Emergent Behaviour in Arrays of Coupled Nonlinear Oscillators

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Abstract

This paper proposes two approaches to the analysis of the different behaviours emerging in arrays of coupled nonlinear oscillators, according to their different parameters of structure, coupling, stiffness or distribution. The first approach is a theoretical analysis of the dynamics of such arrays of coupled nonlinear oscillators, based on a recent tool: *Contraction Theory.* In the second approach, numerical techniques are employed, for such large arrays, to validate conditions for synchronisation of the system, according to its different parameters, as predicted by the theoretical analysis.

Keywords: Coupled Nonlinear Oscillators, Emergent Behaviour, Synchronisation, Contraction Theory

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Introduction

1.1 Background

Mutual synchronisation is a common phenomenon in Nature. It occurs at different levels, ranging from the small scale of the cardiac pace-maker cells of the Sino-Atrial and Atrium Ventricular nodes in the human heart that synchronously fire and give the pace to the whole muscle or of the millions of neurons firing together to control our breathing, to the coordinate behaviours of crickets that chirp in unison and of fireflies that flash together in some parts of southeast Asia, or event to the Moon's spin, which is precisely in synchrony with its orbit [25].

The dynamics of coupled oscillators is a very broad field of research.

The study was, first, initiated by Huygens, who noticed, in the 17^{th} century, that two pendulum clocks on the same shelf would, after a while, reach a stable state, where they would swing with exactly the same frequency, but in anti-phase [2].

It involves, today, a variety of research fields, such as Mathematics [23, 33, 34], Biology [27], Chemistry [8], Neuroscience, Robotics, Electronics, or Nano-Science[28]. Indeed, using the synchronisation properties by imitating the natural oscillating networks is found to be very useful in building such devices as, for instance, artificial pacemakers, olfactory bulbs and breathers, laser arrays, or micro-electro-mechanical devices.

In both the natural and human-made worlds, the important thing is not the behaviour of one oscillator (a single cell or neuron does not have any effect), but of the global system.

Therefore, the question one might want to answer is: what global phenomena could be expected to arise from the rhythmical interactions of whole populations of periodic processes?

Winfree [33] was the first to underline the generality of the problem, fixing the first assumptions for a mathematical model. In his work, each *oscillating species* (cell, cricket, fireflies) is modelled as a nonlinear oscillator with a globally attracting limit-cycle. The oscillators were assumed to be weakly coupled, and their natural frequencies to be randomly distributed across the population.

Kuramoto [8] proposed the first model. His assumptions were that each oscillator is equal to the others, except for the frequency and phase, that the system has a mean-field coupling, and that the amplitudes of the oscillations are all the same.

This kind of model, called a *phase-space* model analyses the behaviour of the phases all the oscillating species, and concludes on the emergence, or not, of a global behaviour.

Another approach of the problem is to perform an analysis directly on the state of the system itself, instead of on the phase of each oscillating species.

This approach uses what are called *state-space* models, which are much closer to physical reality, as they do not set as restrictive conditions on the oscillator and the coupling, as the phase-space models.

However, one cannot say that one model is better than the other, as there still does not exist a general and explicit analysis tool to study the state-space models.

1.2 Motivation

This paper intends to analyse the different behaviours emerging in arrays of coupled nonlinear oscillators, according to their different parameters of structure, coupling, stiffness or distribution, using both a phase-space model and a state-space model.

The importance for the oscillators to be nonlinear is crucial: as it will be demonstrated numerically at the end of the paper, coupled linear oscillators do not reach synchrony; the only global phenomenon which can be observed in networks of coupled linear oscillators is resonance, which is the response of a system that is non-active, *id est* that demonstrates no oscillations without external driving: without an external force, the oscillations die. In practice, the analysis will mainly be applied to the case of Van der Pol oscillators, which present the main interest of being autonomous oscillators, *id est* which do not need to be externally driven to continue oscillating.

First, the state-space model approach will be driven by a recent nonlinear system analysis tool, based on studying convergence between two arbitrary system trajectories: *Contraction Theory.*

Three different behaviours - synchronisation, anti-synchronisation, and oscillator-death - will be spotted, according to conditions on the coupling-strengths, for arrays, of various sizes, of coupled identical oscillators.

Then, the phase-space model approach will be based on a numerical analysis of simulations results for large arrays of coupled non-identical oscillators, displaying the dependencies of the synchronisability of the system on different parameters of the array, such as the coupling-strength, the configuration of the coupling, the number of connections, the stiffness of each oscillator, or the distribution of the natural frequencies.

Finally, a conclusion will draw some remarks, advices and ideas, for those who would intend to physically build such arrays.

Chapter 2

Contraction Theory Approach of the Dynamics of Networks of Coupled Identical Nonlinear Oscillators

This chapter carries out a theoretical analysis of the dynamics of coupled identical Van der Pol oscillators, using a recent method: *Contraction Theory* [20, 21, 29, 30, 31, 32].

After a brief summary of the basic definitions and results of contraction theory, a contraction analysis is performed on two coupled identical Van der Pol oscillators. The behaviour of simple patterns of coupled identical Van der Pol oscillators is then studied, according to the different symmetries of the couplings. Finally, larger arrays of coupled identical Van der Pol oscillators are analysed. The chapter ends with a critical analysis of the contraction theory approach towards the coupled nonlinear oscillators problem.

2.1 Contraction Theory

This section briefly summarises basic definitions and results of contraction theory.

Consider an m-dimension nonlinear system:

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}, t), \tag{2.1}$$

where $\mathbf{f}$ is an $m \times 1$ nonlinear function vector and $\mathbf{x}$ is an $m \times 1$ state vector.

Assuming $\mathbf{f}(\mathbf{x}, t)$ is continuously differentiable, one has:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta\mathbf{x}^T\delta\mathbf{x}\right) &= \delta\dot{\mathbf{x}}^T\delta\mathbf{x} + \delta\mathbf{x}^T\delta\dot{\mathbf{x}} \\
&= \left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\delta\mathbf{x}\right)^T\delta\mathbf{x} + \delta\mathbf{x}^T\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\delta\mathbf{x}\right) \\
&= \delta\mathbf{x}^T\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\delta\mathbf{x} \\
&= 2\,\delta\mathbf{x}^T\mathbf{J}_s\delta\mathbf{x} \\
&= 2\,\delta\mathbf{x}^T\delta\mathbf{x}\,\frac{\delta\mathbf{x}^T\mathbf{J}_s\delta\mathbf{x}}{\delta\mathbf{x}^T\delta\mathbf{x}} \\
&\leq 2\,\delta\mathbf{x}^T\delta\mathbf{x}\,\max_{\mathbf{v}}\frac{\mathbf{v}^T\mathbf{J}_s\mathbf{v}}{\mathbf{v}^T\mathbf{v}} \\
&= 2\,\lambda_{max}(\mathbf{J}_s)\,\delta\mathbf{x}^T\delta\mathbf{x},
\end{aligned} \tag{2.2}$$

where $\delta\mathbf{x}$ is a virtual displacement between two neighbouring solution trajectories of the system, $\mathbf{J}_s = \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right)$ is the symmetric part of the Jacobian matrix:

$$\mathbf{J} \;=\; \frac{\partial\mathbf{f}}{\partial\mathbf{x}},$$

and $\lambda_{max}(\mathbf{J}_s)$ is the largest eigenvalue of $\mathbf{J}_s$.

Hence, if $\mathbf{J}_s$ is uniformly negative definite, all its eigenvalues are uniformly strictly negative, thus so is its largest eigenvalue λ_{max}, and the virtual displacement vector $\delta\mathbf{x}$ converges exponentially to the nil vector [30], implying, in turn, that all the solutions of the system (2.1) converge exponentially to a single trajectory, regardless of the initial conditions.

More generally, consider a coordinate transformation:

$$\delta\mathbf{z} \;=\; \mathbf{\Theta}\delta\mathbf{x},$$

where $\mathbf{\Theta}(\mathbf{x}, t)$ is a uniformly invertible square matrix.

One has:

$$
\begin{aligned}
\frac{d}{dt}\left(\delta\mathbf{z}^T\delta\mathbf{z}\right) &= \delta\dot{\mathbf{z}}^T\delta\mathbf{z} + \delta\mathbf{z}^T\delta\dot{\mathbf{z}} \\
&= \left(\dot{\boldsymbol{\Theta}}\delta\mathbf{x} + \boldsymbol{\Theta}\delta\dot{\mathbf{x}}\right)^T\delta\mathbf{z} + \delta\mathbf{z}^T\left(\dot{\boldsymbol{\Theta}}\delta\mathbf{x} + \boldsymbol{\Theta}\delta\dot{\mathbf{x}}\right) \\
&= \delta\mathbf{x}^T\left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)^T\delta\mathbf{z} + \delta\mathbf{z}^T\left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\delta\mathbf{x} \\
&= \delta\mathbf{z}^T\boldsymbol{\Theta}^{-1T}\left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)^T\delta\mathbf{z} + \delta\mathbf{z}^T\left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\boldsymbol{\Theta}^{-1}\delta\mathbf{z} \\
&= \delta\mathbf{z}^T\left(\left(\left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\boldsymbol{\Theta}^{-1}\right)^T + \left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\boldsymbol{\Theta}^{-1}\right)\delta\mathbf{z} \\
&= 2\,\delta\mathbf{z}^T\mathbf{F}_s\delta\mathbf{z},
\end{aligned}
$$

where $\mathbf{F}_s = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right)$ is the symmetric part of the generalised Jacobian matrix:

$$
\mathbf{F} = \left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\boldsymbol{\Theta}^{-1}.
$$

Hence, if $\mathbf{F}_s$ is uniformly negative definite, the vector $\delta\mathbf{z}$ converges exponentially to the nil vector, implying, in turn, that all the solutions of the system(2.1) converge exponentially to a single trajectory, regardless of the initial conditions.

By convention, if $\mathbf{F}_s$ is uniformly negative definite, the system (2.1) is called *contracting* and $\mathbf{f}(\mathbf{x}, t)$ is called a *contracting function*.

2.2 Dynamics of Two Coupled Identical Van der Pol Oscillators

This section investigates the dynamics of networks composed of two coupled identical Van der Pol oscillators, with respect to their coupling strengths and signs.

Three behaviours are studied: synchronisation, anti-synchronisation, and oscillator-death; first individually, on different basic examples, then altogether on a more general system.

2.2.1 General Results

Before getting started with the coupled identical Van der Pol oscillators case, a few general results about coupled identical systems should be set.

Theorem 1 *Consider a pair of one-way coupled identical oscillators:*

$$\begin{cases} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, t) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, t) + \mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_2) \end{cases},$$

where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$ are the state vectors, $\mathbf{f}(\mathbf{x}, t)$ the dynamics of the uncoupled oscillators, and $\mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_2)$ the coupling force. If the function $\mathbf{f} - \mathbf{u}$ is contracting, the two oscillators will reach synchrony exponentially, regardless of the initial conditions.

Proof The second subsystem, with $\mathbf{u}(\mathbf{x}_1)$ as input, is contracting, and $\mathbf{x}_1(t) - \mathbf{x}_2(t)$ is a particular solution. $\square$

Theorem 2 *Consider two coupled subsystems. If the dynamics equations verify:*

$$\dot{\mathbf{x}}_1 - \mathbf{h}(\mathbf{x}_1, t) \;=\; \dot{\mathbf{x}}_2 - \mathbf{h}(\mathbf{x}_2, t),$$

where the function $\mathbf{h}$ is contracting, then $\mathbf{x}_1$ and $\mathbf{x}_2$ will converge to each other exponentially, regardless of the initial conditions.

Proof Denote by $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ the solutions of the two coupled subsystems. Define:

$$\begin{aligned} \mathbf{g}(\mathbf{x}_1, \mathbf{x}_2, t) &= \dot{\mathbf{x}}_1 - \mathbf{h}(\mathbf{x}_1, t) \\ &= \dot{\mathbf{x}}_2 - \mathbf{h}(\mathbf{x}_2, t), \end{aligned}$$

and construct the auxiliary system:

$$\dot{\mathbf{y}} \;=\; \mathbf{h}(\mathbf{y}) + \mathbf{g}(\mathbf{x}_1(t), \mathbf{x}_2(t), t). \tag{2.3}$$

The system (2.3) is contracting since the function $\mathbf{h}$ is contracting. Thus, all the solutions for $\mathbf{y}$ converge together exponentially. Since $\mathbf{y} = \mathbf{x}_1(t)$ and $\mathbf{y} = \mathbf{x}_2(t)$ are two particular solutions, one gets that $\mathbf{x}_1(t)$ and $\mathbf{x}_1(t)$ converge to each other exponentially, regardless of the initial conditions. $\square$

To be applied to the coupled identical Van der Pol oscillators case, the conditions in Theorem 1 and Theorem 2 need to be relaxed to uniformly negative semi-definite systems, introducing the concept of semi-contraction.

Theorem 3 *Consider an m-dimensional nonlinear system:*

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}, t). \tag{2.4}$$

If the symmetric part $\mathbf{F}_s$ *of the Jacobian* $\mathbf{F} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ *is uniformly negative semi-definite, the system (2.4) is, by convention, called* semi-contracting, *and any virtual displacement vector* $\delta\mathbf{x}$ *between two solutions converge exponentially to a constant vector.*

Proof Using the same argument as in the previous section, one gets, for the virtual displacement $\delta\mathbf{x}$:

$$\frac{d}{dt}\left(\delta\mathbf{x}^T\delta\mathbf{x}\right) \;\leq\; 2\,\lambda_{max}(\mathbf{F}_s)\;\delta\mathbf{x}^T\delta\mathbf{x}.$$

where $\lambda_{max}(\mathbf{F}_s)$ is the largest eigenvalue of the matrix $\mathbf{F}_s$. Thus, if $\mathbf{F}_s$ is uniformly negative semi-definite, its largest eigenvalue λ_{max} is either uniformly strictly negative, or zero, which means, from what precedes, that $\delta\mathbf{x}$ converge exponentially to a constant vector. $\square$

Applied to the Van der Pol oscillator case, Theorem 3 gives the following result:

Lemma 4 *Consider a Van der Pol oscillator*

$$\ddot{x} + (\beta + \alpha x^2)\dot{x} + \omega^2 x \;=\; u(t), \tag{2.5}$$

driven by an external input $u(t)$*, where* α*,* β *and* ω *are strictly positive constants. All the solutions of the system (2.5) converge exponentially to a single trajectory, regardless of the initial conditions.*

Proof Introducing a new variable y, the system (2.5) can be written:

$$\left\{\begin{array}{lcl} \dot{x} & = & \omega y - \frac{\alpha}{3}x^3 - \beta x \\ \dot{y} & = & -\omega x + \frac{u(t)}{\omega} \end{array}\right. . \tag{2.6}$$

The corresponding Jacobian matrix is:

$$\mathbf{F} = \begin{bmatrix} -(\beta+\alpha x^2) & \omega \\ -\omega & 0 \end{bmatrix},$$

and its symmetric part:

$$\mathbf{F}_s = \begin{bmatrix} -(\beta+\alpha x^2) & 0 \\ 0 & 0 \end{bmatrix}$$

is negative semi-definite. Thus, the system is semi-contracting, and according to Theorem 3, the virtual displacement vector $\delta\mathbf{z} = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$ between two solutions of the system converges exponentially to the constant vector $\begin{bmatrix} 0 \\ \delta y_\infty \end{bmatrix}$.
Looking at the first subsystem of the system (2.6), one gets:

$$\begin{aligned} \delta\dot{x} &= \omega\delta y - (\beta+\alpha x^2)\delta x \\ &\to \omega\delta y_\infty. \end{aligned}$$

Thus, $\delta\dot{x}$ has a limit l.
Suppose $l \neq 0$. Then, after a certain period of time, δx will be increasing or decreasing constantly, depending on the sign of l, and will, therefore, never converge, which is a contradiction with the fact that δx converges exponentially to 0. Thus, $l = 0$, which implies: $\delta y_\infty = 0$.
One can conclude that the virtual displacement vector $\delta\mathbf{z}$ converges exponentially to the nil vector, which is equivalent to saying that all the solutions of the system (2.6), thus of the system (2.5), converge exponentially to a single trajectory, regardless of the initial conditions. □

In the rest of the paper, the terms *oscillator* and *nonlinear system* will only apply to the Van der Pol oscillator case whenever they refer to a semi-contracting system.

