A SEMI-ANALYTIC FINITE ELEMENT APPROACH TO THE STRESS ANALYSIS OF CIRCULAR PLATES

BARRINGTON DELVES

A thesis submitted in fulfilment of the requirements for the degree of Master of Philosophy

Department of Mechanical Engineering

The University of Aston in Birmingham

September 1977

A SEMI-ANALYTIC FINITE ELEMENT APPROACH TO THE STRESS ANALYSIS OF CIRCULAR PLATES

BARRINGTON DELVES

A thesis submitted in fulfilment of the requirements for the degree of Master of Philosophy

Department of Mechanical Engineering
The University of Aston in Birmingham

1977

SUMMARY

A review of techniques for the calculation of deflection and stresses in circular plates has been made with the object of ascertaining the most suitable general approach for the analysis of plates under conditions of asymmetric loading. The semi-analytic finite element method was selected and has been successfully developed for use in this application.

Computer programs are presented for the analysis of any annular or complete circular plate which is axi-symmetric in its geometric and material properties, but is loaded either symmetrically or asymmetrically. The programs have been applied to the analysis of several test cases and the results compared with those obtained by other means of analysis and, in some cases, from practical tests. Satisfactory correlation of the results indicates that the programs are sufficiently accurate for most practical purposes, and that they are computationally efficient when compared with other techniques.

The problems associated with the analysis of plates stiffened by the use of radial ribs are discussed, together with the feasability of extending the semi-analytic finite element technique to handle this particular application.

Key words: CIRCULAR PLATES / STRESS ANALYSIS / FINITE ELEMENTS

A C K N O W L E D G M E N T S

Firstly, I should like to express my gratitude to Mr. T.H. Richards whose lectures provided the initial insight and inspiration necessary for this work, and whose subsequent supervision has never ceased to be a model of constructive criticism and a source of limitless enthusiasm and encouragement.

Secondly, my thanks are due to Mr. D. Leighton, Mr. G.L. Edwards and Mr. E.F. Wilson. As former students under my supervision, they carried out much of the practical work contained herein and were at all times impressive in their perseverance and meticulous attention to detail.

To Mrs. S. Clifton I owe special thanks for typing what must have been an almost unintelligible, and seemingly never ending manuscript.

Finally, the help of library and computer centre staff must be acknowledged, for without their considerable assistance life would have been very hard indeed.

CONTENTS

LIST OF SYMBOLS

This list is intended as a reference to the most commonly used symbols contained herein. Symbols may be found in the text which are not referred to here, in which case they are defined on introduction.

Subscripts are normally used to identify the component of a variable in the co-ordinate direction indicated by the subscript. Subscripts having meanings other than this are defined as and when required.

Matrix quantities are defined by a letter within the symbols [ ] for a rectangular matrix, $\lceil\diagdown\rfloor$ for a diagonal matrix and { } for a column vector. The transpose of a matrix is indicated by the superscript,t.

x, y, z or 1, 2, 3	Cartesian co-ordinates
r, θ, z	Cylindrical co-ordinates
u, v, w	Displacements in directions x, y, z respectively
ρ	Radius of curvature
ζ	Distance from neutral surface
h	Plate thickness
σ	Direct stress
τ	Shear stress
ϵ	Direct strain
γ	Engineering shear strain
E	Modulus of elasticity
ν	Poissons ratio
D	Flexural rigidity of a plate
G	Modulus of rigidity
λ	Lamés constant
M	Moment
Q, P, W, F	Force

p	Intensity of loading
U	Strain energy
Ω	Potential energy of loading
V	Total potential energy
N, ϕ	Shape functions
[B]	Strain - displacement relationship
[D]	Stress - strain relationship
$\{\delta\}$	Nodal displacements
[k]	Element stiffness matrix
[K]	System stiffness matrix
$\{F\}$	Force vector
$\int_S dS$	Surface integral over area S
$\int_V dV$	Volume integral over volume V

1

A REVIEW OF ANALYTICAL TECHNIQUES AND PUBLISHED LITERATURE ON THE STRESS ANALYSIS OF CIRCULAR PLATES IN BENDING

1.1 STATEMENT OF THE PROBLEM

Many engineering structures make use of circular plates under various loading conditions. Common examples are the end plates of rotating machines, cover plates for apertures in pressure vessels, machine tool bedplates, and bulkheads in cylindrical tubes. The analysis of the deformation of circular plates is comprehensively presented by Timoshenko and Woinowsky-Krieger [3] and McFarland, Smith and Bernhardt [4] where it is shown that the general equation governing the bending of thin plates is a fourth order partial differential equation (the bi-harmonic equation). An analytical solution to this equation is readily available in simple cases such as when the loading is axi-symmetric and the plate is of constant thickness. In the case of asymmetric loading or variable plate geometry, however, an analytical solution is usually impossible due to the difficulty of finding functions which satisfy the governing equation within the plate and yet simultaneously fulfil all of the boundary conditions. Under these circumstances approximate methods become essential.

In many practical applications the loading on a particular plate may be of such magnitude that the deflection and stresses become unacceptably large if the plate is thin. The functional requirements demanded of such a plate may be fulfilled by the use of a thick, plain plate but this may not be the most economical solution to the problem either in terms of cost or weight. A stiffened thin plate will often provide an acceptable design as it is lighter and may be less costly than a plain, thick plate and yet may be made sufficiently rigid by careful design of the stiffening. Plate stiffening is achieved in practice by the use of orthogonal, circumferential or radial ribs; the latter form probably being the most commonly used, though possibly the least well documented.

The aim of the current investigation is therefore to attempt to develop a general method of analysis for asymmetrically loaded circular plates, with the secondary objective that the method should, if possible, be capable of future extension to incorporate the effects of radial stiffening.

The following review of analytical techniques and published literature was carried out with the purpose of ascertaining the form of approach most likely to achieve the desired aims.

1.2 ANALYTICAL TECHNIQUES FOR PLAIN PLATE PROBLEMS

The general governing equation for the bending of thin plates is, as mentioned above, a fourth order partial differential equation. Circular plates are best defined in terms of cylindrical co-ordinates r, θ and z in which case the governing equation is of the form

$$\nabla^4 w = \frac{p}{D}$$

where ∇^2 is the operator $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\right)$

p is the intensity of loading on the plate.

w is the transverse deflection of the plate.

D is the flexural rigidity of the plate.

In general p, w and D are all functions of position on the plate.

Direct analytical solution of this equation is rarely possible; one exception being when the loading is axi-symmetric and the plate is homogeneous and of constant thickness, in which case the equation reduces to a relatively simple ordinary differential equation where the loading and deflection are functions of radius only. In almost all other cases approximate methods of solution are required. Various approximate methods have been devised and the basic outline of some of these methods is discussed below.

1.2.1 Galerkin's method

This is a method of direct solution of the governing equation based on the selection of an approximate displacement function $\overline{w}$ in the form

$$\overline{w} = \sum_{i=1}^{n} C_i \phi_i$$

where the C_i are constants to be determined and the ϕ_i are co-ordinate functions that satisfy all the boundary conditions on the plate.

A virtual displacement field may be defined by $\delta\overline{w}$, in which case, by consideration of both sides of the governing equation, the virtual work done in moving through these virtual displacements is given by

$$\int_S (D\nabla^4 \overline{w})\delta\overline{w}\, dS \qquad \underline{or} \qquad \int_S (p)\delta\overline{w}\, dS$$

where the integral is the surface integral over the whole plate.

As $\overline{w}$ is not the true displacement field these two expressions for virtual work are not automatically equal, however equality may be forced by suitable adjustment of the C_i.

The condition for equality is therefore

$$\int_S (D\nabla^4 \overline{w})\phi_i\, \delta C_i\, dS = \int_S (p)\phi_i \delta C_i\, dS \qquad \text{for } i = 1, 2, ----, n$$

or $$\int_S (\nabla^4 \overline{w} - \frac{p}{D})\phi_i\, dS = 0 \qquad \text{for } i = 1, 2, ---, n$$

Application of this equation leads to a set of simultaneous equations from which the C_i may be evaluated

This method is successfully used for the analysis of clamped plates in references [8] and [27] but the restriction that the assumed displacement function must satisfy all the boundary conditions (both kinematic and natural) makes the technique more difficult to use for plates with simply supported or free edges.

1.2.2 The Ritz method

Like the method of Galerkin, the Ritz technique also uses an assumed approximate displacement function. The Ritz method differs from Galerkin's approach however in that it is not a direct solution of the governing equation but relies on the existence of a functional which is to be minimised. In the case of plate bending the usual functional is the total potential energy of the system and the assumed displacement function need satisfy only the kinematic boundary constraints. The variational ideas associated with the functional and its minimisation imply that the Ritz method effectively generates a substitute finite degree of freedom problem from the original continuum problem.

The Ritz and Galerkin methods can be shown to give identical results in many solid mechanics applications. The Galerkin approach is arguably the more fundamental as it is a direct solution of the governing equation and does not rely on the equation having a functional. The Ritz technique is often easier to apply however, due to the requirement that the assumed displacement function need not satisfy all the boundary constraints but only the kinematic ones.

The detailed basis of the Ritz method and its application to circular plate problems are discussed extensively in appendices B and D.

1.2.3 The use of series

The use of trigonometric series forms the basis of a direct method of solution to the governing equation by assuming that the deflected shape of any circular plate may be represented by the infinite series

$$w = \sum_{n=0}^{\infty} f_n(r)\cos n\theta + \sum_{n=1}^{\infty} g_n(r)\sin n\theta$$

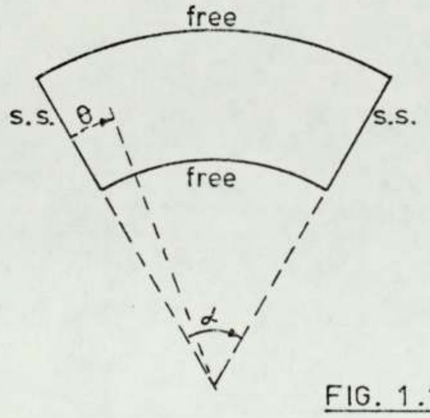
where $f_n(r)$ and $g_n(r)$ are functions of radius only.

This displacement function is substituted into the governing equation and the resulting expression is then made to satisfy all the boundary conditions, thus resulting in the evaluation of the arbitrary constants which are associated with $f_n(r)$ and $g_n(r)$, and arise from the original substitution of the displacement function into the governing equation.

The method is approximate because in practice only a finite number of terms can be handled and the accuracy of the solution depends on taking a suitable number of terms to give convergence of the series to the required degree of accuracy.

A general description of solutions using series is given in references [3] and [4] whilst appendix C discusses an application to the particular problem of a clamped-free annular plate under concentrated edge loading.

Coull and Das [18] have used the method in a slightly more sophisticated manner in the analysis of curved bridge slabs which are in the form of annular sectors with boundary conditions as shown in figure 1.1 below



Displacement function is :-

$$w = \sum_{n=1}^{\infty} R \sin \frac{n\pi\theta}{\alpha}$$

(R is a funtion of radius only)

FIG. 1.1

The general solution to the governing equation $\nabla^4 w = \frac{p}{D}$ is of the form

$$w = w_p + w_h$$

where w_h is the homogeneous part and w_p the particular integral. The evaluation of w_p is very difficult for complex load patterns and the feature of their paper is that the loading on the slab is represented by circumferential line loads expressed in the form of Fourier series. This makes the particular integral zero and the solution is given by the homogeneous part only, the load being represented by a discontinuity in shear force at the load line in a manner similar to that used in the step function method for the solution of beam problems.

The method of solution by series gives a satisfactory means of analysis for many problems but has the disadvantage that it is a fairly lengthy technique to apply and every type of problem has to be treated as an individual case.

1.2.4 Finite difference and finite element methods

The main features and relative merits of these techniques are discussed in section 3.1. With respect to the current application, however, the rapid development of finite element techniques in the late 1960's and early 1970's, especially with the introduction of new elements with curved boundaries, has made this approach very attractive for handling circular plate problems. Contributions of note in the development of such elements are those of Olson, Lindberg and Tulloch [22] , Sawko and Merriman [23] and Singh and Ramaswamy [24] all of whom have developed annular sector elements with varying degrees of sophistication and which are discussed in more detail in section 3.5. These elements enable a satisfactory analysis to be made for most circular plate problems, and could possibly be extended to incorporate the effects of stiffening, but have the disadvantage of requiring considerable computer time and storage requirements.

Many types of component and structure exist in which the geometrical properties are constant in one specified direction. In certain cases it may be possible to simplify the analysis of such components or structures by taking advantage of this geometrical characteristic. Zienkiewicz [13] describes the basis of a semi-analytic finite element method which takes advantage of such geometrical properties and which may result in considerable economies in computational effort. This approach is discussed in section 3.4. Zienkiewicz and Too [17] use the semi-analytic technique in the solution of bridge deck problems similar to that previously analysed by Coull and Das, but application of the technique to circular plate problems appears to be a valuable application that has not yet been explored in detail.

1.3 ANALYTICAL TECHNIQUES FOR STIFFENED PLATE PROBLEMS

A typical application of a stiffened plate is illustrated in figure 1.2a which shows the use of a radially ribbed plate in a crushing mill whilst figure 1.2b shows the effective loading on the plate.

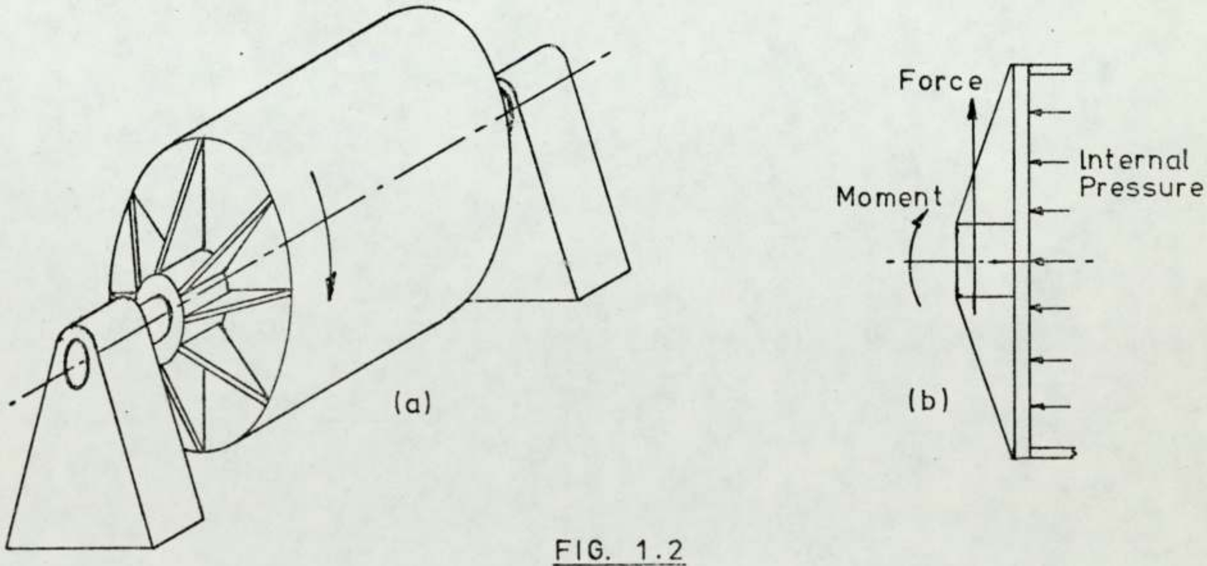


FIG. 1.2

One of the earliest references to the strength of radially ribbed plates was the discussion by Biezeno and Grammel [25] of a plate carrying a sealing gland as shown in figure 1.3

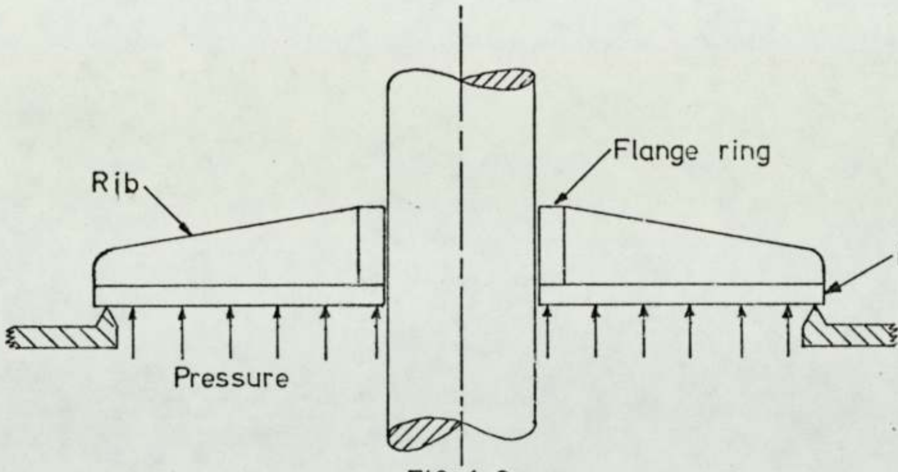


FIG. 1.3

Their solution to this particular problem was very approximate and intended only as a means of assessing the stresses in the flange ring carrying the seal. There was no real attempt to calculate stress distribution in the plate but their analysis did show, in very forceful terms, the adverse effect of the ribs in the form of stress concentrations in the flange ring.

In the decade 1960-70 a considerable effort was made to analyse the vibrational behaviour of stiffened plates, of which the contributions by Kirk [26] for rectangular plates and Desiderati & Laura [27] for elliptical and circular plates, were notable. The Rayleigh-Ritz method was widely used and gave satisfactory results provided that the ribs were small and closely spaced. Although the overall elasticity of the plate needed to be considered, the effect of the discontinuities in plate geometry on the stress distribution was ignored. The idea of incorporating the effect of the rib by means of using orthotropic plate theory was also introduced in this period. The characteristic of all of these papers however was that they only investigated plates with orthogonal ribs, and although they contained some useful principles they were only of limited direct use in the context of the present application.

A paper by Harvey and Duncan [28] dealt directly with the problems of radially stiffened circular plates and was probably one of the earliest comprehensive assessments of the problem. Their contribution was mainly in the form of experimental deflection results that were obtained by interferometry techniques. They pointed out the difficulties in theoretical analysis and used as their mathematical model a system of thin sectorial plates that served to collect the loading (in their case only internal pressure was considered) and transmit it to the ribs. Their theoretical predictions of deflection showed errors of at least fifty per cent and presumably any attempt to calculate stresses from deflections with this order of error would be virtually useless. The large errors in their results were due to the fact that the plate and ribs cannot be treated in isolation,as the plate stiffens the rib and therefore a portion of the plate should have been associated with each rib thus forming a T-beam.

Independently of Harvey and Duncan,and apparently unknown to them probably because the paper was not available outside Russia at that time, Rubach [29] had been working on the problem of a radially ribbed plate with a point load at its centre. He introduced the use of orthotropic plate theory in cylindrical co-ordinates as a means of representing the ribs although he particularised the ribs, for mathematical convenience, into the form where their circumferential width was proportional to radius. His work was followed by Mlotkowski [30] who studied the effect of a central moment on the plate and subsequently by Leyko et.al. [31] who attempted to further generalise the geometry of the plate considered. Although all of these authors claim to produce a satisfactory analysis, the use of orthotropic theory does not give a result that can be presented with conviction, especially if the ribs are not small and closely spaced. Some of the methods used, notably the latter, may be lengthy due to the plate being represented by a series of annular rings and the need for subsequent matching of the boundary conditions at common radii. This was not a finite element approach in the usual sense as it was not a variational technique although it did mean that final solution involved solving many simultaneous equations.

2

INTRODUCTION TO THE CURRENT INVESTIGATION

2.1 AIMS OF THE INVESTIGATION

The main aim of the investigation was the development of a general technique for the analysis of deflection and stresses in annular and complete circular plates under the action of both axi-symmetric and asymmetric transverse loading systems of the type which commonly arise in practical applications. Attention was to be paid to the minimisation of any computational requirements both in terms of storage and processing time.

Many applications of circular plates require the plate to be stiffened by the incorporation of radial ribs. The possibility of extending any proposed technique in order to include this configuration was a feature to be kept under review as development of the technique proceeded.

2.2 SELECTION OF THE METHOD OF ANALYSIS

Some of the early researchers mentioned in chapter 1 had achieved a measure of success, especially in vibration problems, by using the Ritz method. This technique is attractive in its basic simplicity of approach and was therefore explored further in terms of its suitability for use in this particular application. Appendix D describes the work carried out using the Ritz method and shows that a limit of usefulness was reached at a stage somewhat short of that required due to the complexity of algebraic manipulation and the unsatisfactory accuracy of the stress calculations. It was accepted, however, that the method may still be useful in some limited applications.

The use of Galerkin's method was discounted on the grounds that it would suffer from similar, but probably more severe restrictions than the Ritz method. This was due to the difficulty in defining an approximate displacement field in terms of a single expression which satisfied all boundary conditions and that did not result in extreme algebraic complexity in the subsequent calculations.

Analytical solutions using trigonometric series for the description of the displacement field have proved very successful in many unstiffened plate applications, but the problems associated with applying this type of analysis to stiffened plates gave rise to some reservations as to its accuracy. The commonly used approach, whereby the effect of the ribs was introduced as an orthotropic property of the flexural rigidity of the plate, was of questionable validity for plates stiffened by the use of radial ribs due to the situation that with the relatively small number of ribs generally used in practice it was not acceptable to regard the ribs as being closely spaced. Even assuming that a satisfactory overall picture of deflection could be achieved by this technique it is doubtful whether the stress field in a stiffened plate could be calculated with sufficient accuracy to be of practical use.

Finite element and finite difference methods were considered; the relative merits of the two techniques being discussed in section 3.1. On the basis of the points made in that discussion the finite difference method was discarded as being no easier to apply, and also unlikely to yield a more accurate solution than the finite element approach. The finite element approach was also attractive in that once a computer program had been developed a whole range of problems could be solved simply by specifying the plate and loading details as input data.

One of the original aims was to minimise the demands made on computer time and storage. When the possibilities of a semi-analytic finite element analysis became apparent, this approach was selected in preference to conventional finite elements on the grounds that considerable savings in computational effort may be achieved. The investigation therefore developed into the application of the semi-analytic finite element method to the analysis of circular plate problems.

2.3 AN OUTLINE OF THE DEVELOPMENT OF THE INVESTIGATION

Any investigation of this nature must obviously make continual reference to the basic behaviour of thin plates in flexure. For this reason the basic theory and the resulting expressions describing this behaviour were studied and are presented in appendix A.

All finite element methods have their foundations in variational principles and energy methods. Many formulations of the finite element method have been proposed, for example the displacement, force and hybrid formulations, but a common feature of them all is that they are based on the use of variational principles for investigating the existence of stationary values for some particular functional which is defined in terms of the energy quantities associated with the plate and its loading. A discussion of variational principles and energy methods with particular reference to plate bending is given in appendix B.

The displacement formulation of the finite element method is the formulation most commonly used, largely due to the fact that assumed displacement fields and inter-element continuity of displacements are a simpler concept to envisage than a formulation involving assumed stress fields and equilibrium criteria. A detailed discussion of the displacement formulation of the finite element method is given in chapter 3 together with a description of the principles involved in the use of the semi-analytic technique. Particular features associated with the displacement formulation in its application to problems of plate flexure are also discussed.

As a first stage in the development of plate analysis, the problem of the flexure of plates under axi-symmetric loading was considered. The value of analysing this relatively simple problem was that it served to highlight those aspects of the analysis which were likely to produce further difficulties as development progressed. The basic problems of defining element shape functions, together with the computational aspects of element stiffness matrix generation,and assembly and solution of the final equations could all be overcome at this stage in an application where classical analytical solutions were available for comparison. This stage of the investigation is described in chapter 4 together with full documentation for use of the computer program 'SYMPLAT' which analyses the deflection and stresses in any axi-symmetrically loaded,complete or annular plate. This program is of considerable practical use in its own right as it enables an analysis to be made in cases where classical analysis may be extremely difficult. A typical example of this is in applications where there is a radial variation of either the plate thickness or the material properties. A feature of the axi-symmetric nature of the loading was that the problem was quasi-unidimensional, and a conventional finite element approach using annular elements was adequate and computationally efficient. The satisfactory development of 'SYMPLAT',and confirmation of its viability by comparison with classical solutions to several test cases,meant that the next stage in the investigation was to extend the analysis to accept asymmetric loading on the plate.

The immediate effect of considering asymmetric loads was that the problem became fully two-dimensional. Several authors have previously analysed the problem using conventional finite element techniques but in the case where the loading was asymmetric, provided that the plate geometry and material properties were axi-symmetric, it was apparent that the semi-analytic approach may be possible. The development of this semi-analytic approach is presented in chapter 5 together with documentation for use of the program 'ASYMPLAT'. This program analyses the deflection and stresses in axi-symmetric plates with asymmetric loading. A rigorous analytical solution for problems of this type is not generally possible but approximate, series solutions have been used extensively. A particular problem used as a test case for 'ASYMPLAT' was that of a clamped-free annular plate with a concentrated edge load. An analytical,series solution to this problem is presented in appendix C for purposes of comparison. This problem was investigated in some depth in that a test rig was also constructed in order to obtain experimental confirmation of the validity of the theoretical deflection and stress predictions. The design, construction and test procedures for this rig were all carried out under the direct supervision of the author and have been described in detail by Wilson [20]. An abridged description of this experimental work is presented in appendix C. The final form of 'ASYMPLAT' has been proved to give deflection and stress predictions of satisfactory accuracy for most practical purposes by comparison of its results with the solutions to several test cases which have been analysed previously using other techniques. The program is now in a condition where it can make a useful contribution to problems which fall within its range of application and would appear to be of such computational efficiency as to justify its use in preference to most other forms of analysis.

The application of the semi-analytic finite element approach to the problem of radially stiffened plates has not yet been explored in detail but some comments on the possibilities of this application are discussed in chapter 6. An introductory analysis of radially stiffened plates has been attempted by using the Ritz method in the case of a complete plate stiffened by a single diametral rib. The analysis of this plate is discussed in appendix D and again shows that the Ritz method is capable of producing adequate predictions of deflection but is seriously in error in its stress predictions. Since no other form of analysis was available against which the Ritz predictions could be compared, a program of experimental work was initiated on the measurement of deflection and strain in radially ribbed plates. The use of small scale metal models of stiffened structures often results in models which are too stiff to allow accurate measurement of the deflections under load. For this reason the application of thermoplastics for the manufacture of stress analysis models was investigated, the work being carried out by Leighton under the supervision of the author. The results of this investigation were published in reference[34] and are summarised in appendix E where it is proposed that Vybak is a suitable material,and guidance for its use in the current application is given. The experimental work on ribbed plates fabricated from Vybak sheet was performed by Edwards under the supervision of the author and the results have been published in reference [35] . A selection of these results for comparison with the theoretical predictions of the Ritz method is given in appendix D.

3

THE FINITE ELEMENT METHOD

3.1 THE DEVELOPMENT OF NUMERICAL METHODS IN SOLID MECHANICS

In the field of solid mechanics the description of deformation and equilibrium of simple systems can often be achieved in terms of relatively straightforward equations which have well established solutions. As systems become more complex, either in terms of their geometry or applied loading, the mathematical problems associated with an analytical solution may become very severe.

Many common engineering structures are too complicated to be regarded as a simple collection of interconnected rigid bodies with a finite number of degrees of freedom. The study of continuum mechanics, with the implication of an infinite number of degrees of freedom, then becomes essential. The use of continuum theory does, however, lead to system behaviour being defined in the form of partial differential equations. The major problem is then that even if solutions to these equations are available, it is only in very few special cases that they will exactly satisfy the required boundary conditions of load or geometry.

In the absence of analytical solutions to the majority of continuum problems the approach is either:

(a) to attempt to solve the describing equations of the actual system by approximate methods or

(b) to devise an approximate system, the describing equations of which are known to be amenable to solution.

The advent of high speed digital computers with large storage capacity has hastened the application of numerical solutions to continuum problems. The two major techniques that have been developed are the finite difference and finite element methods; corresponding to the above solutions of types (a) and (b) respectively. In recent years the finite element method appears to have established itself as the more important of the two and the outline of the techniques given below provides some indications as to the reasons for this popularity.

Historically the finite difference method is the older of the two, having first been used in the solution of solid mechanics problems approximately seventy years ago, although one of the first successful applications to plate theory was not until that made by Marcus in 1919. The method, which is outlined in references [1],[3] &[4], provides an approximate numerical solution to the overall describing equation and is based on dividing the region over which the equation is applicable into a mesh. The derivatives or partial derivatives in the equation are then replaced by finite difference expressions which approximately relate the value of the derivatives at any given mesh point to the value of the field variable at the point in question, together with its value at neighbouring mesh points. The original equation is therefore replaced by a set of simultaneous finite difference equations, the solution of which gives the values of the field variable at the mesh points.

The finite element method is the subject of very extensive literature with references such as [13], [14], [15] and [16] being typical of the many books available. Martin and Carey [14] present a comprehensive history of the development of the finite element method from its initial use over thirty years ago, through its very rapid development in the 1960's as computer facilities advanced and up to the very sophisticated state of development that it has reached today. The finite element approach is a variational one based on sub-dividing a continuum into discrete elements, connected together only at nodal points, and then defining the behaviour of each element in terms of adjustable nodal parameters. The values of the nodal parameters are then adjusted so as to minimise a prescribed functional. The advantages of the finite element method over the finite difference method may be summarised as

(a) a graded or irregular mesh generally presents few problems.

(b) discontinuities can be handled relatively easily.

(c) irregular boundaries are less of a problem.

(d) the analysis of each element is an independent process, which is not the case with finite differences.

Henshell [12] also claims that since the finite element method is a variational approach it is potentially more accurate than the finite difference method. This claim would appear to be difficult to substantiate however, as there is little reason to suppose that the mathematical modelling ideas used in the finite element approach are necessarily superior to those used in a finite difference analysis purely because they are based on variational principles.

The main limitation on the use of the finite element method is the speed and storage limits of the computer. For this reason a considerable amount of effort must be directed at the efficiency of the elements and numerical processes.

3.2 THE CONCEPT OF FINITE ELEMENTS AS A VARIATIONAL METHOD IN SOLID MECHANICS

Any continuum may be regarded as an assembly of discrete elements of any desired shape and size. The most common form of the finite element technique in solid mechanics is the so-called displacement or stiffness formulation in which the displacement field within each element is approximately defined in terms of assumed degrees of freedom at a selected number of nodal points, the elements being interconnected only at these points. The total potential energy for the element is then determined as a function of the nodal freedoms. The elements are assembled and the relative sizes of the nodal freedoms adjusted to minimise the total potential energy of the system hence, by the principle of stationary total potential energy, the equilibrium configuration is defined. This energy principle is discussed in detail in Appendix B and it becomes apparent that the displacement formulation of the finite element method is in effect a piecewise application of the Ritz method.

3.3 THE DISPLACEMENT FORMULATION IN DETAIL

3.3.1 Strain energy of an element in a solid continuum

The displacement field within any element in a general three dimensional case can be described in terms of three orthogonal displacements of any point within the element. Each of these displacements will be a function of the three Cartesian co-ordinates that define the position of the point, i.e.

Displacement field, $\{f\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$ where u, v and w are each functions of x, y and z

In a displacement formulation it is required that the displacement field within an element should be described in terms of adjustable nodal parameters. These parameters become the degrees of freedom that are allowed for each element and are usually the displacements or derivatives of displacements at the nodes. If a continuous displacement field is to be described by the use of discrete nodal freedoms it is implicit that a set of co-ordinate functions needs to be devised thus:-

$$\{f\} = \left[[N_1]\ [N_2]\text{---}[N_i]\text{---}[N_p]\right] \left\{ \begin{matrix} \left\{\begin{matrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_q\end{matrix}\right\} 1^{st} \text{ node} \\ \vdots \\ \left\{\begin{matrix}\vdots\end{matrix}\right\} i^{th} \text{ node} \\ \vdots \\ \left\{\begin{matrix}\vdots\\ \delta_r\end{matrix}\right\} p^{th} \text{ node} \end{matrix} \right\}$$

or $\{f\} = [N]\{\delta\}^e$ (3.1)

Where $\{\delta\}^e$ is a column vector of the nodal degrees of freedom for the element with p nodes and q degrees of freedom per node giving a total of r freedoms per element. [Note that r is not necessarily $(p \times q)$ as q need not be the same for all nodes]. The $[N_i]$ are, in general, fully populated $(3 \times q)$ matrices each element of which is a function of x, y and z and so chosen that if the nodal co-ordinates are inserted for x, y and z then $\{f\}$ becomes the appropriate nodal displacement. The elements of $[N_i]$ are called shape functions and must be continuous functions within the region of the element. The accuracy of the whole solution is dependent on the degree of correspondence between the chosen shape functions and the true deformed shape of the element.

If compatibility between strain and displacement within the element is analysed, the relationship between these two quantities may be summarised as

$$\{\epsilon\}^e = [G]\{f\} \qquad (3.2)$$

Where, in a general three dimensional situation, $\{\epsilon\}$ is a (9×1) column vector consisting of three direct strains and six shear strains. Also in this general case [G] will be a (9×3)

matrix whose elements are all partial derivatives with respect to the three co-ordinate directions.

By combining (3.1) and (3.2)

$$\{\epsilon\}^e = [G][N]\{\delta\}^e$$

or $$\{\epsilon\}^e = [B]\{\delta\}^e \quad \text{where} \quad [B] = [G][N] \tag{3.3}$$

If the material from which the element is made behaves in a linear, elastic manner under load then the internal element stresses may be related to the strains by the equation

$$\{\sigma\}^e = [D]\{\epsilon\}^e \tag{3.4}$$

Where $\{\sigma\}^e$ consists of three direct stresses and six shear stresses. [D] is, in general, a (9 x 9) matrix, i.e. 81 elements of elastic constants. Due to symmetry of the stress and strain tensors only 21 of these constants need to be independent and if the material is isotropic these 21 can be reduced to 2.

The total strain energy of the element can be shown to be given by

$$U^e = \frac{1}{2}\int_V \{\epsilon\}^{e^t} \{\sigma\}^e dV \tag{3.5}$$

Note that in equation (3.5), if engineering shear strains γ are used in place of the shear strains ϵ, where $\gamma = 2\epsilon$, then due to the symmetry of the stress and strain tensors, the size of $\{\epsilon\}^e$ and $\{\sigma\}^e$ may be reduced to (6 x 1) as each pair of shear components need be listed once only.

Equations (3.3) and (3.4) may now be substituted in equation (3.5) to give

$$U^e = \frac{1}{2}\int_V \{\delta\}^{e^t}[B]^t[D][B]\{\delta\}^e dV$$

Noting that in general [B] and [D] are now only (6 x 3) and (6 x 6) respectively.

$$\therefore \quad U^e = \frac{1}{2}\left(\{\delta\}^{e^t}[k]\{\delta\}^e\right) \tag{3.6}$$

Where $$[k] = \int_V [B]^t[D][B]dV \tag{3.7}$$

[k] is called the element stiffness matrix.

[Note the similarity of form between equation (3.6) and the strain energy expression for a simple spring where $U = \frac{1}{2}kx^2$]

Strains and stresses due to initial values and temperature effects can also be incorporated if required but have not been included in the discussion as they are not relevant to this investigation.

3.3.2 Potential energy of the applied loading on an element

The forces on an element arise from two sources, namely body forces and surface tractions.

Body forces are defined in terms of a force per unit volume which may vary throughout the element and are produced by phenomena such as gravitational, electromagnetic and centrifugal effects.

Surface forces are defined in terms of force per unit area and may vary over the surface of the element. These forces are caused by applied loads, fluid pressure, etc.

In matrix form the body forces and surface tractions may be written as

Body force $\{R\} = \begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix}$ and Surface traction $\{T\} = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}$

Where the elements of $\{R\}$ and $\{T\}$ are all functions of x, y and z.

The work done by these forces on any element in a virtual displacement $d\{f\}$ is given by

$$\text{Virtual work} = \int_V \{R\}^t d\{f\} dV + \int_S \{T\}^t d\{f\} dS$$

But from equation (3.1) $d\{f\} = [N] d\{\delta\}^e$

$$\therefore \text{Virtual work} = \int_V \{R\}^t [N] d\{\delta\}^e dV + \int_S \{T\}^t [N] d\{\delta\}^e dS$$

The potential energy of the loading on the element, $d\Omega^e$ is -(virtual work)

Hence $$d\Omega^e = -\int_V \{R\}^t [N] d\{\delta\}^e dV - \int_S \{T\}^t [N] d\{\delta\}^e dS \qquad (3.8)$$

The volume integral for the body forces is taken over the whole volume of the element.

The surface integral for the tractions is taken over the area for which the tractions are prescribed.

It is convenient to express the potential energy of the loading in terms of a set of equivalent nodal forces moving through their respective virtual nodal displacements.

or $$d\Omega^e = -\{P\}^t d\{\delta\}^e \qquad (3.9)$$

Where $\{P\}$ is a column vector of equivalent nodal loads which, by comparison with equation (3.8) are derived from

$$\{P\} = \int_V [N]^t \{R\} dV + \int_S [N]^t \{T\} dS \qquad (3.10)$$

3.3.3 <u>The principle of stationary total potential energy</u>

If this principle is to be applied to the whole continuum it is necessary to collect together the energies for all the elements.

From equation (3.6) the total strain energy for the continuum is given by

$$U = \sum_{k=1}^{m} \frac{1}{2}\left(\{\delta\}_k^{e^t} [k]_k \{\delta\}_k^e\right)$$

Where $\{\delta\}_k^e$ and $[k]_k$ are respectively the nodal freedoms and the stiffness matrix for the k^{th} element in a system consisting of m elements.

This may be rewritten as

$$U = \frac{1}{2}\{\delta\}^{e^t} [K]^e \{\delta\}^e \qquad (3.11)$$

Where $\{\delta\}^e$ is now a list of the nodal freedoms for <u>all</u> the elements i.e. it is a $[(r \times m) \times 1]$ column vector, and $[K]^e$ is a diagonal matrix of the individual element stiffness matrices as follows

$$[K]^e = \begin{bmatrix} [k]_1 & & & & O \\ & \ddots & & & \\ & & [k]_k & & \\ & & & \ddots & \\ O & & & & [k]_m \end{bmatrix}$$

If equation (3.11) is multiplied out it may be expressed as

$$U = \frac{1}{2} \sum_{k=1}^{m} \sum_{i=1}^{r} \sum_{j=1}^{r} \delta_{ik}\, \delta_{jk}\, k_{ijk} \qquad (3.12)$$

Equation (3.10) defined the manner in which distributed loads on the elements may be apportioned to the nodes. If concentrated loads act on the continuum the simplest way to allow for their effect is to select the size and shape of the elements to ensure that the concentrated loads are positioned at nodes. The potential energy of the loading on the continuum is then given by

$$\Omega = -\left[\sum_{i=1}^{m} \left(\{P\}_i^t \{\delta\}_i^e \right) + \left(\{Q\}^t \{\delta\} \right) \right]$$

Where $\{Q\}$ are the concentrated loads acting at the nodes of the continuum which have displacements $\{\delta\}$. Both $\{Q\}$ and $\{\delta\}$ are therefore (n x 1) in size where n is the number of global nodal freedoms in the continuum.

Contiguous elements are connected only at the nodes and hence any summation over all elements will, at any node, consist of contributions to the quantity being summed from all elements that meet at the node. The expression for the potential energy of the loading may therefore be written as

$$\Omega = -\left[\{P'\}^t \{\delta\} + \{Q\}^t \{\delta\} \right]$$

Where $\{P'\}$ is a vector of equivalent nodal loads <u>on the continuum</u>, the elements of which consist of the summation of the appropriate elements from the $\{P\}$ vectors.

or

$$\Omega = -\{F\}^t \{\delta\} \qquad (3.13)$$

Where $\{F\} = \{P'\} + \{Q\}$ and is the total equivalent nodal loading on the continuum.

From equations (3.11) and (3.13) the total potential energy of the continuum is

$$V = U + \Omega = \frac{1}{2}\{\delta\}^{e^t}[K]^e\{\delta\}^e - \{F\}^t\{\delta\} \tag{3.14}$$

The displacement of the continuum is defined in terms of the n global nodal freedoms. If the principle of stationary potential energy is to be applied it is now required that

$$\delta(V) = 0$$

or $\frac{\partial V}{\partial \delta_1} = \frac{\partial V}{\partial \delta_2} = \text{---------} \frac{\partial V}{\partial \delta_n} = 0$ since the δ_i are arbitrary

Considering the s^{th} global nodal freedom, from equation (3.14), the contribution to $\delta(V)$ from the potential energy of the loading is

$$\frac{\partial \Omega}{\partial \delta_s} = F_s \tag{3.15}$$

The strain energy contribution will consist of the derivatives of terms of the form $\frac{1}{2}\delta_i^2 k_{ii}$ for $i = j$ and $\delta_i \delta_j k_{ij}$ for $i \neq j$ (due to the symmetry of [k]). Since several elements may meet at the s^{th} global node, $\frac{\partial U}{\partial \delta_s}$ is given by

$$\frac{\partial U}{\partial \delta_s} = \sum \frac{\partial}{\partial \delta_s}\left(\frac{1}{2}\delta_i^2 k_{ii} + \delta_i \delta_j k_{ij}\right) \tag{3.16}$$

Where the summation is over all elements having one of its δ_i equal to the global δ_s

Collecting together the results from equations (3.15) and (3.16) for the s^{th} node, the condition $\delta(V) = 0$ for the continuum may be expressed in terms of n equations thus

$$[K]\{\delta\} = \{F\} \tag{3.17}$$

Where the elements of the assembled system stiffness matrix [K] consist of the summation of the appropriate elements from the element stiffness matrices [k] . [K] will be a symmetric, banded matrix whose bandwidth is dependent on the node numbering system.

For a given continuum under a specified load system equation (3.17) may now be solved to give the magnitude of the nodal freedoms.

3.4 THE SEMI-ANALYTIC FINITE ELEMENT METHOD

3.4.1 General Theory

Many problems occur in practice in which the geometry and material properties of a particular continuum are constant with respect to one of the chosen co-ordinate directions. In a few special cases, such as that for plane strain when there is no variation in transverse loading with respect to this particular co-ordinate direction, it may be possible to simplify the analysis of deformation and stress by considering a reduced problem using fewer co-ordinates. In general however even if the continuum has a direction with constant properties of geometry and material behaviour the transverse loading is often not so simply defined and a reduction in the dimensions of the problem is not possible.

The semi-analytic finite element technique has been developed largely by Zienkiewicz [13] and [17] for solid mechanics applications and is a method which takes advantage of the directional properties of the continuum but represents the transverse loading in the form of a series of orthogonal functions. It will be shown that it then becomes possible to replace the original problem by a series of substitute problems of reduced size because they do not involve the particular co-ordinate along which the continuum properties do not vary. The complete behaviour of the continuum is then approximately described by the superposition of the results of the substitute problems.

Consider a three dimensional continuum whose geometry and material properties are constant with respect to the z direction, the continuum being contained within the range

$$0 \leqslant z \leqslant c$$

The displacement field within an element may be written as before $\{f\} = [\bar{N}]\{\delta\}^e$

where the $\bar{N}$ are functions of x, y and z

But now $[\bar{N}]\{\delta\}^e$ may be expressed in such a form as to give

$$\{f\} = \sum_{\ell=1}^{L} \left([\ulcorner \phi^{\ell} \lrcorner][N]\{\delta^{\ell}\}^e \right) \qquad (3.18)$$

Where $[N] = \left[[N_1],[N_2]---[N_i]---[N_p]\right]$ as before but the elements of the $[N_i]$ are functions of x and y only

And $$[\ulcorner \phi^{\ell} \lrcorner] = \begin{bmatrix} \phi_1^{\ell} & 0 & 0 \\ 0 & \phi_2^{\ell} & 0 \\ 0 & 0 & \phi_3^{\ell} \end{bmatrix}$$

ϕ_1^{ℓ}, ϕ_2^{ℓ} and ϕ_3^{ℓ} are functions of z only and satisfy at least the geometric boundary conditions on z = 0 and z = c. The summation of the ϕ^{ℓ} must also represent a continuous function over the range of z. The δ^{ℓ} now become the contribution to the total displacement field due to each of the 'shapes' ϕ^{ℓ}

If equation (3.18) is now used in the formation of an element stiffness matrix following the same procedure that was used to derive equation (3.7), the following types of term may be generated :-

$$\left.\begin{array}{ll}
\int_0^c \phi_i^{\ell} \ \phi_j^h \, dz & \\
\int_0^c \phi_i^{\ell} \ (\phi_j^h)' dz & \int_0^c (\phi_i^{\ell})' \ \phi_j^h \, dz \\
\int_0^c \phi_i^{\ell} \ (\phi_j^h)'' dz & \int_0^c (\phi_i^{\ell})'' \ \phi_j^h \, dz \\
\int_0^c (\phi_i^{\ell})' (\phi_j^h)'' dz & \int_0^c (\phi_i^{\ell})'' (\phi_j^h)' dz \\
\int_0^c (\phi_i^{l})'' (\phi_j^h)'' dz &
\end{array}\right\} , \qquad (3.19)$$

Where the prime signifies the derivative with respect to z and

$i, j = 1, 2$ or 3

$\ell, h = 1, 2, ----, L$

In the derivation of equivalent nodal loads if only the body force component is considered as an example then body force distribution on an element may now be written as

$$\{R\} = \sum_{\ell=1}^{L} [\ulcorner \phi^{\ell} \lrcorner] \{R^{\ell}\} \tag{3.20}$$

where the elements of $\{R^{\ell}\}$ are functions of x and y only.

The virtual work done by forces $\{R^{\ell}\}$ moving through displacements d $\{f\}$ may now be calculated in the same way in which equation (3.10) was formed to give an equivalent nodal loading

$$\{P^h\} = \int_V \sum_{\ell=1}^{L} \sum_{h=1}^{L} [N]^t [\ulcorner \phi^h \lrcorner]^t [\ulcorner \phi^{\ell} \lrcorner] \{R^{\ell}\} dV \tag{3.21}$$

The contribution to the nodal loading of surface tractions and other loads may be considered in a similar manner.

If ϕ_i^{ℓ} and ϕ_i^h are chosen to be orthogonal functions over the range of z then

$$\int_0^c \phi_i^{\ell} \; \phi_i^h dz \begin{array}{ll} = 0 & \text{for } \ell \neq h \\ \neq 0 & \text{for } \ell = h \end{array}$$

In many practical cases, and also due to the fact that sines and cosines are often used for the ϕ_i, it will be found that for $i \neq j$ and $\ell \neq h$ the expressions (3.19) are either zero or do not appear during the element stiffness formation. The orthogonal nature of the ϕ_i also ensures, by consideration of the nodal loading equations of the form (3.21), that the nodal loads are only associated with nodal freedoms $\{\delta^{\ell}\}$

The assembly of the complete system therefore reduces to the form

$$\begin{bmatrix} [K^1] & & & 0 \\ & [K^2] & & \\ & & \ddots & \\ 0 & & & [K^L] \end{bmatrix} \begin{Bmatrix} \{\delta^1\} \\ \{\delta^2\} \\ \vdots \\ \{\delta^L\} \end{Bmatrix} \quad \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \vdots \\ \{F^L\} \end{Bmatrix} \tag{3.22}$$

The significance of this form is the complete decoupling of the general three dimensional problem into L separate two dimensional problems. Provided that the z variation in load and displacement can be adequately described by the summation of relatively few terms of the orthogonal series then this method may give a considerable reduction in both the time and storage required in computation.

3.4.2 A note on orthogonal functions and generalised Fourier series

(i) The concept of orthogonality applied to vectors

(a) Two vectors $\overline{A}$ and $\overline{B}$ are said to be orthogonal if their scalar product is zero

i.e. if $\overline{A}.\overline{B} = |A||B|\cos\theta = 0$ since θ is then $\frac{\pi}{2}$

Alternatively the vectors may be written as

$\overline{A} = A_1 i + A_2 j + A_3 k$ and $\overline{B} = B_1 i + B_2 j + B_3 k$

where i, j and k are a set of mutually perpendicular unit vectors.

Then for orthogonality of $\overline{A}$ and $\overline{B}$

$\overline{A}.\overline{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = 0$

or $\sum_{i=1}^{3} A_i B_i = 0$

(b) A vector $\overline{A}$ is called a unit or normalised vector if $|A| = 1$

It follows from above that

$\overline{A}.\overline{A} = 1$ since $|A| = 1$ and $\theta = 0$

Alternatively

$\overline{A}.\overline{A} = {A_1}^2 + {A_2}^2 + {A_3}^2 = 1$

or $\sum_{i=1}^{3} A_i^2 = 1$

(c) Extending these ideas into three dimensional space then any three vectors ϕ_n (where n = 1, 2 or 3) having components $\phi_n(r)$ (where r = 1, 2 or 3) are orthogonal if

$$\sum_{r=1}^{3} \phi_m(r) \ \phi_n(r) = 0 \quad \text{for } m,n = 1,2 \text{ or } 3$$

but $m \neq n$

The vectors are also normal if

$$\sum_{r=1}^{3}\left[\phi_n(r)\right]^2 = 1 \quad \text{for } n = 1,\ 2 \text{ or } 3$$

The set is both orthogonal and normal i.e. ORTHONORMAL

if

$$\sum_{r=1}^{3} \phi_m(r)\ \phi_n(r) = \delta_{mn} \qquad (3.23)$$

Where m,n = 1,2 or 3 and δ_{mn} is the Kronecker delta.

(ii) Extension of vector concepts to functions

In the same way that a vector in three dimensional space can be expressed in terms of an orthnormal set, it may be possible to express any function within a specified region by an orthonormal set of functions in an infinite number of dimensions within the same region.

In mathematical terms, if a vector f has components f(r) that can be expressed as

$$f(r) = \sum_{r=1}^{3} C_n\phi_n(r)$$ where $n = 1, 2, 3$ and C_n is a constant coefficient

Then it may be possible to express a function f(x) as

$$f(x) = \sum_{n=1}^{\infty} C_n\phi_n(x) \quad \text{in the region } a \leqslant x \leqslant b \qquad (3.24)$$

The $\phi_n(x)$ are now a set of orthonormal functions in the region $a \leqslant x \leqslant b$ and the series becomes an orthonormal series.

The series is known as the generalised Fourier series.

The practical problem in this analysis is to determine the coefficients C_n if a specified function f(x) is to be described in terms of any desired set of $\phi_n(x)$. This problem may be overcome as follows -

Provided the 'completeness' criterion is fulfilled, i.e. the series of equation (3.24) does actually converge to f(x), then if each side of the equation is multiplied by $\phi_m(x)$ and subsequently integrated over the region a to b we have

$$\int_a^b f(x)\phi_m(x)dx = \int_a^b \sum_{n=1}^{\infty} C_n\,\phi_n(x)\,\phi_m(x)dx$$

But if the $\phi_n(x)$ are chosen to be an orthonormal set of functions they must satisfy the condition (obtained by extending the concept of equation (2.23) from vectors to functions) that

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = \delta_{mn}$$

This means that

$$\int_a^b \sum_{n=1}^{\infty} C_n \phi_n(x)\,\phi_m(x)\,dx \quad \text{is in fact simply } C_m$$

Hence

$$C_m = \int_a^b f(x)\,\phi_m(x)\,dx \qquad (3.25)$$

The C_m are called the generalised Fourier coefficients.

In practice it may be observed that provided $f(x)$ and $d\left[f(x)\right]/d_x$ are at least piecewise continuous over the specified region then it is possible to represent $f(x)$ by the use of an orthonormal series and the series will converge. Also in practice it is often found that trigonometric series provide an orthonormal set of functions that are both convenient to handle and at the same time ensure the disappearance of many of the terms of type (3.19).

3.5 ASPECTS OF THE FINITE ELEMENT METHOD AS APPLIED TO PROBLEMS OF PLATE BENDING

3.5.1 Characteristics required of the approximating function defining the element displacements

Approximate displacement functions generally consist of a number of terms from a polynomial or trigonometric series, each term having associated with it an initially unknown coefficient. The accuracy of the final solution depends on the degree of correspondence between the approximating function and the true displacement of the element but the choice of terms used in the function is not completely arbitrary as certain requirements should be fulfilled as follows:-

(a) There must be at least as many terms (coefficients) in the function as there are nodal degrees of freedom for the element.

(b) The function should be such as to provide continuity between elements both at the nodes and along common boundaries. This compatibility condition is often difficult to attain if condition (a) is also to be satisfied. The result may be the formation of non-conforming elements but, as will be seen later, these elements are not necessarily discarded as useful results may still be obtained, although problems of lack of convergence may arise.

(c) It is intuitively apparent that rigid body modes of element displacement may arise in practice and the approximating function must allow for their presence. Also, if the discretization of a continuum is progressively refined by an increase in number, and decrease in size of the elements it is to be expected that the general conditions of strain within any element will tend to become constant. The approximating function must therefore include terms defining a constant strain condition if convergence of the solution is to be achieved.

(d) The approximating function must be continuous within the region of the element and be differentiable to at least an order equivalent to that which is present in the functional being used for the variational formulation of the problem.

3.5.2 Illustration of the above characteristics in relation to simple plate bending elements

(a) Simple rectangular element

Figure 3.1 below shows a rectangular element defined in a Cartesian co-ordinate system

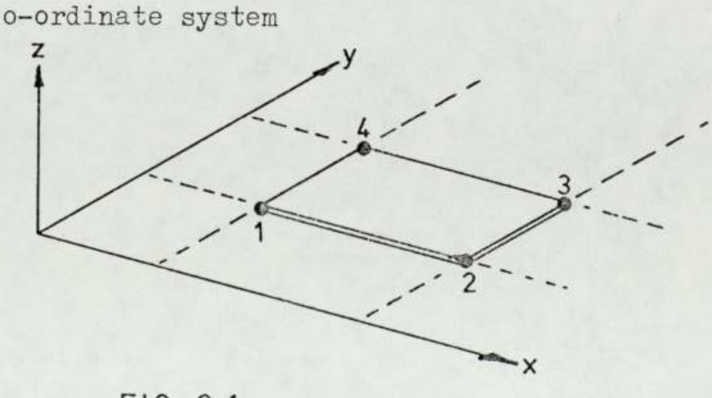


FIG. 3.1

The displacement of the element may be defined in terms of the geometrical conditions at the four nodes (1, 2, 3, 4). If continuity between contiguous elements is to be achieved at the nodes then the transverse deflection, w, and the slopes $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ must be specified as the degrees of freedom at each node; the element therefore having a total of twelve degrees of freedom. If a polynomial in x and y is to be used for the approximating function it must therefore contain at least twelve terms.

Possible rigid body modes of displacement consist of translation in the z direction and rotation with respect to the x and y axes. Linear terms in x and y, together with a constant term will allow for the possibility of these displacements.

