *Secondary Steady-State Solutions (a)-(b)*

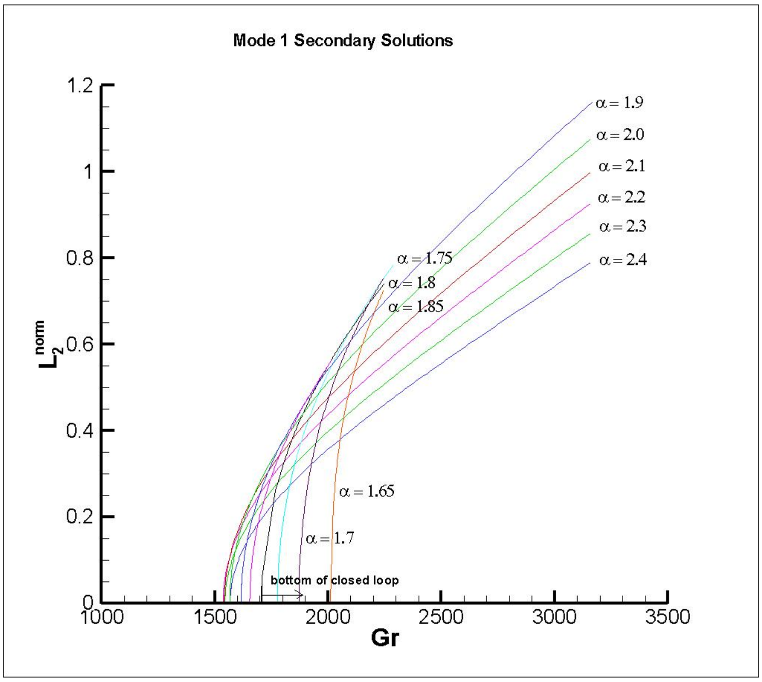


Figure 3.12: *Secondary Solutions for* α > *1.7.*

two secondary solutions overlap inside the island. This leads on to investigation into the nature of the interaction between the secondary solutions within this region (we may well have mixed mode solutions here), as well as the behaviour of the secondary solutions as they approach from either side of this closed region. Beginning by investigating the secondary solutions as α increases from 1.6 and seeing how the solutions behave as they approach the island. Figure 3.13 is a close-up of the solutions as they approach the LHS of the island. For $\alpha = 1.63$ the secondary solution itself loops (as seen in figure 2.8 for $\alpha = 1.6$) and does not rejoin the neutral curve at any point (perhaps another 2D solution mixed mode is bifurcating from around here). Incrementing α by very small amounts it can be seen that as α approaches the very LHS of the neutral curve a transition from the looping and diverging away from the island to a point where the secondary solution joins the island at $\alpha = 1.63175$. Continuing to increase α the secondary solution moves around the LHS of the island upwards towards the top of the island boundary where $\alpha = 1.65$ the secondary solution does join the top of the island. At this point no mixed mode is evident. But there is an overlap of the secondary solutions found at these points, a 3D diagram of the overlap illustrates this clearly, see figure 3.14. A 2D plot of Grashof versus L_2^{norm} for $\alpha = 1.75$ shows clearly how the secondary solutions overlap within the enclosed region of the island, see figure 3.15. It is necessary to further consider the non-ordered eigenvalues at points close to the neutral curve as indicated on figure 3.17 in order to ascertain the

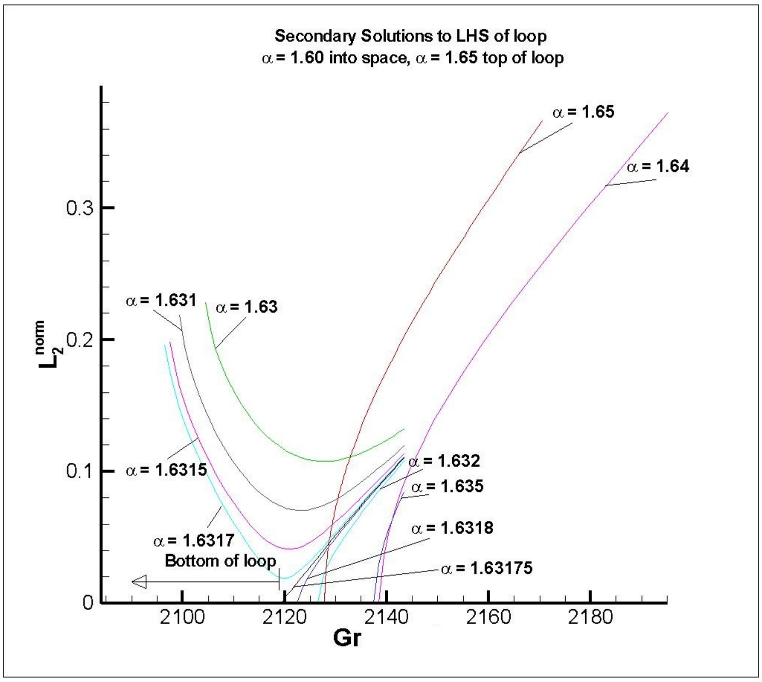


Figure 3.13: *Secondary Solutions to LHS of Loop.*

matrix position of the most dangerous eigenvalues to determine whether indeed there exists a uniquely different secondary solutions bifurcating from points either side of the loop. The results are shown in table 3.3.

Table 3.3: **Nonlinear Stability Analysis**

Ref.	α	Grashof	Matrix Position	Eigenvalue	Eigenvalue
(i)	1.90	1385	110	-0.591111378764E-01	0.293127646276E+02
(ii)	1.83	1616	110	-0.145627782627E+00	0.330157641622E+02
(iii)	1.83	1673	114	0.271460352413E+00	0.209514482847E+02
(iv)	2.03	1556	114	0.404905612569E+01	0.171967682529E+02

As can be seen from table 3.3 the solutions originating from the RHS of the island are different from those originating from the LHS (below) of the island (matrix positions are 110 and 114). In fact as one approaches the loop intersection point the real parts of the eigenvalues converge. After inspecting and comparing the non-zero coefficients for the secondary solutions (keeping the same α and Gr) it is seen that the coefficients are exactly the same but with opposite signs (+/-) and so these solutions are invariant (the same single solution) which again complies with the linear eigenvalues at that value of α, i.e. a single bifurcating secondary solution. However, inside the island region there are

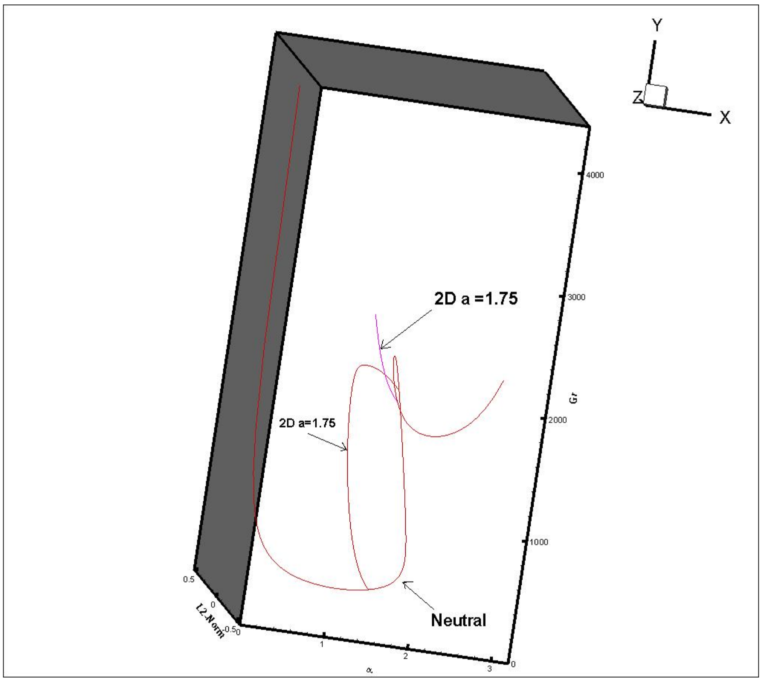


Figure 3.14: *Secondary Solutions for* α = *1.75.*

more secondary solutions bifurcating from the lower branch of the loop and continuing upwards with the same L_2^{norm} values as those secondary solutions which have been traced from the upper RHS of the neutral curve. The leading coefficients are different but some higher harmonic coefficients are exactly the same but with opposite signs (invariant). It would seem that there are indeed two solutions bifurcating from the lower section of the island and passing through the upper section of the island, which complies with the linear eigenvalues at those wavenumbers. So there are two separate linear solutions, one from the LHS and one from the RHS which intersect at $\alpha = 1.8$, cross over to produce the island which connects at $\alpha = 1.63$. It is also found that there are the correct number of bifurcating secondary solutions within the closed loop.

In figure 3.18 I have called the solutions bifurcating from the LHS mode 1 solutions and those from the RHS mode 2 solutions.

There is now a need to investigate the nature of the steady solutions by visualising the temperature, stream functions and total flow at varying α and Gr. It is known that for Pr=0 [27] the even modes are the most dangerous as shown in figure 3.16, where the stream flow is included at $Gr_c = 1721, \alpha_c = 2.05$ to illustrate the benchmark testing against the results of [27]. Note that [27] only did linear analysis, the non-linear diagrams were produced from the eigenvalues they calculated just above the neutral curve.

As the pertubation wave travels through the fluid we would expect transverse rolls

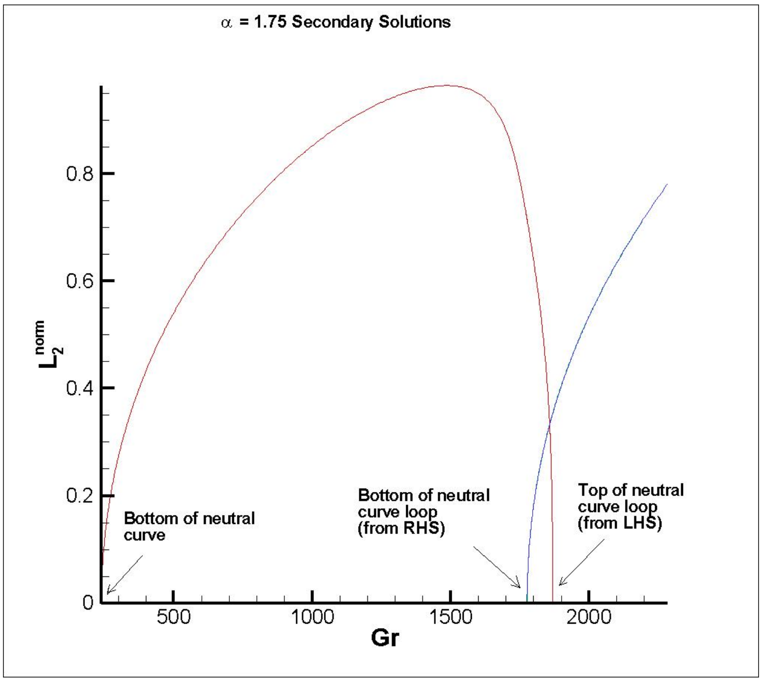


Figure 3.15: *Secondary Solutions for* α = *1.75.*

to form as illustrated in figure3.19. The fluid should be isotropic and have the same rolls throughout with no preference to roll direction. Here we are not forcing a break in isotropy by applying an odd basic flow profile as in [32]. However, the rolls will become distorted as we increase Grashof as will be shown. By looking at the mode 1 solution at $\alpha = 1.75$ and low Gr, very close to the bifurcating point on the neutral curve, we see ordered transverse vortices forming aligned along the vertical axis figure 3.21(a) and the initiation of deformation of the stream function figure 3.21(b). Figure 3.21(c) shows the total flow. Continuing to increase the internal heating for mode 1, figure 3.22 shows how the steady state solutions develop just below the island in the neutral curve, we clearly see deformations of the temperature profile 3.22(a) caused by the presence of more than one simple mode. In 3.22(b) we see the onset of well formed vortices offset about the vertical axis, which is in line with the findings of [27], see figure 3.16. In [50] we see that the experimental results, although with one boundary having a constant temperature gradient, we still see the same vortex structures. Further increasing Gr it is found that the solution is lost at the top of the island in the neutral curve as outlined previously. This effect is seen in figure 3.23(b), as well as seeing that the temperature profile reverts back to a more ordered form figure 3.23(a) as does the total flow profile in figure 3.23(b). Figure 3.24(a) shows the total mean flow and the total mean temperature in figure 3.24(b) for various Grashof and fixed wavenumber $\alpha = 1.75$ across the channel.

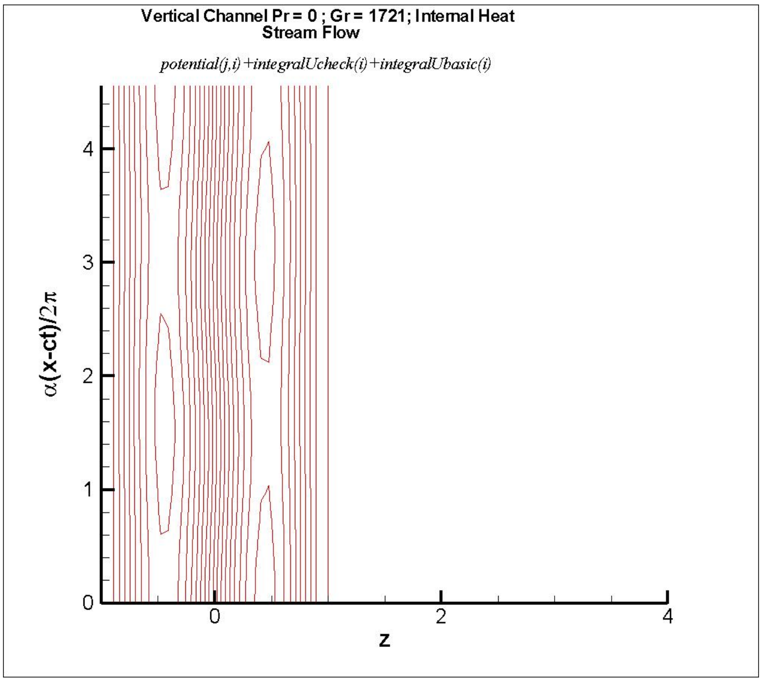


Figure 3.16: *Pr = 0 Stream Flow.*

Moving on to investigate the nature of the secondary steady state solutions that are traced from the RHS of the neutral curve (mode 2) and continue to the left bifurcating from the bottom of the island in the neutral curve. A value of Grashof just above the bifurcation point is taken. From previously mentioned observations it follows to now focus on the nature of the steady mode 2 solutions and compare the two solutions that bifurcate at the same points on the neutral curve. Although the strengths of the stream function steady state solutions perfectly match and follow each other it is noted that many of the coefficients in the solutions were indeed different. Figure 3.25(a) and (b) show clearly how the solutions differ. The temperature profiles show an inversion of the temperature rolls , whereas the stream flows remain identical 3.26(c) and (d). Figure 3.27(e) and (f) illustrate that the total mean temperatures for both mode 2 solutions are identical also. Temperature inversion is a physical phenomenon when heating fluids from below as discussed in the beginning of this study. As a point of reference we now focus on the solutions to the left of the closed loop in the neutral curve by starting with solutions at $Gr_{crit} = 211.323, \alpha_{crit} = 1.37$. Figure 3.28 shows the profiles as we increase Grashof. It was noted that deformation of the temperature rolls begins at low Grashof, as well as the formation of well defined vortices in the stream fluctuations. It was seen that the characteristic meandering effect of the fluctuating stream function present in the total flow similar to that seen in [39]. Ideally the mean flow (or perturbed flow) should be flat as it

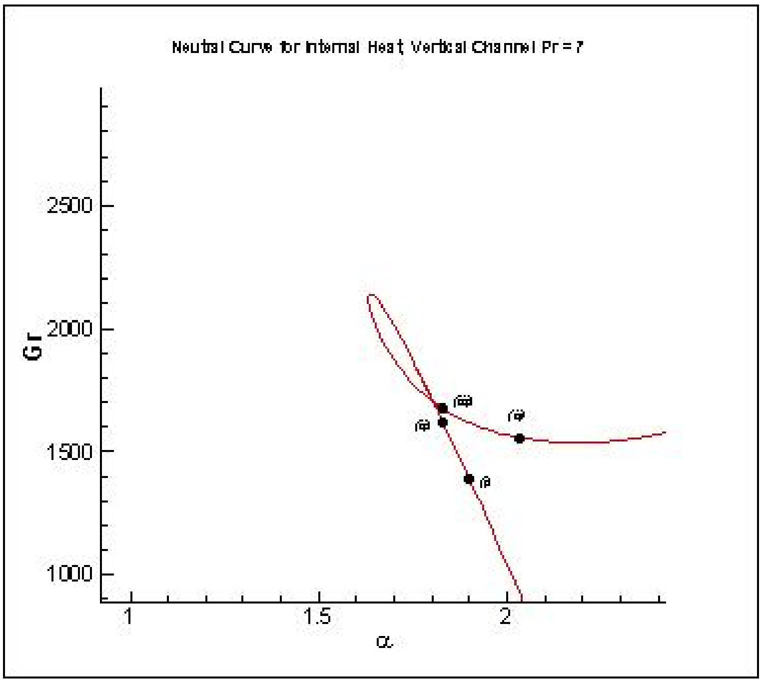


Figure 3.17: *2D Bifurcations on Linear Curve.*

is an average flow, however we do see a slight bend and seems to meander or fluctuate, see figure 3.20.

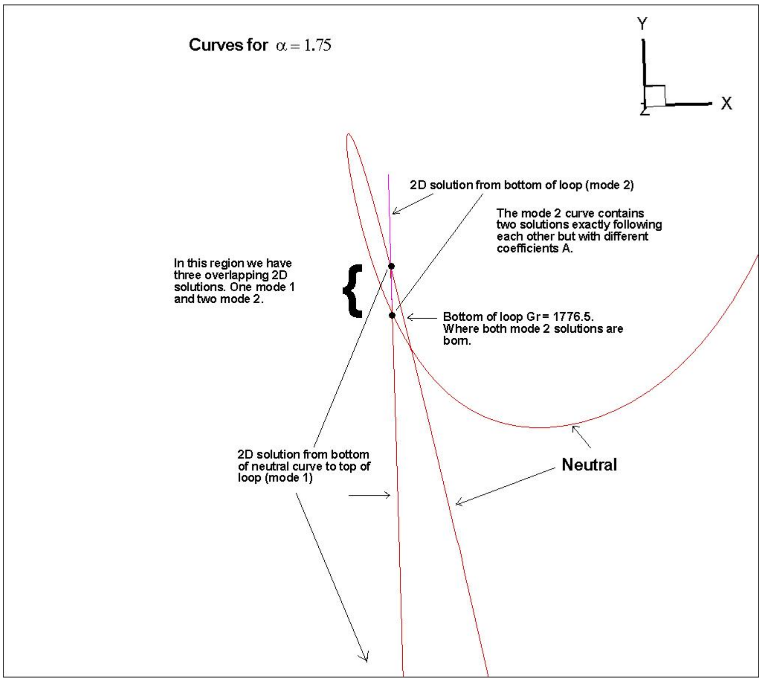


Figure 3.18: *Secondary Solutions* α = *1.75.*

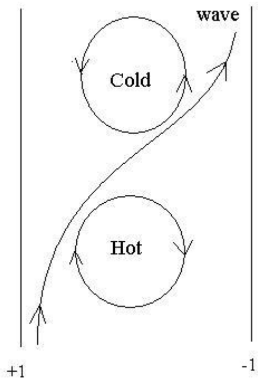


Figure 3.19: *Evolution of Convection Rolls.*

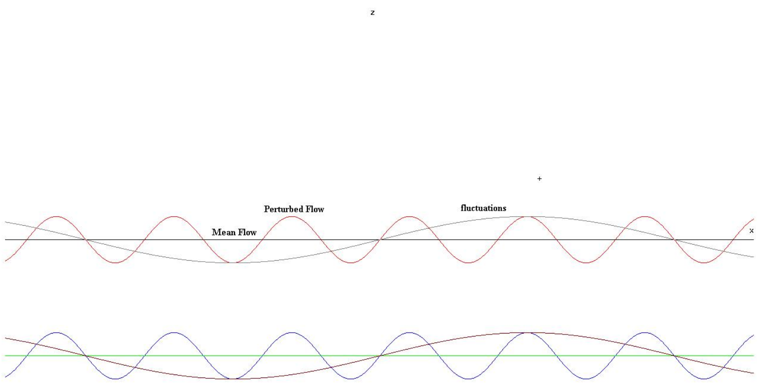


Figure 3.20: *Effect of Constant Terms on Mean Flow.*

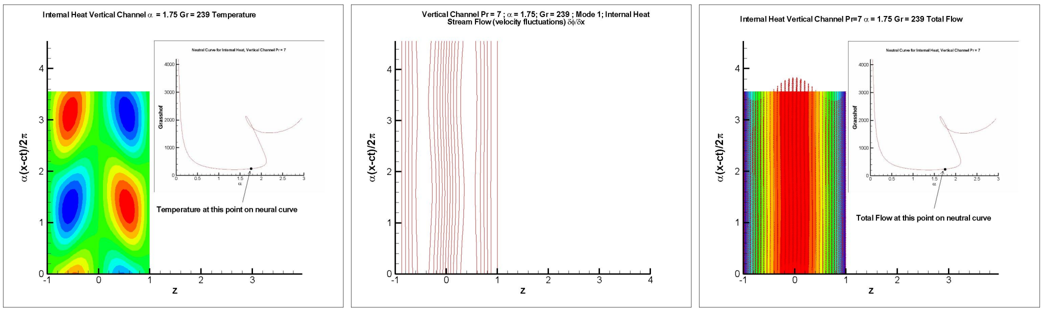


Figure 3.21: *a = 1.75 Gr = 239 Pure Mode Solutions (a)-(b)-(c)*

This effect is caused by the total flow comprising of the mean temperature and the perturbations, which are distorted by the nonlinear advection terms. This is a physical effect seen over the bulk flow similar to the meander of a river. These effects are further amplified as we increase Grashof further as the temperature perturbations have a harmonic and hence the meander. Figure 3.31 shows the associated total mean flow and temperatures. Focusing now on the nature of the steady state 2D solutions as we approach the island increasing α from the critical value. We focus on the solutions for $\alpha = 1.60$ in figures 3.32 - 3.35. At the bifurcation point there are well defined temperature profiles and the onset of vortices in the stream flow figures 3.32(a) - (c). Increasing Grashof and very quickly we begin to see a more pronounced deformation of the temperature and stream profiles as well as a very pronounced meandering structure in the total flow figure 3.33(d) to (f). These characteristics are maintained as we continue to increase Gr in figures 3.32 - 3.35, however, as on entering the island in the steady state equilibrium solution curve (figure 3.11) we notice that the temperature and stream profiles begin to become more regular again and the meandering contribution of the velocity fluctuations in the total flow become less periodic. Figure 3.37 illustrates the total mean flow and temperature for the ranges of Grashof used in figures 3.32 - 3.36. For completeness illustration of the nature of the steady solutions for mode 2 solutions at the RHS of the island in the neutral curve, figures 3.38 - 3.39. Notice that the temperature vortices are well defined and increasing Grashof again results in the formation of rolls in the stream function (e) as well as the characteristic meandering effect of the velocity fluctuations in the total flow (f).

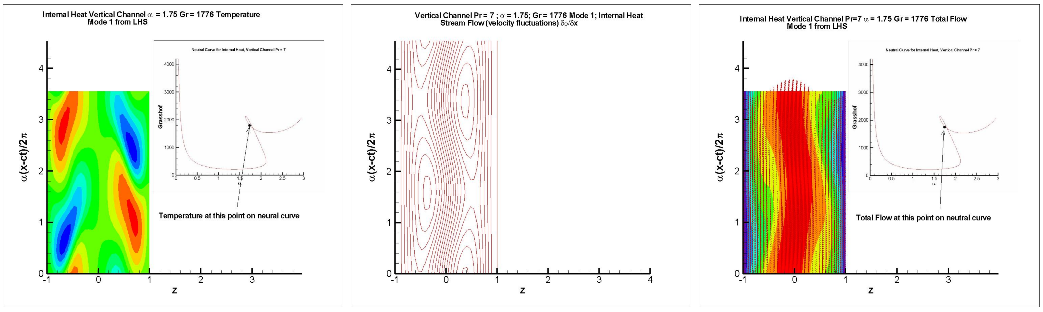


Figure 3.22: *a = 1.75 Gr = 1776 Pure Mode Solutions (a)-(b)-(c)*

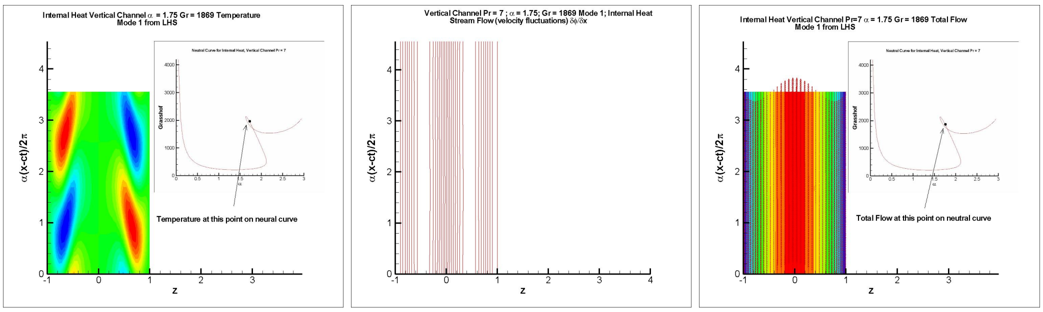


Figure 3.23: *a = 1.75 Gr = 1869 Pure Mode Solutions (a)-(b)-(c)*

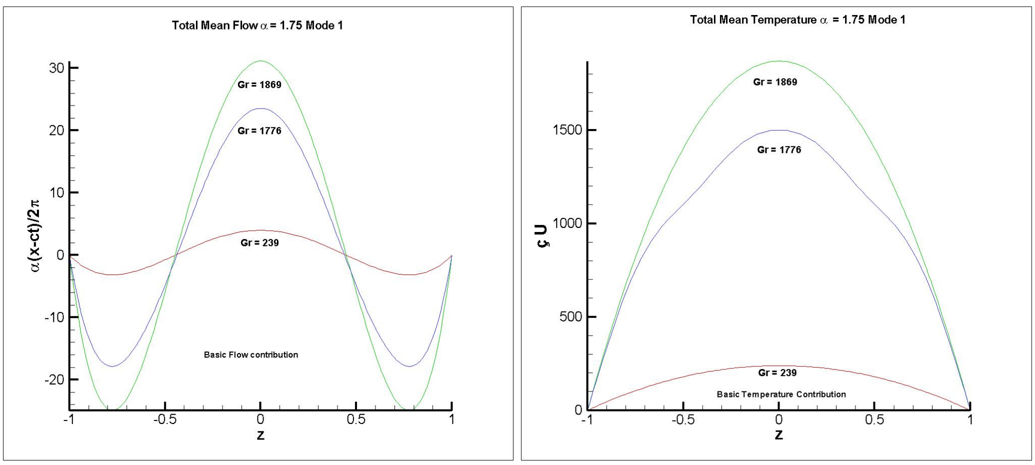


Figure 3.24: *a = 1.75 Mean Flow and Temperatures Mode 1 (a)-(b)*

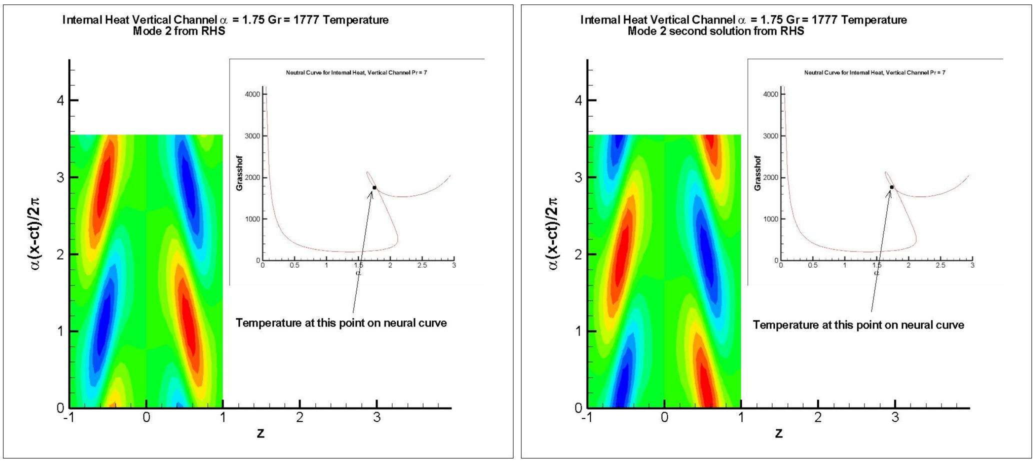


Figure 3.25: *a = 1.75 Temperatures Mode 2 First and Second Solutions (a)-(b)*

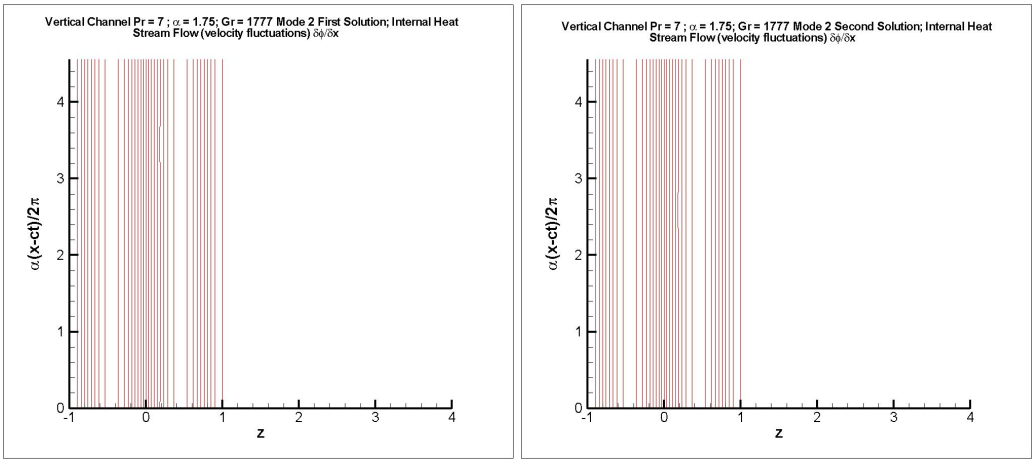


Figure 3.26: *a = 1.75 Steam Flows Mode 2 First and Second Solutions (c)-(d)*

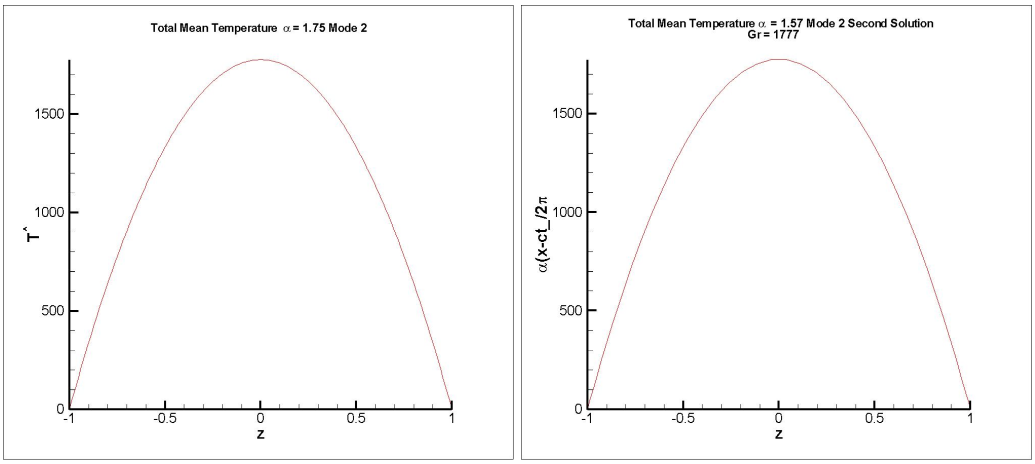


Figure 3.27: *a = 1.75 Mode 2 Mean Temperatures (e)-(f)*

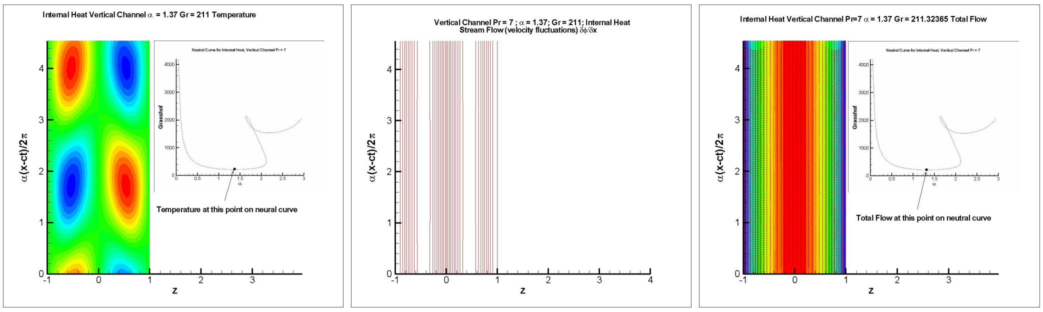


Figure 3.28: *a = 1.37 Gr = 211 Pure Mode Solutions (a)-(b)-(c)*

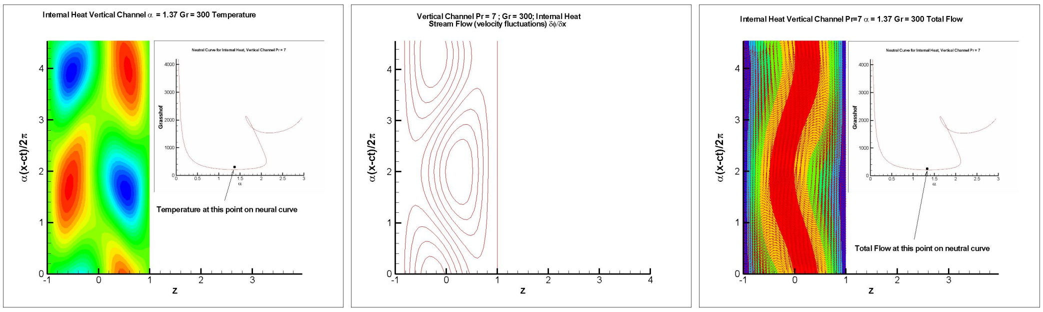


Figure 3.29: *a = 1.37 Gr = 300 Pure Mode Solutions (d)-(e)-(f)*

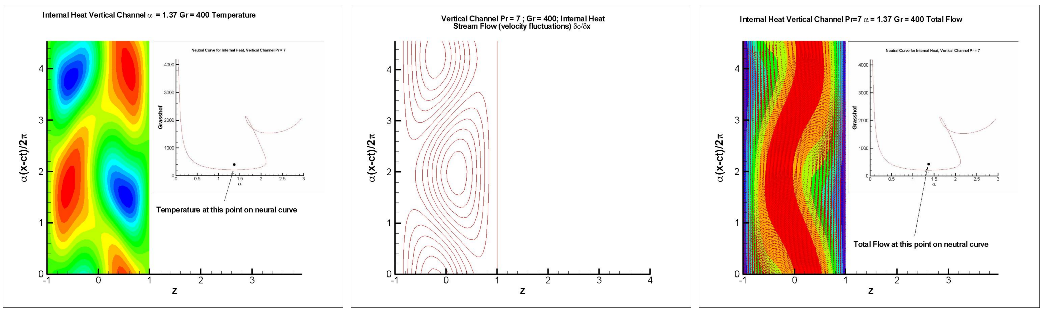


Figure 3.30: *a = 1.37 Gr = 400 Pure Mode Solutions (g)-(h)-(i)*

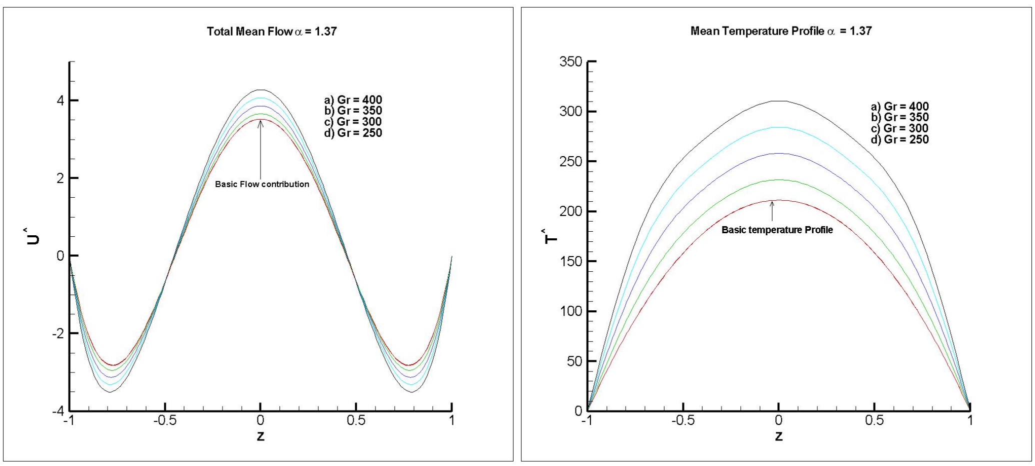


Figure 3.31: *a = 1.37 Mean Flow and Temperatures Pure Mode (a)-(b)*

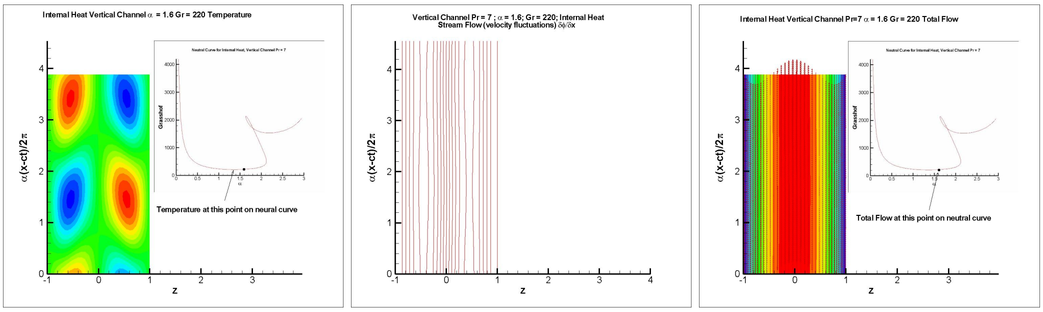


Figure 3.32: *a = 1.6 Gr = 220 Pure Mode Solutions (a)-(b)-(c)*

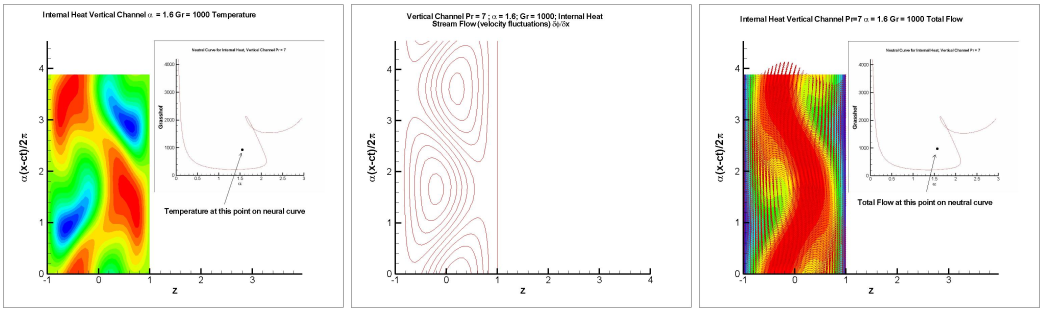


Figure 3.33: *a = 1.6 Gr = 1000 Pure Mode Solutions (d)-(e)-(f)*

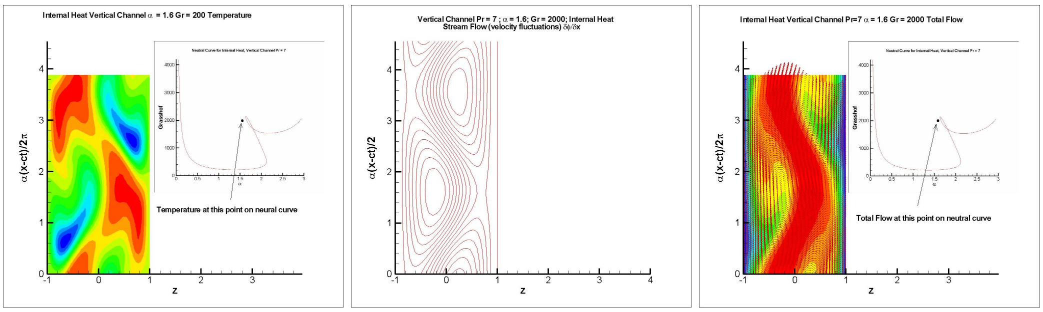


Figure 3.34: *a = 1.6 Gr = 2000 Pure Mode Solutions (g)-(h)-(i)*

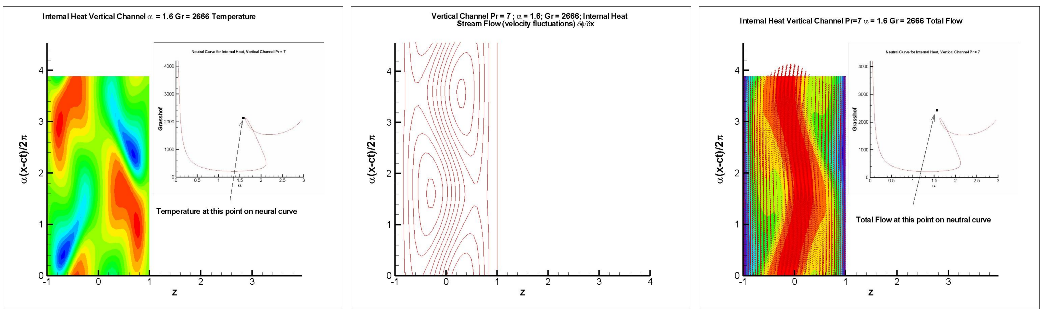


Figure 3.35: *a = 1.6 Gr = 2666 Pure Mode Solutions (j)-(k)-(l)*

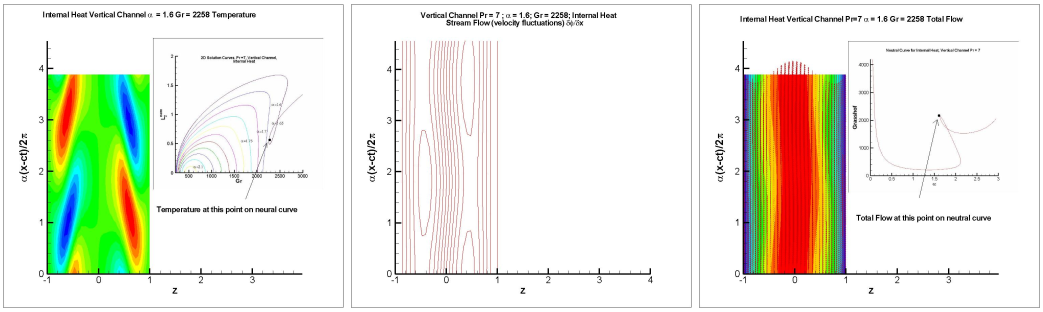


Figure 3.36: *a = 1.6 Gr = 2258 Pure Mode Solutions (m)-(n)-(o)*

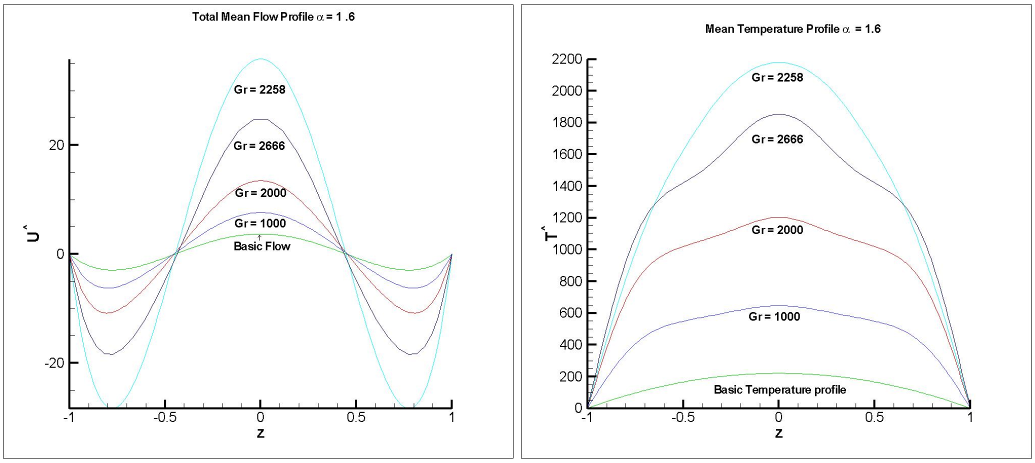


Figure 3.37: *a = 1.6 Mean Flow and Temperatures Pure Mode (a)-(b)*

The solutions found around the island are as expected because the second solution that bifurcates from the bottom of the loop is the one which comes from the second eigenvalue and will dominate the first solution that comes from the bottom of the neutral curve (first eigenvalue). The third solution is NOT a mixed mode solution as it is invariant as there are swapped ϕ_R and ϕ_I coefficients only, so it is not unique. This may explain the temperature inversion in the loop and hence no mixed mode.