From what precedes, Theorem (1) and Theorem (2) can be generalised as follows.

Corollary 5 *Consider a pair of one-way coupled identical oscillators:*

$$\begin{cases} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, t) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, t) + \mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_2) \end{cases},$$

where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$ *are the state vectors,* $\mathbf{f}(\mathbf{x}, t)$ *the dynamics of the uncoupled, and* $\mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_2)$ *the coupling force. If the function* $\mathbf{f} - \mathbf{u}$ *is contracting (or semi-contracting), the two oscillators will reach synchrony exponentially, regardless of the initial conditions.*

Corollary 6 *Consider two coupled subsystems. If the dynamics equations verify:*

$$\dot{\mathbf{x}}_1 - \mathbf{h}(\mathbf{x}_1, t) \;=\; \dot{\mathbf{x}}_2 - \mathbf{h}(\mathbf{x}_2, t),$$

where the function $\mathbf{h}$ *is contracting (or semi-contracting), then* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *will converge to each other exponentially, regardless of the initial conditions.*

The contraction theory results are now fully ready to be applied to the coupled identical Van der Pol oscillators case.

2.2.2 Synchronisation

Consider a system of two-way coupled identical oscillators:

$$\begin{cases} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, t) + \mathbf{u}_1(\mathbf{x}_2) - \mathbf{u}_1(\mathbf{x}_1) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, t) + \mathbf{u}_2(\mathbf{x}_1) - \mathbf{u}_2(\mathbf{x}_2) \end{cases}. \tag{2.7}$$

Suppose $\mathbf{f} - (\mathbf{u}_1 + \mathbf{u}_2)$ is contracting (or semi-contracting).

One gets:

$$\begin{aligned} \mathbf{u}_1(\mathbf{x}_2) + \mathbf{u}_2(\mathbf{x}_1) &= \dot{\mathbf{x}}_1 - (\mathbf{f}(\mathbf{x}_1, t) - (\mathbf{u}_1(\mathbf{x}_1) + \mathbf{u}_2(\mathbf{x}_1))) \\ &= \dot{\mathbf{x}}_2 - (\mathbf{f}(\mathbf{x}_2, t) - (\mathbf{u}_1(\mathbf{x}_2) + \mathbf{u}_2(\mathbf{x}_2))) \end{aligned}$$

According to Corollary 6, $\mathbf{x}_1$ and $\mathbf{x}_2$ will converge to each other exponentially, regardless of the initial conditions.

Furthermore, the coupling forces in system (2.7) vanish exponentially, so, for non-zero initial conditions, both oscillators tend to their original limit-cycle behaviour, but with a common phase.

Hence, if the system:

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}, t)$$

has a stable limit-cycle, both oscillators will continue oscillating, with a common phase.

By convention, the two oscillators are said to be reaching *synchrony*.

Applying this result to the coupled identical Van der Pol oscillators case, one gets as follows.

Consider two coupled identical Van der Pol oscillators:

$$\left\{ \begin{array}{rcl} \ddot{x}_1 + \alpha(x_1^2 - 1)\dot{x}_1 + \omega^2 x_1 &=& \alpha\kappa_1(\dot{x}_2 - \dot{x}_1) \\ \ddot{x}_2 + \alpha(x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 &=& \alpha\kappa_2(\dot{x}_1 - \dot{x}_2) \end{array} \right. , \tag{2.8}$$

where α and ω are both strictly positive constants.

Defining the following functions:

$$\begin{aligned} \mathbf{f}(\mathbf{x}, t) &= \begin{bmatrix} -\alpha(\frac{x^3}{3} - x) + \omega y \\ -\omega x \end{bmatrix}, \\ \mathbf{u}_1(\mathbf{x}) &= \begin{bmatrix} \alpha\kappa_1 x \\ 0 \end{bmatrix}, \\ \mathbf{u}_2(\mathbf{x}) &= \begin{bmatrix} \alpha\kappa_2 x \\ 0 \end{bmatrix}, \end{aligned}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, one gets that the system (2.8) is equivalent to the system (2.7).

The symmetric part of the Jacobian matrix of $\mathbf{f} - (\mathbf{u}_1 + \mathbf{u}_2)$ is:

$$\mathbf{F}_s \;=\; \begin{bmatrix} -\alpha(x^2 + (\kappa_1 + \kappa_2 - 1)) & 0 \\ 0 & 0 \end{bmatrix},$$

which is uniformly negative semi-definite if $\kappa_1 + \kappa_2 - 1 > 0$.

Therefore, for $\kappa_1 + \kappa_2 > 1$, $\mathbf{f} - (\mathbf{u}_1 + \mathbf{u}_2)$ is semi-contracting, and the system:

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}, t)$$

has a stable limit-cycle (according to Appendix A), which implies that the two oscillators will reach synchrony exponentially, for non-zero initial conditions.

This result confirms what one might have intuited: synchronisation takes place for strong enough coupling forces.

2.2.3 Anti-Synchronisation

Consider a system of two-way coupled identical oscillators:

$$\left\{\begin{array}{lcl} \dot{\mathbf{x}}_1 & = & \mathbf{h}(\mathbf{x}_1,t)+\mathbf{u}(\mathbf{x}_2)-\mathbf{u}(\mathbf{x}_1) \\ \dot{\mathbf{x}}_2 & = & \mathbf{h}(\mathbf{x}_2,t)+\mathbf{u}(\mathbf{x}_1)-\mathbf{u}(\mathbf{x}_2) \end{array}\right. . \tag{2.9}$$

Suppose $\mathbf{h}$ is contracting (or semi-contracting) and odd in $\mathbf{x}$.

One gets:

$$\begin{array}{rcl} \dot{\mathbf{x}}_1-\mathbf{h}(\mathbf{x}_1,t) & = & \mathbf{u}(\mathbf{x}_2)-\mathbf{u}(\mathbf{x}_1) \\ & = & -(\mathbf{u}(\mathbf{x}_1)-\mathbf{u}(\mathbf{x}_2)) \\ & = & -(\dot{\mathbf{x}}_2-\mathbf{h}(\mathbf{x}_2,t)) \\ & = & (-\dot{\mathbf{x}}_2)-\mathbf{h}(-\mathbf{x}_2,t) \end{array}$$

According to Corollary 6, $\mathbf{x}_1$ and $-\mathbf{x}_2$ will converge to each other exponentially, regardless of the initial conditions.

Furthermore, considering the system:

$$\left\{\begin{array}{lcl} \dot{\mathbf{x}}_1 & = & (\mathbf{h}(\mathbf{x}_1,t)-2\mathbf{u}(\mathbf{x}_1))+(\mathbf{u}(\mathbf{x}_2)+\mathbf{u}(\mathbf{x}_1)) \\ \dot{\mathbf{x}}_2 & = & (\mathbf{h}(\mathbf{x}_2,t)-2\mathbf{u}(\mathbf{x}_2))+(\mathbf{u}(\mathbf{x}_1)+\mathbf{u}(\mathbf{x}_2)) \end{array}\right. \tag{2.10}$$

equivalent to the system (2.9), one gets that, if $\mathbf{u}$ is odd in $\mathbf{x}$, the term $\mathbf{u}(\mathbf{x}_1)+\mathbf{u}(\mathbf{x}_2)$ in system (2.10) vanishes exponentially, so, for non-zero initial conditions, if the system:

$$\dot{\mathbf{x}} \ = \ \mathbf{h}(\mathbf{x},t)-2\mathbf{u}(\mathbf{x})$$

has a stable limit-cycle, both oscillators will continue oscillating, but with an opposite phase.

By convention, the two oscillators are said to be reaching *anti-synchrony*.

Applying this result to the coupled identical Van der Pol oscillators case, one gets as follows.

Consider two coupled identical Van der Pol oscillators:

$$\left\{\begin{array}{lcl} \ddot{x}_1+\alpha(x_1^2-1)\dot{x}_1+\omega^2 x_1 & = & \alpha\kappa(-\dot{x}_2-\dot{x}_1) \\ \ddot{x}_2+\alpha(x_2^2-1)\dot{x}_2+\omega^2 x_2 & = & \alpha\kappa(-\dot{x}_1-\dot{x}_2) \end{array}\right. , \tag{2.11}$$

where α and ω are both strictly positive constants.

The system (2.11) is equivalent to:

$$\left\{\begin{array}{rcl} \ddot{x}_1+\alpha(x_1^2+(2\kappa-1))\dot{x}_1+\omega^2 x_1 &=& \alpha\kappa(-\dot{x}_2+\dot{x}_1) \\ \ddot{x}_2+\alpha(x_2^2+(2\kappa-1))\dot{x}_2+\omega^2 x_2 &=& \alpha\kappa(-\dot{x}_1+\dot{x}_2) \end{array}\right. .$$

Defining the following functions:

$$\begin{array}{rcl} \mathbf{h}(\mathbf{x},t) &=& \left[\begin{array}{c} -\alpha(\frac{x^3}{3}+(2\kappa-1)x)+\omega y \\ -\omega x \end{array}\right], \\ \mathbf{u}(\mathbf{x}) &=& \left[\begin{array}{c} -\alpha\kappa x \\ 0 \end{array}\right], \end{array}$$

where $\mathbf{x}=\left[\begin{array}{c} x \\ y \end{array}\right]$, one gets that the system (2.11) is equivalent to the system (2.9).

The symmetric part of the Jacobian matrix of $\mathbf{h}$ is:

$$\mathbf{F}_s = \left[\begin{array}{cc} -\alpha(x^2+(2\kappa-1)) & 0 \\ 0 & 0 \end{array}\right],$$

which is uniformly negative semi-definite if $2\kappa-1>0$.

Therefore, for $\kappa>\frac{1}{2}$, $\mathbf{h}$ is semi-contracting, and $\mathbf{h}$ is odd in $\mathbf{x}$, so $\mathbf{x}_1+\mathbf{x}_2$ vanishes exponentially, regardless of the initial conditions.

Moreover, $\mathbf{u}$ is odd in $\mathbf{x}$, and the system:

$$\begin{array}{rcl} \dot{\mathbf{x}} &=& \mathbf{h}(\mathbf{x},t)-2\mathbf{u}(\mathbf{x}) \\ &=& \left[\begin{array}{c} -\alpha(\frac{x^3}{3}-x)+\omega y \\ -\omega x \end{array}\right] \end{array}$$

has a stable limit-cycle (according to Appendix A), which implies that the two oscillators will reach anti-synchrony exponentially, for non-zero initial conditions.

This result confirms what one might have intuited: anti-synchronisation takes place for strong enough inhibitory coupling forces.

2.2.4 Oscillator-Death

Consider an autonomous non-linear system of oscillators:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) \tag{2.12}$$

Suppose $\mathbf{f}$ is contracting (or semi-contracting).

One knows, by definition, that all the solutions of the system (2.12) converge exponentially to a single trajectory, regardless of the initial conditions. The system being autonomous, *id est* with no external force, the nil vector is a particular solution, which implies that all the solutions of the system (2.12) will tend exponentially to the nil vector, regardless of the initial conditions.

By convention, the oscillators are said to be reaching *oscillator-death.*

Applying this result to the coupled identical Van der Pol oscillators case, one gets as follows.

Consider two coupled identical Van der Pol oscillators:

$$\left\{ \begin{array}{rcl} \ddot{x}_1 + \alpha(x_1^2 - 1)\dot{x}_1 + \omega^2 x_1 & = & \alpha\kappa(\dot{x}_2 - \dot{x}_1) \\ \ddot{x}_2 + \alpha(x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 & = & \alpha\kappa(-\dot{x}_1 - \dot{x}_2) \end{array} \right. , \tag{2.13}$$

where α and ω are both strictly positive constants.

Defining the following function:

$$\mathbf{f}(\mathbf{x}, t) = \left[\begin{array}{c} -\alpha(\frac{x_1^3}{3} + (\kappa - 1)x_1) + \omega y_1 + \kappa x_2 \\ -\omega x_1 \\ -\alpha(\frac{x_2^3}{3} + (\kappa - 1)x_2) + \omega y_2 - \kappa x_1 \\ -\omega x_2 \end{array} \right],$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}$, one gets that the system (2.13) is equivalent to the system (2.12).

The symmetric part of the Jacobian matrix of $\mathbf{f}$ is:

$$\mathbf{F}_s = \left[\begin{array}{cccc} -\alpha(x_1^2 + (\kappa - 1)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(x_2^2 + (\kappa - 1)) & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

which is uniformly negative semi-definite if $\kappa - 1 > 0$.

Therefore, for $\kappa > 1$, the whole system is semi-contracting, as well as autonomous, which implies that the oscillations of the two oscillators will die.

This result confirms what one might have intuited: oscillator-death takes place for strong enough coupling forces: one in the excitatory way, and the other in the inhibitory way.

2.2.5 General Case

Consider two identical Van der Pol oscillators coupled in a general way:

$$\left\{ \begin{array}{rcl} \ddot{x}_1 + \alpha(x_1^2 - 1)\dot{x}_1 + \omega^2 x_1 & = & \alpha(\gamma \dot{x}_2 - \kappa \dot{x}_1) \\ \ddot{x}_2 + \alpha(x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 & = & \alpha(\gamma \dot{x}_1 - \kappa \dot{x}_2) \end{array} \right. , \tag{2.14}$$

where α and ω are both strictly positive constants, and κ is a positive constant.

According to what precedes, one gets the following results:

- if $\gamma > 1 - \kappa$, x_1 converges exponentially to x_2;
- if $\gamma < \kappa - 1$, x_1 converges exponentially to $-x_2$.

A study of the stable behaviour of the coupled system needs to be performed, in order to analyse whether it keeps oscillating or tends to a stationary equilibrium.

Assuming $\gamma > 1 - \kappa$, one gets:

$$\ddot{x}_i + \alpha(x_i^2 - 1)\dot{x}_i + \omega^2 x_i \;\sim\; \alpha(\gamma - \kappa)\dot{x}_i, \qquad i = 1, 2,$$

which gives the stable dynamics of x_1 and x_2 as:

$$\ddot{x}_i + \alpha(x_i^2 + (\kappa - \gamma - 1))\dot{x}_i + \omega^2 x_i \;=\; 0, \qquad i = 1, 2.$$

According to Appendix A, the dynamic equation has a stable limit-cycle if $\gamma > \kappa - 1$, and a stable equilibrium point at the origin otherwise.