The constant strain condition for a bent plate implies constant curvature, and curvature is defined in terms of the partial second derivatives of w. The approximating function must therefore contain quadratic terms in x and y.

At this stage therefore, a polynomial may be formed thus:-

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

It is apparent that a further six terms are required. These extra terms may be selected from the general polynomial expression

$$\begin{aligned} w = {} & b_1 && \text{(constant)} \\ & + b_2x + b_3y && \text{(linear)} \\ & + b_4x^2 + b_5xy + b_6y^2 && \text{(quadratic)} \\ & + b_7x^3 + b_8x^2y + b_9xy^2 + b_{10}y^3 && \text{(cubic)} \\ & + b_{11}x^4 + b_{12}x^3y + b_{13}x^2y^2 + b_{14}xy^3 + b_{15}y^4 && \text{(quartic)} \quad (3.26) \end{aligned}$$

etc.

The selection of the cubic terms results in a further four terms and the remaining two must be chosen from the five available quartic terms. The two that are generally selected are the x^3y and xy^3 terms as it is found that use of the x^4 and y^4 terms gives an even greater degree of non-conformability than that which will be shown to already exist.

The final approximating function is therefore

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$
$$+ a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 \qquad (3.27)$$

If the form of the deflection and the normal and tangential slopes along any edge of the element are investigated it will be found that the deflection and tangential slopes are compatible between contiguous elements but that the normal slope is discontinuous. The element is therefore non-conforming.

The fact that the element is non-conforming does not preclude its use however. Huebner [16] states that although conformability and completeness [as defined by compliance with conditions (c) and (d) of section 3.5.1.] are the only guarantee of convergence of the solution, it is still possible to have convergence with non-conforming elements and, in some cases, they may even be superior to conforming elements in terms of rate of convergence.

The rectangular element discussed above is the simplest formulation available. Further refinement is possible by increasing the degrees of freedom; a common addition in this respect is the inclusion of $\frac{\partial^2 w}{\partial x \partial y}$ at each node thus giving the element a total of sixteen degrees of freedom. It may be shown that this element is fully conforming.

(b) Triangular elements

Rectangular elements have been successfully used in practice but are somewhat restricted in their application due to difficulty in the discretization of plates of unusual shape. Most shapes may be represented with a reasonable degree of accuracy by sub-division into triangles. For this reason triangular elements have been extensively developed. The simplest form of triangular element is shown in figure 3.2 below

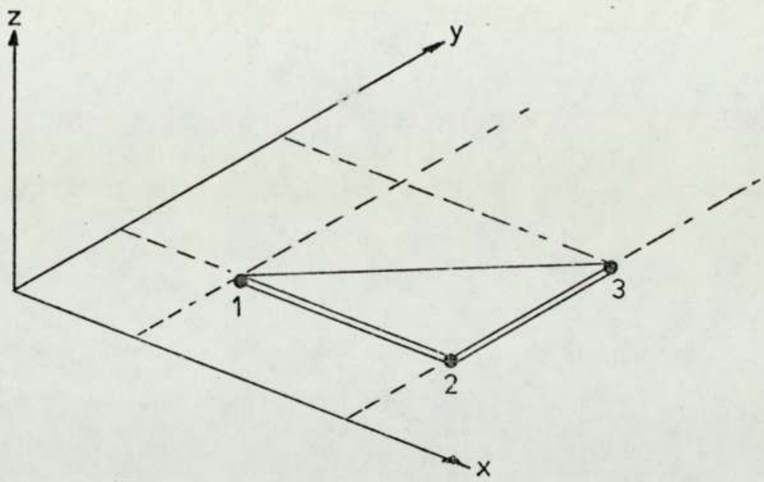


FIG. 3.2

If, as in the case of the rectangular element, nodal freedoms w, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are chosen, then the element has nine degrees of freedom. The general cubic of equation (3.26) has ten terms, which implies that one term must be deleted, and the polynomial is therefore incomplete. Deleting one of the cubic terms on an arbitrary basis would appear to be satisfactory, but in fact causes problems which range from a lack of convergence to a complete inability to provide a solution. A common refinement to this element is to include the normal slope at the mid-point of each side as an additional freedom. It may be shown that this refinement results in a conforming element.

3.5.3 Elements with curved boundaries

Many plates exist in practice where the boundaries are in the form of circular arcs. Although discretization of such plates by the use of triangular elements is possible, a much more satisfactory solution results from the development of elements with circular boundaries.

Olson, Lindberg and Tulloch [22] have developed an annular sector element which is, in effect, the cylindrical co-ordinate equivalent of the rectangular element described above in section 3.5.2. They used the deflection and radial and circumferential slopes at each corner of the element as the nodal freedoms. The displacement function is equation (3.27) with x and y simply replaced by r and θ. The element is non-conforming but has the advantage that it could be used in conjunction with rectangular elements to describe plates of unusual shape. Olson and Lindberg also developed a circular sector element in order to close the central space formed by the use of annular elements. This circular sector element presents considerable problems in its formulation which are discussed in detail in section 5.3. Their eventual solution to the problem resulted in an element which was non-conforming both with similar elements and with their annular element. They admit that their results using this element were disappointing but, rather surprisingly, did not try representing complete plates by the use of annular elements and leaving a very small central hole.

Further refinement of the annular sector element has been made by Sawko and Merriman [23] and Singh and Ramaswamy [24] who successively introduced more degrees of freedom which made the element into a conforming type. (Further details of their elements are given in section 5.1).

4

A FINITE ELEMENT ANALYSIS OF THE BENDING OF SYMMETRICALLY LOADED, UNSTIFFENED CIRCULAR PLATES

4.1 INTRODUCTION

The purpose of this chapter is to explain the basic formulation of a finite element program to analyse the deformation and stress distribution in a symmetrically loaded unstiffened circular plate. The reason for choosing this particular problem is that it provides a means for developing the basic computational requirements in a finite element program whilst at the same time solving a problem with well known theoretical results against which the validity of the computed results may be compared.

Axial symmetry of both the plate and the loading means that the problem is quasi - unidimensional; the transverse deflection and in-plane stress distribution being functions of radius only. With this in mind the plate is discretized into elements that are either annular or complete central discs. The elements are fully conforming in terms of slopes and deflections.

4.2 THE STIFFNESS MATRIX FOR AN ANNULAR ELEMENT

4.2.1 Description of the displacement field and the development of shape functions

Consider a complete plate in which the general annulus may be defined as in figure 4.1a. The inner and outer circumferences of the annulus become nodal lines and the transverse displacement and radial slope at these lines become the nodal degrees of freedom as in figure 4.1b.

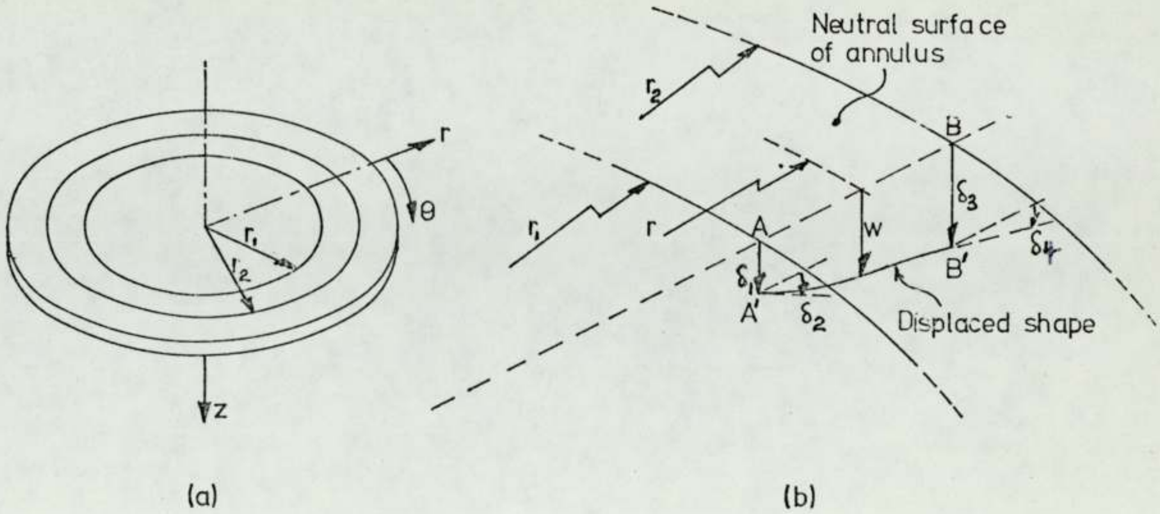


FIG. 4.1

The initially straight radial line AB on the annulus is displaced to position A′ B′ and it is now required that the displacement, w , of any point on the line be described in terms of the nodal freedoms δ_1, δ_2, δ_3 and δ_4 .

The displacement field may be expressed in terms of the nodal freedoms as follows

$$\{f\} = w = [N_1 \ \ N_2 \ \ N_3 \ \ N_4]\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{Bmatrix} \qquad (4.1)$$

Where w and the shape functions N are functions of r only. The N may be derived by the process of interpolation to produce functions which satisfy specified geometric conditions at given boundaries.

4.2.2. <u>A note on the process of interpolation in the generation of shape functions</u>

The basis of interpolation is to generate a set of functions, which for simplicity are often polynomials, that will approximate to an unknown function within a specified region and will satisfy given conditions as to the value of the function and/or its derivatives at selected points called tabular points.

The books by Ralston (10) and Martin & Carey (14) are especially informative in this field; Ralston for his pure mathematical aspects and Martin & Carey for their application to finite element theory.

Let $f(x)$ be the unknown function, values of which, together with some of its derivatives, are known at the tabular points

$y(x)$ is a function we wish to generate in order to represent $f(x)$ within a specified region.

Define an error function $E(x)$ such that

$$E(x) = f(x) - y(x)$$

It is required that $E(x)$ be zero at the tabular points and as small as possible everywhere else.

For general polynomial interpolation, $y(x)$ may be written as

$$y(x) = \sum_{j=1}^{n}\sum_{i=0}^{m} A_{ij}(x)\, f^{(i)}(x_{ij})$$

Where $A_{ij}(x)$ are polynomials in x

Also $f^{(i)}(x_{ij})$ is the value of the i^{th} derivative of $f(x)$ at the j^{th} tabular point in an interpolation where there are n tabular points with derivatives of $f(x)$ up to the m^{th} order being known

Hence
$$E(x) = f(x) - \sum_{j=1}^{n}\sum_{i=0}^{m} A_{ij}(x)\, f^{(i)}(x_{ij}) \qquad (4.2)$$

The simplest form of interpolation is that in which only the value of the function itself is specified at the tabular points. This is called Lagrangian interpolation and the Lagrange polynomials may be developed as follows -

Using only the function itself implies that m = 0. The A_{0j} now become the Lagrange polynomials $\ell_j(x)$ and equation (4.2) becomes

$$E(x) = f(x) - \sum_{j=1}^{n} \ell_j(x)\, f(x_j)$$

Now $E(x)$ must be zero at the tabular points, therefore when

$x = x_k$:-

$$f(x_k) = \sum_{j=1}^{n} \ell_j(x_k) f(x_j) \qquad \text{for } k = 1,2,\text{-----}n.$$

$$\text{or} \quad \ell_j(x_k) = \delta_{jk} \qquad \text{for } j,k = 1,2,\text{----}n. \ (\delta_{jk} \text{ is the Kronecker delta})$$

A polynomial which fits this requirement is

$$\ell_j(x) = \frac{(x-x_1)\text{-------}(x-x_{j-1})(x-x_{j+1})\text{-------}(x-x_n)}{(x_j-x_1) \qquad (x_j-x_{j-1})(x_j-x_{j+1}) \qquad (x_j-x_n)} \tag{4.3}$$

Note the absence of the $(x - x_j)$ term. The $\ell_j(x)$ are therefore polynomials of degree $(n - 1)$.

Lagrange polynomials find practical applications as shape functions for axial stiffness elements and plate stretching problems. Plate bending problems require definition of deflections <u>and</u> slopes at tabular points and therefore equation (4.2) with $m = 1$ is used.

The interpolation process with $m = 1$ is called Hermitian interpolation and Hermitian polynomials are generated as follows.

Using $m = 1$, equation (4.2) may be re-written as

$$E(x) = f(x) - \left[\sum_{j=1}^{n} h_{0j}(x) f(x_j) + \sum_{j=1}^{r} h_{1j}(x) f'(x_j) \right]$$

The interpretation of which is that there are n tabular points at which $f(x)$ is known and at r of these n points the first derivative $f'(x)$ is also known. The tabular points are at $x = x_j$ and $h_{0j}(x)$ and $h_{1j}(x)$ are the Hermitian polynomials.

If $E(x_j) = 0$ for $j = 1, \ldots, n$ and $E'(x_j) = 0$ for $j = 1, \ldots, r$ then the necessary conditions are:-

For zero function error

$$h_{0j}(x_k) = \delta_{jk} \quad j,k = 1,\ldots,n \tag{4.4a}$$

$$h_{1j}(x_k) = 0 \quad j = 1,\ldots,r\ ; \quad k = 1,\ldots,n \tag{4.4b}$$

For zero first derivative error

$$h'_{0j}(x_k) = 0 \quad j = 1,\ldots,r\ : \quad k = 1,\ldots,n \tag{4.4c}$$

$$h'_{1j}(x_k) = \delta_{jk} \quad j,k = 1,\ldots,r \tag{4.4d}$$

Noting that there are $(n + r)$ conditions to satisfy, then the required function will be of degree $(n + r - 1)$

In many practical instances, including the applications in this investigation, the first derivative is known at all of the tabular points hence $n = r$. If this is the case then a polynomial which may fulfil the required conditions for $h_{0j}(x)$ is

$$h_{0j}(x) = t_j(x)\left[\ell_j(x)\right]^2$$

Where $\ell_j(x)$ is a Lagrange polynomial of degree n

and $t_j(x)$ is a polynomial inserted for the purpose of making $h_{0j}(x)$ of degree $(2n - 1)$

This form of h_{0j} will satisfy the requirement of equation (4.4a) if:-

$$t_j(x_j) = 1 \qquad \text{for } j = 1,2,\text{-----},n \tag{4.5a}$$

and that of equation (4.4c) if:-

$$t_j'(x_j) + 2\ell_j'(x_j) = 0 \qquad \text{for } j = 1,2,\text{-----},n \tag{4.5b}$$

A form for $t_j(x)$ that satisfies equations (4.5a) and (4.5b) is

$$t_j(x) = 1 - 2(x - x_j)\ell_j'(x_j)$$

Hence $$h_{0j}(x) = \left[1 - 2(x - x_j)\ell_j'(x_j)\right]\left[\ell_j(x)\right]^2 \qquad \text{for } j = 1,2,\text{-----},n \tag{4.6}$$

In a similar manner, try a form for $h_{1j}(x)$ thus :-

$$h_{1j}(x) = s_j(x)\left[\ell_j(x)\right]^2$$

This will satisfy equation (4.4b) if

$$s_j(x_j) = 0 \qquad \text{for } j = 1,2,\text{-----},n \tag{4.7a}$$

and will also satisfy equation (4.4d) if

$$s_j'(x_j) = 1 \qquad \text{for } j = 1,2,\text{------},n \tag{4.7b}$$

A form for $s_j(x)$ that satisfies equations (4.7a) and (4.7b) is

$$s_j(x) = (x - x_j)$$

Hence $$h_{1j}(x) = (x - x_j)\left[\ell_j(x)\right] \qquad \text{for } j = 1,2,\text{-----},n \tag{4.8}$$

Equations (4.6) and (4.8) thus make possible the formation of a set of polynomials which may be summed to give a polynomial that will represent the required function within the specified region and satisfy exactly the function and its first derivative at the tabular points.

Applying these ideas to the subject of the current investigation and referring back to figure 4.1b there are two tabular points, therefore $n = 2$. The requirements of the polynomials in r are:-

At $r = r_1$ the function is δ_1 and the first derivative is δ_2

At $r = r_2$ the function is δ_3 and the first derivative is δ_4

Now from equation (4.3)

for $j=1$, $\ell_1(r) = \dfrac{r - r_2}{r_1 - r_2}$ hence $\ell_1'(r) = \dfrac{1}{r_1 - r_2}$

for $j=2$, $\ell_2(r) = \dfrac{r - r_1}{r_2 - r_1}$ hence $\ell_2'(r) = \dfrac{1}{r_2 - r_1}$

Therefore from equation (4.6)

$$h_{01}(r) = \left[1 - 2(r - r_1)\left(\frac{1}{r_1 - r_2}\right)\right]\left[\frac{r - r_2}{r_1 - r_2}\right]^2$$

$$= \frac{1}{(r_2 - r_1)^3}\left[2r^3 - 3(r_2 + r_1)r^2 - (6r_1 r_2)r + (r_2 - 3r_1)r_2^2\right] \qquad (4.9a)$$

$$h_{02}(r) = \left[1 - 2(r - r_2)\left(\frac{1}{r_2 - r_1}\right)\right]\left[\frac{r - r_1}{r_2 - r_1}\right]^2$$

$$= \frac{1}{(r_2 - r_1)^3}\left[-2r^3 + 3(r_2 + r_1)r^2 - (6r_1 r_2)r + (3r_2 - r_1)r_1^2\right] \qquad (4.9b)$$

And from equation (4.8)

$$h_{11}(r) = (r - r_1)\left[\frac{r - r_2}{r_1 - r_2}\right]^2$$

$$= \frac{1}{(r_2 - r_1)^2}\left[r^3 - (2r_2 + r_1)r^2 + r_2(r_2 + 2r_1)r - r_1 r_2^2\right] \qquad (4.9c)$$

$$h_{12}(r) = (r - r_2)\left[\frac{r - r_1}{r_2 - r_1}\right]^2$$

$$= \frac{1}{(r_2 - r_1)^2}\left[r^3 - (r_2 + 2r_1)r^2 + r_1(2r_2 + r_1)r - r_2 r_1^2\right] \qquad (4.9d)$$

Thus giving the final interpolated shape $w(r)$ in the region

$r_1 \leqslant r \leqslant r_2$ as

$$w(r) = h_{01}(r)\delta_1 + h_{11}(r)\delta_2 + h_{02}(r)\delta_3 + h_{12}(r)\delta_4$$

Comparing this form for $w(r)$ with that of equation (4.1) shows that the Hermitian polynomials that have been generated are, in fact, the required shape functions. The 'shape' of the polynomials is plotted in figure 4.2

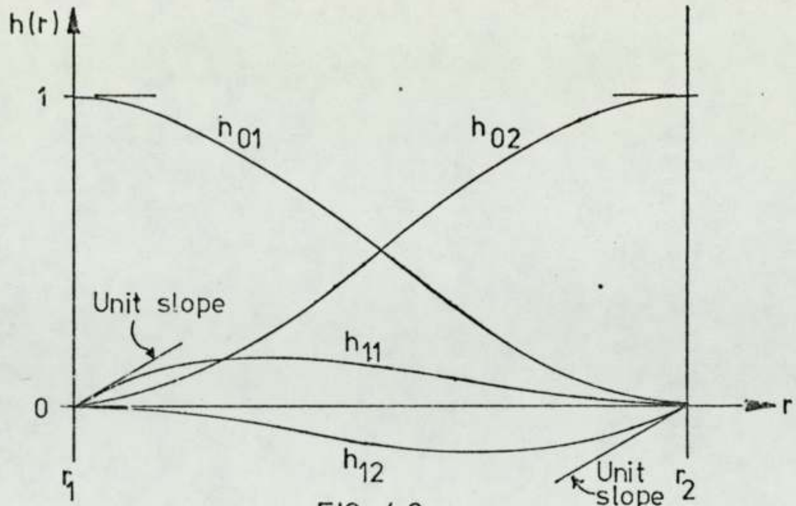


FIG. 4.2

4.2.3 The stress-strain relationship

The stress-strain relationship for a plate in bending is discussed in some detail in Appendix A, section A2.3. The relationship is summarised in equations (A4.2) and may be expressed in matrix form as follows:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix}^e = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix}^e$$

Due to the symmetry of the plate with axisymmetric loading there are no in-plane shear effects and the relationship reduces to

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix}^e = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \end{Bmatrix}^e$$

or $$\{\sigma\}^e = [D]\{\epsilon\}^e \qquad (4.10)$$

Where $$[D] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} = \frac{E}{(1-\nu^2)}[C] \qquad (4.11)$$

Where $[C] = \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$

4.2.4 The strain-displacement relationship

This also is discussed in Appendix A, section A2.2. The relationship is summarised in equations (A4.1) and may be expressed in matrix form thus:-

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix}^e = -z \begin{bmatrix} \frac{\partial^2}{\partial r^2} \\ \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2} \\ -\frac{2}{r^2}\cdot\frac{\partial}{\partial \theta} + \frac{2}{r}\cdot\frac{\partial^2}{\partial r \partial \theta} \end{bmatrix} \{w\}$$

Again, due to the symmetry of the loading, there is no variation in displacement with respect to θ and the relationship reduces to

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \end{Bmatrix}^e = -z \begin{bmatrix} \frac{\partial^2}{\partial r^2} \\ \frac{1}{r}\cdot\frac{\partial}{\partial r} \end{bmatrix} \{w\}$$

But $\{w\} = [N] \{\delta\}^e$

Therefore
$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \end{Bmatrix}^e = -z \begin{bmatrix} \frac{\partial^2 N_1}{\partial r^2} & \frac{\partial^2 N_2}{\partial r^2} & \frac{\partial^2 N_3}{\partial r^2} & \frac{\partial^2 N_4}{\partial r^2} \\ \frac{1}{r}\cdot\frac{\partial N_1}{\partial r} & \frac{1}{r}\cdot\frac{\partial N_2}{\partial r} & \frac{1}{r}\cdot\frac{\partial N_3}{\partial r} & \frac{1}{r}\cdot\frac{\partial N_4}{\partial r} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}^e$$

If the Hermitian polynomials of equations (4.9) are now substituted for the N and the appropriate differentiations performed we have

$$\{\epsilon\}^e = \frac{-z}{(r_2 - r_1)^3} [A] \{\delta\}^e$$

or
$$\{\epsilon\}^e = [B] \{\delta\}^e \qquad (4.12)$$

Where
$$[B] = \frac{-z}{(r_2 - r_1)^3} [A] \qquad (4.13)$$

[A] is (2 x 4) matrix whose elements are listed below

$$A_{11} = 12r - 6(r_2 + r_1)$$
$$A_{12} = 6(r_2 - r_1)r - 2(r_2 - r_1)(2r_2 + r_1)$$
$$A_{13} = -12r + 6(r_2 + r_1)$$
$$A_{14} = 6(r_2 - r_1)r - 2(r_2 - r_1)(r_2 + 2r_1)$$
$$A_{21} = 6r - 6(r_2 + r_1) + 6r_2 r_1 \cdot \frac{1}{r}$$
$$A_{22} = 3(r_2 - r_1)r - 2(r_2 - r_1)(2r_2 + r_1) + (r_2 - r_1)(r_2 + 2r_1)r_2 \cdot \frac{1}{r}$$
$$A_{23} = -6r + 6(r_2 + r_1) - 6r_2 r_1 \cdot \frac{1}{r}$$
$$A_{24} = 3(r_2 - r_1)r - 2(r_2 - r_1)(r_2 + 2r_1) + (r_2 - r_1)(2r_2 + r_1)r_1 \cdot \frac{1}{r}$$

4.2.5 <u>Formation of the element stiffness matrix</u>

From equation (3.7) the element stiffness matrix is given by

$$[k] = \int_V [B]^t [D] [B] \, dV$$

Expressions for [D] and [B] are given in equations (4.11) and (4.13) respectively.

The geometry of the element is described in cylindrical co-ordinates therefore the volume integral becomes a triple integral in r, θ and z.

i.e. $$[k] = \iiint [B]^t [D][B]\, r.dr.d\theta.dz$$

Since [B] and [D] are independent of θ and the element is a complete annulus the θ integral is simply 2π

The z integral consists of $\int z^2 dz$ from the product of $[B]^t$ and [B], and $-\frac{h}{2} \leqslant z \leqslant \frac{h}{2}$ therefore the integral becomes $\frac{h^3}{12}$

Hence $$[k] = \frac{2\pi E h^3}{12(r_2 - r_1)^6 (1-\nu^2)} \int_{r_1}^{r_2} [A]^t [C][A] r.dr$$

But flexural rigidity, $D = \dfrac{Eh^3}{12(1-\nu^2)}$

$$\therefore \quad [k] = \frac{2\pi D}{(r_2 - r_1)^6} \int_{r_1}^{r_2} [A]^t [C][A]\, r.dr \tag{4.14}$$

It is possible to carry out this matrix manipulation and subsequent integration manually to give a general form for [k] in terms of general values of r_1 and r_2, but the algebra is extremely laborious and [k] is more conveniently formed numerically for each particular element as and when required in the final computer program.

4.3 THE STIFFNESS MATRIX FOR A DISC ELEMENT

If a complete plate is to be analysed a disc element is required in order to close the central hole in the annular elements. Due to the symmetry of the loading the slope at the centre of the disc must be zero and the number of nodal freedoms is consequently reduced to three as shown in figure 4.3

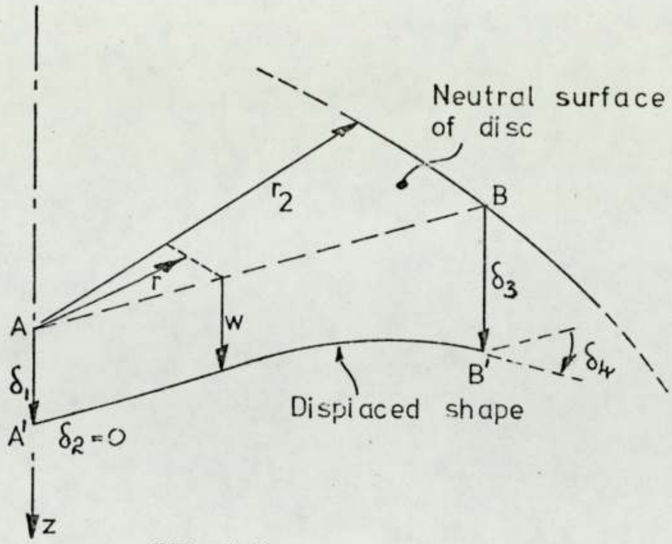


FIG. 4.3

If the initially straight radial line AB on the disc is displaced to A'B' the displacement w at radius r may be expressed in terms of the nodal freedoms as follows

$$w = \begin{bmatrix} N_1 & 0 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}^e$$

Where the N are the same Hermitian polynomials used for the annular element but with $r_1 = 0$ hence

$$N_1 = \frac{1}{r_2^3}\left[2r^3 - 3r_2 r^2 + r_2^3\right]$$

$$N_3 = \frac{1}{r_2^3}\left[-2r^3 + 3r_2 r^2\right]$$

$$N_4 = \frac{1}{r_2^2}\left[r^3 - r_2 r^2\right]$$

From this stage the formulation of $[k]$ is an identical procedure to that for the annular element. It is possible to express $[k]$ for the disc as a (3 x 3) matrix but it is convenient for assembly purposes to have $[k]$ the same size for all elements. For this reason $[k]$ for the disc has been preserved in (4 x 4) form by putting $N_2 = 0$.

Therefore by putting $r_1 = 0$ in equation (4.14), $[k]$ for the disc is given by

$$[k] = \frac{2\pi D}{r_2^6}\int_0^{r_2} [A]^{t}[C][A]\, r.dr \tag{4.15}$$

Where $[C]$ is the same as for the annulus but the elements of $[A]$ are now

$$A_{11} = 12r - 6r_2$$

$$A_{12} = 0$$

$$A_{13} = -12r + 6r_2$$

$$A_{14} = 6r_2 r - 2r_2^2$$

$$A_{21} = 6r - 6r_2$$

$$A_{22} = 0$$

$$A_{23} = -6r + 6r_2$$

$$A_{24} = 3r_2 r - 2r_2^2$$

4.4 ASSEMBLY OF THE SYSTEM STIFFNESS MATRIX

The application of the principle of stationary potential energy for a continuum demonstrated, in equation (3.16), the need to collect together at each global node the stiffness contributions from all elements meeting at that node.

In this particular application the collecting together of the element contributions is relatively easy as the nodal freedoms at the outer circumference of one element must match the freedoms at the inner circumference of the adjoining element as shown in figure 4.4 which depicts a radial section through the plate and illustrates the correlation between the element and global freedoms.

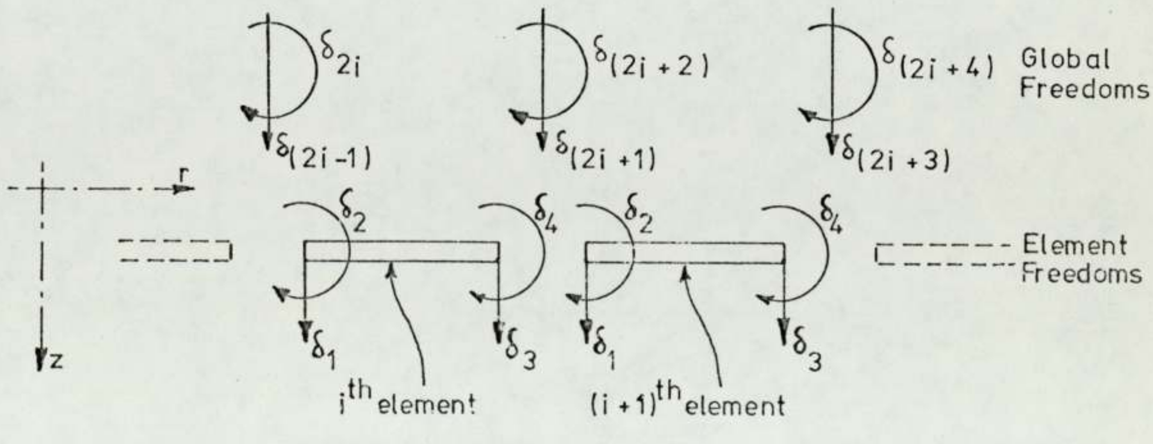
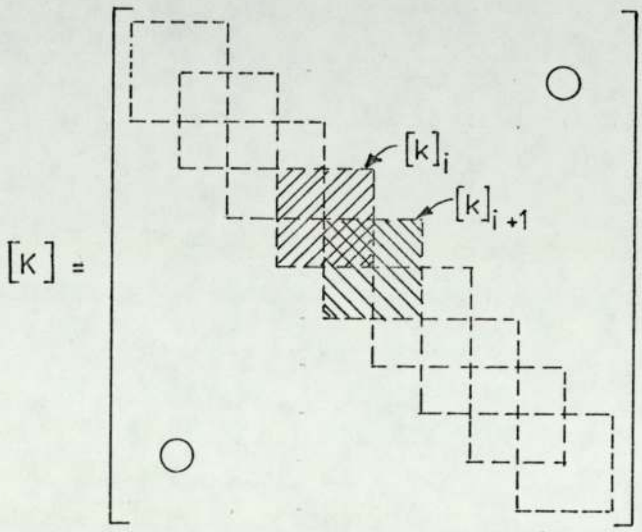


FIG. 4.4

The stiffness to be associated with, say, global freedom $\delta_{(2i+1)}$ will be the sum of the stiffnesses associated with element freedoms δ_3 for the i^{th} element and δ_1 for the $(i + 1)^{th}$ element. This process is achieved in practice simply by 'overlapping' the element stiffness matrices and adding the individual stiffnesses within the overlaps thus:-



The system stiffness matrix is therefore symmetric, relatively sparse and of half bandwidth 4, due to the [k] being (4 x 4),

4.5 FORMATION OF THE SYSTEM FORCE VECTOR

4.5.1 Concentrated forces and moments

The treatment of concentrated forces and moments is straightforward provided that the force or moment acts on a ring which is then chosen as one of the nodal rings. In this case the loading is not associated with any particular element but contributes directly to the global force vector.

4.5.2 Distributed pressures

The most effective way of treating distributed pressure on the plate would be to make use of equation (3.10) to form nodal loads which produce equivalent virtual work when moving through the assumed displacements. In this particular application there are no body forces and, due to symmetry, the surface traction vector $\{T\}$ becomes simply the pressure, $p(r)$, which is acting on the plate and is a function of radius only. Equation (3.10) therefore becomes

$$\{P\}^t = \int_0^{2\pi}\int_{r_1}^{r_2} p(r)[N]\,r.dr.d\theta$$

The elements of $[N]$ are cubics in r which, even if $p(r)$ is a constant pressure, results in the elements of $\{P\}^t$ being fifth order polynomials with considerable algebraic manipulation required for their evaluation.

Gallagher [15] states that the replacement of distributed loads by statically equivalent nodal loads will normally give acceptable results in practice. This approach has been adopted in the current investigation due to the difficulties outlined above and does appear to give an acceptable solution even though it is not so theoretically satisfying. Equivalent nodal loads are therefore calculated as follows.

Consider the distributed pressure on a typical element to vary linearly with radius as shown in figure 4.5

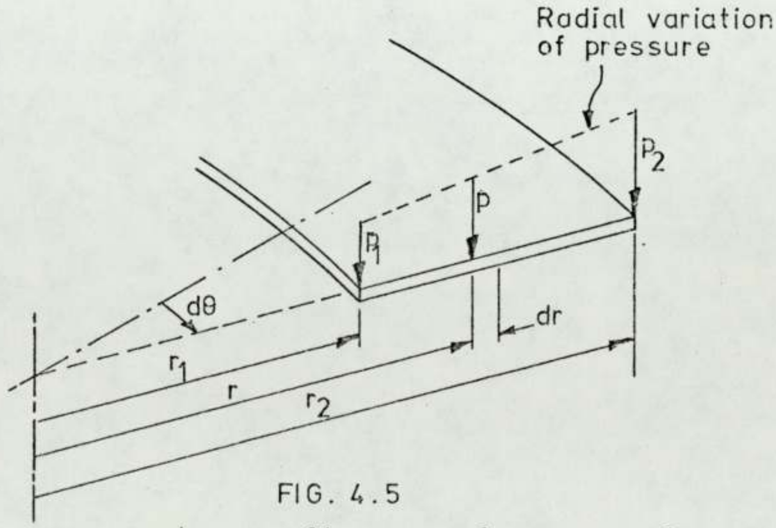


FIG. 4.5

The pressure p at any radius r may be expressed as

$$p = mr + c \qquad \text{where } m = \frac{P_2 - P_1}{r_2 - r_1} \qquad \text{and } c = \frac{P_1 r_2 - P_2 r_1}{r_2 - r_1}$$

The total load on the element is therefore given by

$$\text{total load} = \int_{r_1}^{r_2} p.2\pi r.dr$$

$$= 2\pi \int_{r_1}^{r_2} (mr^2 + cr)dr$$

$$= 2\pi \left[\frac{m}{3}(r_2^3 - r_1^3) + \frac{c}{2}(r_2^2 - r_1^2)\right]$$

If the effect of this load is to be represented by nodal ring loads P_1 at r_1 and P_2 at r_2 then

$$P_1 + P_2 = 2\pi \left[\frac{m}{3}(r_2^3 - r_1^3) + \frac{c}{2}(r_2^2 - r_1^2)\right]$$

$$= \frac{\pi}{3(r_2 - r_1)} 2(p_2 - p_1)(r_2^3 - r_1^3) + 3(p_1 r_2 - p_2 r_1)(r_2^2 - r_1^2)$$

$$= \frac{\pi}{3(r_2 - r_1)}\left[p_2(2r_2^3 - 3r_2^2 r_1 + r_1^3) + p_1(r_2^3 - 3r_2 r_1^2 + 2r_1^3)\right] \qquad (4.16)$$

Also by taking moments about the inner radius for a small sector subtending angle $d\theta$ at the centre. For static equivalence:-

$$\frac{P_2}{2\pi r_2} r_2 d\theta (r_2 - r_1) = \int_{r_1}^{r_2} p r d\theta dr (r - r_1)$$

$$P_2(r_2 - r_1) = 2\pi \int_{r_1}^{r_2} (mr + c)(r - r_1) r dr$$

$$= \frac{\pi}{6}\left[m(3r_2^4 - 4r_2^3 r_1 + r_1^4) + c(4r_3^3 - 6r_2^2 r_1 + 2r_1^3)\right]$$

Substituting for m and c eventually gives

$$P_2 = \frac{\pi}{6}(r_2 - r_1)\left[p_2(3r_2 + r_1) + p_1(r_2 + r_1)\right] \qquad (4.17)$$

Substitution of equation (4.17) in equation (4.16) leads to the evaluation of force P_1 as

$$P_1 = \frac{\pi}{6}(r_2 - r_1)\left[p_2(r_2 + r_1) + p_1(r_2 + 3r_1)\right] \qquad (4.18)$$

4.5.3 Assembly of the system force vector

The assembly of the force vector is simply the process of collecting together all the forces that act at each nodal ring. These forces are either the forces or moments that actually act at the ring together with the summation of the contributions from the pressure on adjacent elements as indicated by the forces evaluated in equations (4.17) and (4.18).

4.6 SYSTEM CONSTRAINTS AND THE FINAL EVALUATION OF SYSTEM DISPLACEMENTS

4.6.1 The incorporation of system constraints

The system stiffness matrix and force vector having previously been assembled, the problem may now be expressed as

$$[K]\{\delta\} = \{F\}$$

In any given problem the support conditions for the plate require that some of the global freedoms, δ, are zero. The formal incorporation of these constraints simply requires that these freedoms, together with the appropriate rows and columns in $[K]$ and $\{F\}$ be deleted. In practice the constrained freedoms are accounted for at the system assembly stage so that the unconstrained system matrices are never generated. This results in a saving of computational effort and will be discussed in more detail in the section dealing with the development of the computer program.

4.6.2 Evaluation of the system displacements

Following the imposition of the system constraints, the evaluation of the nodal displacements requires the solution of the reduced equations

$$[K]\{\delta\} = \{F\}$$

The formal solution to this problem is simply

$$\{\delta\} = [K]^{-1}\{F\}$$

The elements of $\{\delta\}$ are the required nodal displacements but formal inversion of $[K]$ is very lengthy and may be numerically inaccurate. In practice this is also very inefficient because $[K]$ is symmetric and very sparse; properties which may be used to advantage if other methods are used.

Commonly used techniques for solving the equations are:-

(a) Direct methods such as Gaussian elimination or Cholesky's method.

(b) Iterative methods, of which Gauss-Seidel is probably the one most often used.

Direct methods are to be preferred in general as they give an exact solution whose only inaccuracy is rounding-off errors in computation, although iterative methods can be very valuable especially in non-linear problems. In applications such as this particular investigation where the stiffness matrix is symmetric and very sparse the Cholesky method is more economical than Gaussian elimination; Martin and Carey [14] claiming that it may need as little as one quarter the number of arithmetic operations.

The basis of the Cholesky method is as follows:-

Any symmetric, positive definite matrix may be decomposed into the product of a lower triangular matrix and its transpose thus

$$[K] = [L][L]^t$$ where $[L]$ is a lower triangular matrix

Hence $[K]\{\delta\} = \{F\}$ may be expressed as

$$[L][L]^t\{\delta\} = \{F\}$$

or $[L]\{u\} = \{F\}$ where $\{u\} = [L]^t\{\delta\}$

This set of equations is easily solved for $\{u\}$ by forward reduction.

$\{\delta\}$ is then found by back substitution in the equations

$$[L]^t\{\delta\} = \{u\}$$

4.7 CALCULATION OF BENDING STRESSES

The use of cubic polynomials for the shape functions ensures continuity of the stress field within any given element and the choice of deflection and slope as the nodal freedoms ensures geometric continuity with adjacents elements. These specifications do not, however, guarantee continuity of stress between elements. The way that is used to overcome this problem is to calculate the stresses in contiguous elements at the common boundary and then take a nodal average.

The calculation of stresses within any element is carried out thus:-

From equations (4.10), (4.11), (4.12) and (4.13)

$$\{\sigma\}^e = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \frac{E}{1-\nu^2} [C]\{\epsilon\}^e$$

and $$\{\epsilon\}^e = -\frac{z}{(r_2 - r_1)^3} [A]\{\delta\}^e$$

$$\therefore \quad \{\sigma\}^e = -\frac{E z}{(r_2 - r_1)^3} [C][A]\{\delta\}^e$$

At any particular radius the maximum stress occurs at the plate surface where $z = \frac{h}{2}$. These maximum stresses are therefore given by

$$\{\sigma\}^e = -\frac{6D}{(r_2 - r_1)^3 h^2} [C][A]\{\delta\}^e \tag{4.19}$$

The elements of [A] are functions of radius only and as $\{\delta\}$ has previously been calculated, equation (4.19) can now be used to generate the stress field within the elements if required. If r_1 and r_2 are substituted for the general radius r, the stresses at the inner and outer radii of each element are formed. These values may then be averaged with the values from contiguous elements as indicated previously.

4.8 DEVELOPMENT OF THE COMPUTER PROGRAM 'SYMPLAT'

4.8.1 The main computational tasks

The program is written in ALGOL and can be subdivided into five major sections.

(a) the input of basic data, element details and loading details

(b) generation of element stiffness matrices

(c) assembly of these matrices to give the system stiffness matrix together with the incorporation of constraints

(d) solution of the equilibrium equations to give the displacements

(e) calculation of stresses.

The computational aspects of these sections will now be discussed. ALGOL identifiers are introduced where convenient or different from symbols previously used.

4.8.2 Input of basic data

At the commencement of any analysis the basic decision must be taken as to the number of elements (NELEM) that are to be used to represent the plate. Since the elements are arranged consecutively along the plate radius they may be numbered as such by starting with the smallest radius and working outwards. It also follows that the number of nodal rings (NNODE) will be given by

NNODE = NELEM + 1

At each global nodal ring there are two degrees of freedom, namely the transverse deflection and radial slope, which means that the maximum possible total number of degrees of freedon (NDEGF) will be

NDEGF = NNODE x 2

The boundary conditions for the plate mean that some of these global freedoms will be constrained. The number and position of these constraints is known at the start of the analysis and the number (NCON) may be read in as initial data. This means that the actual number of degrees of freedom is given by

NDEGF = (NNODE x 2) - NCON

The relative sizes of the elements are left for the program user to decide. This means that an array of nodal radii (NODRAD) consisting of NNODE values must be read in.

The information which must be supplied for each element is its modulus of elasticity (E), Poissons ratio (V) and thickness (T).

The loading on the plate is read in directly as nodal loads if the loading is concentrated or as the values of pressure at the nodal radii if the loading is distributed, in which case equations (4.17) and (4.18) are used to calculate equivalent nodal loads. These loadings are processed in 'procedures' (CONFOUT and DISFOUT) and then stored as an array of applied nodal forces (APFO).

4.8.3 Generation of element stiffness matrices

It has already been shown in equation (4.14) that the stiffness matrix (ESTF) for an annular element bounded by nodal radii R1 and R2 is given by

$$[ESTF] = \frac{2\pi D}{(R2-R1)^6}\int_{R1}^{R2}[A]^t[C][A]\,r\,dr$$

The elements of [C] are easily read in as they are either unity or Poissons ratio. The elements of [A] however are functions of R1, R2 and r . The radii R1 and R2 are already available in NODRAD but the presence of the variable, r means that the elements of [A] fall under the effect of the integration. This integration may need to be done numerically in general,but is possible formally in this particular application by adopting the following steps:-

Inspection of the elements of [A] shows that [A] may be re-written as

$$[A] = [A1]r + [A2] + [A3]\frac{1}{r}$$

where

$$[A1] = \begin{bmatrix} 12 & 6(R2-R1) & -12 & 6(R2-R1) \\ 6 & 3(R2-R1) & -6 & 3(R2-R1) \end{bmatrix}$$

$$[A2] = \begin{bmatrix} -6(R2-R1) & -2(R2-R1)(2R2+R1) & 6(R2+R1) & -2(R2-R1)(R2+2R1) \\ -6(R2-R1) & -2(R2-R1)(2R2+R1) & 6(R2+R1) & -2(R2-R1)(R2+2R1) \end{bmatrix}$$

$$[A3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6.R1.R2 & (R2-R1)(R2+2.R1)R2 & -6.R1.R2 & (R2+R1)(2.R2+R1)R1 \end{bmatrix}$$

Since R1 and R2 have already been read these matrices are easily formed.

$[A]^t[C][A]$ is now evaluated firstly by forming

$$[C][A] = [C][A1]r + [C][A2] + [C][A3]\frac{1}{r}$$

These separate matrix multiplications are done by the use of a 'procedure' (MATMULT) and the products stored in matrices [CA1], [CA2] and [CA3].

Now $[A]^t = [A1]^t r + [A2]^t + [A3]^t \frac{1}{r}$

Hence $[A]^t[C][A] = \left[[A1]^t r + [A2]^t + [A3]^t \frac{1}{r}\right]\left[[CA1]r + [CA2] + [CA3]\frac{1}{r}\right]$

$$= \left[[A1]^t[CA1]\right] r^2 + \left[[A1]^t[CA2] + [A2]^t[CA1]\right] r$$
$$+ \left[[A1]^t[CA3] + [A2]^t[CA2] + [A3]^t[CA1]\right]$$
$$+ \left[[A2]^t[CA3] + [A3]^t[CA2]\right]\frac{1}{r} + \left[[A3]^t[CA3]\right]\frac{1}{r^2}$$

These matrix transpositions and multiplications are also done by the use of a 'procedure' (TRMAMULT) and the results stored in matrices [B1], [B2] etc.

Thus $[A]^t[C][A] = [B1]r^2 + \left[[B2] + [B3]\right] r + \left[[B4] + [B5] + [B6]\right]$
$$+ \left[[B7] + [B8]\right]\frac{1}{r} + [B9]\frac{1}{r^2}$$

Noting that the matrices [B1] etc are functions of R1 and R2 only, the integration with respect to r is now easily performed to give

$$\int_{R1}^{R2} [A]^t[C][A]r\,dr = \frac{R2^4 - R1^4}{4}[B1] + \frac{R2^3 - R1^3}{3}\left[[B2] + [B3]\right]$$
$$+ \frac{R2^2 - R1^2}{2}\left[[B4] + [B5] + [B6]\right]$$
$$+ (R2 - R1)\left[[B7] + [B8]\right] + \ln\left(\frac{R2}{R1}\right)[B9]$$

Multiplication of this final expression by $2\pi D/(R2-R1)^6$ thus gives the complete matrix [ESTF].

Since [A] is (2 x 4) then [ESTF] is a (4 x 4) symmetric matrix whose elements are functions of R1, R2 and D.

A similar, although slightly simpler procedure is used to generate [ESTF] for a disc element should the plate be a complete one.

4.8.4 Formation of the system stiffness matrix and the incorporation of constraints

Section 4.4 has already shown that the assembly of the element stiffness matrices to give the system stiffness matrix (SSTF) is relatively straightforward in this particular application. The process is best achieved by the use of a nodal connection matrix (NODC). This matrix forms the basis of a simple numerical coding technique whereby a local freedom at a given node of a particular element may be identified as being the same kind of freedom as others at the same global node but from contiguous elements. Once this identification is made then the quantities associated with this freedom such as the stiffness, loading etc may be combined with those from the other elements to give the total quantity to be associated with the global freedom. To avoid repetition, the exact details for the construction of [NODC] will be given later in the instructions for use of the program. During the formation of [NODC] the global freedoms that are to be constrained are identified and incorporated as zeros in [NODC].

As each element stiffness matrix is formed, reference is made to [NODC] and each element is then correctly located in [SSTF]. If a zero is encountered in [NODC] it means that the particular freedom is constrained and any element stiffness contributions to it are discarded at this stage. The unconstrained [SSTF] is therefore never formed which results in a considerable saving of storage. The process whereby each [ESTF] is assembled into [SSTF] as it is formed means that the steps used in forming [ESTF] may be re-used and [ESTF] itself overwritten as successive elements are assembled.

The program is also written so as to economise on the storage of [SSTF]. This matrix is symmetric and sparse. Considerable saving of space is possible if only half of the matrix is stored and that half is further condensed in the following way:

$$[SSTF] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & & & & & & \\ a_{21} & a_{22} & a_{23} & a_{24} & & & & & & \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & & & & \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & & & & \\ & & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & & \\ & & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & & \\ & & & & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{7,10} \\ & & & & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} & a_{8,10} \\ & & & & & & a_{97} & a_{98} & a_{99} & a_{9,10} \\ & & & & & & & & \text{ETC.} & \\ \vdots & & & & & & & & & \vdots \end{bmatrix} \text{ is stored as } \begin{bmatrix} 0 & 0 & 0 & a_{11} \\ 0 & 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & a_{53} & a_{54} & a_{55} \\ a_{63} & a_{64} & a_{65} & a_{66} \\ 0 & a_{75} & a_{76} & a_{77} \\ a_{85} & a_{86} & a_{87} & a_{88} \\ 0 & a_{97} & a_{98} & a_{99} \\ & \text{ETC.} & & \\ \vdots & & & \vdots \end{bmatrix}$$

[NODC] is also used to discard constrained freedoms from the applied load vector {APFO} and to form a vector of nett effective loads (FORCE).

4.8.5 Solution of the equilibrium equations

The principle behind the solution of these equations using the Cholesky method has been discussed in section 4.6.2.

The programming of this method has been carried out by making direct use of the program BANDSOL devised by Wilkinson and described in detail in reference [11] . BANDSOL consists of the two 'procedures' CHOBANDDET AND CHOBANDSOL which must be used together and in that order.

CHOBANDDET takes the matrix [SSTF], which must have been previously condensed into the form shown in section 4.8.4, and decomposes it into a lower triangular form [L] including the diagonal such that

$$[SSTF] = [L][L]^t$$

If the elements of [SSTF] are designated k_{ij} and those of L , ℓ_{ij} then the decomposition is carried out using the algorithm

$$\ell_{ii} = \left(k_{ii} - \sum_{j=1}^{i-1} \ell_{ij}^2 \right)^{\frac{1}{2}} \qquad \text{for } i = 1,2, \text{-----} ,\text{NDEGF}$$

$$\ell_{mi} = \frac{1}{\ell_{ii}}\left(k_{mi} - \sum_{j=1}^{i-1} \ell_{ij}\, \ell_{mj}\right) \qquad \text{for } m = (i+1), \text{---}, \text{NDEGF} \quad i = 1,2, \text{-----}, \text{NDEGF}$$

The output matrix [L] is overwritten on [SSTF] and [SSTF] in its original form is therefore lost at this stage. It can be arranged for a failure message to be given if the determinant of [SSTF] is zero but this has not been included in this particular program.

CHOBANDSOL now solves [SSTF] $\{\delta\}$ = $\{FORCE\}$ which may now be expressed as $[L][L]^t\{\delta\} = \{FORCE\}$ firstly by solving

$$[L]\{u\} = \{FORCE\} \text{ to give } \{u\}$$

and then solving $[L]^t\{\delta\} = \{u\}$ to give $\{\delta\}$

The global displacement vector $\{\delta\}$ is overwritten on $\{FORCE\}$ which is therefore lost at this stage.

4.8.6 Computation of stresses

The stress field within any annular element is given by equation (4.19) as

$$\{\sigma\}^e = -\frac{6D}{(R2-R1)^3 T^2}[C][A]\{\delta\}^e$$

$\{\delta\}^e$ has not been available until this stage of the calculations but [C] and [A] were generated, and successively overwritten, as each element stiffness matrix was formed. This implies that rather than regenerating [C] and [A] at this stage in the computation, it is preferable to make use of them at the point when [ESTF] was being formed and to store the information that is now required for the stress calculation.

The stresses in each element at the nodal radii are required and, with this in view, after [C], [A1], [A2] and [A3] have been formed for use in the computation of [ESTF] the radii R1 and R2 are substituted for r in [A] and matrices are formed of the product

$$\frac{6D}{(R2-R1)^3 T^2}[C][A]$$

These products are identified as [CAINT] for r = R1 and [CAEXT] for r = R2. As each element is assembled the numbers in [CAINT] and [CAEXT] are stored as 'layers' in a three dimensional array (ELCA). [CAINT] and [CAEXT] themselves are then overwritten as each element is processed. The fact that the numbers from [CAINT] and [CAEXT] are stored is not too wasteful of storage as they are only (2 x 4) matrices.

The nodal displacements that have been calculated and subsequently overwritten on the matrix {FORCE} are only the unconstrained displacements. In the calculation of nodal stresses all the nodal freedoms must be considered and not just the unconstrained ones. The displacements in {FORCE} are therefore expanded, using [NODC], to form a matrix of element displacements (ELDISP) which includes both constrained and unconstrained freedoms.

The appropriate elements of [ELCA] and [ELDISP] are now multiplied together to give the radial and tangential stresses at inner and outer radii of all the elements. These stresses are formed in [NODSTRESS] and are printed as part of the output.

The stresses at the common nodal radii of contiguous elements are averaged in [AVSTRESS] and then printed out.

4.8.7 Program flow chart and listing

The final form of the flow chart is shown on pages 73 & 74 and the listing on pages 75 to 82 inclusive.

4.9 DOCUMENTATION FOR USE OF THE COMPUTER PROGRAM 'SYMPLAT'

4.9.1. Program specification

The program analyses the bending of annular or complete circular plates subjected to axi-symmetric loading and boundary conditions. S.I. units are used throughout.

Radial variation of loading, plate thickness and material properties can be accommodated.

The program outputs the radial variation of deflection, slope, radial stress and tangential stress due to the specified loading.

4.9.2 Preparation and presentation of data

Data must be prepared for input to the program as follows and be presented strictly in this order:

(a) The number of plates to be analysed in this particular run.

There is an overall loop in the program for this purpose and each analysis requires its own complete set of data.

(b) The number of elements to be used to represent the plate.

In order to keep computing time and cost to a minimum it is important to use as few elements as possible, consistent with obtaining satisfactory accuracy of solution. (A simple test program showed that doubling the number of elements resulted in the program run time being multiplied by approximately two and a half). Five elements appear to give results of sufficient accuracy for most practical purposes.

(c) The number of constraints on the plate.

This is required at this stage because it affects the array sizes in the program. Any imposed zero slope or deflection at any nodal radius is classified as a constraint. (Note that for the purposes of this program the zero slope at the centre of a complete plate is also classified as a constraint).

(d) The nodal radii.

These must be given in metres and in increasing order of magnitude. A zero radius must be input for the centre of a complete plate as this value is used to select the correct element stiffness subroutine. The choice of the values of the radii has been left to the user. It should be noted however, that elements of equal radial width are not particularly satisfactory and an empirical scheme of making the radial width of the elements approximately proportional to their external radii appears to give better results. If any regions of severe curvature are apparent a finer element mesh in this region will give higher accuracy, particularly in the stress calculations.

(e) Details of each element.

The modulus of elasticity (N/mm^2) and Poissons ratio for the material together with the plate thickness (mm) must be entered as data for each element, commencing with the element nearest to the centre of the plate. In the case of continuous radial variation of any of these three quantities, a stepped approximation must be used.