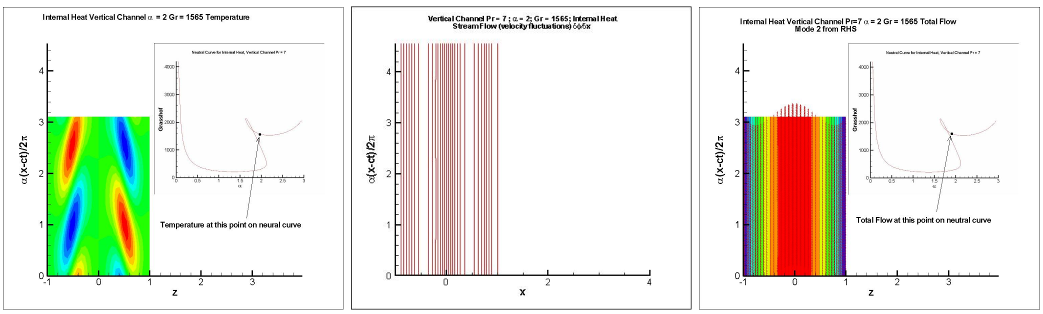


Figure 3.38: *a = 2 Gr = 1565 Pure Mode Solutions (a)-(b)-(c)*

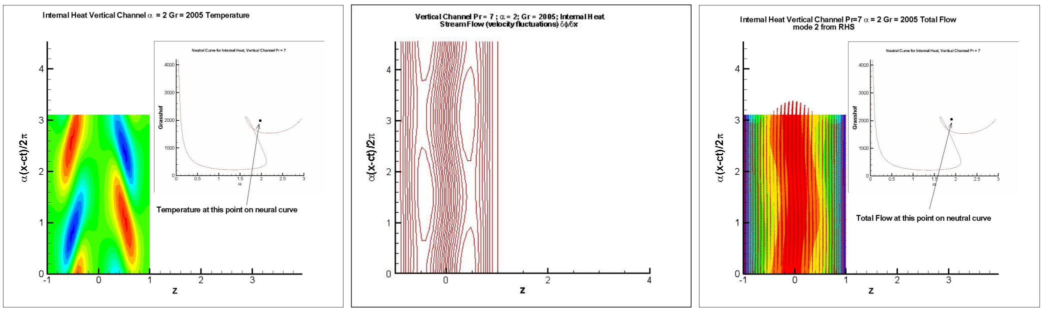


Figure 3.39: *a = 2 Gr = 2005 Pure Mode Solutions (d)-(e)-(f)*

3.2.2 Secondary Flow Stability Analysis

In wall bounded shear flows it has been shown that 2D finite amplitude waves are unstable to 3D infinitesimal disturbances [40] for Couette flow. This is also found to be the case for HHF [39] with Prandlt Number zero. For the stability analysis of the secondary equilibrium solutions the method used by Nagata and Generalis [39] is adopted, where Floquet theory is employed, to search for stability changes in the eigenvalues. Floquet theory is a linear stability (perturbation) analysis on every mode of the strongly nonlinear state. In this sense it is possible to mimic the linear (primary) stability analysis but on a nonlinear state comprising of many modes. In Floquet theory the perturbation is split into a finite amplitude contribution and an infinitesimally small secondary perturbation. As the disturbances augment to a finite amplitude the flow develops fully to some higher transitional state. The imaginary part of the eigenvalue is the most critical if the result is zero we have stationary waves and this means that we may have vortices forming in the fluid layer. Note that as per [48] Squire's Theorem applies to our problem, hence the most dangerous modes are 2D and periodic in the streamwise direction thus 2D-infinitesimal perturbations are applied to the secondary flow $\breve{U}\hat{\mathbf{i}} + \breve{\mathbf{u}}$ as

$$\tilde{\mathbf{u}} = \delta\tilde{\phi} + \varepsilon\tilde{\psi}, \tag{3.35}$$

where the tilde is complex. We use the following secondary linear stability theory to find the bifurcations that arise from the associated secondary equilibrium solutions:

$$\tilde{\phi} = \sum_{m=-\infty}^{\infty} \sum_{l=0}^{\infty} \exp\{im\alpha(x-ct)+id(x-ct)+iby+\sigma t\} \times (1-z^2)^2 \tilde{a}_{lm} T_l(z), \quad (3.36)$$

$$\tilde{\psi} = \sum_{m=-\infty}^{\infty} \sum_{l=0}^{\infty} \exp\{im\alpha(x-ct)+id(x-ct)+iby+\sigma t\} \times (1-z^2) \tilde{b}_{lm} T_l(z), \quad (3.37)$$

$$\tilde{\theta} = \sum_{m=-\infty}^{\infty} \sum_{l=0}^{\infty} \exp\{im\alpha(x-ct)+id(x-ct)+iby+\sigma t\} \times (1-z^2) \tilde{c}_{lm} T_l(z), \quad (3.38)$$

where d and b are Floquet parameters applied in the span and streamwise direction respectively. In some texts [16] these parameters are often referred to as Characteristic Values.

As per [39] $\sigma_r < 0$ is a stable solution and $\sigma_r = 0$ is a possible bifurcation of the secondary flow. We maintain the same boundary conditions applied in the earlier linear stability model. In order to derive the corresponding equations for the disturbance field $\{\tilde{\phi}, \tilde{\psi}\}$, we replace $\{\phi, \psi\}$ with $\{\phi + \tilde{\phi}, \psi + \tilde{\psi}\}$ in eqs.(2.13-2.14) and subtract the equations for the secondary solutions at the same time we ignore the nonlinear terms in the perturbations, arriving at:

$$\frac{\partial}{\partial t}\nabla^2\Delta_2\tilde{\phi} - \nabla^4\Delta_2\tilde{\phi} + \hat{U}\partial_x\nabla^2\Delta_2\tilde{\phi} = \partial_z^2\hat{U}\Delta_2\partial_x\tilde{\phi} + \partial x\partial z\tilde{\theta} + c\partial x\nabla^2\Delta_2\tilde{\phi} - \delta\cdot\left\{(\delta\tilde{\phi}+\varepsilon\tilde{\psi})\cdot\nabla(\delta\phi) + (\delta\phi)\cdot\nabla(\delta\tilde{\phi}+\varepsilon\tilde{\psi})\right\} \quad (3.39)$$

$$\frac{\partial}{\partial t}\Delta_2\tilde{\psi} - \nabla^2\Delta_2\tilde{\psi} = \partial_z\hat{U}\Delta_2\partial_y\tilde{\phi} - \hat{U}\partial_x\Delta_2\tilde{\psi} + c\partial x\Delta_2\tilde{\psi} + \partial y\tilde{\theta} - \varepsilon\cdot\left\{(\delta\tilde{\phi}+\varepsilon\tilde{\psi})\cdot\nabla(\delta\phi) + (\delta\phi)\cdot\nabla(\delta\tilde{\phi}+\varepsilon\tilde{\psi})\right\} \quad (3.40)$$

$$\frac{\partial}{\partial t}\tilde{\theta} = -2Gr(r\cdot\hat{k})\Delta_2\tilde{\theta} + \Delta_2\tilde{\phi}\partial_z\check{T} - \hat{U}\partial_x\tilde{\theta} + c\partial x\tilde{\theta} + Pr^{-1}\nabla^2\tilde{\theta} + (\delta\tilde{\phi}+\varepsilon\tilde{\psi})\cdot\nabla\theta + (\delta\phi+\varepsilon\psi)\cdot\nabla\tilde{\theta} \quad (3.41)$$

As per [39] the resulting generalised eigenvalue problem is solved numerically. The same truncation levels were used as for the previous section. The real part of σ_{1r} of the leading eigenvalue σ_1 defines the damping rate of amplification of the perturbation. The stability boundary is found by applying the condition $\sigma_{1r} = 0$. The imaginary part defines the phase velocity of the propagating disturbances in the flow ($\sigma_{1r} = -\alpha Re[c]$). Finally we note that in all cases examined the symmetry relations $\sigma_{1r}(b,d) = \sigma_{1r}(b,\pm d)$ (b fixed), $\sigma_{1r}(b,d) = \sigma_{1r}(\pm b,d)$ (d fixed), were always observed.

3.2.3 Eckhaus Stability

Following both [17] and [16] (p.417-18) we calculate the sideband stability also known as the Eckhaus stability.

Comparing figure 3.40 with the diagram on p.417 of [16] and taking a value of $Gr =$ 214 it is possible to verify that the Eckhaus boundary shown is in accordance with the theory laid out in [16], that is;

$$\alpha_c - \alpha_3(Gr) \approx (\alpha_c - \alpha_1(Gr))/\sqrt{3} \rightarrow 1.37 - 1.3 \approx (1.37 - 1.25)/\sqrt{3}$$

and

$$\alpha_4(Gr) - \alpha_c \approx (\alpha_2(Gr) - \alpha_c)/\sqrt{3} \rightarrow 1.44 - 1.37 \approx (1.49 - 1.37)/\sqrt{3}$$

Numerically we have indeed met the Eckhaus criterion that $d \ll \alpha$ in the case above,i.e. $0.07 \ll \alpha$ in both cases.

Figure 3.40 shows the Eckhaus ($b = 0$) stability boundary in relation to the Neutral curve for $0.53 \leq \alpha \leq 1.61$. Several values of the Grashof number were studied and the maximum real part of the leading eigenvalue was found. The value of Gr, which determines the boundary of the curve , was calculated by interpolation. From the curve we see that the sideband instability is extremely close to the neutral curve.

As can be seen in figure 3.41 the highest growth rate of σ_{1r} occurs around $b = 0.5$ and hence we use this value to calculate the Hopf bifurcation.

3.2.4 Hopf Bifurcation

A hopf bifurcation will occur at a point where $\sigma_r = 0$ and $\sigma_i \neq 0$. The resulting limit cycle may or may not be phased-locked with the associated 2D state, if the limit cycle is not phase locked to the 2D state it is not possible to proceed to analyse the 3D state as the model for this study does not have time evolution capability. Instead, if the imaginary part of the output eigenvalue is equal the phase velocity the bifurcation is phase locked. For time dependent stability analysis without equilibrium points direct spectral numerical

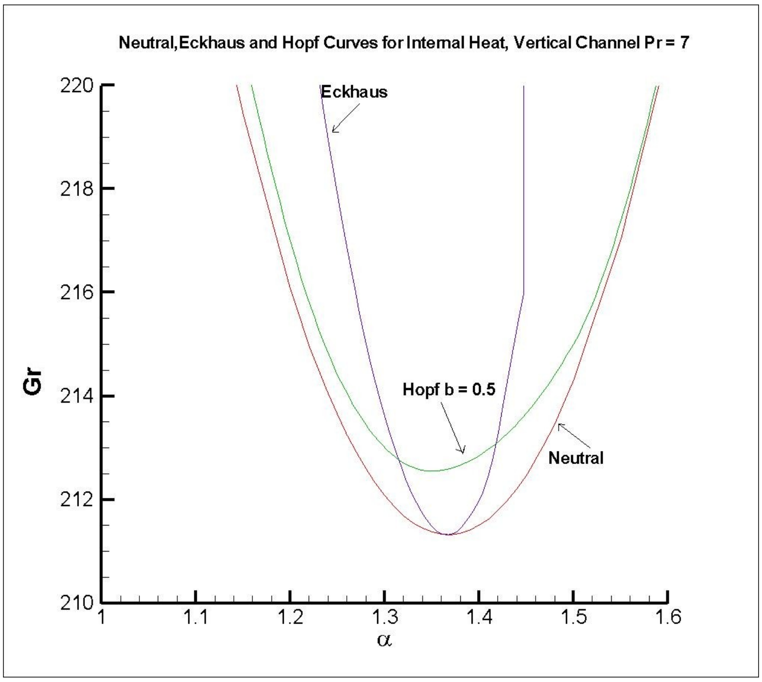


Figure 3.40: *Stability boundaries of secondary TWs.*

calculation of the Navier Stokes equations would have to be used. However, as the initial conditions are never exactly known this leads to problems when performing time evolving calculations. We proceed to slowly reduce d (the Floquet Parameter) to zero to see if the limit cycle is phase locked to the phase velocity of the 2D state. If phase velocity in the 2D state is phase locked at $d = 0$ then it is the condition required proceed to find a 3D state. In order to find the boundaries for a Hopf bifurcation it is necessary to look for the values of the Floquet parameters (characteristic values [29]) b and d at the highest growth rate of σ_r, then look for a change of stability for those parameter values. With the use of a linear interpolation it is possible to ascertain the specific value of Grashof for the two distinct changes of stability. The stability software program allows for stability analysis on many contiguous secondary equilibrium solutions contained in the input file. This enables a rapid search for changing σ_r for many given values of Grashof, it was noticed that the values for the maximum σ_r are identical for any Grashof for a fixed α perturbation. Once the Hopf bifurcation is obtained it's phase velocity (σ_i) is compared at that point to the phase velocity of the 2D solution. If they are equal it is possible to proceed to investigate the tertiary state arising as the phase velocities are phase-locked. If these values are not equal investigation is possible by slowly reducing d whilst fixing b to see if the phase velocity becomes the same for the associated 2D state. Immediately it is seen that the 2D state is very unstable as the Hopf bifurcation curve is very close to

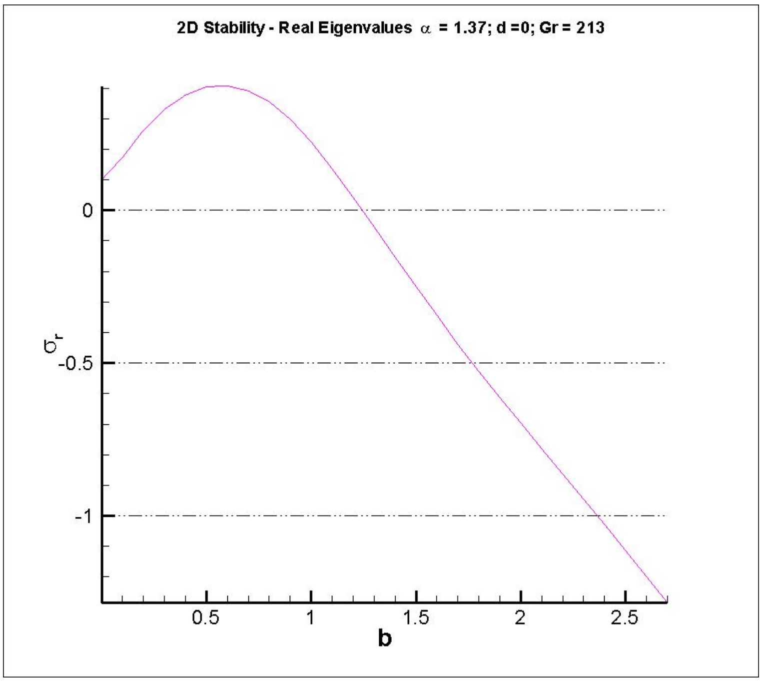


Figure 3.41: *2D Stability.*

the neutral curve. It was also found that in the output data for σ_r the value is decreasing as the Grashof increases for the 2D state (i.e. at $d = 0$), this is unexpected but this can be accounted for by looking at the plots of σ_r versus d where we notice the shape of the peaks and troughs in the curves as they pass through $\sigma_r = 0$. Because of the small stable region it is wise to proceed to look for a 3D (tertiary) state that bifurcates from a 2D state.

3.2.5 Resonant 1:3 Analysis

Having found that a 1:3 higher harmonic resonance exists in the linear results, the nature of the 1:3 resonant secondary equilibrium solutions require investigation as well as seeing whether these solutions can be found for $\alpha < 0.5$. It was found that well converged 1:3 resonant secondary solutions do exist. Note that for the 1:3 resonant secondary solutions the collocation points are not altered in the nonlinear code. Changing the collocation points makes no difference to the results of the secondary equilibrium solutions, which is what one would expect. Tables 3.4 and 3.5 compares the first few coefficients of the pure mode and 1:3 mode solutions. Incidently, it is noted that the initial 1:3 mode solution was found by taking a converged pure solution at $\alpha = 1.2$ with a value of Grashof just above the bifurcation point on the neutral curve and changing the value of $\alpha = 0.4$ (i.e. a third) in the output file and rerunning the program until a converged 1:3 solution was

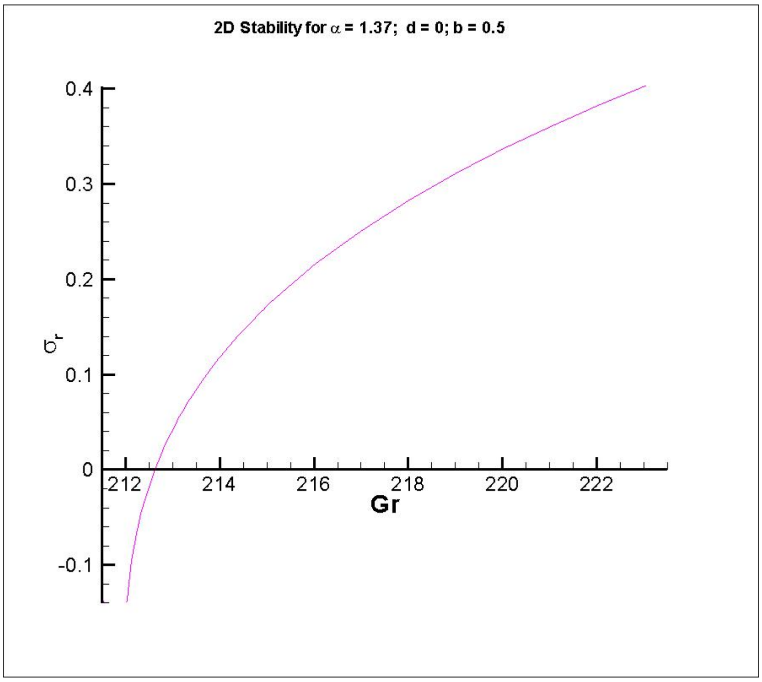


Figure 3.42: *2D Stability Gr.*

obtained. It was found that there exist specific points where the solution is able to "jump" (or burst) from the pure mode 2D solution to a 1:3 mode resonant solution directly but not visa versa, the output files where a jump was first found are shown in tables 3.4 and 3.5 for the $\alpha = 1.2$ fundamental to the $\alpha = 0.4$ resonant 1:3 mode.

Figure 3.47 shows how narrow the bandwidth is within which these bursts were possible for varying values of α in the context of the neutral stability curve. There is now a need to investigate whether there exist mixed mode solutions that would show us a less abrupt transition between the modes found. In order to do this further benchmarking is required. Firstly it is necessary to refer to the important work completed by Mizushima and Fujimura [36] where 1:3 resonances and mixed mode solutions where found in Rayleigh-Benard convection. In order to investgate further the 1:3 resonant solutions for homogeneously heated fluid reproduction of the work in [36] is necessary. Figure 3.44 shows the results for the neutral stability curve for Rayleigh-Benard convection which confers with the results of [36].

The results obtained for the weakly non-linear secondary solutions in [36] use $\omega_i = i\alpha\phi_1$ (the imaginary part of the fourier expansions) as a measure of the strength of the solution whereas we employ the L_2^{norm} as previously stated. We too find the bridge or mixed mode secondary solution that joins the pure to resonant mode found in [36], our visual representation is shown in figure 3.45. We clearly see the bridge between the 1:3

Table 3.4: **Fundamental** $\alpha = 1.2$ **Mode Output File for Gr = 223**

Chebyshev	Harmonic M	Harmonic N	a_{lmn}
0	1	0	0.7953887748949154E-01
0	2	0	-0.1166467388361531E-18
0	3	0	-0.1852912932184240E-04
0	4	0	-0.3896269096767016E-22
0	5	0	0.3452185936253233E-07
0	6	0	-0.5246338355593326E-23
0	7	0	0.3671510636526089E-09
0	8	0	-0.3050638699850794E-25
0	9	0	-0.9310976940711341E-12
0	10	0	0.3630870221220881E-27
1	1	0	0.3802045898695263E-17
1	2	0	-0.4708098708559432E-02
1	3	0	0.2435179707607905E-19
1	4	0	-0.1398334364158642E-05
1	5	0	-0.3698501294042346E-22
1	6	0	0.8419488952170249E-08
1	7	0	-0.1068865765849753E-23
1	8	0	0.2626134339053322E-10
1	9	0	0.2260324770725527E-26
1	10	0	-0.3043522127262350E-12

and pure mode as found in [36]. Incidentally, a 1:4 mode 2D solution was found in the course of this study but further investigation is not warranted here.

Using the replicated methods for reproducing the Rayleigh-Benard results as per [36] it is possible to proceed to search for any mixed mode solutions that bridge the pure and resonant modes for the HHF model considered in this work. After extensive investigations mixed mode solutions could not be found, even close to the intersection point of the 1:3 and pure neutral curves. Figure 3.46 shows the pure and 1:3 resonant secondary solutions for the homogeneously heated fluid model where there is no bridge, or mixed mode solution is evident.

Table 3.5: $\alpha = 0.4$ **Resonant 1:3 Mode Output File for Gr = 223**

Chebyshev	Harmonic M	Harmonic N	a_{lmn}
0	1	0	0.1309594746364094E-32
0	2	0	0.9380770742754546E-34
0	3	0	-0.1453681675131011E+00
0	4	0	-0.1131626816760070E-34
0	5	0	0.2620412006853043E-33
0	6	0	0.9881216609480334E-18
0	7	0	-0.7193437679856988E-34
0	8	0	-0.8482412631052800E-37
0	9	0	0.1756146758213647E-04
0	10	0	0.3288602257454599E-37
1	1	0	0.2919666903997230E-34
1	2	0	-0.2602685552042643E-32
1	3	0	-0.2504951072821455E-17
1	4	0	-0.1413567221694325E-32
1	5	0	-0.2343378749948379E-35
1	6	0	0.2267271864392766E-02
1	7	0	0.6105499758273396E-36
1	8	0	-0.6099346312829144E-34
1	9	0	-0.2155682894314998E-20
1	10	0	0.8181808023707753E-35

In fact it is possible to find further jump phenomena, where it is possible to jump directly from pure mode solutions to 1:3 mode resonant solutions. [36] does mention that transition between mode occurs abruptly for high Prandtl Number in Rayleigh-Benard convection.

Of particular interest is the the relationship between the strength of the secondary solution where the bursts can occur. Figure 3.48 shows that we see sudden troughs in curve at a point where the burst can occur. This is better seen in figure 3.49 where we superimpose figure 3.48 onto the curves showing the pure and 1:3 mode secondary solutions.

Comparing the stream flow and the temperature profiles of the pure mode solution at

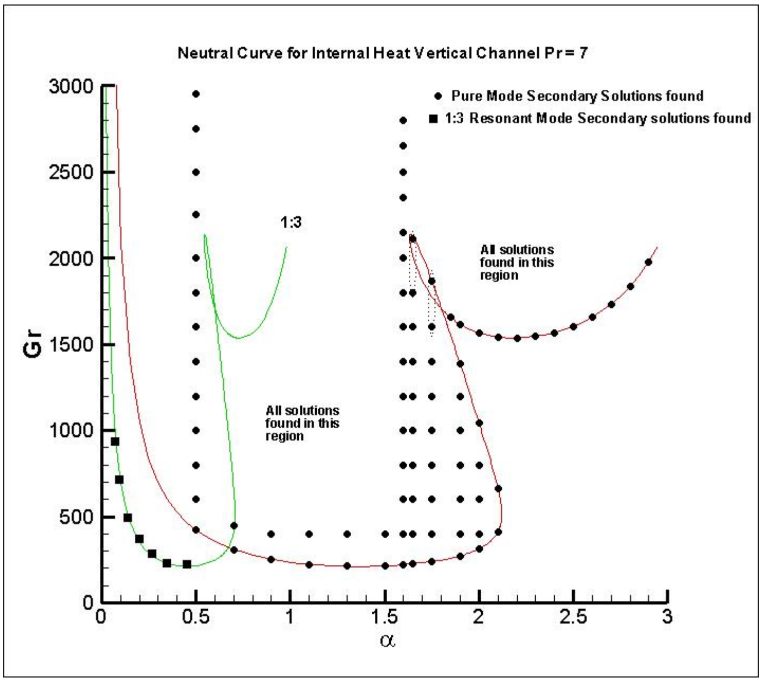


Figure 3.43: *Pure and Resonant 1: 3 Secondary Solutions.*

$\alpha = 1.2$ and the 1:3 resonant solution at $\alpha = 0.4$ for the same Grashof where a jump, or burst is possible it can be seen that the stream flows and temperatures are congruent but with a small phase shift, hence the ability to jump from one solution to the other directly, which is illustrated in figures 3.50 and 3.51. If the shift becomes too large the ability to jump between said solutions is lost.

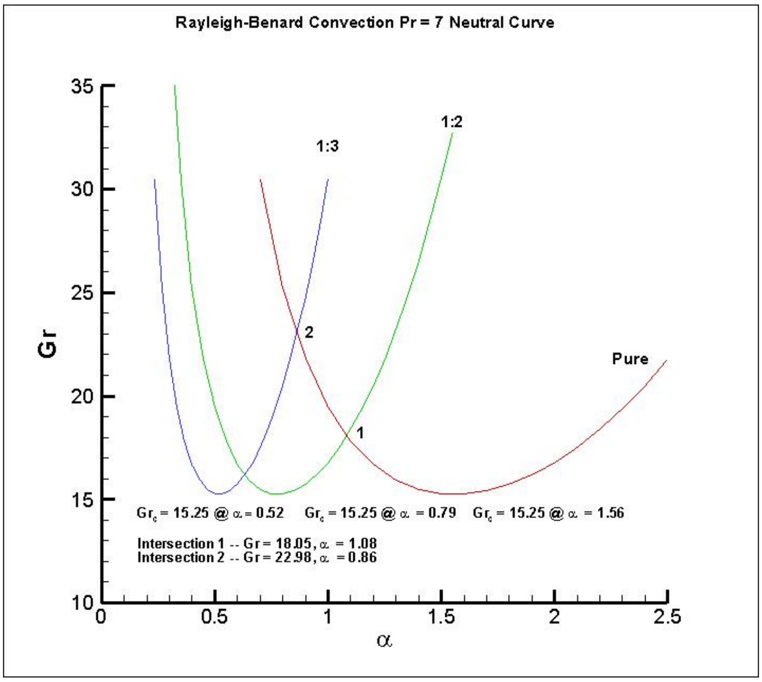


Figure 3.44: *Rayleigh-Benard Pure and Resonant Linear Stability*

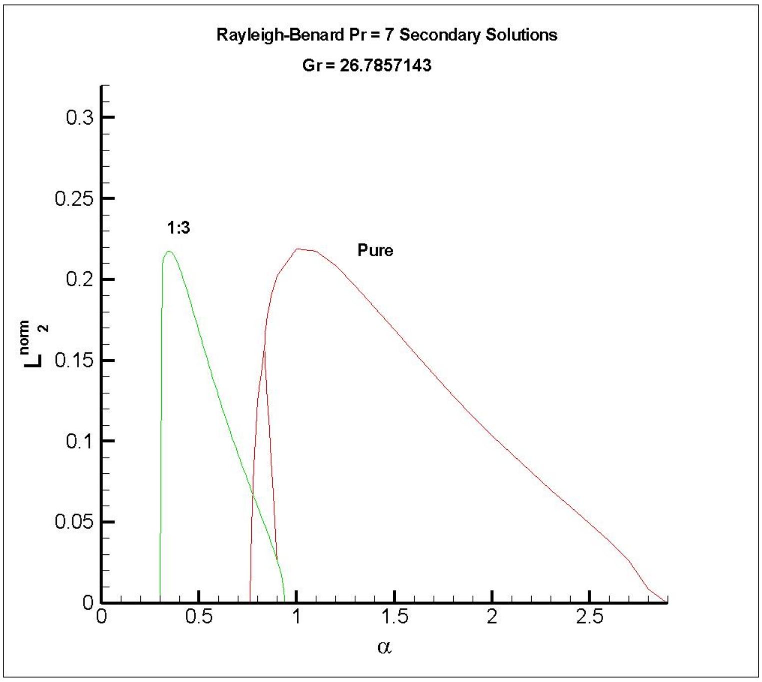


Figure 3.45: *Rayleigh-Benard Pure and Resonant 1: 3 Secondary Solutions.*

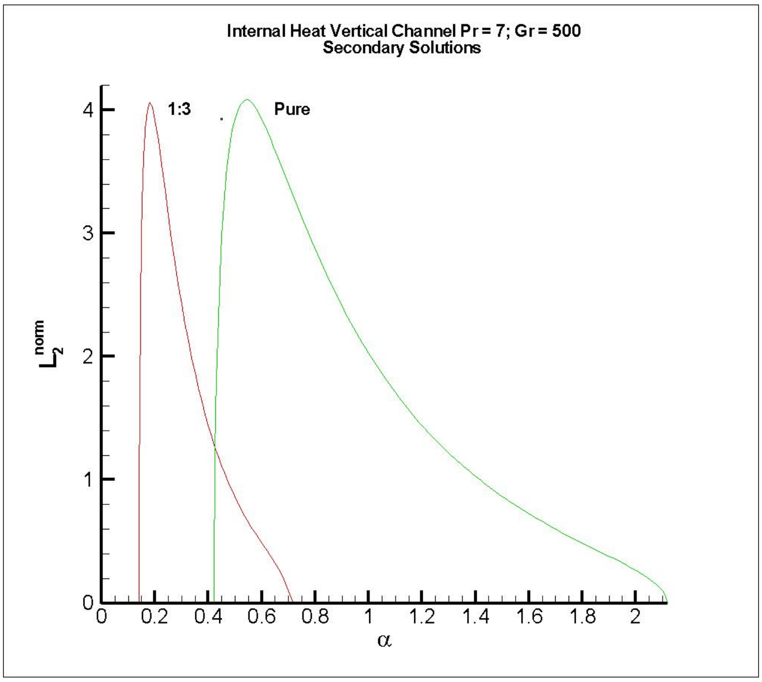


Figure 3.46: *Pure and Resonant 1: 3 Secondary Solutions.*

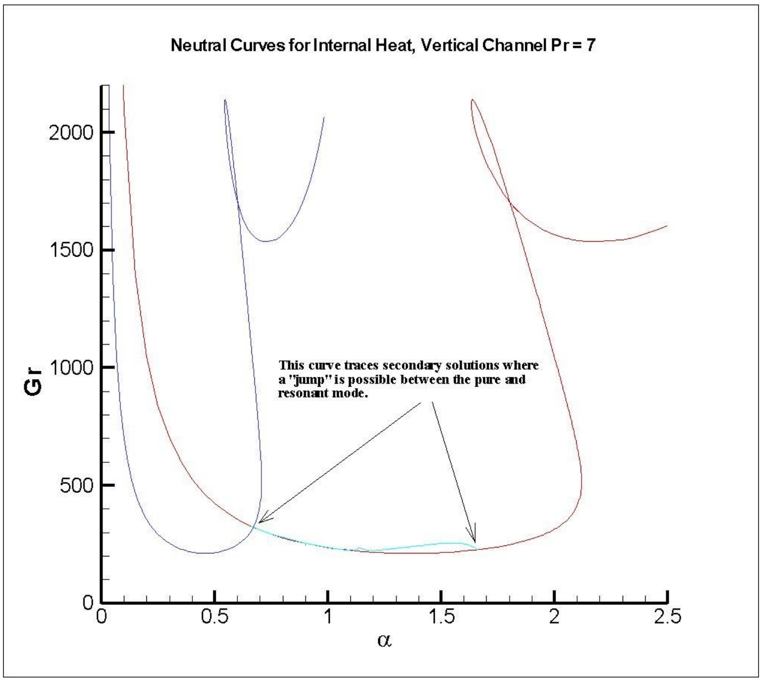


Figure 3.47: *Pure and Resonant 1: 3 Secondary Solutions.*

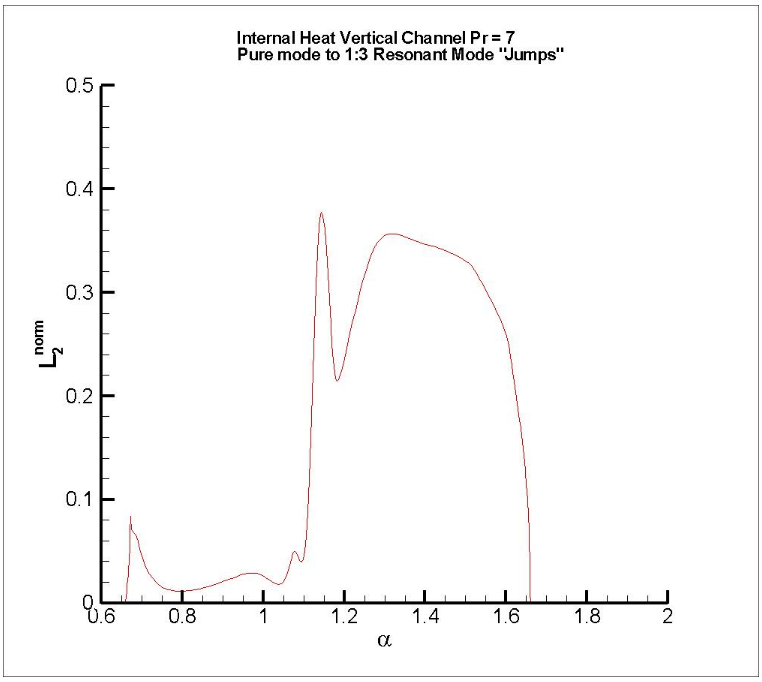


Figure 3.48: *Jump Profile.*

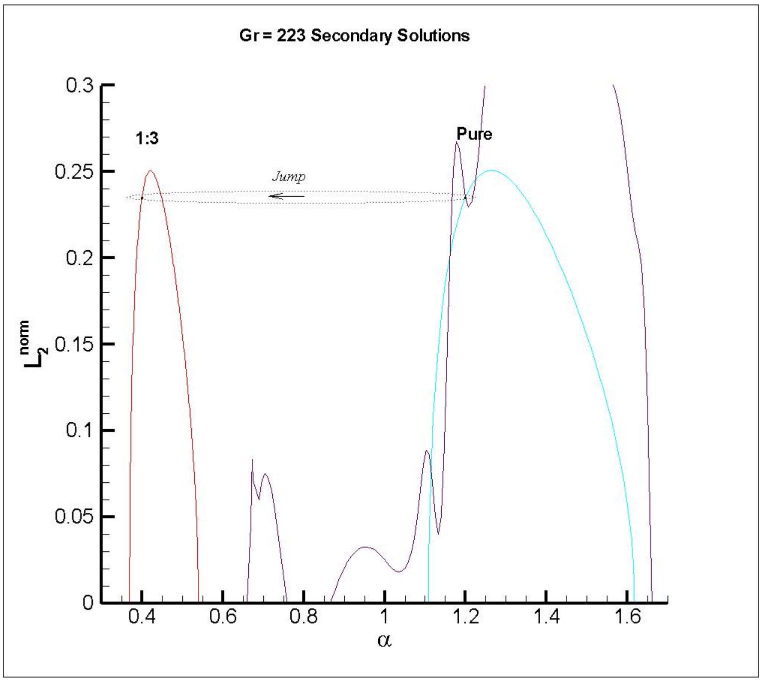


Figure 3.49: *Pure and Resonant 1: 3 Secondary Solutions.*

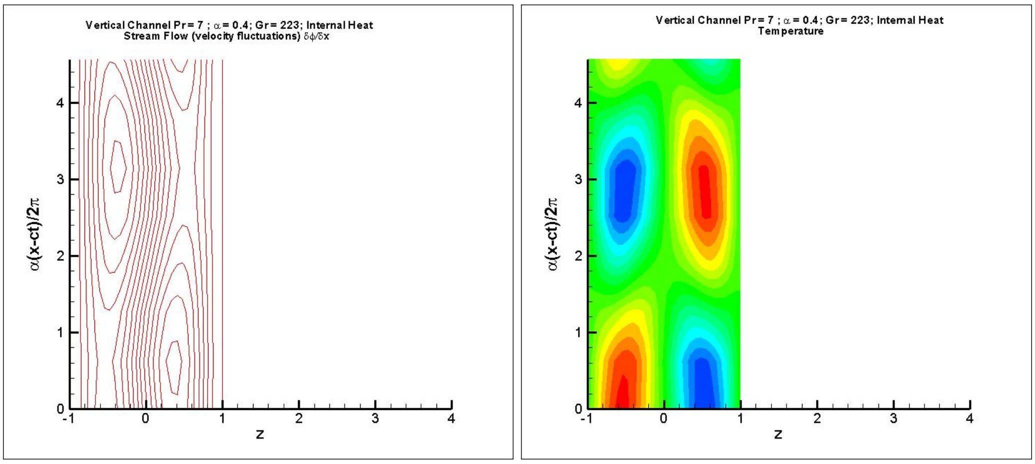


Figure 3.50: *a = 0.4 1:3 Stream Flow and Temperature*

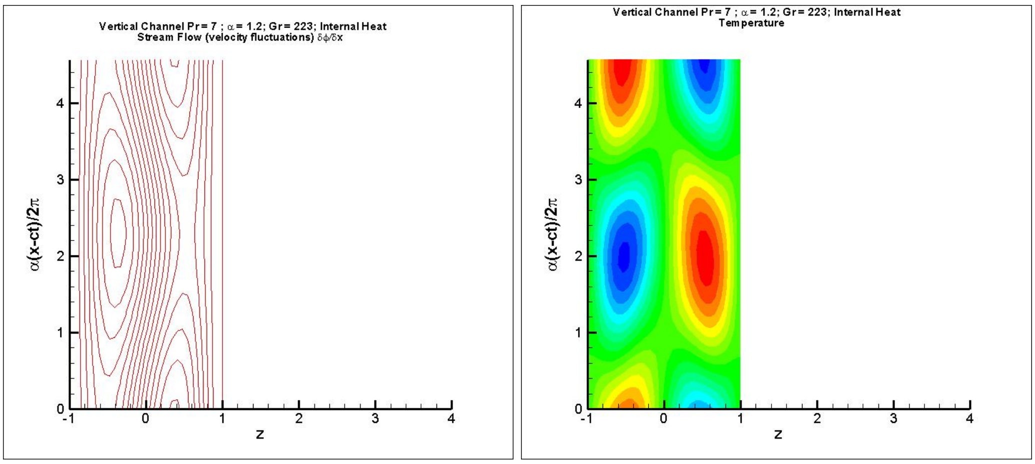


Figure 3.51: *a = 1.2 Stream Flow and Temperature Pure Mode*

3.2.6 Tertiary Equilibrium States

For the 3D calculations we need to decompose the non-linear terms of the N.S. equations in 3.18 as before in section 3.2.1 but this time keeping all the components $\phi-\phi$, $\phi-\psi$ and $\psi-\psi$ and retaining terms in y also, which are; In addition we need to decompose the toroidal equation 3.19 by taking the curl, $\varepsilon\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$ using eqn 3.16;

$$
\begin{aligned}
&\tilde{\varepsilon}\cdot\left(\tilde{u}\cdot\tilde{\nabla}\tilde{u}\right)\\
&=\tilde{\varepsilon}\cdot\left[\left(\tilde{\delta}\phi+\tilde{\varepsilon}\psi\right)\cdot\tilde{\nabla}\left(\tilde{\delta}\phi+\tilde{\varepsilon}\psi\right)\right]\\
&=\varepsilon_i\left(u_j\nabla_j u_i\right)\\
&=\varepsilon_i\left[\left(\delta_j\phi+\varepsilon_j\psi\right)\nabla_j\left(\delta_i\phi+\varepsilon_i\psi\right)\right]\\
&=\varepsilon_i\left\{\left(\delta_j\phi+\varepsilon_j\psi\right)\partial_j\left[\left(\partial_i\partial_z-\lambda_i\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\psi\right]\right\}\\
&=\varepsilon_i\left\{\left(\delta_j\phi+\varepsilon_j\psi\right)\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}\\
&=\varepsilon_i\left\{\left[\left(\partial_j\partial_z-\lambda_j\Delta\right)\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}\\
&=\left(\varepsilon_{irs}\partial_s\lambda_r\right)\left\{\left[\left(\partial_j\partial_z-\lambda_j\Delta\right)\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}
\end{aligned}
$$