Assuming $\gamma < \kappa - 1$, one gets:

$$\ddot{x}_i + \alpha(x_i^2 - 1)\dot{x}_i + \omega^2 x_i \;\sim\; -\alpha(\gamma + \kappa)\dot{x}_i, \qquad i = 1, 2,$$

which gives the stable dynamics of x_1 and x_2 as:

$$\ddot{x}_i + \alpha(x_i^2 + (\kappa + \gamma - 1))\dot{x}_i + \omega^2 x_i \;=\; 0, \qquad i = 1, 2.$$

According to Appendix A, the dynamic equation has a stable limit-cycle if $\gamma < 1-\kappa$, and a stable equilibrium point at origin otherwise.

From this study, one can conclude that:

- the two oscillators will reach synchrony if $\gamma > 0$ and $\gamma > |1-\kappa|$;
- the two oscillators will reach anti-synchrony if $\gamma < 0$ and $|\gamma| > |1-\kappa|$.

This result agrees with the common intuition that excitatory coupling leads to synchrony, while inhibitory coupling leads to anti-synchrony, for a strong enough coupling force.

Furthermore, if $\gamma = 0$, the system (2.14) is equivalent to two independent stable subsystems. Therefore, if $\kappa > 1$, both x_1 and x_2 tend to the origin, which can be considered as a continuous connection between $\gamma > 0$ and $\gamma < 0$. If $\kappa = 1$, x_1 and x_2 will keep oscillating for all $\gamma \neq 0$. Oscillator-death as a transition state between synchronised and anti-synchronised solutions does not exist except when $\gamma = 0$.

Consider now a coupled system with non-symmetric couplings:

$$\left\{ \begin{array}{rcl} \ddot{x}_1 + \alpha(x_1^2 - 1)\dot{x}_1 + \omega^2 x_1 & = & \alpha(\gamma_1 \dot{x}_2 - \kappa_1 \dot{x}_1) \\ \ddot{x}_2 + \alpha(x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 & = & \alpha(\gamma_2 \dot{x}_1 - \kappa_2 \dot{x}_2) \end{array} \right. , \tag{2.15}$$

where α and ω are both strictly positive constants.

Defining the following function:

$$\mathbf{f}(\mathbf{x}, t) = \begin{bmatrix} -\alpha(\frac{x_1^3}{3} + (\kappa_1 - 1)x_1) + \omega y_1 + \gamma_1 x_2 \\ -\omega x_1 \\ -\alpha(\frac{x_2^3}{3} + (\kappa_2 - 1)x_2) + \omega y_2 + \gamma_2 x_1 \\ -\omega x_2 \end{bmatrix},$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}$, one gets that the system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

is equivalent to the system (2.15).

Therefore, the symmetric part of the Jacobian of the system (2.15) is:

$$\mathbf{F}_s = \begin{bmatrix} -\alpha(x_1^2+(\kappa_1-1)) & 0 & \frac{\gamma_1+\gamma_2}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\gamma_1+\gamma_2}{2} & 0 & -\alpha(x_2^2+(\kappa_2-1)) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which can be shown, by a cofactor/minor analysis, to be uniformly negative semi-definite if:

$$\begin{aligned} \kappa_1 &\geq 1, \\ \kappa_2 &\geq 1, \\ (\kappa_1-1)(\kappa_2-1) &\geq \frac{(\gamma_1+\gamma_2)^2}{4}. \end{aligned}$$

From what precedes, if $\kappa_1 = 1$ and $\kappa_2 = 1$, the only way for the system to reach oscillator-death is $\gamma_1 = 0$ and $\gamma_2 = 0$.

Therefore, the condition for oscillator-death of a truly coupled system (2.15) is:

$$\begin{aligned} \kappa_1 &> 1, \\ \kappa_2 &> 1, \\ (\kappa_1-1)(\kappa_2-1) &\geq \frac{(\gamma_1+\gamma_2)^2}{4}. \end{aligned}$$

2.3 Dynamics of Simple Patterns of Coupled Identical Van der Pol Oscillators

This section analyses how the symmetries of the coupling forces can affect the synchronisation of simple arrays of coupled identical Van der Pol oscillators.

Four patterns often encountered in large networks in the natural world are studied: chain structure, one-way coupled ring, two-way coupled ring, all-to-all coupled star, and their threshold coupling values for synchronisation are compared.

In order to avoid repetitions, in this section, as well as in the rest of the paper, results for contracting functions or systems will also apply to semi-contracting functions or systems, in the coupled identical Van der Pol oscillators case, without further mention about it.

2.3.1 Chain Structure

Before getting started with the coupled identical Van der Pol oscillators case, one can easily extend the result of Theorem 1 to a network containing n oscillators with a chain structure.

Theorem 7 *Consider a chain of one-way coupled identical oscillators:*

$$\begin{cases} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, t) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, t) + \mathbf{u}_2(\mathbf{x}_1) - \mathbf{u}_2(\mathbf{x}_2) \\ & \dots \\ \dot{\mathbf{x}}_n &= \mathbf{f}(\mathbf{x}_n, t) + \mathbf{u}_n(\mathbf{x}_{n-1}) - \mathbf{u}_n(\mathbf{x}_n) \end{cases}, \tag{2.16}$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ are the state vectors, $\mathbf{f}(\mathbf{x}, t)$ describes the dynamics of the uncoupled oscillators, and $\mathbf{u}_i(\mathbf{x}_{i-1}) - \mathbf{u}_i(\mathbf{x}_i)$, for $i = 2, \ldots, n$, the coupling forces. If the functions $\mathbf{f} - \mathbf{u}_i$, for $i = 2, \ldots, n$, are contracting, all the oscillators will reach synchrony exponentially, regardless of the initial conditions.

Proof Applying the result of Theorem 1 to the first two subsystems of the system (2.16) with $\mathbf{f} - \mathbf{u}_2$ contracting, one gets that the second subsystem converges exponentially to the system:

$$\dot{\mathbf{x}}_2 = \mathbf{f}(\mathbf{x}_2, t). \tag{2.17}$$

One can, then, apply again the result of Theorem 1 to the system (2.17) and the third subsystem of the system (2.16) with $\mathbf{f} - \mathbf{u}_3$ contracting, and so on, until the last subsystem. $\square$

Applying Theorem 7 to the coupled identical Van der Pol oscillators case, one gets as follows.

Consider a chain of one-way coupled identical Van der Pol oscillators:

$$\begin{cases} \ddot{x}_1 + \alpha(x_1^2 - 1)\dot{x}_1 + \omega^2 x_1 &= 0 \\ \ddot{x}_2 + \alpha(x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 &= \alpha\kappa_2(\dot{x}_1 - \dot{x}_2) \\ & \dots \\ \ddot{x}_n + \alpha(x_n^2 - 1)\dot{x}_n + \omega^2 x_n &= \alpha\kappa_n(\dot{x}_{n-1} - \dot{x}_n) \end{cases}, \tag{2.18}$$

where α and ω are both strictly positive constants.

Defining the following functions:

$$
\begin{aligned}
\mathbf{f}(\mathbf{x},t) &= \begin{bmatrix} -\alpha(\frac{x^3}{3}-x)+\omega y \\ -\omega x \end{bmatrix}, \\
\mathbf{u}_j(\mathbf{x}) &= \begin{bmatrix} \alpha\kappa_j x \\ 0 \end{bmatrix},
\end{aligned}
$$

for $j = 2,\ldots,n$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, one gets that the system (2.18) is equivalent to the system (2.16).

The symmetric part of the Jacobian matrix of $\mathbf{f} - \mathbf{u}_j$, for $j = 2,\ldots,n$, is:

$$
\mathbf{F}_{s_j} = \begin{bmatrix} -\alpha(x^2+(\kappa_j-1)) & 0 \\ 0 & 0 \end{bmatrix},
$$

which is uniformly negative semi-definite if $\kappa_j - 1 > 0$.

Therefore, for $\kappa_j > 1$, $\mathbf{f} - \mathbf{u}_j$ is semi-contracting, for $j = 2,\ldots,n$, and the system:

$$
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)
$$

has a stable limit-cycle (according to Appendix A), which implies that all the oscillators will reach synchrony exponentially, for non-zero initial conditions.

2.3.2 All-to-All Coupled Star: S_n

Networks with S_n symmetry can be analysed using an immediate extension of Theorem 2.

Theorem 8 *Consider n coupled oscillators. If there exists a contracting function $\mathbf{h}(\mathbf{x},t)$ such that:*

$$
\dot{\mathbf{x}}_1 - \mathbf{h}(\mathbf{x}_1,t) = \cdots = \dot{\mathbf{x}}_n - \mathbf{h}(\mathbf{x}_n,t),
$$

then all the oscillators will synchronise exponentially, regardless of the initial conditions.

For instance, consider the following network of n identical oscillators coupled with diffusion type force (*id est* coupled on the amplitudes, and not on the velocities):

$$\dot{\mathbf{x}}_i \;=\; \mathbf{f}(\mathbf{x}_i, t) + \sum_{j=1}^{n}(\mathbf{u}(\mathbf{x}_j) - \mathbf{u}(\mathbf{x}_i)), \qquad i = 1, \ldots, n. \tag{2.19}$$

Contraction of $\mathbf{f} - n\mathbf{u}$ guarantees the synchronisation of the whole network.

Applying this result to the coupled identical Van der Pol oscillators case with S_4 symmetry, one gets as follows.

Consider the following S_n network of n coupled identical Van der Pol oscillators:

$$\ddot{x}_i + \alpha(x_i^2 - 1)\dot{x}_i + \omega^2 x_i \;=\; \alpha\kappa \sum_{j=1}^{n}(\dot{x}_j - \dot{x}_i), \qquad i = 1, \ldots, n, \tag{2.20}$$

where α and ω are both strictly positive constants.

Defining the following functions:

$$\begin{aligned} \mathbf{f}(\mathbf{x}, t) &= \begin{bmatrix} -\alpha(\frac{x^3}{3} - x) + \omega y \\ -\omega x \end{bmatrix}, \\ \mathbf{u}(\mathbf{x}) &= \begin{bmatrix} \alpha\kappa x \\ 0 \end{bmatrix}, \end{aligned}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, one gets that the system (2.20) is equivalent to the system (2.19).

The symmetric part of the Jacobian matrix of $\mathbf{f} - n\mathbf{u}$ is:

$$\mathbf{F}_s \;=\; \begin{bmatrix} -\alpha(x^2 + (n\kappa - 1)) & 0 \\ 0 & 0 \end{bmatrix},$$

which is uniformly negative semi-definite if $n\kappa - 1 > 0$.

Therefore, for $\kappa > \frac{1}{n}$, $\mathbf{f} - n\mathbf{u}$ is semi-contracting and the system:

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}, t)$$

has a stable limit-cycle (according to Appendix A), which implies that all the oscillators will reach synchrony exponentially, for non-zero initial conditions.

2.3.3 Two-Way Coupled Ring: D_n

To remain simple, the only case when $n = 4$ will be investigated here.

Consider the following coupled network with D_4 symmetry:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + (\mathbf{u}(\mathbf{x}_{i-1}) - \mathbf{u}(\mathbf{x}_i)) + (\mathbf{u}(\mathbf{x}_{i-1}) - \mathbf{u}(\mathbf{x}_i)), \qquad i = 1, 2, 3, 4, \tag{2.21}$$

where the subscripts $i-1$ and $i+1$ are computed circularly.

Combining these four oscillators into two groups $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$ and $\begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix}$, one gets the following system:

$$\begin{aligned} \begin{bmatrix} \mathbf{u}(\mathbf{x}_2) + \mathbf{u}(\mathbf{x}_4) \\ \mathbf{u}(\mathbf{x}_1) + \mathbf{u}(\mathbf{x}_3) \end{bmatrix} &= \begin{bmatrix} \dot{\mathbf{x}}_1 - \mathbf{f}(\mathbf{x}_1, t) + 2\mathbf{u}(\mathbf{x}_1) \\ \dot{\mathbf{x}}_2 - \mathbf{f}(\mathbf{x}_2, t) + 2\mathbf{u}(\mathbf{x}_2) \end{bmatrix} \\ &= \begin{bmatrix} \dot{\mathbf{x}}_3 - \mathbf{f}(\mathbf{x}_3, t) + 2\mathbf{u}(\mathbf{x}_3) \\ \dot{\mathbf{x}}_4 - \mathbf{f}(\mathbf{x}_4, t) + 2\mathbf{u}(\mathbf{x}_4) \end{bmatrix}. \end{aligned}$$

According to Theorem 8, if the function $\mathbf{f} - 2\mathbf{u}$ is contracting, $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$ and $\begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix}$ will converge to each other exponentially, regardless of the initial conditions.

Thus, if $\mathbf{f} - 2\mathbf{u}$ is contracting, one gets:

$$\begin{cases} \dot{\mathbf{x}}_1 - \mathbf{f}(\mathbf{x}_1, t) + 2\mathbf{u}(\mathbf{x}_1) & \sim \; 2\mathbf{u}(\mathbf{x}_2) \\ \dot{\mathbf{x}}_2 - \mathbf{f}(\mathbf{x}_2, t) + 2\mathbf{u}(\mathbf{x}_2) & \sim \; 2\mathbf{u}(\mathbf{x}_1) \end{cases},$$

which gives the following stable dynamics system:

$$\begin{cases} \dot{\mathbf{x}}_1 - \mathbf{f}(\mathbf{x}_1, t) + 4\mathbf{u}(\mathbf{x}_1) & = \; 2(\mathbf{u}(\mathbf{x}_1) + \mathbf{u}(\mathbf{x}_2)) \\ \dot{\mathbf{x}}_2 - \mathbf{f}(\mathbf{x}_2, t) + 4\mathbf{u}(\mathbf{x}_2) & = \; 2(\mathbf{u}(\mathbf{x}_1) + \mathbf{u}(\mathbf{x}_2)) \end{cases}.$$

Once again, according to Theorem 8, if the function $\mathbf{f} - 4\mathbf{u}$ is contracting, $\mathbf{x}_1$ and $\mathbf{x}_2$ will converge to each other exponentially, regardless of the initial conditions.

Hence, if $\mathbf{f} - 2\mathbf{u}$ and $\mathbf{f} - 4\mathbf{u}$ are contracting, the four oscillators will reach synchrony exponentially, regardless of the initial conditions.

Applying this result to the coupled identical Van der Pol oscillators case with D_4 symmetry, one gets as follows.

Consider the following D_4 network of four coupled identical Van der Pol oscillators:

$$\ddot{x}_i + \alpha(x_i^2 - 1)\dot{x}_i + \omega^2 x_i \;=\; \alpha\kappa((\dot{x}_{i-1} - \dot{x}_i) + (\dot{x}_{i+1} - \dot{x}_i)), \qquad i = 1,2,3,4 \quad (2.22)$$

where α and ω are both strictly positive constants, and where the subscripts $i-1$ and $i+1$ are computed circularly.