(f) <u>The nodal connection matrix</u>

This relates the element freedoms to the global freedoms and also defines the positions of the constraints. The matrix is formed as follows:-

ELEMENT Nº. Starting with the element nearest to the centre.	ELEMENT FREEDOMS			
	INTERNAL RADIUS		EXTERNAL RADIUS	
	Defl.	Slope	Defl.	Slope
1				
2				
3				
↓				
NELEM				

The matrix consists of the numbers in the spaces bounded by the solid lines. The first column consists of the element reference numbers. The designatory numbers of the global freedoms are then put into the other spaces <u>after</u> zeros have been inserted at any constrained freedoms.

The designatory numbers for the global freedoms must be consecutive integers from 1 to NDEGF and should be fed into the spaces in the matrix from left to right and row by row.

Note that for continuity between elements, the numbers entered in the external radius columns for the n^{th} element must be the same as those in the internal radius columns for the $(n + 1)^{th}$ element where $1 \leqslant n \leqslant (NELEM-1)$

The size of the matrix must be (NELEM x 5).

(g) Details of the loading on the plate.

The program will accept any combination of constant force on a nodal circle, constant radial moment per unit length along a nodal circle or distributed pressure over the plate, provided they are axisymmetric. A switch, operated by a load code number, is incorporated in order that the appropriate procedures may be called up to assemble the force vector.

Load code number = 1

This is for use when all the forces and moments on the plate are concentrated at the nodal radii. The code number should be followed by a list of the total force and moment per unit length at each nodal radius starting with the smallest radius. If no force or moment acts, a zero should be entered, i.e. the list should consist of (2 x NNODE) terms. Note that the program requires the force to be entered as the total load on the nodal circle, whilst any moment must be expressed as a moment per unit length measured along the circle.

Load code number = 2

This is for use when the loading is due to distributed pressure only. The code number should be followed by a list of the pressures at the internal and external radii of each element starting with the element nearest to the centre of the plate. If no pressure acts on any particular element, a pair of zero's should be entered i.e. the list should consist of (2 x NELEM) terms.

The distributed pressure procedure assumes a linear radial variation of pressure across the width of each element, computes statically equivalent concentrated nodal forces and then sums these forces on each element at common radii.

Load code number = 3

This is for use when the loading is a combination of the above cases. The code number should be followed by lists of the loading as described above for when the code number was 1 & 2, and in that order.

4.9.3 Summary of data presentation

(a) The number of plates to be analysed.

(b) The number of elements followed by the number of constraints.

(c) A list of nodal radii (m) commencing with the smallest.

(d) A list of the modulus of elasticity (N/m^2), Poisson's ratio and plate thickness (mm) for each element commencing with the element nearest to the centre of the plate.

(e) The nodal connection matrix.

(f) The load code number.

(g) A list of forces (N) and moments (Nm/m) at each nodal radius commencing with the smallest radius.

and/or

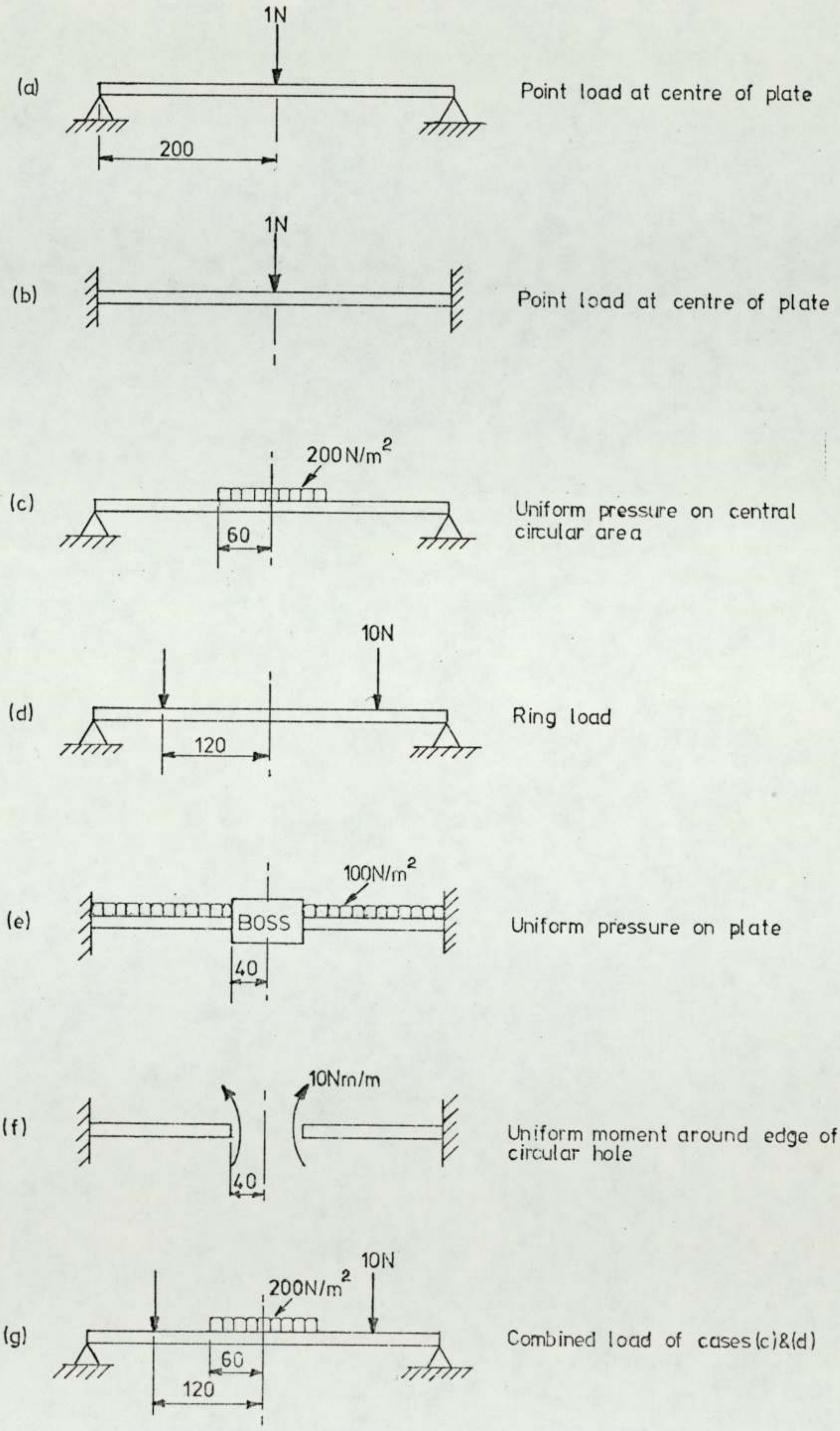
A list of pressures (N/m^2) at the boundaries of each element commencing with the element nearest to the centre of the plate.

4.10 RESULTS AND DISCUSSION OF VARIOUS TEST PROGRAMS

Several test cases have been analysed in order to prove the viability of the program and are shown in figure 4.6 on page 66 These particular cases were chosen because they have known classical solutions and provide the range of loading and boundary conditions that most commonly occur in practice. All of the calculations were made for a Vybak plate 0.2 m radius and 3 mm thick as it was envisaged that any experimental work would be

FIG. 4.6

TEST CASES USED TO PROVE THE PROGRAM 'SYMPLAT'



All dimensions in mm.

carried out on plates of this material and size.

A summary of the results for the various test cases using five elements of varying width is given below and a typical input and printout of results, in this instance for case (g), is shown on pages 83 to 86 inclusive.

TEST CASE	TYPE OF SOLUTION	MAXIMUM DEFL. (mm)	STRESS AT CENTRE OR INNER RAD. (kN/m^2)		STRESS AT OUTER RAD (kN/m^2)	
			RADIAL	TANG.	RADIAL	TANG.
(a)	CLASSICAL SYMPLAT	0.2647 0.2646	∞ -353.8	∞ -353.8	0 1.6	-32.9 -32.3
(b)	CLASSICAL SYMPLAT	0.1081 0.1080	∞ -300.8	∞ -300.8	53.0 54.6	20.1 20.8
(c)	CLASSICAL SYMPLAT	0.5398 0.5378	-317.7 -325.2	-317.7 -325.2	0 2.6	-71.1 -69.8
(d)	CLASSICAL SYMPLAT	1.2966 1.2963	-478.8 -479.5	-478.8 -479.5	0 24.4	-210.5 -201.2
(e)	CLASSICAL SYMPLAT	0.2396 0.2304	-257.2 -262.7	-97.7 -99.8	297.0 296.8	112.9 112.8
(f)	CLASSICAL SYMPLAT	3.6350 3.6336	-6660.0 -5982.0	5571.0 5829.8	-790.0 -778.7	-300.2 -295.9
(g)	CLASSICAL SYMPLAT	1.8358 1.8340	-796.5 -804.6	-796.5 -804.6	0 26.9	-281.5 -271.1

The classical results are calculated from expressions quoted in Roark [2] and the comparison shows that in general the predictions by SYMPLAT of deflection are accurate to less than 4% and those of stress by 12%. The superior accuracy of the deflection predictions is to be expected from a displacement formulation of the finite element method, as the displacements are regarded as the primary variables and the stresses are then calculated in what is implicitly a differentiation process with all the accompanying magnification of error associated with differentation.

Test cases (a) and (b) were analysed using a variety of numbers and sizes of elements. The results are illustrated in graphs 4.1 and 4.2 on pages 70 & 71 and would appear to indicate that the number and size of elements has little effect on the accuracy of the deflection but some effect on the stress prediction. Five elements gives a satisfactory estimate of stresses provided that various element widths are used; the basis being that of making the radial width of each element approximately proportional to its outside radius. Increasing the number of elements to ten did not significantly improve the results.

SYMPLAT does not predict with accuracy the infinite stresses at the location of the point load in cases (a) and (b) or the zero radial stresses at the simply supported edges in cases (a), (c), (d) and (g). Of these discrepancies the former is not too important as the classical theory is also inaccurate in regions of concentrated loading due to the violation of the plate bending assumptions, and the latter could either be overcome by using a finer element mesh at the boundary or disregarded because the stress levels are so low.

In order to demonstrate the practical usefulness of SYMPLAT, a problem has been solved in a situation that would present considerable difficulty if classical theory were used. The problem is that of a clamped plate of variable thickness with a rigid central boss, the plate carrying uniform pressure over its entire surface. The results of the analysis of this plate are shown in graph 4.3 on page 72.

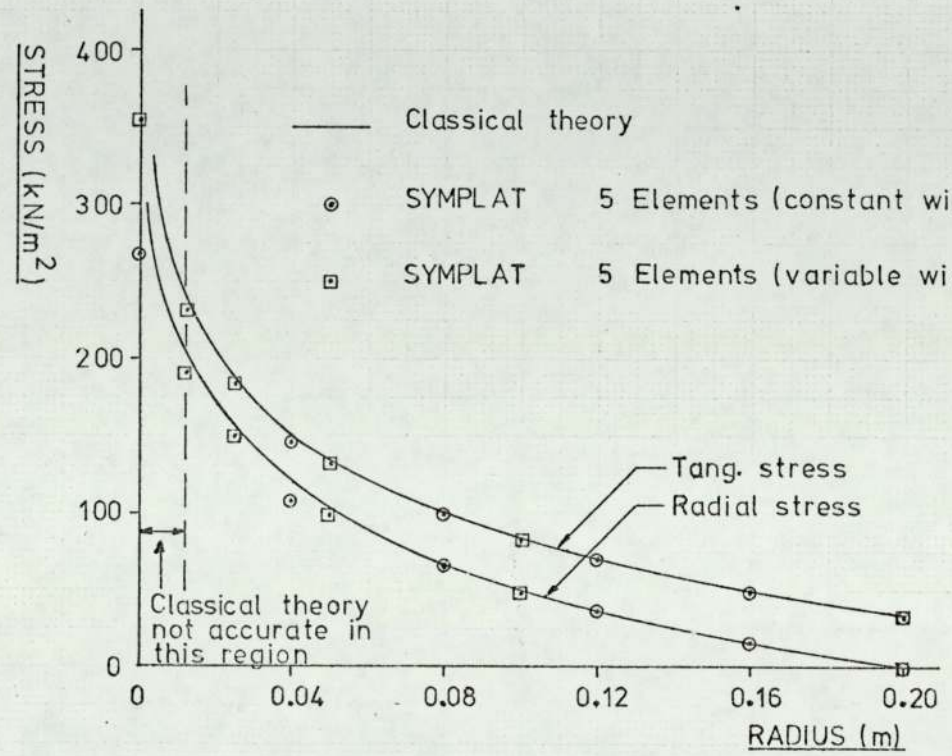
A possible improvement to SYMPLAT would be the use of more sophisticated shape functions that the Hermitian polynomials. Pardoen and Hagen [19] have derived such a set of functions that satisfy identically the governing equilibrium equation for symmetrical bending. The use of these functions would undoubtedly give some improvement in the predictions of SYMPLAT but as this program has only been devised as a first stage in the solution of the asymmetric problem, the extra complexity of these shapes was not thought to be justifiable at this stage.

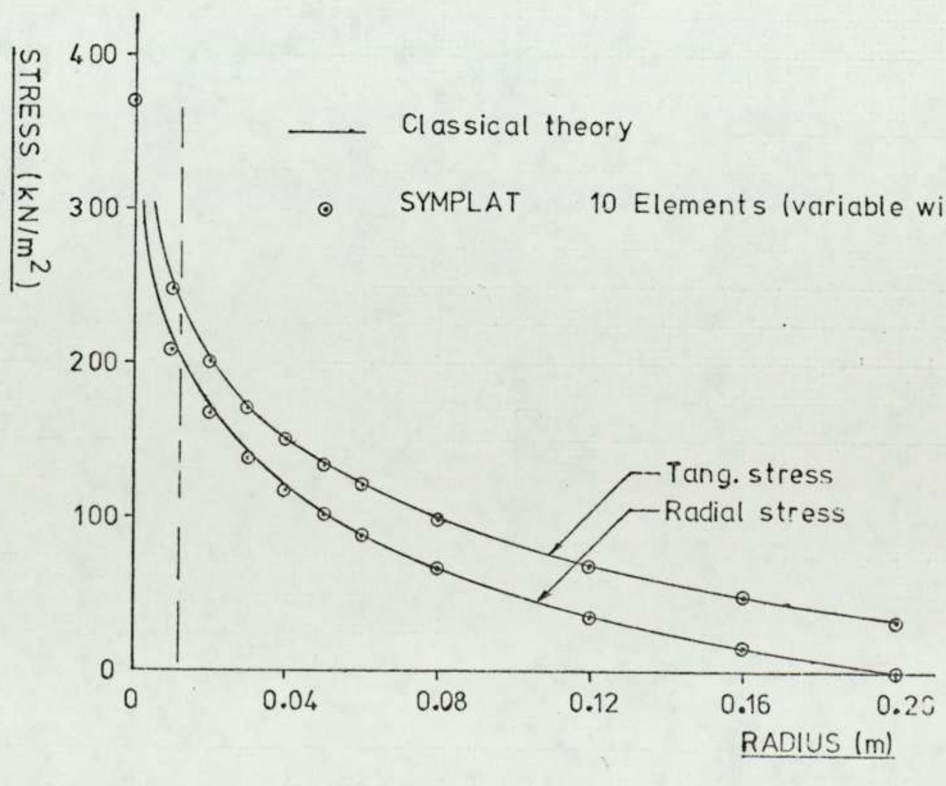
GRAPH 4.1

TEST CASE (a) Simply supported plate - central point load
Material - Vybak ($E = 2.8\ GN/m^2$; $\nu = 0.38$)
Radius = 0.2m ; Thickness = 3mm ; Load = 1 N .

Central deflections -	
Classical theory	0.2647 mm
SYMPLAT 5 Elements (constant width)	0.2642 mm
SYMPLAT 5 Elements (variable width)	0.2646 mm
SYMPLAT 10 Elements (variable width)	0.2646 mm

Stresses



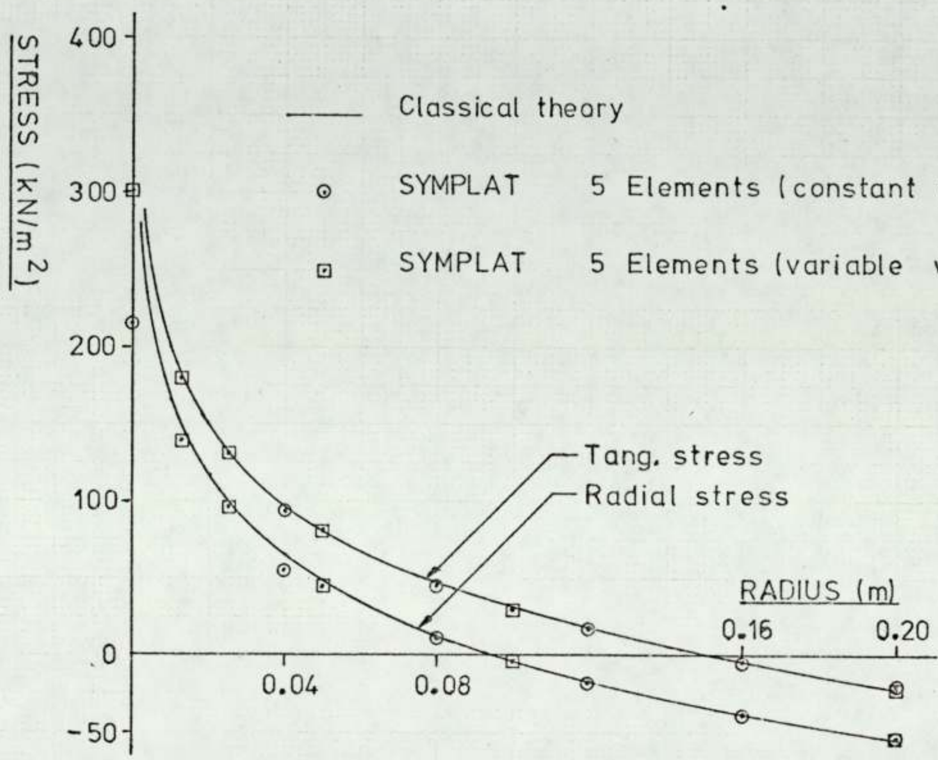


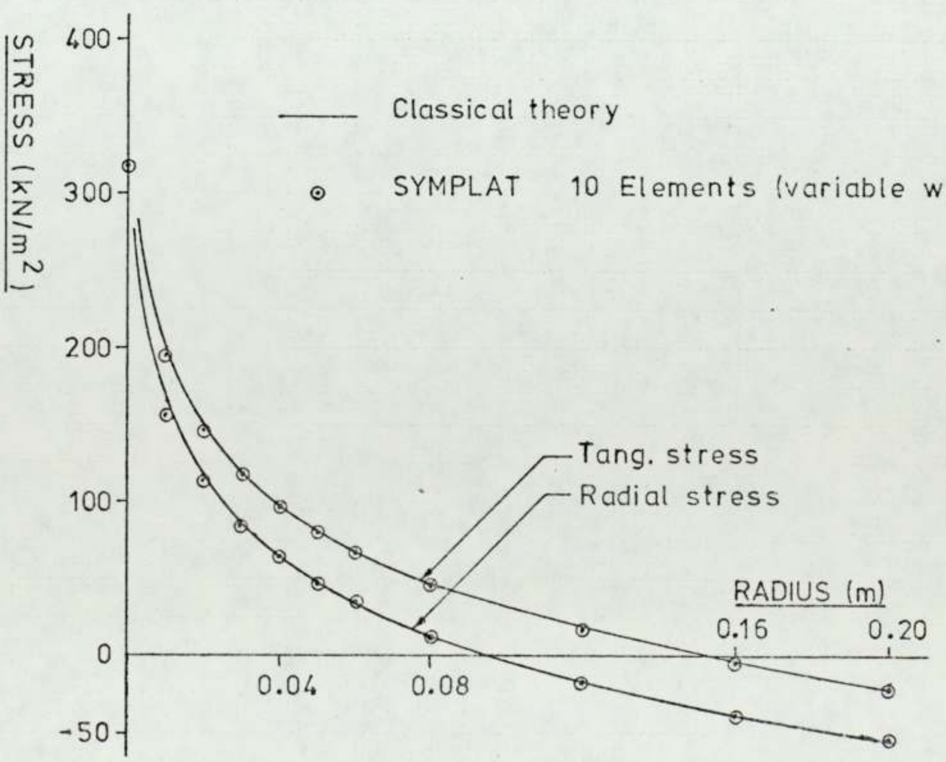
GRAPH 4.2

TEST CASE (b) Clamped plate – central point load
Material - Vybak (E = 2.8 GN/m^2 ; ν = 0.38)
Radius = 0.2m ; Thickness = 3mm ; Load = 1 N

Central deflections - Classical theory – – – – – – – – 0.1081 mm
SYMPLAT 5 Elements (constant width) - 0.1076 mm
SYMPLAT 5 Elements (variable width) - 0.1080 mm
SYMPLAT 10 Elements (variable width) - 0.1080 mm

Stresses





GRAPH. 4.3

ANALYSIS OF A CLAMPED PLATE OF LINEARLY VARYING THICKNESS

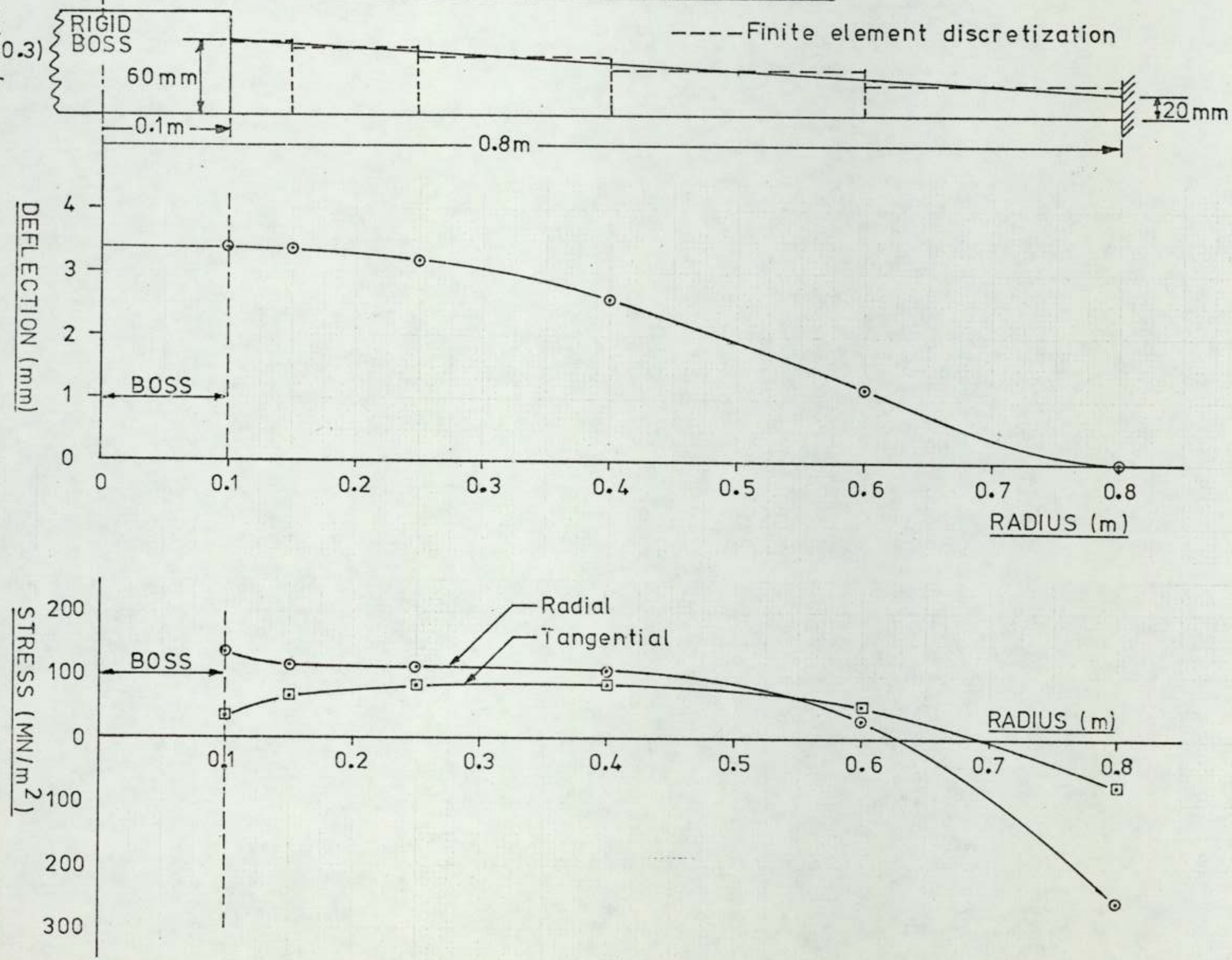
HALF-SECTION THROUGH PLATE

Material - Steel ($E = 200\ GN/m^2$; $\nu = 0.3$)

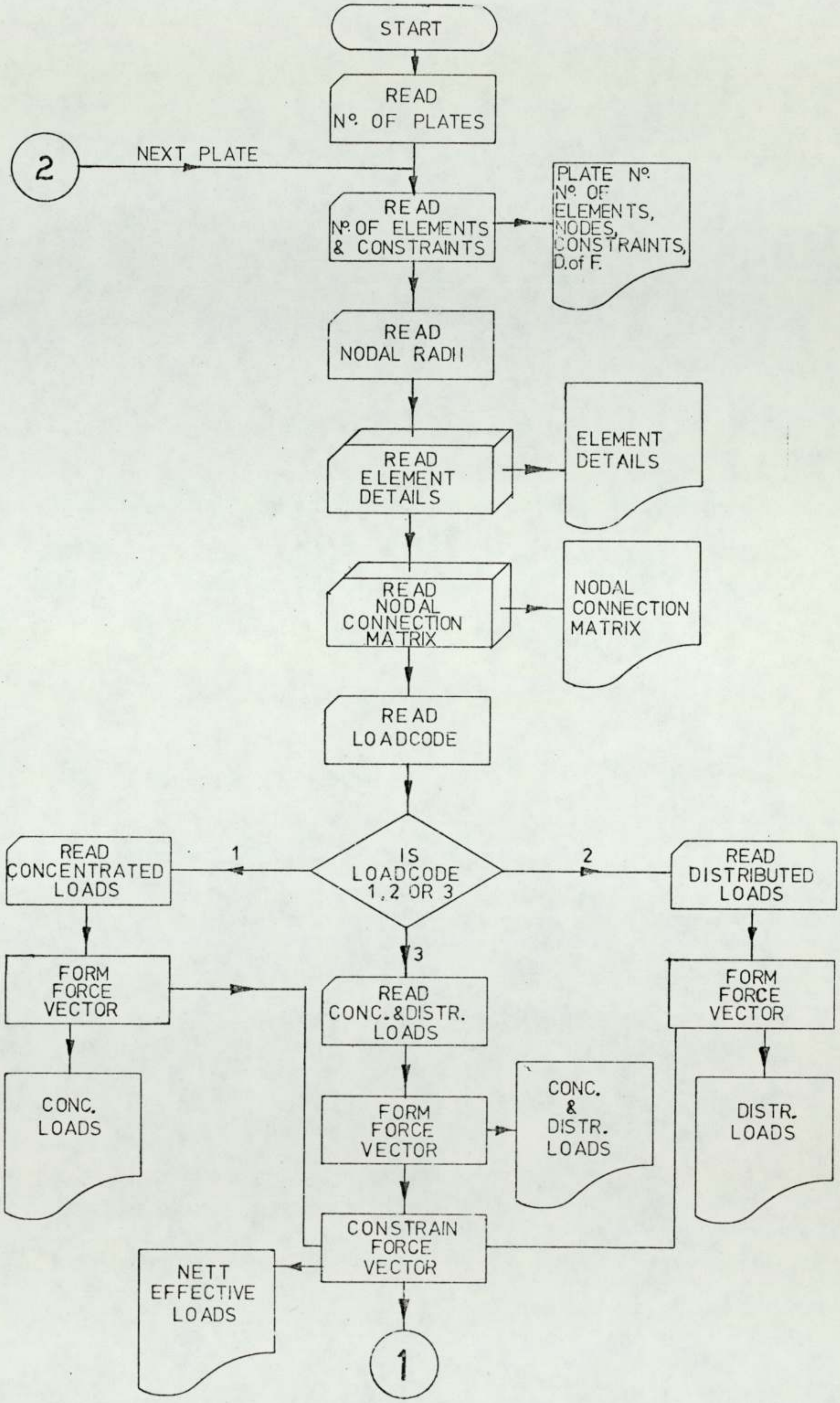
Load - $0.5\ MN/m^2$ pressure over whole surface of plate and boss.

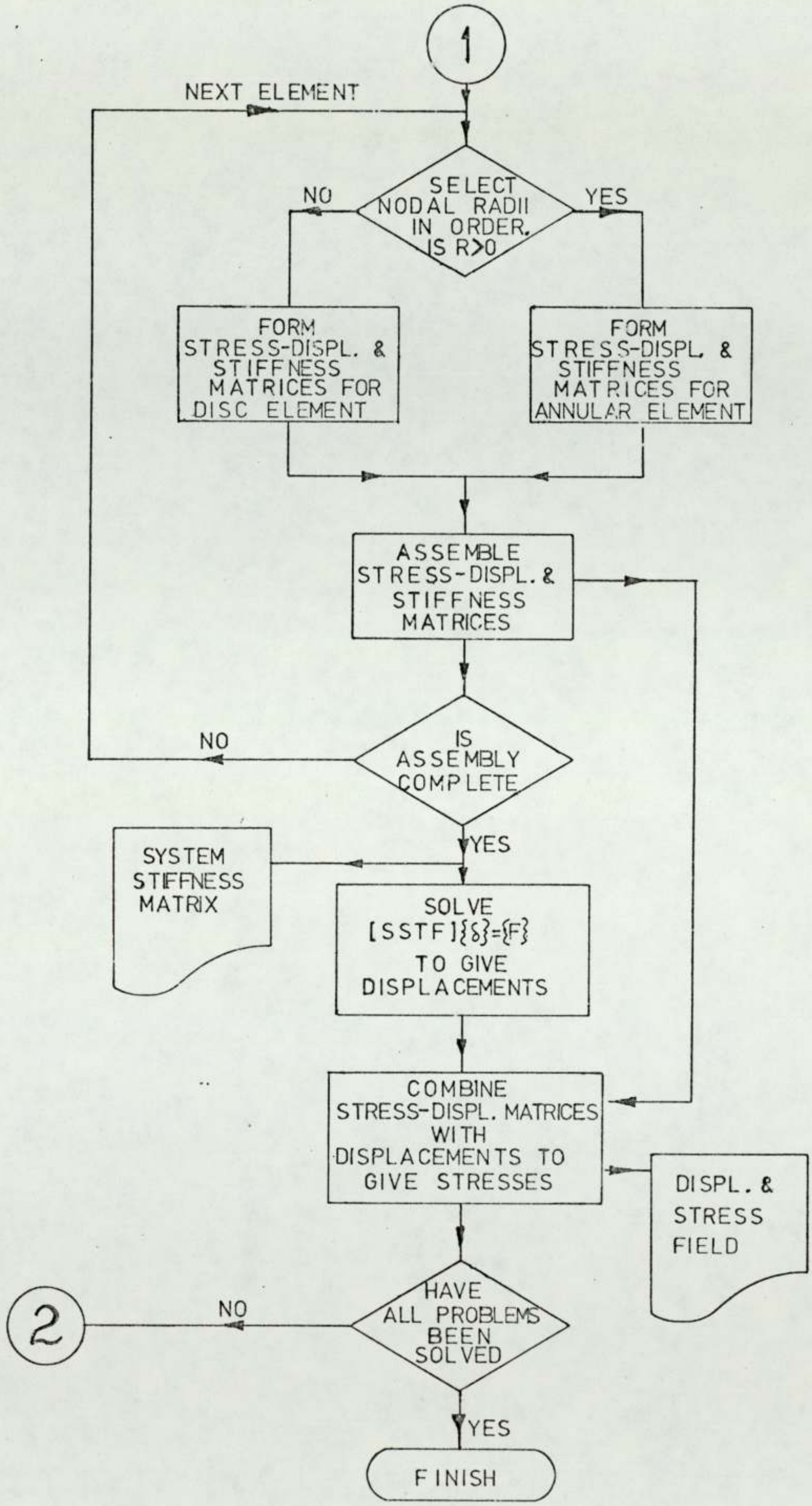
DEFLECTION

STRESSES



PROGRAM FLOWCHART FOR 'SYMPLAT'




1
NEXT ELEMENT
SELECT NODAL RADII IN ORDER. IS R>0
NO
YES
FORM STRESS-DISPL. & STIFFNESS MATRICES FOR DISC ELEMENT
FORM STRESS-DISPL. & STIFFNESS MATRICES FOR ANNULAR ELEMENT
ASSEMBLE STRESS-DISPL. & STIFFNESS MATRICES
IS ASSEMBLY COMPLETE
NO
YES
SYSTEM STIFFNESS MATRIX
SOLVE [SSTF]{δ}={F} TO GIVE DISPLACEMENTS
COMBINE STRESS-DISPL. MATRICES WITH DISPLACEMENTS TO GIVE STRESSES
DISPL. & STRESS FIELD
HAVE ALL PROBLEMS BEEN SOLVED
NO
2
YES
FINISH

A LISTING OF THE PROGRAM 'SYMPLAT'

```
'BEGIN''COMMENT'ANALYSIS OF SYMMETRICALLY LOADED PLATES;
 'INTEGER'NELEM,NNODE,NCON,NDEGF,HBW,PROBNO,NOPROB,
          I,J,K,N,G,L,H,W,Y,Z,LC,F1,F2,F3;
     F1:=FORMAT('('5S-D.DDDD@+ND')');
     F2:=FORMAT('('-ND.DDDD')');
     F3:=FORMAT('('5SND')');
     HBW:=4;
     PROBNO:=0;
     NOPROB:=READ(60);
  START:PROBNO:=PROBNO+1;
  PAGE(30,1);
  WRITETEXT(30,'('ANALYSIS""OF""SYMMETRICALLY""LOADED""PLATES%%%%')');
  WRITETEXT(30,'('PLATE""NO.""""""""""""""""""""""""""""""""""""""')');
     WRITE(30,F3,PROBNO);
     NEWLINE(30,3);
     NELEM:=READ(60);
     NCON:=READ(60);
  WRITETEXT(30,'('NUMBER""OF""ELEMENTS""""""""""""""""""')');
     WRITE(30,F3,NELEM);
     NEWLINE(30,3);
     NNODE:=NELEM+1;
  WRITETEXT(30,'('NUMBER""OF""NODES"""""""""""""""""""""""')');
     WRITE(30,F3,NNODE);
     NEWLINE(30,3);
  WRITETEXT(30,'('NUMBER""OF""CONSTRAINTS""""""""""""""')');
     WRITE(30,F3,NCON);
     NEWLINE(30,3);
     NDEGF:=2*NNODE-NCON;
  WRITETEXT(30,'('NUMBER""OF""DEGREES""OF""FREEDOM""""')');
     WRITE(30,F3,NDEGF);
     NEWLINE(30,3);
 'BEGIN'
    'INTEGER''ARRAY'NODC(/1:NELEM,1:5/);
    'ARRAY'NODRAD(/1:NNODE/),FORCE(/1:NDEGF/),SSTF(/1:NDEGF,0:HBW-1/),
           DEFL,SLOPE(/1:NNODE/),ELCA(/1:NELEM,1:4,1:4/),
           E,V,T(/1:NELEM/),
           ELDISP,NODSTRESS(/1:4,1:NELEM/),AVSTRESS,APFO(/1:2*NNODE/);
    'SWITCH'LOADCODE:=CONF,DISF,COMBF;
    'PROCEDURE'CONFOUT(NO,RAD,FO);
         'VALUE'NO,RAD;
         'ARRAY'RAD,FO;
         'INTEGER'NO;
         'BEGIN'
            WRITETEXT(30,'('CONCENTRATED""LOADS%%
                           NODAL""RADIUS(M)""""""APPLIED""FORCE(N)
                           """"""APPLIED""MOMENT(NM/M)%%')');
            'FOR'I:=1'STEP'1'UNTIL'NO'DO'
            'BEGIN'
               SPACE(30,2);   WRITE(30,F2,RAD(/I/));
               SPACE(30,6);   FO(/2*I-1/):=READ(60);
                              WRITE(30,F1,FO(/2*I-1/));
               SPACE(30,4);FO(/2*I/):=READ(60);WRITE(30,F1,FO(/2*I/));
                              FO(/2*I/):=FO(/2*I/)*6.284*RAD(/I/);
               NEWLINE(30,1);
            'END',
           NEWLINE(30,2);
         'END'OF CONFOUT;
    'PROCEDURE'DISFOUT(NO,RAD,FO);
         'VALUE'NO,RAD;
```

```
        'ARRAY'RAD,FO;
        'INTEGER'NO;
        'BEGIN'
        'REAL'IR,ER,IP,EP,Q1,Q2;
           WRITETEXT(30,'('DISTRIBUTED"LOADS%%ELEMENT"NO.""""
                     INT.RAD(M)""""EXT.RAD(M)""""INT.PRESS(N/SQ.M)
                     """"EXT.PRESS(N/SQ.M)%%')');
          'FOR'I:=1'STEP'1'UNTIL'NO'DO'
          'BEGIN'
             IR:=RAD(/I/);  ER:=RAD(/I+1/);
             IP:=READ(60);  EP:=READ(60);
             Q1:=0.5236*(ER-IR)*(IP*(ER+3*IR)+EP*(ER+IR));
             Q2:=0.5236*(ER-IR)*(IP*(ER+IR)+EP*(3*ER+IR));
                             WRITE(30,F3,I);
             SPACE(30,8);    WRITE(30,F2,IR);
             SPACE(30,6);    WRITE(30,F2,ER);
             SPACE(30,2);    WRITE(30,F1,IP);
             SPACE(30,5);    WRITE(30,F1,EP);
             FO(/2*I-1/):=FO(/2*I-1/)+Q1;
             FO(/2*I+1/):=FO(/2*I+1/)+Q2;
             NEWLINE(30,1);
          'END';
         NEWLINE(30,2);
        'END'OF DISFOUT;
    'PROCEDURE'MATMULT(M1,RO1,M2,RO2,CO2,M3);
        'ARRAY'M1,M2,M3;
        'INTEGER'RO1,RO2,CO2;
        'BEGIN''FOR'I:=1'STEP'1'UNTIL'RO1'DO'
            'BEGIN''FOR'J:=1'STEP'1'UNTIL'CO2'DO'
                'BEGIN'M3(/I,J/):=0;
                    'FOR'N:=1'STEP'1'UNTIL'RO2'DO'
                     M3(/I,J/):=M3(/I,J/)+M1(/I,N/)*M2(/N,J/);
                'END';
            'END';
        'END';
    'PROCEDURE'TRMAMULT(M4,CO4,M5,RO5,CO5,M6);
        'ARRAY'M4,M5,M6;
        'INTEGER'CO4,RO5,CO5;
        'BEGIN''FOR'I:=1'STEP'1'UNTIL'CO4'DO'
            'BEGIN''FOR'J:=1'STEP'1'UNTIL'CO5'DO'
                'BEGIN'M6(/I,J/):=0;
                    'FOR'N:=1'STEP'1'UNTIL'RO5'DO'
                     M6(/I,J/):=M6(/I,J/)+M4(/N,I/)*M5(/N,J/);
                'END';
            'END';
        'END';
'PROCEDURE'CHOBANDDET(N,M,A);
    'VALUE'N,M;
'INTEGER'N,M;
    'ARRAY'A;
    'BEGIN'
      'INTEGER'I,J,K,P,Q,R,S;
      'REAL'Y;
      'FOR'I:=1'STEP'1'UNTIL'N'DO'
        'BEGIN'
           P:=('IF'I>M'THEN'0'ELSE'M-I+1);
           R:=I-M+P;
```

```
            'FOR'J:=P'STEP'1'UNTIL'M'DO'
               'BEGIN'
                  S:=J-1;
                  Q:=M-J+P;
                  Y:=A(/I,J/);
                 'FOR'K:=P'STEP'1'UNTIL'S'DO'
                    'BEGIN'
                       Y:=Y-A(/I,K/)*A(/R,Q/);
                       Q:=Q+1;
                    'END';
                 'IF'J=M'THEN'A(/I,J/):=1/SQRT(Y)
                 'ELSE'A(/I,J/):=Y*A(/R,M/);
                  R:=R+1;
               'END';
          'END';
     'END';
'PROCEDURE'CHOBANDSOL(N,M,R,A,B);
     'VALUE'N,M,R;
     'INTEGER'N,M,R;
     'ARRAY'A,B;
     'BEGIN'
       'INTEGER'I,J,K,P,Q,S;
       'REAL'Y;
        S:=M-1;
       'FOR'J:=1'STEP'1'UNTIL'R'DO'
          'BEGIN'
            'FOR'I:=1'STEP'1'UNTIL'N'DO'
               'BEGIN'
                  P:=('IF'I>M'THEN'0'ELSE'M-I+1);
                  Q:=I;
                  Y:=B(/I/);
                 'FOR'K:=S'STEP'-1'UNTIL'P'DO'
                    'BEGIN'
                       Q:=Q-1;
                       Y:=Y-A(/I,K/)*B(/Q/);
                    'END';
                  B(/I/):=Y*A(/I,M/);
               'END';
            'FOR'I:=N'STEP'-1'UNTIL'1'DO'
               'BEGIN'
                  P:=('IF'N-I>M'THEN'0'ELSE'M-N+I);
                  Y:=B(/I/);
                  Q:=I;
                 'FOR'K:=S'STEP'-1'UNTIL'P'DO'
                    'BEGIN'
                       Q:=Q+1;
                       Y:=Y-A(/Q,K/)*B(/Q/);
                    'END';
                  B(/I/):=Y*A(/I,M/);
               'END';
          'END';
     'END';
     'FOR'I:=1'STEP'1'UNTIL'NNODE'DO'
     NODRAD(/I/):=READ(60);
     'FOR'I:=1'STEP'1'UNTIL'NELEM'DO'
     'BEGIN'
        E(/I/):=READ(60);
```

```
      V(/I/):=READ(60);
      T(/I/):=READ(60);
   'END';
WRITETEXT(30,'('DETAILS_OF_ELEMENTS%%')');
WRITETEXT(30,'('ELEMENT_NO._____INT.RAD(M)_____EXT.RAD(M)_____MOD.OF_
                  ELAST(N/SQ.M)_____POISSONS_RATIO_____PLATE_THICKNESS
                  (MM)%%')');
   'FOR'I:=1'STEP'1'UNTIL'NELEM'DO'
   'BEGIN'
   WRITE(30,F3,I);  SPACE(30,8);
   WRITE(30,F2,NODRAD(/I/));  SPACE(30,6);
   WRITE(30,F2,NODRAD(/I+1/));  SPACE(30,2);
   WRITE(30,F1,E(/I/));  SPACE(30,14);
   WRITE(30,F2,V(/I/));  SPACE(30,14);
   WRITE(30,F2,T(/I/));  SPACE(30,15);  T(/I/):=T(/I/)/1000;
   NEWLINE(30,1);
   'END';
    NEWLINE(30,2);
WRITETEXT(30,'('NODAL_CONNECTION_MATRIX%%')');
   'FOR'I:=1'STEP'1'UNTIL'NELEM'DO'
   'BEGIN'
        'FOR'J:=1'STEP'1'UNTIL'5'DO'
        'BEGIN'
              NODC(/I,J/):=READ(60);
              WRITE(30,F3,NODC(/I,J/));
        'END';
         NEWLINE(30,1);
   'END';
    NEWLINE(30,2);
WRITETEXT(30,'('DETAILS_OF_LOADING%%')');
   'FOR'I:=1'STEP'1'UNTIL'2*NNODE'DO'APFO(/I/):=0;
LC:=READ(60);     'GOTO'LOADCODE(/LC/);
    CONF:CONFOUT(NNODE,NODRAD,APFO);
        'GOTO'LOCON;
    DISF:DISFOUT(NELEM,NODRAD,APFO);
        'GOTO'LOCON;
    COMBF:CONFOUT(NNODE,NODRAD,APFO);
          DISFOUT(NELEM,NODRAD,APFO);
    LOCON:WRITETEXT(30,'('NET_EFFECTIVE_LOADS%%
                         NODAL_RADIUS(M)_________FORCE(N)
                         __________MOMENT(NM)%%')');
   'FOR'I:=1'STEP'1'UNTIL'NNODE-1'DO'
   'BEGIN'
      SPACE(30,2);WRITE(30,F2,NODRAD(/I/));SPACE(30,3);
      'FOR'J:=1'STEP'1'UNTIL'2'DO'
      'BEGIN'
         G:=NODC(/I,J+1/);
         'IF'G=0'THEN'WRITETEXT(30,'('__________CONSTRAINT')')
               'ELSE''BEGIN'
                        FORCE(/G/):=APFO(/2*I-2+J/);
                        SPACE(30,3 );WRITE(30,F1,FORCE(/G/));
                     'END';
      'END';
    NEWLINE(30,1);
    'END';
      SPACE(30,2);WRITE(30,F2,NODRAD(/NNODE/));SPACE(30,3);
      'FOR'J:=1'STEP'1'UNTIL'2'DO'
```

```
      'BEGIN'
         G:=NODC(/NELEM,J+3/);
         'IF'G=0'THEN'WRITETEXT(30,'('__________CONSTRAINT')')
                'ELSE''BEGIN'
                       FORCE(/G/):=APFO(/2*I-2+J/);
                       SPACE(30,3 );WRITE(30,F1,FORCE(/G/));
                    'END';
      'END';
     NEWLINE(30,3);
'COMMENT'FORMATION AND ASSEMBLY OF STIFFNESS MATRIX;
   'BEGIN'
        'REAL'R1,R2,DR,RATR,D;
        'ARRAY'A1,A2,A3,CA1,CA2,CA3,CAINT,CAEXT(/1:2,1:4/),
               C(/1:2,1:2/),
               ESTF,B1,B2,B3,B4,B5,B6,B7,B8,B9(/1:4,1:4/);
        C(/1,1/):=C(/2,2/):=1;
        'FOR'I:=1'STEP'1'UNTIL'NDEGF'DO'
         'FOR'J:=0'STEP'1'UNTIL'HBW-1'DO'
               SSTF(/I,J/):=0;
 'FOR'K:=1'STEP'1'UNTIL'NELEM'DO'
   'BEGIN'
        D:=(E(/K/)*T(/K/)**3)/(12*(1-V(/K/)**2));
        R1:=NODRAD(/K/);
        R2:=NODRAD(/K+1/);
        'IF'R1>0.001'THEN''GOTO'L1;
 'COMMENT'FORM STIFFNESS MATRIX FOR DISC ELEMENT;
        C(/1,2/):=C(/2,1/):=V(/K/);
    A1(/1,1/):=12;
    A1(/1,2/):=0;
    A1(/1,3/):=-12;
    A1(/1,4/):=6*R2;
    A1(/2,1/):=6;
    A1(/2,2/):=0;
    A1(/2,3/):=-6;
    A1(/2,4/):=3*R2;
    A2(/1,1/):=A2(/2,1/):=-6*R2;
    A2(/1,2/):=A2(/2,2/):=0;
    A2(/1,3/):=A2(/2,3/):=6*R2;
    A2(/1,4/):=A2(/2,4/):=-2*R2**2;
        MATMULT(C,2,A1,2,4,CA1);
        MATMULT(C,2,A2,2,4,CA2);
        TRMAMULT(A1,4,CA1,2,4,B1);
        TRMAMULT(A1,4,CA2,2,4,B2);
        TRMAMULT(A2,4,CA1,2,4,B3);
        TRMAMULT(A2,4,CA2,2,4,B4);
   'FOR'I:=1'STEP'1'UNTIL'2'DO'
   'FOR'J:=1'STEP'1'UNTIL'4'DO'
   'BEGIN'
       CAINT(/I,J/):=6*D*CA2(/I,J/)/((T(/K/)**2)*R2**3);
       CAEXT(/I,J/):=6*D*(CA1(/I,J/)*R2+CA2(/I,J/))/
                     ((T(/K/)**2)*R2**3);
   'END';
   'FOR'I:=1'STEP'1'UNTIL'4'DO'
         'FOR'J:=1'STEP'1'UNTIL'4'DO'
        ESTF(/I,J/):=6.284*D*(B1(/I,J/)/(4*R2**2)+
                     (B2(/I,J/)+B3(/I,J/))/(3*R2**3)+
                     B4(/I,J/)/(2*R2**4));
```

```
'GOTO'L2;
'COMMENT'FORM STIFFNESS MATRIX FOR ANNULAR ELEMENT;
     L1;C(/1,2/);=C(/2,1/);=V(/K/);
   A1(/1,1/);=12;
   A1(/1,2/);=A1(/1,4/);=6*(R2-R1);
   A1(/1,3/);=-12;
   A1(/2,1/);=6;
   A1(/2,2/);=A1(/2,4/);=3*(R2-R1);
   A1(/2,3/);=-6;
   A2(/1,1/);=A2(/2,1/);=-6*(R2+R1);
   A2(/1,2/);=A2(/2,2/);=-2*(R2-R1)*(2*R2+R1);
   A2(/1,3/);=A2(/2,3/);=6*(R2+R1);
   A2(/1,4/);=A2(/2,4/);=-2*(R2-R1)*(R2+2*R1);
   A3(/1,1/);=A3(/1,2/);=A3(/1,3/);=A3(/1,4/);=0;
   A3(/2,1/);=6*R1*R2;
   A3(/2,2/);=R2*(R2-R1)*(R2+2*R1);
   A3(/2,3/);=-6*R1*R2;
   A3(/2,4/);=R1*(R2-R1)*(2*R2+R1);
          MATMULT(C,2,A1,2,4,CA1);
          MATMULT(C,2,A2,2,4,CA2);
          MATMULT(C,2,A3,2,4,CA3);
          TRMAMULT(A1,4,CA1,2,4,B1);
          TRMAMULT(A1,4,CA2,2,4,B2);
          TRMAMULT(A2,4,CA1,2,4,B3);
          TRMAMULT(A1,4,CA3,2,4,B4);
          TRMAMULT(A2,4,CA2,2,4,B5);
          TRMAMULT(A3,4,CA1,2,4,B6);
          TRMAMULT(A2,4,CA3,2,4,B7);
          TRMAMULT(A3,4,CA2,2,4,B8);
          TRMAMULT(A3,4,CA3,2,4,B9);
    DR;=(R2-R1)**6;
    RATR;=R2/R1;
   'FOR'I;=1'STEP'1'UNTIL'2'DO'
   'FOR'J;=1'STEP'1'UNTIL'4'DO'
   'BEGIN'
          CAINT(/I,J/);=6*D*(CA1(/I,J/)*R1+CA2(/I,J/)+CA3(/I,J/)/R1)
                          /((T(/K/)**2)*(R2-R1)**3);
          CAEXT(/I,J/);=6*D*(CA1(/I,J/)*R2+CA2(/I,J/)+CA3(/I,J/)/R2)
                          /((T(/K/)**2)*(R2-R1)**3);
   'END';
   'FOR'I;=1'STEP'1'UNTIL'4'DO'
           'FOR'J;=1'STEP'1'UNTIL'4'DO'
          ESTF(/I,J/);=6.284*D*((((R2**4-R1**4)*B1(/I,J/)/(4*DR))+
                    ((R2**3-R1**3)*(B2(/I,J/)+B3(/I,J/))/(3*DR))+
                    ((R2**2-R1**2)*(B4(/I,J/)+B5(/I,J/)+B6(/I,J/))/(2*DR))+
                    ((R2-R1)*(B7(/I,J/)+B8(/I,J/))/DR)+
                    (LN(RATR)*B9(/I,J/)/DR));
'COMMENT'ASSEMBLE ELEMENTS;
 L2;'FOR'I;=1'STEP'1'UNTIL'2'DO'
        'FOR'J;=1'STEP'1'UNTIL'4'DO'
        'BEGIN'ELCA(/K,I,J/);=CAINT(/I,J/);
                ELCA(/K,I+2,J/);=CAEXT(/I,J/);
        'END';
     'FOR'I;=1'STEP'1'UNTIL'4'DO'
        'FOR'J;=1'STEP'1'UNTIL'4'DO'
          'BEGIN'
               G;=NODC(/K,I+1/);
```

```
                L:=NODC(/K,J+1/);
               'IF'G=0'THEN''GOTO'L3;
               'IF'L=0'THEN''GOTO'L3;
               'IF'G>=L'THEN'
                 'BEGIN'
                     H:=L-G+HBW-1;
                     SSTF(/G,H/):=SSTF(/G,H/)+ESTF(/I,J/);
                 'END';
        L3:'END';
   'END';
WRITETEXT(30,'('SYSTEM_STIFFNESS_MATRIX%')');
   'FOR'I:=1'STEP'1'UNTIL'NDEGF'DO'
     'BEGIN'
         NEWLINE(30,1);
        'FOR'J:=0'STEP'1'UNTIL'HBW-1'DO'
          WRITE(30,F1,SSTF(/I,J/));
     'END';
   'END';
    NEWLINE(30,3);
'COMMENT'SOLVE MATRIX EQUATIONS;
    CHOBANDDET(NDEGF,HBW-1,SSTF);
    CHOBANDSOL(NDEGF,HBW-1,1,SSTF,FORCE);
'COMMENT'CALC. STRESSES,CORRECT UNITS THEN OUTPUT ALL RESULTS;
     'FOR'K:=1'STEP'1'UNTIL'NELEM'DO'
        'BEGIN'
          'FOR'J:=2'STEP'1'UNTIL'5'DO'
             'BEGIN' G:=NODC(/K,J/);
                     'IF'G=0'THEN'ELDISP(/J-1,K/):=0
                     'ELSE'ELDISP(/J-1,K/):=FORCE(/G/);
             'END';
          'FOR'I:=1'STEP'1'UNTIL'4'DO'
             'BEGIN' NODSTRESS(/I,K/):=0;
             'FOR'J:=1'STEP'1'UNTIL'4'DO'
                     NODSTRESS(/I,K/):=NODSTRESS(/I,K/)+ELCA(/K,I,J/)
                                      *ELDISP(/J,K/);
             'END';
        'END';
WRITETEXT(30,'('NODAL_STRESS_MATRIX(N/SQ.M)%')');
   'FOR'I:=1'STEP'1'UNTIL'4'DO'
     'BEGIN'
           NEWLINE(30,1);
          'FOR'J:=1'STEP'1'UNTIL'NELEM'DO'
           WRITE(30,F1,NODSTRESS(/I,J/));
    'END';
     NEWLINE(30,3);
         AVSTRESS(/1/):=NODSTRESS(/1,1/);
         AVSTRESS(/2/):=NODSTRESS(/2,1/);
         'FOR'J:=1'STEP'1'UNTIL'NELEM-1'DO'
         'FOR'I:=1'STEP'1'UNTIL'2'DO'
         AVSTRESS(/2*J+I/):=(NODSTRESS(/I+2,J/)+
                             NODSTRESS(/I,J+1/))/2;
         AVSTRESS(/2*NNODE-1/):=NODSTRESS(/3,NELEM/);
         AVSTRESS(/2*NNODE/):=NODSTRESS(/4,NELEM/);
 WRITETEXT(30,'('NODAL_RADIUS(M)____DEFLECTION(MM)_______
                  SLOPE(RAD)____RADIAL_STRESS(N/SQ.M)____
                  TANG._STRESS(N/SQ.M)%%')');
    'FOR'W:=1'STEP'1'UNTIL'NNODE'DO'
```

```
        'BEGIN'
           'IF'W=NELEM+1'THEN''GOTO'L4;
           Y:=NODC(/W,2/);
           Z:=NODC(/W,3/);
           'GOTO'L5;
        L4:Y:=NODC(/W-1,4/);
           Z:=NODC(/W-1,5/);
        L5:'IF'Y=0'THEN'DEFL(/W/):=0'ELSE'
           DEFL(/W/):=(FORCE(/Y/)*1000);
           'IF'Z=0'THEN'SLOPE(/W/):=0'ELSE'
           SLOPE(/W/):=FORCE(/Z/);
           SPACE(30,2);
           WRITE(30,F2,NODRAD(/W/));
           SPACE(30,6);
           WRITE(30,F1,DEFL(/W/));
           SPACE(30,4);
           WRITE(30,F1,SLOPE(/W/));
           SPACE(30,2);
           WRITE(30,F1,AVSTRESS(/2*W-1/));
           SPACE(30,9);
           WRITE(30,F1,AVSTRESS(/2*W/));
           NEWLINE(30,1);
        'END';
 'END';
  'IF'PROBNO<NOPROB'THEN''GOTO'START;
'END'
```

TYPICAL INPUT DATA FOR 'SYMPLAT'

This particular data is for test case (g) of figure 4.6

Plate N°.