Then expand, simplify, apply the necessary tensor algebra and collect sets of partial derivatives in terms of $\phi-\phi$, $\phi-\psi$ and $\psi-\psi$, as shown in Appendix C which for the poloidal terms are;

Poloidal $\phi-\phi$:

$$
\begin{aligned}
&(\partial_x\partial_z\phi)\left(\partial_x^5\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x\partial_y^4\phi\right)+2\,(\partial_x\partial_z\phi)\left(\partial_x^3\partial_y^2\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x^3\partial_z^2\phi\right)\\
&+(\partial_x\partial_z\phi)\left(\partial_x\partial_y^2\partial_z^2\phi\right)+(\partial_y\partial_z\phi)\left(\partial_x^4\partial_y\phi\right)+(\partial_y\partial_z\phi)\left(\partial_y^5\phi\right)+2\,(\partial_y\partial_z\phi)\left(\partial_x^2\partial_y^3\phi\right)\\
&+(\partial_y\partial_z\phi)\left(\partial_x^2\partial_y\partial_z^2\phi\right)+(\partial_y\partial_z\phi)\left(\partial_y^3\partial_z^2\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)\\
&-\left(\partial_x^3\phi\right)\left(\partial_x\partial_z^3\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_z^3\phi\right)\\
&-\left(\partial_x^2\partial_y\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y\partial_z^3\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)\\
&-\left(\partial_y^3\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_y\partial_z^3\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^4\partial_z\phi\right)\\
&-2\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_x^4\partial_z\phi\right)\\
&-\left(\partial_y^2\phi\right)\left(\partial_y^4\partial_z\phi\right)-2\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)\\
&-\left(\partial_y^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^4\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^4\phi\right)\\
&+\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)+\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^4\phi\right)+\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)\\
&+4\,(\partial_x\partial_y\partial_z\phi)\left(\partial_x^3\partial_y\phi\right)+4\,(\partial_x\partial_y\partial_z\phi)\left(\partial_x\partial_y^3\phi\right)+4\,(\partial_x\partial_y\partial_z\phi)\left(\partial_x\partial_y\partial_z^2\phi\right)\\
&+\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^4\phi\right)
\end{aligned}
$$

Poloidal $\phi - \psi$:

$$
\begin{aligned}
&= \left(\partial_x^3\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_y^2\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_z\psi\right)\\
&-2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) + 2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) - 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_y^2\partial_z\psi\right)\\
&-\left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_z^2\psi\right) - \left(\partial_y^3\phi\right)\left(\partial_x\partial_z^2\psi\right) + \left(\partial_x^4\partial_y\phi\right)\left(\partial_x\psi\right) + \left(\partial_y^5\phi\right)\left(\partial_x\psi\right)\\
&+2\left(\partial_x^2\partial_y^3\phi\right)\left(\partial_x\psi\right) + \left(\partial_x^2\partial_y\partial_z^2\phi\right)\left(\partial_x\psi\right) + \left(\partial_y^3\partial_z^2\phi\right)\left(\partial_x\psi\right) - \left(\partial_x^5\phi\right)\left(\partial_y\psi\right)\\
&-\left(\partial_x\partial_y^4\phi\right)\left(\partial_y\psi\right) - 2\left(\partial_x^3\partial_y^2\phi\right)\left(\partial_y\psi\right) - \left(\partial_x^3\partial_z^2\phi\right)\left(\partial_y\psi\right) - \left(\partial_x\partial_y^2\partial_z^2\phi\right)\left(\partial_y\psi\right)\\
&-\left(\partial_x^3\phi\right)\left(\partial_x^2\partial_y\psi\right) - \left(\partial_x^3\phi\right)\left(\partial_y^3\psi\right) - \left(\partial_x\partial_y^2\phi\right)\left(\partial_y^3\psi\right) - \left(\partial_x\partial_y^2\phi\right)\left(\partial_x^2\partial_y\psi\right)\\
&+\left(\partial_x^2\partial_y\phi\right)\left(\partial_x^3\psi\right) + \left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_y^2\psi\right) + \left(\partial_y^3\phi\right)\left(\partial_x^3\psi\right) + \left(\partial_y^3\phi\right)\left(\partial_x\partial_y^2\psi\right)\\
&+2\left(\partial_x^3\partial_y\phi\right)\left(\partial_x^2\psi\right) + 2\left(\partial_x\partial_y^3\phi\right)\left(\partial_x^2\psi\right) + 2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right) - 2\left(\partial_x^4\phi\right)\left(\partial_x\partial_y\psi\right)\\
&-2\left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) + 2\left(\partial_y^4\phi\right)\left(\partial_x\partial_y\psi\right) + 2\left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) - 2\left(\partial_x^3\partial_y\phi\right)\left(\partial_y^2\psi\right)\\
&-2\left(\partial_x\partial_y^3\phi\right)\left(\partial_y^2\psi\right) - 2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right)
\end{aligned}
$$

Poloidal $\psi - \psi$:

$$
= 4(\partial_x\partial_y\partial_z\psi)(\partial_x\partial_y\psi) - 2(\partial_x^2\psi)(\partial_y^2\partial_z\psi) - 2(\partial_y^2\psi)(\partial_x^2\partial_z\psi)
$$

Then collect sets of partial derivatives in terms of $\phi - \phi$, $\phi - \psi$ and $\psi - \psi$, as shown in Appendix C which for the toloidal equation are;

Toroidal $\phi - \phi$ $\left(\partial_x\partial_z^2\phi\right)\left(\partial_x^2\partial_y\phi\right) + \left(\partial_x\partial_z^2\phi\right)\left(\partial_y^3\phi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x^3\phi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_y^2\phi\right)$

Toroidal $\psi - \psi$ $\left(\partial_x^2\partial_y\psi\right)\left(\partial_x\psi\right) + \left(\partial_y^3\psi\right)\left(\partial_x\psi\right) - \left(\partial_x^3\psi\right)\left(\partial_y\psi\right) - \left(\partial_x\partial_y^2\psi\right)\left(\partial_y\psi\right)$

Toroidal $\phi - \psi$ $\left(\partial_x^2\psi\right)\left(\partial_x^2\partial_z\phi\right) + \left(\partial_y^2\psi\right)\left(\partial_y^2\partial_z\phi\right)$

We need to now find the associated output symmetry group for the 3D problem. Appendix F.3 outlines the method employed which is similar to that for the linear and 2D problem but taking into account the toroidal component of the perturbed flow and the partial derivatives in y. Identifying the closed symmetry set emerging from the non-linear terms facilitates the construction of a suitable non-converged solution used by the software program to find a converged "seed", or solution which can then be traced.

After 3D non-linear analysis it was found that the full symmetry set becomes;

$$\begin{aligned}
&T_l^{+}\cos m^{+}\cos n^{+}, T_l^{+}\sin m^{+}\sin n^{+},\\
&T_l^{+}\cos m^{+}\sin n^{+}, T_l^{+}\sin m^{+}\cos n^{+},\\
&T_l^{++}\cos m^{++}\cos n^{++}, T_l^{++}\sin m^{++}\sin n^{++},\\
&T_l^{++}\cos m^{++}\sin n^{++}, T_l^{++}\sin m^{++}\cos n^{++},\\
&T_l^{++}\cos m^{+}\cos n^{+}, T_l^{++}\sin m^{+}\sin n^{+},\\
&T_l^{++}\cos m^{+}\sin n^{+}, T_l^{++}\sin m^{+}\cos n^{+},\\
&T_l^{+}\cos m^{++}\cos n^{++}, T_l^{+}\sin m^{++}\sin n^{++},\\
&T_l^{+}\cos m^{++}\sin n^{++}, T_l^{+}\sin m^{++}\cos n^{++},.
\end{aligned} \tag{3.42}$$

We can see that for $m+n=$ odd we have no members, hence

$$m+n=odd \qquad a_{lmn}=b_{lmn}=c_{lmn}=0 \tag{3.43}$$

Obviously as this is the full symmetry group one must realise that not all symmetries will be evident in any particular 3D solution, or any higher degree (quaternary) solution. Symmetries will be different for each higher degree solution.

In order to find any 3D equilibrium solution modification of a converged 2D solution is required (here $\alpha = 1.37$ and $Gr = 211.423$ was used). Along with the addition of the necessary harmonics into the input file and then perturb the system by some small amount (0.1 in this case) governed by the symmetries for a possible converged 3D state. It was found that for $d \neq 0$ it was not possible to obtain phase-locked 3D solutions, so the condition $d = 0, b \neq 0$ was fixed.

According to the 2D stablity analysis for $\alpha = 1.37$, figure 3.41 we see that the highest stable point is around $b = 0.5$ and this value is chosen for β and hence a converged 3D equilibrium solution is found. Once a converged 3D solution is found this is used as a seed to find others as required. Figures 3.54 and 3.53 show how the 3D solution bifurcates from the 2D solution for $\alpha = 1.37$ and $\alpha = 1.6$ respectively.

Unfortunately 3D stability analysis tells us that the 3D states are not phase locked and hence as we do not have programs that are time evolving we cannot proceed. Our models are steady state only.

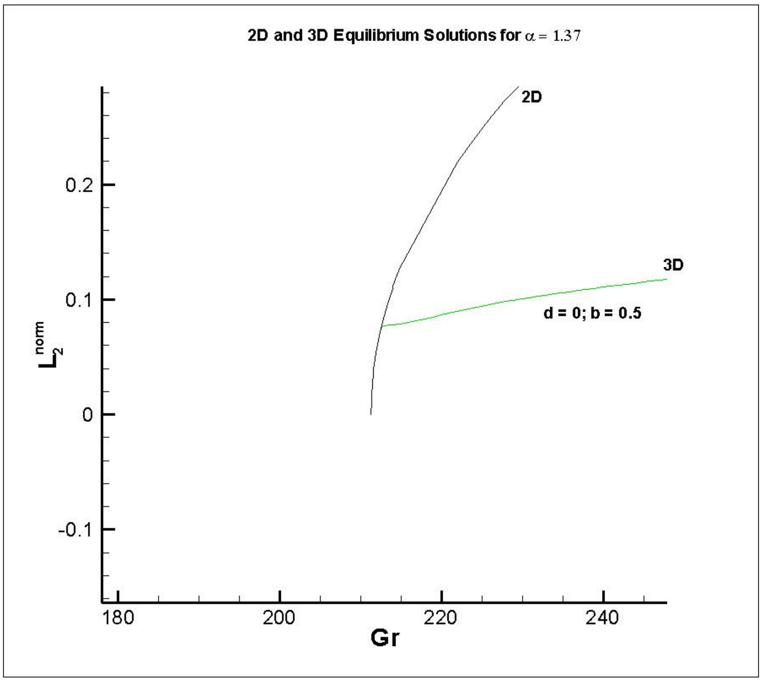


Figure 3.52: *3D Bifurcations.*

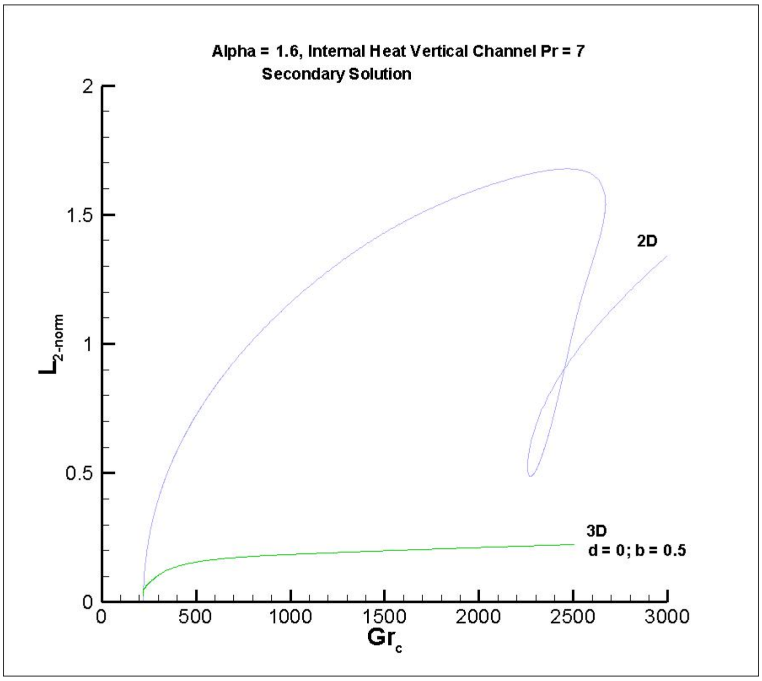


Figure 3.53: *3D Bifurcation* α = *1.6.*

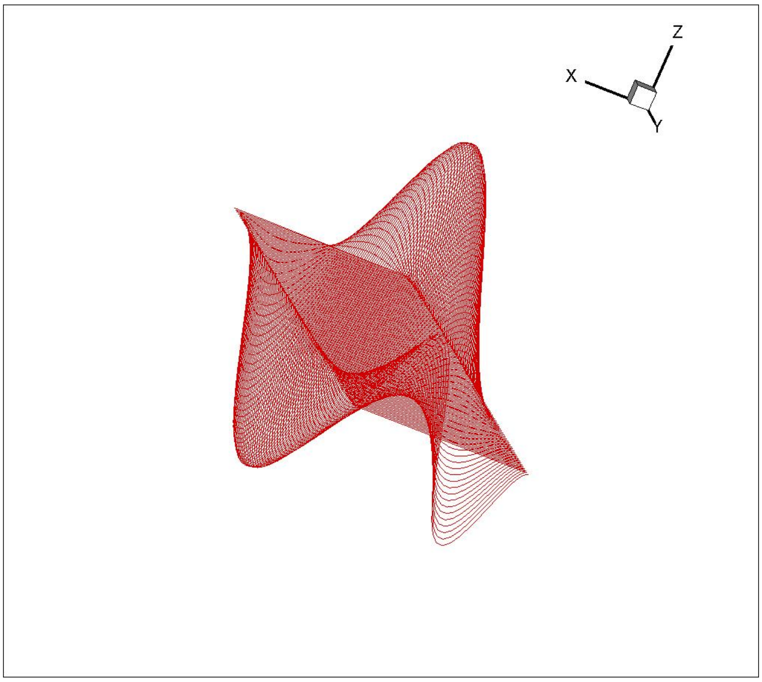


Figure 3.54: *3D Contour Plot* α = *1.37*

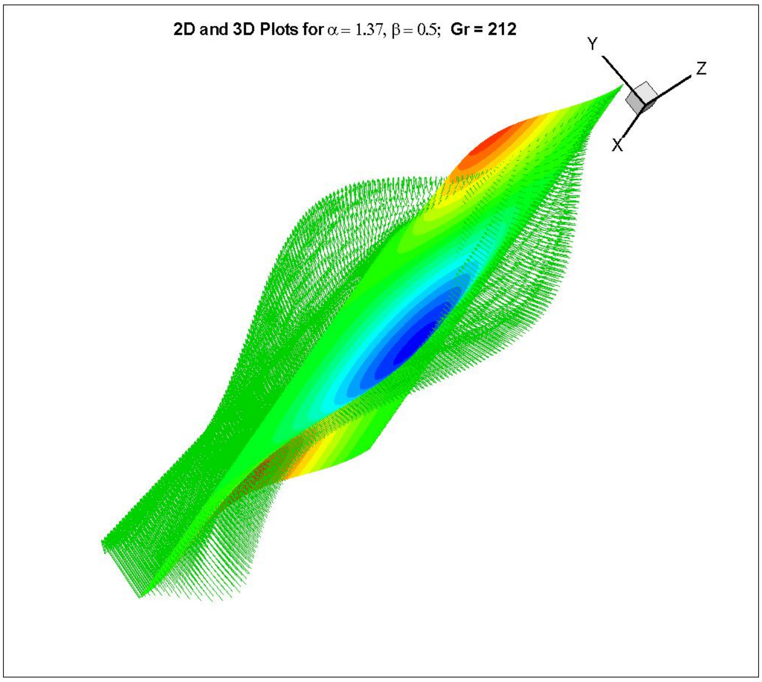


Figure 3.55: *3D Contour Plot* α = *1.37*

4 Vertical Channel Laterally Heated

CONTENTS

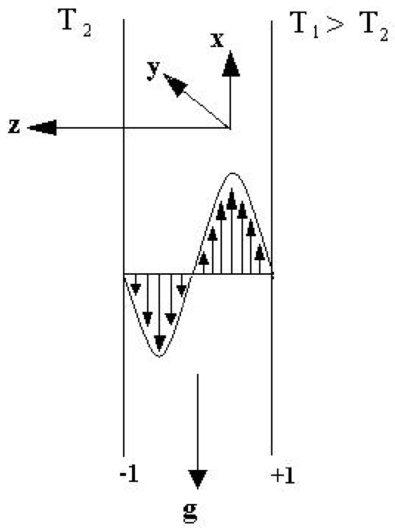


Figure 4.1: *Co-ordinate system for the laterally heated vertical fluid layer with LAR.*

4.1 Linear Analysis

In order to understand and verify the mechanisms involved in the homogeneously heated model the need is to begin with a known problem with known solutions, this enables benchmark testing, beginning with the classic, well researched Rayleigh-Benard model.

Next it is necessary to incline the channel to the vertical orientation and benchmark further against the work by Chait and Korpela [12] and Nagata Busse [38] who took a vertical channel laterally heated with Prandtl Number 0.71, 1000 and 0 respectively. Ensuring that our results for the critical Grashoff and wavenumber corresond to those given by [12] and [38].

4.1.1 Problem Modeling

The problem is modeled in exactly the same way as that for the homogeneously heated model. The system of equations is the same also, except that the temperature boundary condition is modified and hence the term for T in eq.(3.5) is represented by $T = T_1 - T_2$ for the temperatures of the boundaries, and no-slip condition at the boundaries for the velocity field. After applying the changed boundary conditions to the linear parts of the N.S. equations to derive the basic flow and temperature following the same methods for

Internal heat as described in Appendix B.2:

$$U_o(z)\hat{\mathbf{i}} = \tfrac{Gr}{6}(z - z^3) \tag{4.1}$$

$$T_o(z) = \pm Grz. \tag{4.2}$$

The basic velocity profile of eq.(4.1) gives an inflection point and hence linear instability is expected. Deriving the perturbation equations is exactly the same as that for the internal heat model and hence it is not necessary to repeat the theory here, but instead move onto the results of the linear analysis.

4.2 Linear Results

Well converged solutions where found for $l = 30$ Fig. (4.2) shows the pure mode neutral curve with the critical parameters $(\alpha_c, Gr_c) = (1.38, 491.78)$.

It was found that there also exists higher harmonic resonant linear solutions for both a 1:2 and 1:3 resonance where the second most dangerous ordered eigenvalue couple with its specific harmonic was used. Note that the values of Grashof found for the 1:2 and 1:3 resonant neutral curves are exactly the same as those for the pure mode curve but we must half and third the perturbation wavenumber (α) respectively.

Figure 4.2 shows the interaction between the pure, 1:2 and 1:3 neutral curves. The intersection of the 1:2 resonance curve and the pure mode curve is found at $\alpha = 0.88, Gr = 619$. Similarly, the intersection of the 1:3 resonant curve and the pure mode is found at $\alpha = 0.62, Gr = 825$. Also noting that the critical values at $\alpha_c = 0.69, Gr_c = 491.78$ and $\alpha_c = 0.46, Gr_c = 491.78$ for the 1:2 and 1:3 modes respectively.

4.3 Secondary Equilibrium States

As a measure of the numerical convergence we employ the vector $l_2 - norm$ as before. Well converged secondary solutions are obtained at L=23 and M=9 for all wavenumbers, we choose M=9 to ensure we include the 1:3 solutions in our amplitude coefficients output file.

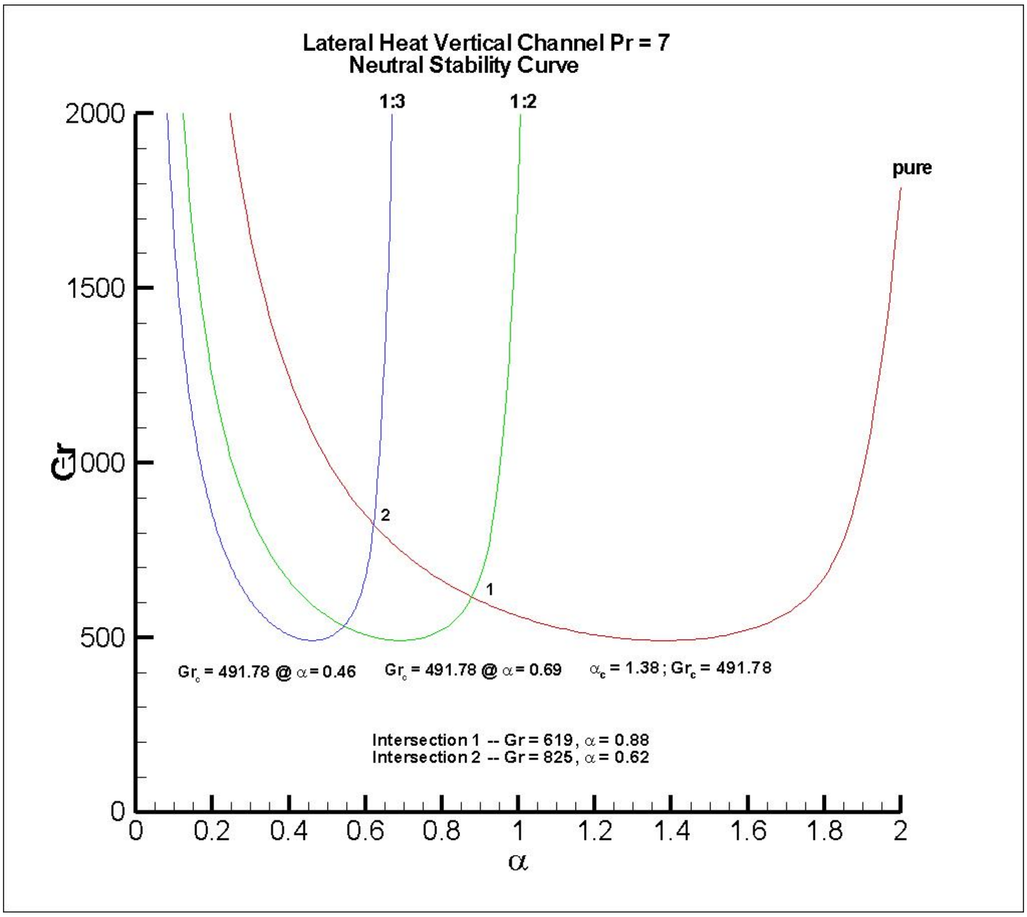


Figure 4.2: *Neutral Stability Curves.*

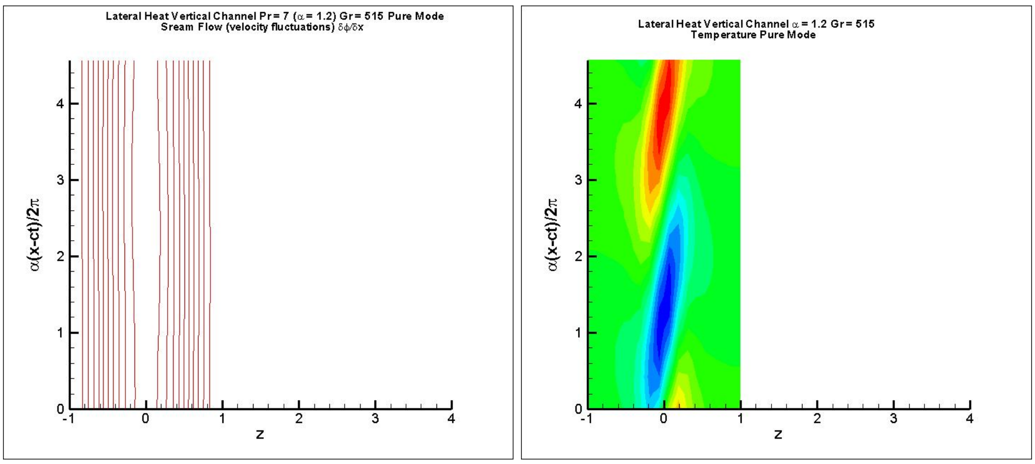


Figure 4.3: *a = 1.2 Stream Flow and Temperature Pure Mode*

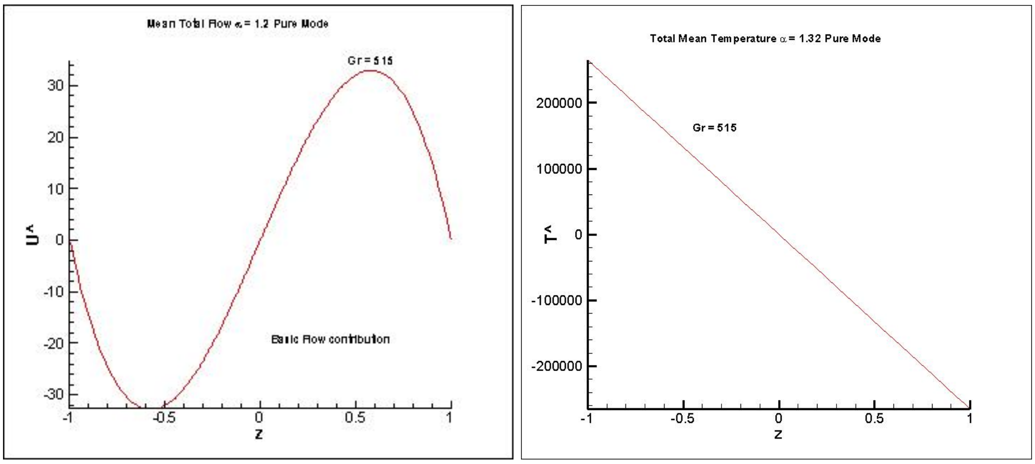
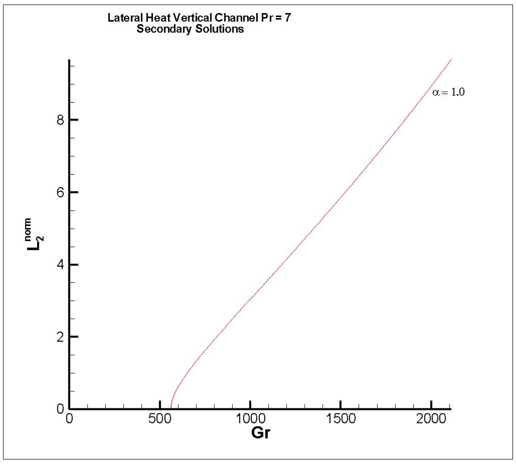


Figure 4.4: *a = 1.2 Mean Flow and Temperature Pure Mode*

4.3.1 Nonlinear Mode Interactions

For $\alpha \geq 0.9$ pure mode secondary solutions bifurcate supercritically from the neutral curve and the strength of the solutions continues to rise as the strength of the temperature gradient rises, see figure 4.5. As we decrease α and approach the intersection point of the pure and 1:2 resonant modes, figure 4.6, we see the emergence of the 1:2 mode with zero phase velocity bifurcating from it's associated 1:2 neutral curve, the strength of the 1:2 secondary solution continues to rise as the temperature parameter rises. In figure 4.6 we note that the pure mode bifurcates from the neutral as expected but as we decrease Gr the pure model secondary solution has a turning point (Gr = 640) that bifurcates to a mixed mode solution (Gr = 652) if we then increase Gr or if we continue to decrease Gr the pure mode bifurcates from the 1:2 mode solution (Gr = 578). The pure mode also has a zero phase velocity. It is worth noting that for the mixed mode solution (c) on figure 4.6 all coefficients of the solution are non-zero ($a_{mn} \neq 0$) and the phase velocity is non-zero indicating that this mixed mode is not stationary, i.e a traveling wave TW. Further increase of Gr with the mixed mode solution (c) show another bifurcation to another distinct mixed mode solution (d) where the odd coefficients of m in $a_{mn} \neq 0$. It was found that the mixed mode solution (d) also bifurcates from the 1:2 mode solution at a high Gr, with zero phase velocity. So we see many interactions between the 1:2 and pure modes that give rise to two distinct mixed mode solutions. As we continue to decease α to 0.7, figure 4.7 we see that we gradually lose the mixed mode solution (d) in figure 4.6 but are still left with the mixed mode solution where $a_{mn} \neq 0$ which bifurcates from a turning point on the pure mode as for $\alpha = 0.8$, the pure mode also bifurcates from the 1:2 mode as for $\alpha = 0.8$. In this section we now continue to reduce α to explore what happens to the secondary solutions as we approach the intersection point where the pure, 1:2 and 1:3 mode solutions overlap. Figure 4.8 shows the secondary solutions obtained for $\alpha = 0.6$, where we clearly see the all the mode interactions, which are complex. For $\alpha = 0.6$ we find that the pure mode bifurcates now from the 1:3 mode and we see a bridge between the 1:2 and 1:3 resonant mode solutions. It is interesting to see that the mixed mode at high Gr bifurcates into two solutions as we began to see previously for $\alpha = 0.7$, also we note that the 1:2 solution does not bifurcate with the pure mode.



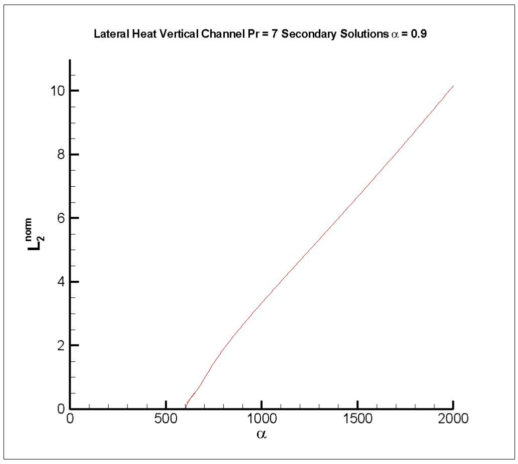


Figure 4.5: *Secondary Steady-State Solutions*

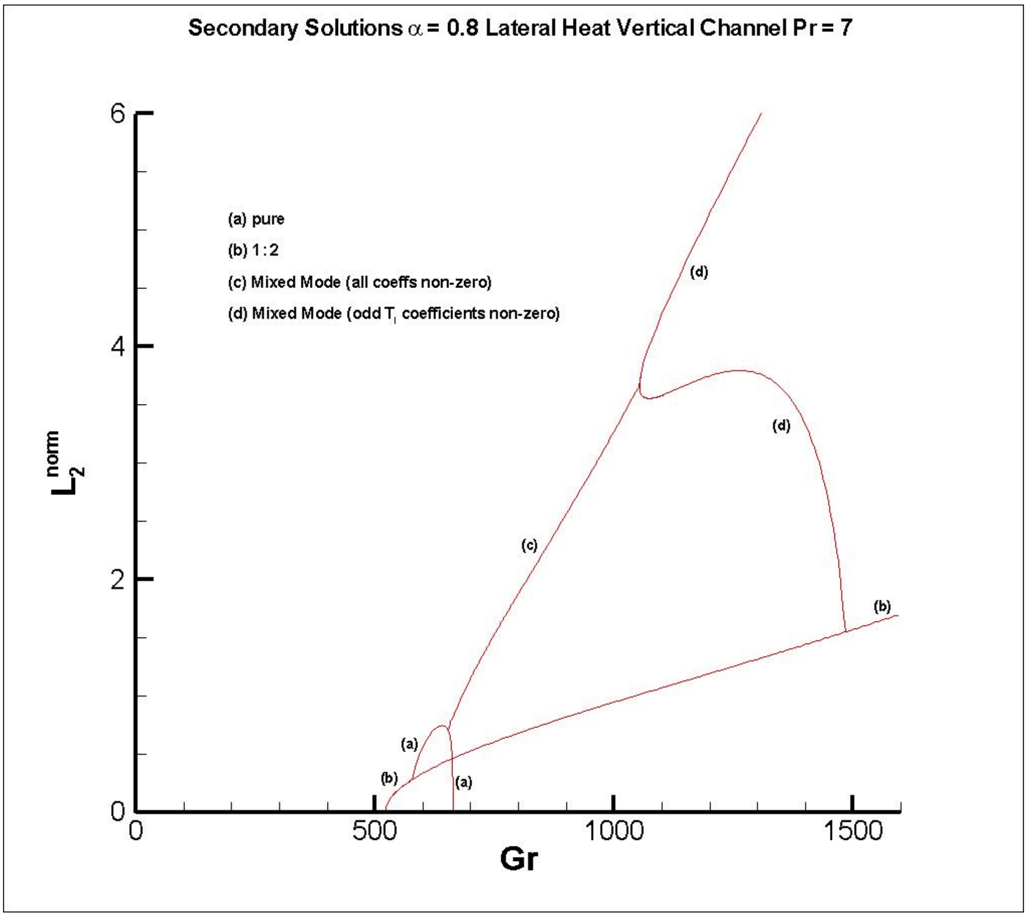


Figure 4.6: *Secondary Solutions* $\alpha = 0.8$

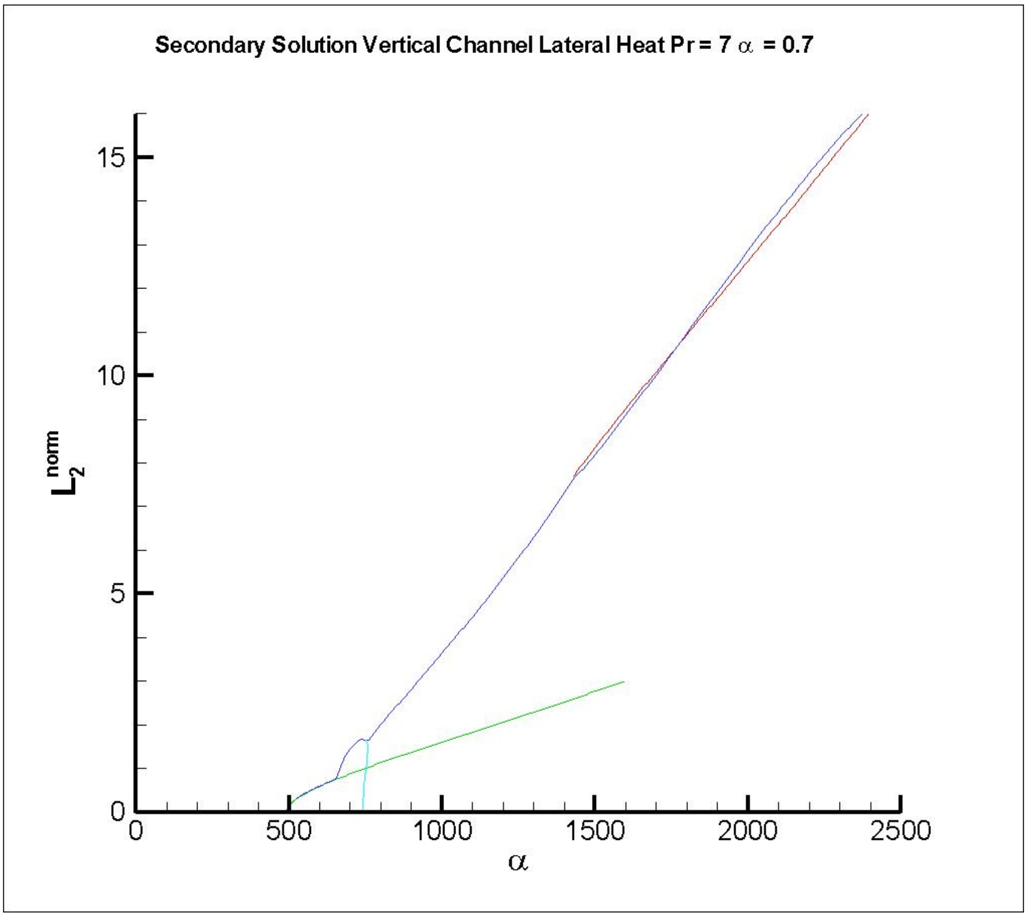


Figure 4.7: *Secondary Solutions* $\alpha = 0.7$

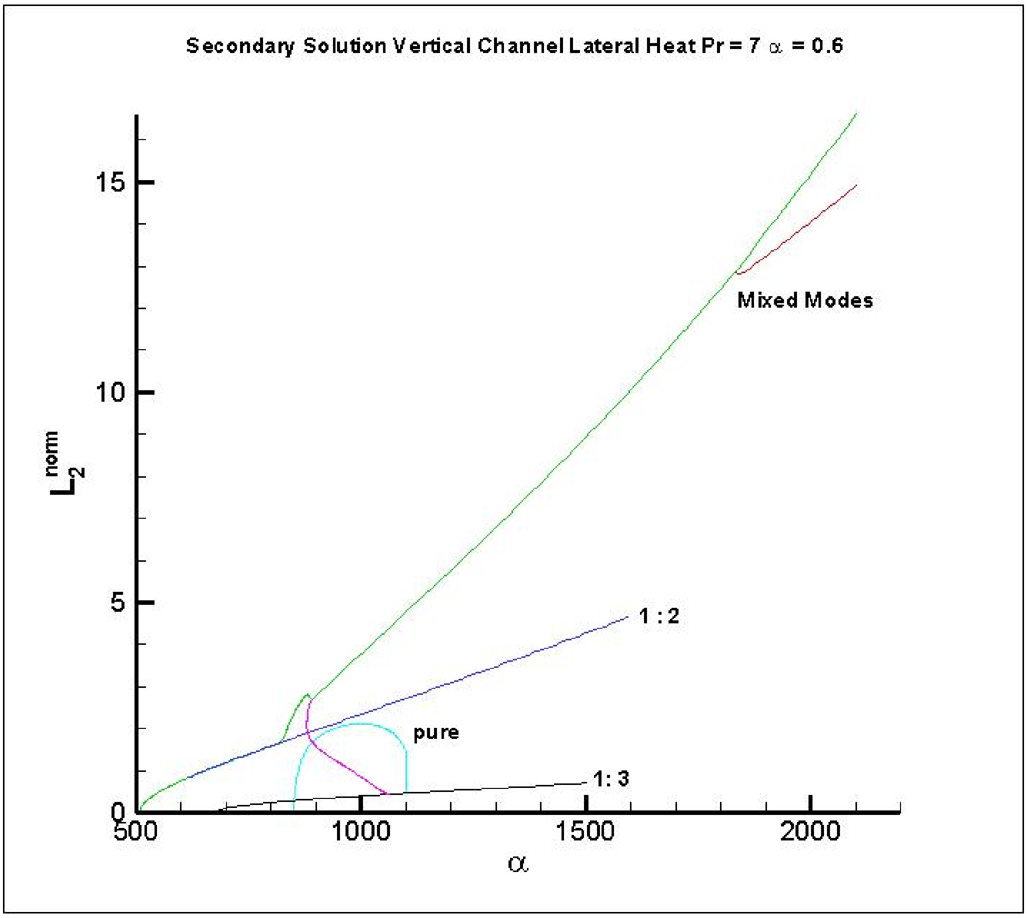


Figure 4.8: *Secondary Solutions* $\alpha = 0.6$

5 Conclusions and Discussions

In summary, this study found that for both heating models (HHF and LHF) there exist higher harmonic resonances for low perturbation wavenumber. For LHF the study has identified that there is a smooth transition between the fundamental and 1:3 resonant mode via mixed modes. This smooth transition is not evident for the homogeneously heated model, instead it is possible to have abrupt transitions between the modes found.

It was also found that for the homogeneously heated model there exists more than one fundamental mode for higher perturbation wavenumbers in conjunction with a higher internal heating parameter. Where the two modes interact there exist overlapping secondary states that appear to be invariant but results in a temperature inversion of the secondary convection rolls. Although mixed mode solutions were expected for higher wavenumber perturbations in HHF none were found as the solutions for these wavenumbers were not resonant.

In investigating the LHF model a tapestry of resonant solutions was found with mixed mode solutions bridging them. The resonant and mixed mode secondary solutions were found to be of stationary and travelling wave types.

Ideally the author would like to further research the models investigated by introducing a parameter that allowed the gradual imposition of the HHF basic flow profile whilst reducing the LHF basic flow profile and slowly inclining the channel from horizontal to the vertical orientation in order to better understand the mechanics of transition from mixed mode bridges to the jumps or bursts found. The author has completed linear analysis of the HHF model when the channel is inclined and found that an exchange of stability took place at an angle of 69°, where LW perturbations become the most dangerous and TW perturbations the least.

Further investigation is required in the stability analysis as the 3D stability model . Much work was undertaken to construct a parallel version of the 3D stability software, which gave identical results as the serial program for σ_r but inconclusive results for σ_i.

In parallel to this study the author has replicated the linear results produced by Kropp and Busse [34] for a differentially heated rotating annulus using the same software models as used in this study. The results are not included in this study because inclusion of the spanwise momentum would have introduced additional symmetries outside the aims of this thesis.

It is envisaged that the author would develop the non-linear model in future research. The implications of any research into a rotating, differentially heated annulus would be far reaching. The research would add to the understanding of atmospheric modeling of the Earth, and move forward the understanding of weather prediction.

The author would also like to persue 3D solutions in line with the research findings of Generalis and Itano [32] who in a recent analysis of heirarchical organisation of coherent structures in turbulent shear flow identified a hairpin vortex structure in Plane Couette Flow (PCF) by superimposing spanwise vortex solutions of Laterally Heated Flow in a vertical channel. They reduced the LHF solutions from the model by means of a homotopy parameter and found the associated symmetries by using the Floquet parameters. As demonstrated 3D solutions were found in this study but this could be extended in future research. I believe the secret to arriving at an analytical solution to the NS equations lies in the nature of the symmetry groups and how they interact with the NS equations and the partial derivatives contained therein. With careful adaption of abstract algebra theory it may be possible to analytically predict the structure of the symmetry groups for higher order bifurcations thus allowing research into higher dimensional bifurcations towards fully developed turbulent flow.

One may ask, "how do we test the theory in an experiment?". In practice one would perturb the system with slow changes in temperature. As the system of equations (Navier-Stokes) are coupled the changes in temperature would emulate the pertubations in the theory. However, due to the fact that the system is HHF it would be difficult to modify the temperature manually, one would have to use a self-heating nuclear plasma or chemical reaction.

Further investigation into the nonlinear stability characteristics for the LHF is required, with a view to benchmarking any results obtained against the weakly nonlinear solutions documented by [36] for Rayleigh-Benard convection. The author would like to note that any possible future collaboration with Prof. K. Fujimura would be an excellent opportunity and feels that the methods adopted in this study into the characteristics of the bifurcations in the highly nonlinear region of the phase space would complement Prof. Fujimura's already considerable research into the nature of higher order harmonic resonances and there interactions with the fundamental mode. Already nonlinear stability programs have been developed and are in current testing by the author in collaboration with Dr. Generalis.