Defining the following functions:

$$\begin{aligned}
\mathbf{f}(\mathbf{x},t) &= \begin{bmatrix} -\alpha(\frac{x^3}{3} - x) + \omega y \\ -\omega x \end{bmatrix}, \\
\mathbf{u}(\mathbf{x}) &= \begin{bmatrix} \alpha\kappa x \\ 0 \end{bmatrix},
\end{aligned}$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, one gets that the system (2.22) is equivalent to the system (2.21).

The symmetric part of the Jacobian matrix of $\mathbf{f} - j\mathbf{u}$, for $j \in \mathbb{N}^*$ is:

$$\mathbf{F}_s \;=\; \begin{bmatrix} -\alpha(x^2 + (j\kappa - 1)) & 0 \\ 0 & 0 \end{bmatrix},$$

which is uniformly negative semi-definite if $j\kappa - 1 > 0$.

In particular, if $2\kappa - 1 > 0$ and $4\kappa - 1 > 0$, the functions $\mathbf{f} - 2\mathbf{u}$ and $\mathbf{f} - 4\mathbf{u}$ are semi-contracting, and the system:

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x},t)$$

has a stable limit-cycle (according to Appendix A), which means that a sufficient condition for the four oscillators to reach synchrony is: $\kappa > \frac{1}{2}$, for non-zero initial conditions.

2.3.4 One-Way Coupled Ring: Z_n

In order to analyse the case of the one-way coupled ring, the previous results need to be extended, using the concept of *partial contraction.*

Theorem 9 (Partial Contraction) *Consider a nonlinear system of the form:*

$$\dot{\mathbf{x}} \;=\; \mathbf{c}(\mathbf{x},t) + \mathbf{d}(\mathbf{x},t),$$

where $\mathbf{c}(\mathbf{x},t)$ *is contracting. If a particular solution verifies a specific property independent of the explicit form of* $\mathbf{d}(\mathbf{x},t)$*, then all system trajectories will verify this property exponentially, regardless of the initial conditions. The system is said to be* partially contracting.

Proof Construct an auxiliary system driven by the input $\mathbf{d}(\mathbf{x}(t),t)$:

$$\dot{\mathbf{y}} \;=\; \mathbf{c}(\mathbf{y},t) + \mathbf{d}(\mathbf{x}(t),t).$$

Since $\mathbf{c}(\mathbf{y},t)$ is contracting, the auxiliary system is contracting. Therefore, since:

$$\dot{\mathbf{y}}(t) - \mathbf{c}(\mathbf{y}(t),t) \;=\; \dot{\mathbf{x}}(t) - \mathbf{c}(\mathbf{x}(t),t),$$

Theorem 2 guarantees the exponential convergence of $\mathbf{x}(t)$ and $\mathbf{y}(t)$ towards each other, regardless of the initial conditions. If a particular solution $\mathbf{x}_0(t)$ verifies a specific property independent of the explicit form of $\mathbf{d}(\mathbf{x},t)$, then a particular solution $\mathbf{y}_0(t)$ verifies the same property for every possible input $\mathbf{d}(\mathbf{x}(t),t)$. Since the auxiliary system is contracting, all trajectories $\mathbf{y}(t)$ tend exponentially to $\mathbf{y}_0(t)$, so they will verify the property exponentially. In turn, since $\mathbf{x}(t)$ and $\mathbf{y}(t)$ tend to each other exponentially, this implies that all trajectories $\mathbf{x}(t)$ will verify the property exponentially. $\square$

Before getting started with the one-way coupled ring case, one may, first, want to define a few matrix notations, which will be re-used in the next sections as well. In what follows, "> 0" (respectively, "< 0", "≥ 0", "≤ 0") stands for: "is positive definite" (respectively, "is negative definite", "is positive semi-definite", "is negative semi-definite").

Definition Consider n square matrices $\mathbf{K}_i$ of identical dimensions, and define:

$$\mathbf{I}^n_{\mathbf{K}_i} \;=\; \begin{bmatrix} \mathbf{K}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{K}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{K}_n \end{bmatrix}$$

One has $\mathbf{I}^n_{\mathbf{K}_i} > 0$ if, and only if, $\mathbf{K}_i > 0, \forall i$.

Definition Consider a square symmetric matrix K, and define:

$$\mathbf{U}^n_{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & \mathbf{K} & \cdots & \mathbf{K} \\ \mathbf{K} & \mathbf{K} & \cdots & \mathbf{K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K} & \mathbf{K} & \cdots & \mathbf{K} \end{bmatrix}_{n\times n}$$

and

$$\mathbf{T}^n_{\mathbf{K}}(ij) = \begin{bmatrix} \ddots & \vdots & & \vdots & \\ \cdots & \mathbf{K} & \cdots & -\mathbf{K} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & -\mathbf{K} & \cdots & \mathbf{K} & \cdots \\ & \vdots & & \vdots & \ddots \end{bmatrix}_{n\times n},$$

where all the elements in $\mathbf{T}^n_{\mathbf{K}}(ij)$ except those already displayed in the four intersection points of the i^{th} and j^{th} rows and i^{th} and j^{th} columns are zero. It can be shown that:

$$\mathbf{K} \geq 0 \;\;\Rightarrow\;\; \mathbf{U}^n_{\mathbf{K}} \geq 0 \text{ and } \mathbf{T}^n_{\mathbf{K}}(ij) \geq 0.$$

To remain simple, the only case when $n = 4$ will be investigated here.

Consider the following coupled network with Z_4 symmetry:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \mathbf{K}(\mathbf{x}_{i-1} - \mathbf{x}_i), \qquad i = 1, 2, 3, 4, \tag{2.23}$$

where $\mathbf{K}$ is a square symmetric matrix, and the subscripts $i-1$ and $i+1$ are computed circularly.

The system (2.23) is equivalent to the following system:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) - \mathbf{K}(2\mathbf{x}_i + \mathbf{x}_{i+1} + \mathbf{x}_{i+2}) + \mathbf{K}\sum_{j=1}^{4} \mathbf{x}_j, \qquad i = 1, 2, 3, 4.$$

One may now construct an auxiliary system driven by the input $\mathbf{K}\sum_{j=1}^{4} \mathbf{x}_j(t)$:

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i, t) - \mathbf{K}(2\mathbf{y}_i + \mathbf{y}_{i+1} + \mathbf{y}_{i+2}) + \mathbf{K}\sum_{j=1}^{4} \mathbf{x}_j(t), \qquad i = 1, 2, 3, 4,$$

which admits the particular solution: $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{y}_3 = \mathbf{y}_4 = \mathbf{y}_\infty$, with:

$$\dot{\mathbf{y}}_\infty = \mathbf{f}(\mathbf{y}_\infty, t) - 4\mathbf{K}\mathbf{y}_\infty + \mathbf{K}\sum_{j=1}^{4}\mathbf{x}_j(t).$$

To apply Theorem 9 for the specific property $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = \mathbf{x}_4$, and prove that all solutions of the system (2.23) will verify this property exponentially, *id est* that all the $\mathbf{x}_i$ will synchronise exponentially, regardless of the initial conditions, there only remains to study the symmetric part of the Jacobian matrix of the auxiliary system:

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 - 2\mathbf{K} & -\mathbf{K} & -\mathbf{K} & 0 \\ 0 & \mathbf{F}_2 - 2\mathbf{K} & -\mathbf{K} & -\mathbf{K} \\ -\mathbf{K} & 0 & \mathbf{F}_3 - 2\mathbf{K} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{K} & 0 & \mathbf{F}_4 - 2\mathbf{K} \end{bmatrix},$$

where $\mathbf{F}_i = \frac{\partial \mathbf{f}(\mathbf{y}_i,t)}{\partial \mathbf{y}_i}$, and whose symmetric part is:

$$\begin{aligned} \mathbf{F}_s &= \begin{bmatrix} \mathbf{F}_{1_s} - 2\mathbf{K} & -\frac{\mathbf{K}}{2} & -\mathbf{K} & -\frac{\mathbf{K}}{2} \\ -\frac{\mathbf{K}}{2} & \mathbf{F}_{2_s} - 2\mathbf{K} & -\frac{\mathbf{K}}{2} & -\mathbf{K} \\ -\mathbf{K} & -\frac{\mathbf{K}}{2} & \mathbf{F}_{3_s} - 2\mathbf{K} & -\frac{\mathbf{K}}{2} \\ -\frac{\mathbf{K}}{2} & -\mathbf{K} & -\frac{\mathbf{K}}{2} & \mathbf{F}_{4_s} - 2\mathbf{K} \end{bmatrix}, \\ &= \mathbf{I}^4_{\mathbf{F}_{i_s}-\mathbf{K}} - \frac{1}{2}\mathbf{U}^4_{\mathbf{K}} - \frac{1}{2}\mathbf{F}_+, \end{aligned}$$

where:

$$\mathbf{F}_+ = \begin{bmatrix} \mathbf{K} & 0 & \mathbf{K} & 0 \\ 0 & \mathbf{K} & 0 & \mathbf{K} \\ \mathbf{K} & 0 & \mathbf{K} & 0 \\ 0 & \mathbf{K} & 0 & \mathbf{K} \end{bmatrix}.$$

One knows that if, for $i = 1, 2, 3, 4$, $\mathbf{F}_{i_s} - \mathbf{K} < 0$, then $\mathbf{I}^4_{\mathbf{F}_{i_s}-\mathbf{K}} < 0$, and if $\mathbf{K} \geq 0$, then $\mathbf{U}^4_{\mathbf{K}} \geq 0$ and $\mathbf{F}_+ \geq 0$. If both conditions are satisfied, the Jacobian $\mathbf{F}_s$ is negative definite, thus the auxiliary system is contracting, and all the $\mathbf{x}_i$ will synchronise exponentially.

Applying this result to the coupled identical Van der Pol oscillators case with Z_4 symmetry, one gets as follows.

Consider the following Z_4 network of four coupled identical Van der Pol oscillators:

$$\ddot{x}_i + \alpha(x_i^2 - 1)\dot{x}_i + \omega^2 x_i \;=\; \alpha\kappa(\dot{x}_{i-1} - \dot{x}_i), \qquad i = 1, 2, 3, 4, \tag{2.24}$$

where α and ω are both strictly positive constants, and where the subscripts $i-1$ and $i+1$ are computed circularly.

Defining the following function:

$$\mathbf{f}(\mathbf{x}, t) \;=\; \begin{bmatrix} -\alpha(\frac{x^3}{3} - x) + \omega y \\ -\omega x \end{bmatrix},$$

and coupling matrix:

$$\mathbf{K} \;=\; \begin{bmatrix} \alpha\kappa & 0 \\ 0 & \alpha\kappa \end{bmatrix},$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, one gets that the system (2.24) is equivalent to the system (2.23).

One gets that, for $i = 1, 2, 3, 4$, the matrix:

$$\mathbf{F}_{i_s} - \mathbf{K} \;=\; \begin{bmatrix} -\alpha(x_i^2 + (\kappa - 1)) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathbf{F}_{i_s}$ is the symmetric part of the matrix $\mathbf{F}_i = \frac{\partial \mathbf{f}(\mathbf{x}_i, t)}{\partial \mathbf{x}_i}$, is uniformly negative semi-definite if $\kappa - 1 > 0$.

Moreover, if $\kappa \geq 0$, one gets: $K \geq 0$.

Noticing, once again, that the system:

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}, t)$$

has a stable limit-cycle (according to Appendix A), one gets that a sufficient condition for the four oscillators to reach synchrony is: $\kappa > 1$, for non-zero initial conditions.

2.4 Dynamics of Larger Arrays of Coupled Identical Van der Pol Oscillators

The aim of this section is to find a synchronisation condition on the coupling force, by performing a partial contraction analysis of the dynamics of larger arrays of identical oscillators, with general coupling structures, before applying it to the particular case of identical Van der Pol oscillators, coupled with nearest-neighbour and sparse structures.

This analysis introduces links towards graph theory, that further research may want to explore.

2.4.1 General Coupling Structure

Consider the following network containing n identical oscillators coupled with diffusion forces:

$$\dot{\mathbf{x}}_i \;=\; \mathbf{f}(\mathbf{x}_i, t) + \mathbf{K} \sum_{j \in \mathcal{N}_i} (\mathbf{x}_j - \mathbf{x}_i), \qquad i = 1, \ldots, n, \tag{2.25}$$

where the coupling matrix $\mathbf{K}$ is symmetric and positive definite, and $\mathcal{N}_i$ denotes the set of indices of the active links of oscillator i.

In what follows, all the $\mathcal{N}_i$ are considered to be having the same number of elements, for $i = 1, \ldots, n$.

The system (2.25) is equivalent to the following system:

$$\dot{\mathbf{x}}_i \;=\; \mathbf{f}(\mathbf{x}_i, t) + \mathbf{K} \sum_{j \in \mathcal{N}_i} (\mathbf{x}_j - \mathbf{x}_i) - \mathbf{K}_0 \sum_{j=1}^{n} \mathbf{x}_j + \mathbf{K}_0 \sum_{j=1}^{n} \mathbf{x}_j, \qquad i = 1, \ldots, n,$$

where $\mathbf{K}_0$ is chosen to be a constant symmetric positive definite matrix.

As usual, one may now construct an auxiliary system driven by the input $\mathbf{K} \sum_{j=1}^{n} \mathbf{x}_j(t)$:

$$\dot{\mathbf{y}}_i \;=\; \mathbf{f}(\mathbf{y}_i, t) + \mathbf{K} \sum_{j \in \mathcal{N}_i} (\mathbf{y}_j - \mathbf{y}_i) - \mathbf{K}_0 \sum_{j=1}^{n} \mathbf{y}_j + \mathbf{K}_0 \sum_{j=1}^{n} \mathbf{x}_j(t), \qquad i = 1, \ldots, n,$$

which admits the particular solution: $\mathbf{y}_1 = \cdots = \mathbf{y}_n = \mathbf{y}_\infty$, with:

$$\dot{\mathbf{y}}_\infty \;=\; \mathbf{f}(\mathbf{y}_\infty, t) - n\mathbf{K}_0 \mathbf{y}_\infty + \mathbf{K}_0 \sum_{j=1}^{n} \mathbf{x}_j(t).$$

To apply Theorem 9 for the specific property $\mathbf{x}_1 = \cdots = \mathbf{x}_n$, and prove that all solutions of the system (2.25) will verify this property exponentially, *id est* that all the $\mathbf{x}_i$ will synchronise exponentially, regardless of the initial conditions, there only remains to study the symmetric part of the Jacobian matrix of the auxiliary system, which is:

$$\mathbf{F}_s = \mathbf{I}^n_{\mathbf{F}_{i_s}} - \frac{1}{2} \sum_{\substack{j \in \mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij) - \mathbf{U}^n_{\mathbf{K}_0},$$

where $\mathbf{F}_{i_s}$ is the symmetric part of: $\mathbf{F}_i = \frac{\partial \mathbf{f}(\mathbf{y}_i,t)}{\partial \mathbf{y}_i}$.

One may introduce the following definition, making reference to graph theory.

Definition Let:

$$\mathbf{J}_r = -\frac{1}{2} \sum_{\substack{j \in \mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij) - \mathbf{U}^n_{\mathbf{K}_0},$$

The network is said to be *connected* if: for $\mathbf{K}_0 > 0$ and $\mathbf{K} > 0$, $\mathbf{J}_r < 0$.