N°. of elements &
N°. of constraints

Nodal radii

Element details

Nodal connection matrix

Load code

Concentrated loads

Distributed loads

```
// DATA A60
1;
5;2;
0;0.015;0.03;0.06;0.12;0.20;
2.8@+9;0.38;3.0;
2.8@+9;0.38;3.0;
2.8@+9;0.38;3.0;
2.8@+9;0.38;3.0;
2.8@+9;0.38;3.0;
1;1;0;2;3;
2;2;3;4;5;
3;4;5;6;7;
4;6;7;8;9;
5;8;9;0;10;
3;
0;0;0;0;0;0;0;0;10;0;0;0;
200;200;200;200;200;200;0;0;0;0;
```

TYPICAL OUTPUT FROM 'SYMPLAT'

This particular solution is for test case (g) of figure 4.6

```
PLATE NO.                          1

NUMBER OF ELEMENTS                 5

NUMBER OF NODES                    6

NUMBER OF CONSTRAINTS              2

NUMBER OF DEGREES OF FREEDOM      10

DETAILS OF ELEMENTS
```

ELEMENT NO.	INT.RAD(M)	EXT.RAD(M)	MOD.OF ELAST(N/SQ.M)	POISSONS RATIO	PLATE THICKNESS(MM)
1	0.0000	0.0150	2.8000@ +9	0.3800	3.0000
2	0.0150	0.0300	2.8000@ +9	0.3800	3.0000
3	0.0300	0.0600	2.8000@ +9	0.3800	3.0000
4	0.0600	0.1200	2.8000@ +9	0.3800	3.0000
5	0.1200	0.2000	2.8000@ +9	0.3800	3.0000

NODAL CONNECTION MATRIX

1	1	0	2	3
2	2	3	4	5
3	4	5	6	7
4	6	7	8	9
5	8	9	0	10

DETAILS OF LOADING

CONCENTRATED LOADS

NODAL RADIUS(M)	APPLIED FORCE(N)	APPLIED MOMENT(NM/M)
0.0000	0.0000	0.0000
0.0150	0.0000	0.0000
0.0300	0.0000	0.0000
0.0600	0.0000	0.0000
0.1200	1.0000@ +1	0.0000
0.2000	0.0000	0.0000

DISTRIBUTED LOADS

ELEMENT NO.	INT.RAD(M)	EXT.RAD(M)	INT.PRESS(N/SQ.M)	EXT.PRESS(N/SQ.M)
1	0.0000	0.0150	2.0000@ +2	2.0000@ +2
2	0.0150	0.0300	2.0000@ +2	2.0000@ +2
3	0.0300	0.0600	2.0000@ +2	2.0000@ +2
4	0.0600	0.1200	0.0000	0.0000
5	0.1200	0.2000	0.0000	0.0000

NET EFFECTIVE LOADS

NODAL RADIUS(M)	FORCE(N)	MOMENT(NM)
0.0000	4.7124@ -2	CONSTRAINT
0.0150	2.8274@ -1	0.0000
0.0300	9.8960@ -1	0.0000
0.0600	9.4248@ -1	0.0000
0.1200	1.0000@ +1	0.0000
0.2000	CONSTRAINT	0.0000

SYSTEM STIFFNESS MATRIX

0.0000	0.0000	0.0000	1.8508@ +6
0.0000	0.0000	-1.8508@ +6	5.7197@ +6
0.0000	1.3881@ +4	1.0926@ +4	3.8689@ +2
0.0000	-3.8689@ +6	-2.4808@ +4	4.8361@ +6
3.1121@ +4	1.3770@ +2	-1.8717@ +4	5.6404@ +2
0.0000	-9.6722@ +5	-1.2404@ +4	1.2090@ +6
1.5561@ +4	1.3770@ +2	-9.3587@ +3	5.6404@ +2
0.0000	-2.4181@ +5	-6.2019@ +3	4.1970@ +5
7.7803@ +3	1.3770@ +2	-1.3969@ +3	6.5504@ +2
0.0000	7.5551@ +3	1.8428@ +2	4.3681@ +2

NODAL STRESS MATRIX(N/SQ.M)

-8.0464@ +5	-7.9529@ +5	-7.6864@ +5	-6.8520@ +5	-5.2139@ +5
-8.0464@ +5	-7.9823@ +5	-7.8252@ +5	-7.2726@ +5	-6.1429@ +5
-7.9670@ +5	-7.7617@ +5	-6.9285@ +5	-5.6121@ +5	2.6948@ +4
-7.9877@ +5	-7.8538@ +5	-7.3017@ +5	-6.2942@ +5	-2.7106@ +5

NODAL RADIUS(M)	DEFLECTION(MM)	SLOPE(RAD)	RADIAL STRESS(N/SQ.M)	TANG. STRESS(N/SQ.M)
0.0000	1.8340	0.0000	-8.0464@ +5	-8.0464@ +5
0.0150	1.8207	-1.7715@ -3	-7.9600@ +5	-7.9850@ +5
0.0300	1.7811	-3.5031@ -3	-7.7241@ +5	-7.8395@ +5
0.0600	1.6275	-6.6698@ -3	-6.8903@ +5	-7.2872@ +5
0.1200	1.0645	-1.1890@ -2	-5.4130@ +5	-6.2185@ +5
0.2000	0.0000	-1.3395@ -2	2.6948@ +4	-2.7106@ +5

5

A SEMI-ANALYTIC FINITE ELEMENT ANALYSIS OF THE BENDING OF ASYMMETRICALLY LOADED, UNSTIFFENED CIRCULAR PLATES

5.1 INTRODUCTION

The previous chapter has shown how symmetrically loaded plates may be analysed by means of a fairly simple finite element program. Many practical problems however, result in asymmetry of loading which leads to asymmetry of displacement and stress distribution even though the geometry of the plate is still axisymmetric. Solution of the asymmetric loading problem was thought to be a necessary pre-requisite to the solution of the stiffened plate which introduces asymmetry of both geometry and loading. Even very simple and easily defined asymmetry of loading makes classical analysis cumbersome in all but a few cases and approximate techniques become virtually essential.

The simplest finite element approach that may be used for circular plate problems is probably the use of triangular plate bending elements. These are well documented in almost all of the basic finite element texts and are relatively simple to use but have the disadvantage in this application that the circular plate boundaries must be represented by a series of straight line approximations. High accuracy will therefore necessitate a large number of elements with the consequent very great increase in program size.

Several authors have investigated the use of annular sector elements such as that shown in figure 5.1

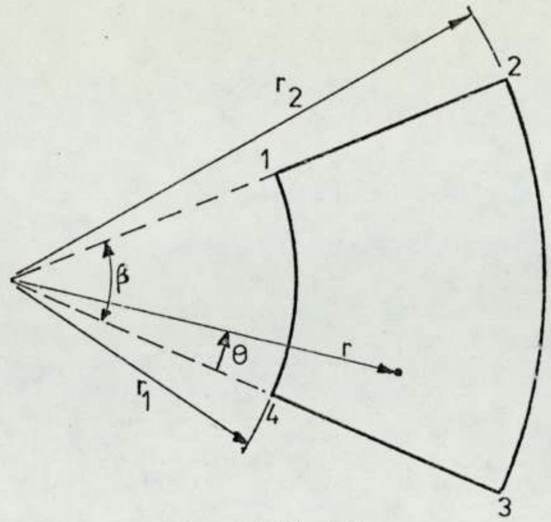


FIG. 5.1

Olson and Lindberg [22] have developed a twelve degree of freedom element using the transverse deflection and the radial and tangential slopes at each corner as the nodal freedoms. The displacement functions used are cubic in r for the radial variation and cubic in θ for the tangential variation.

Sawko and Merriman [23] have used the same shape of element but increased the number of degrees of freedom to sixteen by including the twist at each corner. They used a cubic for the radial distribution of deflection and trigonometric functions for the tangential variation.

The same type of element has been developed further still by Singh and Ramaswamy [24] who incorporated extra nodes at the mid point of both curved sides and used the deflection and radial slope as the freedoms at these additional nodes. The element therefore had twenty degrees of freedom.

It is claimed that each of these elements gives successively more accurate results but the major criticism of their use is that even the simplest of them uses twelve degrees of freedom which means that the representation of a complete plate very quickly builds up into a problem involving several hundred degrees of freedom with the consequent computational problems of storage and efficient solution of the equations. From the previous discussion of the semi-analytic finite element method in chapter 3 it is apparent that provided the geometry of an annular element remains constant in the circumferential direction then the application of this technique should be feasible provided that the circumferential variation of load and displacement can be adequately defined by the use of an orthogonal series. The great advantage of the method is the considerable saving in computational effort that can be achieved provided that sufficiently accurate results can be obtained by the use of a reasonably small number of terms in the series.

The development of the semi-analytic method for handling the asymmetrically loaded plate draws on many ideas already developed in chapter 4 but with the added complexity that the problem is now fully two dimensional and allowance must be made for incorporating a sufficient number of terms of the series.

5.2 THE STIFFNESS MATRIX FOR AN ANNULAR ELEMENT

5.2.1 Description of the displacement field

Referring to figure 5.2 the displacement at any point on an annulus bounded by the nodal rings $r = r_1$, and $r = r_2$ is a function of both r and θ.

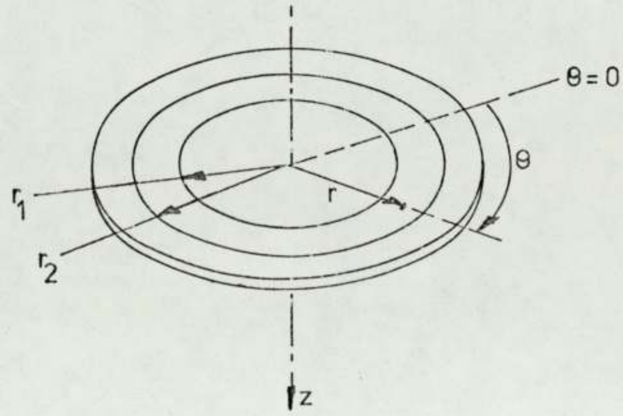


FIG. 5.2

From equation (3.18) the displacement field within the element may be defined as

$$w = \sum_{\ell=1}^{L}\left(\left[[\sim\phi^{\ell}\sim]N_1, [\sim\phi^{\ell}\sim]N_2, [\sim\phi^{\ell}\sim]N_3, [\sim\phi^{\ell}\sim]N_4\right]\left\{\delta^{\ell}\right\}^{e}\right)$$

Where the N are the same Hermitian polynomials as used previously in the symmetric problem, and are functions of r only. $[\sim\phi^{\ell}\sim]$ is a function of θ only, where $0 \leqslant \theta \leqslant 2\pi$

The $\{\delta^{\ell}\}^{e}$ are the nodal displacements of the element appropriate to the ℓ^{th} term of the series, hence for any given value of ℓ, $\{\delta^{\ell}\}^{e}$ is a four element column vector of the nodal deflection and slopes.

In many practical applications the loading asymmetry is not completely general but evenly distributed about some particular diameter. If this not unreasonable restriction is accepted then $[\sim\phi^{\ell}\sim]$ may be replaced by the single term $\cos\ell\theta$ which implies that the loading is symmetric with respect to $\theta = 0$ and that both the loading and displacement can be adequately represented by a cosine series of L terms.

The final form of the assumed displacement function is therefore

$$w = \sum_{\ell=1}^{L}\left(\left[N_1\cos\ell\theta \quad N_2\cos\ell\theta \quad N_3\cos\ell\theta \quad N_4\cos\ell\theta\right]\left\{\delta^{\ell}\right\}^{e}\right) \qquad (5.1)$$

For each term of the series this function will ensure inter-element continuity of deflection and also the radial and tangential slopes.

5.2.2 The stress-strain relationship

A plane stress condition is still assumed but as the loading is now asymmetric it is no longer permissible to delete the in-plane shear stresses and strains. The full relationship described in equations (A4.2) must be used as follows

$$\{\sigma\}^e = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix}^e = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix}^e$$

or $\{\sigma\}^e = [D]\{\epsilon\}^e$ (5.2)

where $$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} = \frac{E}{1-\nu^2}[C]$$

where $$[C] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

5.2.3 <u>The strain-displacement relationship</u>

Again, due to asymmetry, the full form of the equations (A4.1) must be used thus:-

$$\{\epsilon\}^e = -z \begin{bmatrix} \frac{\partial^2}{\partial r^2} \\ \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2} \\ \frac{2}{r^2}\cdot\frac{\partial}{\partial\theta} + \frac{2}{r}\cdot\frac{\partial^2}{\partial r\partial\theta} \end{bmatrix} \{w\}$$

Substitution of w from equation (5.1) gives

$$\{\epsilon\}^e = \sum_{\ell=1}^{L} -z \begin{bmatrix} \frac{\partial^2 N_1}{\partial r^2}\cos\ell\theta & \\ \left(\frac{1}{r}\cdot\frac{\partial N_1}{\partial r} - \frac{\ell^2}{r^2}N_1\right)\cos\ell\theta & \text{Three similar columns operating on } N_2, N_3 \,\&\, N_4 \text{ in turn} \\ \left(\frac{2\ell}{r^2}N_1 - \frac{2\ell}{r}\cdot\frac{\partial N_1}{\partial r}\right)\cos\ell\theta & \end{bmatrix} \begin{Bmatrix} \delta_1^\ell \\ \delta_2^\ell \\ \delta_3^\ell \\ \delta_4^\ell \end{Bmatrix}^e$$

Substitution for N_1, N_2, N_3 and N_4 from equations (4.9) gives, after extensive manipulation, an expression of the form

$$\{\epsilon\}^e = \sum_{\ell=1}^{L} \frac{-z}{(r_2-r_1)^3}[A^\ell]\{\delta^\ell\}^e \qquad (5.3)$$

Where $[A^\ell]$ is a (3 x 4) matrix whose elements are functions of r_1, r_2, ℓ , r and θ.

For convenience the matrix $[A^\ell]$ is expanded and re-written as

$$[A] = [A^\ell 1]r + [A^\ell 2] + [A^\ell 3]\frac{1}{r} + [A^\ell 4]\frac{1}{r^2} \qquad (5.4)$$

The elements of these matrices are listed in figure 5.3 on the following page.

FIG. 5.3

SUB-MATRICES OF THE STRAIN-DISPLACEMENT RELATIONSHIP

$$\left[A^{\ell}{}_1\right] \begin{bmatrix} 12\cos\ell\theta & 6(r_2-r_1)\cos\ell\theta & -12\cos\ell\theta & 6(r_2-r_1)\cos\ell\theta \\ 2(3-\ell^2)\cos\ell\theta & (r_2-r_1)(3-\ell^2)\cos\ell\theta & -2(3-\ell^2)\cos\ell\theta & (r_2-r_1)(3-\ell^2)\cos\ell\theta \\ -8\ell\sin\ell\theta & -4(r_2-r_1)\ell\sin\ell\theta & 8\ell\sin\ell\theta & -4(r_2-r_1)\ell\sin\ell\theta \end{bmatrix}$$

$$\left[A^{\ell}{}_2\right] \begin{bmatrix} -6(r_2+r_1)\cos\ell\theta & -2(r_2-r_1)(2r_2+r_1)\cos\ell\theta & 6(r_2+r_1)\cos\ell\theta & -2(r_2-r_1)(r_2+2r_1)\cos\ell\theta \\ -3(r_2+r_1)(2-\ell^2)\cos\ell\theta & -(r_2-r_1)(2r_2+r_1)(2-\ell^2)\cos\ell\theta & 3(r_2+r_1)(2-\ell^2)\cos\ell\theta & -(r_2-r_1)(r_2+2r_1)(2-\ell^2)\cos\ell\theta \\ 6(r_2+r_1)\ell\sin\ell\theta & 2(r_2-r_1)(2r_2+r_1)\ell\sin\ell\theta & -6(r_2+r_1)\ell\sin\ell\theta & 2(r_2-r_1)(r_2+2r_1)\ell\sin\ell\theta \end{bmatrix}$$

$$\left[A^{\ell}{}_3\right] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6r_2r_1(1-\ell^2)\cos\ell\theta & (r_2-r_1)(r_2+2r_1)r_2(1-\ell^2)\cos\ell\theta & -6r_2r_1(1-\ell^2)\cos\ell\theta & (r_2-r_1)(2r_2+r_1)r_1(1-\ell^2)\cos\ell\theta \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[A^{\ell}{}_4\right] \begin{bmatrix} 0 & 0 & 0 & 0 \\ -(r_2-3r_1)r_2^2\ell^2\cos\ell\theta & (r_2-r_1)r_2^2r_1\ell^2\cos\ell\theta & -(3r_2-r_1)r_1^2\ell^2\cos\ell\theta & (r_2-r_1)r_2r_1^2\ell^2\cos\ell\theta \\ 2(r_2-3r_1)r_2^2\ell\sin\ell\theta & -2(r_2-r_1)r_2^2r_1\ell\sin\ell\theta & 2(3r_2-r_1)r_1^2\ell\sin\ell\theta & -2(r_2-r_1)r_2r_1^2\ell\sin\ell\theta \end{bmatrix}$$

5.2.4 Formation of the stiffness matrix

Equation (3.5) defines the strain energy of an element as

$$U^e = \frac{1}{2}\int_V \{\epsilon\}^{e^t} \{\sigma\}^e dV$$

If the expressions for $\{\sigma\}^e$ and $\{\epsilon\}^e$ from equations (5.2) and (5.3) are substituted we have

$$U^e = \frac{1}{2}\cdot\frac{D}{(r_2 - r_1)^6}\int_0^{2\pi}\int_{r_1}^{r_2}\left(\left[\sum_{\ell=1}^{L}\{\delta^\ell\}^{e^t}[A^\ell]^t\right][C]\left[\sum_{h=1}^{L}[A^h]\{\delta^h\}^e\right]\right) r\,dr\,d\theta$$

On expansion, the matrix multiplication will give terms containing the submultiple

$$[A^\ell]^t\,[C]\,[A^h]$$

When multiplied out and integrated over the range $0 \leqslant \theta \leqslant 2\pi$ this submultiple produces terms such as:-

$\int_0^{2\pi}\cos(\ell\theta)\sin(h\theta)\,d\theta$ ------- which is zero for all ℓ and h

$\int_0^{2\pi}\cos(\ell\theta)\cos(h\theta)\,d\theta$ and $\int_0^{2\pi}\sin(\ell\theta)\sin(h\theta)\,d\theta$ ------- which are zero for $\ell \neq h$ and equal to π for $\ell = h$

The consequence of these results is that the contribution to the total strain energy of the element from the separate terms of the series is such that no cross products occur and the energy decouples into the sum of the individual contributions from each term thus

$$U^e = \sum_{\ell=1}^{L}\frac{1}{2}\cdot\frac{D}{(r_2 - r_1)^6}\int_0^{2\pi}\int_{r_1}^{r_2}\{\delta^\ell\}^{e^t}[A^\ell]^t\,[C][A^\ell]\{\delta^\ell\}^e r\,dr\,d\theta$$

The element may therefore be regarded as having separate stiffness matrices $[k^\ell]$ that are associated with each of the displacements $\{\delta^\ell\}^e$ such that

$$U^e = \frac{1}{2}\sum_{\ell=1}^{L}\{\delta^\ell\}^{e^t}[k^\ell]\{\delta^\ell\}^e$$

where

$$[k^{\ell}] = \frac{D}{(r_2 - r_1)^6} \int_0^{2\pi}\int_{r_1}^{r_2} [A^{\ell}]^t [C][A^{\ell}]\, r\, dr\, d\theta \qquad (5.5)$$

It is not convenient to formulate the $[k^{\ell}]$ in general terms due to the complexity of the algebra and so they are formed numerically as and when required in the computer program.

5.3 PROBLEMS ASSOCIATED WITH THE STIFFNESS MATRIX FOR A DISC ELEMENT

Due to asymmetry, the slope at the centre of the disc is no longer zero as was the case for symmetrical deformation.

It is still possible to describe the displacement of the disc in a form similar to equation (5.1) thus

$$w = \sum_{\ell=1}^{L}\left(\left[N_1 \quad N_2\cos\ell\theta \quad N_3\cos\ell\theta \quad N_4\cos\ell\theta\right]\{\delta^{\ell}\}^e\right)$$

where the N are the same as those for the annular element but with $r_1 = 0$

$$\therefore \quad N_1 = \frac{1}{r_2^3}\left(r_2^3 - 3r_2 r^2 + 2r^3\right)$$

$$N_2 = \frac{1}{r_2^2}\left(r_2^2 r - 2r_2 r^2 + r^3\right)$$

$$N_3 = \frac{1}{r_2^3}\left(3r_2 r^2 - 2r^3\right)$$

$$N_4 = \frac{1}{r_2^2}\left(-r_2 r^2 + r^3\right)$$

Note that the shape function associated with δ_1^{ℓ} is only N_1 and does not require a $\cos\ell\theta$ term as it is describing the effect of the displacement of a single point at the centre of the disc and therefore has no tangential variation.

The problem that arises with this displacement function is that it gives rise to infinite bending moments at $r = 0$.

Equations (A4.4), giving the expressions for bending moments M_r, M_θ and $M_{r\theta}$, show that these moments depend on the derivatives

$$\frac{\partial^2 w}{\partial r^2} \quad , \quad \frac{1}{r}\cdot\frac{\partial w}{\partial r} \quad , \quad \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2} \quad , \quad \frac{1}{r^2}\cdot\frac{\partial w}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial\theta}$$

The shape functions N_1, $N_3\cos\ell\theta$ and $N_4\cos\ell\theta$ are all well behaved at $r = 0$ for all these derivatives, but the linear term in r of the function $N_2\cos\ell\theta$ causes infinite values of M_r, M_θ and $M_{r\theta}$ to be generated at $r = 0$. The explanation for this is that $N_2\cos\ell\theta$ defines the displacement field caused by the central slope and unless the value of this function at angle θ is exactly equal in magnitude but opposite in sign to its value at angle $(\theta + \pi)$ then there will be no continuity of slope along a diameter of the disc as it passes through the centre and a 'kink' is formed with the consequent infinite values of bending moment.

Olson and Lindberg [22] overcame this problem in developing their central sector element by using only $\cos\theta$ or $\sin\theta$ for the tangential variation portion of the shape function, in which case the troublesome terms are self cancelling when bending moments are calculated. This is not however an acceptable solution to the problem here as the very nature of the semi-analytic solution requires the full series to be preserved.

Sawko and Merriman [23] also refer to this problem but do not propose any specific solution other than that of representing any complete plate by a collection of annular sector elements and accepting a small hole at the centre.

At the time of writing, no suitable shape function has been found for use with a disc element in the semi-analytic formulation of the problem. Complete plate problems are therefore analysed by using annular elements only, and leaving a small central hole surrounded by an annulus of small radial width but of high stiffness to ensure approximate continuity of slope along any diameter at the centre.

5.4 COMPLETION OF THE ANALYSIS

Previous discussion in section 3.4 has shown how the semi-analytic method results in a de-coupling of the contributions of the orthogonal series. Element stiffness matrices can be formed for the case when the orthogonal series is a simple cosine series, as has been demonstrated in section 5.2. (It is possible to analyse a completely general problem by using a sine series in addition to the cosine series but this has not been done in this particular analysis).

The complete analysis has therefore been reduced to the solution of L separate problems; one for each term of the series. The solution of each of the problems for the separate harmonics is similar in form to that previously discussed in the analysis of symmetrical bending in chapter 4.

The procedure for each harmonic is therefore:-

(a) to assemble the element stiffness matrices $[k^{\ell}]$ into a constrained system stiffness matrix $[K^{\ell}]$

(b) by the use of Fourier analysis, decompose the applied loading into a series of cosine components acting at the nodal rings. i.e. form the force vectors $\{F^{\ell}\}$

(c) solve the set of equations $[K^{\ell}]\{\delta^{\ell}\} = \{F^{\ell}\}$ to give the global nodal displacements $\{\delta^{\ell}\}$

(d) use $\{\delta^{\ell}\}$ and the stress-displacement relationship to give the element boundary stresses $\{\sigma^{\ell}\}$

After this procedure has been followed for each harmonic the $\{\delta^{\ell}\}$ and $\{\sigma^{\ell}\}$ are summed for all L harmonics to give the total displacement and stress at each nodal ring. The size of L must be chosen such that the results are to the required degree of accuracy.

5.5 THE COMPUTER PROGRAM 'ASYMPLAT'

5.5.1 Basic program structure

The general form of the program is, in effect, a looped version of the program SYMPLAT whereby the force vector is read, the stiffness matrix generated and the resulting equations solved for each of the terms of the required series. The program in its present form is less sophisticated than SYMPLAT in that the loading must be defined in terms of equivalent nodal forces as an external manual operation, but the generation of the element stiffness matrices is done in a more efficient manner. The following paragraphs describe various aspects of the program, the discussion being confined to those parts that are substantially different from the equivalent parts of SYMPLAT.

5.5.2 Formation of the element stiffness matrices

Equation (5.5) has previously shown that the stiffness matrix (ESTFL) for the ℓth term of the series is given by

$$[ESTFL] = \frac{D}{(R2 - R1)^6} \int_0^{2\pi} \int_{R1}^{R2} [AL]^t [C][AL] \, r \, dr \, d\theta$$

where [AL] may be written, from equation (5.4) as

$$[AL] = [AL1]r + [AL2] + [AL3]\frac{1}{r} + [AL4]\frac{1}{r^2}$$

(Note that the Algol identifiers used in the program are now being incorporated where possible for ease of cross reference with the program).

The matrix multiplication may be carried out and grouped in powers of r thus:-

$$\begin{aligned}[AL]^t[C][AL] = {} & [BL1]r^2 + \big[[BL2] + [BL5]\big]r + \big[[BL3] + [BL6] + [BL9]\big] \\ & + \big[[BL4] + [BL7] + [BL10] + [BL13]\big]\frac{1}{r} \\ & + \big[[BL8] + [BL11] + [BL14]\big]\frac{1}{r^2} + \big[[BL12] + [BL15]\big]\frac{1}{r^3} + [BL16]\frac{1}{r^4}\end{aligned}$$

Where:-

$[BL1] = [AL1]^t[C][AL1]$, $[BL5] = [AL2]^t[C][AL1]$

$[BL2] = [AL1]^t[C][AL2]$, $[BL6] = [AL2]^t[C][AL2]$

$[BL3] = [AL1]^t[C][AL3]$, $[BL7] = [AL2]^t[C][AL3]$

$[BL4] = [AL1]^t[C][AL4]$, $[BL8] = [AL2]^t[C][AL4]$

$$[BL9] = [AL3]^t[C][AL1] \quad , \quad [BL13] = [AL4]^t[C][AL1]$$

$$[BL10] = [AL3]^t[C][AL2] \quad , \quad [BL14] = [AL4]^t[C][AL2]$$

$$[BL11] = [AL3]^t[C][AL3] \quad , \quad [BL15] = [AL4]^t[C][AL3]$$

$$[BL12] = [AL3]^t[C][AL4] \quad , \quad [BL16] = [AL4]^t[C][AL4]$$

Hence

$$[ESTFL] = \frac{D}{(R2-R1)^6}\int_0^{2\pi}\Bigg[\left(\frac{R2^4-R1^4}{4}\right)[BL1] + \left(\frac{R2^3-R1^3}{3}\right)\Big[[BL2]+[BL5]\Big]$$

$$+\left(\frac{R2^2-R1^2}{2}\right)\Big[[BL3]+[BL6]+[BL9]\Big]$$

$$+(R2-R1)\Big[[BL4]+[BL7]+[BL10]+[BL13]\Big]$$

$$+\ln\left(\frac{R2}{R1}\right)\Big[[BL8]+[BL11]+[BL14]\Big]$$

$$-\left(\frac{1}{R2}-\frac{1}{R1}\right)\Big[[BL12]+[BL15]\Big]-\frac{1}{2}\left(\frac{1}{R2^2}-\frac{1}{R1^2}\right)[BL16]\Bigg]d\theta$$

At this stage it was realised that it may be possible to form a typical [BL] matrix and perform the subsequent integration with respect to θ by the use of a simple algorithm rather than by the formal matrix multiplication and transposition used in SYMPLAT. The relative simplicity of [C] and the fact that the matrix multiplication gives the θ terms as either $\int_0^{2\pi}\sin^2\ell\theta\, d\theta$ or $\int_0^{2\pi}\cos^2\ell\theta\, d\theta$, both of which equal π, make a simple algorithm possible and this was therefore incorporated in the form of a 'procedure' (FORMKA). A slightly simpler 'procedure' (FORMKS) was also developed for the special case where $\ell = 0$. In the actual program the various matrices [BL1], [BL2] etc are all stored as layers in a three dimensional array identified as [BL].

5.5.3 Calculation of stresses

Combination of equations (5.2) and (5.3) shows that the surface stresses in the plate for each term of the series is given by

$$\{\sigma^{\ell}\}^{e} = \frac{6D}{(R2-R1)^{3}T^{2}}[C][AL]\{\delta^{\ell}\}^{e}$$

The matrices [C] and [AL] are available at the time of formation of the stiffness matrix for each element and it is therefore convenient to make use of them at that stage. The stresses are required only at the nodal radii and therefore for values of radius equal to R1 and then R2 the product

$$\frac{6D}{(R2-R1)^{3}T^{2}}[C][AL]$$

is calculated for each element and the result stored in the matrix[ELCAL]. This is a three dimensional array with the portion associated with any one element assigned to a particular layer.

After the displacements $\{\delta^{\ell}\}$ have been calculated,[ELCAL] is recalled and the two matrices multiplied together to give the nodal stresses (radial, tangential and shear) at the nodal radii for all of the elements, these results being stored in the matrix[NODSTRL].

Continuity between elements is only guaranteed for deflection and slope. This means that the stresses at the outer radius of one element are not the same as those for the inner radius of the adjacent element. Nodal average stresses are therefore calculated and stored in the matrix[AVSTRL]. The numbers stored in[AVSTRL]are the coefficients of each cosine term of the series.

The final part of the program sums these contributions and associates them with the appropriate harmonic in order that the variation of stress with both radius and angle may be printed out.

5.5.4 Program flowchart and listing

The final flowchart is shown on page 120&121 and the listing on pages 122 - 129 inclusive.

5.6 DOCUMENTATION FOR USE OF THE PROGRAM 'ASYMPLAT'

5.6.1 Program specification

The program analyses the bending of annular plates supported axi-symmetrically but the applied loading need only be symmetric about the radial line defining the position $\theta = 0$.

Complete plates may be approximated by accepting the presence of a small central hole.

Radial variation of plate thickness and material properties can be accommodated.

The program outputs the radial and tangential variation of deflection and slope together with the radial, tangential and shear stresses at any specified points on the plate surface.

S.I. units are used throughout.

5.6.2 Preparation and presentation of data

The data required for the operation of the program is very similar to that required for 'SYMPLAT', the major difference being in the form in which the loading is presented.

The required data, in order, is:-

(a) The number of plates to be analysed in this particular run.

There is an overall loop in the program for this purpose and each plate requires its own complete set of data.

(b) The number of elements to be used to represent the plate.

Five elements appear to give results of sufficient accuracy for most practical purposes but a finer mesh may be advisable if regions of severe curvature or stress concentration are anticipated.

(c) The number of constraints on the plate.

Any imposed zero slope or deflection at any nodal radius is classified as a constraint.

(d) The number of terms that are required in the cosine series.

This varies depending on the form of the loading. For example an axi-symmetric load requires only one term whereas a point load may require up to twenty or more before satisfactory convergence of the solution is obtained.

(e) The number of values of θ that are to be used in the printout of the displacement and stress field.

(f) The nodal radii.

These must be given in metres and in increasing order of magnitude. The choice of values for the radii is left to the user although an empirical scheme of making the radial width of the elements approximately proportional to their external radii appears to produce generally satisfactory results.

(g) Details of each element.

The modulus of elasticity (N/m^2) and Poissons ratio for the material together with the plate thickness (mm) must be entered as data for each element, commencing with the element nearest to the centre of the plate. In the case of continuous radial variation of any of these three quantities, a stepped approximation must be used.

(h) The nodal connection matrix.

This relates the element freedoms to the global freedoms and also defines the positions of the constraints. The matrix is formed as follows:-

ELEMENT Nº Starting with the element nearest to the centre	ELEMENT FREEDOMS			
	INTERNAL RADIUS		EXTERNAL RADIUS	
	Defl.	Slope	Defl.	Slope
1				
2				
3				
↓				
NELEM				

The matrix consists of the numbers in the spaces bounded by the solid lines. The first column consists of the element reference numbers. The designatory numbers of the global freedoms are then put into the other spaces after zeros have been inserted at any constrained freedoms.

The designatory numbers for the global freedoms must be consecutive integers from 1 up to the number of degrees of freedom and should be fed into the spaces in the matrix from left to right and row by row.

Note that for continuity between elements, the numbers entered into the external radius columns for the n^{th} element must be the same as those in the internal radius columns for the $(n + 1)^{th}$ element, where $1 \leqslant n \leqslant$ (Nº of elements − 1)

The size of the matrix must be (Nº of elements x 5).

(i) Details of the applied loading

The applied loading must be capable of being expressed in the form of one or more terms of the series

$$F = F_0 + \sum_{\ell=1}^{L} F_\ell \cos \ell\theta$$

Where F is the force (or moment) per unit length that is to be allocated to a particular nodal circle.

Point or ring loads can be expressed in this form using conventional Fourier analysis methods. Distributed loads must first be redefined in the form of statically equivalent nodal loads by using methods such as those described in section 4.5.2.

The loading is then presented as a series of lists; each list quoting the order of the term followed by the loads allocated to each nodal circle (starting with the smallest radius) and associated with that order of term.

Examples of the way in which various types of loading may be presented are given in section 5.7

(j) The values of θ at which the displacement and stresses are required.

5.6.3 Summary of data presentation

(a) The number of plates to be analysed.

(b) The number of elements followed by the number of constraints.

(c) The number of terms in the cosine series.

(d) The number of values of θ that are to be used in the printout.

(e) A list of nodal radii (m) commencing with the smallest.

(f) A list of the modulus of elasticity (N/m^2), Poissons ratio and plate thickness (mm) for each element commencing with the element nearest to the centre of the plate.

(g) The nodal connection matrix.

(h) Lists of term order and associated nodal loading.

(i) The values of θ at which the displacement and stresses are required.

5.7 RESULTS AND DISCUSSION OF VARIOUS TEST PROGRAMS

5.7.1 Clamped-free annular plate with a point load on its free boundary

The general arrangement of such a plate is shown in figure 5.4

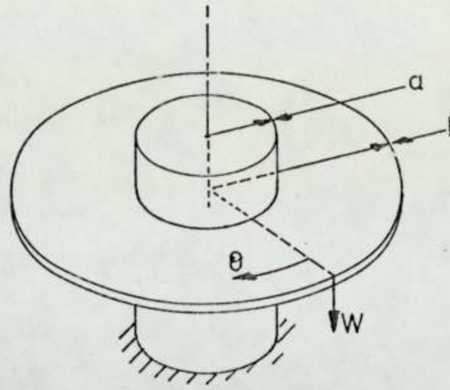


FIG. 5.4

This particular case was chosen because it represents a severe test of the program due to the need to represent the point load by means of a cosine series.

The Fourier analysis of the point load and its breaking down into equivalent nodal loads at the various harmonics is now discussed in detail.

If the nodal ring at which the load W acts is 'unwrapped' then the load may be represented by a distributed load of W/δ per unit length over a total length δ and defined in the range $0 \leqslant x \leqslant \ell$ as shown in figure 5.5 where δ is small and ℓ is the circumference of the nodal ring.

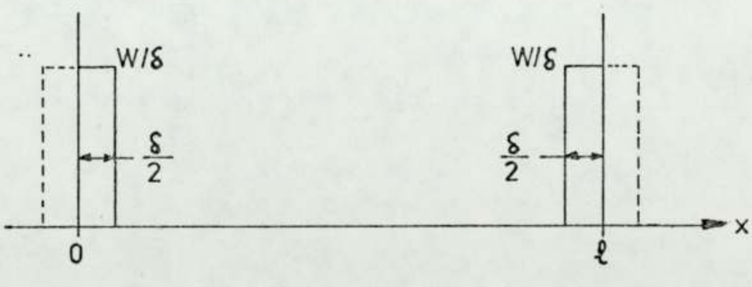


FIG. 5.5

A half range cosine series is to be used and thus the required function is given by

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

where

$$a_n = \frac{2}{\ell} \int_0^{\ell} \left(\frac{W}{\delta}\right) \cos \frac{n\pi x}{\ell} \, dx \qquad \text{for } n = 0,1,2, ---- , \infty$$

$\therefore$ for n = 0 $\qquad a_0 = \frac{2W}{\delta \ell}\int_0^{\frac{\delta}{2}} dx + \frac{2W}{\delta \ell}\int_{\ell-\frac{\delta}{2}}^{\ell} dx$

$$= \frac{2W}{\ell}$$

for n = 1, 2, ---, ∞ $\qquad a_n = \frac{2W}{n\pi\delta}\left(\left[\sin\frac{n\pi x}{\ell}\right]_0^{\frac{\delta}{2}} + \left[\sin\frac{n\pi x}{\ell}\right]_{\ell-\frac{\delta}{2}}^{\ell}\right)$

$$\therefore \quad a_n = \frac{2W}{n\pi\delta}\left(\sin\frac{n\pi\delta}{2\ell} + \sin n\pi - \sin n\pi.\cos\frac{n\pi\delta}{2\ell} + \sin\frac{n\pi\delta}{2\ell}.\cos n\pi\right)$$

but δ is small

$$\therefore \quad a_n = \frac{2W}{n\pi\delta}\left(\frac{n\pi\delta}{2\ell} + \frac{n\pi\delta}{2\ell}\cos n\pi\right)$$

$$= \frac{W}{\ell}(1 + \cos n\pi)$$

Hence $\qquad f(x) = \frac{W}{\ell} + \sum_{n=1}^{\infty}\frac{W}{\ell}(1 + \cos n\pi)\cos\frac{n\pi x}{\ell}$

In terms of θ, $\qquad x = b\theta$ and $\ell = 2\pi b$

$$\therefore f(\theta) = \frac{W}{2\pi b} + \sum_{n=1}^{\infty}\frac{W}{2\pi b}(1 + \cos n\pi)\cos\frac{n\theta}{2}$$

But $(1 + \cos n\pi)$ is zero for n odd, and 2 for n even

$$\therefore f(\theta) = \frac{W}{2\pi b} + \sum_{n=2,4,6}^{\infty}\frac{W}{\pi b}\cdot\cos\frac{n\theta}{2}$$

or $\qquad f(\theta) = \frac{W}{2\pi b} + \sum_{\ell=1,2,3}^{\infty}\frac{W}{\pi b}\cdot\cos\ell\theta \qquad (5.6)$

Equation (5.6) indicates that the effect of a point load may be replaced by the sum of the effects due to a constant ring load of total value W together with the cosine distributions of the load in the infinite series.

The ring load causes symmetrical deformation which has previously been discussed in chapter 4 and in this context will be seen to be handled by the simple device of putting the term order equal to zero in the data input, which then calls up the part of 'ASYMPLAT' concerned with symmetrical loading.

Each term in the infinite series causes asymmetric deformation and before it can be used in the equation $[K^\ell]\{\delta^\ell\} = \{F^\ell\}$

the nodal force vector $\{F^{\ell}\}$ must be formed. The most satisfactory way to do this is to use the equivalent virtual work ideas expressed in equation (3.10). With reference to equation (3.10), in this instance there are no body forces and the surface traction becomes simply the cosine distribution of load along the nodal circle. The integral of the surface traction therefore becomes a line integral along the nodal circle and the contribution to $\{F^{\ell}\}$ at a particular nodal circle becomes

$$F^{\ell} = \oint N^{\ell} T^{\ell} ds$$

where N^{ℓ} is the circumferential shape function, i.e. $\cos \ell\theta$

T^{ℓ} is the nodal circle loading, i.e. $(W/\pi b)\cos\ell\theta$ per unit length

Hence
$$F^{\ell} = \int_0^{2\pi} (\cos \ell\theta)\left(\frac{W}{\pi b}\cos \ell\theta\right)(b\ d\theta)$$

$$= \frac{W}{\pi}\int_0^{2\pi} \cos^2 \ell\theta\ d\theta$$

$$\therefore F^{\ell} = W$$

Each harmonic of the asymmetric portion of the load may therefore be represented by a nodal force of value equal to the load itself.

During the initial stages of development, 'ASYMPLAT' would predict only displacements and at that point some comparisons were made with the results of conventional finite element analysis in order to assess the viability of the semi-analytic approach. Olson and Lindberg [22] have analysed the particular case when $b/a = 1.5$ using various assemblages of their annular sector element and comparisons will be made with their results. The details of the problem are:-

Steel plate (E = 200 GN/m^2, $\nu = 0.3$) of thickness 1 mm

Internal radius 1 m,

External radius 1.5 m,

Edge load 1 N.

Five elements were used and graph 5.1 shows the effect on the deflection at the load point of taking various numbers of terms in the Fourier series representing the load. It can be seen that using only the first eleven terms brings the result within 5% of the exact answer whilst using 21 terms gives an accuracy of approximately 2%. Increasing the number of elements to ten gave no significant improvement in accuracy.

Graph 5.2 shows the variation in deflection at various points on the plate.

Direct comparisons with Olson and Lindberg's results are shown in figure 5.6. Their very simple 1 x 6 grid gives a very satisfactory result at the load point but is inaccurate over the rest of the plate. The results from ASYMPLAT using 5 elements and 21 terms compare very favourably with their analysis using a 4 x 24 grid. The great advantage of ASYMPLAT in this latter comparison is that ASYMPLAT arrived at its solution by solving 21 sets of 10 simultaneous equations which is considerably more economical in computational terms than the single set of 292 equations that are solved in Olson and Lindberg's analysis. This advantage can be expressed in quantitative terms firstly in terms of storage locations for the system stiffness matrix where ASYMPLAT requires only 10^2 locations which are successively overwritten as each term in the series is processed, whereas Olson and Lindberg require 292^2 locations. Secondly, the number of arithmetic operations in the solution of the equations using Cholesky's method is given by Martin and Carey [14] as

$$\frac{n^3}{6} + \frac{3n^2}{2} + \frac{7n}{3} \quad \text{for a set of n equations}$$

On this basis the Olson and Lindberg solution requires approximately 600 times the number of operations required by ASYMPLAT for this particular analysis.

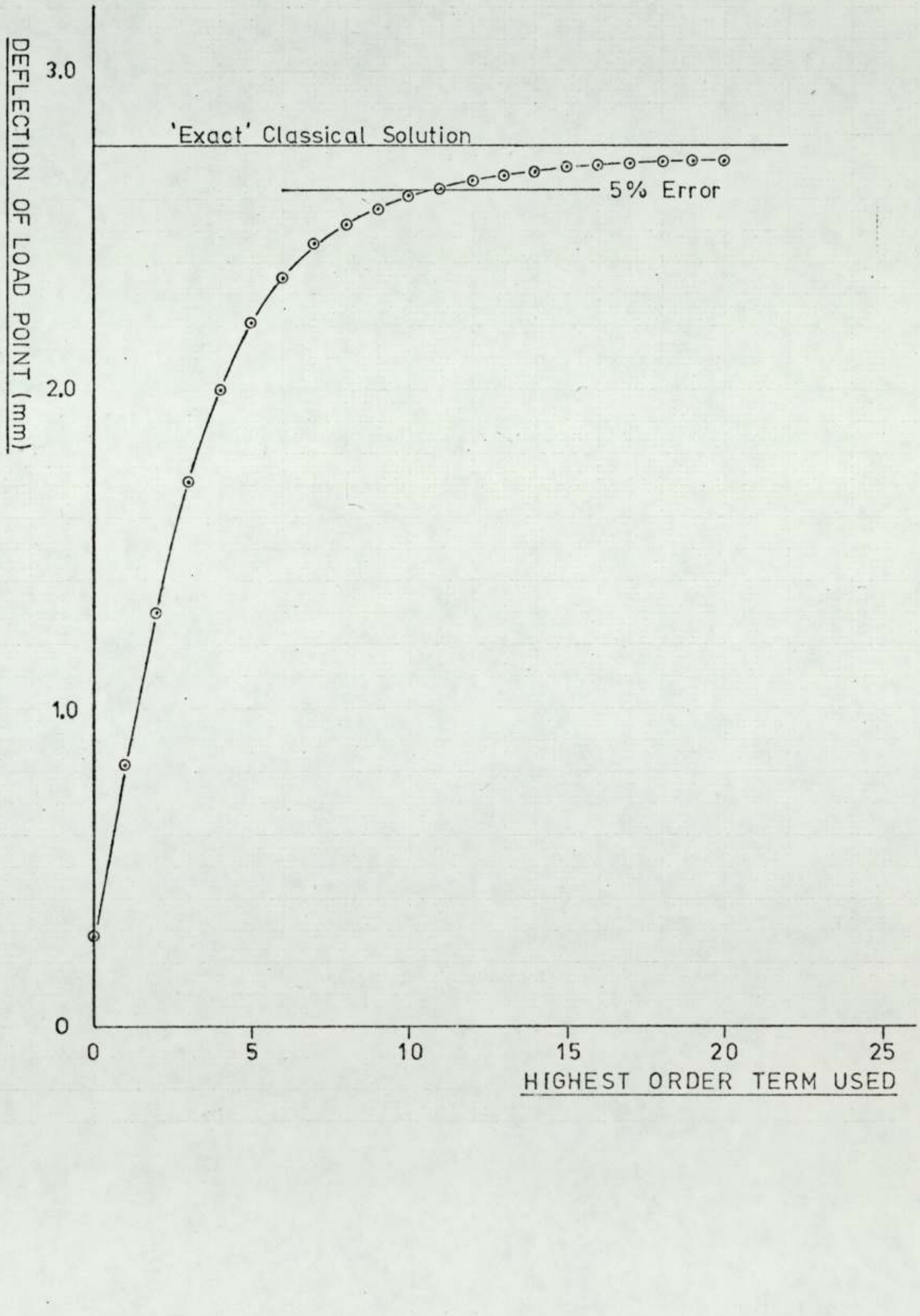
GRAPH 5.1

THE EFFECT OF THE NUMBER OF TERMS ON THE ACCURACY OF SOLUTION

Clamped - free annular plate with point load at edge

Five elements of varying width

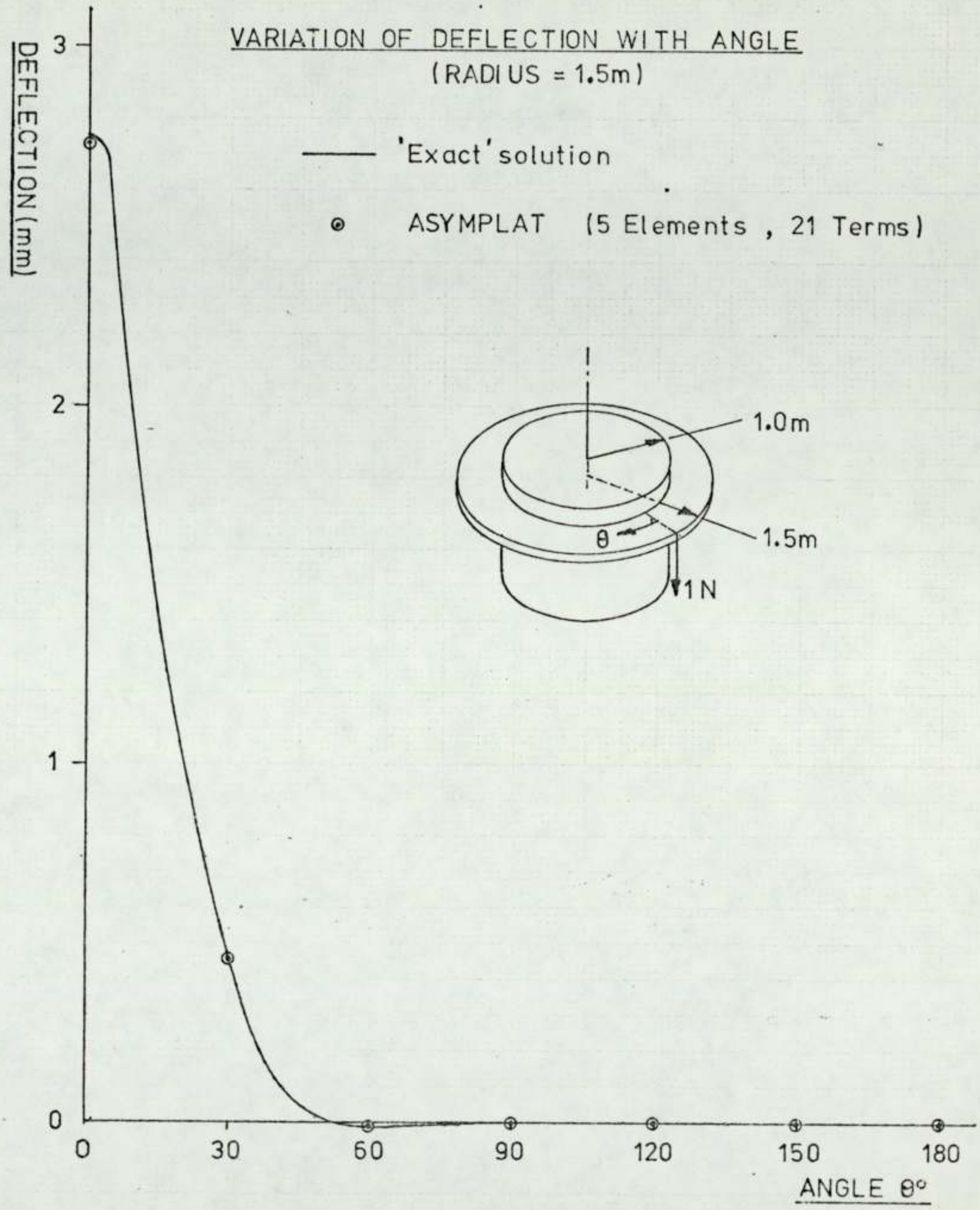
Load = 1 N



GRAPH 5.2

CLAMPED-FREE ANNULAR PLATE WITH POINT LOAD

Steel plate - 1mm thick



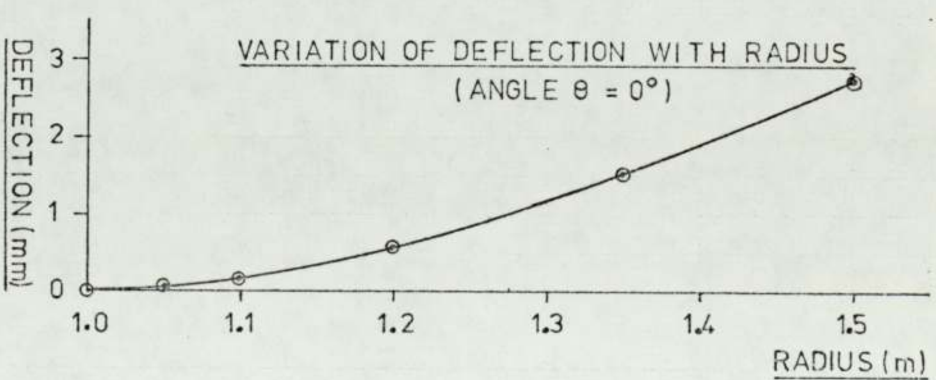


Fig. 5.6

CLAMPED-FREE ANNULAR PLATE WITH POINT LOAD AT EDGE

COMPARISON OF RESULTS FROM 'ASYMPLAT' WITH THOSE FROM OLSON & LINDBERG [22]

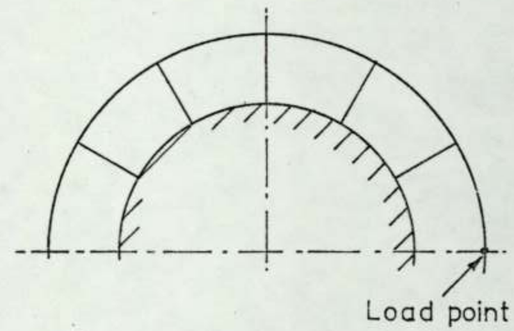
'ASYMPLAT' results are for 5 elements and :-

(a) 11 terms in the series (11 x 10 degrees of freedom)

(b) 21 terms in the series (21 x 10 degrees of freedom)

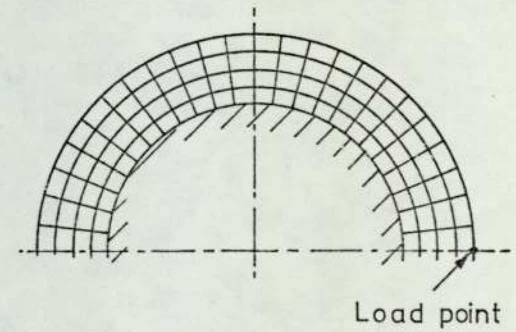
Olson & Lindberg results are for :-

(a) 1 x 6 grid



(19 degrees of freedom)