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A Appendix - Non-Dimensionalisation of the N.S. Equations

A.1 Non-dimensionalisation of the momentum equation

The motion equation is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla \pi}{\rho} + g\gamma T\hat{\mathbf{i}} + \nu \nabla^2 \mathbf{u}. \tag{A.1}$$

For the non-dimensional description of the problem the following parameters are used:

- d for length,
- $\frac{d^2}{\nu}$ for time,
- $\frac{\nu}{d}$ for velocity field($\mathbf{u}$),
- $\frac{1}{d}$ for ∇,
- $\frac{1}{d^2}$ for ∇^2,

- $\frac{qd^2}{2\kappa Gr}$ for temperature,
- $\frac{g\gamma qd^5}{2\nu^2\kappa}$ for Gr, and
- $-\frac{d^3\nabla\pi}{2\nu^2\rho}$ for R.

Using the parameters defined above, we can rewrite the motion equation as

$$\frac{\nu}{d^2}\cdot\frac{\nu}{d}(\frac{\partial\mathbf{u}^*}{\partial t^*})+\frac{\nu}{d}\cdot\frac{1}{d}\cdot\frac{\nu}{d}(\mathbf{u}^*\cdot\nabla^*\mathbf{u}^*)=-\frac{1}{\rho}\pi\frac{1}{d}(-\frac{1}{\rho}\pi\nabla^*)+$$
$$g\gamma\hat{\mathbf{i}}\cdot\frac{\nu^2}{g\gamma d^3}(g\gamma\hat{\mathbf{i}}T^*)+\nu\cdot\frac{1}{d^2}\cdot\frac{\nu}{d}(\nabla^{2*}\mathbf{u}^*). \tag{A.2}$$

Simplifying the above equation gives

$$\frac{\nu^2}{d^3}(\frac{\partial\mathbf{u}^*}{\partial t^*})+\frac{\nu^2}{d^3}(\mathbf{u}^*\cdot\nabla^*\mathbf{u}^*)=-\frac{\pi}{\rho d}+\frac{\nu^2}{d^3}(T^*\hat{\mathbf{i}})+\frac{\nu^2}{d^3}(\nabla^{2*}\mathbf{u}^*). \tag{A.3}$$

Note that the constants are not non-dimensionalised and that the (*) indicates dimension-full parameters .

Multiplying by $\frac{d^3}{\nu^2}$ gives

$$(\frac{\partial\mathbf{u}^*}{\partial t^*})+(\mathbf{u}^*\cdot\nabla^*\mathbf{u}^*)=-\frac{d^2}{\nu^2}\frac{\pi}{\rho}+T^*\hat{\mathbf{i}}+\nabla^{2*}\mathbf{u}^*. \tag{A.4}$$

But $-\frac{d^2}{\nu^2}\frac{\pi}{\rho}=2R$, therefore, the non-dimensional momentum equation is

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=2R+T\hat{\mathbf{i}}+\nabla^2\mathbf{u}. \tag{A.5}$$

A.2 Non-dimensionalisation of energy equation

The non-dimensionalisation of the energy equation starts off by substituting the parameters defined above into the temperature equation, which is equation (3.2).

The resulting equation is

$$\frac{\nu}{d^2}\cdot\frac{qd^2}{\kappa}\cdot\frac{\nu^2\kappa}{g\gamma qd^5}(\frac{\partial T^*}{\partial t^*})+\frac{\nu}{d}\cdot\frac{1}{d}\cdot\frac{qd^2}{\kappa}\cdot\frac{\nu^2\kappa}{g\gamma gd^5}(\mathbf{u}^*\cdot\nabla^*T^*)=$$
$$\kappa\cdot\frac{1}{d^2}\cdot\frac{qd^2}{\kappa}\cdot\frac{\nu^2\kappa}{g\gamma qd^5}(\nabla^2*T^*)+q, \tag{A.6}$$

which after simplification becomes

$$\frac{\nu^3}{g\gamma d^5}(\frac{\partial T^*}{\partial t^*}) + \frac{\nu^3}{g\gamma d^5}(\mathbf{u}^* \cdot \nabla^* T^*) = \frac{\kappa\nu^2}{g\gamma d^5}(\nabla^2 * T^*) + q. \tag{A.7}$$

By multiplying by $(\frac{g\gamma d^5}{\nu^3})$, we get

$$\frac{\partial T^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* T^* = \frac{\kappa}{\nu}(\nabla^2 * T^*) + \frac{g\gamma q d^5}{\nu^2\kappa}. \tag{A.8}$$

To simplify the equation further, we multiply by the Pandtl number[1], which is $\frac{\nu}{\kappa}$, to get

$$\frac{\nu}{\kappa}(\frac{\partial T^*}{\partial t^*}) + \frac{\nu}{\kappa}(\mathbf{u}^* \cdot \nabla^* T^*) = \nabla^{2*} T^* + \frac{g\gamma q d^5}{\nu^2\kappa} \tag{A.9}$$

Since $\frac{g\gamma q d^5}{\nu^2\kappa}$ is 2Gr,we can divide by the Prandtl number, $\frac{\nu}{\kappa}$, so that we obtain

$$(\frac{\partial T^*}{\partial t^*}) + (\mathbf{u}^* \cdot \nabla^* T^*) = Pr^{-1}(\nabla^{2*} T^* + 2Gr), \tag{A.10}$$

where Gr already non- dimensional.

Therefore, the non- dimensional temperature equation is

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = Pr^{-1}(\nabla^2 T + 2Gr) \tag{A.11}$$

[1] Pr (the Prandtl number) measures strength of shear or viscous forces against convective forces.

B Appendix - Basic Flow and Temperature Derivations

B.1 Internal Heat Basic Flow and Temperature

In order to analyse the stability of the system against infinitesimal perturbations, we need to find the basic flow ($\mathbf{u}_0$) and basic temperature ($\mathbf{T}_0$), which satisfy the no-slip condition for the velocity, $\mathbf{u} = 0$, and the fixed temperature, $T = T_0$, on the boundary at $z = \pm 1$. We choose the narrow-gap boundaries between +1 and -1 as this allows us to use cartesian coordinates. All along the vertical channel, we assume there is laminar flow of the heated fluid, as shown in figure 1.2.

B.1.1 Basic Temperature

In the non-dimensional equation for temperature:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = Pr^{-1}(\nabla^2 T + 2Gr), \tag{B.1}$$

$\frac{\partial T}{\partial t} = o$ because of no time dependence due to the steady state of temperature, and also $\mathbf{u} \cdot \nabla T = 0$.

This is because

(B.2)

As there is no flow in either the y or z direction and applying the boundary condition ($T_0(z) = 0$) eventually make $\mathbf{u} \cdot \nabla T = 0$ due to the constant internal heating between the two plates.

Hence,

$$0 = \nabla^2 T + 2Gr, \tag{B.3}$$

which implies

$$-2Gr = \frac{\partial T_0{}^2}{\partial z^2}, \tag{B.4}$$

and thus

$$T''_0(z) = -2Gr. \tag{B.5}$$

Now, to get $T_0(z)$, we solve the ordinary differential equation [B.5]. First, we integrate equation [B.5] once to get

$$T'_0(z) = -2Grz + C. \tag{B.6}$$

Then, we integrate equation [B.6] and get

$$T_0(z) = -Grz^2 + Cz + D, \tag{B.7}$$

where C and D are arbitrary constants.
We now apply the boundary condition, which is $T_0(z) = 0$ at $z = \pm 1$, to equation [B.7]:
at z = 1

$$0 = -Gr + C + D, \tag{B.8}$$

at z = -1

$$0 = -Gr - C + D. \tag{B.9}$$

Adding equation [B.8] and equation [B.9], gives $0 = -2Gr + 2D$, which implies $D = Gr$ and subtracting equation [B.9] from equation [B.8] gives $0 = 2C$,which implies $C = 0$.

Thus, substituting $D = Gr$ and $C = 0$ in equation [B.7] we get the basic temperature:

$$T_0(z) = -Grz^2 + Gr = Gr(1 - z^2). \tag{B.10}$$

B.1.2 Basic Flow

The basic flow is the time independent steady state laminar flow. So, we take the basic flow of the non- dimensionalised momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 2R + T\hat{\mathbf{i}} + \nabla^2 \mathbf{u}. \tag{B.11}$$

First of all, $\frac{\partial \mathbf{u}}{\partial t}$ goes to zero because the velocity field does not depend on time as the fluid is in steady state for the basic flow. Secondly, $(\mathbf{u} \cdot \nabla)\mathbf{u}$ also goes to zero because

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{u}_0(z)\hat{\mathbf{i}}.(\partial_x \mathbf{u}_0(z)\hat{\mathbf{i}} + \partial_y \mathbf{u}_0(z)\hat{\mathbf{i}} + \partial_z \mathbf{u}_0(z)\hat{\mathbf{i}}), \tag{B.12}$$

where $\partial_x \mathbf{u}_0(z)\hat{\mathbf{i}} = 0$, $\partial_y \mathbf{u}_0(z)\hat{\mathbf{i}} = 0$, and only $\partial_z \mathbf{u}_0(z)\hat{\mathbf{i}}$ is left. But, we do not expect flow in the z-direction.

Therefore, we get

$$0 = 2R + T_0(z)\hat{\mathbf{i}} + \nabla^2 \mathbf{u}_0(z)\hat{\mathbf{i}}. \tag{B.13}$$

But, $2R = 0$ because there is no pressure gradient or forced flow in our system, however it is required to allow for no net vertical flow in the plane i.e.

$$\int_{-1}^{1} \mathbf{u} dz = 0 \tag{B.14}$$

This can be allowed for by adding a new unknown term X to equation B.13 also, $T_0(z)\hat{\mathbf{i}} = Gr(1 - z^2)$. Thus, we get

$$\frac{\partial^2}{\partial z^2}\mathbf{u}_0(z) = -Gr + Grz^2 - X, \tag{B.15}$$

$$\mathbf{u}_0^{''}(z) = -Gr + Grz^2 - X. \tag{B.16}$$

As this is a second order ordinary differential equation, we need to integrate equation [B.16] to get $\mathbf{u}_0(z)$. Firstly, we integrate equation [B.16] once, which gives

$$\mathbf{u}_0^{'}(z) = -Grz + \frac{Grz^3}{3} - Xz + C, \tag{B.17}$$

and then, we integrate equation[B.17], to get

$$\mathbf{u}_0(z) = -\frac{Grz^2}{2} + \frac{Grz^4}{12} - \frac{Xz^2}{2} + Cz + D. \tag{B.18}$$

The boundary conditions state that $\mathbf{u}_0(z) = 0$ for $z = 1$ and -1 at the boundaries only. So, we apply these boundary conditions to equation [B.18]:
at $z = 1$

$$0 = -\frac{Gr}{2} + \frac{Gr}{12} - \frac{X}{2} + C + D, \tag{B.19}$$

while at $z = -1$

$$0 = -\frac{Gr}{2} + \frac{Gr}{12} - \frac{X}{2} - C + D. \tag{B.20}$$

We must now find the two constants. Adding equation [B.20] to equation [B.19] gives $C = 0$ and $D = \frac{5Gr}{12} + \frac{X}{2}$.
Hence

$$\mathbf{u}_0(z) = -\frac{Grz^2}{2} + \frac{Grz^4}{12} - \frac{Xz^2}{2} + \frac{5Gr}{12} + \frac{X}{2}$$

$$\mathbf{u}_0(z) = -\frac{Grz^2}{2} + \frac{Grz^4}{12} + \frac{X}{2}(1 - z^2) + \frac{5Gr}{12} \tag{B.21}$$

In order to find X it is necessary to allow for no net vertical flow in the plane i.e.

$$\int_{-1}^{1} \mathbf{u} dz = 0$$

So,

$$\left[Gr(-\frac{z^3}{6} + \frac{z^5}{60} + \frac{5z}{12}) + \frac{X}{2}(z - \frac{z^3}{3}) \right]_{-1}^{1} = 0$$

$$2 \left[Gr(-\frac{1}{6} + \frac{1}{60} + \frac{5}{12}) + \frac{X}{2}(1 - \frac{1}{3}) \right] = 0$$

$$2 \left[Gr(\frac{16}{60}) + \frac{X}{2}(\frac{2}{3}) \right] = 0$$

Hence,

$$X = \frac{-4Gr}{5}.$$

Substituting into equation [B.21] gives the basic flow as;

$$U_0(z) = \frac{Gr}{60}(5z^4 - 6z^2 + 1) \tag{B.22}$$

B.2 Lateral Heat Basic Flow and Temperature

B.2.1 Basic Temperature

We know that we have a linear relationship between z and T_0, so we can begin with

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T, \tag{B.23}$$

where as before we are only interested in the linear terms in z:

$$\nabla^2 T = 0,$$

so;

$$T_0''(z) = 0,\ T_0'(z) = A,\ T_0(z) = Az + B.$$

Using the boundary conditions;

$$\begin{aligned} T_0(1) = T_1 &= A + B \\ T_0(-1) = T_2 &= -A + B \\ T_1 - T_2 &= 2A \\ T_2 + T_1 &= 2B \end{aligned}$$

we arrive at

$$T_0(z) = (T_1 - T_2)z + (T_1 + T_2).$$

Removing the arbitrary constants an writing in non-dimensional form [27];

$$T_0(z) = Grz. \tag{B.24}$$

B.2.2 Basic Flow

From B.13 we remove the non-linear terms to get;

$$U_0''(z) = -Grz,\ U_0'(z) = -\frac{Gr}{2}z^2 + C,\ U_0(z) = -\frac{Gr}{6}z^3 + Cz + D.$$

Using the boundary conditions;

$$\begin{aligned} U_0(1) = 0 &= -\frac{Gr}{6} + C + D \\ U_0(-1) = 0 &= \frac{Gr}{6} - C + D \end{aligned}$$

Solving, gives the Basic Flow;

$$U_0(z) = \frac{Gr}{6}(z - z^3). \tag{B.25}$$

C Appendix - Derivation of the Perturbation Equations

C.1 Preliminary Notes on Tensor Algebra

Notes on Tensors

For more information consult Kendall's book on Cartesian Tensors [8] and Rutherford Aris's book on Fluid Tensors [3].

1. Permutation Tensor (Anti-symmetric)

$\varepsilon_{ijk} = 1$ if ijk is cyclic 123, 312 etc. $\varepsilon_{ijk} = -1$ if ijk is anticyclic 321,132 etc. and $\varepsilon_{ijk} = 0$ if any two i,j or k are the same.

The anti-symmetric rule states that : $\varepsilon_{ijk}\varepsilon_{jmn} = -\varepsilon_{jik}\varepsilon_{jmn}$.

2. The Kronecker Delta (Symmetric)

$\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq 0$. Therefore $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$.

The Kronecker Delta is used when combining a pair of Permutation Tensors with differing indices. It is worth taking the reader through the process of contracting indices as this is required later.

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{lm}) - \delta_{im}(\delta_{jl}\delta_{ln} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}). \quad \text{(C}$$

One Index Contracted ($i = l$) Equation (D.1) becomes: $\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, hence we obtain a widely used identity;

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{C.2}$$

Two Indices Contracted ($i = l, j = m$) Equation (D.1) becomes $2\delta_{kn}$

$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \tag{C.3}$$

Three Indices Contracted ($i = l, j = m, k = n$) Equation (D.1) becomes 6

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{C.4}$$

3. $\varepsilon_{jlm}\lambda_l\partial_m\psi \cdot \varepsilon_{ipq}\lambda_i\lambda_p\partial_q\partial_j\partial_t^2\psi = \varepsilon_{jlm}\lambda_l\partial_m\psi \cdot \varepsilon_{33q}\lambda_i\lambda_p\partial_q\partial_j\partial_t^2\psi = 0$

4. $\lambda_i\lambda_i = \lambda_1\lambda_1 + \lambda_2\lambda_2 + \lambda_3\lambda_3 = 0 + 0 + 1 = 1$ where $\lambda = \hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

5. $\left(\lambda_j\Delta\phi\right)\left(\lambda_i\partial_j\partial_t^2\Delta\phi\right) = (\Delta\phi)\left(\lambda_i\lambda_j\partial_j\partial_t^2\Delta\phi\right) = (\Delta\phi)\left(\lambda_i\partial_z\partial_t^2\Delta\phi\right)$ as λ is a multiplier and can be taken out of the brackets to act on a derivative.

6. $\varepsilon_{ipq}\lambda_p\partial_j\partial_q\partial_z\psi \cdot \varepsilon_{jlm}\lambda_l\partial_i\partial_m\psi$ $l = p = 3$; ***[so i, q, j and m$\neq$ 3 (do δz)]***

1st index contracted so;

$\varepsilon_{i3q}\varepsilon_{j3m} = \left[\delta_{ij}\delta_{qm} - \delta_{im}\delta_{qj}\right]$

7. $\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\partial_i\psi = \varepsilon_{i3q}\partial_1\partial_j\partial_z\partial_2\psi + \varepsilon_{i3q}\partial_2\partial_j\partial_z\partial_1\psi = -1+1 = 0$

8. $\varepsilon_{jlm}\cdot\varepsilon_{ipq}\lambda_l\lambda_p\partial_m\partial_q\partial_j\partial_i\partial_z\psi = \varepsilon_{j3m}\cdot\varepsilon_{i3q}\partial_m\partial_q\partial_j\partial_z\psi = 0$ for reason in 5.

9. $\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi \rightarrow \varepsilon_{i3q}\partial_q\partial_j\partial_z\psi$ the index i is outside the contraction. If we had ∂_i we could contract i and eliminate all with antisymmetric properties.

10. $\partial_t\partial_t = \partial_t^2 = \Delta$ as does $\partial_m\partial_m = \partial_m^2 = \Delta$ etc. but not $\partial_z\partial_z = \partial_z^2 \neq \Delta$ as z already allocated.

11. $\partial_1 = \partial_x; \partial_2 = \partial_y; \partial_3 = \partial_z$

12. Example of procedure for eliminating permutation tensor notation by considering the assignment of indexes to ensure non-zero status of ε_{ijk};

$$\begin{aligned}
&\left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi \\
&= \partial_i\partial_j\left(\partial_z^2\phi\right)\left[\varepsilon_{231}\partial_1\partial_i\psi + \varepsilon_{132}\partial_2\partial_i\psi\right] \\
&= \partial_i\partial_j\left(\partial_z^2\phi\right)\left[1\times\partial_1\partial_i - 1\times\partial_2\partial_i\right]\psi \qquad , \\
&= \partial_i\partial_j\left(\partial_z^2\phi\right)\left[\partial_x\partial_i - \partial_y\partial_i\right]\psi \\
&= \partial_i\partial_y\left(\partial_z^2\phi\right)\left(\partial_x\partial_i\psi\right) - \partial_i\partial_x\left(\partial_z^2\phi\right)\partial_y\partial_i\psi
\end{aligned}$$

when $\partial_1\partial_i \rightarrow j = 2; \partial_2\partial_i \rightarrow j = 1$

C.2 Derivation of ε_i and δ_i

The solenoidal fluid velocity field is $u = \hat{k}\cdot\nabla\times\nabla\times\phi + \hat{k}\cdot\nabla\times\psi$, we shall derive the curl curl operator firstly for the poloidal component of u.

$$\hat{k}\cdot\nabla\times\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \phi \end{vmatrix} = \hat{i}(\partial_y\phi) - \hat{j}(\partial_x\phi) + \hat{k}(0). \tag{C.5}$$

Now take curl again,

$$\nabla \times \nabla \times \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_y \phi & -\partial_x \phi & \phi \end{vmatrix} = \hat{i}(\partial_x \partial_z \phi) + \hat{j}(\partial_y \partial_z \phi) + \hat{k}(-\partial_x^2 - \partial_y^2)\phi. \quad \text{(C.6)}$$

Now we can say,

$$u_i = (\partial_i \partial_z - \lambda_i \Delta)\phi \quad \text{(C.7)}$$

Where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ and $\lambda_i \cdot \Delta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot (\partial_x^2 + \partial_y^2 + \partial_z^2)$, $\lambda_x = \lambda_y = 0$ with $\lambda_z = 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Working through using values of i = 1,2,3=x,y,z respectively

$i = 1 \rightarrow \partial_x \partial_z - 0$, $i = 2 \rightarrow \partial_y \partial_z - 0$ and $i = 3 \rightarrow \partial_z^2 - (\partial_x^2 + \partial_y^2 + \partial_z^2) = \Delta_2$.

So we can define $\delta_i = \partial_i \partial_z - \lambda_i \Delta$ as our curl curl operator.

Now let us derive our curl operator for the toroidal component of the velocity field

$$\hat{k} \times \nabla \times \psi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \psi \\ \partial_x & \partial_y & \partial_z \end{vmatrix} = -\hat{i}(\partial_y \psi) + \hat{j}(\partial_x \psi) + \hat{k}(0). \quad \text{(C.8)}$$

We can make use of the permutation tensor outlined in the previous section in this appendix to say,

$$\varepsilon_i = \varepsilon_{ijk} \lambda_j \partial_k \psi \quad \text{(C.9)}$$

Working through using values of i = 1,2,3=x,y,z respectively

$i = 1 \rightarrow \varepsilon_{123} \lambda_2 \partial_z - \varepsilon_{132} \lambda_3 \partial_y = -\partial_y$,

$i = 2 \rightarrow -\varepsilon_{213} \lambda_1 \partial_z + \varepsilon_{231} \lambda_2 \partial_x = \partial_x$,

$i = 3 \rightarrow -\varepsilon_{321} \lambda_2 \partial_x + \varepsilon_{312} \lambda_1 \partial_y = 0$.

So we can define $\varepsilon_i = \varepsilon_{ijk} \lambda_j \partial_k$ as our curl operator.

C.3 Proof of Incompressibilty Condition

$$\begin{aligned}
0 &= \nabla \cdot u \\
&= \nabla \cdot (\delta\phi + \varepsilon\psi) \\
&= \partial_x(\delta\phi + \varepsilon\psi)_x + \partial_y(\delta\phi + \varepsilon\psi)_y + \partial_z(\delta\phi + \varepsilon\psi)_z \\
&= \partial_i[\partial_i\partial_z\phi - \lambda_i\Delta\phi + \varepsilon_{ijk}\lambda_j\partial_k\psi] \\
&= \partial_i\partial_i\partial_z\phi + \varepsilon_{ijk}\lambda_j\partial_i\partial_k\psi - \lambda_i\Delta\phi\partial_i \\
&= \partial_i\partial_i\partial_z\phi - \varepsilon_{1j3}\lambda_j\partial_1\partial_3\psi + \varepsilon_{3j1}\lambda_j\partial_3\partial_1\psi - \lambda_i\Delta\phi\partial_i \\
&= \Delta\partial_z\phi - \lambda_i\Delta\partial_i\phi \\
&= 0.
\end{aligned} \tag{C.10}$$

C.4 Curl and curl curl of the motion equation

C.4.1 Curl of the motion equation

We take the non-dimensionalised motion equation and apply the ε operator on each part separately;

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 2R + T\hat{\mathbf{i}} + \nabla^2\mathbf{u}. \tag{C.11}$$

For convenience I shall drop the use of the bold font.

1. $\varepsilon \cdot u$:

$$\varepsilon_i u_i = \varepsilon_i(\delta_i\phi + \varepsilon_i\psi) = \varepsilon_i\delta_i\phi + \varepsilon_i\varepsilon_i\psi \tag{C.12}$$

In this equation, $\varepsilon_i\delta_i\phi = 0$. This is because ε_i and δ_i are orthogonal to each other, and also

$$\begin{aligned}
\delta_i\varepsilon_i &= (\partial_i\partial_z - \lambda_i\Delta)\varepsilon_{ijk}\lambda_j\partial_k \\
&= \partial_i\partial_z E_{ijk}\lambda_j\partial_k - \Delta\varepsilon_{ijk}\lambda_i\lambda_j\partial_k.
\end{aligned} \tag{C.13}$$

However, λ_i, λ_j andλ_k are always $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore, $i = j$ and therefore according to the property of permutation tensor $\varepsilon_{ijk}\lambda_i\lambda_j\partial_k$ will be 0 due to the fact that two indices are the same.

Also, if we fix $j = 3$ in $\partial_i \partial_z \varepsilon_{ijk} \lambda_j \partial_k$, we get $\partial_i \partial_z \varepsilon_{i3k} \lambda_j \partial_k$ which is

$$\begin{aligned} \varepsilon_{i3k} \partial_i \partial_z \partial_k &= \varepsilon_{132} \partial_1 \partial_z \partial_2 + \varepsilon_{231} \partial_2 \partial_z \partial_1 \\ &= -\partial_x \partial_z \partial_y + \partial_y \partial_z \partial_x \\ &= 0. \end{aligned} \tag{C.14}$$

Therefore,

$$\begin{aligned} \varepsilon_i u_i &= \varepsilon_i \varepsilon_i \psi \\ &= \varepsilon_{ijk} \lambda_j \partial_k \varepsilon_{ilm} \lambda_l \partial_m \psi \\ &= \varepsilon_{ijk} \varepsilon_{ilm} \lambda_j \lambda_l \partial_k \partial_m \psi \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) \lambda_j \lambda_l \partial_k \partial_m \psi \\ &= \delta_{jl} \delta_{km} \lambda_j \lambda_l \partial_k \partial_m \psi - \delta_{jm} \delta_{kl} \lambda_j \lambda_l \partial_k \partial_m \psi \end{aligned} \tag{C.15}$$

Now, for δ_{jl} to be 1, j must equal to l. Similarly, $k = m$ for $\delta_{km} = 1$, $j = m$ for $\delta_{jm} = 1$, and $k = l$ for $\delta_{kl} = 1$.

Therefore, $\varepsilon_i u_i = \lambda_j \lambda_j \partial_k \partial_k \psi - \lambda_j \partial_j \lambda_l \partial_l \psi$.

But, $\lambda_j \partial_j = \lambda_l \partial_l = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \partial_z$, and $\lambda_j \lambda_j = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$.

Therefore,

$$\begin{aligned} \varepsilon_i u_i &= \partial_k^2 \psi - \partial_z^2 \psi \\ &= (\partial_x^2 + \partial_y^2 + \partial_z^2) \psi - \partial_z^2 \psi \\ &= \partial_x^2 \psi + \partial_y^2 \psi + \partial_z^2 \psi - \partial_z^2 \psi \\ &= \partial_x^2 \psi + \partial_y^2 \psi \\ &= (\partial_x^2 + \partial_y^2) \psi \end{aligned} \tag{C.16}$$

Here, we can apply the Planform Laplacian, Δ_2. Thus, $\varepsilon_i u_i = \Delta_2 \psi$.

2. The Reynolds number is a scalar. Therefore, we cannot apply the curl on R.

3. Likewise we cannot apply the curl on temperature as it is also scalar and hence this term disappears.

4. The curl on $\varepsilon \cdot \nabla^2 u$ is simply the curl on u because the curl on ∇^2 is ∇^2 is itself. And, the curl on **u** was already found out in (1). So,

$$\begin{aligned} \varepsilon \cdot \nabla^2 u &= \varepsilon \cdot \nabla^2(\varepsilon \cdot u) \\ &= \nabla^2 \delta_2 \psi. \end{aligned} \tag{C.17}$$

5. The curl of the linear components of $u \cdot \nabla u$.

The velocity field, u is comprised of 2 components:
$u = u_0(z)\hat{\imath} + \breve{u}$,
where

- $u_0(z)$ is the basic flow, and
- $\breve{u}$ is the deviation.

Further, $\breve{u} = \hat{u} + \bar{u}$, where

- $\hat{u}$ is the perturbation, and
- $\bar{u}$ is the mean flow.

So,

$$u \cdot \nabla u = (u_0(z)\hat{\imath} + \hat{u} + (u_0(z)\hat{\imath} + \hat{u} + \bar{u}) \tag{C.18}$$

giving $u_0(z)\hat{\imath} \cdot \nabla u_0(z)\hat{\imath} +$ other terms which may be linear in perturbations.
i.e:

$$u \cdot \nabla u = \bar{u} \cdot \nabla \breve{u} + \breve{u} \cdot \nabla \bar{u} + \breve{u} \cdot \nabla \breve{u}, \tag{C.19}$$

where $\breve{u} \cdot \nabla \breve{u}$ is the non-linear term, which we ignore because my system is linear.

The curl of $(u \cdot \nabla)u$:

$$\varepsilon \cdot (u \cdot \nabla u) = \varepsilon \cdot (\bar{u} \cdot \nabla \breve{u}) + \varepsilon \cdot (\breve{u} \cdot \nabla \bar{u}) \tag{C.20}$$

$$\begin{aligned}
\varepsilon\cdot(\bar{u}\cdot\nabla\breve{u}) &= \varepsilon_i(\bar{u}\cdot\nabla u_i)\\
&= \varepsilon_{ijk}\lambda_k\partial_j[\nabla(\partial_i\partial_z-\lambda_i\Delta)\phi-\varepsilon_{ilm}\lambda_m\partial_l\psi]\bar{u}\\
&= \varepsilon_{ijk}\lambda_k\partial_j[\partial_i\nabla\partial_z\phi-\nabla\Delta\lambda_i\phi-\varepsilon_{ilm}\lambda_m\partial_l\nabla\psi]\bar{u}\\
&= [\varepsilon_{1j3}\partial_j\partial_i\nabla\partial_z\phi-\varepsilon_{3j3}\partial_j\nabla\Delta\lambda_i\lambda_k\phi-\varepsilon_{ljk}\varepsilon_{ilm}\lambda_k\lambda_m\partial_j\partial_l\nabla\psi]\bar{u}\\
&= 0-0(-\varepsilon_{ijk}\varepsilon_{ilm}\lambda_k\lambda_m\partial_j\partial_l\nabla\psi)\bar{u}\\
&= -(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\lambda_k\lambda_m\partial_j\partial_k\nabla\psi\bar{u}\\
&= (-\nabla^3\psi+\nabla\partial_z^2\psi)\bar{u}\\
&= \Delta_2\nabla\bar{u}\psi\\
&= \bar{u}\cdot\nabla\Delta_2\psi
\end{aligned} \tag{C.21}$$

$$\begin{aligned}
\varepsilon\cdot(\breve{u}\cdot\nabla\bar{u}) &= \varepsilon_i(u_j\cdot\nabla\bar{u})_i\\
&= \varepsilon_{ipq}\lambda_q\partial_p(\breve{u}_j\cdot\nabla_j u_0(z))\\
&= \varepsilon_{123}\partial_p(\breve{u}_j\cdot\nabla_j u_0(z))\\
&= \partial_y(\breve{u}_j\cdot\partial_j u_0(z))
\end{aligned} \tag{C.22}$$

Here, $\breve{u}_j\cdot\partial_j u_0(z)=(u_x\partial_x+u_y\partial_y+u_z\partial_z)u_0(z)$. But, we want only the z-component. Therefore,

$$\begin{aligned}
\varepsilon\cdot(\breve{u}\cdot\nabla\bar{u}) &= \partial_y(\partial_j\partial_z-\lambda_j\Delta)\phi u_0(z)\\
&= \partial_y(\partial_{z^2}-\Delta)\phi u_0(z)\\
&= \partial_y\Delta_2\phi u_0(z)
\end{aligned} \tag{C.23}$$

Therefore,

$$\varepsilon\cdot(u\cdot\nabla u)=\bar{u}\cdot\nabla\Delta_2\psi+\partial_y\Delta_2\phi u_0(z) \tag{C.24}$$

C.4.2 The curl curl of the motion equation

Let us take the double curl of each component individually.

1.

$$\begin{aligned}
\delta \cdot \mathbf{u} &= \delta_i u_i \\
&= \delta_i(\delta_i \phi + \varepsilon_i \psi) \\
&= \delta_i \delta_i \phi + \delta_i \varepsilon_i \psi \\
&= \delta_i \delta_i \phi \quad (\delta_i \text{ is orthogonal to } \varepsilon_i \text{ and thus, } \delta_i \varepsilon_i \psi \text{ goes to zero}) \\
&= (\partial_i \partial_z - \lambda_i \Delta)(\partial_i \partial_z - \lambda_i \Delta)\phi \\
&= (\partial_i^2 \partial_z^2 - 2\partial_i \partial_z \lambda_i \Delta + \lambda_i^2 \Delta^2)\phi
\end{aligned} \tag{C.25}$$

Now,

$\partial_i^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = \Delta$, $\partial_i \lambda_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \partial_z$, and $\lambda_i^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$

Therefore,

$$\begin{aligned}
\delta \cdot \mathbf{u} &= (\Delta \partial_z^2 - 2\partial_z^2 \Delta + \Delta^2)\phi \\
&= (\Delta^2 - \partial_z^2 \Delta)\phi \\
&= \Delta(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_z^2)\phi \\
&= \Delta(\partial_x^2 + \partial_y^2)\phi \\
&= \Delta \Delta_2 \phi \\
&= \nabla^2 \Delta_2 \phi
\end{aligned} \tag{C.26}$$

2. Again, we cannot apply the curl operator to R as it is scalar and so can be excluded.

3.

$$\begin{aligned}
\delta \cdot T\hat{\mathbf{i}} &= \delta_i T\hat{\mathbf{i}} \\
&= (\partial_i \partial_z - \lambda_i \Delta)\hat{\mathbf{i}} T \\
&= (\partial_i \partial_z \hat{\mathbf{i}} - \lambda_i \Delta_i)T
\end{aligned} \tag{C.27}$$

Here, $\partial_i \partial_z = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \partial_x$ and $\lambda_i \Delta_i = 1$. Therefore,

$$\delta \cdot T\hat{\mathbf{i}} = (\partial_x \partial_z - \Delta)T. \tag{C.28}$$

4.

$$\delta \cdot \nabla^2 \mathbf{u} = (\delta \cdot \nabla^2)(\delta \cdot \mathbf{u}) \tag{C.29}$$

But, $(\delta \cdot \nabla^2)$ gives ∇^2 and $(\delta \cdot \mathbf{u})$. Therefore,

$$\begin{aligned} \delta \cdot \nabla^2 \mathbf{u} &= \nabla^2 \nabla^2 \Delta_2 \phi \\ &= \nabla^4 \Delta_2 \phi \end{aligned} \tag{C.30}$$

5. The curl curl of the linear components of $u \cdot \nabla u$.

- The curl curl of $(\mathbf{u} \cdot \nabla \mathbf{u})$

$$\delta \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = \delta \cdot (\breve{\mathbf{u}} \nabla \bar{\mathbf{u}}) + \delta \cdot (\bar{\mathbf{u}} \nabla \breve{\mathbf{u}}) \tag{C.31}$$

$$\begin{aligned} \delta \cdot (\breve{\mathbf{u}} \nabla \bar{\mathbf{u}}) &= \delta_i (\breve{\mathbf{u}}_j \cdot \nabla_j u_0(z)_i) \\ &= (\partial_i \partial_z - \lambda_i \Delta)[((\partial_j \partial_z - \lambda_j \Delta)\phi \\ &+ \varepsilon_{jlm} \lambda_m \partial_l \psi) \cdot \partial_j - u_0(z)_i] \end{aligned} \tag{C.32}$$

But, $\delta_i \varepsilon_{jlm} \lambda_m \partial_l \psi = 0$, $\lambda_x = \lambda_y = 0$ and $\lambda_z = 1$. To make things easy, let us define $\nabla_j u_0(z)$ as u_0'.

Therefore, we get

$$\begin{aligned} \delta \cdot (\breve{\mathbf{u}} \nabla \bar{\mathbf{u}}) &= (\partial_x \partial_z)[(\partial_{z^2}) - \Delta)\phi \cdot u_0'] \\ &= \partial_x(-\partial_z \Delta_2 \phi \cdot u_0' - \Delta_2 \phi \cdot u_0'') \\ &= -\partial_x \partial_z \Delta_2 \phi \cdot u_0' - \partial_x \Delta_2 \phi \cdot u_0'' \end{aligned} \tag{C.33}$$

$$\begin{aligned} \delta \cdot (\bar{\mathbf{u}} \nabla \breve{\mathbf{u}}) &= \delta_i (u_0(z) \cdot \nabla_j \breve{\mathbf{u}}_i) \\ &= (\partial_i \partial_z - \lambda_i \Delta)[u_o(z) \cdot \partial_j((\partial_i \partial_z - \lambda_i \Delta)\phi + \varepsilon_{ilm} \lambda_m \partial_l \psi) \breve{\mathbf{u}}_i] \end{aligned} \tag{C.34}$$

But, $\delta_i \varepsilon_{ilm} \lambda_m \partial_l \psi = 0$ and $u_i = u_x + u_y + u_z$ where $u_x = \partial_x \partial_y \phi$, $u_y = \partial_y \partial_z$ and $u_z = -\Delta_2 \phi$. Hence,

$$\begin{aligned} \delta \cdot (\bar{\mathbf{u}} \nabla \breve{\mathbf{u}}) &= (\partial_i \partial_z - \lambda_i \Delta)(u_0(z) \cdot \partial_x \breve{\mathbf{u}}_i) \\ &= \partial_i \partial_z (u_0 \cdot \partial_i u_i) + \Delta(u_0 \cdot \partial_x \Delta_2 \phi) \\ &= \partial_x \partial_z (u_0 \cdot \partial_{x^2} \partial_z \phi) + \partial_y \partial_z (u_0 \cdot \partial_x \partial_y \partial_z \phi) - \partial_{z^2}(u_0 \cdot \partial_x \Delta_2 \phi) \\ &+ \Delta(u_0 \cdot \partial_x \Delta_2 \phi) \end{aligned} \tag{C.35}$$

But, $-\partial_{z^2}(u_0\cdot\partial_x\Delta_2\phi)+\Delta(u_0\cdot\partial_x\Delta_2\phi)=\Delta_2(u_0\partial_x\Delta_2\phi)$. Hence,

$$\begin{aligned}\delta\cdot(\bar{\mathbf{u}}\nabla\breve{u}) &= \partial_x(u_0'\cdot\partial_{x^2}\partial_z\phi)+\partial_x(u_o\cdot\partial_{x^2}\partial_{z^2}\phi)+\partial_y(u_o'\cdot\partial_x\partial_y\partial_z\phi)\\ &\quad+\partial_y(u_0\cdot\partial_x\partial_y\partial_{z^2}\phi)+\Delta_2(u_0\cdot\partial_x\Delta_2\phi)\\ &= u_0'\cdot\partial_{x^3}\partial_z\phi+u_0'\cdot\partial_x\partial_{y^2}\partial_z\phi\\ &= u_0'\cdot\partial_x\partial_z\Delta_2\phi\end{aligned} \tag{C.36}$$

Therefore,

$$\begin{aligned}\delta\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) &= -\partial_x\partial_z\Delta_2\phi\cdot u_0'-\partial_x\Delta_2\phi\cdot u_0''+u_0'\cdot\partial_x\partial_z\Delta_2\phi\\ &= u_0\partial_x\partial_{z^2}\Delta_2+\Delta_2(u_0\partial_x\Delta_2\phi)-\partial_x\Delta_2\phi\cdot u_0''\\ &= u_0\partial_x\nabla^2\Delta_2\phi-u_0''\partial_x\Delta_2\phi.\end{aligned} \tag{C.37}$$

$$\delta\cdot(\mathbf{u}\cdot\nabla\mathbf{u})=u_0\partial_x\nabla^2\Delta_2\phi-u_0''\partial_x\Delta_2\phi. \tag{C.38}$$

Now, replacing each term back into the momentum equation, we get

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2\Delta_2\phi+u_0\partial_x\nabla^2\Delta_2\phi-u_0''\partial_x\Delta_2\phi &= (\partial_x\partial_z-\Delta)T\\ &\quad+\nabla^4\Delta_2\phi.\end{aligned} \tag{C.39}$$

C.5 The curl and curl curl of the non-linear components of $\mathbf{u}\cdot\nabla\mathbf{u}$

<u>Phi Equation</u>

$$\begin{aligned}&\tilde{\delta}\cdot\left(\tilde{u}\cdot\tilde{\nabla}\tilde{u}\right)\\ &=\tilde{\delta}\cdot\left[\left(\tilde{\delta}\phi+\tilde{\varepsilon}\psi\right)\cdot\tilde{\nabla}\left(\tilde{\delta}\phi+\tilde{\varepsilon}\psi\right)\right]\\ &=\delta_i\left(u_j\nabla_j u_i\right)\\ &=\delta_i\left[\left(\delta_j\phi+\varepsilon_j\psi\right)\nabla_j\left(\delta_i\phi+\varepsilon_i\psi\right)\right]\\ &=\delta_i\left\{\left(\delta_j\phi+\varepsilon_j\psi\right)\partial_j\left[\left(\partial_i\partial_z-\lambda_i\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\psi\right]\right\}\\ &=\delta_i\left\{\left(\delta_j\phi+\varepsilon_j\psi\right)\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}\\ &=\delta_i\left\{\left[\left(\partial_j\partial_z-\lambda_j\Delta\right)\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}\\ &=\left(\partial_i\partial_z-\lambda_i\Delta\right)\left\{\left[\left(\partial_j\partial_z-\lambda_j\Delta\right)\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}\end{aligned}$$

Now apply the product rule twice for $\partial_i\partial_z$ and $\lambda_i\Delta$ letting $\Delta = \partial_t\partial_t$ in second expansion.

$$= \partial_i\partial_z \left\{ \left[\left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi \right] \quad \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi \right] \right\}$$
$$-\lambda_i\partial_t\partial_t \left\{ \left[\left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi \right] \quad \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi \right] \right\}$$

$$= \partial_i \left\{ \begin{array}{l} \left[\left(\partial_j\partial_z - \lambda_i\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi \right] \quad \left[\partial_z \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi \right] \\ + \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi \right] \quad \left[\partial_z \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi \right] \end{array} \right\}$$
$$-\lambda_i\partial_t \left\{ \begin{array}{l} \left[\left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi \right] \quad \left[\partial_t \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi \right] \\ + \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi \right] \quad \left[\partial_t \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi \right] \end{array} \right\}$$

$$= \left[\left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi \right] \quad \left[\partial_i\partial_z \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_i\partial_z\psi \right]$$
$$+ \left[\partial_z \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi \right] \quad \left[\partial_i \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi \right]$$
$$+ \left[\left(\partial_i\partial_j\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi \right] \quad \left[\partial_i\partial_z \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_z\partial_i\psi \right]$$
$$+ \left[\partial_z \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi \right] \quad \left[\partial_i \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_i\psi \right]$$
$$- \left[\left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi \right] \quad \left[\lambda_i\partial_t\partial_t \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t^2\psi \right]$$
$$- \left[\partial_t \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi \right] \quad \left[\lambda_i\partial_t \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi \right]$$
$$- \left[\left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi \right] \quad \left[\lambda_i\partial_t^2 \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi \right]$$
$$- \left[\partial_t \left(\partial_j\partial_z - \lambda_j\Delta\right)\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi \right] \quad \left[\partial_t\lambda_i \left(\partial_j\partial_i\partial_z - \lambda_i\partial_j\Delta\right)\phi + \varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t\psi \right]$$

Notes on Tensors

1. $\varepsilon_{jlm}\lambda_l\partial_m\psi \cdot \varepsilon_{ipq}\lambda_i\lambda_p\partial_q\partial_j\partial_t^2\psi = \varepsilon_{jlm}\lambda_l\partial_m\psi \cdot \varepsilon_{33q}\lambda_i\lambda_p\partial_q\partial_j\partial_t^2\psi = 0$

2. $\lambda_i\lambda_i = \lambda_1\lambda_1 + \lambda_2\lambda_2 + \lambda_3\lambda_3 = 0 + 0 + 1 = 1$

3. $\left(\lambda_j\Delta\phi\right)\left(\lambda_i\partial_j\partial_t^2\Delta\phi\right) = \left(\Delta\phi\right)\left(\lambda_i\lambda_j\partial_j\partial_t^2\Delta\phi\right) = \left(\Delta\phi\right)\left(\lambda_i\partial_z\partial_t^2\Delta\phi\right)$ as λ is a multiplier and can be taken out of the brackets to act on a derivative.