One can notice that the definition of J_r has a physical interpretation: it represents the network's geometric structure, whereas $\mathbf{I}^n_{\mathbf{F}_{i_s}}$ represents the oscillators' internal dynamics.

The purpose of the following analysis is to get a condition on the coupling force, in order to get:

$$\lambda_{max}(\mathbf{F}_s) < 0,$$

where the notation $\lambda_{max}(\mathbf{M})$ refers to the largest eigenvalue of a matrix $\mathbf{M}$.

Noticing that $\mathbf{F}_s = \mathbf{I}^n_{\mathbf{F}_{i_s}} + \mathbf{J}_r$, and that $\mathbf{K}_0$ and $\mathbf{K}$ are both symmetric, one gets the equality:

$$\lambda_{max}(\mathbf{F}_s) = \lambda_{max}(\mathbf{I}^n_{\mathbf{F}_{i_s}}) + \lambda_{max}(\mathbf{J}_r).$$

It is evident that, if $\mathbf{J}_r$ is only negative semi-definite, no condition on the coupling force will ever be found. Therefore, one has to prove that $\mathbf{K}_0$ can be chosen large enough to make $\mathbf{J}_r < 0$, and still not affect the *synchronisation rate*, defined as $\lambda_{max}(\mathbf{F}_s)$.

The largest eigenvalue of $\mathbf{J}_r$ can be calculated as follows [12]:

$$\begin{aligned}\lambda_{max}(\mathbf{J}_r) &= \max_{\|\mathbf{v}\|=1} \mathbf{v}^T\mathbf{J}_r\mathbf{v} \\ &= \max_{\|\mathbf{v}\|=1}\left(-\frac{1}{2}\mathbf{v}^T \sum_{\substack{j\in\mathcal{N}_i\\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij)\mathbf{v} - \mathbf{v}^T\mathbf{U}^n_{\mathbf{K}_0}\mathbf{v}\right).\end{aligned}$$

Since $\mathbf{K}_0 > 0$, $-\mathbf{v}^T\mathbf{U}^n_{\mathbf{K}_0}\mathbf{v}$ keeps decreasing as $\mathbf{K}_0$ increases, for all vector $\mathbf{v} = [v_i]_{i=1,\dots,n}$, except on the set $\sum_{i=1}^n v_i = 0$, so one can choose $\mathbf{K}_0$ large enough, and still get a condition on the coupling force:

$$\begin{aligned}\lambda_{max}(\mathbf{J}_r) &= \max_{\substack{\|\mathbf{v}\|=1\\ \sum_{i=1}^n v_i=0}}\left(-\frac{1}{2}\mathbf{v}^T \sum_{\substack{j\in\mathcal{N}_i\\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij)\mathbf{v}\right) \\ &= -\frac{1}{2}\min_{\substack{\|\mathbf{v}\|=1\\ \sum_{i=1}^n v_i=0}}\left(\mathbf{v}^T \sum_{\substack{j\in\mathcal{N}_i\\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij)\mathbf{v}\right) \\ &= -\lambda_{m+1}\left(\sum_{\substack{j\in\mathcal{N}_i\\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij)\right),\end{aligned}$$

according to the Courant-Fischer Theorem [11, 22], where the eigenvalues are arranged in an increasing order, and:

$$\lambda_1\left(\sum_{\substack{j\in\mathcal{N}_i\\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij)\right) = \cdots = \lambda_m\left(\sum_{\substack{j\in\mathcal{N}_i\\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij)\right) = 0,$$

where m is the dimension of the individual oscillator.

Note that, in the particular case $m = 1$, eigenvalue $\lambda_2 \left(\sum_{\substack{j \in \mathcal{N}_i \\ i=1,\ldots,n}} \mathbf{T}^n_{\mathbf{K}}(ij) \right)$ is a fundamental quantity in graph theory, named *algebraic connectivity*, which is equal to zero if and only if the graph is not connected.

From this analysis, one gets the following theorem:

Theorem 10 *Consider a network containing identical oscillators coupled with diffusion forces which are positive definite and symmetric in different directions. Assuming the network is connected, and the largest eigenvalue of* $\mathbf{F}_{i_s}$ *is bounded, all the coupled oscillators will reach synchrony exponentially if the coupling forces are strong enough.*

The two conditions to guarantee synchrony in Theorem 10 are the requirements to both the oscillators' internal dynamics and the network's geometric structure. The condition that the couplings have to be strong enough means:

$$\lambda_{m+1} \left(\sum_{\substack{j \in \mathcal{N}_i \\ i=1,\ldots,n}} \mathbf{T}^n_{\mathbf{K}}(ij) \right) > \max_{i=1,\ldots,n} \left(\lambda_{max}(\mathbf{F}_{i_s}) \right)$$

uniformly. An upper bound on the corresponding threshold can be computed through eigenvalue analysis if a special network is given.

2.4.2 Nearest-Neighbour and Sparse-Coupling in the Coupled Identical Van der Pol Oscillators Case

Consider the following general network composed of coupled identical Van der Pol oscillators:

$$\ddot{x}_i + \alpha(x_i^2 - 1)\dot{x}_i + \omega^2 x_i = \alpha\kappa \sum_{j \in \mathcal{N}_i} (\dot{x}_j - \dot{x}_i), \qquad i = 1, \ldots, n. \tag{2.26}$$

Defining the following matrices:

$$\begin{aligned}\mathbf{F}_{i_s} &= \begin{bmatrix} -\alpha(x_i^2-1) & 0 \\ 0 & 0 \end{bmatrix}, \qquad i=1,\dots,n,\\ \mathbf{K} &= \begin{bmatrix} \alpha\kappa & 0 \\ 0 & 0 \end{bmatrix},\end{aligned}$$

one gets, according to the previous partial contraction analysis, the Jacobian matrix:

$$\begin{aligned}\mathbf{F}_s &= \mathbf{I}^n_{\mathbf{F}_{i_s}} + \mathbf{J}_r \\ &= \mathbf{I}^n_{\mathbf{F}_{i_s}} - \frac{1}{2}\sum_{\substack{j\in\mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_{\mathbf{K}}(ij) - \mathbf{U}^n_{\mathbf{K}_0}.\end{aligned} \tag{2.27}$$

The problem with the system (2.27) is that the coupling matrix $\mathbf{K}$ is positive semi-definite, and not positive definite.

To apply Theorem 10, and thus get a condition on the coupling strength κ, one must, therefore, consider, first, the following system, obtained by ruling out the even rows and columns in the system (2.27):

$$\bar{\mathbf{F}}_s = \mathbf{I}^n_{\bar{\mathbf{F}}_{i_s}} - \frac{1}{2}\sum_{\substack{j\in\mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_{\bar{\mathbf{K}}}(ij) - \mathbf{U}^n_{\bar{\mathbf{K}}_0}, \tag{2.28}$$

with:

$$\begin{aligned}\bar{\mathbf{F}}_{i_s} &= -\alpha(x_i^2-1), \qquad i=1,\dots,n,\\ \bar{\mathbf{K}} &= \alpha\kappa.\end{aligned}$$

Thus, one gets the following condition for the negative definiteness of $\bar{\mathbf{F}}_s$:

$$
\begin{aligned}
\lambda_2 \left(\sum_{\substack{j \in \mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_{\bar{\mathbf{K}}}(ij) \right) &> \max_{i=1,\dots,n} \left(\lambda_{max}(\bar{\mathbf{F}}_{i_s}) \right) \\
\alpha\kappa \, \lambda_2 \left(\sum_{\substack{j \in \mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_1(ij) \right) &> \max_{i=1,\dots,n} \left(-\alpha(x_i^2 - 1) \right) \\
\kappa &> \frac{1}{\lambda_2 \left(\sum_{\substack{j \in \mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_1(ij) \right)},
\end{aligned} \tag{2.29}
$$

which guarantees the system (2.28) to be contracting, and, therefore, simultaneously, the system (2.27) to be semi-contracting, which ensures the exponential synchronisation of all the oscillators.

The previous result allows one to draw the following conclusions on the dependencies of the critical coupling strength K_c, which will be assessed in Chapter 3:

- The critical coupling strength depends on the non-linearity of the system.

  First of all, one should notice that the real coupling strength is not κ, but $K = \alpha\kappa$. Therefore, the condition (2.29) is actually:

  $$
  K > \frac{\alpha}{\lambda_2 \left(\sum_{\substack{j \in \mathcal{N}_i \\ i=1,\dots,n}} \mathbf{T}^n_1(ij) \right)},
  $$

  which means that the critical coupling strength K_c increases with the non-linearity α.

- The critical coupling strength depends on the configuration of the coupling.

  For instance, one might want to compare two systems:

- one using a *nearest-neighbour connectivity*, where each oscillator is coupled equally to its four nearest neighbours;
- one using a *sparse connectivity*, where each oscillator is coupled to a fixed number n_c of neighbouring oscillators, with a certain probability of connection depending on the distance between two oscillators - this probability will be fully described in Section 3.1.2.

Computing the values of the algebraic connectivity of the Laplacian matrix $\left(\sum_{j \in \mathcal{N}_i} \mathbf{T}_1^n(ij) \right)_{i=1,\ldots,n}$ for both cases, for the same number n of oscillators, one gets that the algebraic connectivity in the nearest-neighbour connectivity case is significantly smaller than in the sparse connectivity case, which means that the critical coupling strength is larger in the nearest-neighbour connectivity case than in the sparse connectivity case.

For example, for $n = 5^2$ and $n_c = 4$, for a given probability function, one gets: $K_{c_{nn}} = 2.61803$ for the nearest-neighbour connectivity case, and $K_{c_{sp}} = 0.341104$ for the sparse connectivity case.

More generally, one will see, in the next chapter, that the critical coupling strength depends on the average distance between two oscillators coupled to one another.

- The critical coupling strength depends on the size n of the system.

  Computing the values of the algebraic connectivity of the Laplacian matrix for different values of n, one gets, according to Figure 2.1, that the critical coupling strength increases logarithmically with n, within a certain range of n.

- The critical coupling strength depends on the number n_c of connections per oscillator.

  Computing the values of the algebraic connectivity of the Laplacian matrix, in the sparse connectivity case, for a given probability function, for different values of

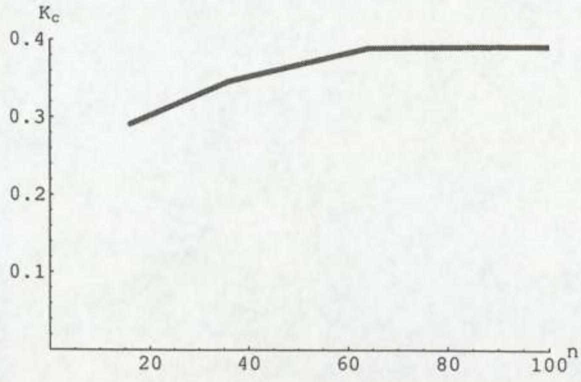

Figure 2.1: Theoretical results for the critical coupling-strength as a function of n, in the sparse connectivity case.

n_c, one gets, according to Figure 2.2, that the critical coupling strength decreases asymptotically to a positive constant as n_c increases.

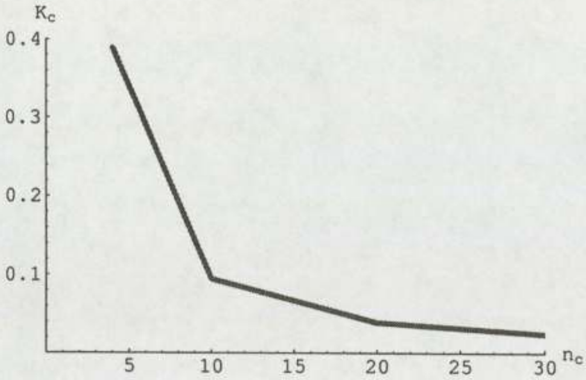

Figure 2.2: Theoretical results for the critical coupling-strength as a function of n_c, in the sparse connectivity case.

All these conclusions confirm what one might have intuited in the first place: synchronisation is easier for small networks of not too stiff oscillators, coupled with a lot of connections scanning the whole area of the network.

Chapter 3 will show that, in some cases, intuition might be wrong...

2.5 Limits of the Contraction Theory Approach for the Coupled Nonlinear Oscillators Problem

This section carries out a brief critical analysis of the contraction theory approach towards the coupled nonlinear oscillators problem, showing its restrictiveness and complexity.

2.5.1 Non-Optimality of the Contraction Analysis

From the results of Section 2.1, one sees that the fact that contraction of the dynamics implies synchronisation of the system does not mean that synchronisation of the system implies contraction of the dynamics, which can be summarised by saying that contraction theory is an implication, and not an equivalence.

Therefore, one might have a synchronising system which is not contracting. Thus, the conditions on the coupling strengths found during the contraction analysis performed all through Chapter 2 are not optimal, and one may find systems, with coupling strengths lying below the so-found thresholds, which might still synchronise.

Consider, for instance, the system (2.8) of two coupled identical Van der Pol oscillators, studied in Section 2.2.2.

The condition for contraction was found to be $\kappa_1 + \kappa_2 > 1$.

Consider, now, the representation through the time of the oscillations of the two coupled identical Van der Pol oscillators, taking $\kappa_1 = \kappa_2 = 0.1$.

One can see, in Figure 2.3, that the two oscillators reach synchrony, even though the values of the coupling strengths do not satisfy the conditions required for the system to be contracting.

Therefore, one might want to get more accurate conditions on the coupling strengths, performing numerical simulations. This will be the subject of Chapter 3.

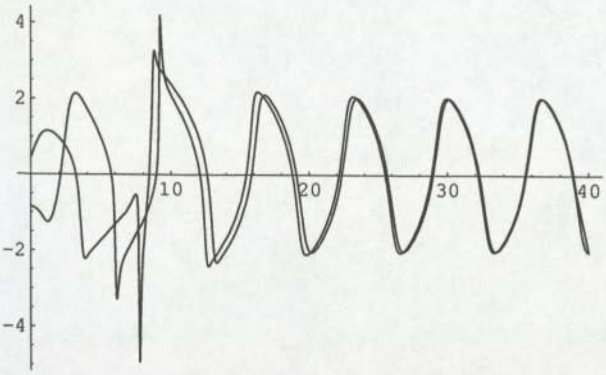

Figure 2.3: Simulation result of two coupled identical Van der Pol oscillators, with parameters: $\alpha = 1$, $\omega = 1$, $\kappa_1 = \kappa_2 = 0.1$, and random initial conditions.

2.5.2 Non-Generality of the oscillators dynamics

The analysis carried out all through Chapter 2 only deal with the case of coupled identical nonlinear oscillators. Contraction Theory does not give any result for coupled non-identical nonlinear oscillators.

Consider, for instance, the following system of two coupled non-identical Van der Pol oscillators:

$$\left\{ \begin{array}{rcl} \ddot{x}_1 + \alpha_1(x_1^2 - 1)\dot{x}_1 + \omega_1^2 x_1 & = & \alpha_1 \kappa_1 (\dot{x}_2 - \dot{x}_1) \\ \ddot{x}_2 + \alpha_2(x_2^2 - 1)\dot{x}_2 + \omega_2^2 x_2 & = & \alpha_2 \kappa_2 (\dot{x}_1 - \dot{x}_2) \end{array} \right. , \qquad (2.30)$$

and consider the representation through the time of the oscillations of the two coupled non-identical Van der Pol oscillators, taking distinct values for α_1 and α_2, and ω_1 and ω_2.