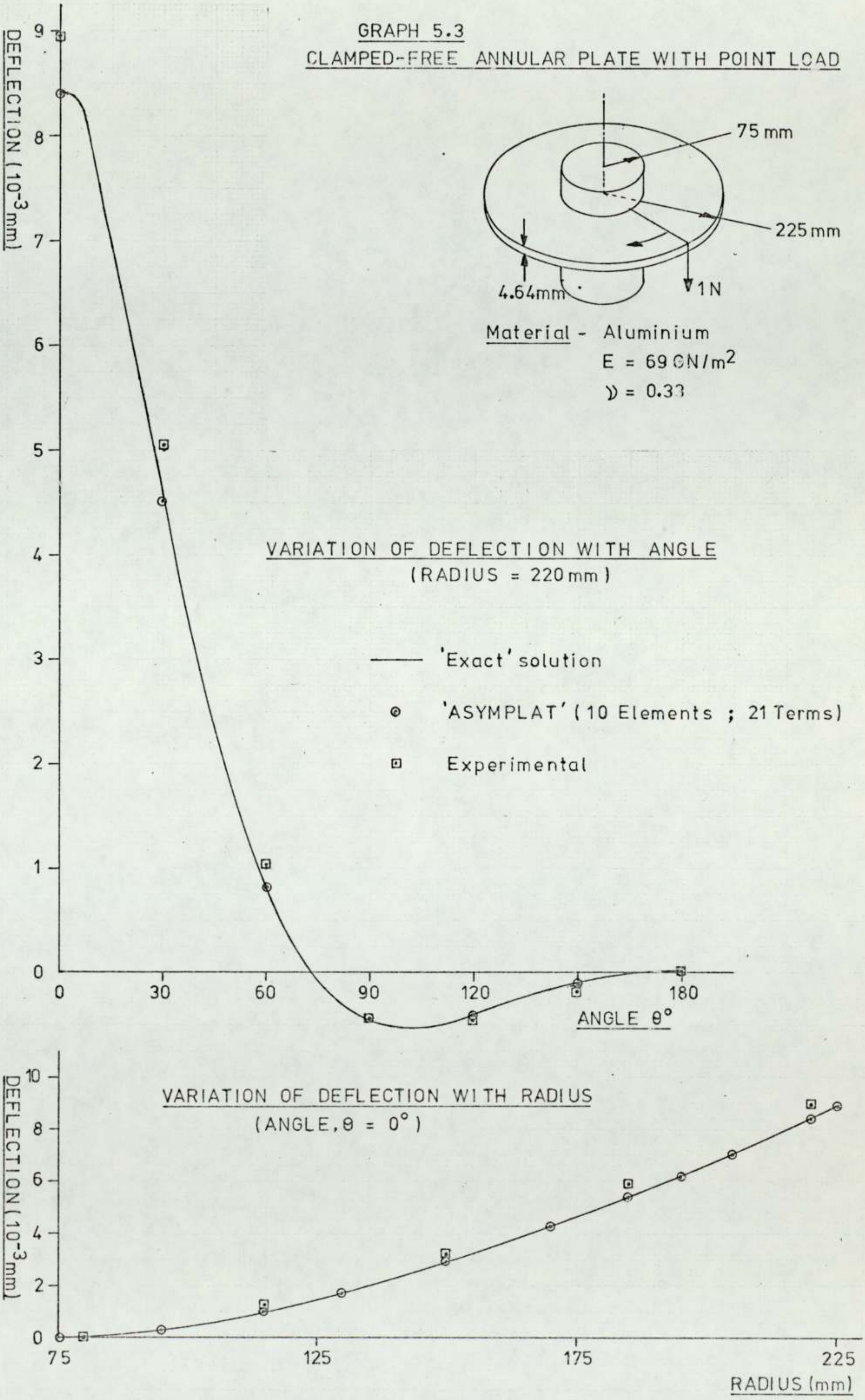
(b) 4 x 24 grid



(292 degrees of freedom)

DEFLECTION (mm) AT FREE EDGE FOR A POINT LOAD OF 1N

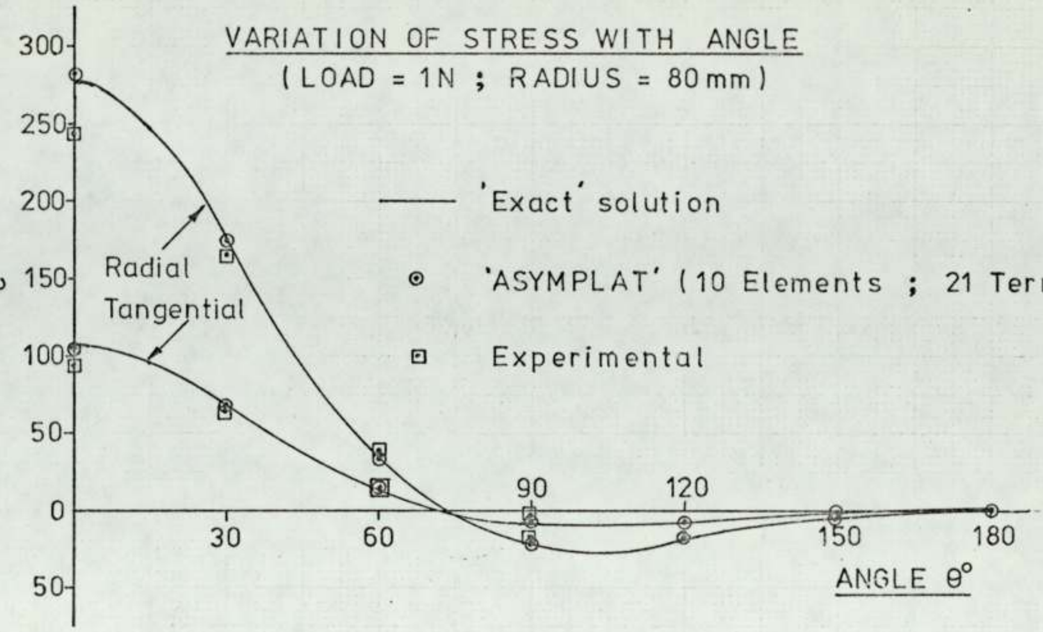
TYPE OF ANALYSIS	NUMBER OF DEGREES OF FREEDOM	ANGLE θ (DEG.)						
		0	30	60	90	120	150	180
OLSON & LINDBERG 1 x 6 GRID	19	2.7789	4.5646×10^{-1}	-1.6296×10^{-1}	-1.9364×10^{-2}	9.3568×10^{-3}	7.2225×10^{-4}	-1.0452×10^{-3}
OLSON & LINDBERG 4 x 24 GRID	292	2.7783	4.5503×10^{-1}	-2.1462×10^{-2}	-5.5343×10^{-3}	1.6893×10^{-4}	6.5564×10^{-5}	2.0079×10^{-6}
'ASYMPLAT' 5 Elements x 11 terms	11 x 10	2.6089	4.1412×10^{-1}	-4.1856×10^{-2}	-1.7997×10^{-2}	1.3815×10^{-3}	1.1624×10^{-2}	1.5316×10^{-2}
'ASYMPLAT' 5 Elements x 21 terms	21 x 10	2.7276	4.5554×10^{-1}	-1.0837×10^{-2}	-3.2134×10^{-3}	-2.0437×10^{-3}	-4.6312×10^{-4}	2.0258×10^{-3}
'EXACT'	(30 terms in series)	2.7508	4.6204×10^{-1}	-1.2721×10^{-2}	-5.6677×10^{-3}	6.1657×10^{-4}	-5.5351×10^{-4}	6.1120×10^{-4}

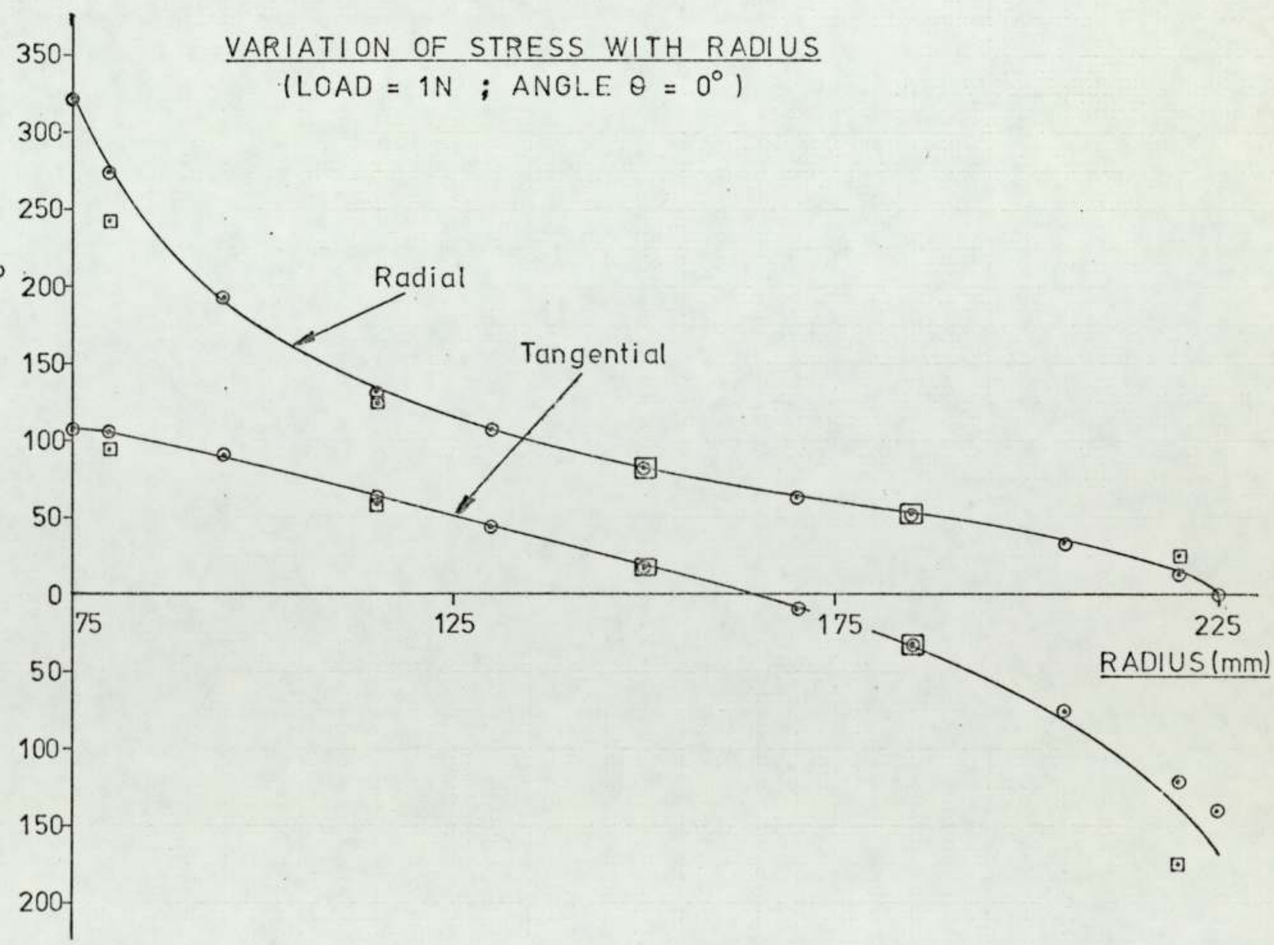

GRAPH 5.3
CLAMPED-FREE ANNULAR PLATE WITH POINT LOAD
75 mm
225 mm
4.64mm
1 N
Material - Aluminium
E = 69 GN/m²
ν = 0.33
DEFLECTION (10⁻³ mm)
VARIATION OF DEFLECTION WITH ANGLE
(RADIUS = 220 mm)
'Exact' solution
'ASYMPLAT' (10 Elements ; 21 Terms)
Experimental
ANGLE θ°
0
30
60
90
120
180
1
2
3
4
5
6
7
8
9
VARIATION OF DEFLECTION WITH RADIUS
(ANGLE, θ = 0°)
DEFLECTION (10⁻³ mm)
10
75
125
175
225
RADIUS (mm)

GRAPH 5.4

CLAMPED-FREE ANNULAR PLATE WITH POINT LOAD

PLATE AS SHOWN IN GRAPH 5.3





ASYMPLAT was further developed to include the analysis of plate stresses and at that stage it was thought desirable to obtain some practical test results with which to compare the theoretical analysis. Appendix C contains details of the construction of a suitable rig and its instrumentation. It also describes a classical analysis which was carried out using a method outlined by Timoshenko and Woinowski-Krieger [3] . Full details of the test results are given by Wilson [20] and a selection of these results is presented here for the purpose of comparison. Graph 5.3 shows the correlation between deflections and graph 5.4 between stresses. No major discrepancies are apparent; the largest deviations being in the values of the low measured stress at the inner boundary and the high measured stress at the load point. These deviations may be explained by lack of fixity at the clamp, and stress concentration in the region of the load respectively.

A typical set of input data for the solution of this problem using 'ASYMPLAT' is given on page 130.

5.7.2 Clamped circular plate with a point load

The general arrangement of such a plate is shown in figure 5.7.

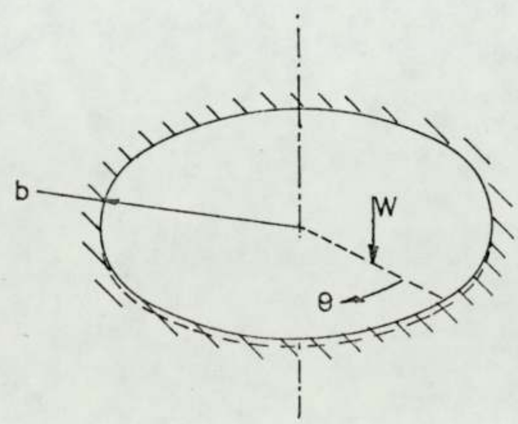


FIG. 5.7

Details of the particular plate analysed are:-

Steel plate ($E = 200\ GN/m^2$, $\nu = 0.3$)

Radius = 1 m, with load situated at 0.5 m

Thickness = 1 mm

Load = 1 N

The analysis was performed using a 10 element representation of the plate, the elements being of varying radial width and the centre of the plate approximated by using a narrow annulus with a very small inner radius (0.1 mm) and a modulus of elasticity 1000 times greater than that for the remainder of the plate.

The results of the analysis are illustrated in graph 5.5 on page 116. The displacements show very good agreement with 'exact' theory and compare favourably with Olson and Lindberg's solution. ASYMPLAT again has the advantage of using considerably less computational effort than Olson and Lindberg's approach. The graph also shows the stress distribution across the diameter on which the load acts, although no exact solution was available for comparison.

5.7.3 Clamped annular plate with applied moment on central boss

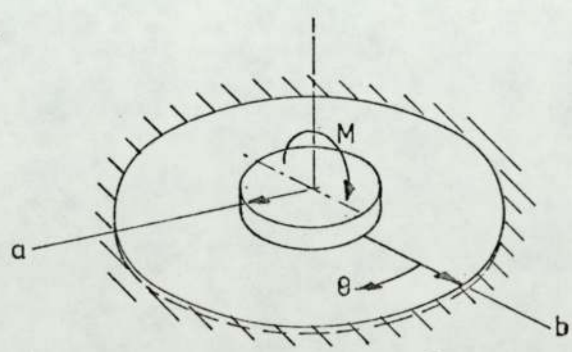


FIG. 5.8

Details of the particular plate analysed are:

Steel plate ($E = 200\ GN/m^2$, $\nu = 0.3$)

Plate radius, $b = 1.667$ m

Boss radius, $a = 1.0$ m

Thickness = 1 mm

Moment = 1 Nm

In the finite element formulation, two representations of loading were used. The first formulation represented the moment by a point moment at the edge of the boss using an infinite series identical to that used for the point load in section 5.7.1. In the second formulation the moment was represented by ring load W cos Θ around the edge of the boss where W was the force per unit length of circumference. The value of W was then adjusted such that the moment due to the ring load was statically equivalent to the applied moment M.

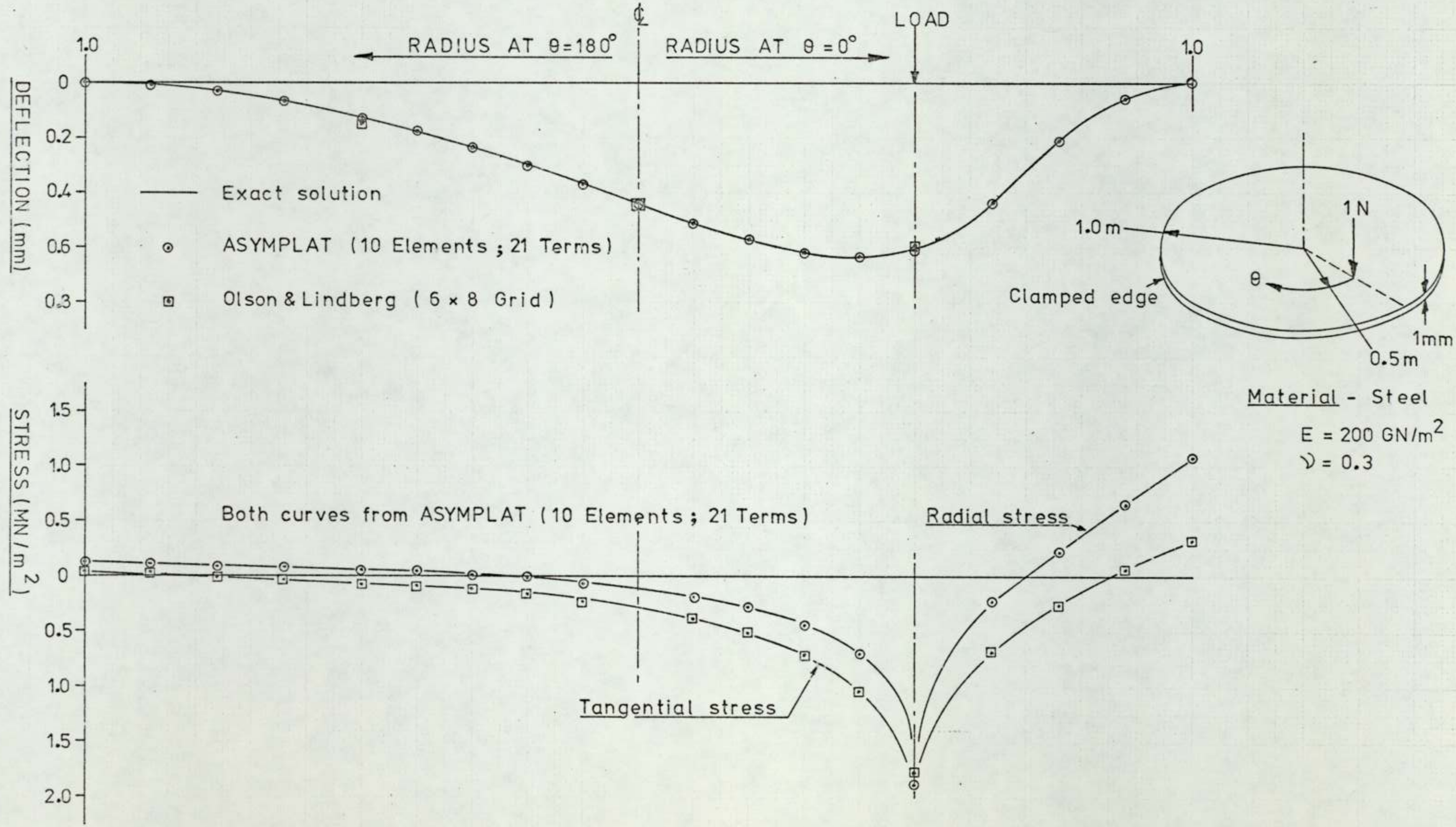
The method of representation of the boss caused some difficulty in the interpretation of the computed stresses. The program calculates nodal average stresses, and is generally satisfactory, except that where a boss exists the averaging process at a thick boss - thin plate interface does not give a correct result for the plate stresses. This problem was overcome by representing the rigidity of the boss by an annulus with the same thickness as the plate, and with a small central hole, but with a very much increased modulus of elasticity. (10^6 times that of the plate).

The two methods of representing the moment produced almost identical results but the use of the ring load, W cos Θ, was computationally much more economical as only one term was required in the series representation. The ring load form of representation was used, however, in the knowledge that a simple cosine wave was in fact the exact solution. Such knowledge is obviously not always available and the infinite series form of point moment representation should be regarded as a more general approach.

The deflection and stress distribution is illustrated in graph 5.6 on page 117 and comparisons are made with exact results where available.

GRAPH 5.5

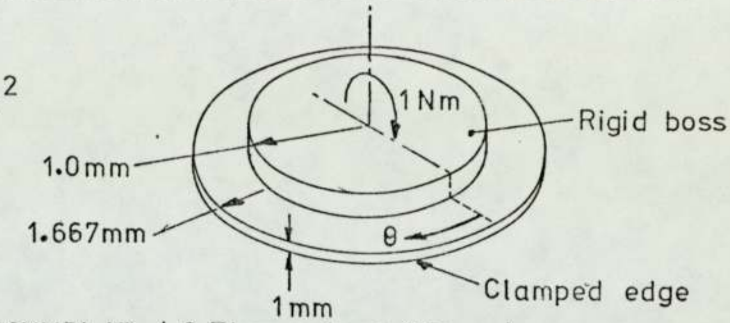
CLAMPED CIRCULAR PLATE WITH POINT LOAD



GRAPH 5.6

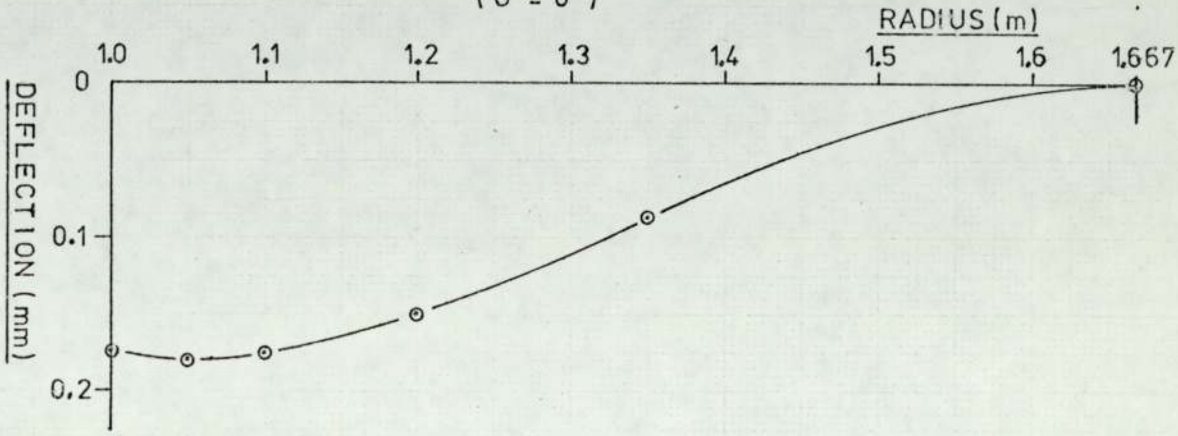
CLAMPED CIRCULAR PLATE WITH CENTRAL BOSS CARRYING A MOMENT

Material - Steel
$E = 200\ GN/m^2$
$\nu = 0.3$

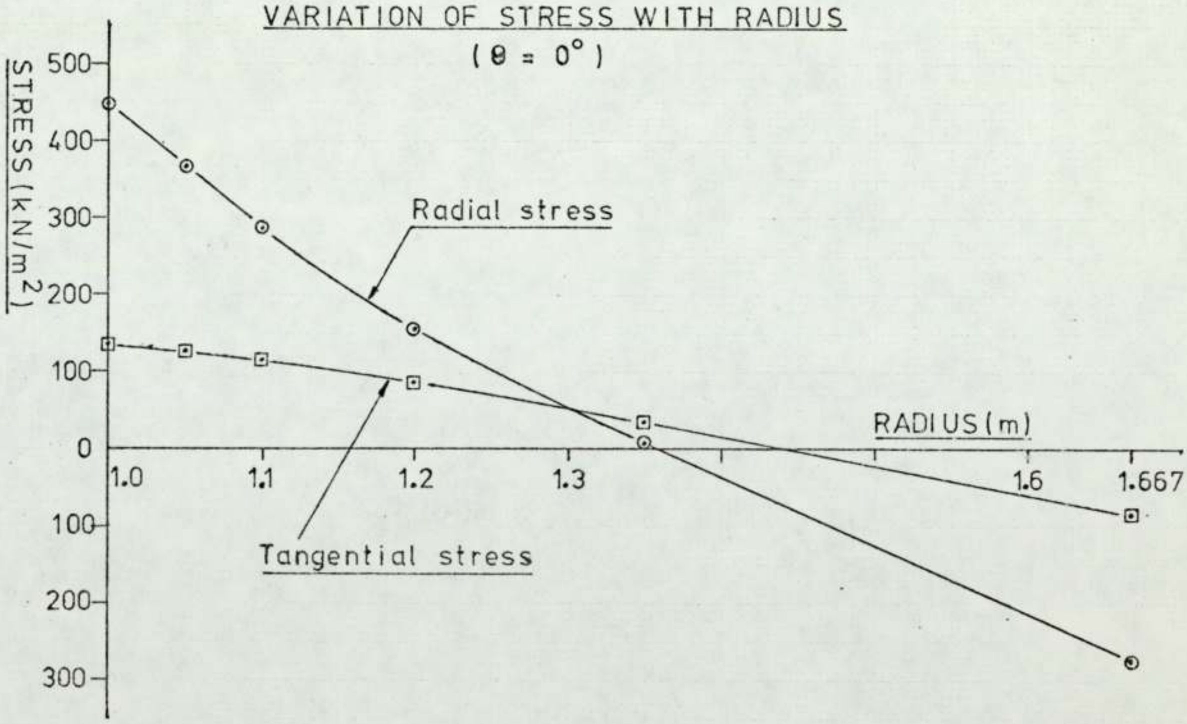


Results obtained from ASYMPLAT (6 Elements ; 1 Term)

VARIATION OF DEFLECTION WITH RADIUS
(θ = 0°)



VARIATION OF STRESS WITH RADIUS
(θ = 0°)



Comparison with 'exact' results (taken from Timoshenko [3])

Method of Analysis	Slope of boss (rad.)	STRESS (kN/m²)			
		Inner Radius		Outer Radius	
		σ_r	σ_θ	σ_r	σ_θ
Exact	1.7556×10^{-4}	411.57	123.47	270.66	81.20
ASYMPLAT	1.7497×10^{-4}	448.49	134.55	275.74	82.73

5.7.4 Variable thickness plate with distributed pressure and central moment

This plate was analysed to illustrate the type of problem that is amenable to solution using 'ASYMPLAT' and which would be extremely difficult, if not impossible, to solve using exact methods.

The plate is the same as the one previously shown in graph 4.3 on page 72 but with the addition of a moment of 80 kNm applied to the central boss.

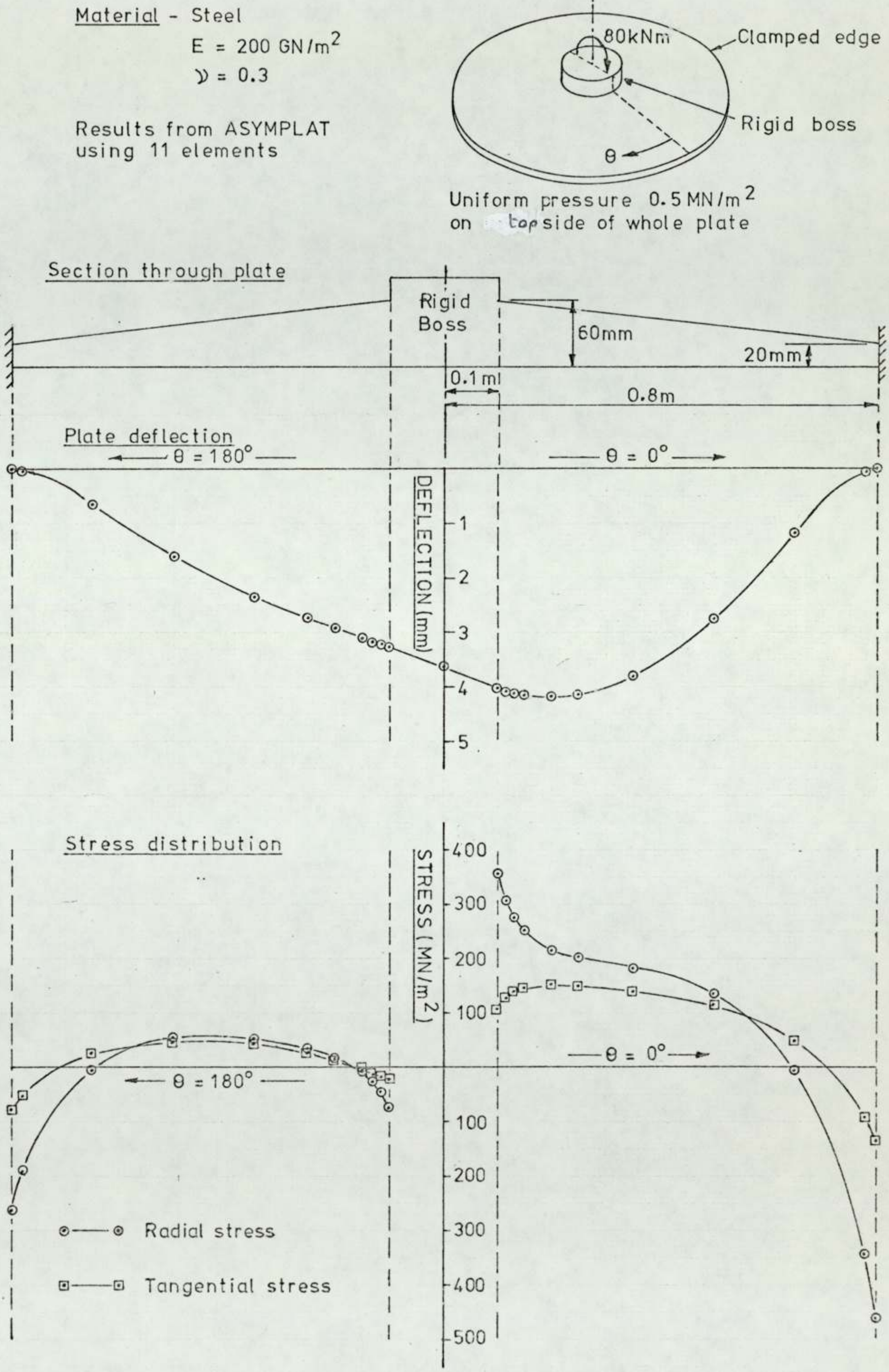
The distributed pressure was allocated to the nodal circles using equations (4.17) and (4.18) and then read into the program as zero order terms.

The moment on the boss was represented by the ring load technique described in section 5.7.3 above and read into the program as a first order term.

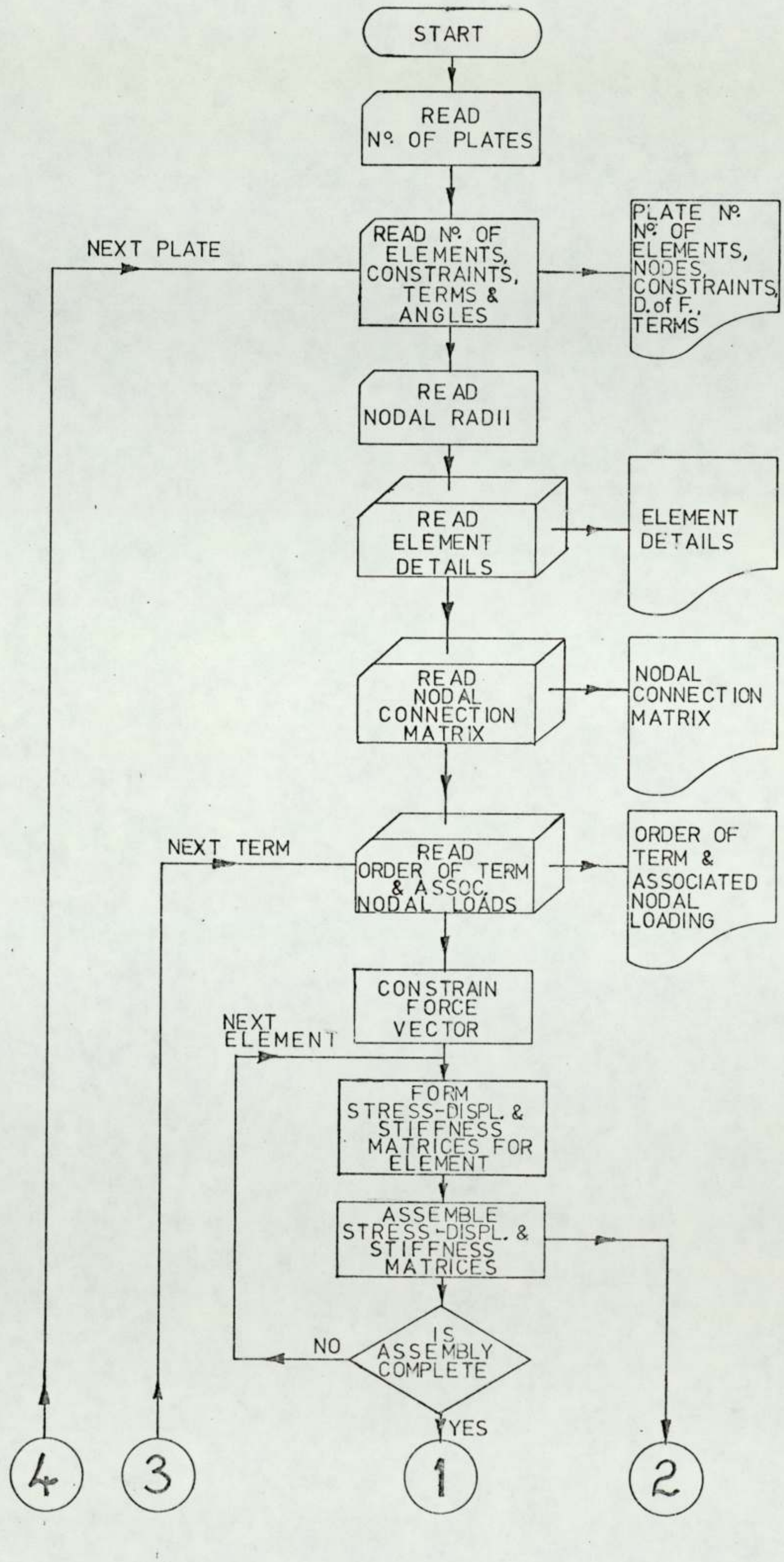
The deflection and stress distribution are illustrated in graph 5.7 on the following page.

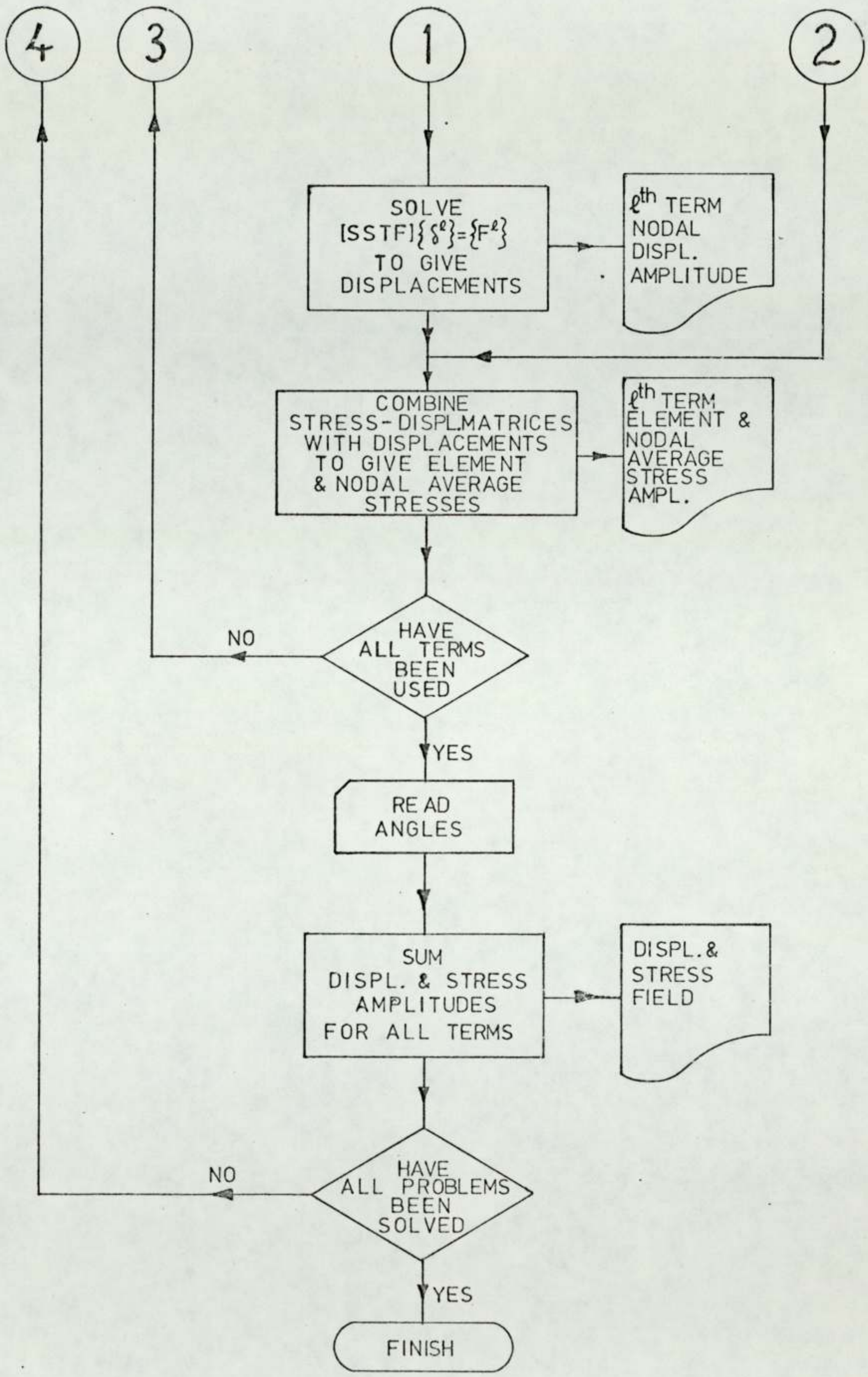
GRAPH 5.7

VARIABLE THICKNESS PLATE CARRYING BOTH A MOMENT & UNIFORM PRESSURE



PROGRAM FLOWCHART FOR 'ASYMPLAT'





4
3
1
2
SOLVE
[SSTF]{δ^ℓ}={F^ℓ}
TO GIVE
DISPLACEMENTS
ℓth TERM
NODAL
DISPL.
AMPLITUDE
COMBINE
STRESS-DISPL.MATRICES
WITH DISPLACEMENTS
TO GIVE ELEMENT
& NODAL AVERAGE
STRESSES
ℓth TERM
ELEMENT &
NODAL
AVERAGE
STRESS
AMPL.
NO
HAVE
ALL TERMS
BEEN
USED
YES
READ
ANGLES
SUM
DISPL. & STRESS
AMPLITUDES
FOR ALL TERMS
DISPL. &
STRESS
FIELD
NO
HAVE
ALL PROBLEMS
BEEN
SOLVED
YES
FINISH

A LISTING OF THE PROGRAM 'ASYMPLAT'

```
'BEGIN''COMMENT'ANALYSIS OF ASYMMETRICALLY LOADED PLATES;
    'REAL'A;
    'INTEGER'NELEM,NNODE,NCON,NDEGF,HBW,PROBNO,NPROB,NTERM,NANGL,
            F,G,H,I,J,K,L,M,N,W,Y,Z,F1,F2,F3,F4;
     F1:=FORMAT('('5S-D.DDDD@+ND')');
     F2:=FORMAT('('-ND.DDD')');
     F3:=FORMAT('('5SND')');
     F4:=FORMAT('('NDD.D')');
     HBW:=4;
     PROBNO:=0;
     NPROB:=READ(60);
 START:PROBNO:=PROBNO+1;
     PAGE(30,1);
  WRITETEXT(30,'('ANALYSIS_OF_ASYMMETRICALLY_LOADED_PLATES%%%%')');
  WRITETEXT(30,'('PLATE_NO.______________________________')');
     WRITE(30,F3,PROBNO);
     NEWLINE(30,3);
'COMMENT' READ AND OUTPUT THE GENERAL PROBLEM DETAILS;
     NELEM:=READ(60);
     NCON:=READ(60);
  WRITETEXT(30,'('NUMBER_OF_ELEMENTS_______________')');
     WRITE(30,F3,NELEM);
     NEWLINE(30,3);
     NNODE:=NELEM+1;
  WRITETEXT(30,'('NUMBER_OF_NODES__________________')');
     WRITE(30,F3,NNODE);
     NEWLINE(30,3);
  WRITETEXT(30,'('NUMBER_OF_CONSTRAINTS____________')');
     WRITE(30,F3,NCON);
     NEWLINE(30,3);
     NDEGF:=2*NNODE-NCON;
  WRITETEXT(30,'('NUMBER_OF_DEGREES_OF_FREEDOM_____')');
     WRITE(30,F3,NDEGF);
     NEWLINE(30,3);
     NTERM:=READ(60);
  WRITETEXT(30,'('NUMBER_OF_TERMS__________________')');
     WRITE(30,F3,NTERM);
     NEWLINE(30,3);
     NANGL:=READ(60);
 'BEGIN'
    'INTEGER''ARRAY'NODC(/1:NELEM,1:5/),TERMORD(/1:NTERM/),
    'ARRAY'NODRAD(/1:NNODE/),FORCE(/1:NDEGF/),THETA(/1:NANGL/),
           E,V,T(/1:NELEM/),SSTF(/1:NDEGF,0:HBW-1/),
           DISPL(/1:NTERM,1:2*NNODE/),APLOL(/1:2*NNODE/),
           ELCAL(/1:NELEM,1:6,1:4/), NODSTRL(/1:6,1:NELEM/),
           AVSTRL(/1:NTERM,1:3*NNODE/),
           DEFL,RSLOPE,TSLOPE,RSTR,TSTR,SSTR(/1:NNODE,1:NANGL/);
'COMMENT' LIST THE PROCEDURES;
    'PROCEDURE' FORMKS(M,Q,P,A1,A2);
         'ARRAY'M,A1,A2;
         'REAL'P;
         'INTEGER'Q;
         'BEGIN''FOR'I:=1'STEP'1'UNTIL'4'DO'
              'BEGIN''FOR'J:=1'STEP'1'UNTIL'4'DO'
                   'BEGIN'
                        M(/I,J,Q/):=0;
                        M(/I,J,Q/):=(A1(/1,I/)*(A2(/1,J/)+P*A2(/2,J/)))
                                   +(A1(/2,I/)*(P*A2(/1,J/)+A2(/2,J/)));
                   'END';
```

```
              'END';
           'END' OF FORMKS;
    'PROCEDURE' FORMKA(M,Q,P,A1,A2);
         'ARRAY'M,A1,A2;
         'REAL'P;
         'INTEGER'Q;
         'BEGIN''FOR'I;=1'STEP'1'UNTIL'4'DO'
               'BEGIN''FOR'J;=1'STEP'1'UNTIL'4'DO'
                   'BEGIN'
                        M(/I,J,Q/);=0;
                        M(/I,J,Q/);=(A1(/1,I/)*(A2(/1,J/)+P*A2(/2,J/)))
                                    +(A1(/2,I/)*(P*A2(/1,J/)+A2(/2,J/)))
                                    +(A1(/3,I/)*(1-P)*A2(/3,J/)/2);
                   'END';
              'END';
           'END' OF FORMKA;
'PROCEDURE'CHOBANDDET(N,M,A);
     'VALUE'N,M;
'INTEGER'N,M;
     'ARRAY'A;
     'BEGIN'
       'INTEGER'I,J,K,P,Q,R,S;
       'REAL'Y;
       'FOR'I;=1'STEP'1'UNTIL'N'DO'
          'BEGIN'
             P;=('IF'I>M'THEN'0'ELSE'M-I+1);
             R;=I-M+P;
            'FOR'J;=P'STEP'1'UNTIL'M'DO'
               'BEGIN'
                  S;=J-1;
                  Q;=M-J+P;
                  Y:=A(/I,J/);
                 'FOR'K;=P'STEP'1'UNTIL'S'DO'
                    'BEGIN'
                       Y;=Y-A(/I,K/)*A(/R,Q/);
                       Q;=Q+1;
                    'END';
                 'IF'J=M'THEN'A(/I,J/);=1/SQRT(Y)
                 'ELSE'A(/I,J/);=Y*A(/R,M/);
                  R;=R+1;
               'END';
          'END';
     'END' OF CHOBANDDET;
'PROCEDURE'CHOBANDSOL(N,M,R,A,B);
     'VALUE'N,M,R;
     'INTEGER'N,M,R;
     'ARRAY'A,B;
     'BEGIN'
       'INTEGER'I,J,K,P,Q,S;
       'REAL'Y;
        S;=M-1;
       'FOR'J;=1'STEP'1'UNTIL'R'DO'
          'BEGIN'
            'FOR'I;=1'STEP'1'UNTIL'N'DO'
               'BEGIN'
                  P;=('IF'I>M'THEN'0'ELSE'M-I+1);
                  Q;=I;
```

```
                Y;=B(/I/);
               'FOR'K;=S'STEP'-1'UNTIL'P'DO'
                  'BEGIN'
                     Q;=Q-1;
                     Y;=Y-A(/I,K/)*B(/Q/);
                  'END';
                B(/I/);=Y*A(/I,M/);
             'END';
          'FOR'I;=N'STEP'-1'UNTIL'1'DO'
             'BEGIN'
                P;=('IF'N-I>M'THEN'0'ELSE'M-N+I);
                Y;=B(/I/);
                Q;=I;
               'FOR'K;=S'STEP'-1'UNTIL'P'DO'
                  'BEGIN'
                     Q;=Q+1;
                     Y;=Y-A(/Q,K/)*B(/Q/);
                  'END';
                B(/I/);=Y*A(/I,M/);
             'END';
        'END';
    'END' OF CHOBANDSOL;
'COMMENT' INPUT DETAILS OF ELEMENTS AND LOADING;
    'FOR'I;=1'STEP'1'UNTIL'NNODE'DO'
    NODRAD(/I/);=READ(60);
    'FOR'I;=1'STEP'1'UNTIL'NELEM'DO'
    'BEGIN'
       E(/I/);=READ(60);
       V(/I/);=READ(60);
       T(/I/);=READ(60);
    'END';
  WRITETEXT(30,'('DETAILS_OF_ELEMENTS%%')');
  WRITETEXT(30,'('ELEMENT_NO.____INT.RAD(M)____EXT.RAD(M)____MOD.OF_
                  ELAST(N/SQ.M)____POISSONS_RATIO____PLATE_THICKNESS
                  (MM)%%')');
    'FOR'I;=1'STEP'1'UNTIL'NELEM'DO'
    'BEGIN'
    WRITE(30,F3,I);  SPACE(30,8);
    WRITE(30,F2,NODRAD(/I/));  SPACE(30,7);
    WRITE(30,F2,NODRAD(/I+1/));  SPACE(30,5);
    WRITE(30,F1,E(/I/));  SPACE(30,13);
    WRITE(30,F2,V(/I/));  SPACE(30,13);
    WRITE(30,F2,T(/I/));   T(/I/);=T(/I/)/1000;
    NEWLINE(30,1);
    'END';
     NEWLINE(30,2);
  WRITETEXT(30,'('NODAL_CONNECTION_MATRIX%%')');
    'FOR'I;=1'STEP'1'UNTIL'NELEM'DO'
    'BEGIN'
        'FOR'J;=1'STEP'1'UNTIL'5'DO'
        'BEGIN'
              NODC(/I,J/);=READ(60);
              WRITE(30,F3,NODC(/I,J/));
        'END';
         NEWLINE(30,1);
    'END';
     NEWLINE(30,2);
```

```
'FOR'M:=1'STEP'1'UNTIL'NTERM'DO'
  'BEGIN'
     WRITETEXT(30,'('ORDER_OF_TERM____')');
     L:=READ(60);
     WRITE(30,F3,L);
     TERMORD(/M/):=L;
     NEWLINE(30,3);
     WRITETEXT(30,'('APPLIED_NODAL_LOADING%%')');
    'FOR'J:=1'STEP'1'UNTIL'2*NNODE'DO'  APLOL(/J/):=READ(60);
    'FOR'J:=1'STEP'2'UNTIL'2*NNODE-1'DO'
    'BEGIN'   SPACE(30,5);
        WRITE(30,F2,APLOL(/J/));     SPACE(30,5);
        WRITE(30,F2,APLOL(/J+1/));  NEWLINE(30,1);
    'END';
     NEWLINE(30,3);
    'FOR'I:=1'STEP'1'UNTIL'NNODE-1'DO'
    'BEGIN'
       'FOR'J:=1'STEP'1'UNTIL'2'DO'
       'BEGIN'
          G:=NODC(/I,J+1/);
         'IF'G>0'THEN'FORCE(/G/):=APLOL(/2*I-2+J/);
       'END';
    'END';
       'FOR'J:=1'STEP'1'UNTIL'2'DO'
       'BEGIN'
          G:=NODC(/NELEM,J+3/);
         'IF'G>0'THEN'FORCE(/G/):=APLOL(/2*I-2+J/);
       'END';
'COMMENT' FORM ELEMENT STIFFNESS MATRICES;
        'BEGIN''REAL'R1,R2,D,DR,DR6,RATR,NU;
               'ARRAY'AL,AL1,AL2,AL3,AL4(/1:3,1:4/),
                      BL(/1:4,1:4,1:16/),ESTFL(/1:4,1:4/);
               'FOR'I:=1'STEP'1'UNTIL'NDEGF'DO'
               'FOR'J:=0'STEP'1'UNTIL'HBW-1'DO'
               SSTF(/I,J/):=0;
             'FOR'K:=1'STEP'1'UNTIL'NELEM'DO'
             'BEGIN'
                R1:=NODRAD(/K/);
                R2:=NODRAD(/K+1/);
                DR:=R2-R1;   DR6:=DR**6;    RATR:=R2/R1;
                D:=(E(/K/)*T(/K/)**3)/(12*(1-V(/K/)**2));
                NU:=V(/K/);
   AL1(/1,1/):=12;
   AL1(/1,2/):=AL1(/1,4/):=6*DR;
   AL1(/1,3/):=-12;
   AL1(/2,1/):=2*(3-L**2);
   AL1(/2,2/):=AL1(/2,4/):=(3-L**2)*DR;
   AL1(/2,3/):=-2*(3-L**2);
   AL2(/1,1/):=-6*(R2+R1);
   AL2(/1,2/):=-2*(2*R2+R1)*DR;
   AL2(/1,3/):=6*(R2+R1);
   AL2(/1,4/):=-2*(R2+2*R1)*DR;
   AL2(/2,1/):=-3*(2-L**2)*(R2+R1);
   AL2(/2,2/):=-(2-L**2)*(2*R2+R1)*DR;
   AL2(/2,3/):=3*(2-L**2)*(R2+R1);
   AL2(/2,4/):=-(2-L**2)*(R2+2*R1)*DR;
   AL3(/1,1/):=AL3(/1,2/):=AL3(/1,3/):=AL3(/1,4/):=0;
```

```
AL3(/2,1/):=6*(1-L**2)*R1*R2;
AL3(/2,2/):=(1-L**2)*(R2+2*R1)*DR*R2;
AL3(/2,3/):=-6*(1-L**2)*R1*R2;
AL3(/2,4/):=(1-L**2)*(2*R2+R1)*DR*R1;
    'IF'L=0'THEN'
    'BEGIN'
      FORMKS(BL,1,NU,AL1,AL1);
      FORMKS(BL,2,NU,AL1,AL2);
      FORMKS(BL,3,NU,AL1,AL3);
      FORMKS(BL,4,NU,AL2,AL1);
      FORMKS(BL,5,NU,AL2,AL2);
      FORMKS(BL,6,NU,AL2,AL3);
      FORMKS(BL,7,NU,AL3,AL1);
      FORMKS(BL,8,NU,AL3,AL2);
      FORMKS(BL,9,NU,AL3,AL3);
'FOR'I:=1'STEP'1'UNTIL'4'DO'
'FOR'J:=1'STEP'1'UNTIL'4'DO'
     'BEGIN'
      ESTFL(/I,J/):=6.284*D*(((R2**4-R1**4)*BL(/I,J,1/)/4)+
           ((R2**3-R1**3)*(BL(/I,J,2/)+BL(/I,J,4/))/3)+
           ((R2**2-R1**2)*(BL(/I,J,3/)+BL(/I,J,5/)+BL(/I,J,7/))/2)+
           (DR*(BL(/I,J,6/)+BL(/I,J,8/)))+
           (LN(RATR)*BL(/I,J,9/)))/DR6;
     'END';
                          'FOR'J:=1'STEP'1'UNTIL'4'DO'
                          'BEGIN'
                             'FOR'I:=1'STEP'1'UNTIL'2'DO'
                     AL(/I,J/):=6*D*(AL1(/I,J/)*R1+AL2(/I,J/)+AL3(/I,J/)
                                /R1)/(T(/K/)**2*DR**3);
                     ELCAL(/K,1,J/):=AL(/1,J/)+NU*AL(/2,J/);
                     ELCAL(/K,2,J/):=NU*AL(/1,J/)+AL(/2,J/);
                     ELCAL(/K,3,J/):=0;
                             'FOR'I:=1'STEP'1'UNTIL'2'DO'
                     AL(/I,J/):=6*D*(AL1(/I,J/)*R2+AL2(/I,J/)+AL3(/I,J/)
                                /R2)/(T(/K/)**2*DR**3);
                     ELCAL(/K,4,J/):=AL(/1,J/)+NU*AL(/2,J/);
                     ELCAL(/K,5,J/):=NU*AL(/1,J/)+AL(/2,J/);
                     ELCAL(/K,6,J/):=0;
                           'END';
                           'GOTO' L4;
    'END';
AL1(/3,1/):=-8*L;          AL1(/3,3/):=8*L;
AL1(/3,2/):=AL1(/3,4/):=-4*L*DR;
AL2(/3,1/):= 6*L*(R2+R1);
AL2(/3,2/):=2*L*(2*R2+R1)*DR;
AL2(/3,3/):= -6*L*(R2+R1);
AL2(/3,4/):=2*L*(R2+2*R1)*DR;
AL3(/3,1/):=AL3(/3,2/):=AL3(/3,3/):=AL3(/3,4/):=0;
AL4(/1,1/):=AL4(/1,2/):=AL4(/1,3/):=AL4(/1,4/):=0;
AL4(/2,1/):=-(L**2)*(R2-3*R1)*R2**2;
AL4(/2,2/):=(L**2)*DR*R1*R2**2;
AL4(/2,3/):=-(L**2)*(3*R2-R1)*R1**2;
AL4(/2,4/):=(L**2)*DR*R2*R1**2;
AL4(/3,1/):=2*L*(R2-3*R1)*R2**2;
AL4(/3,2/):=-2*L*DR*R1*R2**2;
AL4(/3,3/):=2*L*(3*R2-R1)*R1**2;
AL4(/3,4/):=-2*L*DR*R2*R1**2;
```

```
        FORMKA(BL,1,NU,AL1,AL1);
        FORMKA(BL,2,NU,AL1,AL2);
        FORMKA(BL,3,NU,AL1,AL3);
        FORMKA(BL,4,NU,AL1,AL4);
        FORMKA(BL,5,NU,AL2,AL1);
        FORMKA(BL,6,NU,AL2,AL2);
        FORMKA(BL,7,NU,AL2,AL3);
        FORMKA(BL,8,NU,AL2,AL4);
        FORMKA(BL,9,NU,AL3,AL1);
        FORMKA(BL,10,NU,AL3,AL2);
        FORMKA(BL,11,NU,AL3,AL3);
        FORMKA(BL,12,NU,AL3,AL4);
        FORMKA(BL,13,NU,AL4,AL1);
        FORMKA(BL,14,NU,AL4,AL2);
        FORMKA(BL,15,NU,AL4,AL3);
        FORMKA(BL,16,NU,AL4,AL4);
    'FOR'I:=1'STEP'1'UNTIL'4'DO'
    'FOR'J:=1'STEP'1'UNTIL'4'DO'
        'BEGIN'
        ESTFL(/I,J/):=3.142*D*(((R2**4-R1**4)*BL(/I,J,1/)/4)+
            ((R2**3-R1**3)*(BL(/I,J,2/)+BL(/I,J,5/))/3)+
            ((R2**2-R1**2)*(BL(/I,J,3/)+BL(/I,J,6/)+BL(/I,J,9/))/2)+
            (DR*(BL(/I,J,4/)+BL(/I,J,7/)+BL(/I,J,10/)+BL(/I,J,13/)))+
            (LN(RATR)*(BL(/I,J,8/)+BL(/I,J,11/)+BL(/I,J,14/)))-
            (((1/R2)-(1/R1))*(BL(/I,J,12/)+BL(/I,J,15/)))-
            (((1/(2*R2**2))-(1/(2*R1**2)))*BL(/I,J,16/)))/DR6;
        'END';
                        'FOR'J:=1'STEP'1'UNTIL'4'DO'
                        'BEGIN'
                            'FOR'I:=1'STEP'1'UNTIL'3'DO'
                AL(/I,J/):=6*D*(AL1(/I,J/)*R1+AL2(/I,J/)+AL3(/I,J/)
                            /R1+AL4(/I,J/)/R1**2)/(T(/K/)**2*DR**3);
                ELCAL(/K,1,J/):=AL(/1,J/)+NU*AL(/2,J/);
                ELCAL(/K,2,J/):=NU*AL(/1,J/)+AL(/2,J/);
                ELCAL(/K,3,J/):=(1-NU)*AL(/3,J/)/2;
                            'FOR'I:=1'STEP'1'UNTIL'3'DO'
                AL(/I,J/):=6*D*(AL1(/I,J/)*R2+AL2(/I,J/)+AL3(/I,J/)
                            /R2+AL4(/I,J/)/R2**2)/(T(/K/)**2*DR**3);
                ELCAL(/K,4,J/):=AL(/1,J/)+NU*AL(/2,J/);
                ELCAL(/K,5,J/):=NU*AL(/1,J/)+AL(/2,J/);
                ELCAL(/K,6,J/):=(1-NU)*AL(/3,J/)/2;
                            'END';
'COMMENT' ASSEMBLE ELEMENTS;
            L4:'FOR'I:=1'STEP'1'UNTIL'4'DO'
                'FOR'J:=1'STEP'1'UNTIL'4'DO'
                    'BEGIN'
                        F:=NODC(/K,I+1/);
                        G:=NODC(/K,J+1/);
                       'IF'F=0'THEN''GOTO'L3;
                       'IF'G=0'THEN''GOTO'L3;
                       'IF'F>=G'THEN'
                          'BEGIN'
                              H:=G-F+HBW-1;
                              SSTF(/F,H/):=SSTF(/F,H/)+ESTFL(/I,J/);
                          'END';
                L3:'END';
              'END' OF K LOOP;
```

```
    .'END';
'COMMENT 'SOLVE MATRIX EQUATIONS;
               CHOBANDDET(NDEGF,HBW-1,SSTF);
               CHOBANDSOL(NDEGF,HBW-1,1,SSTF,FORCE);
               WRITETEXT(30,'('NODAL_DISPLACEMENTS%%')');
              'FOR'W:=1'STEP'1'UNTIL'NNODE'DO'
              'BEGIN'
                 'IF'W=NELEM+1'THEN''GOTO'L5;
                    Y:=NODC(/W,2/);
                    Z:=NODC(/W,3/);
                 'GOTO'L6;
                  L5:Y:=NODC(/W-1,4/);
                    Z:=NODC(/W-1,5/);
              L6:'IF'Y=0'THEN'DISPL(/M,2*W-1/):=0
                       'ELSE'DISPL(/M,2*W-1/):=FORCE(/Y/);
                        WRITE(30,F1,DISPL(/M,2*W-1/));  SPACE(30,5);
                 'IF'Z=0'THEN'DISPL(/M,2*W/):=0
                       'ELSE'DISPL(/M,2*W/):=FORCE(/Z/);
                        WRITE(30,F1,DISPL(/M,2*W/));
                 NEWLINE(30,1);
              'END';
               NEWLINE(30,3);
'COMMENT' CALCULATE NODAL STRESSES;
          WRITETEXT(30,'('NODAL_STRESSES%%')');
              'FOR'K:=1'STEP'1'UNTIL'NELEM'DO'
              'BEGIN'
                   'FOR'I:=1'STEP'1'UNTIL'6'DO'
                   'BEGIN'
                    NODSTRL(/I,K/):=0;
                   'FOR'J:=1'STEP'1'UNTIL'4'DO'
                    NODSTRL(/I,K/):=NODSTRL(/I,K/)+ELCAL(/K,I,J/)*
                                        DISPL(/M,2*K-2+J/);
                    WRITE(30,F1,NODSTRL(/I,K/));  SPACE(30,5);
                   'END';
                    NEWLINE(30,1);
              'END';
               NEWLINE(30,3);
               AVSTRL(/M,1/):=NODSTRL(/1,1/);
               AVSTRL(/M,2/):=NODSTRL(/2,1/);
               AVSTRL(/M,3/):=NODSTRL(/3,1/);
              'FOR'I:=1'STEP'1'UNTIL'3'DO'
              'FOR'J:=1'STEP'1'UNTIL'NELEM-1'DO'
              AVSTRL(/M,3*J+I/):=(NODSTRL(/I+3,J/)+NODSTRL(/I,J+1/))/2;
               AVSTRL(/M,3*NNODE-2/):=NODSTRL(/4,NELEM/);
               AVSTRL(/M,3*NNODE-1/):=NODSTRL(/5,NELEM/);
               AVSTRL(/M,3*NNODE/):=NODSTRL(/6,NELEM/);
              'FOR'I:=1'STEP'3'UNTIL'3*NNODE-2'DO'
              'BEGIN'
                 WRITE(30,F1,AVSTRL(/M,I/));       SPACE(30,5);
                 WRITE(30,F1,AVSTRL(/M,I+1/));     SPACE(30,5);
                 WRITE(30,F1,AVSTRL(/M,I+2/));     NEWLINE(30,1);
              'END';
               NEWLINE(30,3);
      'END' OF M LOOP;
'COMMENT' CALCULATE AND OUTPUT THE DISPL. AND STRESS FIELD;
       WRITETEXT(30,'('DISPLACEMENT_AND_STRESS_FIELD%%')');
    'FOR'I:=1'STEP'1'UNTIL'NANGL'DO'
```

```
      THETA(/I/):=READ(60);
      'FOR'W:=1'STEP'1'UNTIL'NNODE'DO'
      'FOR'J:=1'STEP'1'UNTIL'NANGL'DO'
      'BEGIN'
          DEFL(/W,J/):=0;
          RSLOPE(/W,J/):=0;
          TSLOPE(/W,J/):=0;
          RSTR(/W,J/):=0;
          TSTR(/W,J/):=0;
          SSTR(/W,J/):=0;
      'END';
      'FOR'W:=1'STEP'1'UNTIL'NNODE'DO'
      'BEGIN'
          WRITETEXT(30,'('NODAL„RADIUS(M)„„„')');
          WRITE(30,F2,NODRAD(/W/));
          NEWLINE(30,2);
         WRITETEXT(30,'('ANGLE'('10S')'DEFLECTION'('10S')'RAD.„SLOPE
                   '('9S')'TANG.„SLOPE'('9S')'RAD.„STRESS'('9S')'
                   TANG.„STRESS'('8S')'SHEAR„STRESS%
                   (DEG)'('13S')'(MM)'('15S')'(RAD)'('15S')'(RAD)'('13S
                   ')'(N/SQ.M)'('13S')'(N/SQ.M)'('12S')'(N/SQ.M)%%')');
        'FOR'J:=1'STEP'1'UNTIL'NANGL'DO'
        'BEGIN'
            WRITE(30,F4,THETA(/J/));
           'FOR'M:=1'STEP'1'UNTIL'NTERM'DO'
           'BEGIN'
              A:=TERMORD(/M/)*THETA(/J/)*3.142/180;
              DEFL(/W,J/):=DEFL(/W,J/)+DISPL(/M,2*W-1/)*COS(A)*1000;
              RSLOPE(/W,J/):=RSLOPE(/W,J/)+DISPL(/M,2*W/)*COS(A);
              TSLOPE(/W,J/):=TSLOPE(/W,J/)+DISPL(/M,2*W-1/)*SIN(A);
              RSTR(/W,J/):=RSTR(/W,J/)+AVSTRL(/M,3*W-2/)*COS(A);
              TSTR(/W,J/):=TSTR(/W,J/)+AVSTRL(/M,3*W-1/)*COS(A);
              SSTR(/W,J/):=SSTR(/W,J/)+AVSTRL(/M,3*W/)*SIN(A);
           'END';
                SPACE(30,4);   WRITE(30,F1,DEFL(/W,J/));
                SPACE(30,4);   WRITE(30,F1,RSLOPE(/W,J/));
                SPACE(30,4);   WRITE(30,F1,TSLOPE(/W,J/));
                SPACE(30,4);   WRITE(30,F1,RSTR(/W,J/));
                SPACE(30,4);   WRITE(30,F1,TSTR(/W,J/));
                SPACE(30,4);   WRITE(30,F1,SSTR(/W,J/));
                NEWLINE(30,1);
        'END';
      NEWLINE(30,2);
     'END';
  'END';
    'IF'PROBNO<NPROB'THEN''GOTO'START;
'END'
```