4. $\varepsilon_{ipq}\lambda_p\partial_j\partial_q\partial_z\psi \cdot \varepsilon_{jlm}\lambda_l\partial_i\partial_m\psi$ $l = p = 3$; ***[so i, q, j and m$\neq$ 3 (do δz)]***

One index contracted so;

$\varepsilon_{i3q}\varepsilon_{j3m} = \left[\delta_{ij}\delta_{qm} - \delta_{im}\delta_{qj}\right]$

5. $\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\partial_i\psi = \varepsilon_{i3q}\partial_1\partial_j\partial_z\partial_2\psi + \varepsilon_{i3q}\partial_2\partial_j\partial_z\partial_1\psi = -1+1 = 0$

6. $\varepsilon_{jlm}\cdot\varepsilon_{ipq}\lambda_l\lambda_p\partial_m\partial_q\partial_j\partial_i\partial_z\psi = \varepsilon_{j3m}\cdot\varepsilon_{i3q}\partial_m\partial_q\partial_j\partial_z\psi = 0$ for reason in 5.

7. $\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi \rightarrow \varepsilon_{i3q}\partial_q\partial_j\partial_z\psi$ the index i is outside the contraction. If we had ∂_i we could contract i and eliminate all with antisymmetric properties.

8. $\partial_t\partial_t = \partial_t^2 = \Delta$ as does $\partial_m\partial_m = \partial_m^2 = \Delta$ etc. but not $\partial_z\partial_z = \partial_z^2 \neq \Delta$ as z already allocated.

9. $\partial_1 = \partial_x; \partial_2 = \partial_y; \partial_3 = \partial_z$

10. Example of procedure for eliminating permutation tensor notation by considering the assignment of indexes to ensure non-zero status ofε_{ijk};

$$
\begin{aligned}
&\left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi \\
&= \partial_i\partial_j\left(\partial_z^2\phi\right)\left[\varepsilon_{231}\partial_1\partial_i\psi + \varepsilon_{132}\partial_2\partial_i\psi\right] \\
&= \partial_i\partial_j\left(\partial_z^2\phi\right)\left[1\times\partial_1\partial_i - 1\times\partial_2\partial_i\right]\psi \qquad , \\
&= \partial_i\partial_j\left(\partial_z^2\phi\right)\left[\partial_x\partial_i - \partial_y\partial_i\right]\psi \\
&= \partial_i\partial_y\left(\partial_z^2\phi\right)\left(\partial_x\partial_i\psi\right) - \partial_i\partial_x\left(\partial_z^2\phi\right)\partial_y\partial_i\psi
\end{aligned}
$$

when $\partial_1\partial_i \rightarrow j = 2; \partial_2\partial_i \rightarrow j = 1$

11. For code put terms collected by order of z derivatives.

$$
\begin{aligned}
&= \left[\partial_j\partial_z\phi - \lambda_j\Delta\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi\right] \quad \left[\partial_i^2\partial_z^2\partial_j\phi - \partial_j\partial_z^2\Delta\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_i\partial_z\psi\right] \\
&+ \left[\partial_i\partial_j\partial_z^2\phi - \lambda_i\partial_j\partial_z\Delta\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi\right] \quad \left[\partial_i\partial_j\partial_z\phi - \lambda_j\partial_i\Delta\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi\right] \\
&+ \left[\partial_i\partial_j\partial_z\phi - \lambda_i\partial_j\Delta\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right] \quad \left[\partial_i\partial_j\partial_z^2\phi - \lambda_j\partial_i\partial_z\Delta\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_z\partial_i\psi\right] \\
&+ \left[\partial_j\partial_z^2\phi - \lambda_j\partial_z\Delta\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi\right] \quad \left[\partial_i^2\partial_j\partial_z\phi - \partial_j\partial_z\Delta\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_i\psi\right] \\
&- \left[\partial_j\partial_z\phi - \lambda_j\Delta\phi + \varepsilon_{jlm}\lambda_l\partial_m\psi\right] \quad \left[\partial_t^2\partial_z^2\partial_j\phi - \lambda_i\lambda_i\partial_j\partial_t^2\Delta\phi + \varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t^2\psi\right] \\
&- \left[\partial_i\partial_j\partial_t\partial_z\phi - \lambda_i\partial_j\partial_t\Delta\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi\right] \quad \left[\lambda_i\partial_j\partial_z\partial_t\phi - \lambda_i\lambda_j\partial_t\Delta\phi + \varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi\right] \\
&- \left[\partial_i\partial_j\partial_z\phi - \lambda_i\partial_j\Delta\phi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right] \quad \left[\lambda_i\partial_j\partial_z\partial_t^2\phi - \lambda_i\lambda_j\partial_t^2\Delta\phi + \varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi\right] \\
&- \left[\partial_j\partial_z\partial_t\phi - \lambda_j\partial_t\Delta\phi + \varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi\right] \quad \left[\partial_j\partial_t\partial_z^2\phi - \lambda_i\lambda_i\partial_j\partial_t\Delta\phi + \varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t\psi\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\partial_j\partial_z\phi\right)\left(\partial_i^2\partial_z^2\partial_j\phi\right) - \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_z^2\Delta\phi\right) + \left(\partial_j\partial_z\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\partial_i\psi \\
&+ \left(\lambda_j\Delta\phi\right)\left(\partial_z^2\partial_j\Delta\phi\right) - \left(\lambda_j\Delta\phi\right)\left(\partial_i^2\partial_z^2\partial_j\phi\right) - \left(\lambda_j\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\partial_i\psi \\
&+ \left(\partial_i^2\partial_z^2\partial_j\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi - \left(\partial_j\partial_z^2\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi + \varepsilon_{jlm}\lambda_l\partial_m\psi\cdot\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\partial_i\psi \\
&+ \left(\partial_i\partial_j\partial_z^2\phi\right)\left(\partial_i\partial_j\partial_z\phi\right) - \left(\partial_i\partial_j\partial_z^2\phi\right)\left(\lambda_j\partial_i\Delta\phi\right) + \left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi \\
&- \left(\lambda_i\partial_j\partial_z\Delta\phi\right)\left(\partial_i\partial_j\partial_z\phi\right) + \left(\lambda_i\partial_j\partial_z\Delta\phi\right)\left(\lambda_j\partial_i\Delta\phi\right) - \left(\lambda_i\partial_j\partial_z\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi \\
&+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi\cdot\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi + \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi - \left(\lambda_j\partial_i\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi \\
&+ \left(\partial_j\partial_i\partial_z\phi\right)\left(\partial_i\partial_j\partial_z^2\phi\right) - \left(\partial_i\partial_j\partial_z\phi\right)\left(\lambda_j\partial_i\partial_z\Delta\phi\right) + \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_z\partial_i\psi \\
&- \left(\lambda_i\partial_j\Delta\phi\right)\left(\partial_i\partial_j\partial_z^2\phi\right) + \left(\lambda_i\partial_j\Delta\phi\right)\left(\lambda_j\partial_i\partial_z\Delta\phi\right) - \left(\lambda_i\partial_j\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\partial_z\psi \\
&+ \left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi - \left(\lambda_j\partial_i\partial_z\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\cdot\varepsilon_{jlm}\lambda_l\partial_m\partial_i\partial_z\psi \\
&+ \left(\partial_j\partial_z^2\phi\right)\left(\partial_i^2\partial_j\partial_z\phi\right) - \left(\partial_j\partial_z^2\phi\right)\left(\partial_j\partial_z\Delta\phi\right) + \left(\partial_j\partial_z^2\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_i\psi \\
&- \left(\lambda_j\partial_z\Delta\phi\right)\left(\partial_i^2\partial_j\partial_z\phi\right) + \left(\lambda_j\partial_z\Delta\phi\right)\left(\partial_j\partial_z\Delta\phi\right) - \left(\lambda_j\partial_z\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_i\psi \\
&+ \left(\partial_i^2\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi - \left(\partial_j\partial_z\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi + \varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi\cdot\varepsilon_{ipq}\lambda_p\partial_i\partial_q\partial_j\psi \\
&- \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_t^2\partial_z^2\phi\right) + \left(\partial_j\partial_z\phi\right)\left(\lambda_i\lambda_i\partial_j\partial_t^2\Delta\phi\right) + \left(\lambda_j\Delta\phi\right)\left(\partial_t^2\partial_z^2\partial_j\phi\right) \\
&- \left(\partial_j\partial_z\phi\right)\varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t^2\psi + \left(\lambda_j\Delta\phi\right)\varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t^2\psi - \left(\lambda_j\Delta\phi\right)\left(\lambda_i\lambda_i\partial_j\partial_t^2\Delta\phi\right) \\
&- \left(\lambda_i\partial_t^2\partial_z^2\partial_j\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi + \left(\lambda_i\lambda_i\partial_j\partial_t^2\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi + \varepsilon_{jlm}\lambda_l\partial_m\psi\cdot\varepsilon_{ipq}\lambda_i\lambda_p\partial_q\partial_j\partial_t^2\psi \\
&- \left(\partial_i\partial_j\partial_t\partial_z\phi\right)\left(\lambda_i\partial_j\partial_z\partial_t\phi\right) + \left(\partial_i\partial_j\partial_t\partial_z\phi\right)\left(\lambda_i\lambda_j\partial_t\Delta\phi\right) - \left(\partial_i\partial_j\partial_t\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi \\
&+ \left(\lambda_i\partial_j\partial_t\Delta\phi\right)\left(\lambda_i\partial_j\partial_t\partial_z\phi\right) - \left(\lambda_i\partial_j\partial_t\Delta\phi\right)\left(\lambda_i\lambda_j\partial_t\Delta\phi\right) + \left(\lambda_i\partial_j\partial_t\Delta\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi \\
&- \left(\lambda_i\partial_j\partial_t\partial_z\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi + \left(\lambda_i\lambda_j\partial_t\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi\cdot\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi \\
&- \left(\partial_i\partial_j\partial_z\phi\right)\left(\lambda_i\partial_j\partial_t^2\partial_z\phi\right) + \left(\partial_i\partial_j\partial_z\phi\right)\left(\lambda_i\lambda_j\partial_t^2\Delta\phi\right) - \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi \\
&+ \left(\lambda_i\partial_j\Delta\phi\right)\left(\lambda_i\partial_j\partial_z\partial_t^2\Delta\phi\right) - \left(\lambda_i\partial_j\Delta\phi\right)\left(\lambda_i\lambda_j\partial_t^2\Delta\phi\right) + \left(\lambda_i\partial_j\Delta\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi \\
&- \left(\lambda_i\partial_j\partial_z\partial_t^2\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi + \left(\lambda_i\lambda_j\partial_t^2\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\cdot\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi \\
&- \left(\partial_j\partial_z\partial_t\phi\right)\left(\partial_j\partial_t\partial_z^2\phi\right) + \left(\partial_j\partial_z\partial_t\phi\right)\left(\lambda_i\lambda_i\partial_j\partial_t\Delta\phi\right) - \left(\partial_j\partial_z\partial_t\phi\right)\varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t\psi \\
&+ \left(\lambda_j\partial_t\Delta\phi\right)\left(\partial_j\partial_t\partial_z^2\phi\right) - \left(\lambda_j\partial_t\Delta\phi\right)\left(\lambda_i\lambda_i\partial_j\partial_t\Delta\phi\right) - \left(\partial_j\partial_t\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi \\
&- \left(\lambda_j\partial_t\Delta\phi\right)\varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t\psi + \left(\lambda_i\lambda_i\partial_j\partial_t\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi - \varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi\cdot\varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t\psi
\end{aligned}
$$

Now eliminate where possible using the previously outlined notes.

$$= \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_z^2\Delta\phi\right) - \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_z^2\Delta\phi\right) + 0$$
$$+ (\Delta\phi)\left(\partial_z^3\Delta\phi\right) - (\Delta\phi)\left(\partial_z^3\Delta\phi\right) - 0$$
$$+ \left(\Delta\partial_z^2\partial_j\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi - \left(\partial_j\partial_z^2\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi + \varepsilon_{jlm}\lambda_l\partial_m\psi\cdot\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\partial_i\psi$$
$$+ \left(\partial_i\partial_j\partial_z^2\phi\right)\left(\partial_i\partial_j\partial_z\phi\right) - \left(\partial_i\partial_z^3\phi\right)(\partial_i\Delta\phi) + \left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi$$
$$- \left(\partial_j\partial_z\Delta\phi\right)\left(\partial_j\partial_z^2\phi\right) + \left(\partial_z^2\Delta\phi\right)(\partial_z\Delta\phi) - \left(\lambda_i\partial_j\partial_z\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi$$
$$+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi\cdot\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi + \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi - \left(\lambda_j\partial_i\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi$$
$$+ \left(\partial_j\partial_i\partial_z\phi\right)\left(\partial_i\partial_j\partial_z^2\phi\right) - \left(\partial_i\partial_z^2\phi\right)\left(\partial_i\partial_z\Delta\phi\right) + \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_z\partial_i\psi$$
$$- \left(\partial_j\Delta\phi\right)\left(\partial_j\partial_z^3\phi\right) + (\partial_z\Delta\phi)\left(\partial_z^2\Delta\phi\right) - \left(\lambda_i\partial_j\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\partial_z\psi$$
$$+ \left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi - \left(\lambda_j\partial_i\partial_z\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\cdot\varepsilon_{jlm}\lambda_l\partial_m\partial_i\partial_z\psi$$
$$+ \left(\partial_j\partial_z^2\phi\right)\left(\Delta\partial_j\partial_z\phi\right) - \left(\partial_j\partial_z^2\phi\right)\left(\partial_j\partial_z\Delta\phi\right) + 0$$
$$- (\partial_z\Delta\phi)\left(\Delta\partial_z^2\phi\right) + (\partial_z\Delta\phi)\left(\partial_z^2\Delta\phi\right) - 0$$
$$+ \left(\Delta\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi - \left(\partial_j\partial_z\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi + \varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi\cdot\varepsilon_{ipq}\lambda_p\partial_i\partial_q\partial_j\psi$$
$$- \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_t^2\partial_z^2\phi\right) + \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_t^2\Delta\phi\right) + (\Delta\phi)\left(\partial_t^2\partial_z^3\phi\right)$$
$$-0 + 0 - (\Delta\phi)\left(\partial_z\partial_t^2\Delta\phi\right)$$
$$- \left(\partial_t^2\partial_z^2\partial_j\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi + \left(\partial_j\partial_t^2\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi + \varepsilon_{jlm}\lambda_l\partial_m\psi\cdot\varepsilon_{ipq}\lambda_i\lambda_p\partial_q\partial_j\partial_t^2\psi$$
$$- \left(\partial_j\partial_t\partial_z^2\phi\right)\left(\partial_j\partial_z\partial_t\phi\right) + \left(\partial_t\partial_z^3\phi\right)(\partial_t\Delta\phi) - \left(\partial_i\partial_j\partial_t\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi$$
$$+ \left(\partial_j\partial_t\Delta\phi\right)\left(\partial_j\partial_t\partial_z\phi\right) - \left(\partial_z\partial_t\Delta\phi\right)(\partial_t\Delta\phi) + \left(\lambda_i\partial_j\partial_t\Delta\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi$$
$$- \left(\lambda_i\partial_j\partial_t\partial_z\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi + \left(\lambda_i\lambda_j\partial_t\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi\cdot\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi$$
$$- \left(\partial_j\partial_z^2\phi\right)\left(\partial_j\partial_t^2\partial_z\phi\right) + \left(\partial_z^3\phi\right)\left(\partial_t^2\Delta\phi\right) - \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi$$
$$+ \left(\partial_j\Delta\phi\right)\left(\partial_j\partial_z\partial_t^2\Delta\phi\right) - (\partial_z\Delta\phi)\left(\partial_t^2\Delta\phi\right) + \left(\lambda_i\partial_j\Delta\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi$$
$$- \left(\lambda_i\partial_j\partial_z\partial_t^2\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi + \left(\lambda_i\lambda_j\partial_t^2\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi + \varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\cdot\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi$$
$$- \left(\partial_j\partial_z\partial_t\phi\right)\left(\partial_j\partial_t\partial_z^2\phi\right) + \left(\partial_j\partial_z\partial_t\phi\right)\left(\partial_j\partial_t\Delta\phi\right) - 0$$
$$+ (\partial_t\Delta\phi)\left(\partial_t\partial_z^3\phi\right) - (\partial_t\Delta\phi)\left(\partial_z\partial_t\Delta\phi\right) - \left(\partial_j\partial_t\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi$$
$$-0 + \left(\partial_j\partial_t\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi - 0$$

Now eliminate or collect like terms where we can and separate into derivatives of phi-phi , phi-psi and psi-psi:-

<u>**ϕ - ϕ Terms**</u>

$$
\begin{aligned}
&= \left(\partial_i\partial_j\partial_z^2\phi\right)\left(\partial_i\partial_j\partial_z\phi\right) - \left(\partial_i\partial_z^3\phi\right)\left(\partial_i\Delta\phi\right) - \left(\partial_j\partial_z\Delta\phi\right)\left(\partial_j\partial_z^2\phi\right) + \left(\partial_z^2\Delta\phi\right)\left(\partial_z\Delta\phi\right)\\
&+ \left(\partial_i\partial_j\partial_z\phi\right)\left(\partial_i\partial_j\partial_z^2\phi\right) - \left(\partial_i\partial_z^2\phi\right)\left(\partial_i\partial_z\Delta\phi\right) - \left(\partial_j\Delta\phi\right)\left(\partial_j\partial_z^3\phi\right) + \left(\partial_z\Delta\phi\right)\left(\partial_z^2\Delta\phi\right)\\
&- \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_t^2\partial_z^2\phi\right) + \left(\partial_j\partial_z\phi\right)\left(\partial_j\partial_t^2\Delta\phi\right) + \left(\Delta\phi\right)\left(\partial_t^2\partial_z^3\phi\right) - \left(\Delta\phi\right)\left(\partial_z\partial_t^2\Delta\phi\right)\\
&- \left(\partial_j\partial_t\partial_z^2\phi\right)\left(\partial_j\partial_z\partial_t\phi\right) + \left(\partial_t\partial_z^3\phi\right)\left(\partial_t\Delta\phi\right) + \left(\partial_j\partial_t\Delta\phi\right)\left(\partial_j\partial_z\partial_t\phi\right)\\
&- \left(\partial_t\partial_z\Delta\phi\right)\left(\partial_t\Delta\phi\right) - \left(\partial_j\partial_z^2\phi\right)\left(\partial_j\partial_t^2\partial_z\phi\right) + \left(\partial_z^3\phi\right)\left(\partial_t^2\Delta\phi\right)\\
&+ \left(\partial_j\Delta\phi\right)\left(\partial_j\partial_t^2\partial_z\phi\right) - \left(\partial_z\Delta\phi\right)\left(\partial_t^2\Delta\phi\right) - \left(\partial_j\partial_z\partial_t\phi\right)\left(\partial_j\partial_t\partial_z^2\phi\right)\\
&+ \left(\partial_j\partial_t\partial_z\phi\right)\left(\partial_j\partial_t\Delta\phi\right) + \left(\partial_t\Delta\phi\right)\left(\partial_t\partial_z^3\phi\right) - \left(\partial_t\Delta\phi\right)\left(\partial_z\partial_t\Delta\phi\right)
\end{aligned}
$$

<u>ϕ - ψ Terms</u>

$$
\begin{aligned}
&= \left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi - \left(\lambda_i\partial_j\partial_z\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi\\
&+ \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi - \left(\lambda_j\partial_i\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi\\
&+ \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_z\partial_i\psi - \left(\lambda_i\partial_j\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_i\partial_z\psi\\
&+ \left(\partial_i\partial_j\partial_z^2\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi - \left(\lambda_j\partial_i\partial_z\Delta\phi\right)\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\\
&- \left(\partial_t^2\partial_z^2\partial_j\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi + \left(\partial_j\partial_t^2\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi\\
&- \left(\partial_i\partial_j\partial_t\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi + \left(\lambda_i\partial_j\partial_t\Delta\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi\\
&- \left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi + \left(\lambda_i\partial_j\Delta\phi\right)\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi\\
&- \left(\partial_j\partial_t\partial_z^2\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi + \left(\partial_j\partial_t\Delta\phi\right)\varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi
\end{aligned}
$$

<u>ψ - ψ Terms</u>

$$
\begin{aligned}
&\varepsilon_{jlm}\lambda_l\partial_m\psi\cdot\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_i\partial_z\psi(=0)\\
&+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\psi\cdot\varepsilon_{jlm}\lambda_l\partial_m\partial_i\psi\\
&+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\cdot\varepsilon_{jlm}\lambda_l\partial_m\partial_i\partial_z\psi\\
&-\varepsilon_{jlm}\lambda_l\partial_m\partial_z\psi\cdot\varepsilon_{ipq}\lambda_p\partial_q\partial_i\partial_j\psi(=0)\\
&-\varepsilon_{jlm}\lambda_l\partial_m\psi\cdot\varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t^2\psi(=0)\\
&-\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_t\psi\cdot\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t\psi(=0)\\
&-\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\cdot\varepsilon_{jlm}\lambda_l\lambda_i\partial_m\partial_t^2\psi(=0)\\
&-\varepsilon_{jlm}\lambda_l\partial_m\partial_t\psi\cdot\varepsilon_{ipq}\lambda_p\lambda_i\partial_q\partial_j\partial_t\psi(=0)
\end{aligned}
$$

Now eliminate the permutation tensors where possible.

<u>ϕ - ϕ Terms</u>

$$= -3\left(\partial_j\partial_z\Delta\phi\right)\left(\partial_j\partial_z^2\phi\right)+2\left(\partial_z\Delta\phi\right)\left(\partial_z^2\Delta\phi\right)-\left(\partial_j\partial_z\phi\right)\left(\partial_j\Delta\partial_z^2\phi\right)$$
$$+\left(\partial_j\partial_z\phi\right)\left(\partial_j\Delta\Delta\phi\right)+\left(\Delta\phi\right)\left(\Delta\partial_z^3\phi\right)-\left(\Delta\phi\right)\left(\partial_z\Delta\Delta\phi\right)+\left(\partial_z^3\phi\right)\left(\Delta\Delta\phi\right)$$
$$-\left(\partial_z\Delta\phi\right)\left(\Delta\Delta\phi\right)-\left(\partial_t\Delta\phi\right)\left(\partial_z\partial_t\Delta\phi\right)+2\left(\partial_j\partial_t\partial_z\phi\right)\left(\partial_j\partial_t\Delta\phi\right)$$

Now using note 10 we can simplify the Φ - ψ terms.

ϕ - ψ Terms

$$= \left(\partial_i\partial_y\partial_z^2\phi\right)\left(\partial_i\partial_x\psi\right)-\left(\partial_i\partial_x\partial_z^2\phi\right)\left(\partial_i\partial_y\psi\right)-\left(\partial_y\Delta\partial_z\phi\right)\left(\partial_x\partial_z\psi\right)$$
$$+\left(\partial_x\Delta\partial_z\phi\right)\left(\partial_y\partial_z\psi\right)+\left(\partial_j\partial_y\partial_z\phi\right)\left(\partial_j\partial_x\partial_z\psi\right)-\left(\partial_j\partial_x\partial_z\phi\right)\left(\partial_j\partial_y\partial_z\psi\right)$$
$$-\left(\partial_y\Delta\phi\right)\left(\partial_x\partial_z^2\psi\right)+\left(\partial_x\Delta\phi\right)\left(\partial_y\partial_z^2\psi\right)+\left(\partial_i\partial_y\partial_z\phi\right)\left(\partial_i\partial_x\partial_z\psi\right)$$
$$-\left(\partial_i\partial_x\partial_z\phi\right)\left(\partial_i\partial_y\partial_z\psi\right)-\left(\partial_y\Delta\phi\right)\left(\partial_x\partial_z^2\psi\right)+\left(\partial_x\Delta\phi\right)\left(\partial_y\partial_z^2\psi\right)$$
$$+\left(\partial_j\partial_y\partial_z^2\phi\right)\left(\partial_j\partial_x\psi\right)-\left(\partial_j\partial_x\partial_z^2\phi\right)\left(\partial_j\partial_y\psi\right)-\left(\partial_y\partial_z\Delta\phi\right)\left(\partial_x\partial_z\psi\right)$$
$$+\left(\partial_x\Delta\partial_z\phi\right)\left(\partial_y\partial_z\psi\right)-\left(\partial_y\Delta\partial_z^2\phi\right)\left(\partial_x\psi\right)+\left(\partial_x\Delta\partial_z^2\phi\right)\left(\partial_y\psi\right)+\left(\partial_y\Delta\Delta\phi\right)\left(\partial_x\psi\right)$$
$$-\left(\partial_x\Delta\Delta\phi\right)\left(\partial_y\psi\right)-\left(\partial_t\partial_y\partial_z^2\phi\right)\left(\partial_t\partial_x\psi\right)+\left(\partial_t\partial_x\partial_z^2\phi\right)\left(\partial_t\partial_y\psi\right)$$
$$+\left(\partial_t\partial_y\Delta\phi\right)\left(\partial_t\partial_x\psi\right)-\left(\partial_t\partial_x\Delta\phi\right)\left(\partial_t\partial_y\psi\right)-\left(\partial_y\partial_z^2\phi\right)\left(\partial_x\Delta\psi\right)+\left(\partial_x\partial_z^2\phi\right)\left(\partial_y\Delta\psi\right)$$
$$+\left(\partial_y\Delta\phi\right)\left(\partial_x\Delta\psi\right)-\left(\partial_x\Delta\phi\right)\left(\partial_y\Delta\psi\right)+\left(\partial_t\partial_x\partial_z^2\phi\right)\left(\partial_t\partial_y\psi\right)-\left(\partial_t\partial_y\partial_z^2\phi\right)\left(\partial_t\partial_x\psi\right)$$
$$+\left(\partial_t\partial_y\Delta\phi\right)\left(\partial_t\partial_x\psi\right)-\left(\partial_t\partial_x\Delta\phi\right)\left(\partial_t\partial_y\psi\right)$$

ψ – ψ Terms (Employing Kronecker Delta)

$$\varepsilon_{ipq}\lambda_p(\partial_q\partial_j\partial_z\psi)\cdot\varepsilon_{jlm}\lambda_l(\partial_m\partial_i\psi)$$
$$=\varepsilon_{i3q}(\partial_q\partial_j\partial_z\psi)\cdot\varepsilon_{j3m}(\partial_m\partial_i\psi)$$
$$\varepsilon_{i3q}\varepsilon_{j3m}=\varepsilon_{3iq}\varepsilon_{3jm}=\delta_{ij}\delta_{mq}-\delta_{im}\delta_{jq}$$
$$\Rightarrow\left(\delta_{ij}\delta_{mq}-\delta_{im}\delta_{jq}\right)(\partial_q\partial_j\partial_z\psi)(\partial_m\partial_i\psi)$$
$$=(\partial_q\partial_i\partial_z\psi)(\partial_q\partial_i\psi)-(\Delta\partial_z\psi)(\Delta\psi)$$

And similarly,

$$\varepsilon_{ipq}\lambda_p(\partial_q\partial_j\psi)\cdot\varepsilon_{jlm}\lambda_l(\partial_m\partial_i\partial_z\psi)$$
$$=\varepsilon_{i3q}(\partial_q\partial_j\psi)\cdot\varepsilon_{j3m}(\partial_m\partial_i\partial_z\psi)$$
$$\varepsilon_{i3q}\varepsilon_{j3m}=\varepsilon_{3iq}\varepsilon_{3jm}=\delta_{ij}\delta_{mq}-\delta_{im}\delta_{jq}\qquad \textbf{[i, q, j, m} \neq \textbf{3 so no } \delta\textbf{z]}$$
$$\Rightarrow\left(\delta_{ij}\delta_{mq}-\delta_{im}\delta_{jq}\right)(\partial_q\partial_j\psi)(\partial_m\partial_i\partial_z\psi)$$
$$=(\partial_q\partial_i\psi)(\partial_q\partial_i\partial_z\psi)-(\Delta\psi)(\Delta\partial_z\psi)$$

Combining the two results gives;

$$= 2(\partial_q \partial_i \partial_z \psi)(\partial_q \partial_i \psi) - 2(\Delta\psi)(\Delta\partial_z \psi)$$

Now we need to expand the derivatives employing Einstein's Summation Convention and simplify where possible. We shall expand the ψ - ψ terms first to illustrate the technique and because there are not many terms.

ψ - ψ Terms

$$
\begin{aligned}
&= 2(\partial_q \partial_i \partial_z \psi)(\partial_q \partial_i \psi) - 2(\Delta\psi)(\Delta\partial_z \psi) \\
&= 2\left[\left(\partial_q \partial_x \partial_z \psi\right)\left(\partial_q \partial_x \psi\right) + \left(\partial_q \partial_y \partial_z \psi\right)\left(\partial_q \partial_y \psi\right) - (\Delta\psi)(\Delta\partial_z \psi)\right] \\
&= 2(\partial_x^2 \partial_z \psi)(\partial_x^2 \psi) + 2(\partial_x \partial_y \partial_z \psi)(\partial_x \partial_y \psi) + 2(\partial_x \partial_y \partial_z \psi)(\partial_x \partial_y \psi) \\
&+2(\partial_y^2 \partial_z \psi)(\partial_y^2 \psi) \\
&-2(\partial_x^2 \psi)(\partial_x^2 \partial_z \psi) - (\partial_x^2 \psi)(\partial_y^2 \partial_z \psi) - (\partial_y^2 \psi)(\partial_x^2 \partial_z \psi) - (\partial_y^2 \psi)(\partial_y^2 \partial_z \psi)
\end{aligned}
$$

We are left with;

$$= 4(\partial_x \partial_y \partial_z \psi)(\partial_x \partial_y \psi) - 2(\partial_x^2 \psi)(\partial_y^2 \partial_z \psi) - 2(\partial_y^2 \psi)(\partial_x^2 \partial_z \psi)$$

ϕ - ϕ Terms

Now expand the terms;

$$
\begin{aligned}
&= -3\left(\partial_x \partial_z \Delta\phi\right)\left(\partial_x \partial_z^2 \phi\right) - 3\left(\partial_y \partial_z \Delta\phi\right)\left(\partial_y \partial_z^2 \phi\right) - 3\left(\partial_z^2 \Delta\phi\right)\left(\partial_z^3 \phi\right) \\
&+2\left(\partial_z \Delta\phi\right)\left(\partial_z^2 \Delta\phi\right) - \left(\partial_x \partial_z \phi\right)\left(\partial_x \Delta\partial_z^2 \phi\right) - \left(\partial_y \partial_z \phi\right)\left(\partial_y \Delta\partial_z^2 \phi\right) \\
&- \left(\partial_z^2 \phi\right)\left(\Delta\partial_z^3 \phi\right) + \left(\partial_x \partial_z \phi\right)\left(\partial_x \Delta\Delta\phi\right) + \left(\partial_y \partial_z \phi\right)\left(\partial_y \Delta\Delta\phi\right) + \left(\partial_z^2 \phi\right)\left(\Delta\Delta\partial_z \phi\right) \\
&+ \left(\Delta\phi\right)\left(\Delta\partial_z^3 \phi\right) + \left(\partial_z^3 \phi\right)\left(\Delta\Delta\phi\right) - \left(\partial_x \Delta\phi\right)\left(\partial_x \partial_z \Delta\phi\right) - \left(\partial_y \Delta\phi\right)\left(\partial_y \partial_z \Delta\phi\right) \\
&- \left(\partial_z \Delta\phi\right)\left(\Delta\partial_z^2 \phi\right) - \left(\Delta\phi\right)\left(\Delta\Delta\partial_z \phi\right) - \left(\partial_z \Delta\phi\right)\left(\Delta\Delta\phi\right) \\
&+2\left(\partial_j \partial_t \partial_z \phi\right)\left(\partial_j \partial_t \Delta\phi\right)
\end{aligned}
$$

Now remove the del operators;

Note:

$$\Delta\Delta = (\partial_i^2)^2 = \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)^2 = \partial_x^4 + 2\partial_x^2 \partial_y^2 + 2\partial_x^2 \partial_z^2 + 2\partial_y^2 \partial_z^2 + \partial_y^4 + \partial_z^4$$

So,

$$
\begin{aligned}
&= \\
&-3\left(\partial_x^3\partial_z\phi\right)\left(\partial_x\partial_z^2\phi\right)-3\left(\partial_x\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_z^2\phi\right)-3\left(\partial_x\partial_z^3\phi\right)\left(\partial_x\partial_z^2\phi\right) \\
&-3\left(\partial_x^2\partial_y\partial_z\phi\right)\left(\partial_y\partial_z^2\phi\right)-3\left(\partial_y^3\partial_z\phi\right)\left(\partial_y\partial_z^2\phi\right)-3\left(\partial_y\partial_z^3\phi\right)\left(\partial_y\partial_z^2\phi\right) \\
&-3\left(\partial_x^2\partial_z^2\phi\right)\left(\partial_z^3\phi\right)-3\left(\partial_y^2\partial_z^2\phi\right)\left(\partial_z^3\phi\right)-3\left(\partial_z^4\phi\right)\left(\partial_z^3\phi\right) \\
&+2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)+2\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)+2\left(\partial_x^2\partial_z\phi\right)\left(\partial_z^4\phi\right) \\
&+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_z^4\phi\right) \\
&+2\left(\partial_z^3\phi\right)\left(\partial_x^2\partial_z^2\phi\right)+2\left(\partial_z^3\phi\right)\left(\partial_y^2\partial_z^2\phi\right)+2\left(\partial_z^3\phi\right)\left(\partial_z^4\phi\right) \\
&-\left(\partial_x\partial_z\right)\left(\partial_x^3\partial_z^2\phi\right)-\left(\partial_x\partial_z\right)\left(\partial_x\partial_y^2\partial_z^2\phi\right)-\left(\partial_x\partial_z\right)\left(\partial_x\partial_z^4\phi\right) \\
&-\left(\partial_y\partial_z\right)\left(\partial_x^2\partial_y\partial_z^2\phi\right)-\left(\partial_y\partial_z\right)\left(\partial_y^3\partial_z^2\phi\right)-\left(\partial_y\partial_z\right)\left(\partial_y\partial_z^4\phi\right) \\
&-\left(\partial_z^2\right)\left(\partial_x^2\partial_z^3\phi\right)-\left(\partial_z^2\right)\left(\partial_y^2\partial_z^3\phi\right)-\left(\partial_z^2\right)\left(\partial_z^5\phi\right) \\
&+\left(\partial_x\partial_z\right)\left(\partial_x^5\phi\right)+\left(\partial_x\partial_z\right)\left(\partial_x\partial_y^4\phi\right)+\left(\partial_x\partial_z\right)\left(\partial_x\partial_z^4\phi\right) \\
&+2\left(\partial_x\partial_z\right)\left(\partial_x^3\partial_y^2\phi\right)+2\left(\partial_x\partial_z\right)\left(\partial_x^3\partial_z^2\phi\right)+2\left(\partial_x\partial_z\right)\left(\partial_x\partial_y^2\partial_z^2\phi\right) \\
&+\left(\partial_y\partial_z\right)\left(\partial_x^4\partial_y\phi\right)+\left(\partial_y\partial_z\right)\left(\partial_y^5\phi\right)+\left(\partial_y\partial_z\right)\left(\partial_y\partial_z^4\phi\right) \\
&+2\left(\partial_y\partial_z\right)\left(\partial_x^2\partial_y^3\phi\right)+2\left(\partial_y\partial_z\right)\left(\partial_x^2\partial_y\partial_z^2\phi\right)+2\left(\partial_y\partial_z\right)\left(\partial_y^3\partial_z^2\phi\right) \\
&+\left(\partial_z^2\phi\right)\left(\partial_x^4\partial_z\phi\right)+\left(\partial_z^2\phi\right)\left(\partial_y^4\partial_z\phi\right)+\left(\partial_z^2\phi\right)\left(\partial_z^5\phi\right) \\
&+2\left(\partial_z^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)+2\left(\partial_z^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)+2\left(\partial_z^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right) \\
&+\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)+\left(\partial_x^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)+\left(\partial_x^2\phi\right)\left(\partial_z^5\phi\right) \\
&+\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)+\left(\partial_y^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)+\left(\partial_y^2\phi\right)\left(\partial_z^5\phi\right) \\
&+\left(\partial_z^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)+\left(\partial_z^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)+\left(\partial_z^2\phi\right)\left(\partial_z^5\phi\right) \\
&+\left(\partial_z^3\phi\right)\left(\partial_x^4\phi\right)+\left(\partial_z^3\phi\right)\left(\partial_y^4\phi\right)+\left(\partial_z^3\phi\right)\left(\partial_z^4\phi\right) \\
&+2\left(\partial_z^3\phi\right)\left(\partial_x^2\partial_y^2\phi\right)+2\left(\partial_z^3\phi\right)\left(\partial_x^2\partial_z^2\phi\right)+2\left(\partial_z^3\phi\right)\left(\partial_y^2\partial_z^2\phi\right) \\
&-\left(\partial_x^3\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_z^3\phi\right) \\
&-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_z^3\phi\right) \\
&-\left(\partial_x\partial_z^2\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x\partial_z^2\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x\partial_z^2\phi\right)\left(\partial_x\partial_z^3\phi\right) \\
&-\left(\partial_x^2\partial_y\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y\partial_z^3\phi\right) \\
&-\left(\partial_y^3\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_y\partial_z^3\phi\right) \\
&-\left(\partial_y\partial_z^2\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_y\partial_z^2\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_y\partial_z^2\phi\right)\left(\partial_y\partial_z^3\phi\right) \\
&-\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_z^4\phi\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_z^4\phi\right) \\
&-\left(\partial_z^3\phi\right)\left(\partial_x^2\partial_z^2\phi\right)-\left(\partial_z^3\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-\left(\partial_z^3\phi\right)\left(\partial_z^4\phi\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\partial_x^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^4\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_z^5\phi\right)\\
&-2\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-2\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-2\left(\partial_x^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)\\
&-\left(\partial_y^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_y^4\partial_z\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_z^5\phi\right)\\
&-2\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-2\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-2\left(\partial_y^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)\\
&-\left(\partial_z^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_z^2\phi\right)\left(\partial_y^4\partial_z\phi\right)-\left(\partial_z^2\phi\right)\left(\partial_z^5\phi\right)\\
&-2\left(\partial_z^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-2\left(\partial_z^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-2\left(\partial_z^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^4\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^4\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_z^4\phi\right)\\
&-2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_y^2\phi\right)-2\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)\\
&-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^4\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^4\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_z^4\phi\right)\\
&-2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_y^2\phi\right)-2\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)\\
&-\left(\partial_z^3\phi\right)\left(\partial_x^4\phi\right)-\left(\partial_z^3\phi\right)\left(\partial_y^4\phi\right)-\left(\partial_z^3\phi\right)\left(\partial_z^4\phi\right)\\
&-2\left(\partial_z^3\phi\right)\left(\partial_x^2\partial_y^2\phi\right)-2\left(\partial_z^3\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-2\left(\partial_z^3\phi\right)\left(\partial_x^2\partial_z^2\phi\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^4\phi\right)+2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_y^2\phi\right)+2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)\\
&+2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^3\partial_y\phi\right)+2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x\partial_y^3\phi\right)+2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x\partial_y\partial_z^2\phi\right)\\
&+2\left(\partial_x\partial_z^2\phi\right)\left(\partial_x^3\partial_z\phi\right)+2\left(\partial_x\partial_z^2\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)+2\left(\partial_x\partial_z^2\phi\right)\left(\partial_x\partial_z^3\phi\right)\\
&+2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^3\partial_y\phi\right)+2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x\partial_y^3\phi\right)+2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x\partial_y\partial_z^2\phi\right)\\
&+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_y^2\phi\right)+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^4\phi\right)+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)\\
&+2\left(\partial_y\partial_z^2\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)+2\left(\partial_y\partial_z^2\phi\right)\left(\partial_y^3\partial_z\phi\right)+2\left(\partial_y\partial_z^2\phi\right)\left(\partial_y\partial_z^3\phi\right)\\
&+2\left(\partial_x\partial_z^2\phi\right)\left(\partial_x^3\partial_z\phi\right)+2\left(\partial_x\partial_z^2\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)+2\left(\partial_x\partial_z^2\phi\right)\left(\partial_x\partial_z^3\phi\right)\\
&+2\left(\partial_y\partial_z^2\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)+2\left(\partial_y\partial_z^2\phi\right)\left(\partial_y^3\partial_z\phi\right)+2\left(\partial_y\partial_z^2\phi\right)\left(\partial_y\partial_z^3\phi\right)\\
&+2\left(\partial_z^3\phi\right)\left(\partial_x^2\partial_z^2\phi\right)+2\left(\partial_z^3\phi\right)\left(\partial_y^2\partial_z^2\phi\right)+2\left(\partial_z^3\phi\right)\left(\partial_z^4\phi\right)
\end{aligned}
$$