One can see, in Figure 2.4, that the two oscillators reach synchrony, even though their dynamics are non-identical.

Therefore, one might want to perform numerical simulations on coupled non-identical nonlinear oscillators, and get results regarding the different values of their parameters. This will be dealt with in Chapter 3.

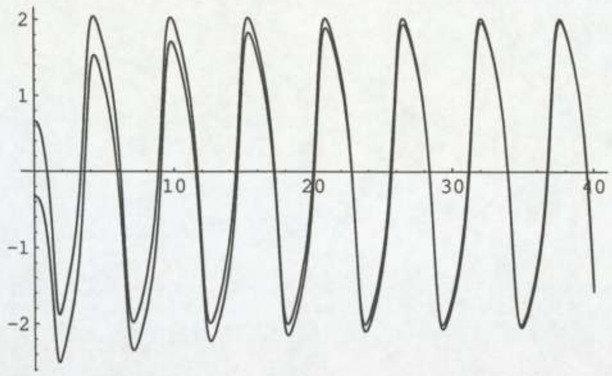

Figure 2.4: Simulation result of two coupled non-identical Van der Pol oscillators, with parameters: $\alpha_1 = 1$, $\alpha_2 = 1.5$, $\omega_1 = 1$, $\omega_2 = 1.25$, $\kappa_1 = 10$ and $\kappa_2 = 1$, and random initial conditions.

2.5.3 Complexity of the High-Dimensional Case

Even though the analysis performed in Section 2.4 is valid for all dimension n, the calculations of the algebraic connectivity of the Laplacian matrix, carried out in Section 2.4.2, show to be very demanding in terms of computational memory, power and speed.

Therefore, the calculations in Section 2.4.2 could not be performed for values of n beyond 12^2, as they had to deal with finding the eigenvalues of a matrix $n^2 \times n^2$...

The problem of finding a general analytical solution for the algebraic connectivity of the Laplacian matrix calls for some heavy analysis in Graph Theory, and has not been solved yet.

Simulations in Chapter 3 do not deal with the Laplacian matrix of the system, and, therefore, are able to look at networks with $n = 25$, or even more.

2.5.4 Exclusivity of the Synchronisation

Reading through Chapter 3, one sees that contraction theory only deals with the convergence of the trajectories of the oscillators, and not about other effects of self-entrainment.

Consider, for instance, the system (2.30) of two coupled non-identical Van der Pol oscillators and the representation through the time of their oscillations, for given values of the parameters.

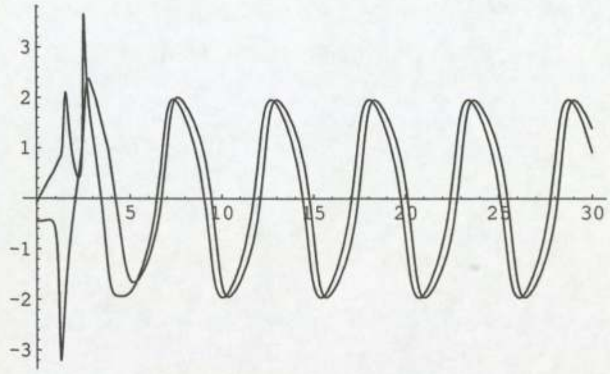

Figure 2.5: Simulation result of two coupled non-identical Van der Pol oscillators, with parameters: $\alpha_1 = 1$, $\alpha_2 = 1.5$, $\omega_1 = 1$, $\omega_2 = 1.5$ and $\kappa_1 = \kappa_2 = 1$, and random initial conditions.

One can see, in Figure 2.5, that the two oscillators do not reach phase-synchrony, but do reach pulse-synchrony, or *phase-locking*.

This phenomenon will be tackled in Section 3.3.1.

2.5.5 Absence of Spatial Behaviour Results

Mainly from the facts that contraction theory does not deal with either coupled non-identical nonlinear oscillators or other self-entrainment effects other than phase-synchronisation it, therefore, cannot say anything about the spatial behaviour of arrays of oscillators that may form different patterns according to the different distributions of their frequencies.

This aspect will be considered in the end of Chapter 3.

Chapter 3

Numerical Approach of the Dynamics of Networks of Coupled Non-Identical Nonlinear Oscillators

This chapter performs simulations of the dynamics of arrays of coupled non-identical Van der Pol oscillators.

After a presentation of the configuration of the network of coupled non-identical Van der Pol oscillators, a numerical analysis of the dependencies of the critical coupling-strength to different parameters is performed. The chapter ends with a brief overview of other self-entrainment aspects.

3.1 Configuration of the System

This section presents the configuration of the network on which the simulations are performed, as well as the different kinds of coupling connectivity used. It defines, as well, the order parameters used in Section 3.2 to analyse the conditions on the coupling-strength to get synchronisation.

3.1.1 Basic Equation of the Oscillator-Net

To make things not too complex, the oscillator-net used to perform the simulations is two-dimensional a square array of coupled non-identical Van der Pol oscillators. The two-dimensionality was chosen for two reasons:

- first, it is much closer to the physical arrays one may want to build, in the MEMS field, for instance;
- then, two dimensions are the best compromise between one-dimensional arrays, which barely show global self-entrainment effects, and three-dimensional arrays, which require bigger computational resources.

The equation of motion of the oscillator-net is written as:

$$\ddot{x}_i + \alpha_i(x_i^2 - 1)\dot{x}_i + \omega_i^2 x_i \;=\; K \sum_{j \in \mathcal{N}_i} (\dot{x}_j - \dot{x}_i), \qquad i = 1, \dots, n,$$

where α_i and ω_i are, respectively, the non-linearity and the natural frequency of the i^{th} oscillator, K is the coupling strength, $\mathcal{N}_i$ denotes the set of indices of the active links of the i^{th} oscillator, $n_c = \mathrm{card}(\mathcal{N}_i)$, for $i = 1, \dots, n$, is the number of active links of each oscillator, and n is the total number of oscillators.

One can notice that, contrary to Chapter 2, Chapter 3 does not consider κ, which was only a convenience for contraction analysis, but directly deals with the effective coupling strength $K = \alpha\kappa$, as this is the actual coupling strength one should set in order to build a physical array of such oscillators.

Moreover, from the definition of n_c, one can see that the number of connections is considered to be the same for each oscillator.

Simulations performed all through Chapter 3 usually use the following values for the parameters, unless the contrary is specified: $n = 25^2$, $n_c = 10$, and $\alpha_i = 1$, for $i = 1, \dots, n$.

The initial conditions for x_i and $\dot{x}_i$, for $i = 1, \dots, n$, are set to random complex numbers included in the square defined by $-(1 + \mathrm{i})$ and $1 + \mathrm{i}$, and are identical for every simulation. Each oscillator is, therefore, bi-dimensional, matching usual physical cases.

Simulations performed in Section 3.2 use a uniform distribution of the ω_i around the central frequency ω_0, with a bandwidth δ, where $\frac{\delta}{\omega_0} \ll 1$, to make the synchronisation easier to occur; Section 3.3.1 will show the effect of a too large δ. The natural frequency of the i^{th} oscillator is, for $i = 1, \ldots, n$, set as: $\omega_i = \omega_0 - \frac{\delta}{2} + \delta\frac{i-1}{n-1}$, and the oscillators are arranged on the square-lattice sites from the bottom left to the top right, in order of increasing natural frequencies. Usually, in what follows, simulations will be performed with: $\omega_0 = 10$ and $\delta = 0.1$.

3.1.2 Configuration of the Coupling Connectivity

Simulations performed all through Chapter 3 use two types of configuration for the coupling connectivity [15]:

- nearest-neighbour connectivity, where each oscillator is coupled equally to its four nearest neighbours;
- sparse connectivity, where each oscillator is coupled equally to a fixed number n_c of neighbouring oscillators, with a probability of connection, decreasing with the distance $d_{i,j}$ between two oscillators i and j, and defined by the following Gaussian probability density function:

$$p_{i,j} = \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{d_{i,j}}{2\sigma^2}},$$

  with mean zero and standard deviation σ.

Both configurations have a different physical meaning.

The nearest-neighbour connectivity represents a short-range coupling: even though it is the least difficult to build, it will also show to be the least effective for synchronisation.

The sparse connectivity is a long-range coupling: not as easy to build as the previous one, it will, nevertheless, prove to be more effective. Furthermore, the advantage of defining a Gaussian probability of connection, in comparison to, for instance, a random probability, is that one can control the average distance between two oscillators coupled to one another, which is proportional to the standard deviation σ: this might be quite helpful, and cost-saving, when time comes to physically build such an array.

3.1.3 Definition of the Order Parameters

In order to analyse whether, or not, synchronisation takes place, one needs to introduce a few parameters [19].

One may, first, define the average of the normalised complex amplitudes:

$$X(t) = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i(t)}{|x_i(t)|}.$$

It is easy to see that, if all the $x_i(t)$, for $i = 1, \ldots, n$, are equal at an instant t, one gets: $|X(t)| = 1$. Therefore, if there is synchronisation, *id est* if all the $x_i(t)$, for $i = 1, \ldots, n$ converge to a single trajectory, the magnitude of $X(t)$ approaches to unity, and the trajectory of $X(t)$ converges to the unity circle. On the opposite, if there is no self-entrainment, the motion of $X(t)$ remains exclusively inside the unity circle, and does not converge to a circle.

To be able to analyse the synchronisation effect, independently of the time, one might want to introduce the order parameter η of the equilibrium state and the fluctuation ζ of $|X(t)|$, defined, respectively, as:

$$\eta = \left(\langle |X(t)|^2\rangle\right)^{\frac{1}{2}},$$

and:

$$\zeta = \left(\langle (|X(t)| - \langle |X(t)|\rangle)^2\rangle\right)^{\frac{1}{2}},$$

where $\langle . \rangle$ is the long-time average.

The average being processed on the long time, the behaviour of the early steps of the simulation does not have an important weight in the values of η and ζ, for the benefit of the long-time behaviour, so it is not too difficult to figure out that, if there is synchronisation, the values of η and ζ will be, respectively, (around) 1 and 0.

One can notice, as well, from the form of its definition, that ζ will be zero even if there is only phase-locking, and not phase-synchronisation.

In Section 3.2, synchronisation will be considered reached if $\eta > 0.95$. The simulation results for ζ do not appear to be very accurate: their global λ-shaped behaviour is intuitively meaningful, but their deviation is too strong and, thus, does not enable one to get a precise result for K_c; therefore, they will not be taken into account for the analysis, which implies that the results for the critical coupling-strength will only deal with conditions for synchronisation, and not any other effect of self-entrainment, like phase-locking. This phenomenon will, nevertheless, be tackled in Section 3.3.1.

However, the results for ζ will still be displayed, for future research to try and find a reason for such a non-accuracy. Its definition might make it too sensitive...

3.2 Dependencies of the Critical Coupling-Strength

This section performs a numerical analysis of the dependencies of the critical coupling-strength to different parameters.

One must be aware that the range for the coupling-strength might vary from one figure to another, because of the different scale of the critical coupling-strength in each case.

3.2.1 Connectivity-Dependency

First, one might wonder if the configuration of the coupling connectivity is a determinant factor for the value of the critical coupling-strength.

This can be analysed by simulating the value of the order-parameter η, for different values of the coupling-strength K, using both nearest-neighbour connectivity and sparse connectivity. To be relevant, the number of connections in both cases must be the same; therefore, as it is equal to 4 in the nearest-neighbour connectivity case, n_c has to be set to 4 in the sparse connectivity case.

Figure 3.1 shows, in the nearest-neighbour connectivity case, that the order parameter η increases slowly as K increases, and barely reaches 0.8 for $K = 3$ (at least for this sample), whereas, in the sparse connectivity case, the transition from the low η state to the high η state is quite sharp, and η goes beyond 0.95 at $K = 0.6$, and stays

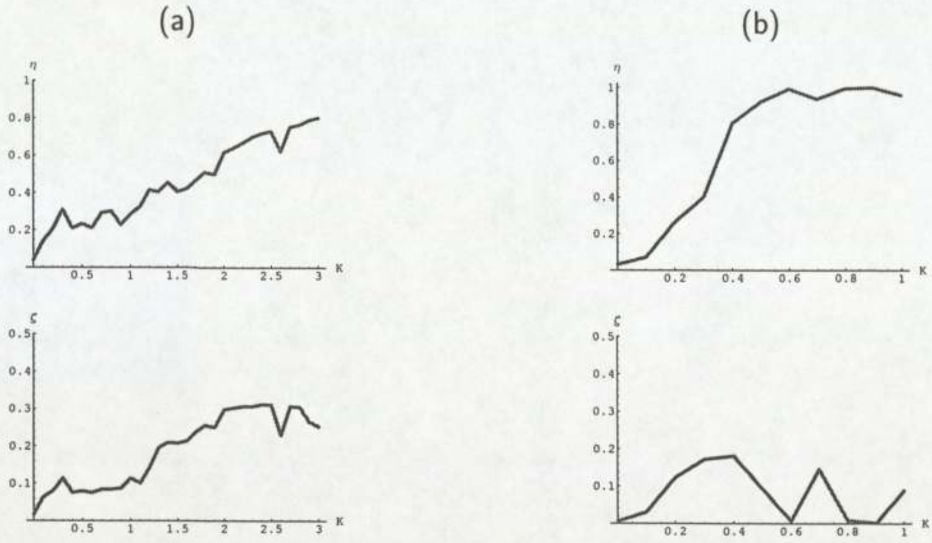

Figure 3.1: Simulations results for η and ζ, in the (a) nearest-neighbour connectivity, (b) sparse connectivity case, with parameters: $n_c = 4$, $\sigma = 6$.

close to 1 from that point.

One can, therefore, conclude that synchronisation occurs more easily, and for a significantly weaker coupling-strength, in the sparse connectivity case than in the nearest-neighbour connectivity case. This result confirms the intuition and the analysis carried out in Section 2.4.2.

Moreover, the same simulations (not displayed here), performed on another sample, *id est* changing the initial conditions, proved η to be quite sample-dependent in the nearest-neighbour connectivity case, and not in the sparse connectivity case.

Further simulations will, thus, be carried out on the sparse connectivity case only.

3.2.2 σ-Dependency

Following the idea, presented in Section 3.1.2, that being able to get a control on the average distance between two oscillators coupled to one another may be quite helpful for building purpose, one may, now, wonder whether the standard deviation σ of the Gaussian probability density function defining the probability of connection between two oscillators in the sparse connectivity case has an impact on the value of the critical coupling-strength.

This can be analysed by simulating the value of the order-parameter η, for different values of the coupling-strength K, using sparse connectivity, with different values of σ. As usual, to be relevant, the number of connections must be the same in each case.