TYPICAL INPUT DATA FOR 'ASYMPLAT'

This data is for the plate shown in graph 5.3 using 5 elements and 21 terms

Labels on first data line (left to right): Plate N°. ; N°. of elements ; N°. of constraints ; N°. of terms ; N°. of angles

Plate N°.; N°. of elements; N°. of constraints; N°. of terms; N°. of angles	1;5;2;21;7;
Nodal radii	0.075;0.115;0.15;0.185;0.22;0.225;
Element details	69@9;0.33;4.64; 69@9;0.33;4.64; 69@9;0.33;4.64; 69@9;0.33;4.64; 69@9;0.33;4.64;
Nodal connection matrix	1;0;0;1;2; 2;1;2;3;4; 3;3;4;5;6; 4;5;6;7;8; 5;7;8;9;10;
Term order & associated nodal forces	0;0;0;0;0;0;0;0;0;0;0;1.0;0; 1;0;0;0;0;0;0;0;0;0;0;1.0;0; 2;0;0;0;0;0;0;0;0;0;0;1.0;0; 3;0;0;0;0;0;0;0;0;0;0;1.0;0; 4;0;0;0;0;0;0;0;0;0;0;1.0;0; 5;0;0;0;0;0;0;0;0;0;0;1.0;0; 6;0;0;0;0;0;0;0;0;0;0;1.0;0; 7;0;0;0;0;0;0;0;0;0;0;1.0;0; 8;0;0;0;0;0;0;0;0;0;0;1.0;0; 9;0;0;0;0;0;0;0;0;0;0;1.0;0; 10;0;0;0;0;0;0;0;0;0;0;1.0;0; 11;0;0;0;0;0;0;0;0;0;0;1.0;0; 12;0;0;0;0;0;0;0;0;0;0;1.0;0; 13;0;0;0;0;0;0;0;0;0;0;1.0;0; 14;0;0;0;0;0;0;0;0;0;0;1.0;0; 15;0;0;0;0;0;0;0;0;0;0;1.0;0; 16;0;0;0;0;0;0;0;0;0;0;1.0;0; 17;0;0;0;0;0;0;0;0;0;0;1.0;0; 18;0;0;0;0;0;0;0;0;0;0;1.0;0; 19;0;0;0;0;0;0;0;0;0;0;1.0;0; 20;0;0;0;0;0;0;0;0;0;0;1.0;0;
Values of angles at which the output is required	0.0;30.0;60.0;90.0;120.0;150.0;180.0;

6

DISCUSSION AND CONCLUSIONS

6.1 GENERAL DISCUSSION

The satisfactory development of the semi-analytic finite element program 'ASYMPLAT' has justified the selection of this technique in preference to others. The solution of a wide range of practical problems has now been made possible with a considerable degree of computational efficiency. During the development of this program however, several points have arisen which are worthy of further discussion.

The use of the conventional Ritz method has been shown to be of some value in the analysis of both plain and stiffened plates, particularly if estimates of deflection and not stresses are all that is required. The technique is of limited application, however, in that many types of plates and boundary conditions which occur in practice lead to calculations of the utmost algebraic complexity.

The conventional finite element program 'SYMPLAT' which was developed at an interim stage for the analysis of plates which are axisymmetric in geometry, material properties,and loading now stands as a useful practical program in its own right. It is comprehensively documented, easy to use and computationally efficient. Many practical problems which would present considerable analytical difficulty if tackled by classical methods may be solved to a satisfactory order of accuracy for most practical purposes using only a few elements and, since the elements are compatible, it may reasonably be expected that increasing the number of elements should at least produce a degree of convergence towards the true solution. The one possible extension to this program which may be of some limited practical use would be the evaluation of the boundary forces required in order to produce specified non-zero geometrical constraints at the plate boundaries, although it must be remembered that these non-zero constraints could not be very large or the basic assumptions of plate bending theory would be violated. This aspect of extending 'SYMPLAT' has not been investigated in depth but may be possible by the use of some form of iterative overall loop in the program based on assumed applied boundary forces and the resulting boundary displacements as the zero constraints are released.

The semi-analytic finite element program 'ASYMPLAT' has been developed to the stage where it provides a very useful contribution to the solution of problems which fall within its range of application. With the program in its present state, this range of application refers to annular or complete circular plates of axi-symmetric geometrical and material properties under the action of loads which may exhibit any form of radial variation but must at least be circumferentially symmetric with respect to some specified radial line. The program has been shown to produce satisfactory results when compared with solutions obtained by other forms of analysis and is computationally more efficient than the conventional finite element approach using annular sector elements. It is accepted however that this latter approach may have a wider range of application, for example in the analysis of plates with off-centre holes. The prediction of stresses by 'ASYMPLAT', particularly at sudden changes in plate section, requires a degree of care in interpretation, and from this point of view it is possible to argue that calculation of the stress levels at mid-element positions may have been more satisfactory than the use of the present nodal averaging technique, but this difficulty may be overcome by using elements of narrow radial width in regions of radial discontinuities of thickness. The incorporation of automatic mesh generation into the program was considered at one stage, but problems such as the one just mentioned led to the belief that the sub-division of the plate was best left to the user. The prediction of tangential stresses is generally more accurate than that of the radial stresses; a possible reason for this being the better representation of tangential moments within the elements due to the use of trigonometric circumferential shapes, whereas the cubic radial variation in shape gives a strict linear variation in radial bending moment which is obviously only an approximate statement of the truth in most cases. One source of inaccuracy in 'ASYMPLAT' was also traced to ill-conditioned terms in the generation of the element stiffness matrices. Much of the early computation was carried out on a computer using 24 bit words and employing double word length for the storage of real numbers. This was equivalent to real numbers being rounded off to eleven significant figures and was the cause of a significant build up of error as the stiffness matrices were formed and assembled. Subsequent work was all performed on a computer using a 32 bit word length (equivalent to real numbers being represented by 16 significant figures) and the rounding off error was thereby reduced to an acceptable level.

The application of 'ASYMPLAT' to the analysis of radially stiffened plates may be possible but it is doubtful whether the application will be as straightforward as was initially anticipated. The representation of the circumferential variation in displacement of a plate by the use of an infinite trigonometric series had originally raised hopes that the same type of series may also be acceptable for the ribbed plate. The problem which now arises is that of all the load forms that have been investigated, only the point load requires the use of an infinite series in its representation and, due to the de-coupling of terms in the semi-analytic approach, an infinite series form for deflection will only be generated if an infinite series form for the loading is available. A logical development plan would have as its first stage an investigation of the behaviour of radially stiffened plates under the action of axi-symmetric loads. This type of loading requires only the zero order term of the trigonometric series for its representation, which immediately precludes direct application of the technique in its present form as a ribbed plate would intuitively be expected to exhibit some form of circumferential variation in deflection. A possible means of overcoming this problem may be to 'detach' the rib from the plate and represent its presence by the introduction of internal point reactions between the rib and the plate at the nodal radii; these point reactions could then be represented by infinite series thus allowing the semi-analytic approach to proceed. This method of analysis, although more complicated than that initially envisaged, may still prove more economical than conventional finite element methods.

6.2 SUGGESTIONS FOR FURTHER WORK

(a) A very brief numerical investigation into the process of element stiffness matrix generation demonstrated the ill-conditioning of some terms resulting in the rounding-off errors discussed in section 6.1. No immediate ways of improving this situation were apparent but a deeper study of the whole process may indicate some possible re-organisation of the arithmetic processes to give greater accuracy.

(b) The program 'ASYMPLAT' in its current state only accepts loads that are axi-symmetric or circumferentially symmetric with respect to a given radial line. This is due to the fact that the program uses only the cosine terms in the infinite series used for load and deflection representation. It is a relatively straight-forward extension of the program to include sine terms in addition to the cosines and therefore to enable an analysis with fully asymmetric loads to be made. This restriction in the present program is only of relatively minor practical importance as most loads occurring in practice exhibit the form of circumferential symmetry that is catered for by use of the cosine series only.

(c) It is still thought to be possible to extend 'ASYMPLAT' to analyse radially stiffened plates along the lines indicated in section 6.1 above. This extension is not as simple as was initially anticipated but may well prove to be viable in terms of computational economy.

6.3 CONCLUSIONS

(a) The Ritz method is of limited use in the study of both stiffened and unstiffened plates where an analysis of deflection and not stress is required.

(b) The program 'SYMPLAT' provides a simple and accurate method for analysing deflection and stresses in annular and complete circular plates in cases where the plate geometry, material properties and loading are all axi-symmetric.

(c) The program 'ASYMPLAT' provides a computationally efficient analysis for deflection and stresses in annular and complete circular plates where the plate geometry and material properties are axi-symmetric but the loading is symmetric only with respect to a specified radial line.

(d) A satisfactory analysis of stiffened plates may prove possible along the lines suggested in section 6.1.

APPENDICES

A

BASIC RELATIONSHIPS FOR THE BENDING OF THIN PLATES

A1 TRANSFORMATION FROM CARTESIAN TO CYLINDRICAL CO-ORDINATES

A1.1 Why the transformation is required and the definition of co-ordinates

The basic theory of plate bending is generally derived in terms of Cartesian co-ordinates as a matter of convenience. The geometry and flexural behaviour of circular plates are, however, easier to define in terms of cylindrical co-ordinates. Basic theory is therefore derived in Cartesian co-ordinates,and then transformed into cylindrical co-ordinates for application to circular plate problems,in preference to working from first principles in cylindrical co-ordinates.

Consider the thin plate shown in figure A1.1

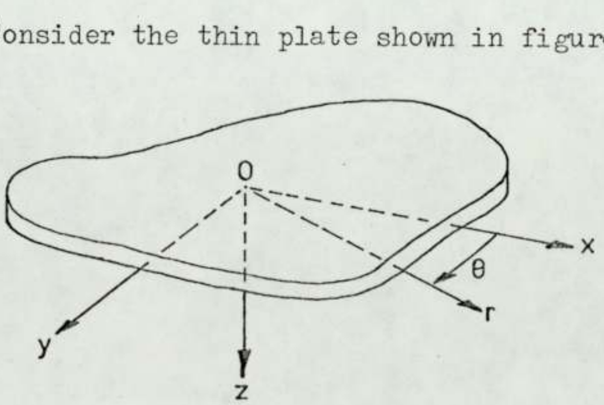


FIG. A1.1

Cartesian and cylindrical co-ordinate systems may be defined such that:

(a) they have a common origin.

(b) the x-y and r-θ planes coincide with the middle surface of the unloaded plate.

(c) the θ = 0 radius coincides with the x-axis.

Under these conditions the transformation between the two co-ordinate systems is expressed by the relationships

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \qquad \text{(A1.1a)}$$

or alternatively

$$x^2 + y^2 = r^2 \quad \text{and} \quad \frac{y}{x} = \tan\theta \qquad \text{(A1.1b)}$$

A1.2 <u>Transformation of functions and their derivatives</u>

The displacement and stress fields within a thin plate are functions of position on the plate. Any function f(x,y) can be transformed into a function f(r,θ) by direct substitution of equations (A1.1a).

Later work will show that the transformation of the first and second derivatives of functions is also required and these may be achieved as follows:

(a) <u>First derivatives.</u>

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

But from equations (A1b) and subsequently (A1a)

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

∴ by substitution

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\cos\theta - \frac{\partial f}{\partial \theta}\cdot\frac{\sin\theta}{r} \qquad \text{(A1.2a)}$$

Similarly

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\cdot\sin\theta + \frac{\partial f}{\partial \theta}\cdot\frac{\cos\theta}{r} \qquad \text{(A1.2b)}$$

(b) <u>Second derivatives.</u>

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \left[\frac{\partial^2 f}{\partial r^2}\cos\theta - \frac{\partial^2 f}{\partial r\partial\theta}\cdot\frac{\sin\theta}{r} + \frac{\partial f}{\partial\theta}\cdot\frac{\sin\theta}{r^2}\right]\cos\theta$$

$$-\left[-\frac{\partial f}{\partial r}\sin\theta + \frac{\partial^2 f}{\partial r\partial\theta}\cos\theta - \frac{\partial f}{\partial\theta}\cdot\frac{\cos\theta}{r} - \frac{\partial^2 f}{\partial\theta^2}\cdot\frac{\sin\theta}{r}\right]\frac{\sin\theta}{r}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 f}{\partial r\partial\theta}\cdot\frac{\sin\theta\cos\theta}{r} + \frac{\partial f}{\partial r}\cdot\frac{\sin^2\theta}{r} + 2\frac{\partial f}{\partial\theta}\cdot\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 f}{\partial\theta^2}\cdot\frac{\sin^2\theta}{r^2} \quad \text{(A1.2c)}$$

Similarly

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2}\sin^2\theta + 2\frac{\partial^2 f}{\partial r\partial\theta}\cdot\frac{\sin\theta\cos\theta}{r} + \frac{\partial f}{\partial r}\cdot\frac{\cos^2\theta}{r} - 2\frac{\partial f}{\partial\theta}\cdot\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 f}{\partial\theta^2}\cdot\frac{\cos^2\theta}{r^2} \quad \text{(A1.2d)}$$

And

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial r^2}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial r\partial\theta}\cdot\frac{\cos 2\theta}{r} - \frac{\partial f}{\partial r}\cdot\frac{\sin\theta\cos\theta}{r} - \frac{\partial f}{\partial\theta}\cdot\frac{\cos 2\theta}{r^2} - \frac{\partial^2 f}{\partial\theta^2}\frac{\sin\theta\cos\theta}{r^2} \quad \text{(A1.2e)}$$

Full details of the transformations are presented in references [3] and [4].

A2 BASIC THEORY OF PLATE BENDING

A2.1 Introduction and assumptions

In its most rigorous form, the theory of bending for plates of arbitrary shape, thickness and loading would necessitate the use of the complete theory of three-dimensional elasticity. The majority of cases that occur in practice neither justify nor require such a comprehensive approach, and for most engineering purposes an extension of ideas developed for the analysis of beams is adequate.

A full discussion of plate bending theory is given in references [3] and [4] ; the theory that is presented here being the framework of that which was considered to be the essential basis for work on thin plate problems. The theory is developed in terms of Cartesian co-ordinates and subsequently converted into cylindrical co-ordinates for application to circular plates by use of the transformations previously developed in Section A1.

Beam theory is based on several assumptions that may be adapted for plate analysis as follows:-

(a) The middle surface of the plate remains unstrained during bending. The restriction that this assumption imposes in practice is that the transverse deflection of the plate should not exceed its thickness, otherwise the in-plane stresses that are induced may not be exclusively due to bending.

(b) Normals to the middle surface of the plate before bending remain straight and normal afterwards. This implies that transverse shear strains in the plate are negligible; a condition that is only satisfied if the plate is thin compared with its other major dimensions.

(c) Direct stresses perpendicular to the plane of the plate are small compared with other stresses. This assumption is only valid for thin plates and even then it does not hold in close proximity to concentrated loads.

(d) The plate material must be homogeneous, isotropic and linearly elastic. This assumption is made in order to simplify the introduction of the elastic constraints.

A2.2 Compatibility between strain and displacement

Consider a plate whose middle surface was originally flat and in the x-y plane, but that is now slightly bent as shown in figure A2.1a.

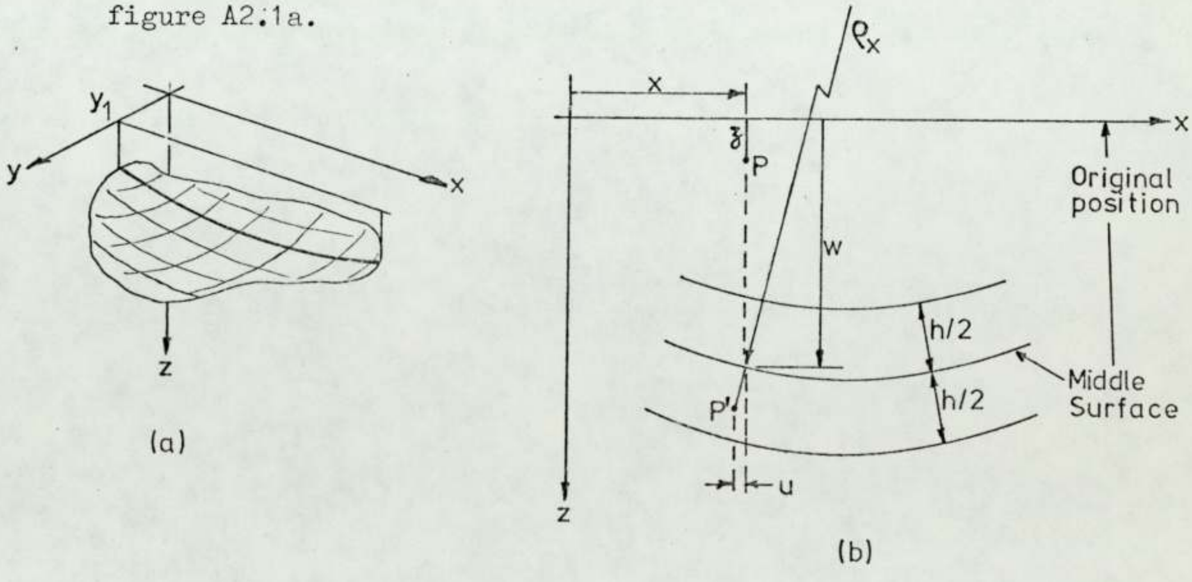


FIG. A2.1

Figure A2.1b shows a section through the plate at $y = y_1$. Provided that the displacement and slope are small, then in a plane parallel to the x-z plane it can be shown that to a first order of approximation:-

The curvature is given by $\frac{1}{\rho_x} = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -\frac{\partial^2 w}{\partial x^2}$ (A2.1a)

The displacement of point P to P$'$ in the x direction is

$$u = -z\cdot\frac{\partial w}{\partial x} \quad \text{(A2.1b)}$$

Similarly if a plane parallel to the y-z plane is considered then

The curvature is $\frac{1}{\rho_y} = -\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -\frac{\partial^2 w}{\partial y^2}$ (A2.1c)

The displacement of point P to P$'$ in the y direction is

$$v = -z\cdot\frac{\partial w}{\partial y} \quad \text{(A2.1d)}$$

Parallel to the x-y plane, a small element ABCD at distance z from the middle surface will undergo deformation to A$'$B$'$C$'$D$'$ as shown in figure A2.2

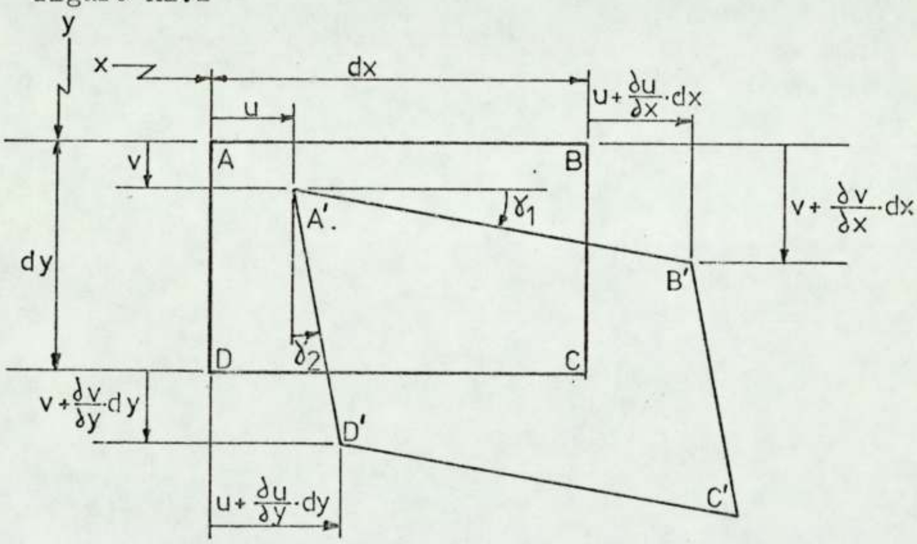


FIG. A2.2

Now direct strain $\epsilon_x = \frac{A'B' - AB}{AB} = \frac{\partial u}{\partial x}$ (A2.2a)

Similarly $\epsilon_y = \frac{A'D' - AD}{AD} = \frac{\partial v}{\partial y}$ (A2.2b)

Also the shear strain $\gamma_{xy} = \gamma_1 + \gamma_2$

But $\gamma_1 = \frac{\partial v}{\partial x}$ and $\gamma_2 = \frac{\partial u}{\partial y}$

$$\therefore \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \text{(A2.2c)}$$

The strains on this surface in terms of the transverse deflection of the surface can be obtained by combining equations (A2.1) and (A2.2) thus:-

$$\epsilon_x = \frac{\partial u}{\partial x} = -z \cdot \frac{\partial^2 w}{\partial x^2} \qquad \text{(A2.3a)}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = -z \cdot \frac{\partial^2 w}{\partial y^2} \qquad \text{(A2.3b)}$$

$$\gamma = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2z \cdot \frac{\partial^2 w}{\partial x \partial y} \qquad \text{(A2.3c)}$$

It may be noticed that putting $z = 0$ gives the condition that the middle surface is unstrained. This is in confirmation of compliance with the basic assumptions (a) and (b).

A2.3 <u>Stress-strain relationships and stress resultants</u>

In accordance with assumption (d) the general stress-strain relationships in three dimensions can be expressed in terms of two independent constants; the modulus of elasticity, E and Poissons ratio, ν. These relationships are:-

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z)$$

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \cdot \tau_{xy}$$

$$\gamma_{yz} = \frac{2(1+\nu)}{E} \cdot \tau_{yz}$$

$$\gamma_{zx} = \frac{2(1+\nu)}{E} \cdot \tau_{zx}$$

From assumptions (b) and (c), γ_{yz}, γ_{zx} and σ_z are negligible therefore these relationships may be manipulated into the following form:-

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x - \nu\epsilon_y) \qquad \text{(A2.4a)}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y - \nu\epsilon_x) \qquad \text{(A2.4b)}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)} \cdot \gamma_{xy} \qquad \text{(A2.4c)}$$

Equations (A2.4) can be expressed in terms of the deflection of the middle surface by substitution of equations (A2.3)

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \quad \text{(A2.5a)}$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \quad \text{(A2.5b)}$$

$$\tau_{xy} = -\frac{E}{1+\nu}\cdot\frac{\partial^2 w}{\partial x \partial y} \quad \text{(A2.5c)}$$

The stresses given by equations (A2.5) act on an element of plate as shown in figure A2.3a and integration of the stress resultants on the element over the plate thickness gives the forces, bending moments and twisting moments per unit length shown in figure A2.3b.

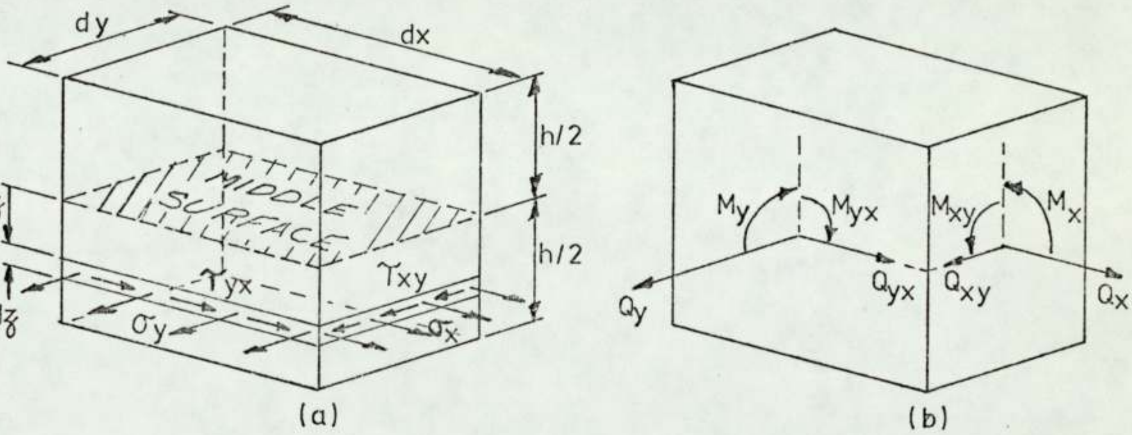


FIG. A2.3

$$M_x = \int_{-h/2}^{h/2} \sigma_x z\,dz = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \quad \text{(A2.6a)}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z\,dz = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \quad \text{(A2.6b)}$$

$$M_{yx} = M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} z\,dz = D(1-\nu)\cdot\frac{\partial^2 w}{\partial x \partial y} \quad \text{(A2.6c)}$$

$$Q_x = Q_y = Q_{xy} = Q_{yx} = 0 \quad \text{(A2.6d)}$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$

A2.4 Equilibrium conditions and the governing differential equation

The conditions for equilibrium are examined by considering a differential element acted upon by a transverse load of intensity

$p(x,y)$. The stress resultants due to this load are shown in figure A2.4. For convenience only the middle surface of the element, of thickness h, is shown.

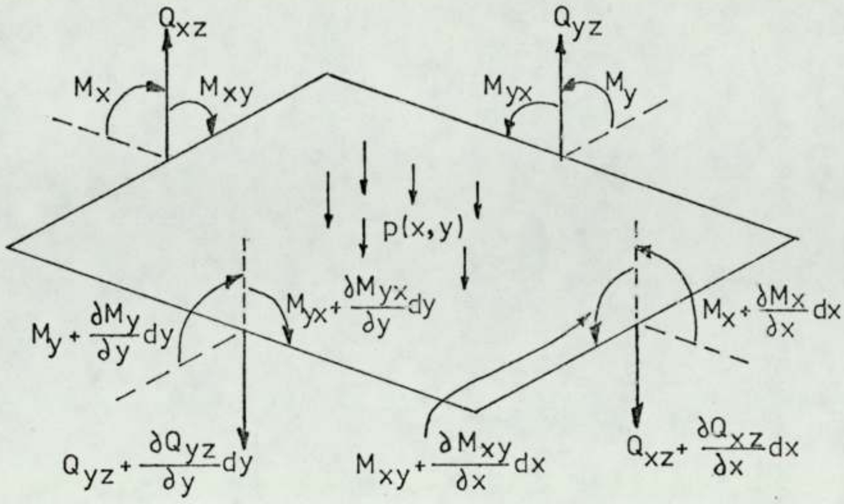


FIG. A2.4

For equilibrium of forces in the z direction

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + p = 0 \qquad \text{(A2.7a)}$$

For equilibrium of moments about an edge parallel to the x axis

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_{yz} = 0 \qquad \text{(A2.7b)}$$

For equilibrium of moments about an edge parallel to the y axis

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_{xz} = 0 \qquad \text{(A2.7c)}$$

If equations (A2.7b) and (A2.7c) are substituted in equation (A2.7a) and noting that $M_{xy} = M_{yx}$ then:-

$$\frac{\partial^2 M_x}{\partial x^2} - 2\cdot\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p \qquad \text{(A2.8)}$$

If equations (A2.6a), (A2.6b) and (A2.6c) are now substituted into equation (A2.8) the governing differential equation is produced

$$\frac{\partial^4 w}{\partial x^4} + 2\cdot\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}$$

or

$$\nabla^4 w = \frac{p}{D} \qquad \text{(A2.9)}$$

where ∇^2 is the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

It is also convenient at this stage to substitute equations (A2.6) into equations (A2.7b) and (A2.7c) in order to derive expressions for the transverse shear forces in terms of the deflection of the middle surface thus:-

$$Q_{xz} = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) = -D\frac{\partial}{\partial x}(\nabla^2 w) \qquad \text{(A2.10a)}$$

$$Q_{yz} = -D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right) = -D\frac{\partial}{\partial y}(\nabla^2 w) \qquad \text{(A2.10b)}$$

A3 STRAIN ENERGY OF A THIN PLATE IN PURE BENDING

In general terms, using indicial notation, the strain energy density U_o is given by

$$U_o = \frac{1}{2}\,\tau_{ij}\,\epsilon_{ij} \qquad \text{(A3.1)}$$

Hooke's Law for isotropic materials states that

$$\tau_{ij} = \lambda\,\delta_{ij}\,\epsilon_{kk} + 2G\epsilon_{ij} \qquad \text{(A3.2)}$$

Where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $G = \frac{E}{2(1+\nu)}$

Hence from equations (A3.1) and (A3.2)

$$U_o = \frac{1}{2}\lambda(\epsilon_{kk})^2 + G\epsilon_{ij}\epsilon_{ij}$$

which, on expansion gives

$$U_o = \frac{1}{2}\lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})^2 + G(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2) + 2G(\epsilon_{12}^2 + \epsilon_{23}^2 + \epsilon_{31}^2)$$

Consider a thin plate with its neutral surface in the x-y plane of a Cartesian co-ordinate system. If the general co-ordinate system is now aligned with the plate system and 'engineering strains' are substituted for the mathematical strains, the expression for U_o becomes

$$U_o = \frac{1}{2}\lambda(\epsilon_x + \epsilon_y + \epsilon_z)^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \qquad \text{(A3.3)}$$

Provided that transverse deflections are small, a thin transversely loaded plate may be considered as a plane stress system in which σ_z, τ_{yz}, τ_{zx}, γ_{yz} and γ_{zx} are all negligible.

It may also be shown that under plane stress conditions

$$\epsilon_z = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)$$

$\therefore$ by substitution into equation (A3.3)

$$U_o = \frac{1}{2}\lambda\left[\epsilon_x + \epsilon_y - \frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)\right]^2 + G\left[\epsilon_x^2 + \epsilon_y^2 + \frac{\nu^2}{(1-\nu)^2}(\epsilon_x^2 + \epsilon_y^2)\right] + \frac{1}{2}G\gamma_{xy}^2$$

Substituting for λ and G gives

$$U_o = \frac{E}{2(1-\nu^2)}\left[\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y\right] + \frac{E}{4(1+\nu)}\gamma_{xy}^2 \qquad \text{(A3.4)}$$

To enable the most use to be made the expression (A3.4) for U_o, it is generally more convenient to work in terms of displacements rather than strains. Plate bending theory has already shown in section A2.2 that:-

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2}\ ;\ \epsilon_y = -z\frac{\partial^2 w}{\partial y^2}\ ;\ \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\partial y}$$

Therefore:-

$$U_o = \frac{E}{2(1-\nu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\right)\right] z^2 + \frac{E}{(1+\nu)}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 z^2$$

To obtain the total strain energy for a plate, this expression for strain energy density must be integrated over the volume of the plate.

Total strain energy, $U = \int_V U_o dV = \iiint U_o\, dx\, dy\, dz = \iint\left[\int U_o dz\right] dx\, dy$

For a plate of constant thickness both z and its limits are independent of x and y

$$\therefore \int_{-h/2}^{h/2} U_o dz = \frac{Eh^3}{24(1-\nu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\right)\right] + \frac{Eh^3}{12(1+\nu)}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2$$

$$= \frac{D}{2}\left\{\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}$$

$$U = \frac{D}{2}\iint\left\{\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\} dx\, dy \qquad \text{(A3.5)}$$

A4 SUMMARY OF IMPORTANT RELATIONSHIPS FOR CIRCULAR PLATES

Sections A2 and A3 gave an abbreviated derivation of the more important parts of the basic theory for the bending of thin plates defined in a Cartesian co-ordinate system. This section now makes use of the transformation equations developed in section A1 to convert the results of the theory previously developed into equivalent expressions using a cylindrical co-ordinate system.

The details of the transformation are not presented here but the results are quoted in concise form for ease of reference.

Strains

$$\epsilon_r = -z \cdot \frac{\partial^2 w}{\partial r^2} \quad \text{(A4.1a)}$$

$$\epsilon_\theta = -z \left(\frac{1}{r} \cdot \frac{\partial w}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2}\right) \quad \text{(A4.1b)}$$

$$\gamma_{r\theta} = 2z \left(\frac{1}{r^2} \cdot \frac{\partial w}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 w}{\partial r \partial \theta}\right) \quad \text{(A4.1c)}$$

Stresses in terms of strains

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) \quad \text{(A4.2a)}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r) \quad \text{(A4.2b)}$$

$$\tau_{r\theta} = \frac{E}{2(1+\nu)} \gamma_{r\theta} \quad \text{(A4.2c)}$$

Stresses in terms of deflection of the middle surface

$$\sigma_r = -\frac{Ez}{1-\nu^2} \left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \cdot \frac{\partial w}{\partial r} + \frac{\nu}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2}\right) \quad \text{(A4.3a)}$$

$$\sigma_\theta = -\frac{Ez}{1-\nu^2} \left(\nu \cdot \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial w}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2}\right) \quad \text{(A4.3b)}$$

$$\tau_{r\theta} = \frac{Ez}{1+\nu} \left(\frac{1}{r^2} \cdot \frac{\partial w}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 w}{\partial r \partial \theta}\right) \quad \text{(A4.3c)}$$

Stress resultants in terms of deflection of the middle surface

$$M_r = -D \left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \cdot \frac{\partial w}{\partial r} + \frac{\nu}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2}\right) \quad \text{(A4.4a)}$$

$$M_\theta = -D \left(\nu \cdot \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial w}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2}\right) \quad \text{(A4.4b)}$$

$$M_{r\theta} = D(1+\nu) \left(\frac{1}{r} \cdot \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \cdot \frac{\partial w}{\partial \theta}\right) \quad \text{(A4.4c)}$$

$$Q_{rz} = -D \frac{\partial}{\partial r}(\nabla^2 w) \quad \text{(A4.4d)}$$

$$Q_{\theta z} = -D \frac{\partial}{\partial \theta}(\nabla^2 w) \quad \text{(A4.4e)}$$

Where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2}$

The governing differential equation

$$\nabla^4 w = \frac{p}{D} \quad \text{(A4.5)}$$

where p is a function of r and θ

Strain energy of the plate

$$U = \frac{D}{2} \iint \left\{ \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \cdot \frac{\partial w}{\partial r}\right]^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r} \cdot \frac{\partial w}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2}\right) - \left(\frac{1}{r} \cdot \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \cdot \frac{\partial w}{\partial \theta}\right)^2\right] \right\} r \, dr \, d\theta$$

------ (A4.6)

B

VARIATIONAL PRINCIPLES AND ENERGY METHODS

B1 INTRODUCTION

The late 17th century saw the science of mechanics developing extensively on two separate fronts.

Newton had formulated his equations of motion which were based on the study of the equilibrium of forces and moments in static systems, or on the changes in momentum of dynamic systems. Due to the vectorial nature of the quantities involved, this approach was also known as vector mechanics.

During the same period, however, Liebnitz was proposing the basic principles of analytical mechanics whereby the state of any system was defined in terms of the scalar quantities of work and energy. The ideas of Liebnitz were extended very considerably in the 18th century by Euler and Lagrange who developed the calculus of variations. The application of these variational principles to the basic concepts of work and energy led to the mathematically elegant theories of analytical mechanics which form the background to many modern computational methods.

The principles of analytical mechanics are very attractive in the study of the deformation of solid continua for the following reasons:

(a) the mathematics of the analytical approach are often simpler than those of the vectorial method; a complete system being described in one equation embodying a particular variational principle rather than the multiplicity of equations that are usually required in a vector mechanics formulation.

(b) forces at rigid constraints are not involved in the analysis, thus reducing the size of the problem.

(c) the use of the principles of analytical mechanics leads directly to the formation of the governing equation for the system under consideration and at the same time automatically generates the natural boundary conditions.

The general principles of analytical mechanics are discussed in detail in references [5], [6], [7], [8] and [9] whilst particular aspects pertinent to the present investigation are described in the following sections.

B2 THE PRINCIPLE OF VIRTUAL WORK

B2.1 Statement of the Principle

"If a system is acted upon by a set of conservative forces that are in equilibrium, then the work done in any virtual displacement of the system is zero".

Conservative forces are defined as forces that do not work in any closed loading cycle.

Virtual displacements are defined as infinitesimal, arbitrary displacements away from the equilibrium configuration of a system which do not violate the geometric boundary constraints on the system.

The forces are considered to have been applied before the virtual displacements are imposed and are assumed not to vary during the displacements.

B2.2 Derivation of the principle and its application to deformable solids

Figure B2.1 shows a set of concurrent forces F_1, F_2, F_3 and F_4 acting at a point O. F_R is the resultant force.

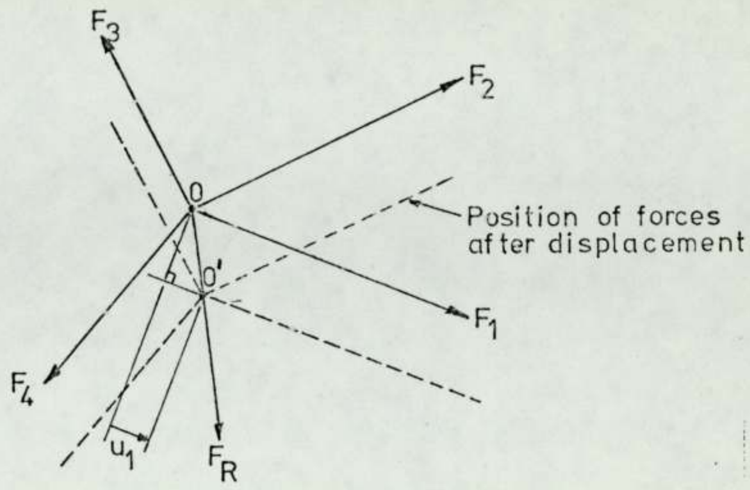


FIG. B2.1

Let point O undergo a virtual displacement, u_R along the line of action of F_R to a point O'. The distances moved along their lines of action by the forces F_1, F_2, F_3 and F_4 are respectively u_1, u_2, u_3 and u_4.

The virtual work done, δW, can be expressed as:-

$$\delta W = F_R u_R = F_1 u_1 + F_2 u_2 + F_3 u_3 + F_4 u_4$$
$$= \sum_{i=1}^{4} F_i u_i$$

Now if, and only if, the system is in equilibrium then F_R is zero.

Hence $$\delta W = \sum F_i u_i = 0 \qquad \text{(B2.1)}$$

It is important to note that neither the size nor the direction of the virtual displacement is of any consequence provided that it is a geometrically admissible displacement and that it does not seriously deform the system shape.

The principle is easily applied to systems that consist of rigid links, as discrete displacements are easily defined and the forces are simply the external loads and reactions. For a system that contains deformable solids the force system needs more careful definition and classification as follows:

(a) the externally applied loading. This may be further subdivided into surface tractions and body forces.

(b) the external reactions to the loading.

(c) the internal forces within the solid.

The concept of the internal forces can be readily understood if the deformable solid is considered to be made up of an infinite number of small ideal deformable elements interconnected at ideal mass points. As an illustration consider the solid shown in figure B2.2a which may be split into deformable elements and mass points as in figure B2.2b. (only three elements are shown for convenience).

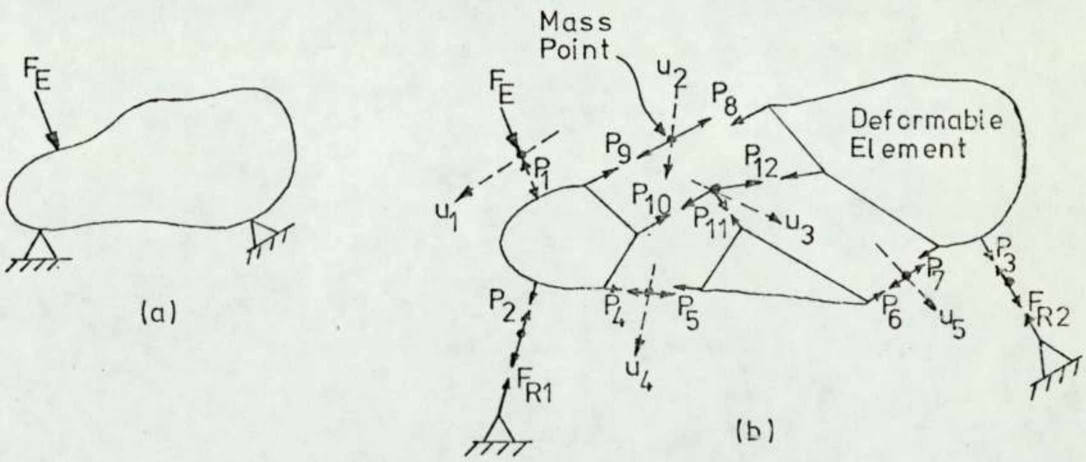


FIG. B2.2

F_E is the external load

F_{R1} and F_{R2} are the external reactions

P_1 .... P_{12} are the internal forces

u_1 .... u_5 are the virtual displacements of the mass points (note that there are no displacements at the supports).

The principle of virtual work applied to the mass points (since it is at these points that equilibrium is defined) gives

$$\delta(W_{EL} + W_{ER} + W_I) = 0$$

Where W_{EL} is the virtual work of the external loads

W_{ER} is the virtual work of the external reactions

W_I is the virtual work of the internal forces at the mass points.

It should be noted that the reason for including the effect of the forces P in the virtual work expression is that if any of the deformable elements is considered, the forces P acting on it all move through different distances and therefore give rise to a nett amount of work being done.

The external reactions do no work in the virtual displacement of a system with rigid constraints since the displacements must not violate the geometric boundary conditions. This means that W_{ER} is zero.

$$\therefore \quad \delta(W_{EL} + W_I) = 0 \tag{B2.2}$$

Provided that the solid is made of an ideal elastic material then the work done by the forces P in deforming the elements is stored as strain energy. This work done on the elements is equal but opposite to that done on the mass points.

$$\therefore \quad W_I = -U$$

Substitution in equation (B2.2) and changing signs gives

$$\delta(U - W_{EL}) = 0 \tag{B2.3}$$

In conservative systems where the external loading has a potential

$$W_{EL} = -\Omega$$

Where Ω is the potential energy of the external loading.

Therefore $\quad \delta(U + \Omega) = 0$

$(U + \Omega)$ is called the total potential energy, V, for the system.

Hence $$\delta V = \delta(U + \Omega) = 0 \tag{B2.4}$$

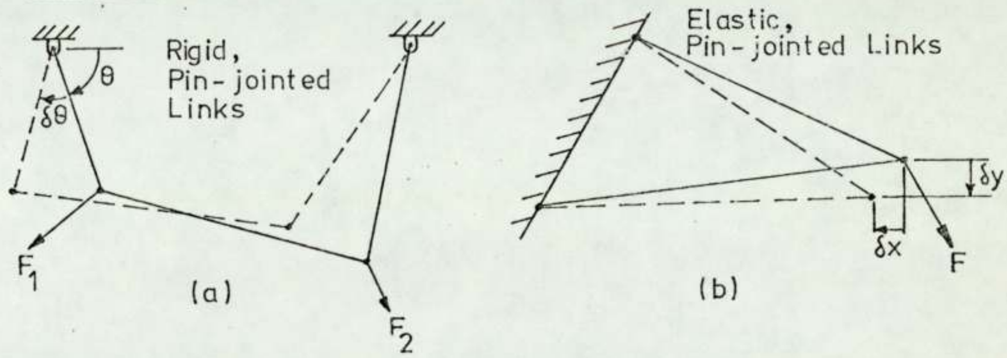
Equation (B2.4) is the mathematical statement of the principle of stationary total potential energy whereby in any virtual displacement of a system away from its equilibrium configuration, the first variation of the total potential energy is zero.

The corollary to this statement is that of all the possible virtual displacements of a system, the one which makes the total potential energy stationary is the equilibrium configuration.

It is possible to show that for stable equilibrium, the total potential energy is in fact a minimum and not just stationary.