ϕ - ϕ Terms

Collect and delete terms:-

$$=$$
$$+(\partial_x\partial_z\phi)\left(\partial_x^5\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x\partial_y^4\phi\right)$$
$$+2(\partial_x\partial_z\phi)\left(\partial_x^3\partial_y^2\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x^3\partial_z^2\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x\partial_y^2\partial_z^2\phi\right)$$
$$+(\partial_y\partial_z\phi)\left(\partial_x^4\partial_y\phi\right)+(\partial_y\partial_z\phi)\left(\partial_y^5\phi\right)$$
$$+2(\partial_y\partial_z\phi)\left(\partial_x^2\partial_y^3\phi\right)+(\partial_y\partial_z\phi)\left(\partial_x^2\partial_y\partial_z^2\phi\right)+(\partial_y\partial_z\phi)\left(\partial_y^3\partial_z^2\phi\right)$$
$$-\left(\partial_x^3\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_z^3\phi\right)$$
$$-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_z^3\phi\right)$$
$$-\left(\partial_x^2\partial_y\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y\partial_z^3\phi\right)$$
$$-\left(\partial_y^3\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_y\partial_z^3\phi\right)$$

$$-\left(\partial_x^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^4\partial_z\phi\right)$$
$$-2\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)$$
$$-\left(\partial_y^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_y^4\partial_z\phi\right)$$
$$-2\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)$$

$$-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^4\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^4\phi\right)+\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)$$
$$-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)$$

$$+\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^4\phi\right)+\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)$$
$$+4(\partial_x\partial_y\partial_z\phi)\left(\partial_x^3\partial_y\phi\right)+4(\partial_x\partial_y\partial_z\phi)\left(\partial_x\partial_y^3\phi\right)+4(\partial_x\partial_y\partial_z\phi)\left(\partial_x\partial_y\partial_z^2\phi\right)$$
$$+\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^4\phi\right)$$

ϕ - ψ Terms

$$
\begin{aligned}
&= \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right) - \left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) + \left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) - \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right)\\
&+ \left(\partial_y\partial_z^3\phi\right)\left(\partial_x\partial_z\psi\right) - \left(\partial_x\partial_z^3\phi\right)\left(\partial_y\partial_z\psi\right) - \left(\partial_y\Delta\partial_z\phi\right)\left(\partial_x\partial_z\psi\right) + \left(\partial_x\Delta\partial_z\phi\right)\left(\partial_y\partial_z\psi\right)\\
&+ \left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_z\psi\right) - \left(\partial_x^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) + \left(\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right)\\
&- \left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_y^2\partial_z\psi\right) + \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_z^2\psi\right) - \left(\partial_x\partial_z^2\phi\right)\left(\partial_y\partial_z^2\psi\right) - \left(\partial_y\Delta\phi\right)\left(\partial_x\partial_z^2\psi\right)\\
&+ \left(\partial_x\Delta\phi\right)\left(\partial_y\partial_z^2\psi\right) + \left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_z\psi\right) - \left(\partial_x^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) + \left(\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right)\\
&- \left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_y^2\partial_z\psi\right) + \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_z^2\psi\right) - \left(\partial_x\partial_z^2\phi\right)\left(\partial_y\partial_z^2\psi\right) - \left(\partial_y\Delta\phi\right)\left(\partial_x\partial_z^2\psi\right)\\
&+ \left(\partial_x\Delta\phi\right)\left(\partial_y\partial_z^2\psi\right) + \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right) - \left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) + \left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right)\\
&- \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right) + \left(\partial_y\partial_z^3\phi\right)\left(\partial_x\partial_z\psi\right) - \left(\partial_x\partial_z^3\phi\right)\left(\partial_y\partial_z\psi\right) - \left(\partial_y\partial_z\Delta\phi\right)\left(\partial_x\partial_z\psi\right)\\
&+ \left(\partial_x\Delta\partial_z\phi\right)\left(\partial_y\partial_z\psi\right) - \left(\partial_y\Delta\partial_z^2\phi\right)\left(\partial_x\psi\right) + \left(\partial_x\Delta\partial_z^2\phi\right)\left(\partial_y\psi\right) + \left(\partial_y\Delta\Delta\phi\right)\left(\partial_x\psi\right)\\
&- \left(\partial_x\Delta\Delta\phi\right)\left(\partial_y\psi\right) - \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right) + \left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) - \left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right)\\
&+ \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right) - \left(\partial_y\partial_z^3\phi\right)\left(\partial_x\partial_z\psi\right) + \left(\partial_x\partial_z^3\phi\right)\left(\partial_y\partial_z\psi\right)\\
&+ \left(\partial_x\partial_y\Delta\phi\right)\left(\partial_x^2\psi\right) - \left(\partial_x^2\Delta\phi\right)\left(\partial_x\partial_y\psi\right) + \left(\partial_y^2\Delta\psi\right)\left(\partial_x\partial_y\phi\right) - \left(\partial_x\partial_y\Delta\phi\right)\left(\partial_y^2\psi\right)\\
&+ \left(\partial_y\Delta\partial_z\phi\right)\left(\partial_x\partial_z\psi\right) - \left(\partial_x\Delta\partial_z\phi\right)\left(\partial_y\partial_z\psi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\Delta\psi\right) + \left(\partial_x\partial_z^2\phi\right)\left(\partial_y\Delta\psi\right)\\
&+ \left(\partial_y\Delta\phi\right)\left(\partial_x\Delta\psi\right) - \left(\partial_x\Delta\phi\right)\left(\partial_y\Delta\psi\right) - \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right) + \left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right)\\
&- \left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) + \left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right) - \left(\partial_y\partial_z^3\phi\right)\left(\partial_x\partial_z\psi\right) + \left(\partial_x\partial_z^3\phi\right)\left(\partial_y\partial_z\psi\right)\\
&+ \left(\partial_x\partial_y\Delta\phi\right)\left(\partial_x^2\psi\right) - \left(\partial_x^2\Delta\phi\right)\left(\partial_x\partial_y\psi\right) + \left(\partial_y^2\Delta\phi\right)\left(\partial_x\partial_y\psi\right) - \left(\partial_x\partial_y\Delta\phi\right)\left(\partial_y^2\psi\right)\\
&+ \left(\partial_y\Delta\partial_z\phi\right)\left(\partial_x\partial_z\psi\right) - \left(\partial_x\Delta\partial_z\phi\right)\left(\partial_y\partial_z\psi\right)
\end{aligned}
$$

Tidy up.

$$
\begin{aligned}
&= 2\left(\partial_x\Delta\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_z\psi\right) - 2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right)\\
&+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) - 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_y^2\partial_z\psi\right) + 2\left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_z^2\psi\right)\\
&-2\left(\partial_x\partial_z^2\phi\right)\left(\partial_y\partial_z^2\psi\right) - 2\left(\partial_y\Delta\phi\right)\left(\partial_x\partial_z^2\psi\right) - \left(\partial_y\Delta\partial_z^2\phi\right)\left(\partial_x\psi\right)\\
&+ \left(\partial_x\Delta\partial_z^2\phi\right)\left(\partial_y\psi\right) + \left(\partial_y\Delta\Delta\phi\right)\left(\partial_x\psi\right) - \left(\partial_x\Delta\Delta\phi\right)\left(\partial_y\psi\right)\\
&- \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\Delta\psi\right) + \left(\partial_x\partial_z^2\phi\right)\left(\partial_y\Delta\psi\right) + \left(\partial_y\Delta\phi\right)\left(\partial_x\Delta\psi\right)\\
&- \left(\partial_x\Delta\phi\right)\left(\partial_y\Delta\psi\right) + 2\left(\partial_x\partial_y\Delta\phi\right)\left(\partial_x^2\psi\right) - 2\left(\partial_x^2\Delta\phi\right)\left(\partial_x\partial_y\psi\right)\\
&+2\left(\partial_y^2\Delta\phi\right)\left(\partial_x\partial_y\psi\right) - 2\left(\partial_x\partial_y\Delta\phi\right)\left(\partial_y^2\psi\right)
\end{aligned}
$$

Now expand the del and del-del terms;

$$= 2\left(\partial_x^3\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_y^2\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_z^2\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_z\psi\right)$$
$$-2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) + 2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) - 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_y^2\partial_z\psi\right)$$
$$+2\left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_z^2\psi\right) - 2\left(\partial_x\partial_z^2\phi\right)\left(\partial_y\partial_z^2\psi\right) - 2\left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_z^2\psi\right) - 2\left(\partial_y^3\phi\right)\left(\partial_x\partial_z^2\psi\right)$$
$$-2\left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_z^2\psi\right) - \left(\partial_x^2\partial_y\partial_z^2\phi\right)\left(\partial_x\psi\right) - \left(\partial_y^3\partial_z^2\phi\right)\left(\partial_x\psi\right)$$
$$-\left(\partial_y\partial_z^4\phi\right)\left(\partial_x\psi\right) + \left(\partial_x^3\partial_z^2\phi\right)\left(\partial_y\psi\right) + \left(\partial_x\partial_y^2\partial_z^2\phi\right)\left(\partial_y\psi\right)$$
$$+\left(\partial_x^4\partial_y\phi\right)\left(\partial_x\psi\right) + \left(\partial_y^5\phi\right)\left(\partial_x\psi\right) + \left(\partial_y\partial_z^4\phi\right)\left(\partial_x\psi\right) + 2\left(\partial_x^2\partial_y^3\phi\right)\left(\partial_x\psi\right) + \left(\partial_x\partial_z^4\phi\right)\left(\partial_y\psi\right)$$
$$+2\left(\partial_x^2\partial_y\partial_z^2\phi\right)\left(\partial_x\psi\right) + 2\left(\partial_y^3\partial_z^2\phi\right)\left(\partial_x\psi\right) - \left(\partial_x^5\phi\right)\left(\partial_y\psi\right) - \left(\partial_x\partial_y^4\phi\right)\left(\partial_y\psi\right)$$
$$-\left(\partial_x\partial_z^4\phi\right)\left(\partial_y\psi\right) - 2\left(\partial_x^3\partial_y^2\phi\right)\left(\partial_y\psi\right) - 2\left(\partial_x^3\partial_z^2\phi\right)\left(\partial_y\psi\right) - 2\left(\partial_x\partial_y^2\partial_z^2\phi\right)\left(\partial_y\psi\right)$$

$$-\left(\partial_y\partial_z^2\phi\right)\left(\partial_x^3\psi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_y^2\psi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_z^2\psi\right) + \left(\partial_x\partial_z^2\phi\right)\left(\partial_x^2\partial_y\psi\right)$$
$$+\left(\partial_x\partial_z^2\phi\right)\left(\partial_y^3\psi\right) + \left(\partial_x\partial_z^2\phi\right)\left(\partial_y\partial_z^2\psi\right) - \left(\partial_x^3\phi\right)\left(\partial_x^2\partial_y\psi\right) - \left(\partial_x^3\phi\right)\left(\partial_y^3\psi\right)$$
$$-\left(\partial_x^3\phi\right)\left(\partial_y\partial_z^2\psi\right) - \left(\partial_x\partial_y^2\phi\right)\left(\partial_y^3\psi\right) - \left(\partial_x\partial_y^2\phi\right)\left(\partial_x^2\partial_y\psi\right) - \left(\partial_x\partial_z^2\phi\right)\left(\partial_x^2\partial_y\psi\right)$$
$$-\left(\partial_x\partial_z^2\phi\right)\left(\partial_y^3\psi\right) - \left(\partial_x\partial_z^2\phi\right)\left(\partial_y\partial_z^2\psi\right) + \left(\partial_x^2\partial_y\phi\right)\left(\partial_x^3\psi\right) + \left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_y^2\psi\right)$$
$$+\left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_z^2\psi\right) + \left(\partial_y^3\phi\right)\left(\partial_x^3\psi\right) + \left(\partial_y^3\phi\right)\left(\partial_x\partial_y^2\psi\right) + \left(\partial_y^3\phi\right)\left(\partial_x\partial_z^2\psi\right)$$
$$+\left(\partial_y\partial_z^2\phi\right)\left(\partial_x^3\psi\right) + \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_y^2\psi\right) + \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_z^2\psi\right)$$
$$+2\left(\partial_x^3\partial_y\phi\right)\left(\partial_x^2\psi\right) + 2\left(\partial_x\partial_y^3\phi\right)\left(\partial_x^2\psi\right) + 2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right) - 2\left(\partial_x^4\phi\right)\left(\partial_x\partial_y\psi\right)$$
$$-2\left(\partial_x^2\partial_y^2\phi\right)\left(\partial_x\partial_y\psi\right) - 2\left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) + 2\left(\partial_x^2\partial_y^2\phi\right)\left(\partial_x\partial_y\psi\right) + 2\left(\partial_y^4\phi\right)\left(\partial_x\partial_y\psi\right)$$
$$+2\left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) - 2\left(\partial_x^3\partial_y\phi\right)\left(\partial_y^2\psi\right) - 2\left(\partial_x\partial_y^3\phi\right)\left(\partial_y^2\psi\right) - 2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right)$$

Now collect terms and cancel down;

$\phi - \psi$ Terms

$$= \left(\partial_x^3\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_y^2\phi\right)\left(\partial_y\partial_z^2\psi\right) + 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_z\psi\right)$$
$$-2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) + 2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right) - 2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_y^2\partial_z\psi\right)$$
$$-\left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_z^2\psi\right) - \left(\partial_y^3\phi\right)\left(\partial_x\partial_z^2\psi\right) + \left(\partial_x^4\partial_y\phi\right)\left(\partial_x\psi\right) + \left(\partial_y^5\phi\right)\left(\partial_x\psi\right)$$
$$+2\left(\partial_x^2\partial_y^3\phi\right)\left(\partial_x\psi\right) + \left(\partial_x^2\partial_y\partial_z^2\phi\right)\left(\partial_x\psi\right) + \left(\partial_y^3\partial_z^2\phi\right)\left(\partial_x\psi\right) - \left(\partial_x^5\phi\right)\left(\partial_y\psi\right)$$
$$-\left(\partial_x\partial_y^4\phi\right)\left(\partial_y\psi\right) - 2\left(\partial_x^3\partial_y^2\phi\right)\left(\partial_y\psi\right) - \left(\partial_x^3\partial_z^2\phi\right)\left(\partial_y\psi\right) - \left(\partial_x\partial_y^2\partial_z^2\phi\right)\left(\partial_y\psi\right)$$

$$-\left(\partial_x^3\phi\right)\left(\partial_x^2\partial_y\psi\right) - \left(\partial_x^3\phi\right)\left(\partial_y^3\psi\right) - \left(\partial_x\partial_y^2\phi\right)\left(\partial_y^3\psi\right) - \left(\partial_x\partial_y^2\phi\right)\left(\partial_x^2\partial_y\psi\right)$$
$$+\left(\partial_x^2\partial_y\phi\right)\left(\partial_x^3\psi\right) + \left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_y^2\psi\right) + \left(\partial_y^3\phi\right)\left(\partial_x^3\psi\right) + \left(\partial_y^3\phi\right)\left(\partial_x\partial_y^2\psi\right)$$
$$+2\left(\partial_x^3\partial_y\phi\right)\left(\partial_x^2\psi\right) + 2\left(\partial_x\partial_y^3\phi\right)\left(\partial_x^2\psi\right) + 2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right) - 2\left(\partial_x^4\phi\right)\left(\partial_x\partial_y\psi\right)$$
$$-2\left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) + 2\left(\partial_y^4\phi\right)\left(\partial_x\partial_y\psi\right) + 2\left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right) - 2\left(\partial_x^3\partial_y\phi\right)\left(\partial_y^2\psi\right)$$
$$-2\left(\partial_x\partial_y^3\phi\right)\left(\partial_y^2\psi\right) - 2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right)$$

Finding Curl of Non-Linear N.S. Terms

Psi Equation.

$$
\begin{aligned}
&\tilde{\varepsilon}\cdot\left(\tilde{u}\cdot\tilde{\nabla}\tilde{u}\right)\\
&=\tilde{\varepsilon}\cdot\left[\left(\tilde{\delta}\phi+\tilde{\varepsilon}\psi\right)\cdot\tilde{\nabla}\left(\tilde{\delta}\phi+\tilde{\varepsilon}\psi\right)\right]\\
&=\varepsilon_i\left(u_j\nabla_j u_i\right)\\
&=\varepsilon_i\left[\left(\delta_j\phi+\varepsilon_j\psi\right)\nabla_j\left(\delta_i\phi+\varepsilon_i\psi\right)\right]\\
&=\varepsilon_i\left\{\left(\delta_j\phi+\varepsilon_j\psi\right)\partial_j\left[\left(\partial_i\partial_z-\lambda_i\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\psi\right]\right\}\\
&=\varepsilon_i\left\{\left(\delta_j\phi+\varepsilon_j\psi\right)\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}\\
&=\varepsilon_i\left\{\left[\left(\partial_j\partial_z-\lambda_j\Delta\right)\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}\\
&=\left(\varepsilon_{irs}\partial_s\lambda_r\right)\left\{\left[\left(\partial_j\partial_z-\lambda_j\Delta\right)\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&=\left[\partial_j\partial_z\phi-\lambda_j\Delta\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\varepsilon_{irs}\lambda_r\partial_s\left(\partial_j\partial_i\partial_z-\lambda_i\partial_j\Delta\right)\phi+\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\\
&+\left[\partial_j\partial_i\partial_z\phi-\lambda_i\partial_j\Delta\phi+\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi\right]\quad\left[\varepsilon_{irs}\lambda_r\partial_s\left(\partial_j\partial_z\phi-\lambda_j\Delta\phi\right)+\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\\
&=\left[\partial_j\partial_z\phi-\lambda_j\Delta\phi+\varepsilon_{jlm}\lambda_l\partial_m\psi\right]\quad\left[\varepsilon_{irs}\lambda_r\partial_i\partial_j\partial_s\partial_z-0+\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\right]\\
&+\left[\partial_i\partial_j\partial_z\phi-\lambda_i\partial_j\Delta\phi+\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\right]\quad\left[\varepsilon_{irs}\lambda_r\partial_j\partial_s\partial_z\phi-\varepsilon_{irs}\lambda_r\lambda_j\partial_s\Delta\phi+\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{jlm}\lambda_l\partial_m\psi\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\partial_j\partial_z\phi\right)\left(\varepsilon_{irs}\lambda_r\partial_i\partial_j\partial_s\partial_z\phi\right)+\left(\partial_j\partial_z\phi\right)\left(\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\right)\\
&-\left(\lambda_j\Delta\phi\right)\left(\varepsilon_{irs}\lambda_r\partial_i\partial_j\partial_s\partial_z\phi\right)-\left(\lambda_j\Delta\phi\right)\left(\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\right)\\
&+\left(\varepsilon_{jlm}\lambda_l\partial_m\psi\varepsilon_{irs}\lambda_r\partial_i\partial_j\partial_s\partial_z\phi\right)+\left(\varepsilon_{jlm}\lambda_l\partial_m\psi\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\right)\\
&+\left(\partial_i\partial_j\partial_z\phi\right)\left(\varepsilon_{irs}\lambda_r\partial_j\partial_s\partial_z\phi\right)-\left(\partial_i\partial_j\partial_z\phi\right)\left(\varepsilon_{irs}\lambda_r\lambda_j\partial_s\Delta\phi\right)\\
&+\left(\partial_i\partial_j\partial_z\phi\right)\left(\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{jlm}\lambda_l\partial_m\psi\right)+\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\varepsilon_{irs}\lambda_r\partial_j\partial_s\partial_z\phi\\
&-\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\varepsilon_{irs}\lambda_r\lambda_j\partial_s\Delta\phi+\left(\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{jlm}\lambda_l\partial_m\psi\right)
\end{aligned}
$$

Another, result of interest;

$$\tilde{\partial}\cdot\tilde{\varepsilon}=\left(\partial_i\partial_z-\lambda_i\Delta\right)\varepsilon_{ijk}\partial_j\lambda_k=\partial_i\partial_z\varepsilon_{ijk}\partial_j\lambda_k-\Delta\varepsilon_{ijk}\lambda_k\lambda_i\partial_j=0$$

We are combining symmetric and anti-symmetric tensors, which eliminate each other.

Kronecker Delta – Notes

δ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$

I.e. $\delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

One Index Contracted

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Two Indices Contracted

$$\varepsilon_{ijk}\varepsilon_{ijn} = \delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj} = 3\delta_{kn} - \delta_{kn} = 2_{kn}$$

Three Indices Contracted

$$\varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{kk} = 6$$

Also, recall the fact that; $\varepsilon_{ijk} = -\varepsilon_{ikj}$ antisymmetric j $\neq$ k.

Another result that we may require is;

$$\begin{aligned} &\varepsilon_{ijk}\varepsilon_{ilm}\varepsilon_{jlm} \\ &= \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)\varepsilon_{jlm} \\ &= \delta_{jl}\delta_{km}\varepsilon_{jlm} - \delta_{jm}\delta_{kl}\varepsilon_{jlm} \\ &= \varepsilon_{llk} - \varepsilon_{mkm} \\ &= 0 \end{aligned}$$

Now we shall extract and expand the terms.

$\phi - \phi$ Terms

$$\left(\partial_j\partial_z\phi\right)\varepsilon_{irs}\lambda_r\partial_i\partial_j\partial_s\partial_z\phi = \left(\partial_j\partial_z\phi\right)\left[\varepsilon_{132}\partial_x\partial_j\partial_y\partial_z\phi + \varepsilon_{231}\partial_y\partial_j\partial_x\partial_z\phi\right] = 0$$

$$-\left(\lambda_j\Delta\phi\right)\left[\varepsilon_{irs}\lambda_r\partial_i\partial_j\partial_s\partial_z\phi\right] = -\left(\Delta\phi\right)\left[\varepsilon_{132}\partial_x\partial_y\partial_z^2\phi + \varepsilon_{231}\partial_y\partial_x\partial_z^2\phi\right] = 0$$

$$\left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{irs}\lambda_r\partial_j\partial_s\partial_z\phi = \left(\partial_i\partial_j\partial_z\phi\right)\left[\varepsilon_{132}\partial_j\partial_y\partial_z\phi + \varepsilon_{231}\partial_j\partial_x\partial_z\phi\right]$$
$$= \left(\partial_j\partial_y\partial_z\phi\right)\left(\partial_j\partial_x\partial_z\phi\right) - \left(\partial_j\partial_x\partial_z\phi\right)\left(\partial_j\partial_y\partial_z\phi\right) = 0$$

$$-\left(\partial_i\partial_z^2\phi\right)\left[\varepsilon_{132}\partial_y\Delta\phi + \varepsilon_{231}\partial_x\Delta\phi\right] = \left(\partial_x\partial_z^2\phi\right)\left(\partial_y\Delta\phi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\Delta\phi\right)$$
$$= \left(\partial_x\partial_z^2\phi\right)\left(\partial_x^2\partial_y\phi\right) + \left(\partial_x\partial_z^2\phi\right)\left(\partial_y^3\phi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x^3\phi\right) - \left(\partial_y\partial_z^2\phi\right)\left(\partial_x\partial_y^2\phi\right)$$

$\psi - \psi$ Terms

$$\varepsilon_{jlm}\lambda_l\partial_m\psi\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi$$
$$= \varepsilon_{j3m}\partial_m\psi\varepsilon_{i3s}\partial_s\varepsilon_{i3q}\partial_j\partial_q\psi$$
$$= \varepsilon_{3jm}\partial_m\psi\varepsilon_{3is}\partial_s\varepsilon_{3iq}\partial_j\partial_q\psi$$
$$= \left[\left(\delta_{ij}\delta_{ms} - \delta_{js}\delta_{im}\right)\left(\partial_m\psi\right)\left(\partial_s\right)\right]\varepsilon_{i3q}\partial_j\partial_q\psi$$
$$= \left(\Delta\psi\right)\varepsilon_{3iq}\partial_i\partial_q\psi - \left(\partial_i\partial_j\psi\right)\varepsilon_{3iq}\partial_j\partial_q\psi$$
$$= 0 - \left(\partial_i\partial_j\psi\right)\left[\varepsilon_{312}\partial_y\partial_j\psi + \varepsilon_{321}\partial_x\partial_j\psi\right]$$
$$= -\left(\partial_x\partial_j\psi\right)\left(\partial_y\partial_j\psi\right) + \left(\partial_y\partial_j\psi\right)\left(\partial_x\partial_j\psi\right)$$
$$= 0$$

$$\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{jlm}\lambda_l\partial_m\psi$$
$$= \left[\left(\delta_{sq}\right)\partial_j\partial_q\partial_s\psi\right]\varepsilon_{jlm}\lambda_l\partial_m\psi$$
$$= \left(\partial_j\Delta\psi\right)\varepsilon_{jlm}\lambda_l\partial_m\psi$$
$$= \left(\partial_j\Delta\psi\right)\left[\varepsilon_{231}\partial_x\psi + \varepsilon_{132}\partial_y\psi\right] \qquad [\text{ Note: j} \neq 3]$$
$$= \left(\partial_y\Delta\psi\right)\left(\partial_x\psi\right) - \left(\partial_x\Delta\psi\right)\left(\partial_y\psi\right)$$
$$= \left(\partial_x^2\partial_y\psi\right)\left(\partial_x\psi\right) + \left(\partial_y^3\psi\right)\left(\partial_x\psi\right)$$
$$- \left(\partial_x^3\psi\right)\left(\partial_y\psi\right) - \left(\partial_x\partial_y^2\psi\right)\left(\partial_y\psi\right)$$

$\phi - \psi$ Terms

$$- (\lambda_j \Delta\phi) [\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_q\partial_j\psi]$$
$$= - (\Delta\phi) [\varepsilon_{i3s}\partial_s\varepsilon_{i3q}\partial_q\partial_z\psi]$$
$$= - (\Delta\phi) [\varepsilon_{3is}\partial_s\varepsilon_{3iq}\partial_q\partial_z\psi]$$
$$= - (\Delta\phi) [(\delta_{ii}\delta_{sq} - \delta_{3q}\delta_{3s}) \partial_q\partial_z] \quad \text{(Two Indices Contracted)}$$
$$\delta_{ii} = \delta_{11} + \delta_{22} = 2$$
$$= - (\Delta\phi) [(2\delta_{sq} - \delta_{sq}) \partial_q\partial_z]$$
$$= - (\Delta\phi) [(\delta_{sq}) \partial_s\partial_q\partial_z\psi]$$
$$= - (\Delta\phi) (\Delta\partial_z\psi)$$

Next;

$$= - (\partial_x^2\phi) (\partial_x^2\partial_z\psi) - (\partial_y^2\phi) (\partial_y^2\partial_z\psi) - (\partial_x^2\phi) (\partial_y^2\partial_z\psi) - (\partial_y^2\phi) (\partial_x^2\partial_z\psi)$$

$$- (\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi) (\varepsilon_{irs}\lambda_r\lambda_j\partial_s\Delta\phi)$$
$$= - (\varepsilon_{ipq}\lambda_p\partial_q\partial_z\psi) (\varepsilon_{irs}\lambda_r\partial_s\Delta\phi)$$
$$= - [\varepsilon_{i3q}\partial_q\partial_z\psi (\varepsilon_{i3s}\partial_s\Delta\phi)]$$
$$= - [(\delta_{qs}) \partial_q\partial_z\psi (\partial_s\Delta\phi)]$$
$$= - (\partial_q\partial_z\psi) (\partial_q\Delta\phi)$$
$$= - (\partial_x\partial_z\psi) (\partial_x\Delta\phi) - (\partial_y\partial_z\psi) (\partial_y\Delta\phi)$$
$$= - (\partial_x\partial_z\psi) (\partial_x^3\phi) - (\partial_x\partial_z\psi) (\partial_x\partial_y^2\phi) - (\partial_y\partial_z\psi) (\partial_x^2\partial_y\phi) - (\partial_y\partial_z\psi) (\partial_y^3\phi)$$

[Note: q $\neq$ 3]

$$(\partial_j\partial_z\phi)\, \varepsilon_{irs}\lambda_r\partial_s\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi$$
$$= (\partial_j\partial_z\phi) [\varepsilon_{i3s}\partial_s\varepsilon_{i3q}\partial_j\partial_q\psi]$$
$$= (\partial_j\partial_z\phi) [(\delta_{sq})\partial_s\partial_j\partial_q\psi]$$
$$= (\partial_j\partial_z\phi) (\partial_j\Delta\psi)$$
$$= (\partial_x\partial_z\phi) (\partial_x^3\psi) + (\partial_x\partial_z\phi) (\partial_x\partial_y^2\psi) + (\partial_y\partial_z\phi) (\partial_x^2\partial_y\psi) + (\partial_y\partial_z\phi) (\partial_y^3\psi)$$

$$\varepsilon_{jlm}\lambda_l\partial_m\psi\varepsilon_{irs}\lambda_r\partial_i\partial_j\partial_s\partial_z\phi$$
$$= [\varepsilon_{j3m}\partial_m\psi\varepsilon_{i3s}\partial_i\partial_j\partial_s\partial_z\phi]$$
$$= [\varepsilon_{3jm}\partial_m\psi\varepsilon_{3is}\partial_i\partial_j\partial_s\partial_z\phi] \quad \text{(One Index Contracted)}$$
$$= [-(\delta_{33}\delta_{jm} - \delta_{3i}\delta_{3m}) \partial_m\psi\partial_i\Delta\partial_z\phi]$$
$$= [(\delta_{ij}\delta_{ms} - \delta_{js}\delta_{im}) (\partial_m\psi)(\partial_i\partial_j\partial_s\partial_z\phi)]$$
$$= (\partial_m\psi) (\Delta\partial_m\partial_z\phi) - (\partial_i\psi) (\Delta\partial_i\partial_z\phi) = 0$$

$$\begin{aligned}
&\left(\partial_i\partial_j\partial_z\phi\right)\varepsilon_{irs}\lambda_r\partial_s\varepsilon_{jlm}\lambda_l\partial_m\psi \\
&= \left(\partial_i\partial_j\partial_z\phi\right)\left[\varepsilon_{i3s}\partial_s\varepsilon_{j3m}\partial_m\psi\right] \\
&= \left(\partial_i\partial_j\partial_z\phi\right)\left[\left(\delta_{ij}\delta_{sm}-\delta_{im}\delta_{js}\right)(\partial_s)(\partial_m)\psi\right] \\
&= \left(\partial_i\partial_j\partial_z\phi\right)\left[(\Delta\psi)-\left(\partial_i\partial_j\psi\right)\right]
\end{aligned}$$

For first expansion i and j cannot be different hence,

$$\begin{aligned}
&= (\Delta\partial_z\phi)(\Delta\psi)-\left(\partial_i\partial_j\partial_z\phi\right)\left(\partial_i\partial_j\psi\right) \\
&= \left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\psi\right)+\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\psi\right)+\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\psi\right)+\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\psi\right) \\
&-\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x\partial_y\psi\right)-\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x\partial_y\psi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\psi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\psi\right) \\
&= \left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\psi\right)+\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\psi\right)-2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x\partial_y\psi\right)
\end{aligned}$$

$$\begin{aligned}
&\varepsilon_{ipq}\lambda_p\partial_j\partial_q\psi\varepsilon_{irs}\lambda_r\partial_j\partial_s\partial_z\phi \\
&= \left[\varepsilon_{i3q}\partial_j\partial_q\psi\varepsilon_{i3s}\partial_j\partial_s\partial_z\phi\right] \\
&= \left[\left(\delta_{sq}\right)\left(\partial_j\partial_q\psi\right)(\partial_i\partial_s\partial_z\phi)\right] \\
&= \left(\partial_j\partial_q\psi\right)\left(\partial_j\partial_q\partial_z\phi\right) \\
&= \left(\partial_x^2\psi\right)\left(\partial_x^2\partial_z\phi\right)+2\left(\partial_x\partial_y\psi\right)\left(\partial_x\partial_y\partial_z\phi\right)+\left(\partial_y^2\psi\right)\left(\partial_y^2\partial_z\phi\right)
\end{aligned}$$

The $2\left(\partial_x\partial_y\psi\right)\left(\partial_x\partial_y\partial_z\phi\right)$ terms cancel in the last two expansions.

Expanding Non-Linear N.S. Terms

Theta Equation

We do not need to find the curl or curl curl of the non-linear parts of the temperature equation (Eqn. 12) as temperature is scalar.

Hence,

$$\begin{aligned}
&\tilde{u}\cdot\tilde{\nabla}\theta \\
&= \left(\tilde{\delta}\phi+\tilde{\varepsilon}\psi\right)\cdot\tilde{\nabla}\theta \\
&= u_i\nabla_i\theta \\
&= \left(\delta_i\phi+\varepsilon_i\psi\right)\nabla_i\theta \\
&= \left[\left(\partial_i\partial_z-\lambda_i\Delta\right)\phi+\left(\varepsilon_{ijk}\lambda_j\partial_k\right)\psi\right]\cdot\partial_i\theta \\
&= \partial_i\partial_z\phi\partial_i\theta-\Delta\phi\partial_z\theta+\varepsilon_{ijk}\lambda_j\partial_k\psi\partial_i\theta
\end{aligned}$$

This gives;

$$
\begin{aligned}
&(\partial_i \partial_z \phi)(\partial_i \theta) - (\Delta\phi)(\partial_z \theta) \\
&= (\partial_x \partial_z \phi)(\partial_x \theta) + (\partial_y \partial_z \phi)(\partial_y \theta) + \left(\partial_z^2 \phi\right)(\partial_z \theta) - \left(\partial_x^2 \phi\right)(\partial_z \theta) - \left(\partial_y^2 \phi\right)(\partial_z \theta) - \left(\partial_z^2 \phi\right)(\partial_z \theta) \\
&= (\partial_x \partial_z \phi)(\partial_x \theta) + (\partial_y \partial_z \phi)(\partial_y \theta) - \left(\partial_x^2 \phi\right)(\partial_z \theta) - \left(\partial_y^2 \phi\right)(\partial_z \theta)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\partial_i \theta)\, \varepsilon_{ijk} \lambda_j \partial_k \psi \\
&= (\partial_i \theta)\left[\varepsilon_{231} \partial_2 \psi + \varepsilon_{132} \partial_1 \psi\right] \\
&= (\partial_i \theta)\left[\partial_x \psi - \partial_y \psi\right] \\
&= (\partial_i \theta)(\partial_x \psi) - (\partial_i \theta)(\partial_y \psi) \\
&= (\partial_x \theta)(\partial_x \psi) + (\partial_y \theta)(\partial_x \psi) + (\partial_z \theta)(\partial_x \psi) - (\partial_x \theta)(\partial_y \psi) - (\partial_y \theta)(\partial_y \psi) - (\partial_z \theta)(\partial_y \psi)
\end{aligned}
$$

D Appendix - Constant Flux Chebyshev Calculations

D.1 Chebyshev Polynomials - Definitions

NOTE: In this Appendix the subscript used is n as opposed to l in the main body of the thesis.

The Constant Flux condition is generalised by

$$\int_{-1}^{+1} U\partial_z = 0$$

which can be readily performed for the basic flow

$$\int_{-1}^{+1} U_0\partial_z = 0.$$

However, we need to find the same integral for the perturbed flow

$$\int_{-1}^{+1} \breve{U}\partial_z = 0.$$

Which can be expressed as

$$\int_{-1}^{+1} a_n(1-z^2)T_n(z)\partial_z = 0.$$

We begin initially by familiarising the reader with an overview of Chebyshev Polynomials.

Let $x = \cos\vartheta$ or $\cos^{-1} x = \vartheta$.

Then the Chebyshev polynomial is defined as $T_n(x) = \cos\left(n\cos^{-1} x\right)$

Expanding the terms we arrive at;

$$\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= 2x^2 - 1\\
T_3(x) &= 4x^3 - 3x\\
T_4(x) &= 8x^4 - 8x^2 + 1\\
T_5(x) &= 16x^5 - 20x^3 + 5x\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1\\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x
\end{aligned}$$

Recurrence relation is ; $T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$

We can derive each term of the sequence using some basic trigonometric identities as follows;

$$\text{Using}\quad \begin{aligned}
&\cos 2\vartheta \equiv 2\cos^2\vartheta - 1\\
&\cos^2\vartheta + \sin^2\vartheta \equiv 1\\
&\sin 2\vartheta \equiv 2\sin\vartheta\cos\vartheta\\
&\cos 2\vartheta \equiv 2\cos^2\vartheta - 1
\end{aligned}\quad ,$$

it follows that

$$T_0(x) = \cos 0 = 1$$

$$T_1(x) = \cos\cos^{-1} x = x$$

$$T_2(x) = \cos\left(2\cos^{-1} x\right) \Rightarrow \cos 2\vartheta = 2\cos^2\vartheta - 1 = 2x^2 - 1$$

$$T_3(x) = \cos\left(3\cos^{-1} x\right) \Rightarrow \cos(\vartheta + 2\vartheta)$$

$$\begin{aligned}
&= \cos\vartheta\cos 2\vartheta - \sin\vartheta\sin 2\vartheta\\
&= \cos\vartheta(2\cos^2\vartheta - 1) - \sin\vartheta(2\sin\vartheta\cos\vartheta)\\
&= 2\cos^3\vartheta - \cos\vartheta - 2\sin^2\vartheta\cos\vartheta\\
&= 2\cos^3\vartheta - \cos\vartheta - 2\cos\vartheta(1 - \cos^2\vartheta)\\
&= 2\cos^3\vartheta - \cos\vartheta - 2\cos\vartheta + 2\cos^3\vartheta\\
&= 4\cos^3\vartheta - 3\sin\vartheta = 4x^3 - 3x
\end{aligned}$$

Derivation of higher orders of Chebyshevs follow in the same manner. We need to also analyse the effect of n < 0 in $T_n(z)$;
For example take $T_{-2}(z)$

$$
\begin{aligned}
&\cos(-2\theta) \\
&= \cos(-\theta + -\theta) \\
&= \cos(-\theta)\cos(-\theta) - \sin(-\theta)\sin(-\theta) \\
&= \cos^2(-\theta) - \sin^2(-\theta) \\
&= \cos^2(-\theta) - (1 - \cos^2(-\theta)) \\
&= 2\cos^2(-\theta) - 1 \\
&= 2x^2 - 1
\end{aligned}
$$

Due the symmetrical nature of cosine function we can summise that;

$$T_{|n|}(z) = T_n(z)$$

because

$$\cos|\theta| = \cos\theta$$

.

D.2 Integrating Recurrence Relationship

The objective of the following section is to find a general formula for the integral of the Chebyshev recurrence formula between the limits of -1, z and ± 1.
We begin by quoting the formula [19]:

$$z^2 T_n = \frac{1}{2^2} \sum_{i=0}^{2} \begin{pmatrix} r \\ i \end{pmatrix} T_{n-2+2i}$$

Where,

$$\begin{pmatrix} r \\ i \end{pmatrix} = \frac{r!}{i!(r-i)!}$$
$$\begin{pmatrix} n \\ 0 \end{pmatrix} = \frac{n!}{0!(n-0)!} = 1$$
$$\begin{pmatrix} n \\ 1 \end{pmatrix} = \frac{n!}{1!(n-1)!}$$
$$\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

So that:

$$z^2T_n = \tfrac{1}{4}\textstyle\sum_{i=0}^{2} \frac{2!}{i!(2-i)!}T_{n-2+2i}$$
$$= \tfrac{1}{4}\left\{\tfrac{2!}{0!2!}T_{n-2} + \tfrac{2!}{1!(2-1)!}T_{n-2+2} + \tfrac{2!}{2!(2-1)!}T_{n-2+4}\right\}$$
$$= \tfrac{1}{4}\{T_{n-2} + 2T_n + T_{n+2}\}$$
$$= \tfrac{1}{4}T_{n-2} + \tfrac{1}{2}T_n + \tfrac{1}{4}T_{n+2}$$

Therefore:

$$\left(1-z^2\right)T_n = T_n - \tfrac{1}{4}T_{n-2} - \tfrac{1}{2}T_n - \tfrac{1}{4}T_{n+2}$$
$$= \tfrac{1}{2}T_n - \tfrac{1}{4}T_{n-2} - \tfrac{1}{4}T_{n+2}$$

Then, $\int_{-1}^{x}\sum_{r=0}^{N} a_rT_rdz = \sum_{r=0}^{N} a_r\int T_rdz = \sum_{r=0}^{N+1} b_rT_r.$

Suppose at this stage that;

$$\textstyle\sum_{r=0}^{N} a_rT_r = a_0T_0 + a_1T_1 + a_2T_2.$$

Then,

$a_0\int_{-1}^{x}T_o + a_1\int_{-1}^{x}T_1 + a_2\int_{-1}^{x}T_2 = a_0T_1|_{-1}^{x} + \frac{a_1}{4}\{T_0+T_2\}_{-1}^{x} + \frac{a_2}{2}\left\{\frac{1}{3}T_3 - T_1\right\}_{-1}^{x}.$

Which shows that;

$$\int a_0T_0 + a_1T_1 + a_2T_2$$

$$= a_0(T_{1+1}) + \tfrac{a_1}{4}(T_0 + T_2 - T_0 - T_0 = T_2 - T_0)$$
$$+\tfrac{a_2}{2}\left(\tfrac{T_3}{3} - T_1 + \tfrac{1}{3} - 1 = \tfrac{T_3}{3} - T_1 - \tfrac{2}{3}T_0\right)$$

$$= a_0(T_1 + T_0) + \frac{a_1}{4}(T_2 - T_0) + \frac{a_2}{2}\left(\frac{T_3}{3} - T_1 - \frac{2}{3}T_0\right) \tag{D.1}$$

$$= \sum_{r=0}^{N+1} b_r T_r = \sum_{r=0}^{3} b_r T_r$$

$$= b_0 T_0 + b_1 T_1 + b_2 T_2 + b_3 T_3 \tag{D.2}$$

Where we have equated coefficients of T_0, T_1, T_2 and T_3 and where $b_1 = a_0 - \frac{a_2}{2}$, $b_2 = \frac{a_1}{4}$ and $b_3 = \frac{a_2}{6}$.

In order to find b_0 we set x = -1 in Eqn. D.1;

$$0 = a_0(-1+1) + \frac{a_1}{4}(1-1) + \frac{a_2}{2}\left(-\frac{1}{3} + 1 - \frac{2}{3}\right)$$

Or in Eqn. D.2;

$$0 = b_0 - b_1 + b_2 - b_3 \Rightarrow b_0 = b_1 - b_2 + b_3$$
$$\Rightarrow b_0 = a_0 - \tfrac{a_2}{2} - \tfrac{a_1}{4} + \tfrac{a_2}{6} = a_0 - a_2\left(\tfrac{1}{2} - \tfrac{1}{6} = \tfrac{1}{3}\right) - \tfrac{a_1}{4} = a_0 - \tfrac{a_2}{3} - \tfrac{a_1}{4}$$

Which is precisely the terms multiplying T_0 in Eqn. D.1.