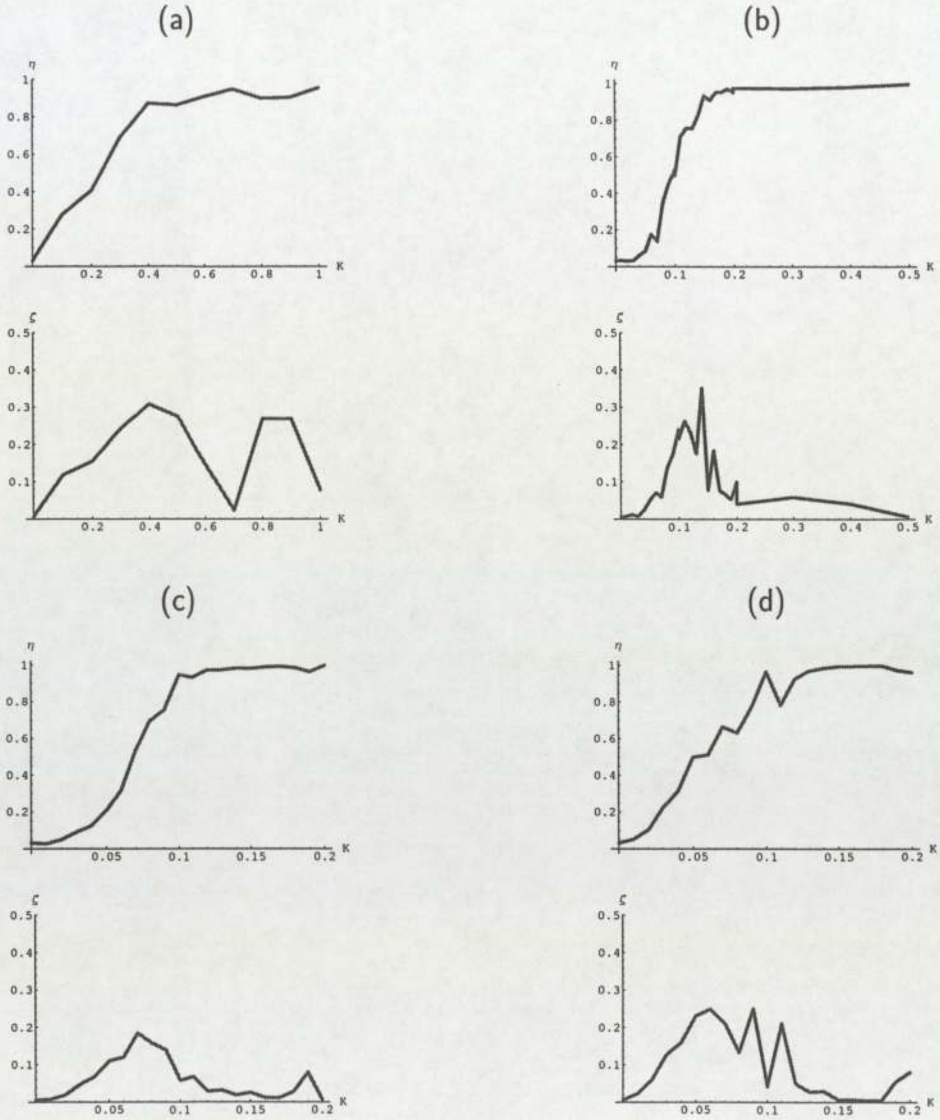

Figure 3.2: Simulations results for η and ζ, in the sparse connectivity case, with parameters: (a) $\sigma = 2$, (b) $\sigma = 6$, (c) $\sigma = 20$, (d) $\sigma = 50$.

Figure 3.2 confirms what one may have intuited: the critical coupling-strength is larger for a small σ (here, $K_c = 1$, for $\sigma = 2$) than for a large σ (here, $K_c = 0.17$, for $\sigma = 6$), which means that synchronisability increases with the average distance between two oscillators coupled to one another, within a certain range of σ.

However, Figure 3.3 shows that keeping increasing σ, beyond a certain threshold, does not have any influence on K_c any more. This can be explained by the fact that the

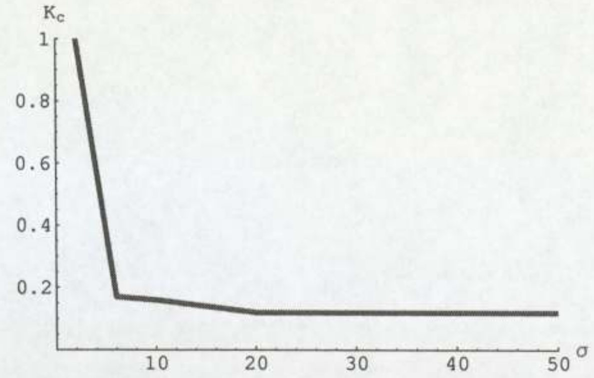

Figure 3.3: Simulations results for the critical coupling-strength as a function of σ, in the sparse connectivity case.

array analysed has a finite dimension, so the average distance between two oscillators coupled to one another cannot go beyond the value of this dimension.

From this analysis, one could conclude that it is possible to choose, easily, an optimal value of σ to have a minimal critical coupling-strength (in the analysed case, this value would be around 20). Nevertheless, one will see, in Chapter 4, that such a choice cannot be made so simply.

3.2.3 n_c-Dependency

A direct question one may, now, want to answer is whether the number of connections per oscillator has an influence on the critical coupling-strength.

This can be analysed by simulating the value of the order-parameter η, for different values of the coupling-strength K, using sparse connectivity, with different values of n_c. As usual, to be relevant, the values of the other parameters must be the same in each case.

Once again, Figure 3.4 proves the intuition, as well as the analysis performed in Section 2.4.2, to be right: the critical coupling-strength decreases significantly with the number of connections, within a certain range of n_c (from $K_c = 0.6$, for $n_c = 4$ to $K_c = 0.17$, for $n_c = 20$.)

However, Figure 3.5 shows that this significant decrease is not valid anymore beyond a certain point: from a certain number of connections, adding some more does

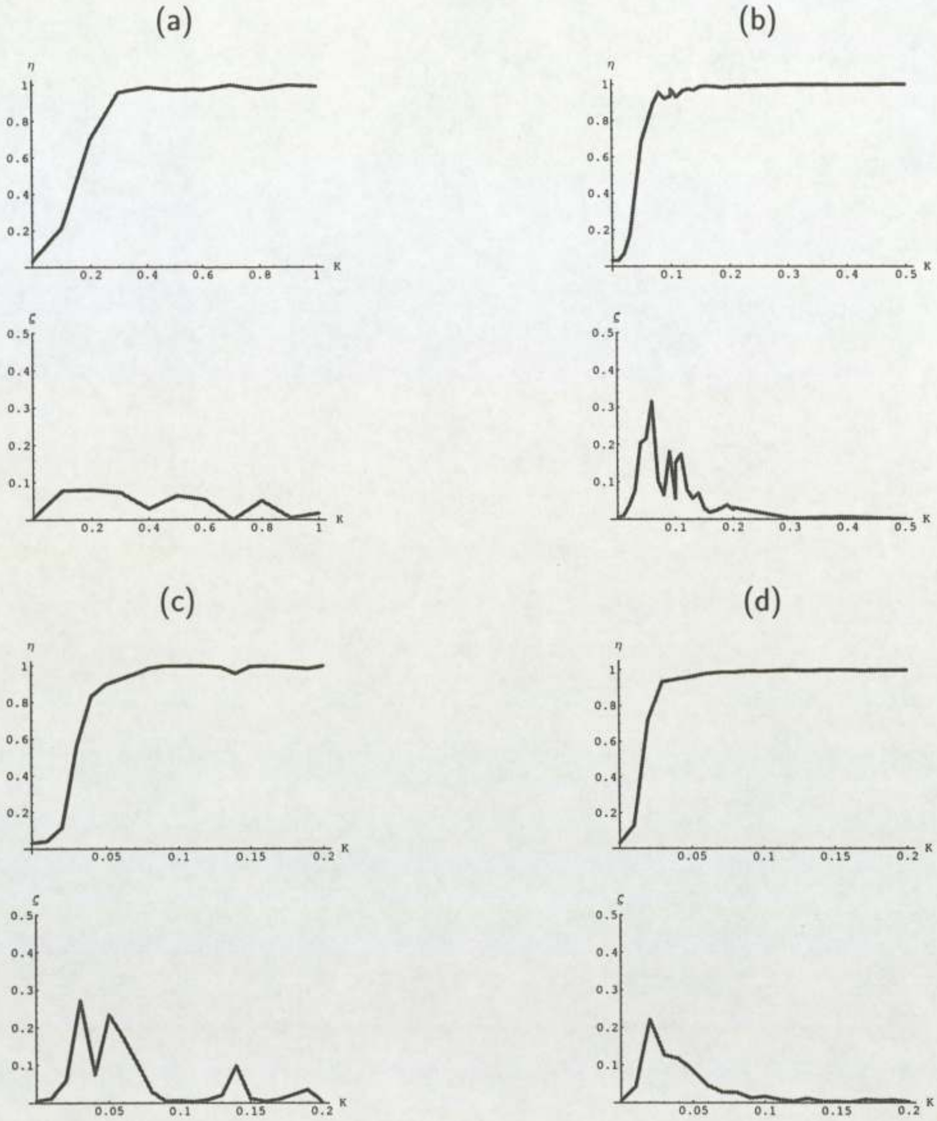

Figure 3.4: Simulations results for η and ζ, in the sparse connectivity case, with parameters: (a) $n_c = 7$, (b) $n_c = 20$, (c) $n_c = 30$, (d) $n_c = 50$.

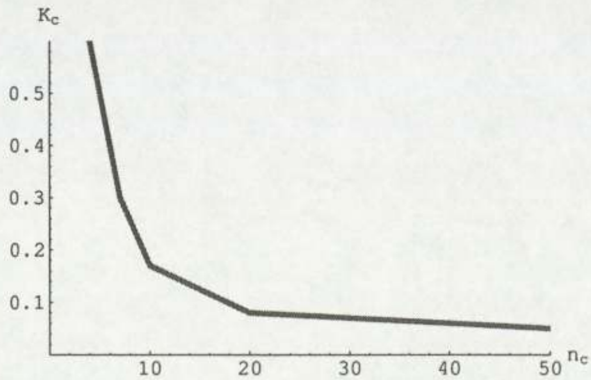

Figure 3.5: Simulations results for the critical coupling-strength as a function of n_c, in the sparse connectivity case.

not improve the synchronisation that much, anymore, and the critical coupling-strength seems to converge to a limit value, below which no synchronisation can occur, no matter how many connections are set.

The consequence of this result will be discussed in Chapter 4.

3.2.4 n-Dependency

A last dependency of the critical coupling-strength on the structure of the array might be interesting to study: the size of the array itself.

This can be analysed by simulating the value of the order-parameter η, for different values of the coupling-strength K, using sparse connectivity, with different values of n. As usual, to be relevant, the values of the other parameters must be the same in each case.

Once again, Figure 3.6 proves the intuition, as well as the analysis performed in Section 2.4.2, to be right: the critical coupling-strength increases with the size of the array, within a certain range of n (from $K_c = 0.12$, for $n = 100$ to $K_c = 0.19$, for $n = 400$.)

However, Figure 3.7 shows something that the analysis in Section 2.4.2 could not predict, because of the too small size of array it could handle with: the increase of the critical coupling-strength is not valid anymore beyond a certain point; from a certain size of the array, adding more oscillators does not affect the synchronisability of the system, and the critical coupling-strength remains constant, no matter how many oscillators are added.

The consequence of this result will be discussed in Chapter 4.

3.2.5 α-Dependency

After analysing the dependencies of the critical coupling-strength on the structure of the array, on may, now, wonder whether the structure of the oscillator itself has an influence or not on K_c. The case of the natural frequency will be dealt with in Section 3.3.1; therefore, the only case of the α-dependency will be exposed here.

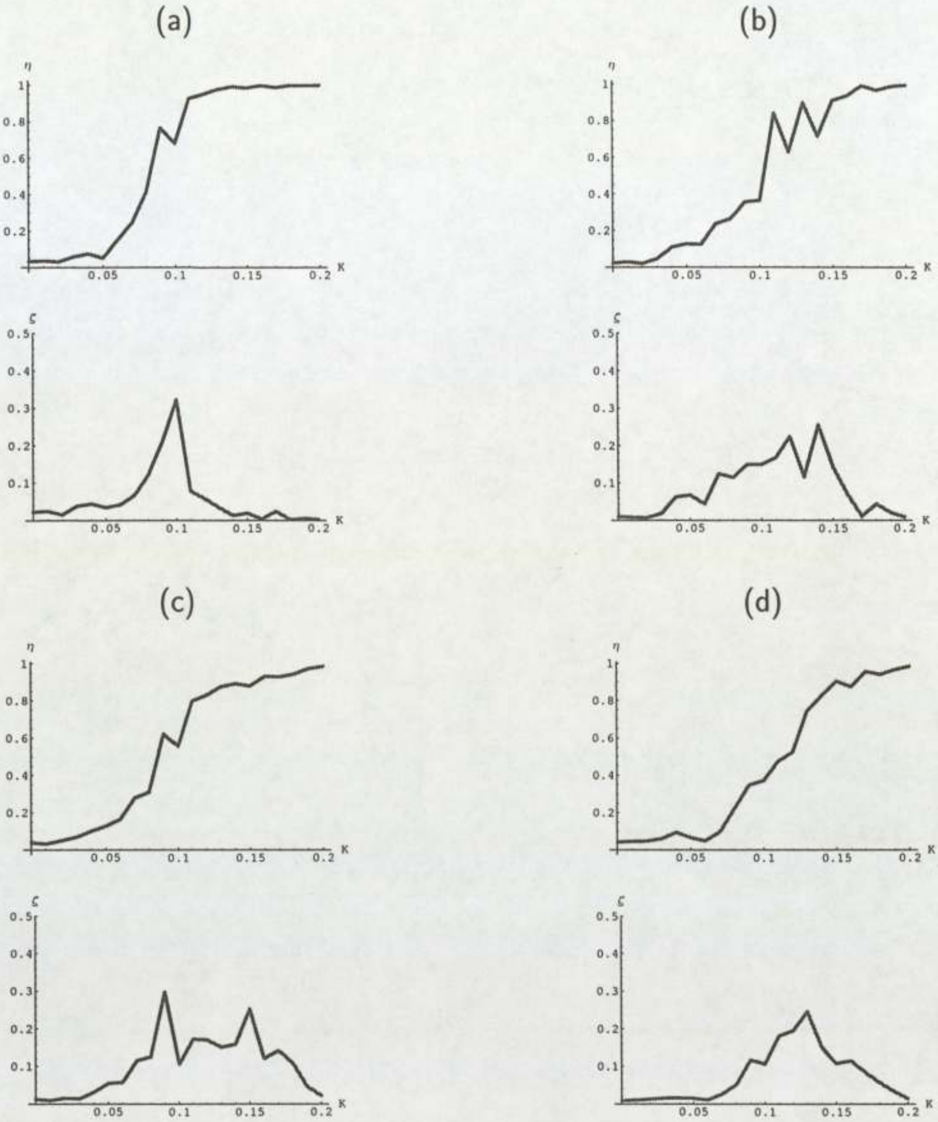

Figure 3.6: Simulations results for η and ζ, in the sparse connectivity case, with parameters: (a) $n = 10$, (b) $n = 15$, (c) $n = 20$, (d) $n = 30$.