B3 APPLICATION OF THE PRINCIPLE OF STATIONARY TOTAL POTENTIAL ENERGY

B3.1 Rigid and simple deformable systems



Dotted lines show virtual displacements

FIG. B3.1

In figure B3.1(a) the potential energy of the system contains only the contribution from the external loads as the links are rigid

$$\therefore \ \delta V = \delta\Omega = 0$$

In figure B3.1(b) the total potential energy contains terms from both the potential energy of the load and the strain energy of the links

$$\therefore \ \delta V = \delta(U + \Omega) = 0$$

In both cases however the first variation of the total potential energy of the system can be easily defined in terms of variations to the generalised co-ordinates; these variations being the virtual displacements of the system. Written mathematically this statement becomes

$$\delta V = \delta V(q_1, q_2, ---, q_n)$$

where the q are the discrete generalised co-ordinates defining the system configuration.

For the total potential energy of the system to be stationary

$$\delta V = \left(\frac{\partial V}{\partial q_1}\right)\delta q_1 + \left(\frac{\partial V}{\partial q_2}\right)\delta q_2 + ------ \left(\frac{\partial V}{\partial q_n}\right)\delta q_n = 0$$

The definition of virtual displacements insists that they are arbitrary. It therefore follows that

$$\frac{\partial V}{\partial q_1} = \frac{\partial V}{\partial q_2} = -------- = \frac{\partial V}{\partial q_n} = 0$$

In simple rigid body systems the evaluation of these partial derivatives usually leads to a direct determination of q_1, q_2, etc. For a finite degree of freedom, linear elastic system the solution is more involved and leads to a set of equations of the form

$$[K]\{q\} = \{F\}$$

The solution of this set of equations then gives the values of q_1, q_2, etc., which define the equilibrium configuration

B3.2 The variational problem in continuum mechanics

For an elastic continuum it is impossible to describe virtual displacements in terms of discrete co-ordinates. Distributed co-ordinates are required and expressions such as polynomials or trigonometric series may be used as co-ordinate functions to describe a displacement field. In such cases the coefficients of the terms of the polynomial or series become the generalised co-ordinates of the system.

In continuum problems it is not always necessary however to define the displacement field in such detail for the initial formation stage of the virtual work expressions. As an example consider the simple beam shown in figure B3.2

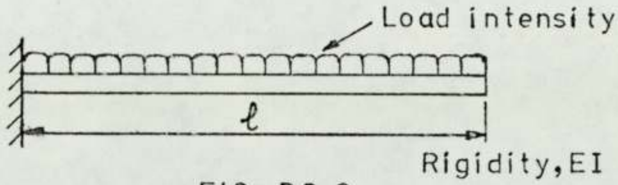


FIG. B3.2

Strain energy, $U = \frac{EI}{2}\int_0^{\ell}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$

Potential energy of the loading, $\Omega = -\int_0^{\ell} pw\,dx$

$$\therefore \quad V = U + \Omega = \int_0^{\ell}\left[\frac{EI}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - pw\right]dx$$

Up to this stage a statement that w is a function of x is all that has been required and its specification in terms of generalised co-ordinates has not been necessary.

The total potential energy of any continuum is a functional that is an integral or multiple integral, for example in the above problem

$$V = \int_0^{\ell} f\left(p, w, \frac{\partial^2 w}{\partial x^2}\right) dx$$

The principle of stationary total potential energy requires the first variation of V to be zero, which implies that an admissible form is sought for w that will extremise the functional for V. There are various ways of achieving this goal, two of which are discussed in the next section.

B4 SOLUTIONS TO THE VARIATIONAL PROBLEM FOR ELASTIC CONTINUA

B4.1 The use of variational calculus

B4.1.1 General principles of variational calculus

It has already been shown in section B3.2 that the principle of stationary total potential energy for a continuum requires the extremization of a functional that, in general, is a multiple integral. The variational calculus approach to this problem is to generate a family of continuous admissible solutions by taking the true extremizing function and adding to it a single parameter variable. The effects on the value of the functional of changes in the single parameter variable are then investigated.

The usefulness of this mathematically elegant process is that in one series of operations the ordinary or partial differential equation for equilibrium of the system is generated (the Euler-Lagrange equation) together with all of the natural boundary constraints on the system.

The Euler-Lagrange equation may, in itself, however, be extremely difficult to solve. For example in plate theory the bi-harmonic equation is generated; direction solution of which may be impossible for peculiar plate shapes or unusual boundary conditions. Powerful numerical techniques, such as the Ritz method, can however be based on the variational principles in order to produce practically acceptable approximate solutions to Euler-Lagrange equation.

The application of variational calculus to the plate bending problem is discussed in the next section, followed by a description of the Ritz method.

B4.1.2 A variational approach to the bending of circular plates

The expression for the total strain energy of a plate as given by equation (A4.6) may be written in functional form as:-

$$U = \iint f_1(r, w_r, w_\theta, w_{rr}, w_{\theta\theta}, w_{r\theta})\, dr\, d\theta$$

Where subscripts are now used to mean partial differentiation with respect to the subscript

Potential energy of the transverse loading is given by:-

$$\Omega = -\iint p\, w\, r\, dr\, d\theta \equiv -\iint f_2(p, r, w)\, dr\, d\theta$$

Hence the total potential energy of the system is

$$V = U + \Omega = \iint f(p, r, w, w_r, w_\theta, w_{rr}, w_{\theta\theta}, w_{r\theta})\, dr\, d\theta \qquad (B4.1)$$

For the plate to be in equilibrium the value of V must be stationary for any virtual displacement of the plate, provided that any such displacement does not violate the geometric constraints that are put on the plate.

Let $w(r,\theta)$ be the equilibrium displacement and let $\overline{w}(r,\theta)$ be any other admissible displacement.

Then $\overline{w}(r,\theta) = w(r,\theta) + \epsilon\eta(r,\theta)$

Where ϵ is a variable parameter that is independent of r and θ, and $\eta(r,\theta)$ is a function that (a) is zero on any boundary where geometric constraints are imposed and (b) has continuous derivatives up to third order.

For V to become stationary as $\overline{w}(r,\theta)$ approaches $w(r,\theta)$ then:-

$\frac{\partial V}{\partial \epsilon}$ must be zero when $\epsilon = 0$

$$\therefore \left.\frac{\partial V}{\partial \epsilon}\right|_{\epsilon=0} = \iint \left(f_w \eta + f_{w_r} \cdot \eta_r + f_{w_\theta} \cdot \eta_\theta + f_{w_{rr}} \cdot \eta_{rr} + f_{w_{\theta\theta}} \cdot \eta_{\theta\theta} + f_{w_{r\theta}} \cdot \eta_{r\theta} \right) dr\, d\theta = 0 \qquad (B4.2)$$

Since the values of the derivatives of η are not defined in general terms within the region of the integration, it is desirable to express equation (B4.2) in a form in which these derivatives are either not present or are only present in terms of their value at the boundary. This can be achieved by integrating each term by parts. Consider each term separately :-

$$\iint f_{w_r} \eta_r\, dr\, d\theta = \int \left\{ f_{w_r} \eta \Big|_{r_1}^{r_2} \right\} d\theta - \iint \eta \frac{\partial}{\partial r}(f_{w_r})\, dr\, d\theta \qquad (B4.3a)$$

$$\iint f_{w_\theta} \eta_\theta\, dr\, d\theta = \int \left\{ f_{w_\theta} \eta \Big|_{\theta_1}^{\theta_2} \right\} dr - \iint \eta \frac{\partial}{\partial \theta}(f_{w_\theta})\, dr\, d\theta \qquad (B4.3b)$$

$$\begin{aligned} \iint f_{w_{rr}} \eta_{rr}\, dr\, d\theta &= \int \left\{ f_{w_{rr}} \eta_r \Big|_{r_1}^{r_2} - \int \eta_r \frac{\partial}{\partial r}(f_{w_{rr}})\, dr \right\} d\theta \\ &= \int \left\{ f_{w_{rr}} \eta_r \Big|_{r_1}^{r_2} - \frac{\partial}{\partial r}(f_{w_{rr}})\eta \Big|_{r_1}^{r_2} + \int \eta \frac{\partial^2}{\partial r^2}(f_{w_{rr}})\, dr \right\} d\theta \end{aligned} \qquad (B4.3c)$$

Similarly:

$$\iint f_{w_{\theta\theta}} \eta_{\theta\theta} dr\, d\theta = \int\left\{ f_{w_{\theta\theta}} \eta_\theta \Big|_{\theta_1}^{\theta_2} - \frac{\partial}{\partial r}(f_{w_{\theta\theta}}) \eta \Big|_{\theta_1}^{\theta_2} + \int \eta \frac{\partial^2}{\partial \theta^2}(f_{w_{\theta\theta}}) d\theta \right\} dr \qquad \text{(B4.3d)}$$

And:

$$\iint f_{w_{r\theta}} \eta_{r\theta} dr\, d\theta = f_{w_{r\theta}} \eta \Big|_{r_1}^{r_2} \Big|_{\theta_1}^{\theta_2} - \int\left\{ \eta \frac{\partial}{\partial \theta}(f_{w_{r\theta}}) \Big|_{r_1}^{r_2} \right\} d\theta - \int\left\{ \eta \frac{\partial}{\partial r}(f_{w_{r\theta}}) \Big|_{\theta_1}^{\theta_2} \right\} dr + \iint \eta \frac{\partial^2}{\partial r \partial \theta}(f_{w_{r\theta}}) dr\, d\theta \qquad \text{(B4.3e)}$$

Back substitution into equation (B4.2) gives:-

$$\iint \eta \left[f_w - \frac{\partial}{\partial r}(f_{w_r}) - \frac{\partial}{\partial \theta}(f_{w_\theta}) + \frac{\partial^2}{\partial r^2}(f_{w_{rr}}) + \frac{\partial^2}{\partial \theta^2}(f_{w_{\theta\theta}}) + \frac{\partial^2}{\partial r \partial \theta}(f_{w_{r\theta}}) \right] dr\, d\theta \; +$$

$$\int\left\{ \eta \left[f_{w_\theta} - \frac{\partial}{\partial \theta}(f_{w_{\theta\theta}}) - \frac{\partial}{\partial r}(f_{w_{r\theta}}) \right]_{\theta_1}^{\theta_2} \right\} dr \; + \; \int\left\{ \eta_\theta f_{w_{\theta\theta}} \Big|_{\theta_1}^{\theta_2} \right\} dr \; +$$

$$\int\left\{ \eta \left[f_{w_r} - \frac{\partial}{\partial r}(f_{w_{rr}}) - \frac{\partial}{\partial \theta}(f_{w_{r\theta}}) \right]_{r_1}^{r_2} \right\} d\theta \; + \; \int\left\{ \eta_r f_{w_{rr}} \Big|_{r_1}^{r_2} \right\} d\theta + f_{w_{r\theta}} \eta \Big|_{r_1}^{r_2} \Big|_{\theta_1}^{\theta_2} = 0 \qquad \text{(B4.4)}$$

Since η and hence η_r and η_θ are arbitrary then each separate integral and the last term must all be zero.

Hence

(i) $f_w - \frac{\partial}{\partial r}(f_{w_r}) - \frac{\partial}{\partial \theta}(f_{w_\theta}) + \frac{\partial^2}{\partial r^2}(f_{w_{rr}}) + \frac{\partial^2}{\partial \theta^2}(f_{w_{\theta\theta}}) + \frac{\partial^2}{\partial r \partial \theta}(f_{w_{r\theta}}) = 0$ (B4.5a)

This is the Euler-Lagrange partial differential equation that must be satisfied within the region.

(ii) Along the boundaries $\theta = \theta_1$ and $\theta = \theta_2$

Either w is prescribed or $f_{w_\theta} - \frac{\partial}{\partial \theta}(f_{w_{\theta\theta}}) - \frac{\partial}{\partial r}(f_{w_{r\theta}}) = 0$ (B4.5b)

<u>And</u> either w_θ is prescribed or $f_{w_{\theta\theta}} = 0$ (B4.5c)

(iii) Along the boundaries $r = r_1$ and $r = r_2$

Either w is prescribed or $f_{w_r} - \frac{\partial}{\partial r}(f_{w_{rr}}) - \frac{\partial}{\partial \theta}(f_{w_{r\theta}}) = 0$ (B4.5d)

<u>And</u> either w_r is prescribed or $f_{w_{rr}} = 0$ (B4.5e)

(iv) The last term refers to the corners of a sectorial plate where either w is prescribed or $f_{w_{r\theta}} = 0$ (B4.5f)

Equations (B4.5b to f) define either the geometric or natural boundary conditions.

Now by expanding the expression for strain energy and combining it with the expression for the potential energy of the loading it can be shown that the functional f is of the form

$$f = \frac{D}{2}\Big[r w_{rr}^2 + \frac{2\nu}{r} w_{rr} w_{\theta\theta} + 2\nu w_{rr} w_r + \frac{1}{r^3} w_{\theta\theta}^2 + \frac{2}{r^2} w_r w_{\theta\theta} + 2(1-\nu)\frac{1}{r} w_{r\theta}^2$$

$$- 4(1-\nu)\frac{1}{r^2} w_{r\theta} w_\theta + \frac{1}{r} w_r^2 + 2(1-\nu)\frac{1}{r^3} w_\theta^2 \Big] - prw$$

Hence:-

$$f_w = - pr$$

$$f_{w_r} = \frac{D}{2}\Big[2\nu w_{rr} + \frac{2}{r^2} w_{\theta\theta} + \frac{2}{r} w_r \Big]$$

$$\therefore \frac{\partial}{\partial r}(f_{w_r}) = \frac{D}{2}\Big[2\nu w_{rrr} + \frac{2}{r^2} w_{r\theta\theta} - \frac{4}{r^3} w_{\theta\theta} + \frac{2}{r} w_{rr} + \frac{2}{r^2} w_r \Big]$$

$$f_{w_\theta} = \frac{D}{2}\Big[-4(1-\nu)\frac{1}{r^2} w_{r\theta} + 4(1-\nu)\frac{1}{r^3} w_\theta \Big]$$

$$\therefore \frac{\partial}{\partial \theta}(f_{w_\theta}) = \frac{D}{2}\Big[-4(1-\nu)\frac{1}{r^2} w_{r\theta\theta} + 4(1-\nu)\frac{1}{r^3} w_{\theta\theta} \Big]$$

$$f_{w_{rr}} = \frac{D}{2}\Big[2 r w_{rr} + \frac{2\nu}{r} w_{\theta\theta} + 2\nu w_r \Big]$$

$$\therefore \frac{\partial}{\partial r}(f_{w_{rr}}) = \frac{D}{2}\Big[2 r w_{rrr} + 2(1+\nu) w_{rr} + \frac{2\nu}{r} w_{r\theta\theta} - \frac{2\nu}{r^2} w_{\theta\theta} \Big]$$

$$\therefore \frac{\partial^2}{\partial r^2}(f_{w_{rr}}) = \frac{D}{2}\Big[2 r w_{rrrr} + 2(2+\nu) w_{rrr} + \frac{2\nu}{r} w_{rr\theta\theta} - \frac{4\nu}{r^2} w_{r\theta\theta} + \frac{4\nu}{r^3} w_{\theta\theta} \Big]$$

$$f_{w_{\theta\theta}} = \frac{D}{2}\Big[\frac{2\nu}{r} w_{rr} + \frac{2}{r^3} w_{\theta\theta} + \frac{2}{r^2} w_r \Big]$$

$$\therefore \frac{\partial^2}{\partial \theta^2}(f_{w_{\theta\theta}}) = \frac{D}{2}\Big[\frac{2\nu}{r} w_{rr\theta\theta} + \frac{2}{r^3} w_{\theta\theta\theta\theta} + \frac{2}{r^2} w_{r\theta\theta} \Big]$$

$$f_{w_{r\theta}} = \frac{D}{2}\Big[4(1-\nu)\frac{1}{r} w_{r\theta} - 4(1-\nu)\frac{1}{r^2} w_\theta \Big]$$

$$\therefore \frac{\partial^2}{\partial r \partial \theta}(f_{w_{r\theta}}) = \frac{D}{2}\Big[4(1-\nu)\frac{1}{r} w_{rr\theta\theta} - 8(1-\nu)\frac{1}{r^2} w_{r\theta\theta} + 8(1-\nu)\frac{1}{r^3} w_{\theta\theta} \Big]$$

Now by substitution, the Euler-Lagrange equation (B4.5a) becomes:-

$$- pr + \frac{D}{2}\Big[2 r w_{rrrr} + \frac{4}{r} w_{rrr} - \frac{2}{r} w_{rr} + \frac{2}{r^2} w_r + \frac{8}{r^3} w_{\theta\theta} - \frac{4}{r^2} w_{r\theta\theta} + \frac{4}{r} w_{rr\theta\theta} + \frac{2}{r^3} w_{\theta\theta\theta\theta} \Big] = 0$$

This expression can easily be shown to be equivalent to

$\nabla^4 w = \frac{p}{D}$ i.e. the differential equation for the deflected shape.

where ∇^2 is the operator $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\right)$

If use is made of equations (A4.4) relating the moments and shear forces to the derivatives of w, the boundary conditions of equations (B4.5b to f) can be shown to reduce to:-

(i) Along the boundaries $\theta = \theta_1$ and $\theta = \theta_2$

Either w is prescribed or $Q_{\theta z} - \frac{\partial}{\partial r}(M_{r\theta}) = 0$

And either $\frac{\partial w}{\partial \theta}$ is prescribed or $M_\theta = 0$

(ii) Along the boundaries $r = r_1$ and $r = r_2$

Either w is prescribed or $Q_{rz} - \frac{1}{r}\cdot\frac{\partial}{\partial \theta}(M_{r\theta}) = 0$

And either $\frac{\partial w}{\partial r}$ is prescribed or $M_r = 0$

(iii) At any corners

Either w is prescribed or $M_{r\theta} = 0$

Where in the above conditions the subscripts have reverted to their normal meaning.

B4.2 The Ritz Method

This is an approximate method for the solution of the variational problem for continua. In describing the deformed shape of a system it has already been suggested that co-ordinate functions in the form of an infinite series may be used. For some very simple structures it may be possible to carry out a complete analysis, using the principle of stationary total potential energy, and maintaining the infinite series intact. Once the analysis is complete, sufficient terms of the series are then considered in order to produce the required degree of accuracy.

For more complex problems, solutions which maintain an infinite series intact are generally not possible and only a finite number of terms may be conveniently handled.

The Ritz method shows that satisfactory, although only approximate, solutions may be obtained by using only a finite number of terms.

The basis of the method is to assume that the displacement field may be represented by a finite number of linearly independent co-ordinate functions. For example in circular plate problems

$$w(r,\theta) = \phi_0(r,\theta) + C_1\phi_1(r,\theta) + C_2\phi_2(r,\theta) + \text{-------} + C_n\phi_n(r,\theta) \qquad \text{(B4.6)}$$

where the ϕ_i are co-ordinate functions and the C_i are constants to be evaluated. The choice of the ϕ_i is arbitrary within the restriction that they must be such that $w(r,\theta)$ satisfies all of the geometric boundary conditions on the plate irrespective of the values of the constants C_i.

If equation (B4.6) is now substituted into the functional for total potential energy which, in the case of a circular plate, is given by equation (B4.1) and the integration is carried out, the functional becomes a simple function of the constants, C_i.

For the total potential energy to be stationary all that is now required is that

$$\frac{\partial V}{\partial C_i} = 0 \qquad \text{for} \quad i = 1, 2, ---, n$$

since the variational problem has been replaced by a simple maximum and minimum problem.

For a linear elastic plate material this set of partial derivatives leads to n simultaneous equations of the form

$$[K]\{C\} = \{F\}$$

The solution of these equations thus evaluates the constants, C_i.

It can be shown that if $(n + 1)$ terms are used in the description of the displacement field then the solution obtained will be better, or at least no worse, than if only n terms are used. The values of C_i are not fixed however and will be readjusted each time a different number of terms is used in order to give the best solution possible for that number of terms.

The practical application of the Ritz method may be described as a combination of art and science, as the judicious selection of the ϕ_i based on experience can result in considerable improvement in the accuracy of solution when compared with a purely arbitrary selection of the ϕ_i.

The description of the finite element method given in chapter 3 shows that it may be regarded as a piecewise application of the Ritz technique. In the finite element method an approximate displacement field is defined only within each element and not for the whole plate, and the nodal freedoms replace the constants C_i as the adjustable parameters which minimise the total potential energy functional.

C

ANALYSIS OF CLAMPED-FREE PLATES WITH CONCENTRATED EDGE LOADING

C1 INTRODUCTION

The general arrangement of the type of plate being considered is shown in figure C1.1 below:

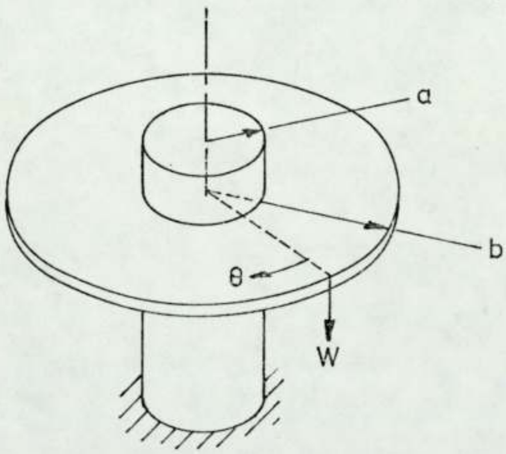


FIG. C1.1

The reasons for investigating this particular plate in detail are

(a) due to the severe conditions which are imposed in the representation of a concentrated load by a limited number of terms from an infinite series, the problem is an extreme test for the viability of the semi-analytic finite element program 'ASYMPLAT'.

(b) it is an asymmetrically loaded plate for which an 'exact' theoretical solution is possible with a relatively small amount of computational effort.

(c) it is one of the test cases used by Olson and Lindberg [22] for proving their conventional finite element approach using annular sector elements.

For the purposes of the current investigation the problem is therefore of great interest in that it is possible to obtain both 'exact' and conventional finite element solutions against which the results of the semi-analytic finite element approach may be compared.

A test rig has also been constructed with the object of providing practical results for comparison with the theoretical predictions of deflections and stresses.

C2 THEORETICAL ANALYSIS

C2.1 The general solution to the governing equation

The original analysis of the problem was by Reissner [21] but the general form of solution to problems of this type is discussed in references [3] and [4]

The governing equation for the bending of thin circular plates is given by equation (A4.5) as

$$\nabla^4 w = \frac{p}{D}$$

where $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}$$

and p is the intensity of loading on the plate (in general a function of both r and θ)

The solution to the governing equation may be assumed to be

$$w = w_h + w_p$$

where w_h is the homogeneous part, given by the solution to the equation

$$\nabla^4 w_h = 0$$

and w_p is the particular part, given by the solution to the equation

$$\nabla^4 w_p = \frac{p}{D}$$

In this particular application the distributed loading is zero, hence the solution to the governing equation is given by the homogeneous part only.

The homogeneous solution is now assumed to take the form

$$w = \sum_{n=0}^{\infty} f_n(r)\cos n\theta + \sum_{n=1}^{\infty} g_n(r)\sin n\theta$$

where the functions $f_n(r)$ and $g_n(r)$ are functions of radius only and are to be evaluated in such a way as to satisfy both the governing equation and the boundary conditions.

For this particular plate the loading and hence the resulting deflection is symmetric with respect to $\theta = 0$, consequently only the cosine terms need to be considered and the solution becomes

$$w = \sum_{n=0}^{\infty} f_n(r)\cos n\theta$$

If this assumed solution is substituted into the governing equation, the resulting equation is the fourth order, ordinary differential equation

$$\frac{d^4 f_n}{dr^4} + \left(\frac{2}{r}\right)\frac{d^3 f_n}{dr^3} - \left(\frac{1+2n^2}{r^2}\right)\frac{d^2 f_n}{dr^2} + \left(\frac{1+2n^2}{r^3}\right)\frac{d f_n}{dr} + n^2\left(\frac{n^2-4}{r^4}\right)f_n = 0$$

$$\text{for} \quad n = 0, 1, 2, ----, \infty$$

The solution to this equation may be shown to be

$$\begin{aligned} w = {} & A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \\ & + \left(A_1 r + B_1 r^3 + C_1 r^{-1} + D_1 r \ln r\right)\cos n\theta \\ & + \sum_{n=2}^{\infty}\left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right)\cos n\theta \end{aligned}$$

where the constants A_n, B_n, C_n and D_n for $n = 0, 1, ---, \infty$ must now be calculated in accordance with the boundary conditions on the plate.

C2.2 Determination of boundary conditions

Referring to figure C1.1 it is apparent that the boundary conditions at the inner radius are those of the geometrical constraints, that is

$$\text{Deflection} = w = 0 \quad \text{at} \quad r = a \qquad \text{(C2.2a)}$$

$$\text{Radial slope} = \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = a \qquad \text{(C2.2b)}$$

At the outer radius where there is a geometrically free boundary, one condition will be that of zero radial bending moment, that is

$$\text{Bending moment} = M_r = 0 \text{ at } r = b \qquad (C2.2c)$$

The remaining boundary condition at the outer radius is not immediately apparent but may be developed as follows.

In the variational formulation of the plate bending equations given in section B4.1.2 the boundary conditions along a boundary of constant radius were derived as:

either w is prescribed or $Q_{rz} - \frac{1}{r}\cdot\frac{\partial}{\partial r}(M_{r\theta}) = 0$

<u>and</u> either $\frac{\partial w}{\partial r}$ is prescribed or $M_r = 0$

At a geometrically free edge neither w nor $\frac{\partial w}{\partial r}$ are prescribed. The condition that $M_r = 0$ has previously been stated in condition (C2.2c). The other condition refers to the quantity

$$Q_{rz} - \frac{1}{r}\cdot\frac{\partial}{\partial r}(M_{r\theta})$$

This quantity may be regarded as the nett plate reaction to the applied shear loading on the edge of the plate. For an edge that is geometrically free <u>and</u> unloaded, which was the general situation under discussion in section B4.1.2, this quantity therefore became zero.

For the plate of figure C1.1 however, the outer boundary is loaded by the force W which, by reference to equation (5.6), may be represented by the Fourier series

$$\frac{W}{2\pi b} + \sum_{n=1}^{\infty}\frac{W}{\pi b}\cos n\theta$$

Hence the boundary condition may be stated as

$$\text{Nett shear load} = Q_{rz} - \frac{1}{r}\cdot\frac{\partial}{\partial r}(M_{r\theta}) = \frac{W}{2\pi b} + \sum_{n=1}^{\infty}\frac{W}{\pi b}\cos n\theta \qquad (C2.2d)$$

C2.3 <u>Imposition of boundary conditions</u>

The general solution (C2.1) may now be substituted into each of the four boundary conditions (C2.2a, b, c and d) in turn, and noting that the resulting equations must be valid for all values of θ between 0 and 2π we obtain:-

Condition (C2.2a)

$w = 0$ at $r = a$

Hence $A_0 + (a^2)B_0 + (\ln a)C_0 + (a^2 \ln a)D_0 = 0$;

$$(a)A_1 + (a^3)B_1 + (a^{-1})C_1 + (a \ln a)D_1 = 0 \;;$$

and $(a^n)A_n + (a^{-n})B_n + (a^{n+2})C_n + (a^{-n+2})D_n = 0$ for $n = 2, ---, \infty$

Condition (C2.2b)

$\frac{\partial w}{\partial r} = 0$ at $r = a$

Hence $(2a)B_0 + (a^{-1})C_0 + a(2\ln a + 1)D_0 = 0$;

$$A_1 + (3a^2)B_1 - (a^{-2})C_1 + (\ln a + 1)D_1 = 0 \;;$$

and $(na^{n-1})A_n - (na^{-n-1})B_n + (n+2)a^{n+1}C_n + (-n+2)a^{-n+1}D_n = 0$ for $n = 2, ---\infty$

Condition (C2.2c)

Equation (A4.4a) defines M_r in terms of w, r and θ. This condition therefore becomes

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}\right) = 0 \quad \text{at } r = b$$

Hence $2(1+\nu)B_0 - (1-\nu)b^{-2}C_0 + [(3+\nu) + 2(1+\nu)\ln b]D_0 = 0$;

$$2(3+\nu)bB_1 + 2(1-\nu)b^{-3}C_1 + (1+\nu)b^{-1}D_1 = 0 \;;$$

and $n(n-1)(1-\nu)b^{n-2}A_n + n(n+1)(1-\nu)b^{-n-2}B_n + (n+1)[(n+2) - \nu(n-2)]b^n C_n$
$+ (n-1)[(n-2) - \nu(n+2)]b^{-n}D_n = 0$ for $n = 2, ----, \infty$

Condition (C2.2d)

Equations (A4.4d) and (A4.4c) define Q_{rz} and $M_{r\theta}$ in terms of w, r and θ

This condition therefore becomes, at $r = b$,

$$Q_{rz} - \frac{1}{r}\cdot\frac{\partial}{\partial r}(M_{r\theta}) = -D\left(\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\cdot\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\cdot\frac{\partial w}{\partial r} - \frac{(3-\nu)}{r^3}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{(2-\nu)}{r^2}\cdot\frac{\partial^3 w}{\partial r\partial \theta^2}\right) = \frac{W}{2\pi b} + \sum_{n=1}^{\infty}\frac{W}{\pi b}\cos n\theta$$

Hence $8D_0 = -\frac{W}{\pi D}$;

$$2(3+\nu)bB_1 + 2(1-\nu)b^{-3}C_1 + (\nu - 3)b^{-1}D_1 = -\frac{W}{\pi D} \;;$$

and $n^2(1-n)(1-\nu)b^{n-2}A_n + n(1+n)(1-\nu)b^{-n-2}B_n + n(n+1)[4-n(1-\nu)]b^n C_n$

$+ n(n-1)[4+n(1-\nu)]b^{-n}D_n = -\frac{W}{\pi D}$ for $n = 2, ----, \infty$

The equations that have resulted from the imposition of the four boundary conditions may be re-grouped and expressed as follows

$$\begin{bmatrix} 1 & a^2 & \ln a & a^2\ln a \\ 0 & 2a & a^{-1} & a(2\ln a + 1) \\ 0 & 2(1+\nu) & -(1-\nu)b^{-2} & (3+\nu) + 2(1+\nu)\ln b \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{Bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\frac{W}{\pi D} \end{Bmatrix}$$

$$\begin{bmatrix} a & a^3 & a^{-1} & a\ln a \\ 1 & 3a^2 & -a^{-2} & (\ln a + 1) \\ 0 & 2(3+\nu)b & 2(1-\nu)b^{-3} & (1+\nu)b^{-1} \\ 0 & 2(3+\nu)b & 2(1-\nu)b^{-3} & (\nu-3)b^{-1} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\frac{W}{\pi D} \end{Bmatrix}$$

$$\begin{bmatrix} a^n & a^{-n} & a^{n+2} & a^{-n+2} \\ n a^{n-1} & -n a^{-n-1} & (n+2)a^{n+1} & (-n+2)a^{-n+1} \\ n(n-1)(1-\nu)b^{n-2} & n(n+1)(1-\nu)b^{-n-2} & (n+1)[(n+2)-\nu(n-2)]b^n & (n-1)[(n-2)-\nu(n+2)]b^{-n} \\ n^2(1-n)(1-\nu)b^{n-2} & n^2(1+n)(1-\nu)b^{-n-2} & n(n+1)[4-n(1-\nu)]b^n & n(n-1)[4+n(1-\nu)]b^{-n} \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\frac{W}{\pi D} \end{Bmatrix}$$

Thus the evaluation of the constants reduces to the solution of a series of groups of four simultaneous equations.

The number of groups of equations that need to be solved is governed by the number of constants that need to be evaluated in order to ensure adequate convergence of the deflection function given by equation (C2.1).

The calculation is easily adapted for computer solution as all that is basically required is a looped program which will form the coefficients of A_n, B_n, C_n and D_n, then invert the resulting (4 x 4) coefficient matrix and multiply by the right hand side column vector to evaluate A_n, B_n, C_n and D_n. This process is repeated for $n = 0, 1, --- N$, where N is the number of terms which ensures satisfactory convergence of the solution. The displacement field is then generated by feeding the coefficients into equation (C2.1), and performing the summation of the terms to give the total displacement for any desired values of r and θ.

Once the constants, and hence the displacement field, have been evaluated it is a relatively simple matter to compute the surface stresses in the plate by making use of equations (A4.3a, b and c) with $z = h/2$ to give

$$\sigma_r = -\frac{6D}{h^2}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}\right)$$

$$\sigma_\theta = -\frac{6D}{h^2}\left(\nu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}\right)$$

$$\tau_{r\theta} = \frac{6D}{h^2}\left(\frac{1}{r^2}\cdot\frac{\partial w}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial \theta}\right)$$

These equations, on substitution of the expression for w, give the stresses at any desired values of r and θ.

The computer program and its documentation are discussed in detail by Wilson [20].

C3 THE TEST RIG AND ITS INSTRUMENTATION

A photograph of the assembled test rig is shown on page 172 The majority of published theoretical work has analysed a plate of proportions b = 1.5a. From a practical point of view these proportions were not very satisfactory because insufficient space was available for the attachment of strain gauges to the plate unless the outside radius was made inconveniently large. A plate of proportions b = 3a was suitable for the attachment of the strain gauges whilst an outside diameter of 0.45 m was chosen as the maximum acceptable for ease of machining with the available lathes. The plate was made of bright rolled aluminium sheet. This material had an adequate surface finish for the attachment of strain gauges without further machining and was preferable to steel because its lower modulus of elasticity would give measurable deflections with relatively smaller loads.

The final details of the plate were therefore

Inside radius	=	0.075 m
Outside radius	=	0.225 m
Plate thickness	=	4.64 mm
Modulus of elasticity	=	69 GN/m^2
Poissons ratio	=	0.33

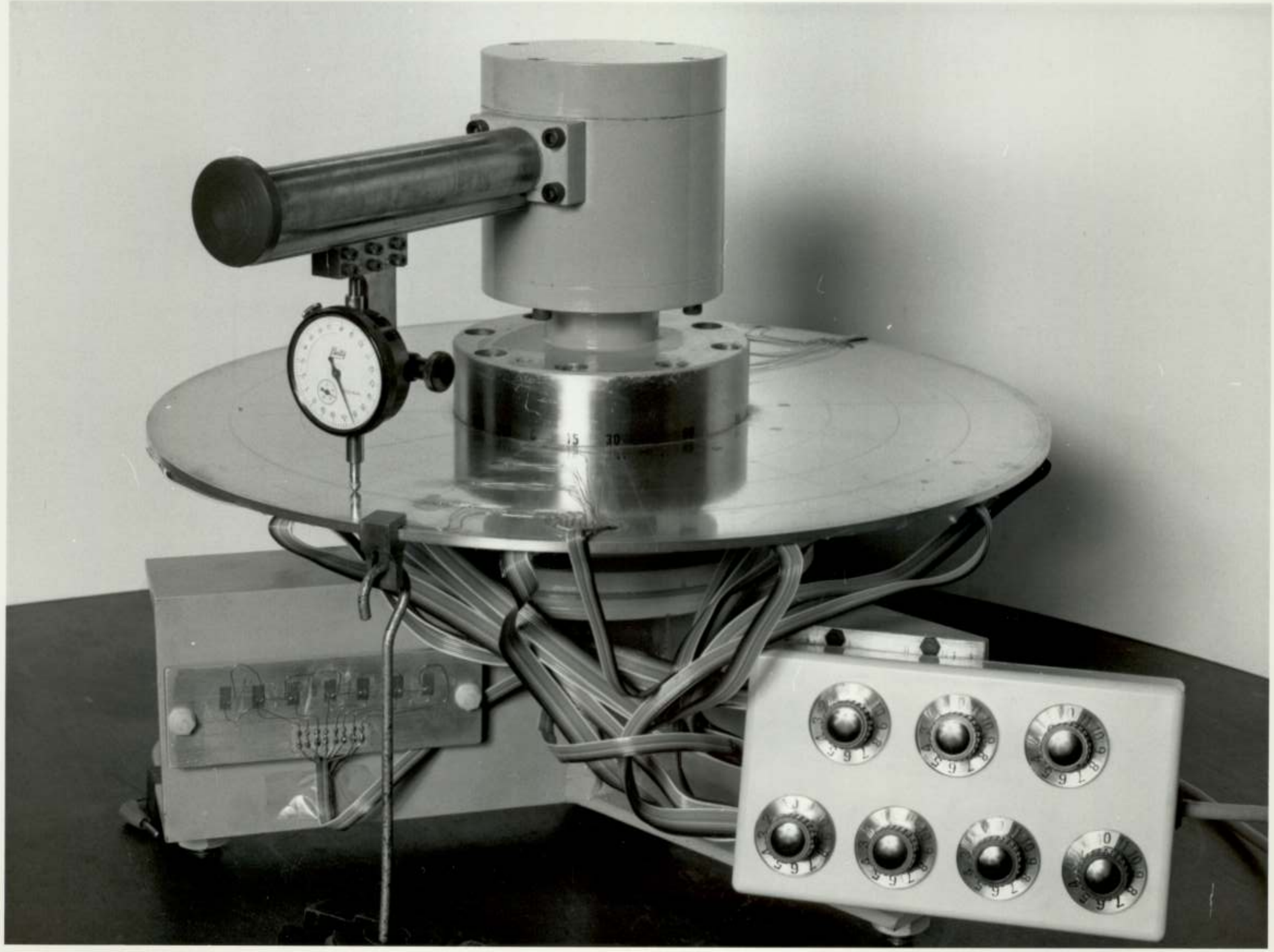
The deflection of the loaded plate was measured using a sensitive dial gauge mounted on a rotating arm. Surface strains in the plate were measured using foil gauges attached to the plate at positions shown in the photograph on page 173 and detailed in figure C3.1 on page 174. The concentrated edge loading was applied by means of weights placed on a special hanger which ensured that the load was situated as near as possible to the edge of the plate.

C4 COMPARISON OF MEASURED AND COMPUTED DEFLECTIONS AND STRESSES

Details of the measurements taken and the way in which they were processed are described by Wilson [20] . A selection of the more important measured values is tabulated in figure C4.1 on page 175 where direct comparisons may be made with the computed theoretical values. The blank spaces in the table of measured values occur wherever the readings were so small as to make their accuracy questionable.

In general it may be seen that the correlation between measured and computed values is reasonable, with typical differences being in the order of 10%. The only major differences that occur are those in the stress values at the load point, and at the inner boundary. The low value of computed stress at the load point is due to the theoretical assumption of plane stress in the plate not being valid in the region of the load point. At the inner boundary the actual degree of fixity may fall short of being perfect, which may result in a relaxation of the assumed zero slope condition and a consequent lowering of the measured stress values.

The computed values were obtained by taking a series of 30 terms which gave a degree of convergence such that the theoretical results were reliable to three significant figures.

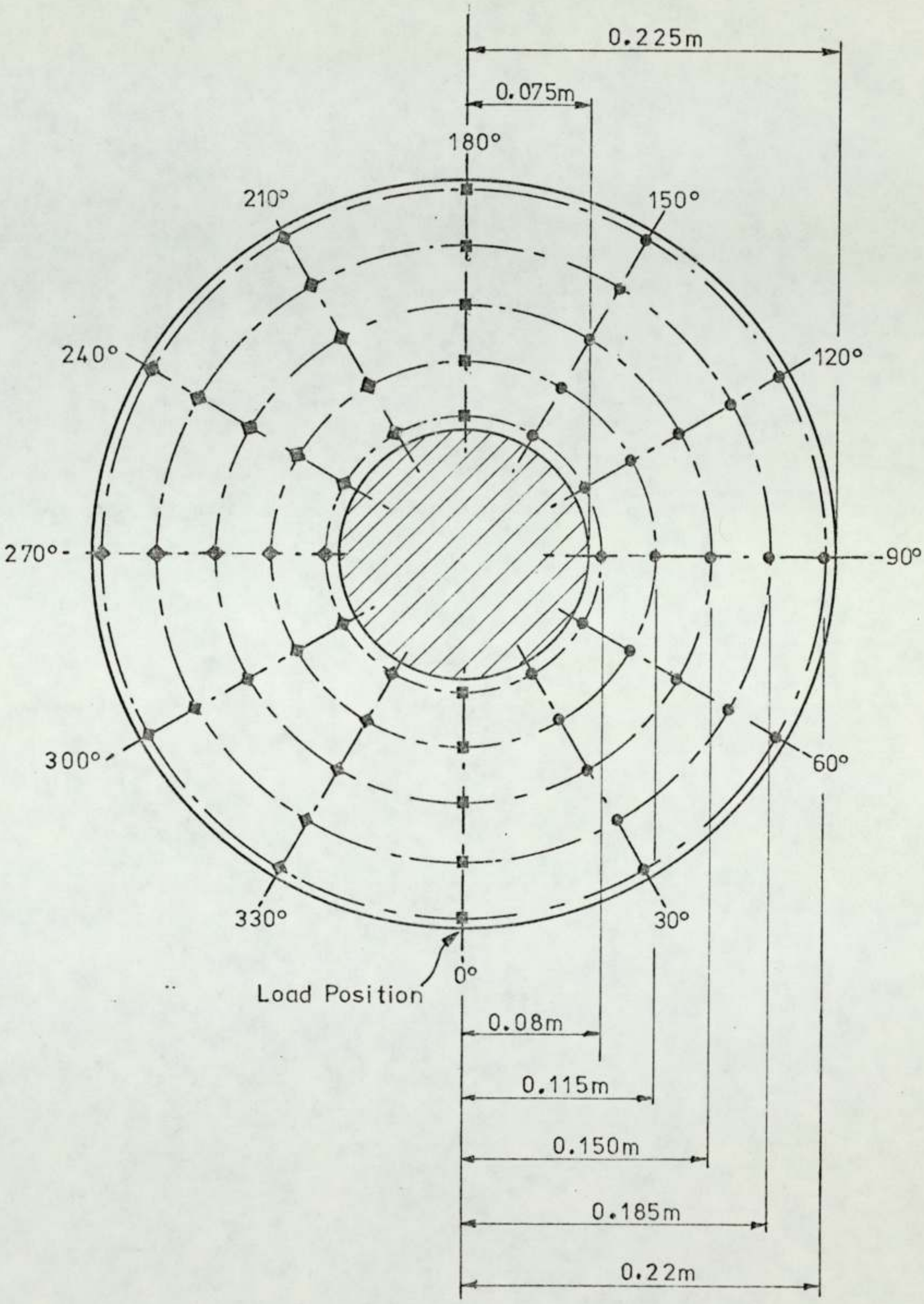


TEST RIG FOR DEFLECTION AND STRAIN MEASUREMENT ON A CLAMPED-FREE ALUMINIUM PLATE

A VIEW OF THE UNDERSIDE OF THE PLATE SHOWING THE STRAIN GAUGES

FIG. C3.1

DETAILS OF STRAIN GAUGE POSITIONS



KEY

(a) Tangential gauge on underside of plate

(b) Radial gauge on topside of plate

(c) Gauges of type (a) and (b)

Fig. C4.1

COMPARISON BETWEEN MEASURED AND COMPUTED DISPLACEMENT AND STRESS FIELDS

LOAD = 1N

RADIUS (m)	ANGLE (deg)	DEFLECTION(.001 mm)		RAD.STRESS(kN/m^2)		TANG.STRESS(kN/m^2)	
		Measured	Computed	Measured	Computed	Measured	Computed
0.220	0	+8.95	+8.41	-26.0	-14.65	+175.0	+139.7
	30	+5.05	+4.53	- 1.92	- 1.15	- 39.2	- 43.4
	60	+1.03	+0.827	- 0.25	- 0.50	- 35.2	- 34.6
	90	-0.42	-0.445	+ 0.23	+ 0.03	- 13.3	- 12.7
	120	-0.47	-0.402		- 0.10		+ 0.91
	150	-0.19	-0.109		+ 0.14		+ 2.17
	180	+0.02	+0.015		- 0.09		+ 3.49
0.185	0	+5.95	+5.41	-52.8	-52.5	+ 32.2	+ 34.0
	30	+3.75	+3.16	-18.2	-16.0	- 48.8	- 45.9
	60	+0.75	+0.63	- 6.02	- 4.65	- 41.7	- 37.5
	90	-0.27	-0.29	- 0.80	- 0.50	- 12.7	- 12.4
	120	-0.35	-0.28		- 0.48		+ 0.55
	150	-0.11	-0.08		+ 0.36		+ 3.13
	180	-0.02	+0.005		+ 0.19		+ 2.98
0.150	0	+3.19	+2.91	-82.6	-83.1	- 17.6	- 18.2
	30	+2.06	+1.79	-40.8	-42.9	- 53.7	- 52.4
	60	+0.54	+0.38	-12.6	-12.7	- 37.0	- 37.9
	90	-0.062	-0.167	- 0.34	- 0.43	- 8.4	- 10.8
	120	-0.082	-0.168		+ 2.02		+ 1.69
	150	-0.043	-0.049		+ 1.14		+ 3.40
	180	-0.010	+0.001		+ 0.52		+ 2.88
0.115	0	+1.30	+1.01	-124.0	-134.3	- 60.6	- 64.1
	30	+0.86	+0.64	- 62.0	-86.6	- 64.0	- 63.3
	60	+0.21	+0.13	- 27.0	-25.7	- 33.0	- 32.6
	90	-0.064	-0.067	+ 1.92	+ 2.16	- 3.57	- 4.65
	120	-0.082	-0.064	+ 4.40	+ 6.17	+ 1.46	+ 4.19
	150	-0.042	-0.018		+ 2.75		+ 3.33
	180	-0.020	+0.001		+ 0.86		+ 2.05
0.080	0		+0.021	-242.0	-277.5	- 94.0	-106.3
	30		+0.013	-166.0	-175.9	- 65.0	- 68.2
	60		+0.002	- 39.0	- 33.1	- 13.2	- 13.4
	90		-0.0018	+ 17.0	+ 21.5	+ 5.6	+ 8.12
	120		-0.0014		+ 18.8		+ 7.24
	150		-0.0003		+ 4.84		+ 1.95
	180		+0.0001		- 0.82		- 0.26

D

ANALYSIS OF PLAIN AND STIFFENED PLATES USING THE RITZ METHOD

D1 DEFLECTION AND STRESS ANALYSIS OF UNSTIFFENED PLATES

The analysis of an unstiffened circular plate was made with the dual purpose of testing the efficacy of both the Ritz theoretical method and the experimental Vybak model technique in an application where a rigorous theoretical solution was already known. (Appendix E gives reasons for the selection of Vybak as a material suitable for this purpose).

The specific case considered was that of a circular, isotropic plate of constant thickness. The plate was simply supported at its circumference with a single concentrated, transverse load at its centre. The reasons for this choice were that it was considered that a simple support was easier to achieve in practice than a truly fixed edge, and that a point load is very easy to apply although it was appreciated that the theoretical analysis of stress in the vicinity of the load would be unreliable.

A diametral section through the plate, defining the co-ordinate axes is shown in figure D1.1

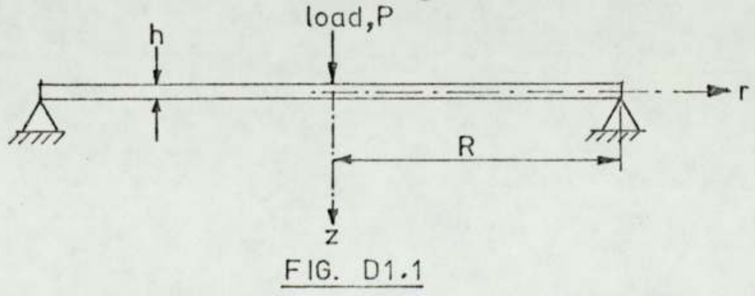


FIG. D1.1

The system is axisymmetric which implies that the transverse deflection of the plate is a function of radius only. A convenient set of co-ordinate functions to describe the deflected shape is a fourth order polynomial of the form

$$w = a + br + cr^2 + dr^3 + er^4$$

The fourth order term was anticipated to be the highest order that could be handled without prohibitive algebraic complexity.

The imposition of the necessary geometric boundary constraints gives:-

(i) the deflection is zero at the support, i.e. $w = 0$ when $r = R$

(ii) the slope is zero at the centre of the plate,

i.e. $\frac{dw}{dr} = 0$ when $r = 0$

Hence $b = 0$ and $a = -(cR^2 + dR^3 + eR^4)$

An admissible displacement function suitable for use in the Ritz method is therefore

$$w = c(r^2 - R^2) + d(r^3 - R^3) + e(r^4 - R^4) \qquad \text{(D1.1)}$$

Due to the symmetry of the system, the expression for strain energy given by equation (A4.6) may be simplified to:

$$U = \frac{D}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{d^2w}{dr^2} + \frac{1}{r}\cdot\frac{dw}{dr}\right)^2 - 2(1-\nu)\frac{1}{r}\cdot\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}\right] r\,dr\,d\theta \qquad \text{(D1.2)}$$

Substitution of equation (D1.1) into (D1.2) and evaluation of the double integral gives, after extensive algebraic complexity

$$U = \pi D\left[4(1-\nu)R^2c^2 + \frac{9(5+4\nu)R^4}{4}d^2 + \frac{16(5+3\nu)R^6}{3}e^2 + 12(1+\nu)R^3cd + \frac{24(7+5\nu)R^5}{5}de + 16(1+\nu)R^4ec\right] \qquad \text{(D1.3)}$$

The potential energy of the load, Ω, is given by

$\Omega = -P\,w_0$ where w_0 is the deflection at $r = 0$

$$\Omega = P(cR^2 + dR^3 + eR^4) \qquad \text{(D1.4)}$$

Addition of equations (D1.3) and (D1.4) gives the total potential energy of the system, V.

Where $V = V(c, d, e)$

Hence for V to be stationary

$$\frac{\partial V}{\partial c} = 0 \quad , \quad \frac{\partial V}{\partial d} = 0 \quad \text{and} \quad \frac{\partial V}{\partial e} = 0$$

If these partial derivatives are obtained and the resulting expressions re-arranged, three simultaneous equations are produced thus:-

$$\begin{bmatrix} 16(1+\nu) & \frac{24}{5}(7+5\nu)R & \frac{32}{3}(5+3\nu)R^2 \\ 12(1+\nu) & \frac{9}{2}(5+4\nu)R & \frac{24}{5}(7+5\nu)R^2 \\ 8(1+\nu) & 12(1+\nu)R & 16(1+\nu)R^2 \end{bmatrix} \begin{Bmatrix} c \\ d \\ e \end{Bmatrix} = -\frac{P}{\pi D} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Solution of these equations gives -

$$c = -\frac{(37+25\nu)P}{96(1+\nu)\pi D}$$

$$d = \frac{5P}{18\pi DR}$$

$$e = -\frac{5P}{64\pi DR^2}$$

Therefore the deflected shape is given by

$$w = \frac{PR^2}{\pi D}\left\{\frac{(37+25\nu)}{96(1+\nu)}\left[1-\left(\frac{r}{R}\right)^2\right] - \frac{5}{18}\left[1-\left(\frac{r}{R}\right)^3\right] + \frac{5}{64}\left[1-\left(\frac{r}{R}\right)^4\right]\right\} \quad \text{(D1.5)}$$

and the maximum deflection is at the centre of the plate where $r = 0$.

$$\text{Central deflection} = \frac{PR^2}{\pi D}\left[\frac{(37+25\nu)}{96(1+\nu)} + \frac{5}{18} + \frac{5}{64}\right]$$

$$= \frac{PR^2}{\pi D}\cdot\frac{(107+35\nu)}{576(1+\nu)}$$

The expression for central deflection derived by classical methods as given by Timoshenko and Woinowski - Krieger [3] is

$$\text{Central deflection} = \frac{PR^2}{\pi D}\cdot\frac{(3+\nu)}{16(1+\nu)}$$

The difference between these two results is only approximately 1.1% for typical values of Poisson's ratio encountered in practice.

A comparison between these theoretical results and practical test results is shown in graph D1.1 on page 180 which presents the load against central deflection characteristics of a 400 mm diameter 'Vybak' plate.

The stresses on the surface of the plate are determined by the use of the stress equations (A4.3a) and (A4.3b) and the substitution of the deformed shape from equation (D1.5). The stresses obtained are given by

Radial stress, $$\sigma_r = \frac{6P}{\pi h^2}\left[-\frac{1}{48}(37+25\nu) + \frac{5}{6}(2+\nu)\left(\frac{r}{R}\right) - \frac{5}{16}(3+\nu)\left(\frac{r}{R}\right)^2\right]$$

Tangential stress, $$\sigma_\theta = \frac{6P}{\pi h^2}\left[-\frac{1}{48}(37+25\nu) + \frac{5}{6}(1+2\nu)\left(\frac{r}{R}\right) - \frac{5}{16}(1+3\nu)\left(\frac{r}{R}\right)^2\right]$$

The expressions produced by the classical analysis are

$$\sigma_r = \frac{3P}{2\pi h^2}\left[(1+\nu)\ln\left(\frac{R}{r}\right)\right]$$

$$\sigma_\theta = \frac{3P}{2\pi h^2}\left[(1+\nu)\ln\left(\frac{R}{r}\right) + 1 + \nu\right]$$

These theoretical stresses, together with practically measured values for the same 'Vybak' plate, are shown in graph D1.2 on page 181.

The results show that the Ritz method is very satisfactory for the assessment of deflection but that the prediction of stresses is not so accurate. The reasons for the discrepancies in stress prediction are:

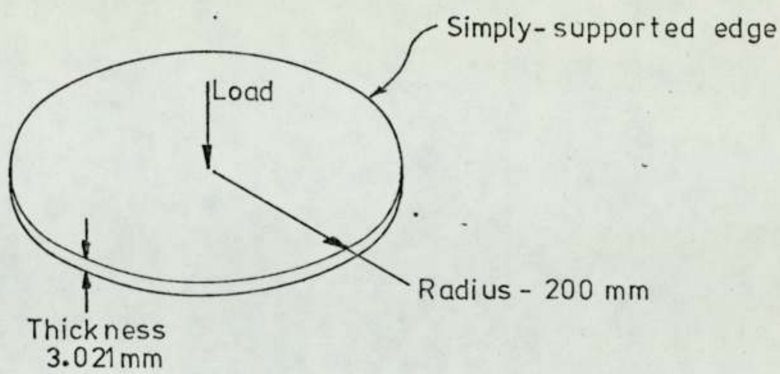
(a) the assumption that stresses normal to the plane of the plate are negligible is violated in the region of the concentrated load.

(b) the assumed deformed shaped used in the Ritz method does not satisfy the condition of zero radial bending moment at the support. This bending moment boundary condition could easily be imposed but it results in considerable algebraic difficulty.