Then;

$$\left(1 - z^2\right) T_n = \frac{1}{2} T_n - \frac{1}{4} T_{n-2} - \frac{1}{4} T_{n+2}$$

So that;

$$\sum_{n=0}^{N} \int_{-1}^{x} a_n (1 - z^2) T_n = \sum_{n=0}^{N} \frac{a_n}{4} \int_{-1}^{x} (2T_n - T_{n-2} - T_{n+2}) = Y \tag{D.3}$$

We shall now proceed to see if the formula $z^r T_n = \frac{1}{2^r} \sum_{i=0}^{r} \begin{pmatrix} r \\ i \end{pmatrix} T_{n-r+2i}(z)$ works;

We need to use $T_{|n|}(z) = T_n(z)$ established previously.

$$(1 - z^2) T_0 = T_0 - z^2 T_0$$
$$z^2 = z^2 T_0 = \tfrac{1}{4} \textstyle\sum_{i=0}^{2} \begin{pmatrix} 2 \\ i \end{pmatrix} T_{0-2+2i} = \tfrac{1}{4}\left\{T_{0-2} + 2T_0 + T_2\right\} = \tfrac{1}{4}\left\{2z^2 - 1 + 2 + 2z^2 = 1\right\} = z^2$$
$$z^2 T_1 = z^3 = \tfrac{1}{4}\left\{T_{-1} + 2T_1 + T_3 = 3z + 4z^3 - 3z = 4z^3\right\} = z^3$$

Going back to Eqn. D.3;

$$Y=\sum_{n=0}^{N}\frac{a_n}{4}\left\{2\int_{-1}^{x}T_n dx-\int_{-1}^{x}T_{n-2}-\int_{-1}^{x}T_{n+2}\right\} \tag{D.4}$$

$$\int_{-1}^{x}T_n dx=\frac{T_{n+1}}{2(n+1)}-\frac{T_{n-1}}{2(n-1)}-\frac{(-1)^{n+1}}{(n+1)(n-1)} \tag{D.5}$$

$$\Rightarrow 2\int_{-1}^{x}T_n dx=\frac{T_{n+1}}{n+1}-\frac{T_{n-1}}{n-1}-\frac{(-1)^{n+1}.2}{(n+1)(n-1)}$$

We need to further consider;

$$\int_{-1}^{x}T_{n+2}dx=\frac{T_{n+3}}{2(n+3)}-\frac{T_{n+1}}{2(n+1)}-\frac{(-1)^{n+3}}{(n+3)(n+1)}$$

and

$$\int_{-1}^{x}T_{n-2}dx=\frac{T_{n-1}}{2(n-1)}-\frac{T_{n-3}}{2(n-3)}-\frac{(-1)^{n-1}}{(n-1)(n-3)}$$

Consider Eqn. D.3 again;

$$Y=\sum_{n=2}^{N}\frac{a_n}{4}\left\{\frac{T_{n+1}}{n+1}-\frac{T_{n-1}}{n-1}-\frac{2(-1)^{n+1}}{(n+1)(n-1)}\right\}+a_0(T_1+T_0)+\frac{a_1}{4}(T_2-T_0)$$

$$-\frac{T_{n-1}}{2(n-1)}+\frac{T_{n-3}}{2(n-3)}-\frac{(-1)^{n-1}}{(n-1)(n-3)}-\frac{T_{n+3}}{2(n+3)}+\frac{T_{n+1}}{2(n+1)}-\frac{(-1)^{n+3}}{(n+3)(n+1)}$$

Now we can deduce;

$$z^2T_0=\tfrac{1}{4}\{T_2+2T_0+T_2\}=\tfrac{1}{4}\{2T_2+2T_0\}=\tfrac{1}{2}(T_2+T_0)$$
$$z^2T_n=\tfrac{1}{4}\{T_{n-2}+2T_n+T_{n+2}\}=\tfrac{1}{4}\{2T_n+2T_{n+2}\}=\tfrac{1}{2}(T_n+T_{n+2})$$
$$z^2T_2=z^2(2z^2-1)=2z^4-z^2=\tfrac{1}{4}\{T_0+2T_2+T_4\}$$

$$=\frac{1}{4}\left\{1+4z^2-2+8z^4-8z^2+1\right\}=2z^4-4z^2$$

$$z^2T_3=z^2(4z^3-3z)=4z^5-3z^3$$

$$=\tfrac{1}{4}\{T_1+2T_3+T_5\}=\tfrac{1}{4}\left(z+8z^3-6z+16z^5-20z^3+5z\right)$$
$$=\tfrac{1}{4}T_1+\tfrac{1}{2}T_3+\tfrac{1}{4}T_5$$

And;

$$(1-z^2)T_0 = T_0 - z^2T_0 = T_0 - \tfrac{1}{4}\{2T_0 + 2T_2\} = \tfrac{1}{2}(T_0 - T_2)$$
$$(1-z^2)T_1 = T_1 - z^2T_1 = T_1 - \tfrac{1}{4}\{3T_1 + T_3\} = \tfrac{1}{4}(T_1 - T_3)$$
$$(1-z^2)T_2 = T_2 - \tfrac{1}{4}T_0 - \tfrac{1}{2}T_2 - \tfrac{1}{4}T_4 = \tfrac{1}{2}T_2 - \tfrac{1}{4}T_0 - \tfrac{1}{4}T_4$$
$$(1-z^2)T_3 = \tfrac{1}{2}T_3 - \tfrac{1}{4}T_1 - \tfrac{1}{4}T_5$$

We can now proceed to a general formula, taking a sufficiently large n for our approximation;

$$\textstyle\sum_{n=0}^{45}\int_{-1}^{z} a_n(1-z^2)T_n(z)dz = a_0\int_{-1}^{z}(1-z^2)T_o dz$$
$$= a_0\textstyle\int_{-1}^{z}(1-z^2)T_o dz + a_1\int_{-1}^{z}(1-z^2)T_1 dz + a_2\int_{-1}^{z}(1-z^2)T_2 dz + a_3\int_{-1}^{z}(1-z^2)T_3 dz$$
$$+\sum_{n=4}^{45} a_n \int_{-1}^{z}(1-z^2)T_n dz$$

$$= a_0\textstyle\int_{-1}^{z}\frac{1}{2}(T_o - T_2)dz + a_1\int_{-1}^{z}\frac{1}{4}(T_1 - T_3)dz$$
$$+a_2\textstyle\int_{-1}^{z}\left(\frac{1}{2}T_2 - \frac{1}{4}T_0 - \frac{1}{4}T_4\right)dz + a_3\int_{-1}^{z}\left(\frac{1}{2}T_3 - \frac{1}{4}T_1 - \frac{1}{4}T_5\right)dz$$
$$+\textstyle\sum_{n=4}^{45} a_n\int_{-1}^{z}\left\{\frac{1}{2}T_n - \frac{1}{4}T_{n-2} - \frac{1}{4}T_{n+2}\right\}$$

Using Eqn. D.5 we have;

$$\frac{a_0}{2}\int_{-1}^{z}(T_0 - T_2)\,dz = \frac{a_0}{2}\left\{T_0 + T_1 - \frac{T_3}{6} + \frac{T_1}{2} + \frac{T_0}{3}\right\} = \frac{a_0}{2}\left\{\frac{4T_0}{3} + \frac{3T_1}{2} - \frac{T_1}{6}\right\}$$

$$\frac{a_1}{4}\int_{-1}^{z}(T_1 - T_3)\,dz = \frac{a_1}{4}\left\{\frac{T_2}{4} - \frac{T_0}{4} - \frac{T_4}{8} + \frac{T_2}{4} - \frac{T_0}{8}\right\} = \frac{a_1}{4}\left\{-\frac{3T_0}{8} + \frac{T_2}{2} - \frac{T_4}{8}\right\}$$

$$a_2\textstyle\int_{-1}^{z}\left(\frac{1}{2}T_2 - \frac{1}{4}T_0 - \frac{1}{4}T_4\right)dz = a_2\left\{\frac{T_3}{12} - \frac{T_1}{4} - \frac{T_0}{6} - \frac{T_0}{4} - \frac{T_1}{4} - \frac{T_5}{40} + \frac{T_3}{24} + \frac{T_0}{60}\right\}$$
$$= a_2\left\{-\tfrac{2T_0}{5} - \tfrac{T_1}{2} + \tfrac{T_3}{8} - \tfrac{T_5}{40}\right\}$$

$$a_3\textstyle\int_{-1}^{z}\left(\frac{1}{2}T_3 - \frac{1}{4}T_1 - \frac{1}{4}T_5\right)dz = a_3\left\{\frac{T_4}{16} - \frac{T_2}{8} + \frac{T_0}{16} - \frac{T_2}{16} + \frac{T_0}{16} - \frac{T_6}{48} + \frac{T_4}{32} - \frac{T_0}{96}\right\}$$
$$= a_3\left\{\tfrac{11T_0}{96} + \tfrac{3T_4}{32} - \tfrac{3T_2}{16} - \tfrac{T_6}{48}\right\}$$

$$+\sum_{n=4}^{45}\frac{a_n}{4}\left\{\begin{array}{l}\frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1} + \frac{2(-1)^{n+1}}{(n+1)(n-1)}T_0 - \frac{T_{n-1}}{2(n-1)} + \frac{T_{n-3}}{2(n-3)} - \frac{(-1)^{n-1}}{(n-1)(n-3)}T_0 \\ -\frac{T_{n+3}}{2(n+3)} + \frac{T_{n+1}}{2(n+1)} - \frac{(-1)^{n+3}}{(n+1)(n+3)}T_0\end{array}\right\}$$

D.3 General Formulae for Definite Integrals

$$\sum_{n=0}^{45} \int_{-1}^{z} a_n(1-z^2)T_n dz$$
$$= a_0\left\{\frac{2T_0}{3}+\frac{3T_1}{4}-\frac{T_3}{12}\right\}$$
$$+a_1\left\{-\frac{3T_0}{32}+\frac{T_2}{8}-\frac{T_4}{32}\right\}$$
$$+a_2\left\{-\frac{2T_0}{5}-\frac{T_1}{2}+\frac{T_3}{8}-\frac{T_5}{40}\right\}$$
$$+a_3\left\{\frac{11T_0}{96}+\frac{3T_4}{32}-\frac{3T_2}{16}-\frac{T_6}{48}\right\}$$
$$+\sum_{n=4}^{45}\frac{a_n}{4}\left\{\begin{array}{l}\frac{T_{n+1}}{n+1}-\frac{T_{n-1}}{n-1}+\frac{2(-1)^{n+1}}{(n+1)(n-1)}T_0-\frac{T_{n-1}}{2(n-1)}+\frac{T_{n-3}}{2(n-3)}-\frac{(-1)^{n-1}}{(n-1)(n-3)}T_0\\ -\frac{T_{n+3}}{2(n+3)}+\frac{T_{n+1}}{2(n+1)}-\frac{(-1)^{n+3}}{(n+1)(n+3)}T_0\end{array}\right\}$$

This formula is applicable to both odd and even values of n.

We shall now derive a general formula for $\int_{-1}^{+1}(1-z^2)T_n dz$.

It is useful to note that;

$T_n 1 = 1$ for all n, and $T_n(-1) = -1$ for odd n $T_n(-1) = 1$ for even n.

So,

$$\int_{-1}^{+1}(1-z^2)T_n dz = \int_{-1}^{+1} T_n dz - \int_{-1}^{+1} z^2 T_n dz$$

Use Eqn. D.5 and split into two parts then analyse odd and even n;

<u>odd n;</u>

$$\int_{-1}^{+1} T_n dz = \left[\frac{T_{n+1}}{2(n+1)}-\frac{T_{n-1}}{2(n-1)}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]-\frac{1}{2}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]=0$$

<u>even n;</u>

$$\int_{-1}^{+1} T_n dz = \left[\frac{T_{n+1}}{2(n+1)}-\frac{T_{n-1}}{2(n-1)}\right]_{-1}^{1}$$
$$= \frac{1}{2}\left[\frac{(n-1)-(n+1)}{(n+1)(n-1)}\right]-\frac{1}{2}\left[\frac{-1}{n+1}-\frac{-1}{n-1}\right]$$
$$= \frac{1}{2}\left[\frac{n-1-n-1}{(n+1)(n-1)}\right]-\frac{1}{2}\left[\frac{-n+1+n+1}{(n+1)(n-1)}\right]$$
$$= -\frac{1}{(n+1)(n-1)}-\frac{1}{(n+1)(n-1)}$$
$$= -\frac{2}{(n+1)(n-1)}$$

<u>odd n;</u>

$$
\begin{aligned}
&-\int_{-1}^{+1} z^2 T_n dz \\
&= -\tfrac{1}{4}\int_{-1}^{+1} T_{n-2}dz - \tfrac{1}{2}\int_{-1}^{+1} T_n dz - \tfrac{1}{4}\int_{-1}^{+1} T_{n+2}dz \\
&= \tfrac{1}{4}\int_{-1}^{+1} T_{n-2}dz - 0 - \tfrac{1}{4}\int_{-1}^{+1} T_{n+2}dz \\
&= -\tfrac{1}{8}\left[\tfrac{T_{n-1}}{n-1} - \tfrac{T_{n-3}}{n-3}\right]_{-1}^{+1} - \tfrac{1}{8}\left[\tfrac{T_{n+3}}{n+3} - \tfrac{T_{n+1}}{n+1}\right]_{-1}^{+1} \\
&= \tfrac{1}{8}\left[\tfrac{1}{n-1} - \tfrac{1}{n-3}\right] - \tfrac{1}{8}\left[\tfrac{1}{n-1} - \tfrac{1}{n-3}\right] + \tfrac{1}{8}\left[\tfrac{1}{n+3} - \tfrac{1}{n+1}\right] - \tfrac{1}{8}\left[\tfrac{1}{n+3} - \tfrac{1}{n+1}\right] = 0
\end{aligned}
$$

Even n;

$$
\begin{aligned}
&-\int_{-1}^{+1} z^2 T_n dz \\
&= -\tfrac{1}{4}\int_{-1}^{+1} T_{n-2}dz - \tfrac{1}{2}\int_{-1}^{+1} T_n dz - \tfrac{1}{4}\int_{-1}^{+1} T_{n+2}dz \\
&= \underbrace{-\tfrac{1}{8}\left[\tfrac{T_{n-1}}{n-1} - \tfrac{T_{n-3}}{n-3}\right]_{-1}^{+1}}_{1} \underbrace{- \tfrac{1}{4}\left[\tfrac{T_{n+1}}{n+1} - \tfrac{T_{n-1}}{n-1}\right]_{-1}^{+1}}_{2} \underbrace{- \tfrac{1}{8}\left[\tfrac{T_{n+3}}{n+3} - \tfrac{T_{n+1}}{n+1}\right]_{-1}^{+1}}_{3}
\end{aligned}
$$

1.
$$
\begin{aligned}
&-\tfrac{1}{8}\left[\tfrac{1}{n-1} - \tfrac{1}{n-3}\right] + \tfrac{1}{8}\left[\tfrac{-1}{n-1} - \tfrac{-1}{n-3}\right] = -\tfrac{1}{8}\left[\tfrac{n-3-n+1}{(n-1)(n-3)}\right] + \tfrac{1}{8}\left[\tfrac{-n+3+n-1}{(n-1)(n-3)}\right] \\
&= -\tfrac{1}{8}\left[\tfrac{-2}{(n-1)(n-3)}\right] + \tfrac{1}{8}\left[\tfrac{2}{(n-1)(n-3)}\right] = \tfrac{1}{4(n-1)(n-3)} + \tfrac{1}{4(n-1)(n-3)} \\
&= \tfrac{1}{2(n-1)(n-3)}
\end{aligned}
$$

2.
$$
\begin{aligned}
&-\tfrac{1}{4}\left[\tfrac{1}{n+1} - \tfrac{1}{n-1}\right] + \tfrac{1}{4}\left[\tfrac{-1}{n+1} - \tfrac{-1}{n-1}\right] = -\tfrac{1}{4}\left[\tfrac{n-1-n-1}{(n+1)(n-1)}\right] + \tfrac{1}{4}\left[\tfrac{-n+1+n+1}{(n+1)(n-1)}\right] \\
&= -\tfrac{1}{4}\left[\tfrac{-2}{(n+1)(n-1)}\right] + \tfrac{1}{4}\left[\tfrac{2}{(n+1)(n-1)}\right] = \tfrac{1}{2(n+1)(n-1)} + \tfrac{1}{2(n+1)(n-1)} \\
&= \tfrac{1}{(n+1)(n-1)}
\end{aligned}
$$

3.
$$
\begin{aligned}
&-\tfrac{1}{8}\left[\tfrac{1}{n+3} - \tfrac{1}{n+1}\right] + \tfrac{1}{8}\left[\tfrac{-1}{n+3} - \tfrac{-1}{n+1}\right] = -\tfrac{1}{8}\left[\tfrac{n+1-n-3}{(n+3)(n+1)}\right] + \tfrac{1}{8}\left[\tfrac{-n-1+n+3}{(n+3)(n+1)}\right] \\
&= -\tfrac{1}{8}\left[\tfrac{-2}{(n+3)(n+1)}\right] + \tfrac{1}{8}\left[\tfrac{2}{(n+3)(n+1)}\right] = \tfrac{1}{4(n+3)(n+1)} + \tfrac{1}{4(n+3)(n+1)} \\
&= \tfrac{1}{2(n+3)(n+1)}
\end{aligned}
$$

Hence, when n is even we have;

$$\int_{-1}^{+1}(1-z^2)T_n dz = \frac{1}{2(n+3)(n+1)} - \frac{1}{(n+1)(n-1)} + \frac{1}{2(n-1)(n-3)} \tag{D.6}$$

If n is odd we have;

$$\int_{-1}^{+1}(1-z^2)T_n dz = 0 \tag{D.7}$$

Testing General Formulae

The derived formulae for the definite integrals containing the Chebyshev functions were tested for accuracy. A mathematical software package (Derive) was used to produce the definite integrals required, some of the integrals were verified correct by hand in order to test the results obtained by Derive. The general formulae were then tested correct by ensuring that the results obtained using the formulae were exactly as those obtained using Derive. I tested for n = 1 to 5 and then for n = 12 and 13, I can conclude that the derived formulae are correct and can be confidently used in the computer model.

E Appendix - Mean Flow Calculations

E.1 Components of $\tilde{u}$

We need to state the components of the flow as follows, as this is required later in this Appendix.

$$Ux(i=1)$$

$$\text{and} \quad \begin{array}{l} \delta_i = \partial_i \partial_z - \lambda_i \Delta \\ \varepsilon_i = \varepsilon_{ijk} \lambda_j \partial_k \end{array} \quad \text{and } \lambda_i = [0,0,1]$$

$$\begin{aligned} u &= \partial_x \partial_z \phi - \lambda_1 \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \phi + \varepsilon_{132} \partial_y \psi \\ u &= \partial_x \partial_z \phi - 0 - \partial_y \psi \\ u &= \partial_x \partial_z \phi - \partial_y \psi \end{aligned}$$

$$Uy(i=2)$$

$$v = \partial_y \partial_z \phi - \lambda_2 \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \phi + \varepsilon_{231} \partial_x \psi$$
$$v = \partial_y \partial_z \phi - 0 + \partial_x \psi$$
$$v = \partial_y \partial_z \phi + \partial_x \psi$$

$$Uz(i = 3)$$

$$w = \partial_z^2 \phi - \lambda_3 \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \phi + \varepsilon_{332} \partial_y \psi + \varepsilon_{331} \partial_x \psi$$
$$w = -\Delta_2 \phi$$

E.2 Mean Flow

We have the N.S. equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = 2R + T\hat{\mathbf{i}} + \nabla^2 \mathbf{u} \tag{E.1}$$

The total flow is given by,

$$\tilde{u}(z,t) = Uo(z) + \overline{u}(z,t) + \breve{u}(x,y,z,t) \tag{E.2}$$

Where, Uo(z) is the basic laminar flow (time independent), $\overline{u}(z,t)$ is the mean flow (it shall be shown in section 3 that there are no toroidal x, or poloidal y parts and also that there is no time dependence) and $\breve{u}(x,y,z,t)$ is the fluctuating flow, where;

$$\breve{u} = \delta\breve{\phi} + \varepsilon\breve{\psi} \tag{E.3}$$

$$\breve{u}(x,y,z,t) = \sum_{m,n=-\infty; m,n\neq 0}^{m,n=\infty} \sum_{l=0}^{\infty} a_{mn} b_{mn} f_l(z) e^{-i(m\alpha\cdot x + n\beta\cdot y) + \sigma t} \tag{E.4}$$

$$\breve{\theta}(x,y,z,t) = \sum_{m,n=-\infty; m,n\neq 0}^{m,n=\infty} \sum_{l=0}^{\infty} c_{mn} f_l(z) e^{-i(m\alpha\cdot x + n\beta\cdot y) + \sigma t} \tag{E.5}$$

Alternatively, we may write the poloidal and toroidal parts of the fluctuating flow as,

$$\breve{\phi}(x,y,z,t) = \sum_{m-\infty; m,n\neq 0}^{m=\infty} \sum_{l=0}^{\infty} a_{mn} f_l(z) e^{-i(m\alpha\cdot x) + \sigma t} \tag{E.6}$$

$$\breve{\psi}(x,y,z,t) = \sum_{m-\infty; m,n\neq 0}^{m=\infty} \sum_{l=0}^{\infty} b_{mn} f_l(z) e^{-i(m\alpha\cdot y) + \sigma t} \tag{E.7}$$

We proceed by substituting Eqn. E.2 into Eqn. E.1 term by term. We can ignore the basic flow substitution in the N.S. equation as the N.S. equation does not exist for laminar flow. We now substitute the mean flow term into the N.S. equation. Because we have flow in the x and y directions we average the flow over x and y by applying the operator

$$\int_0^{\frac{2\pi}{\alpha}} \int_0^{\frac{2\pi}{\beta}} dxdy \tag{E.8}$$

on the x and y components of equation Eqn. E.1, as well as applying the fixed pressure condition $\overline{\partial_{xy}\pi} = 0$. We can show that we are able to eliminate the linear terms of Eqn.E.1, by showing that averaging over 2π is zero, using the Eulerian identity:

$$e^{a+ib} = e^a.e^{ib} = e^a\left[\cos b + i\sin a\right] \Rightarrow e^{ib} = \cos b + i\sin b, \tag{E.9}$$

Taking Eqn. E.4 and E.5 and using the Eulerian and some trigonometric identities we can arrive at,

$$\int_0^{\frac{2\pi}{\alpha}} \int_0^{\frac{2\pi}{\beta}} a_{mn} f_l(z)\left[\cos(m\alpha x + n\beta y) + i\sin(m\alpha x + n\beta y\right] dxdy = 0 \tag{E.10}$$

We must further assert that the waveform is assumed periodic and no phase shift in the period takes place and the wavelengths are equal. So once we have applied the above theory to Eqn. E.1 we are only left with the non-linear term $\tilde{u}.\tilde{\nabla}\tilde{u}$. We need the incompressibility condition:

$$\nabla\tilde{u} = \partial_x U_x + \partial_y U_y + \partial_z U_z = 0 \tag{E.11}$$

We shall now derive the mean flow of the velocity field;

E.2.1 X Component of Flow (using integration by parts)

$$\int\int u.\nabla u_x = \int\int u_x\partial_x u_x + \int\int u_y\partial_y u_x + \int\int u_z\partial_z u_x$$
$$\int\int u_x\partial_x u_x = u_x u_x|_0^{2\pi/\beta} - \int u_x\partial_x u_x = 0 - \int u_x\partial_x u_x = -\int u_x\partial_x u_x$$
$$\int\int u_y\partial_y u_x = u_y u_x|_0^{2\pi/\alpha} - \int u_x\partial_y u_y = 0 - \int u_x\partial_y u_y = -\int u_x\partial_y u_y$$
$$\int\int u_z\partial_z u_x = u_x u_z|_0^{2\pi/\alpha} - \int u_x\partial_z u_z = 0 - \int u_x\partial_z u_z = -\int u_x\partial_z u_z$$

Note that the ∂_x, or ∂_y will cancel the dx or dy operator and its integral, i.e.

$$\int_0^{2\pi/\alpha}\int_0^{2\pi/\beta} u_x\partial_x u_x dxdy = u_x u_x|_0^{2\pi/\beta} - \int_0^{2\pi/\beta} u_x\partial_x u_x dy = -\int_0^{2\pi\beta} u_x\partial_x u_x dy$$

Note also that, $u_x u_x|_0^{2\pi/\alpha} = 0, u_y u_x|_0^{2\pi/\alpha} = 0$ and $u_x u_z|_0^{2\pi/\alpha} = 0$ as again this is periodic over 2π, see Eqn. E.10. Adopting the over-bar notation,

$$\overline{u.\nabla u_x} = -\overline{u_x\partial_x u_x} - \overline{u_x\partial_y u_y} + \overline{u_z\partial_z u_x}$$

Replacing, $-\overline{u_x\partial_x u_x} - \overline{u_x\partial_y u_y}$ for $+\overline{u_x\partial_z u_z}$ using Eqn. E.11, we have,

$$\overline{u.\nabla u_x} = \overline{u_x\partial_z u_z} + \overline{u_z\partial_z u_x} = \overline{\partial_z(u_z u_x)} \neq 0$$
$$\overline{u.\nabla u_x} = \overline{-\partial_z\Delta_2\phi\left(\partial_x\partial_z\phi - \partial_y\psi\right)} \quad \text{(E.12)}$$

E.2.2 Y Component of Flow

$$\int\int u.\nabla u_y = \int\int u_x\partial_x u_y + \int\int u_y\partial_y u_y + \int\int u_z\partial_z u_y$$
$$\int\int u_x\partial_x u_y = u_x u_y\Big|_0^{2\pi/\beta} - \int u_y\partial_x u_x = 0 - \int u_y\partial_x u_x = -\int u_y\partial_x u_x$$
$$\int\int u_y\partial_y u_y = u_y u_y\Big|_0^{2\pi/\alpha} - \int u_y\partial_y u_y = 0 - \int u_y\partial_y u_y = -\int u_y\partial_y u_y$$
$$\int\int u_z\partial_z u_y = u_z u_y\Big|_0^{2\pi/\alpha} - \int u_y\partial_z u_z = 0 - \int u_y\partial_z u_z = -\int u_y\partial_z u_z$$

As before;

$$\overline{u.\nabla u_x} = -\overline{u_y\partial_x u_x} - \overline{u_y\partial_y u_y} + \overline{u_z\partial_z u_y}$$

Replacing, $-\overline{u_y\partial_x u_x} - \overline{u_y\partial_y u_y}$ for $+\overline{u_y\partial_z u_z}$ using Eqn. E.11, we have,

$$\overline{u.\nabla u_y} = \overline{u_y\partial_z u_z} + \overline{u_z\partial_z u_y} = \overline{\partial_z(u_z u_y)u.\nabla u_y} = \overline{-\partial_z\Delta_2\phi\left(\partial_y\partial_z\phi + \partial_x\psi\right)} \quad \text{(E.13)}$$

We now shall show that there is no mean flow in the z-direction as expected.

E.2.3 Z Component of Flow

$$\int\int u.\nabla u_z = \int\int u_x\partial_x u_z + \int\int u_y\partial_y u_z + \int\int u_z\partial_z u_z$$
$$\int\int u_x\partial_x u_z = u_x u_z\Big|_0^{2\pi/\beta} - \int u_z\partial_x u_x = 0 - \int u_z\partial_x u_x = -\int u_z\partial_x u_x$$
$$\int\int u_y\partial_y u_z = u_y u_z\Big|_0^{2\pi/\alpha} - \int u_z\partial_y u_y = 0 - \int u_z\partial_y u_y = -\int u_z\partial_y u_y$$
$$\int\int u_z\partial_z u_z = u_z u_z\Big|_0^{2\pi/\alpha} - \int u_z\partial_z u_z = 0 - \int u_z\partial_z u_z = -\int u_z\partial_z u_z$$

Again,

$$\overline{u.\nabla u_z} = -\overline{u_z\partial_x u_x} - \overline{u_z\partial_y u_y} + \overline{u_z\partial_z u_z}$$

Replacing, $-\overline{u_z\partial_x u_x} - \overline{u_z\partial_y u_y}$ for $+\overline{u_z\partial_z u_z}$ using Eqn. E.11, we have,

$$\overline{u.\nabla u_z} = \overline{u_z\partial_z u_z} + \overline{u_z\partial_z u_z} = \overline{\partial_z(u_z u_z)} = \overline{\partial_z\left(u_z^2\right)}$$
$$\overline{u.\nabla u_z} = \overline{(u_z)^2} = 0 \quad \text{(E.14)}$$

Note:

$$\int_0^{2\pi/\alpha}\int_0^{2\pi/\beta} dxdyu_z\partial_z u_z = \left[u_z^2\right]_0^{2\pi/\alpha,\beta} - \int_0^{2\pi/\alpha} dx \int_0^{2\pi/\beta} dyu_z\partial_z u_z = 0$$

and $\iint \overline{2u_z\partial_z u_z} = \int u_z^2 dy$ and $\frac{d}{dz}(u_z)^2 = 2u_z\partial_z u_z$. Both of the mean flow Equations E.12 and E.13 are also supplemented with $\frac{d\overline{u}}{dz^2}$ and $\frac{d\overline{u}}{dt}$, because we must substitute $\overline{u}$ into all the terms of Eqn. E.1, although $\frac{d\overline{u}}{dt}$ = 0 as we have shown that there is no time dependence, i.e. $\overline{u}(z,t)$becomes $\overline{u}(z)$ in Eqn. E. 2.

Summarizing, and not forgetting the coupled mean temperature term $T\hat{\mathbf{i}}$

$$\overline{T} + \partial_z^2\overline{u} + \partial_z\overline{\Delta_2\phi(\partial_{xy}\partial_z \mp \partial_{yx}\psi)} = \partial t\overline{u} \tag{E.15}$$

E.2.4 Time Averaging Mean Flow

Taking,

$$\int_{t_0}^{T+t_0} -\partial_z\Delta_2\phi(\partial_x\partial_z\phi - \partial_y\psi)dt \tag{E.16}$$

we may eliminate all the non-time elements (constants) for simplicity and focus on only the time dependent elements of ϕ and ψ, which is the eigenvalue σt from Eqns. E.6 and E.7.

$$\int_{t_0}^{T+t_0} e^{\sigma t}(e^{\sigma t}+e^{\sigma t})dt = \int_{t_0}^{T+t_0} e^{\sigma t}(e^{\sigma t}+e^{\sigma t})dt = 2\int_{t_0}^{T+t_0} e^{2\sigma t}dt = \left[\frac{1}{\sigma}e^{2\sigma t}\right]_{t_0}^{T+t_0}$$
$$= \left[e^{2\sigma T}e^{2\sigma t_0} - e^{2\sigma t_0}\right] = \frac{1}{\sigma}e^{2\sigma t_0}\left(e^{2\sigma T}-1\right) \tag{E.17}$$

We can see that if σ = 0 Eqn. E.17 gives zero, and hence $\overline{u}[z]$ and not $\overline{u}[z,t]$, no time dependence. When we are analysing the points of transition from stable in unstable in our bifucation diagram we are looking for critical points where the real part of the eigenvalue σ_R is zero. This is repeated as we continue to analyse the bifurcation diagram moving from secondary to higher solutions. For all our solutions we also use a moving frame of reference hence our mean flow is time independent.

When we arrive at turbulent flow we are constantly approximating the waveform structure by using many cosine harmonics. At the bifurcation points we will see a transition from one fluctuating flow structure to another structure and have to re-approximate the new structure, at these points $\sigma_R = 0$ and hence we assert $\overline{u}[z]$.

The mean of the fluctuating flow $\overline{\breve{u}}$ is also not time dependent [37] (pp. 418-419).

E.3 Mean Temperature

Temperature is a scalar variable hence only one calculation required. From our previous document outlining the Linear Analysis we have:

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = -\frac{1}{Pr}(\nabla^2 T + 2Gr) \tag{E.18}$$

As for mean flow the linear terms are eliminated, hence we are left with only the non-linear term.

$$\begin{aligned}
\mathrm{Pr}\int\int u.\nabla\theta = \mathrm{Pr}\int\int u_x\partial_x\theta + \mathrm{Pr}\int\int u_y\partial_y\theta + \mathrm{Pr}\int\int u_z\partial_z\theta \\
\mathrm{Pr}\int\int u_x\partial_x\theta = u_x\theta|_0^{2\pi/\alpha} - \mathrm{Pr}\int\theta\partial_x u_x = -\mathrm{Pr}\int\theta\partial_x u_x \\
\mathrm{Pr}\int\int u_y\partial_y\theta = u_y\theta|_0^{2\pi/\alpha} - \mathrm{Pr}\int\theta\partial_y u_y = -\mathrm{Pr}\int\theta\partial_y u_y \\
\mathrm{Pr}\int\int u_z\partial_z\theta = u_z\theta|_0^{2\pi/\alpha} - \mathrm{Pr}\int\theta\partial_z u_z = -\mathrm{Pr}\int\theta\partial_z u_z
\end{aligned}$$

Again,

$$\mathrm{Pr}\,\overline{u.\nabla\theta} = -\mathrm{Pr}\,\vartheta\overline{\partial_x u_x} - \mathrm{Pr}\,\theta\overline{\partial_y u_z} + \mathrm{Pr}\,\overline{u_z\partial_z}\theta$$

Replacing,

$$\mathrm{Pr}\,\theta\overline{\partial_x u_x} - \mathrm{Pr}\,\theta\overline{u\partial_y u_y} + \mathrm{Pr}\,\theta\overline{\partial_z u_z}$$

using Eqns. E.4 and E.5, we have,

$$\begin{aligned}
\mathrm{Pr}\,\overline{u.\nabla\theta} = \mathrm{Pr}\,\theta\overline{\partial_z u_z} + \mathrm{Pr}\,\overline{u_z\partial_z\theta} = \mathrm{Pr}\,\overline{\partial_z u_z}\theta \\
\mathrm{Pr}\,\overline{u.\nabla\theta} = -\mathrm{Pr}\,\overline{\partial_z\,(\Delta_2\phi)}\theta
\end{aligned}$$

In Eqn. E.19 this gives $Pr\partial_t\overline{T}$ for the time dependent term and taking the z component of the RHS gives $\partial_z^2\overline{T}$. Hence we can write;

$$\partial_z^2\overline{T} + \mathrm{Pr}\,\overline{\partial_z\,(\Delta_2\phi)}\theta = Pr\partial_t\overline{T} \tag{E.19}$$

F Appendix - Symmetries

We shall now analyse the nature and structure of the output matrix symmetries by looking at the effect of odd and even values of m and n in the coefficient of phi. We intend to produce a set of symmetries that enable us to ascertain structures produced by the output files after calculations. We are going to produce a minimal closed set for a_{mn} (i.e. we will not include trivial elements of the set like "1") and this output set will help us to identify converged solutions within the output data produced by the computer by their symmetries. We attempt to limit the size of the closed symmetry set to also optimise the computing time required and to find the smallest number of set members that still enable us to describe fully the symmetry of the problem.

We start by looking at the set of symmetries produced by the linear equations. We then proceed to see which of these symmetries continue to exist once application of the 2D non-linear part of the equations is applied as this non-linear contribution is the most critical and defines which symmetries survive. We then move onto the 3D symmetries.

We have

$$\phi = \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty} \sum_{l=0}^{\infty} a_{lm} \left(1-z^2\right)^2 T_l(z) e^{-im\alpha x+\sigma t}.$$

We do not require σt as the following analysis will not be affected by it. Hence,

$$\$\phi = \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty} \sum_{l=0}^{\infty} a_{lm} \left(1-z^2\right)^2 T_l(z) \left[\cos(m\alpha x) + i\sin(m\alpha x)\right]$$

We can see that $\left(1-z^2\right)^2$ will always follow T_l, i.e. be the same because of multiplication. For ∂z we only need to take into account derivatives of $T_l(z)$ and we need the derivatives of sine and cosine functions, i.e.

x	sin m	cos m	z	T_l^+	T_l^{++}
∂_x	cos m	-sin m	∂_z	T_l^{++}	T_l^+
∂_x^2	-sin m	-cos m	∂_z^2	T_l^+	T_l^{++}
∂_x^3	-cos m	sin m	∂_z^3	T_l^{++}	T_l^+
∂_x^4	sin m	cos m	∂_z^4	T_l^+	T_l^{++}
∂_x^5	cos m	-sin m	∂_z^5	T_l^{++}	T_l^+

We have adopted the notation + means odd and ++means even, also note that we have started with ***both*** l ***and*** m ***odd*** to include $l,m=1$.

F.1 Symmetries Emerging from the Linear Terms

Taking our homogeneously heated flow profile as an even profile $U(-z)=U(z)$and using the fact that in the linear system we are working with a single harmonic within a single ϕ and hence we only need one function of sine and cosine, i.e. m^+, i.e. $m=1$ only. We use the terms from the poloidal perturbation equation noting that $\breve{U}$ is fixed at an even chebyshev as our initial conditions require an even function in U(z).

$$\nabla^2\Delta_2\phi \to \left(\partial_x^2+\partial_z^2\right)\left(\partial_x^2\right)\phi \to \left(\partial_x^4\right)\phi + \left(\partial_x^2\partial_z^2\right)\phi$$

$$\nabla^4\Delta_2\phi \to \left(\partial_x^4+2\partial_x^2\partial_z^2+\partial_z^4\right)\left(\partial_x^2\right)\phi \to \left(\partial_x^6\right)\phi + \left(2\partial_x^4\partial_z^2\right)\phi + \left(\partial_x^2\partial_z^4\right)\phi$$

$$\breve{U}\partial_x\nabla^2\Delta_2\phi \rightarrow T_l^{++}(\partial_x)\left(\partial_x^2+\partial_z^2\right)\left(\partial_x^2\right)\phi \rightarrow T_l^{++}(\partial_x^5)\phi + T_l^{++}\left(\partial_x^3\partial_z^2\right)\phi$$

$$\breve{U}\partial_z^2\partial_x\Delta_2\phi \rightarrow T_l^{++}(\partial_x)\left(\partial_z^2\right)\left(\partial_x^2\right)\phi \rightarrow T_l^{++}\left(\partial_x^3\right)\left(\partial_z^2\right)\phi$$

We now analyse the output set produced by applying the above derivative operators when starting with the initial conditions l^+, m^+, l^{++}, m^+.

$$l^+, m^+ l^{++}, m^+$$

$$T_l^+ \cos m^+ + T_l^+ \sin m^+ T_l^{++} \cos m^+ + T_l^{++} \sin m^+$$

$$-T_l^+ \cos m^+ - T_l^+ \sin m^+ - T_l^{++} \cos m^+ - T_l^{++} \sin m^+$$

$$-T_l^+ \cos m^+ - T_l^+ \sin m^+ - T_l^{++} \cos m^+ - T_l^{++} \sin m^+$$

$$T_l^+ \cos m^+ + T_l^+ \sin m^+ T_l^{++} \cos m^+ + T_l^{++} \sin m^+$$

$$-T_l^+ \cos m^+ - T_l^+ \sin m^+ - T_l^{++} \cos m^+ - T_l^{++} \sin m^+$$

$$-T_l^+ \sin m^+ + T_l^+ \cos m^+ - T_l^{++} \sin m^+ + T_l^{++} \cos m^+$$

$$T_l^+ \sin m^+ + T_l^+ \cos m^+ T_l^{++} \sin m^+ + T_l^{++} \cos m^+$$

$$T_l^+ \sin m^+ - T_l^+ \cos m^+ T_l^{++} \sin m^+ - T_l^{++} \cos m^+$$

Without proceeding any further we can readily see that the output is a closed set with all possible members;

$$T_l^+ \cos m^+ T_l^{++} \sin m^+$$

$$T_l^+ \sin m^+ T_l^{++} \cos m^+$$

We now have to analyse which members of this set of symmetries will survive after application of the non-linear term.

F.2 2D Non-Linear Symmetries

We shall remind ourselves of the surviving $\phi-\phi$ terms of $\tilde{\delta}\cdot\tilde{u}\cdot\tilde{\nabla}\tilde{u}$, after taking the curl curl of the non-linear part of our N.S. Equation. We only require terms in ∂_x, ∂_z:

$$+\left(\partial_x\partial_z\phi\right)\left(\partial_x^5\phi\right)+\left(\partial_x\partial_z\phi\right)\left(\partial_x^3\partial_z^2\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_z^3\phi\right)$$
$$-\left(\partial_x^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)+\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^4\phi\right)+\left(\partial_x^2\partial z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)$$

We consider that both of the $\phi - \phi$ are the same, this is an important starting premise. We shall only fully work through the symmetries for the first non-linear term leaving the reader to work through the other terms as they wish. Taking the first non-linear term we proceed as follows:

$$(\partial x\partial z\phi)(\partial^5 x\phi)$$

$$(\partial x\partial z\phi) \to -T_l^{++}\sin m^+ + T_l^{++}\cos m^+$$

$$(\partial^5 x\phi) \to -T_l^{+}\sin m^+ + T_l^{+}\cos m^+$$

$$(\partial x\partial z\phi)\left(\partial^5 x\phi\right) \to \left(-T_l^{++}\sin m^+ + T_l^{++}\cos m^+\right)\left(-T_l^{+}\sin m^+ + T_l^{+}\cos m^+\right)$$
$$\to T_l^{++}\sin m^+ T_l^{+}\sin m^+ \to T_l^{+}\sin^2 m^+ \to T_l^{+}\cos 2m^{++} \to T_l^{+}\cos m^{++}$$
$$\because \cos^2 m^+ - \sin^2 m^+ \equiv \cos 2m^{++} \therefore \sin^2 m^+ \propto \cos m^{++}$$

$$\to T_l^{++}\cos m^+ T_l^{+}\cos m^+ \to T_l^{+}\cos^2 m^+ \to T_l^{+}\cos 2m^{++} \to T_l^{+}\cos m^{++}$$
$$\because \cos^2 m^+ - \sin^2 m^+ \equiv \cos 2m^{++} \therefore \cos^2 m^+ \propto \cos m^{++}$$

$$\to -T_l^{++}\cos m^+ T_l^{+}\sin m^+ \to -T_l^{+}\cos m^+ \sin m^+ \to -T_l^{+}\sin 2m^{++} \to -T_l^{+}\sin m^{++}$$
$$\because 2\sin m^+ \cos m^+ \equiv \sin 2m^{++} \therefore 2\sin m^+ \cos m^+ \propto \sin m^{++}$$
$$\to -T_l^{++}\sin m^+ T_l^{+}\cos m^+ \to -T_l^{+}\sin m^+ \cos m^+ \to -T_l^{+}\sin 2m^{++} \to -T_l^{+}\sin m^{++}$$

The negative signs are immaterial as a_{mn} can be positive or negative. Also we are only interested in the element produced by the product as we are finding $\phi - \phi$ terms.

Proceeding as above with the remaining terms we can summarise as follows;

$$(\partial x\partial z\phi)(\partial^3 x\partial^2 z\phi) \to T_l^{+}\cos m^{++}, -T_l^{+}\sin m^{++}$$

In the above result, note that although there are four terms only two are unique so are kept.

$$-(\partial^3 x\phi)(\partial^3 x\partial z\phi) \to -T_l^{+}\cos m^{++}, T_l^{+}\sin m^{++}$$

$$-(\partial^3 x\phi)(\partial x\partial^3 z\phi) \to T_l^{+}\cos m^{++}, -T_l^{+}\sin m^{++}$$

$$-(\partial^2 x\phi)(\partial^4 x\partial z\phi) \to T_l^{+}\cos m^{++}, T_l^{+}\sin m^{++}$$

$$-(\partial^2 x\phi)(\partial^2 x\partial^3 z\phi) \to T_l^{+}\cos m^{++}, -T_l^{+}\sin m^{++}$$

$$(\partial^2 x\partial z\phi)(\partial^4 x\phi) \to -T_l^{+}\cos m^{++}, -T_l^{+}\sin m^{++}$$

$$(\partial^2 x\partial z\phi)(\partial^2 x\partial^2 z\phi) \rightarrow T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

Closed Set

$$T_l^+ \sin m^{++}, T_l^+ \cos m^{++}$$

We shall now analyse different start points for l and m to be thorough. by taking our ***l and m values as both even***, in order to check whether we need to append our symmetry group already found. (l^{++}, m^{++})

$$(\partial x\partial z\phi)(\partial^5 x\phi) \rightarrow T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$(\partial x\partial z\phi)(\partial^3 x\partial^2 z\phi) \rightarrow -T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$-(\partial^3 x\phi)(\partial^3 x\partial z\phi) \rightarrow -T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$-(\partial^3 x\phi)(\partial x\partial^3 z\phi) \rightarrow T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$-(\partial^2 x\phi)(\partial^4 x\partial z\phi) \rightarrow T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$-(\partial^2 x\phi)(\partial^2 x\partial^3 z\phi) \rightarrow T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$(\partial^2 x\partial z\phi)(\partial^4 x\phi) \rightarrow -T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$(\partial^2 x\partial z\phi)(\partial^2 x\partial^2 z\phi) \rightarrow T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

Closed Set

$T_l^+ \sin m^{++}, T_l^+ \cos m^{++}$, as before.

We shall now proceed by taking our ***l and m values as opposites***, in order to check whether we need to append our symmetry group already found. (m^{++}, l^+)

$$(\partial x\partial z\phi)(\partial^5 x\phi) \rightarrow T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$(\partial x\partial z\phi)(\partial^3 x\partial^2 z\phi) \rightarrow -T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$-(\partial^3 x\phi)(\partial^3 x\partial z\phi) \rightarrow -T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$-(\partial^3 x\phi)(\partial x\partial^3 z\phi) \rightarrow T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$-(\partial^2 x\phi)(\partial^4 x\partial z\phi) \rightarrow T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$-(\partial^2 x\phi)(\partial^2 x\partial^3 z\phi) \rightarrow -T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$(\partial^2 x\partial z\phi)(\partial^4 x\phi) \rightarrow -T_{nl}^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$(\partial^2 x\partial z\phi)(\partial^2 x\partial^2 z\phi) \rightarrow T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

Closed Set

$T_l^+ \sin m^{++}, T_l^+ \cos m^{++}$, as before.

At this point it can be induced that if we start with (m^+, l^{++}) we will also have the same closed set, because we will always be left with even sine or cosine functions and even Chebyshev polynomials.

We should now proceed to look at the output set produced when $\phi - \phi$ are ***different***. This will lead to all the members of the minimal closed set required to realise our symmetry group. We only need to analyse when;

$\phi_1 = l^+, m^+; \phi_2 = l^{++}, m^{++}.$ ——- A

$\phi_1 = l^{++}, m^+; \phi_2 = l^+, m^{++}$ ——- B

We shall look at A first, but missing out all the intermediate steps.

$$(\partial x \partial z \phi)(\partial^5 x \phi)$$

$$\begin{aligned}
&T_l^{++} T_l^{++} \left[\sin m^+ + \cos m^+\right] \left[\sin m^{++} + \cos m^{++}\right] \\
&\to T_l^{++} \left[\cos\left(m^+ + m^{++}\right) + \sin\left(m^+ + m^{++}\right)\right] \\
&\to T_l^{++} \cos m^+ \\
&\to T_l^{++} \sin m^+
\end{aligned}$$

Now B.

$$(\partial x \partial z \phi)(\partial^5 x \phi)$$

$$\begin{aligned}
&T_l^{+} T_l^{+} \left[\sin m^+ + \cos m^+\right] \left[\sin m^{++} + \cos m^{++}\right] \\
&\to T_l^{++} \left[\cos\left(m^+ + m^{++}\right) + \sin\left(m^+ + m^{++}\right)\right] \\
&\to T_l^{++} \cos m^+ \\
&\to T_l^{++} \sin m^+
\end{aligned}$$

We can proceed very quickly at this point and providing we start with l and m different we create the above elements for every $\phi - \phi$ term.

We now have to perform the same analysis with temperature included.

We shall remind ourselves of the surviving $\phi - \theta$ terms of $\tilde{\delta} \cdot \tilde{u} \cdot \tilde{\nabla} \tilde{u}$, the non-linear part of our N.S. Equation.

$$(\partial x \partial z \phi)\,(\partial x \theta) - \left(\partial^2 x \phi\right)(\partial z \theta)$$

We have;

$$\theta = \sum_{\substack{m=-\infty \\ m \neq 0}}^{m=\infty} \sum_{l=0}^{\infty} b_{lm}\left(1-z^2\right) T_l(z)\left[\cos(m\alpha x) + i\sin(m\alpha x)\right]$$

The only difference between this function and the velocity function is $\left(1-z^2\right)$, this again will follow the nature of T_l due to multiplication.

We shall begin with l and m odd. (m^+, l^+)

$$(\partial x \partial z \phi)(\partial x \theta) \to T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$-(\partial^2 x \phi)(\partial z \theta) \to T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$\left(m^{++}, l^{++}\right)$$

$$(\partial x \partial z \phi)(\partial x \theta) \to T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$-(\partial^2 x \phi)(\partial z \theta) \to T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

$$\left(m^{++}, l^{+}\right)$$

$$(\partial x \partial z \phi)(\partial x \theta) \to T_l^+ \cos m^{++}, -T_l^+ \sin m^{++}$$

$$-(\partial^2 x \phi)(\partial z \theta) \to T_l^+ \cos m^{++}, T_l^+ \sin m^{++}$$

<u>Closed Set</u>

$$T_l^+ \sin m^{++}, T_l^+ \cos m^{++}.$$

By the same reasoning as used during the $\phi - \phi$ analysis we induce that (m^+, l^{++}) will give the same results as above.

<u>**Conclusion**</u>

We can now conclude that the closed set for our 2D symmetry group is made up of

$$T_l^{+}\sin m^{++}, T_l^{+}\cos m^{++}, T_l^{++}\sin m^{+}, T_l^{++}\cos m^{+}$$

For l + m = even then $a_{lm} = 0$, and we should see zero eigenvalues where this condition is met in the ouput files.

There is a simpler method for finding the 2D symmetries which is outlined here; Referring to Figure 3.1 we can immediately state that $f(x,z) = f(-x,-z)$.

$$\begin{aligned}
f(x,z) = & \\
& \sum a^c_{lm} cos(m\alpha x)T_l(z) + a^s_{lm} sin(m\alpha x)T_l(z) \\
& \Longleftrightarrow \\
& \sum a^c_{lm} cos(-m\alpha x)T_l(-z) + a^s_{lm} sin(-m\alpha x)T_l(-z) \\
& = \sum a^c_{lm} cos(m\alpha x)T_l(z) + a^s_{lm} sin(m\alpha x)T_l(z) \\
& \therefore \\
& \sum [a^c_{lm} cos(m\alpha x)][T_l(-z) - T_l(z)] + a^s_{lm}[sin(-m\alpha x)T_l(-z) - sin(m\alpha x)T_l(z)] = 0
\end{aligned} \tag{F.1}$$

Where $[T_l(-z) - T_l(z)] = 0$ for l = even , and $[T_l(-z)T_l(z)] \neq 0$ for l = odd.
Note: $cos[-m\alpha x] = cos[m\alpha x]$

F.3 3D Symmetries.

For the 3D symmetries there seems to be no easy way to find a closed symmetry group. We begin with the poloidal symmetries first then continue with the toroidal symmetries.

F.3.1 Poloidal Symmetries

We need to proceed with more care as we now have to include all the terms produced by the curl curl of the non-linear $u \cdot \nabla u$. Which are listed below;

φ - φ Terms from curl curl of $u \cdot \nabla u$

$$+(\partial_x\partial_z\phi)\left(\partial_x^5\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x\partial_y^4\phi\right)$$
$$+2(\partial_x\partial_z\phi)\left(\partial_x^3\partial_y^2\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x^3\partial_z^2\phi\right)+(\partial_x\partial_z\phi)\left(\partial_x\partial_y^2\partial_z^2\phi\right)$$
$$+(\partial_y\partial_z\phi)\left(\partial_x^4\partial_y\phi\right)+(\partial_y\partial_z\phi)\left(\partial_y^5\phi\right)$$
$$+2(\partial_y\partial_z\phi)\left(\partial_x^2\partial_y^3\phi\right)+(\partial_y\partial_z\phi)\left(\partial_x^2\partial_y\partial_z^2\phi\right)+(\partial_y\partial_z\phi)\left(\partial_y^3\partial_z^2\phi\right)$$
$$-\left(\partial_x^3\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x^3\phi\right)\left(\partial_x\partial_z^3\phi\right)$$
$$-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x^3\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_y^2\partial_z\phi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x\partial_z^3\phi\right)$$
$$-\left(\partial_x^2\partial_y\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_x^2\partial_y\phi\right)\left(\partial_y\partial_z^3\phi\right)$$
$$-\left(\partial_y^3\phi\right)\left(\partial_x^2\partial_y\partial_z\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_y^3\partial_z\phi\right)-\left(\partial_y^3\phi\right)\left(\partial_y\partial_z^3\phi\right)$$

$$-\left(\partial_x^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^4\partial_z\phi\right)$$
$$-2\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)$$
$$-\left(\partial_y^2\phi\right)\left(\partial_x^4\partial_z\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_y^4\partial_z\phi\right)$$
$$-2\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_y^2\partial_z\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_z^3\phi\right)-\left(\partial_y^2\phi\right)\left(\partial_y^2\partial_z^3\phi\right)$$

$$-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^4\phi\right)-\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^4\phi\right)+\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^2\partial_z^2\phi\right)$$
$$-\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)$$

$$+\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^4\phi\right)+\left(\partial_x^2\partial_z\phi\right)\left(\partial_x^2\partial_z^2\phi\right)$$
$$+4(\partial_x\partial_y\partial_z\phi)\left(\partial_x^3\partial_y\phi\right)+4(\partial_x\partial_y\partial_z\phi)\left(\partial_x\partial_y^3\phi\right)+4(\partial_x\partial_y\partial_z\phi)\left(\partial_x\partial_y\partial_z^2\phi\right)$$
$$+\left(\partial_y^2\partial_z\phi\right)\left(\partial_y^4\phi\right)$$

We have already analysed the terms containing $\partial_x\partial_z$ and need to now look at the terms containing $\partial_y\partial_z$, then finally the terms containing all $\partial_x\partial_y$ and ∂_z.

Looking at the 8 terms containing ∂_y and ∂_z only, they are exactly the same as the 8 terms containing the $\partial_x\partial_z$ terms only and thus we can conclude that the symmetries must be the same for each set.

For terms containing ∂_x and ∂_y with or without ∂_z we need;

$$\sum_{m,n=-\infty m,n\neq 0}^{m,n=\infty}\quad\sum_{l=0}^{\infty} a_{mn}b_{mn}\left(1-z^2\right)^2 T_l(z)e^{-(im\alpha x+n\beta y)}.$$

$$\sum_{\substack{m,n=-\infty\\ m,n\neq 0}}^{m,n=\infty}\quad\sum_{l=0}^{\infty} a_{mn}b_{mn}\left(1-z^2\right)^2 T_l(z)\left[\cos(m\alpha x)+i\sin(m\alpha x)\right]\left[\cos(n\beta y)+i\sin(n\beta y)\right]$$

Now we assign; $\partial x \rightarrow m, \partial y \rightarrow n, \partial z \rightarrow l$.

Start with both $\phi's$ the same in this case l^+, m^+, n^+ all odd

We have $T_l e^{i(m\alpha x+n\beta y)} \rightarrow T_l^+ [\cos m^+ + i\sin m^+] \cdot [\cos n^+ + i\sin n^+]$

Note that we must use the derivative product, where if we have terms in x only and are finding ∂_y, the outcome for that derivative is zero.

$$(\partial x \partial z \phi)\left(\partial x \partial^4 y \phi\right)$$

$$1)\,(\partial x \partial z \phi) \rightarrow T_l^{++}\left[-\sin m^+ + \cos m^+\right] \cdot \left[\cos n^+ + \sin n^+\right]$$

$$2)\,\left(\partial x \partial^4 y \phi\right) \rightarrow T_l^{+}\left[-\sin m^+ + \cos m^+\right] \cdot \left[\cos n^+ + \sin n^+\right]$$

1) is ∂x we now find $\partial^4 y$ of this term to get 2)

We now multiply these terms together:

$$T_l^+ [-\sin m^+ + \cos m^+]^2 \cdot [\cos n^+ + \sin n^+]^2$$
$$\rightarrow T_l^+ [\cos m^{++} + \sin m^{++}] \cdot [\cos n^{++} + \sin n^{++}]$$

The symmetries produced are;

$$T_l^+ \cos m^{++} \cos n^{++}$$
$$T_l^+ \sin m^{++} \sin n^{++}$$
$$T_l^+ \sin m^{++} \cos n^{++}$$
$$T_l^+ \cos m^{++} \sin n^{++}$$

We can see straight away that T_l will always be odd for m^+, n^+ or m^{++}, n^{++} combinations because we always combine an odd with an even power of ∂z in each pair of $\phi - \phi$ combinations.

If we analyse the orders of each derivative for each pair of the non-linear terms, we can quite quickly see that will produce the output set below. The sine and cosine terms will always be even because of the resulting double angle identities.

We should now proceed to look at the output set produced when $\phi - \phi$ are ***different***, i.e. we need to analyse when;

$\phi_1 = l^{++}, m^{++}, n^{++}; \phi_2 = l^+, m^+, n^+$. ——- A

$\phi_1 = l^{+}, m^{+}, n^{++}; \phi_2 = l^{++}, m^{++}, n^+$ ——- B

We shall look at A first.

$$(\partial x \partial z \phi)\left(\partial x \partial^4 y \phi\right)$$

$$(\partial x\partial z\phi) \rightarrow T_l^{+}\left[-\sin m^{++}+\cos m^{++}\right]\cdot\left[\cos n^{++}+\sin n^{++}\right]$$

$$\left(\partial x\partial^4 y\phi\right) \rightarrow T_l^{+}\left[-\sin m^{+}+\cos m^{+}\right]\cdot\left[\cos n^{+}+\sin n^{+}\right]$$

We now multiply these terms together:

$$T_l^{++}\left[\cos(m^{++}+m^{+})+\sin(m^{++}+m^{+})\right]\cdot\left[\cos(n^{++}+n^{+})+\sin(n^{++}+n^{+})\right]$$
$$\rightarrow T_l^{++}\left[\cos m^{+}+\sin m^{+}\right]\cdot\left[\cos n^{+}+\sin n^{+}\right]$$

Which gives the output set

$$\begin{array}{l} T_l^{++}\cos m^{+}\cos n^{+} \\ T_l^{++}\sin m^{+}\sin n^{+} \\ T_l^{++}\sin m^{+}\cos n^{+} \\ T_l^{++}\cos m^{+}\sin n^{+} \end{array}$$

We shall now look at B.

$$(\partial x\partial z\phi)\left(\partial x\partial^4 y\phi\right)$$

$$(\partial x\partial z\phi) \rightarrow T_l^{++}\left[-\sin m^{+}+\cos m^{+}\right]\cdot\left[\cos n^{++}+\sin n^{++}\right]$$

$$\left(\partial x\partial^4 y\phi\right) \rightarrow T_l^{++}\left[-\sin m^{++}+\cos m^{++}\right]\cdot\left[\cos n^{+}+\sin n^{+}\right]$$

We now multiply these terms together:

$$T_l^{++}\left[\cos(m^{++}+m^{+})+\sin(m^{++}+m^{+})\right]\cdot\left[\cos(n^{++}+n^{+})+\sin(n^{++}+n^{+})\right]$$
$$\rightarrow T_l^{++}\left[\cos m^{+}+\sin m^{+}\right]\cdot\left[\cos n^{+}+\sin n^{+}\right]$$

Which gives the same elements as currently in the set.

In summary, our closed output set is made up of 16 terms;

$$\begin{array}{l} T_l^{++}\cos m^{+}\cos n^{+} \\ T_l^{++}\sin m^{+}\sin n^{+} \\ T_l^{++}\sin m^{+}\cos n^{+} \\ T_l^{++}\cos m^{+}\sin n^{+} \end{array} \quad \text{and} \quad \begin{array}{l} T_l^{+}\cos m^{++}\cos n^{++} \\ T_l^{+}\sin m^{++}\sin n^{++} \\ T_l^{+}\sin m^{++}\cos n^{++} \\ T_l^{+}\cos m^{++}\sin n^{++} \end{array}.$$

We must now proceed to analyse the symmetries produced by the $\phi-\psi$ and $\psi-\psi$ terms for completeness.

φ-ψ Terms

$$\begin{aligned}
&\left(\partial_x^3\phi\right)\left(\partial_y\partial_z^2\psi\right)+2\left(\partial_x\partial_y^2\phi\right)\left(\partial_y\partial_z^2\psi\right)+2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_z\psi\right)\\
&-2\left(\partial_x^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right)+2\left(\partial_y^2\partial_z\phi\right)\left(\partial_x\partial_y\partial_z\psi\right)-2\left(\partial_x\partial_y\partial_z\phi\right)\left(\partial_y^2\partial_z\psi\right)\\
&-\left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_z^2\psi\right)-\left(\partial_y^3\phi\right)\left(\partial_x\partial_z^2\psi\right)+\left(\partial_x^4\partial_y\phi\right)\left(\partial_x\psi\right)+\left(\partial_y^5\phi\right)\left(\partial_x\psi\right)\\
&+2\left(\partial_x^2\partial_y^3\phi\right)\left(\partial_x\psi\right)+\left(\partial_x^2\partial_y\partial_z^2\phi\right)\left(\partial_x\psi\right)+\left(\partial_y^3\partial_z^2\phi\right)\left(\partial_x\psi\right)-\left(\partial_x^5\phi\right)\left(\partial_y\psi\right)\\
&-\left(\partial_x\partial_y^4\phi\right)\left(\partial_y\psi\right)-2\left(\partial_x^3\partial_y^2\phi\right)\left(\partial_y\psi\right)-\left(\partial_x^3\partial_z^2\phi\right)\left(\partial_y\psi\right)-\left(\partial_x\partial_y^2\partial_z^2\phi\right)\left(\partial_y\psi\right)
\end{aligned}$$

$$\begin{aligned}
&-\left(\partial_x^3\phi\right)\left(\partial_x^2\partial_y\psi\right)-\left(\partial_x^3\phi\right)\left(\partial_y^3\psi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_y^3\psi\right)-\left(\partial_x\partial_y^2\phi\right)\left(\partial_x^2\partial_y\psi\right)\\
&+\left(\partial_x^2\partial_y\phi\right)\left(\partial_x^3\psi\right)+\left(\partial_x^2\partial_y\phi\right)\left(\partial_x\partial_y^2\psi\right)+\left(\partial_y^3\phi\right)\left(\partial_x^3\psi\right)+\left(\partial_y^3\phi\right)\left(\partial_x\partial_y^2\psi\right)\\
&+2\left(\partial_x^3\partial_y\phi\right)\left(\partial_x^2\psi\right)+2\left(\partial_x\partial_y^3\phi\right)\left(\partial_x^2\psi\right)+2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_x^2\psi\right)-2\left(\partial_x^4\phi\right)\left(\partial_x\partial_y\psi\right)\\
&-2\left(\partial_x^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right)+2\left(\partial_y^4\phi\right)\left(\partial_x\partial_y\psi\right)+2\left(\partial_y^2\partial_z^2\phi\right)\left(\partial_x\partial_y\psi\right)-2\left(\partial_x^3\partial_y\phi\right)\left(\partial_y^2\psi\right)\\
&-2\left(\partial_x\partial_y^3\phi\right)\left(\partial_y^2\psi\right)-2\left(\partial_x\partial_y\partial_z^2\phi\right)\left(\partial_y^2\psi\right)
\end{aligned}$$

Start with both $\phi-\psi$ the same in this case l^+, m^+, n^+ all odd

$$\left(\partial x^3\phi\right)\left(\partial y\partial^2 z\psi\right)$$

This results in the closed set;

$$\begin{aligned}
&T_l^{++}\cos m^{++}\cos n^{++}, T_/^{++}\sin m^{++}\sin n^{++}\\
&T_l^{++}\cos m^{++}\sin n^{++}, T_l^{++}\sin m^{++}\cos n^{++}
\end{aligned}\qquad (F.2).$$

The following terms will also give F.2 by inspection directly;

$$\begin{aligned}
&\left(\partial x\partial y\partial z\phi\right)\left(\partial^2 x\partial z\psi\right),\left(\partial x^3\phi\right)\left(\partial y\partial^2 z\psi\right),\\
&\left(\partial^2 x\partial z\phi\right)\left(\partial x\partial y\partial z\psi\right),\left(\partial^2 y\partial z\phi\right)\left(\partial x\partial y\partial z\psi\right),\left(\partial x\partial y\partial z\phi\right)\left(\partial^2 y\partial z\psi\right),\\
&\left(\partial^2 x\partial y\phi\right)\left(\partial x\partial^2 z\psi\right),\left(\partial^3 y\phi\right)\left(\partial x\partial^2 z\psi\right)
\end{aligned}$$

We can see immediately that all the pairs of derivatives have either no ∂z component or one $\partial^2 z$ component or a pair of ∂z and ∂z. This leads us to the conclusion that for l^+ we will always have T_i^{++} outputs and for l^{++} we will always have T_i^{++} outputs. Continuing we also find that the following also give F.2.

$$\left(\partial^4 x\partial y\phi\right)\left(\partial x\psi\right),\left(\partial^5 y\phi\right)\left(\partial x\psi\right),\left(\partial x\partial y\partial z\phi\right)\left(\partial^2 x\partial z\psi\right)$$

In fact all the pairs will give the same results to F.2 due to the combinations of cosines and sines with all of the harmonics.

Now with (l^{++}, m^{++}, n^{++}) all even. This will always give bracket pairs of $\sin m^{++}, \cos m^{++}$ and $\sin n^{++}, \cos n^{++}$, which on combination will always give the outputs as F.2.

Now make harmonics different (l^{++}, m^{+}, n^{++}).

$$\left(\partial x^3 \phi\right)\left(\partial y \partial^2 z \psi\right)$$

This also results in the closed set F.2.

We again can see that this result is repeated for all pairs given that m^{+}, n^{++} as well as for m^{++}, n^{+} because the pairs of brackets have the same odd or even harmonics. More crucially we need to investigate when $\phi \neq \psi$.

We remind ourselves of the start conditions.

$\phi = l^{++}, m^{++}, n^{++}; \psi = l^{+}, m^{+}, n^{+}$ ——- A

$\phi = l^{+}, m^{+}, n^{++}; \psi = l^{++}, m^{++}, n^{+}$ ——- B

Start with A first

$$\left(\partial^3 x \phi\right)\left(\partial y \partial^2 z \psi\right) \rightarrow T_l^{+} \cos m^{+} \cos n^{+}, T_l^{+} \sin m^{+} \sin n^{+}, T_l^{+} \sin m^{+} \cos n^{+}, T_l^{+} \cos m^{+} \sin n^{+}$$

It can be see that for all pairs of terms this new output symmetry set will emerge.

Start with B now.

$$\left(\partial^3 x \phi\right)\left(\partial y \partial^2 z \psi\right) \rightarrow T_l^{+} \cos m^{+} \cos n^{+}, T_l^{+} \sin m^{+} \sin n^{+}, T_l^{+} \sin m^{+} \cos n^{+}, T_l^{+} \cos m^{+} \sin n^{+}$$

Which gives the same as A.

<u>ψ -ψ Terms</u>

$$(\partial_x \partial_y \partial_z \psi)(\partial_x \partial_y \psi) - (\partial_x^2 \psi)(\partial_y^2 \partial_z \psi) - (\partial_y^2 \psi)(\partial_x^2 \partial_z \psi)$$

Start with both $\psi - \psi$ the same in this case l^{+}, m^{+}, n^{+} all odd

$$\left(\partial x \partial y \partial z \psi\right)\left(\partial x \partial y \psi\right), \left(\partial^2 y \psi\right)\left(\partial^2 x \partial z \psi\right), \left(\partial^2 x \psi\right)\left(\partial^2 y \partial z \psi\right)$$

This results in the closed set;

$$\begin{array}{c} T_l^{+} \cos m^{++} \cos n^{++}, T_/^{+} \sin m^{++} \sin n^{++} \\ T_l^{+} \cos m^{++} \sin n^{++}, T_l^{+} \sin m^{++} \cos n^{++} \end{array} \quad (F.3)$$

With (l^{++}, m^{++}, n^{++}) all even we will also arrive at the same result as F.3 due to the double angle identities always being even in this case.

Now make harmonics different (l^{+}, m^{++}, n^{+}), l^{+} or l^{++} will aways give T_l^{+} due to the odd and even combinations of orders of ∂z in the $\psi - \psi$ pairs.

$(\partial x \partial y \partial z \psi)(\partial x \partial y \psi)$ also gives F.3.

We can quite quickly see that other derivative pairs of $\psi - \psi$ will give the same result as above.

More crucially we need to investigate when $\psi_1 \neq \psi_2$.

Again with the same initial conditions as previous.

$\psi_1 = l^{++}, m^{++}, n^{++}; \psi_2 = l^{+}, m^{+}, n^{+}$. ——- A

$\psi_1 = l^{+}, m^{+}, n^{++}; \psi_2 = l^{++}, m^{++}, n^{+}$ ——- B

Start with A first.

$$(\partial x \partial y \partial z \psi)(\partial x \partial y \psi) \rightarrow T_l^{++} \cos m^{+} \cos n^{+}, T_l^{++} \sin m^{+} \sin n^{+}, T_l^{++} \cos m^{+} \sin n^{+}, T_l^{++} \sin m^{+} \cos n^{+}$$

Start with B.

$$(\partial x \partial y \partial z \psi)(\partial x \partial y \psi)$$

This gives the same output set as that for A. Examining the other derivative pairs we will always arrive at the same output symmetry group

We now need to analyse the terms from the toroidal part of the equations as we are in 3D space.

F.3.2 Toroidal Symmetries

$\varphi - \varphi$ terms from curl of $u \cdot \nabla u$

$$(\partial_x \partial_z^2 \phi)(\partial_x^2 \partial_y \phi) + (\partial_x \partial_z^2 \phi)(\partial_y^3 \phi) - (\partial_y \partial_z^2 \phi)(\partial_x^3 \phi) - (\partial_y \partial_z^2 \phi)(\partial_x \partial_y^2 \phi)$$

Start with both $\phi - \phi$ the same in this case l^{+}, m^{+}, n^{+} all odd and we need only check $(\partial_x \partial_z^2 \phi)(\partial_x^2 \partial_y \phi), (\partial_x \partial_z^2 \phi)(\partial_y^3 \phi)$ as the others have the same generic structure

$$(\partial x \partial^2 z \phi)(\partial^2 x \partial y \phi), (\partial x \partial^2 z \phi)(\partial^3 y \phi)$$

Results in the closed set;

$$T_l^{++}\cos m^{++}\cos n^{++}, T_/^{++}\sin m^{++}\sin n^{++} \quad .(F.4)$$
$$T_l^{++}\cos m^{++}\sin n^{++}, T_l^{++}\sin m^{++}\cos n^{++}$$

Now make harmonics (l^{++}, m^{++}, n^{++}).

We see that T_l^{++} will always remain T_l^{++} for any start point l^{++} or l^{+}, so immediately we see that T_l^{++} is always in the output set.

$$\left(\partial x \partial^2 z \phi\right)\left(\partial^2 x \partial y \phi\right)$$

$$\left(\partial x \partial^2 z \phi\right)\left(\partial^3 y \phi\right)$$

Give the same closed set F.4..

Now make harmonics all different (l^{++}, m^{+}, n^{++}).

$$\left(\partial x \partial^2 z \phi\right)\left(\partial^2 x \partial y \phi\right)$$

$$\left(\partial x \partial^2 z \phi\right)\left(\partial^3 y \phi\right)$$

Give the same closed set F.4.

More crucially we need to investigate when $\phi_1 \neq \phi_2$.

$\phi_1 = l^{++}, m^{++}, n^{++}; \phi_2 = l^{+}, m^{+}, n^{+}$. ——- A

$\phi_1 = l^{+}, m^{+}, n^{++}; \phi_2 = l^{++}, m^{++}, n^{+}$ ——- B

Start with A first.

$$\left(\partial x \partial^2 z \phi_1\right)\left(\partial^2 x \partial y \phi_2\right), \left(\partial x \partial^2 z \phi_1\right)\left(\partial^3 y \phi_2\right)$$

This gives the output symmetry group;

$$T_l^{+}\cos m^{+}\cos n^{+}, T_/^{+}\sin m^{+}\sin n^{+}, T_l^{+}\cos m^{+}\sin n^{+}, T_l^{+}\sin m^{+}\cos n^{+} (F.5)$$

Now start with B.

$$\left(\partial x \partial^2 z \phi_1\right)\left(\partial^2 x \partial y \phi_2\right), \left(\partial x \partial^2 z \phi_1\right)\left(\partial^3 y \phi_2\right)$$

Also result in F.5.

<u>ψ -ψ Terms</u>

$$\left(\partial_x^2\partial_y\psi\right)\left(\partial_x\psi\right)+\left(\partial_y^3\psi\right)\left(\partial_x\psi\right)-\left(\partial_x^3\psi\right)\left(\partial_y\psi\right)-\left(\partial_x\partial_y^2\psi\right)\left(\partial_y\psi\right)$$

We only need to check $\left(\partial_x^2\partial_y\psi\right)\left(\partial_x\psi\right),\left(\partial_y^3\psi\right)\left(\partial_x\psi\right)$ as the others are the same generically.Very quickly if we look at T_l we notice no ∂z terms at all so we will always have;

$T_l^{++}\cdot T_l^{+}\rightarrow T_l^{++}$
$T_l^{++}\cdot T_l^{++}\rightarrow T_l^{++}$ when $\psi_1=\psi_2$ and
$T_l^{+}\cdot T_l^{++}\rightarrow T_l^{+}$ when $\psi_1\neq\psi_2$.

Start with both $\psi-\psi$ the same in this case l^+,m^+,n^+.

$$\left(\partial^2x\partial y\psi\right)\left(\partial x\psi\right)$$

This results in the closed set;

$$\begin{array}{l}T_l^{++}\cos m^{++}\cos n^{++},T_/^{++}\sin m^{++}\sin n^{++}\\ T_l^{++}\cos m^{++}\sin n^{++},T_l^{++}\sin m^{++}\cos n^{++}\end{array}.(F.6)$$

$$\left(\partial^3y\psi\right)\left(\partial x\psi\right)$$

Give F.6, in fact we can see that with $m^{++},n^{++}$and m^{++},n^{+} and m^{++},n^{++} and m^{+},n^{++} we will always arrive at the same set as above. We need to look at the output symmetry group when $\psi_1\neq\psi_2$. Refering to A and B combinations of l, m and n as before we can quite readily conclude that we will always arrive at the same output symmetry set as F.5.

<u>φ-ψ Terms</u>

$$-\left(\partial_x\partial_z\psi\right)\left(\partial_x^3\phi\right)-\left(\partial_x\partial_z\psi\right)\left(\partial_x\partial_y^2\phi\right)-\left(\partial_y\partial_z\psi\right)\left(\partial_x^2\partial_y\phi\right)-\left(\partial_y\partial_z\psi\right)\left(\partial_y^3\phi\right)$$

$$\left(\partial_x\partial_z\phi\right)\left(\partial_x^3\psi\right)+\left(\partial_x\partial_z\phi\right)\left(\partial_x\partial_y^2\psi\right)+\left(\partial_y\partial_z\phi\right)\left(\partial_x^2\partial_y\psi\right)+\left(\partial_y\partial_z\phi\right)\left(\partial_y^3\psi\right)$$

$$\left(\partial_x^2\psi\right)\left(\partial_x^2\partial_z\phi\right)-\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z\psi\right)-\left(\partial_x^2\phi\right)\left(\partial_y^2\partial_z\psi\right)-\left(\partial_y^2\phi\right)\left(\partial_x^2\partial_z\psi\right)$$

$$\left(\partial_x^2\partial_z\phi\right)\left(\partial_y^2\psi\right)+\left(\partial_y^2\partial_z\phi\right)\left(\partial_x^2\psi\right)$$

Of the above we only need analyse
$\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z\psi\right),\left(\partial_x\partial_z\psi\right)\left(\partial_x^3\phi\right),\left(\partial_x\partial_z\psi\right)\left(\partial_x\partial_y^2\phi\right)$
because $\left(\partial_x\partial_z\phi\right)\left(\partial_x^3\psi\right)$ where ϕ and ψ are swapped will give the same results as some

of the terms above, hence $\left(\partial_x^2\phi\right)\left(\partial_x^2\partial_z\psi\right),\left(\partial_x\partial_z\psi\right)\left(\partial_x^3\phi\right),\left(\partial_x\partial_z\psi\right)\left(\partial_x\partial_y^2\phi\right)$ are the only unique combinations to consider.

Start with both $\phi-\psi$ the same in this case l^+,m^+,n^+.

$$\left(\partial^2x\phi\right)\left(\partial^2x\partial z\psi\right),\left(\partial x\partial z\psi\right)\left(\partial^3x\phi\right),\left(\partial x\partial z\psi\right)\left(\partial x\partial^2y\phi\right)$$

Gives the closed set;

$$\begin{array}{c} T_l^+\cos m^{++}\cos n^{++},T_l^+\sin m^{++}\sin n^{++} \\ T_l^+\cos m^{++}\sin n^{++},T_l^+\sin m^{++}\cos n^{++} \end{array}.(F.7)$$

$$l^{++},m^{++},n^{++}$$

$$\left(\partial^2x\phi\right)\left(\partial^2x\partial z\psi\right),\left(\partial x\partial z\psi\right)\left(\partial^3x\phi\right),\left(\partial x\partial z\psi\right)\left(\partial x\partial^2y\phi\right)$$

Also gives F.7.

l^*,m^{++},n^+ l^* this will always be T_l^+.

$$\left(\partial^2x\phi\right)\left(\partial^2x\partial z\psi\right),\left(\partial x\partial z\psi\right)\left(\partial^3x\phi\right),\left(\partial x\partial z\psi\right)\left(\partial x\partial^2y\phi\right)$$

Also gives F.7.

We need to investigate when $\phi\neq\psi$.

$\phi=l^{++},m^{++},n^{++};\psi=l^+,m^+,n^+$. ——- A

$\phi=l^+,m^+,n^{++};\psi=l^{++},m^{++},n^+$ ——- B

Start with A first.

$$\left(\partial^2x\phi\right)\left(\partial^2x\partial z\psi\right),\left(\partial x\partial z\psi\right)\left(\partial^3x\phi\right),\left(\partial x\partial z\psi\right)\left(\partial x\partial^2y\phi\right)$$

This gives the output symmetry group;

$$T_l^{++}\cos m^+\cos n^+,T_l^{++}\sin m^+\sin n^+,T_l^{++}\cos m^+\sin n^+,T_l^{++}\sin m^+\cos n^+(F.8)$$

Now B.

$$\left(\partial^2x\phi\right)\left(\partial^2x\partial z\psi\right),\left(\partial x\partial z\psi\right)\left(\partial^3x\phi\right),\left(\partial x\partial z\psi\right)\left(\partial x\partial^2y\phi\right)$$

Gives F.7.

Because of the interchanging between ϕ and ψ we should check the following;

$$\psi = l^{++}, m^{++}, n^{+}; \phi = l^{+}, m^{+}, n^{++}.$$

$$\left(\partial^2 x\phi\right)\left(\partial^2 x\partial z\psi\right), \left(\partial x\partial z\psi\right)\left(\partial x\partial^2 y\phi\right), \left(\partial x\partial z\psi\right)\left(\partial^3 x\phi\right)$$

Which gives F.8.

$$\phi = l^{++}, m^{+}, n^{++}; \psi = l^{+}, m^{++}, n^{++}$$

$$\left(\partial^2 x\phi\right)\left(\partial^2 x\partial z\psi\right), \left(\partial x\partial z\psi\right)\left(\partial^3 x\phi\right), \left(\partial x\partial z\psi\right)\left(\partial x\partial^2 y\phi\right)$$

Which gives F.8.

3D Symmetries - Summary

Poloidal Symmetries

$$\phi - \phi$$

$$T_l^{++}\cos m^{+}\cos n^{+}, T_l^{++}\sin m^{+}\sin n^{+}$$
$$T_l^{++}\cos m^{+}\sin n^{+}, T_l^{++}\sin m^{+}\cos n^{+}$$

$$T_l^{+}\cos m^{++}\cos n^{++}, T_l^{+}\sin m^{++}\sin n^{++}$$
$$T_l^{+}\cos m^{++}\sin n^{++}, T_l^{+}\sin m^{++}\cos n^{++}$$

$$\phi - \psi$$

$$T_l^{++}\cos m^{++}\cos n^{++}, T_l^{++}\sin m^{++}\sin n^{++}$$
$$T_l^{++}\cos m^{++}\sin n^{++}, T_l^{++}\sin m^{++}\cos n^{++}$$

$$T_l^{+}\cos m^{+}\cos n^{+}, T_l^{+}\sin m^{+}\sin n^{+}$$
$$T_l^{+}\cos m^{+}\sin n^{+}, T_l^{+}\sin m^{+}\cos n^{+}$$

$$\psi - \psi$$

$$T_l^{+}\cos m^{++}\cos n^{++}, T_l^{+}\sin m^{++}\sin n^{++}$$
$$T_l^{+}\cos m^{++}\sin n^{++}, T_l^{+}\sin m^{++}\cos n^{++}$$

$$T_l^{++}\cos m^{+}\cos n^{+}, T_l^{++}\sin m^{+}\sin n^{+}$$
$$T_l^{++}\cos m^{+}\sin n^{+}, T_l^{++}\sin m^{+}\cos n^{+}$$

Toroidal Symmetries

$$\phi - \phi$$

$$T_l^{++}\cos m^{++}\cos n^{++}, T_l^{++}\sin m^{++}\sin n^{++}$$
$$T_l^{++}\cos m^{++}\sin n^{++}, T_l^{++}\sin m^{++}\cos n^{++}$$

$$T_l^{+}\cos m^{+}\cos n^{+}, T_l^{+}\sin m^{+}\sin n^{+}$$
$$T_l^{+}\cos m^{+}\sin n^{+}, T_l^{+}\sin m^{+}\cos n^{+}$$

$$\phi - \psi$$

$$T_l^{+}\cos m^{++}\cos n^{++}, T_l^{+}\sin m^{++}\sin n^{++}$$
$$T_l^{+}\cos m^{++}\sin n^{++}, T_l^{+}\sin m^{++}\cos n^{++}$$

$$T_l^{++}\cos m^{+}\cos n^{+}, T_l^{++}\sin m^{+}\sin n^{+}$$
$$T_l^{++}\cos m^{+}\sin n^{+}, T_l^{++}\sin m^{+}\cos n^{+}$$

$$\psi - \psi$$

$$T_l^{++}\cos m^{++}\cos n^{++}, T_l^{++}\sin m^{++}\sin n^{++}$$
$$T_l^{++}\cos m^{++}\sin n^{++}, T_l^{++}\sin m^{++}\cos n^{++}$$

$$T_l^{+}\cos m^{+}\cos n^{+}, T_l^{+}\sin m^{+}\sin n^{+}$$
$$T_l^{+}\cos m^{+}\sin n^{+}, T_l^{+}\sin m^{+}\cos n^{+}$$

Summary

The complete symmetry group for the complete system is:-

$$T_l^{+}\cos m^{+}\cos n^{+}, T_l^{+}\sin m^{+}\sin n^{+}$$
$$T_l^{+}\cos m^{+}\sin n^{+}, T_l^{+}\sin m^{+}\cos n^{+}$$

$$T_l^{++}\cos m^{++}\cos n^{++}, T_l^{++}\sin m^{++}\sin n^{++}$$
$$T_l^{++}\cos m^{++}\sin n^{++}, T_l^{++}\sin m^{++}\cos n^{++}$$

$$T_l^{++}\cos m^{+}\cos n^{+}, T_l^{++}\sin m^{+}\sin n^{+}$$
$$T_l^{++}\cos m^{+}\sin n^{+}, T_l^{++}\sin m^{+}\cos n^{+}$$

$$T_l^{+}\cos m^{++}\cos n^{++}, T_l^{+}\sin m^{++}\sin n^{++}$$
$$T_l^{+}\cos m^{++}\sin n^{++}, T_l^{+}\sin m^{++}\cos n^{++}$$

We can see that for m+n = odd we have no members of the set, so for m+n=odd we should have zero coefficients for the fundamental pure mode. Finally; when $m+n$ = odd;

$$a_{lmn} = b_{lmn} = c_{lmn} = 0$$

F.4 Imposing the Reality Condition

Real life only deals with real situations so we impose the Reality Condition

$\phi^* = \phi$, where (*) denotes the complex conjugate

$$\phi = \sum a_{lmn} e^{i(m\alpha x + n\beta y)} T_n$$
$$= \sum \left(a^R_{lmn} + i a^I_{lmn}\right) \left[\cos\left(m\alpha x + n\beta y\right) + i \sin\left(m\alpha x + n\beta y\right)\right] T_l$$
$$\phi^* = \phi$$
$$= a^R_{lmn}\cos\left(m\alpha x + n\beta y\right) - a^I_{lmn}\sin\left(m\alpha x + n\beta y\right) + i\left[a^I_{lmn}\cos\left(m\alpha x + n\beta y\right) + a^R_{lmn}\sin\left(m\alpha x + n\beta y\right)\right]$$
$$= a^R_{lmn}\cos\left(m\alpha x + n\beta y\right) - a^{I*}_{lmn}\sin\left(m\alpha x + n\beta y\right) - i\left[a^{I*}_{lmn}\cos\left(m\alpha x + n\beta y\right) + a^R_{lmn}\sin\left(m\alpha x + n\beta y\right)\right]$$