(c) stresses are obtained by a process that involves double

GRAPH D1.1

DEFLECTION OF A SIMPLY-SUPPORTED PLATE WITH CENTRAL POINT LOAD

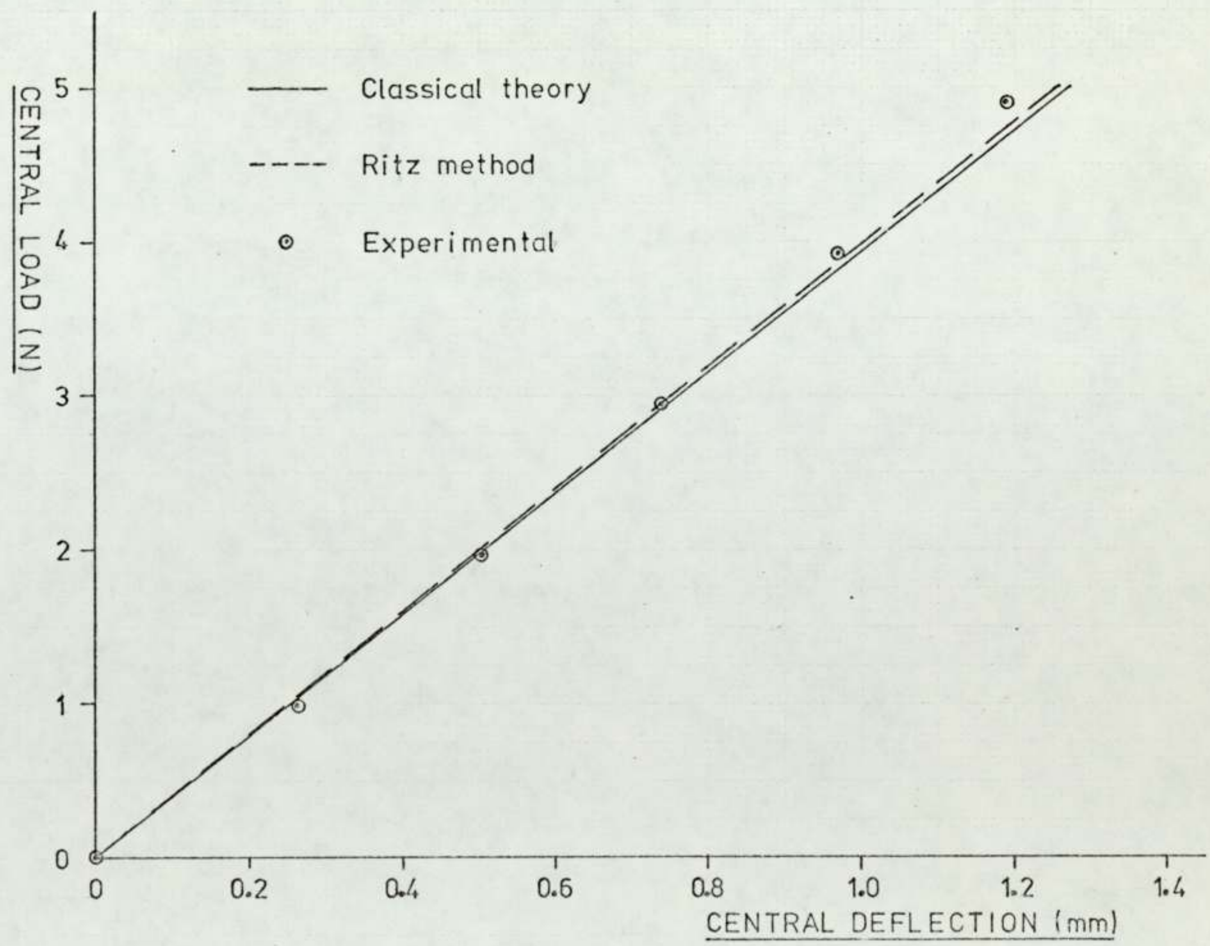


Material - Vybak

Temperature - 19.5°C

$E = 2.88\ GN/m^2$ (Temperature corrected)

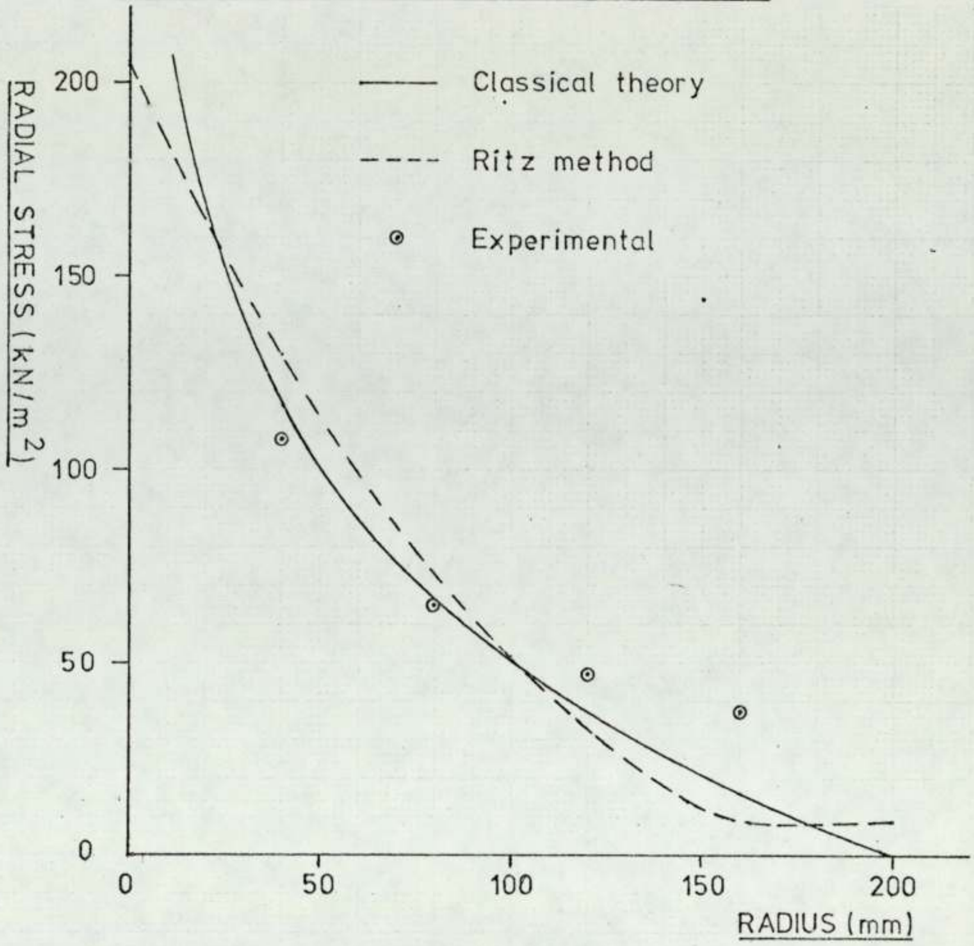
$\nu = 0.37$

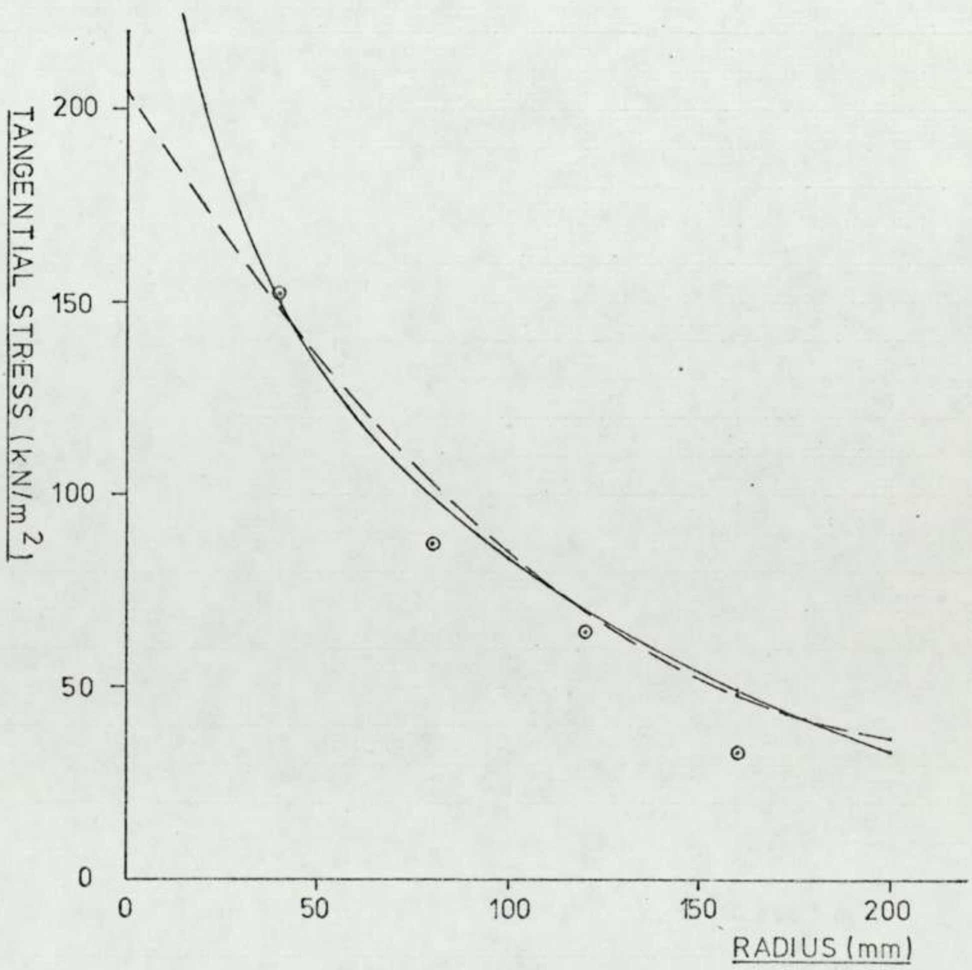


GRAPH D1.2

STRESSES IN A SIMPLY-SUPPORTED PLATE WITH CENTRAL POINT LOAD

PLATE AS IN GRAPH D1.1 WITH LOAD = 1N





differentiation of the deflection function for the plate. The deflection function is in itself only an approximation to the true displaced shape and each differentiation process tends to emphasize the difference between them, thus resulting in considerable inaccuracy of the stress field.

The experimental results give satisfactory confirmation of the central deflection but only an indication as to the trend of the stress variation. The deviation between experimental and theoretical stress values tended to be larger at the edges of the plate than in the body of the plate. This may be explained by errors in the theoretical stresses due to the reasons discussed above or to practical difficulties of ensuring that the plate was originally perfectly flat and resting evenly on the knife-edge support at its boundary.

D2 DEFLECTION AND STRESS ANALYSIS OF STIFFENED PLATES

The provision of radial stiffening on a circular plate has the effect of introducing geometric asymmetry. The displacement field has a tangential as well as a radial variation which must be allowed for in the assumed displacement function.

The simplest case of radial stiffening is that of a single rib of constant depth as shown in figure D2.1

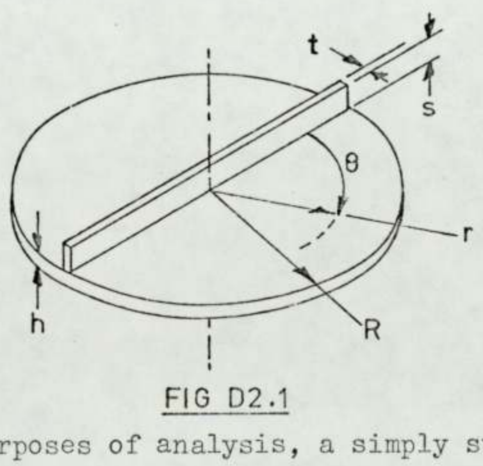


FIG D2.1

For the purposes of analysis, a simply supported plate with a central point load is considered in order that comparisons may be made with the previous analysis of the unstiffened plate.

The assumed displacement function is similar to the one used for the unstiffened plate except for the addition of a further term to allow for tangential variation in displacement. This further term is required to give tangential variation without violation of the boundary constraints previously satisfied. One of the simplest functions to fulfil these requirements is

$$f r^2(r - R)\cos 2\theta$$

The complete displacement function is therefore

$$w = c(r^2 - R^2) + d(r^3 - R^3) + e(r^4 - R^4) + f r^2(r - R)\cos 2\theta \qquad (D2.1)$$

Due to the plate asymmetry the complete expression for strain energy as given by equation (A4.6) is now required, viz.

$$U_p = \frac{D}{2}\int_0^{2\pi}\int_0^R\left\{\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r}\right]^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\cdot\frac{\partial w}{\partial r} \quad \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - \left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\cdot\frac{\partial w}{\partial \theta}\right)^2\right]\right\} r\,dr\,d\theta$$

Substitution of equation (D2.1) into equation (A4.6) and evaluation of the double integral gives the total strain energy for the plate as

$$U_p = \pi D\Big[4(1-\nu)R^2c^2 + \frac{9}{4}(5+4\nu)R^4d^2 + \frac{16}{3}(5+3\nu)R^6e^2 + \frac{1}{8}(21+4\nu)R^4f^2$$
$$+ 12(1+\nu)R^3cd + \frac{24}{5}(7+5\nu)R^5de + 16(1+\nu)R^4ce\Big] \qquad (D2.2)$$

It should be noted however that this stage is reached only after very extensive and complex algebraic manipulation.

From simple beam theory, the strain energy for the rib is given by

$$U_r = 2\left(\frac{EI}{2}\right)\int_0^R\left(\frac{\partial^2 w}{\partial r^2}\right)^2 dr$$

The deflection of the rib is given by putting $\theta = 0$ in equation D2.1 i.e.

$$w = c(r^2 - R^2) + d(r^3 - R^3) + e(r^4 - R^4) + f r^2(r - R)$$

Hence by substitution and evaluation of the integral

$$U_r = EI\Big[4Rc^2 + 12R^3d^2 + \frac{144}{5}R^5e^2 + 4R^3f^2 + 12R^2cd + 36R^4de$$
$$+ 20R^4ef + 4R^2cf + 16R^3ce + 12R^3df\Big] \qquad (D2.3)$$

The potential energy of the loading, Ω, is given by

$\Omega = -Pw_0$ where w_0 is the deflection at r = 0

Hence

$$\Omega = P(cR^2 + dR^3 + eR^4) \qquad (D2.4)$$

Addition of equations (D2.2), (D2.3) and (D2.4) gives the total potential energy of the system, V, where

V = V(c,d,e,f)

Hence for V to the stationary

$$\frac{\partial V}{\partial c} = 0 \quad , \quad \frac{\partial V}{\partial d} = 0 \quad , \quad \frac{\partial V}{\partial e} = 0 \quad \text{and} \quad \frac{\partial V}{\partial f} = 0$$

If these partial derivatives are obtained, four simultaneous equations are produced thus:-

$$\begin{bmatrix} 16\left[(1+\nu)+K\right] & 12\left[\frac{2}{5}(7+5\nu)+3K\right]R & 32\left[\frac{1}{3}(5+3\nu)+\frac{9}{5}K\right]R^2 & 20KR \\ 12\left[(1+\nu)+K\right] & 3\left[\frac{3}{2}(5+4\nu)+8K\right]R & 12\left[\frac{2}{5}(7+5\nu)+3K\right]R^2 & 12KR \\ 8\left[(1+\nu)+K\right] & 12\left[(1+\nu)+K\right]R & 16\left[(1+\nu)+K\right]R^2 & 4KR \\ 4K & 12KR & 20KR^2 & \left[\frac{1}{4}(21+4\nu)+8K\right]R \end{bmatrix} \begin{Bmatrix} c \\ d \\ e \\ f \end{Bmatrix} = -\frac{P}{\pi D} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

Where $K = \frac{EI}{\pi DR}$

The solution of this set of equations to give the constants c, d, e and f is carried out using a small computer program.

Following the evaluation of these constants, the deflection at any point on the plate can be determined from equation (D2.1).

The surface stresses may be obtained by the use of equations (A4.3a) and (A4.3b) which, after substitution of equation (D2.1) gives

$$\sigma_r = -\frac{6D}{h^2}\left\{2(1-\nu)c + 3(2+\nu)rd + 4(3+\nu)r^2e + \left[r(6-\nu) + 2R(\nu-1)\right]f\cos 2\theta\right\}$$

$$\sigma_\theta = -\frac{6D}{h^2}\left\{2(1-\nu)c + 3(1+2\nu)rd + 4(1+3\nu)r^2e + \left[r(6\nu-1) + 2R(1-\nu)\right]f\cos 2\theta\right\}$$

The numerical evaluation of these stresses was incorporated in the computer program that had been written for the calculation of the constants c, d, e and f. The program calculates the stresses for a range of radii and angles.

The computer program was written in BASIC and is suitable for use on a small computer as only a small amount of storage is required and the most complicated operation is the inversion of a 4 x 4 matrix. The program is listed on page 192 and a typical output is shown on page 194.

Experimental results were obtained from a series of tests on a simply supported 'Vybak' plate with a single diametral rib of constant depth. A drawing of the plate is shown in figure D2.2 on page 188 and a photograph of the plate under test on page 189. The plate was subjected to a range of central loads and measurements were recorded of the central deflection and the radial and tangential strains at various points on the plate. The tests were repeated after successive milling operations on the edge of the rib to give a range of rib depths.

Graph D2.1 on page 190 shows the variation of central deflection with rib depth. The curve demonstrates that the Ritz method gives a very satisfactory prediction of deflection. The discrepancy between theory and practice for small rib depths is almost certainly due to unexpected behaviour of the experimental plate as a plain plate had already given good correlation. Possible reasons for the discrepancy are:

(a) the fillet of adhesive between the rib and the plate give an additional second moment of area to the rib which was not allowed for.

(b) the adhesive may alter the properties of the material in the region of the joint.

(c) the machining operations on the rib may affect the material properties.

The effect of all these possibilities would become more pronounced as the rib depth was decreased.

The typical stress-radius curves for one particular rib depth shown in graph D2.2 on page 191 demonstrate that the Ritz method cannot be relied upon for a satisfactory prediction of stress levels.

The reasons for this unreliability are that the calculation of stress levels involves double differentiation of the deflection function. The deflection function is itself only approximate and any error is magnified in the differentiation process. The deflection function that has been used is also in error in that it has only been made to satisfy the geometric boundary constraints and not those of shear force and bending moment. This automatically implies that the calculated stress values at the boundaries may not be correct. A further error in the theoretical predictions is that bending takes place about the middle surface of the plate. This is an inaccurate assumption in the region of the rib and may seriously affect the stress predictions near the rib. The effect on the calculated displacement field is not very significant as the assumption is generally valid except for the localized violation near the ribs.

The experimental results quoted in this appendix are extracted from the extensive program of experimental work reported by Edwards [35] who made use of the expertise previously gained by Leighton [34] in the application of Vybak to stress analysis models.

D3 DISCUSSION AND CONCLUSIONS

In the case of a plain plate or a simple stiffened plate the Ritz method has been shown to give satisfactory predictions of deflections but that the subsequent stress calculations are too unreliable for practical use. The accuracy of the results is related to the choice of the assumed deflection function, but improvements in this direction proved difficult to implement as indicated below.

The method has the advantage that once a particular case has been analysed and computerised, the program is so simple that similar problems may then be solved at little cost.

The disadvantages of the method are considerable, however, and include:

(a) the unreliability of the stress calculations. In principle this may be alleviated to some extent by making the assumed deflection function conform to shear force and bending moment boundary conditions. In practice however this introduced even more complicated algebraic manipulation than that already encountered and was therefore abandoned.

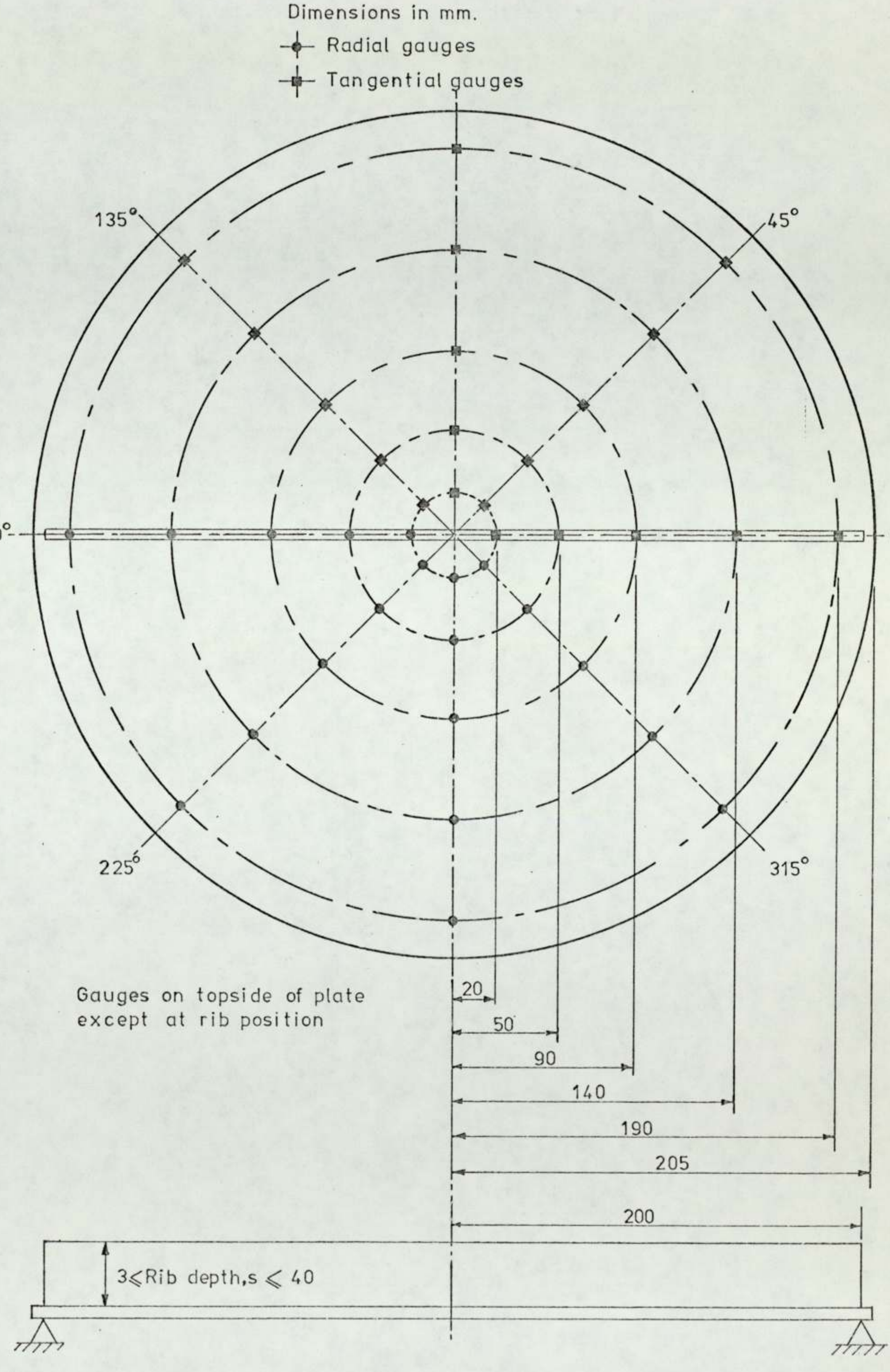
(b) radial variation in plate thickness or rib depth, which often occur in practice, is not easy to incorporate.

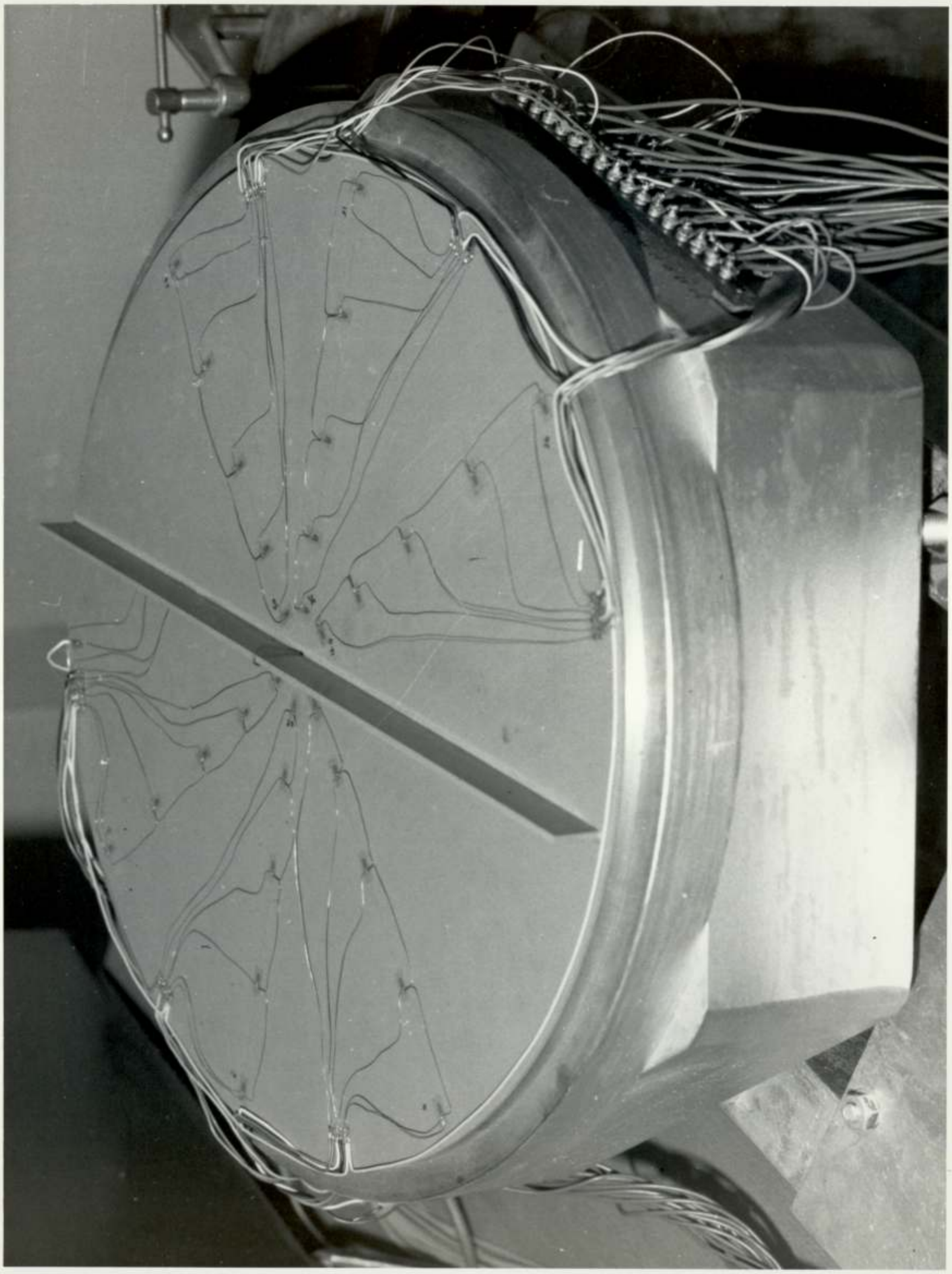
(c) multi-ribbed plates or the provision of a rigid central boss make the deflection function more complex and the consequent evaluation of the strain energy becomes a complicated procedure.

In conclusion, the method appears to be attractive for simple plates if only a deflection analysis is required. For the more complicated plate configurations that occur in practice the method becomes extremely complex in its algebra and the stress predictions become unreliable.

FIG D 2.2

DETAILS OF TEST PLATE SHOWING STRAIN GAUGE POSITIONS

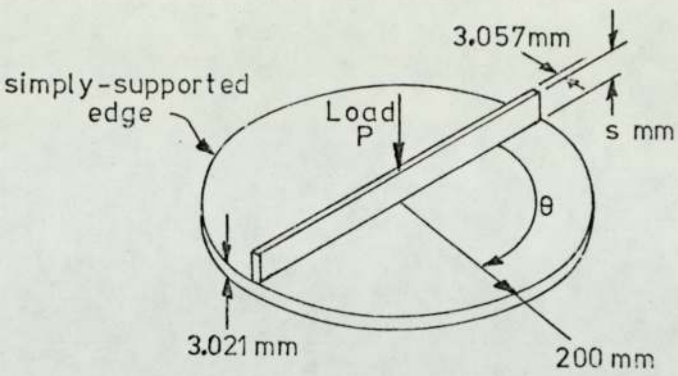




TEST RIG FOR DEFLECTION AND STRAIN MEASUREMENT ON A SIMPLY SUPPORTED, SINGLE RIBBED VYBAK PLATE

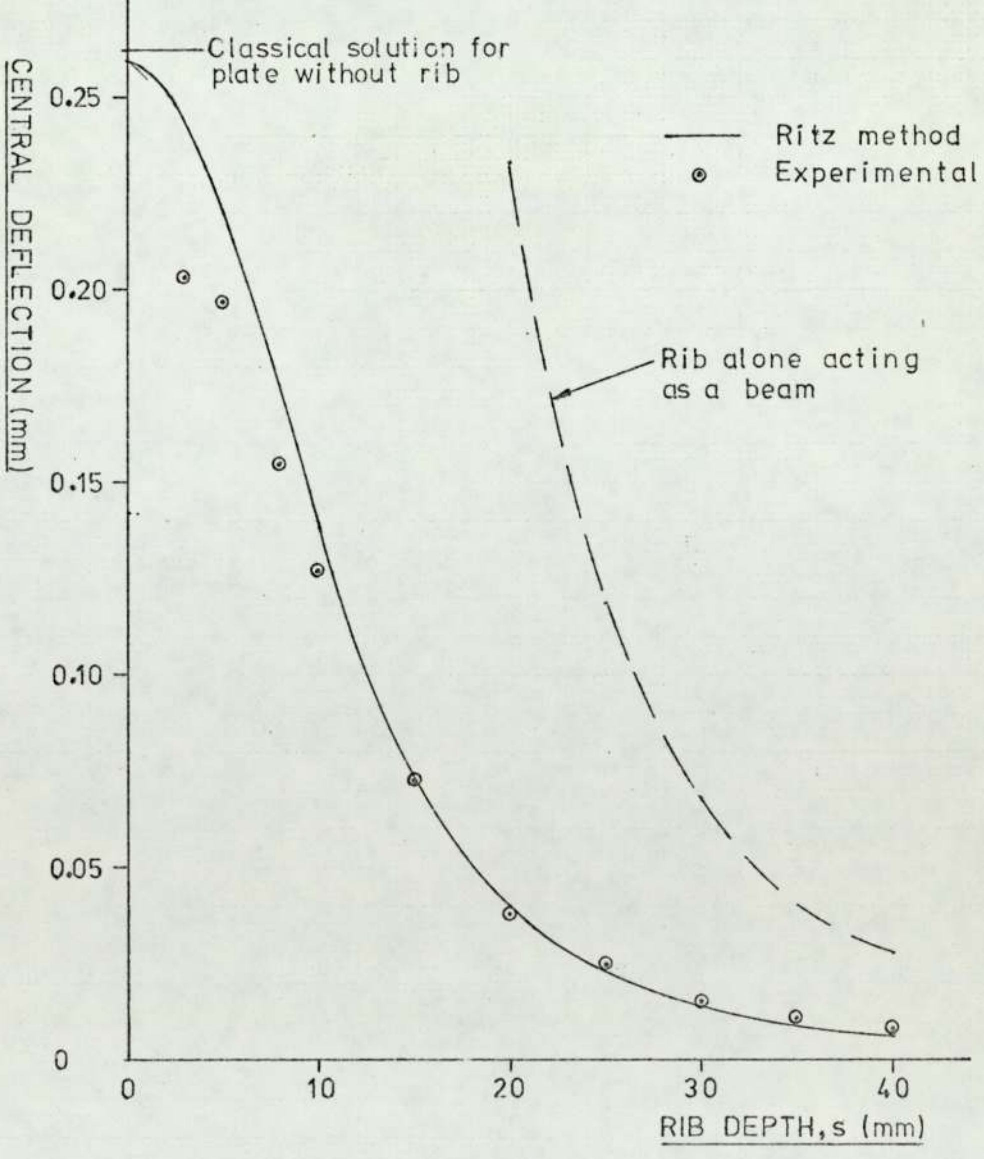
GRAPH D2.1

DEFLECTION OF A PLATE WITH A SINGLE DIAMETRAL RIB



Material - Vybak
Temperature - 23°C
$E = 2.8\ GN/m^2$
$\nu = 0.37$

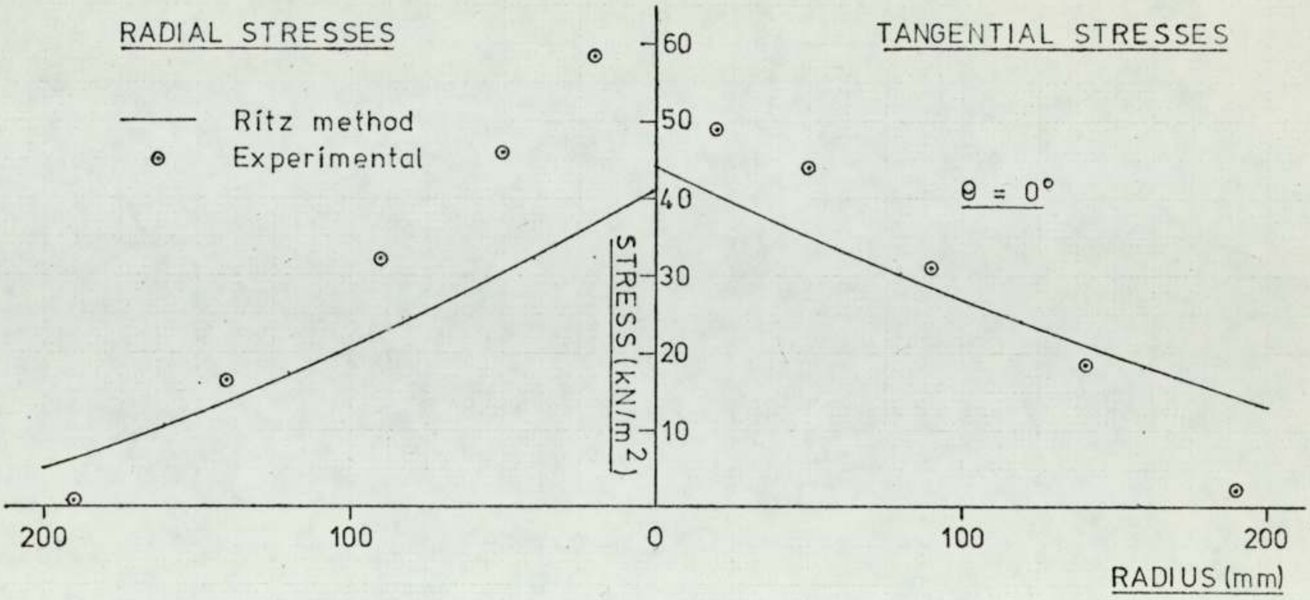
DEFLECTION FOR A CENTRAL LOAD OF 1N

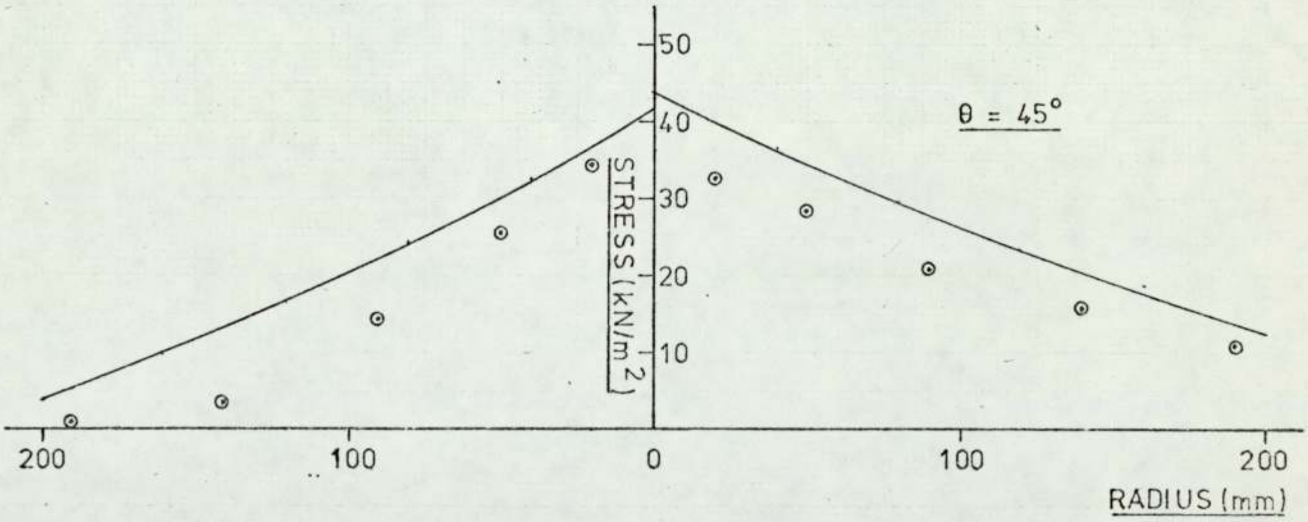


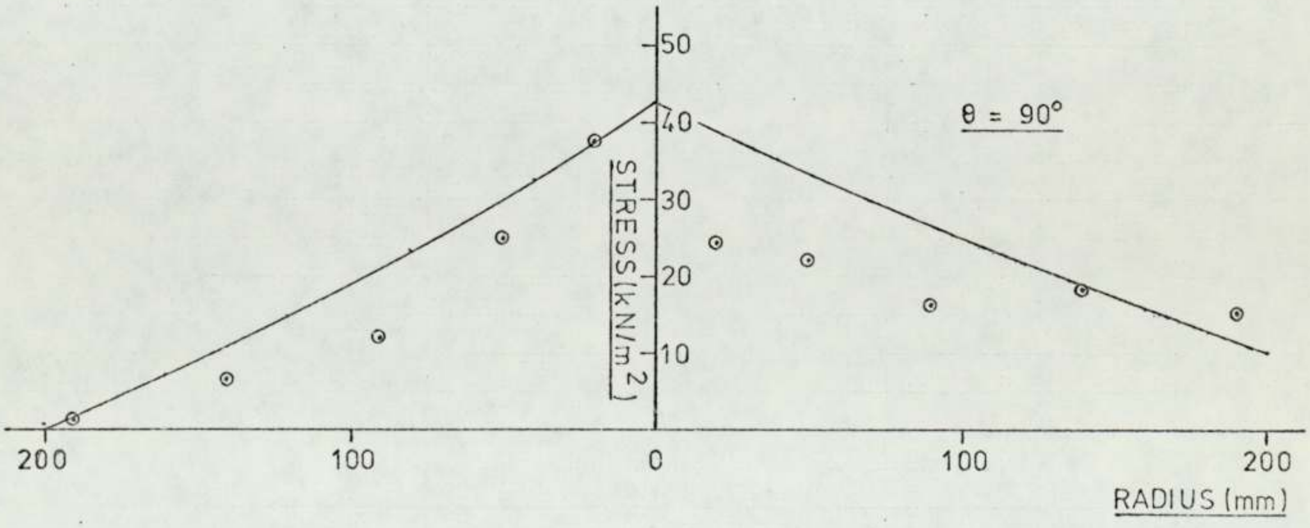
GRAPH D2.2

STRESSES IN A PLATE WITH A SINGLE DIAMETRAL RIB

PLATE AS IN GRAPH D2.1 WITH LOAD = 1N







A LISTING OF THE PROGRAM 'RIBPLT'

This program analyses a simply-supported plate with a single diametral rib & central point load

```
LIST
RIBPLT

10  PRINT "WHAT IS THE PLATE RADIUS(M) AND THICKNESS(MM)"
20  INPUT R,H
30  PRINT "WHAT IS THE RIB DEPTH(MM) AND THICKNESS(MM)"
40  INPUT S,T
50  PRINT "WHAT IS THE MODULUS OF ELASTICITY(N/SQ.M) AND POISSONS RATIO"
60  INPUT E,V
70  I=(T*S↑3/12+T*S*((S+H)/2)↑2)/10↑12
80  D=E*H↑3/(12*(1-V↑2)*10↑9)
90  K=E*I/(3.142*D*R)
100  DIM C[4,4]
110  DIM F[4,1]
120  LET C[1,1]=16*(1+V+K)
130  LET C[1,2]=12*(2*(7+5*V)/5+3*K)*R
140  LET C[1,3]=32*((5+3*V)/3+9*K/5)*R*R
150  LET C[1,4]=20*K*R
160  LET C[2,1]=12*(1+V+K)
170  LET C[2,2]=3*(3*(5+4*V)/2+3*K)*R
180  LET C[2,3]=C[1,2]*R
190  LET C[2,4]=12*K*R
200  LET C[3,1]=3*(1+V+K)
210  LET C[3,2]=C[2,1]*R
220  LET C[3,3]=C[1,1]*R*R
230  LET C[3,4]=4*K*R
240  LET C[4,1]=4*K
250  LET C[4,2]=12*K*R
260  LET C[4,3]=20*K*R*R
270  LET C[4,4]=((21+4*V)/4+3*K)*R
280  LET F[1,1]=-1/(3.142*D)
290  LET F[2,1]=F[1,1]
300  LET F[3,1]=F[1,1]
310  LET F[4,1]=0
```

cont.

```
320  DIM G[4,4]
330  DIM J[4,1]
340  MAT G=INV(C)
350  MAT J=G*F
360  LET W=-(J[1,1]*R↑2+J[2,1]*R↑3+J[3,1]*R↑4)*1000
370  PRINT
380  PRINT "CENTRAL DEFL. PER NEWTON LOAD IS";W;"(MM)"
390  FOR M=0 TO 2 STEP 1
400  PRINT
410  PRINT
420  LET O1=M*45
430  LET O=O1*3.14159/180
440  PRINT "STRESS DISTRIBUTION FOR THETA =";O1;"(DEG)"
450  PRINT "RADIUS          RAD.STRESS       TANG.STRESS"
460  PRINT " (M)             (N/SQ.M)          (N/SQ.M)"
470  FOR N=0 TO 5 STEP 1
480  LET A=.2*N*R
490  LET P1=2*(1+V)*J[1,1]
500  LET P2=3*(2+V)*A*J[2,1]
510  LET P3=4*(3+V)*A↑2*J[3,1]
520  LET P4=(2*(V-1)*R+A*(6-V))*J[4,1]*COS(O)
530  LET S1=-6*D*(P1+P2+P3+P4)*10↑6/H↑2
540  LET Q1=P1
550  LET Q2=3*(1+2*V)*A*J[2,1]
560  LET Q3=4*(1+3*V)*A↑2*J[3,1]
570  LET Q4=(2*(1-V)*R+A*(6*V-1))*J[4,1]*COS(O)
580  LET S2=-6*D*(Q1+Q2+Q3+Q4)*10↑6/H↑2
590  PRINT A,S1,S2
600  NEXT N
610  NEXT M
620  END
```

TYPICAL OUTPUT FROM THE PROGRAM 'RIBPLT'

```
RUN
RIBPLT

WHAT IS THE PLATE RADIUS(M) AND THICKNESS(MM)
?0.2, 3.021
WHAT IS THE RIB DEPTH(MM) AND THICKNESS(MM)
?15, 3.057
WHAT IS THE MODULUS OF ELASTICITY(N/SQ.M) AND POISSONS RATIO
?2.8E9, 0.37

CENTRAL DEFL. PER NEWTON LOAD IS 7.29342E-02   (MM)


STRESS DISTRIBUTION FOR THETA = 0     (DEG)
RADIUS          RAD.STRESS      TANG.STRESS
 (M)            (N/SQ.M)        (N/SQ.M)
 0               41105.5         44021.9
 .04             32442.9         36943.7
 .08             24501.2         30326.9
 .12             17230.7         24156.6
 .16             10731.2         18437.8
 .2              5002.31         13170.4

STRESS DISTRIBUTION FOR THETA = 45    (DEG)
RADIUS          RAD.STRESS      TANG.STRESS
 (M)            (N/SQ.M)        (N/SQ.M)
 0               41532.6         43594.3
 .04             32433.3         36433.9
 .08             24165.          29734.4
 .12             16562.3         23431.4
 .16             9631.63         17679.9
 .2              3521.53         12329.3

STRESS DISTRIBUTION FOR THETA = 90    (DEG)
RADIUS          RAD.STRESS      TANG.STRESS
 (M)            (N/SQ.M)        (N/SQ.M)
 0               42563.7         42563.7
 .04             32597.9         35203.1
 .08             23353.2         28304.
 .12             14329.5         21351.3
 .16             7026.94         15350.1
 .2             -54.6043         10300.3

DONE
```

E

THE USE OF PLASTICS FOR STRESS ANALYSIS MODELS

E1 INTRODUCTION

One of the most common applications of plastics in stress analysis, namely the use of thermosetting resins for the manufacture of photoelastic models, has been developed over many years. The technique has been used with a considerable amount of success, but in some applications the geometry of the structure to be analysed may not be amenable to photoelastic study.

Modern instrumentation has made possible the processing of the results obtained from large numbers of electrical resistance strain gauges in a reasonable period of time. This, coupled with miniaturization of the gauges themselves, has made the use of strain gauge techniques on small scale models a practical proposition. The manufacture of small scale models in metal, however, presents its own problems. Firstly the fabrication of small models may be practically difficult and secondly, small models tend to be very rigid with the consequent problems of measuring deformation under load.

The advent of thermoplastics in commercially available sheet, block and bar form has made the construction of very sophisticated models a relatively simple task, as fabrication using adhesives is fairly straightforward. The low elastic modulus of thermoplastics implies that easily measured deformations may be produced with relatively small loads which are consequently easy to react in a predictable manner.

The properties required of a thermoplastic for use in strain gauged models are basically:

(a) it must be possible to produce sound joints in a fabrication, together with the maintenance of consistent material properties across the joint.

(b) it must be possible to attach strain gauges in a satisfactory manner.

(c) a completed model should have high dimensional stability.

(d) the material should have a linear stress-strain characteristic if possible and a low creep sensitivity.

A considerable amount of work has already been carried out on the selection and characteristics of thermoplastics for model making. The most notable contributions are probably those of Wallace [32] and Swan [33] of the Naval Construction Research Establishment. They recommend Vybak as a suitable material and compare its properties with other thermoplastics. Further investigations have been made by Leighton [34] and Edwards [35] with a view to confirming the suitability of Vybak and developing a 'feel' for its use.

E2 <u>A SURVEY OF SUITABLE THERMOPLASTICS</u>

Three thermoplastics are in common use for stress analysis models. These are marketed under the brand names of Vybak, Xylonite and Perspex. Brief comments on their suitability for this application are as follows:-

<u>Vybak</u> - Humidity has little effect on its mechanical properties but temperature effects are large enough to warrant consideration. The creepbehaviour of this material is superior to the others, which is the main reason for

its use. The ideal material for static stress analysis models would exhibit no creep characteristics at all. Vybak creeps considerably less than the other materials considered and even this can be kept under control with the precautions advocated in section E3 below. It has high dimensional stability, is easily machined and fabricated, and has isotropic properties. It is commonly available only in sheet form.

<u>Xylonite</u> - Dimensional stability is poor and its properties are very susceptible to changes in humidity. It is less sensitive to temperature changes than Vybak but its creep characteristics are inferior. This material is slightly anisotropic and also presents an element of fire risk.

<u>Perspex</u> - This material is inferior to Vybak in both its temperature sensitivity and its creep characteristics. Its main advantage is in being readily available in cast block form.

On the basis of these comments, Vybak was selected as being the most suitable material for the modelling of stiffened plates. The properties of Vybak will now be described in greater detail, the observations being based on the work of Wallace, Leighton and Edwards [op.cit.] and manufacturers' literature [36] .

E3 <u>THE PROPERTIES OF VYBAK</u>

Elastic Modulus - 2.8 GN/m^2 at 23°C

For temperature fluctuations within the normal variations of room temperature, the modulus decreases by 0.85% for every degree Kelvin of temperature rise. The relationship between stress and strain is linear for stresses less than 20 MN/m^2 and strains less than 0.7%

Poissons Ratio - 0.37

Creep - The effects of creep are negligible provided the elastic strains are maintained below 0.5%. A five minute settling period after the application of load and before measurement of deformation is advisable however if great accuracy is desired. This five minute period is generally quite adequate as the elastic modulus is only 4% lower after four hours under load than its value after five minutes.

The attachment of strain gauges to Vybak sheet presents no unusual problems and a satisfactory bond can be obtained using conventional methods. For models made of thin sheet it is preferable to use thin foil gauges. The stiffening effect due to the attachment of small, thin gauges is generally low and, in the case where bending strains are measured, the decrease in strain due to stiffening is compensated to some extent by the gauge being at an increased distance from the neutral surface of the model due to the thickness of the adhesive.

One problem with the use of strain gauges on any thermoplastic is that, due to the low thermal conductivity of the plastic, the material in the vicinity of the gauge will increase in temperature with the consequent lowering of the elastic modulus and increase in the strain recorded. For this reason the current flow in the strain bridge should be kept to a minimum. For stability of temperatures it is desirable that all gauges should be continuously energised. This implies the existence of a dummy gauge for every active one and hence a large number of gauges may be required for a relatively simple model. Satisfactory results may be obtained by having a limited number of dummy gauges which may be used with a selected equal number of active ones and then repeating the loading cycle as readings are taken from each set of active gauges in turn. Time must be allocated for temperature stabilisation as each set of gauges is selected.

R E F E R E N C E S

GENERAL ELASTICITY AND PLATE THEORY

1. Timoshenko, S.P. and Goodier, J.N. - 'Theory of Elasticity', 3rd edition, McGraw-Hill, 1970.

2. Roark, R.J.. 'Formulas for stress and strain', 4th edition, MrGraw-Hill, 1965.

3. Timoshenko, S.P. and Woinowsky-Krieger, S., - 'Theory of plates and shells', 2nd edition, McGraw-Hill, 1959.

4. McFarland, D., Smith, B.L. and Bernhardt, W.D. - 'Analysis of plates', Spartan, 1972.

VARIATIONAL CALCULUS, ENERGY METHODS AND GENERAL MATHEMATICS

5. Charlton, T.M., - 'Energy principles in applied statics', Blackie, 1959.

6. Lanczos, C., - 'The variational principles of mechanics', 2nd edition, University of Toronto Press, 1962.

7. Forray, M.J., - 'Variational calculus in science and engineering', McGraw-Hill, 1968.

8. Dym, C.L. and Shames, I.H., - 'Solid mechanics : a variational approach', McGraw-Hill, 1973.

9. Argyris, J.H., - 'Energy theorems and structural analysis', Aircraft Engineering, October 1954.

10. Ralston, A., - 'A first course in numerical analysis', McGraw-Hill, 1965.

11. Wilkinson, J.H. and Reinsch, C. - 'Handbook for automatic computation. Vol. II - Linear algebra', Springer-Verlag, 1971.

SPECIALISED THEORY FOR UNSTIFFENED PLATES AND FINITE ELEMENT APPLICATIONS

12. Stanley, P., 'Computing developments in experimental and numerical stress analysis', Applied Science Publishers, 1976.

13. Zienkiewicz, O.C., - 'The finite element method in engineering science', McGraw-Hill, 1971.

14. Martin, H.C. and Carey, G.F. - 'Introduction to finite element analysis', McGraw-Hill, 1973.

15. Gallagher, R.H. - 'Finite element analysis fundamentals', Prentice-Hall, 1975.

16. Huebner, K.H., - 'The finite element method for engineers', Wiley, 1975.

17. Zienkiewicz, O.C. and Too, J.J.M., - 'The finite prism in analysis of thick, simply supported bridge boxes', Proc.I.C.E., Sept.1972, Vol.53, Part II - Theory and research, pp 147-172.

18. Coull, A and Das, P.C. - 'Analysis of curved bridge decks', Proc.I.C.E., Vol.37, May 1967, pp 75-85.

19. Pardoen, C.P. and Hagen, R.L., - 'Symmetrical bending of circular plates using finite elements', Computers and Structures, Vol.2, 1972, pp 547-553.

20. Wilson, E., - 'Analysis of clamped-free circular plates under concentrated edge loading', Internal undergraduate project report, North Staffordshire Polytechnic, 1976.

21. Reissner, H., - 'Uber die unsymmetrische biegung dünner kriesring platte', Ing. - Archiv, Vol.1, 1929, p 72 (in German).

22. Olson, M.D., Lindberg, G.M. and Tulloch, H., - 'Finite plate bending elements in polar co-ordinates', National Research Council of Canada Aeronautical Report LR-512, 1968.

23. Sawko, F and Merriman, P.A., - 'An annular segment finite element for plate bending', Int.J.Num.Meth.in Eng., Vol.3, 1971, pp 119 - 129.

24. Singh, S and Ramaswamy, G.S., - 'A sector element for thin plate flexure', Int.J.Num.Meth.in Eng., Vol.4, 1972, pp 134 - 142.

STIFFENED PLATES

25. Biezeno, C.B. and Grammel, R., - 'Engineering Dynamics', Vol.II., Part III, 2nd edition, Blackie, 1973.

26. Kirk, C.L., - 'Vibration characteristics of stiffened plates', J. of Mech.Eng.Sci., Vol.2, No.3, 1960, pp 242-253.

27. Desiderati, F.W. and Laura, P.A. - 'Vibrations of rib-stiffened elliptical and circular plates', J.Acoust.Soc.Am., Vol.48, No.1, 1970, pp 6 - 11.

28. Harvey, J. and Duncan, J.P., - 'The rigidity of rib reinforced cover plates', Proc.I.Mech.E., Vol.177, No.5, 1963, pp 115-121.

29. Rubach, M., - 'Bending of circular plates reinforced with radial ribs', Academia Nauk Ukrainskoj SSR, Sbornik Trudov Istituta Stroitelnoj Mekhaniki, No.20, 1955, pp 132-147 (In Russian).

30. Mlotkowski, A., - 'Approximate calculations on a circular plate, ribbed on one side and loaded asymmetrically', Mechanika Teoretyczna i Stosowana, Vol.8, No.2, 1970, pp 127-136 (in Polish).

31. Leyko, J., Krolak, M. and Mlotkowski, A., - 'Approximate method for strength analysis of circular, axially symmetric plates of variable thickness, with ribs on one side and subject to an axially symmetric load', Archiwum Budowy Maszyn, Vol.19, No.3, 1972, pp 421 - 434. (In Polish but translated by B.L.L. - Translation No. RTS.10426).

PLASTICS FOR STRESS ANALYSIS MODELS

32. Wallace, G. - 'Structural model analysis with thermoplastics', Strain, Vol.3, No.3, July 1967, pp 4-17.

33. Swan, J.W., - 'Resistance strain gauges on thermoplastics', Strain, April 1973, pp 56-59.

34. Leighton, D. - 'The use of thermoplastics for stress analysis models, I', Internal undergraduate project report, North Staffordshire Polytechnic, 1974.

35. Edwards, G. 'The use of thermoplastics for stress analysis models, II', Internal undergraduate project report, North Staffordshire Polytechnic, 1975.

36. Vybak 252 Rigid P.V.C. Sheet - Technical Data Sheet 9A/7, Bakelite-Xylonite Ltd.