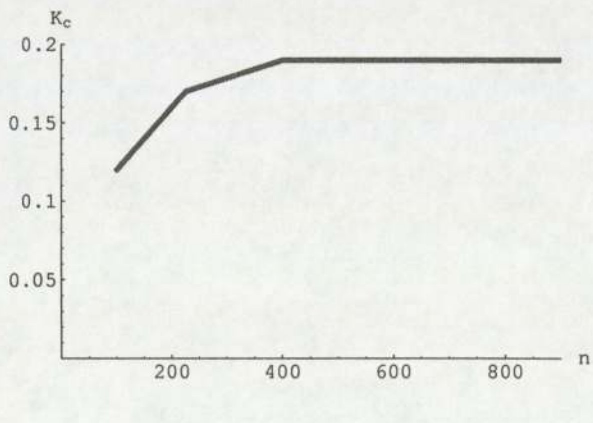

Figure 3.7: Simulations results for the critical coupling-strength as a function of n, in the sparse connectivity case.

As usual, the value of the order-parameter η will be simulated, for different values of the coupling-strength K, using sparse connectivity, with different values of α, and the same values for all the other parameters.

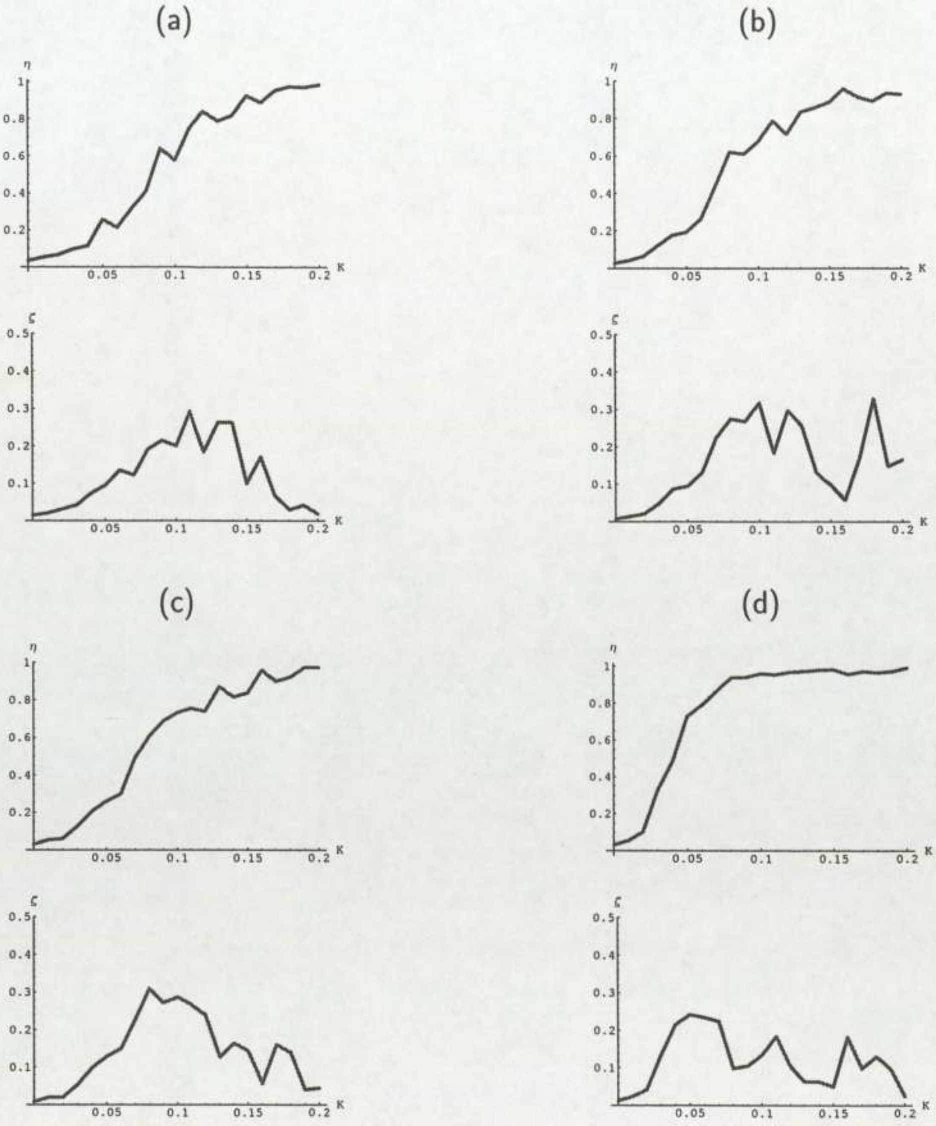

Figure 3.8: Simulations results for η and ζ, in the sparse connectivity case, with parameters: (a) $\alpha = 0.5$, (b) $\alpha = 0.1$, (c) $\alpha = 0.01$, (d) $\alpha = 0$.

Figure 3.8 and Figure 3.9 seem to show that the nonlinearity α has nearly no-influence on the critical coupling-strength K_c.

One must notice that this representation for small nonlinearity is deceiving: indeed, it seems to show that, for $\alpha = 0$, *id est* for a linear oscillator, synchronisation is still present..., which seems weird, given the fact that self-synchronisation only occurs

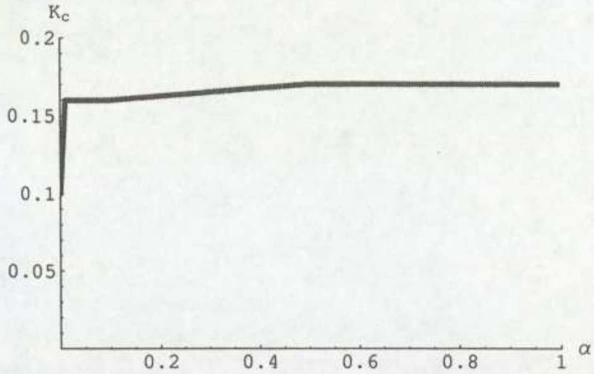

Figure 3.9: Simulations results for the critical coupling-strength as a function of α, in the sparse connectivity case.

within nonlinear systems.

This deceit comes from the definition of the order parameters, especially this of the average of the normalised complex amplitudes $X(t)$. Actually, $X(t)$ will converge to 1 whenever all the trajectories converge to a single, but will not say anything about the average amplitude of this single trajectory.

Therefore, one might want to define the average of the (non-normalised) complex amplitudes:

$$Y(t) \;=\; \frac{1}{n}\sum_{i=1}^{n} x_i(t),$$

and look at its behaviour through time for different values of α.

Figure 3.10 shows two interesting results:

- for α not too small (for example, $0.5 \leq \alpha \leq 1$), there is synchronisation, and the speed it occurs with increases with the nonlinearity; for a smaller α (for instance $\alpha = 0.1$), synchronisation is not reached within the time elapsed, but the increase in the amplitude of the simulations indicates it will be, after a while;

- for a nil α, there is no synchronisation: the convergence of the trajectories to a single one is a kind of self-entrainment which is not synchronisation, but only a resonance phenomenon; that is why the amplitude of the oscillations is very small (around 0.015, against around 2 in the case of synchronisation).

The consequence of this result will be discussed in Chapter 4.

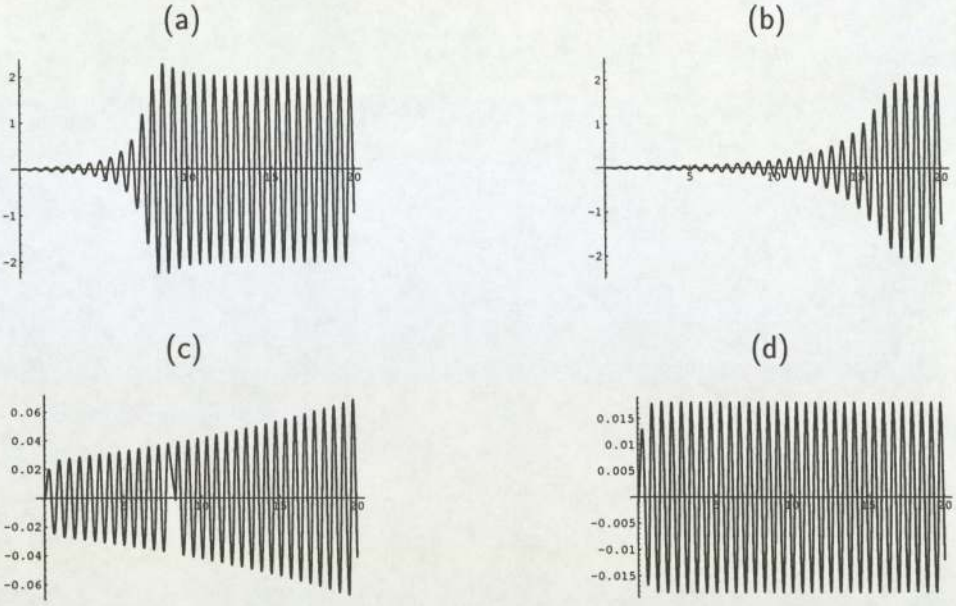

Figure 3.10: Simulations results for $Y(t)$, in the sparse connectivity case, with parameters: (a) $\alpha = 1$, (b) $\alpha = 0.5$, (c) $\alpha = 0.1$, (d) $\alpha = 0$.

3.3 Other Effects of Self-Entrainment

This section presents a brief overview of effects of self-entrainment, other than synchronisation: phase-locking and clusterisation.

3.3.1 Phase-Locking

As presented in Section 2.5.4, there exists a phenomenon when the oscillations are self-entrained, but do not reach phase-synchrony: instead, their frequencies synchronise, so their phases are locked to a constant difference; this phenomenon is called *phase-locking*, and can occur, for instance, for a small enough coupling-strength, when the bandwidth of the initial distribution of the natural frequencies is too large compared to the central frequency, *id est* if the condition $\frac{\delta}{\omega_0} \ll 1$, mentioned in Section 3.1.1, is not satisfied.

As this phenomenon can less easily be spotted using the technique of the order parameters, one might want to look directly at the spatial distribution of the frequencies.

Figure 3.11 shows that the phases of the oscillators are not synchronised, but that their frequencies have reached synchrony and converged to the central frequency ω_0.

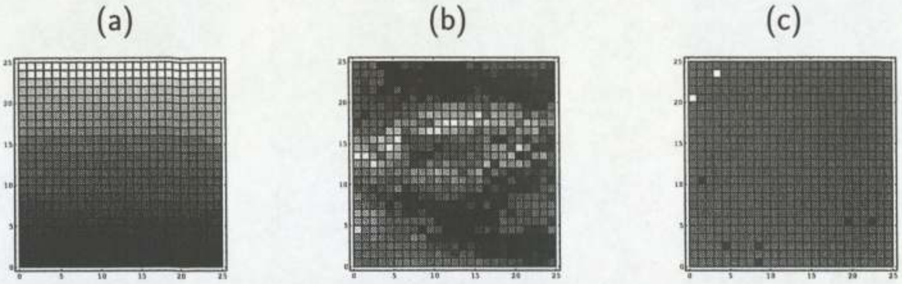

Figure 3.11: Simulations results for the spatial distribution of the (a) natural frequencies, (b) phases, (c) frequencies, with parameters: $\omega_0 = 10$, $\delta = 1$, $\sigma = 2$ and $K = 0.05$.

3.3.2 Clusterisation

Another self-entrainment phenomenon can also be observed when only part of the oscillators reach synchrony, to form different groups, called *clusters*, of coherent oscillators; this phenomenon is called *clusterisation*, and can occur, for instance, for a small enough coupling-strength, when the initial distribution of the natural frequencies shows specific properties, for example one or two Gaussian peaks.

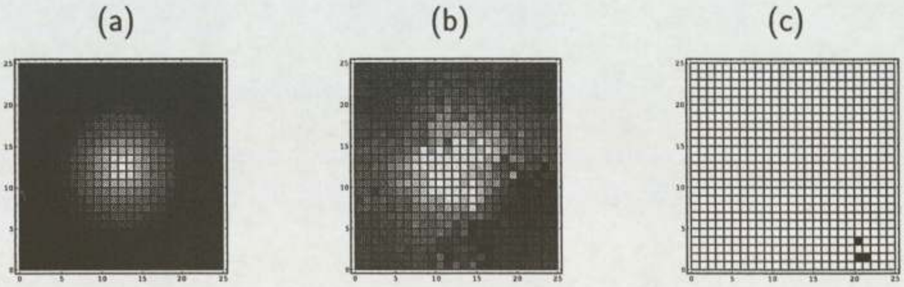

Figure 3.12: Simulations results for the spatial distribution of the (a) natural frequencies, (b) phases, (c) frequencies, in the one-Gaussian-peaked case, with parameters: $\omega_0 = 10$, $\sigma_{Gaussian} = 4$, $\sigma = 2$ and $K = 0.05$.

Figure 3.12 shows a case of particular case of phase-locking, where the oscillators behave like two macro-oscillators - one in the middle, one dispatched in the four corners - that have reached anti-synchronisation.

Figure 3.13 shows a case of particular case of phase-locking, where the oscillators behave like two macro-oscillators - one in the diagonal, one dispatched in two of the corners - that have reached anti-synchronisation.

These behaviours may be interesting to study, in the future, as they might reduce

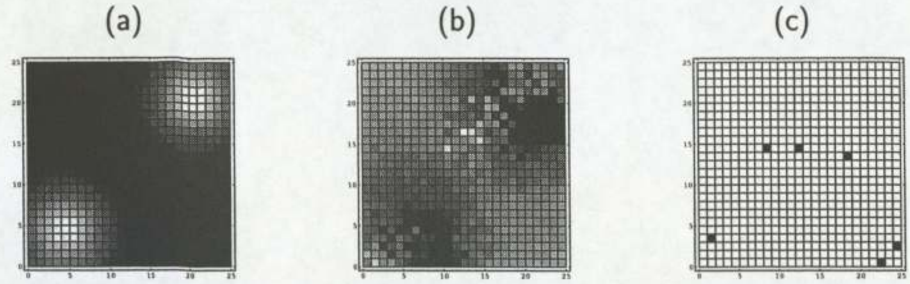

Figure 3.13: Simulations results for the spatial distribution of the (a) natural frequencies, (b) phases, (c) frequencies, in the two-Gaussian-peaked case, with parameters: $\omega_0 = 10$, $\sigma_{Gaussian} = 4$, $\sigma = 2$ and $K = 0.05$.

a large array problem, to a two-oscillators macro-problem.

Chapter 4

Conclusion

4.1 Summary

This paper has been carrying out an analysis, first theoretical, then numerical, of the global behaviour of arrays of coupled nonlinear oscillators, according to the values of its different parameters (coupling-strengths, coupling symmetries and configurations, numbers of oscillators and connections, natural frequencies distributions, nonlinearity).

In a first place, a theoretical analysis, based on Contraction Theory, studied the different behaviours of arrays of coupled identical nonlinear oscillators, according to the different values of the coupling strengths used.

This analysis, simple and general, gave exact and global results, applicable to any type of coupled identical nonlinear oscillators. Although its efficiency for arrays of small numbers of oscillators, arranged in simple patterns, cannot be denied, its application to large networks with general coupling structures proved to be, in practice, equivalent to a very complex problem, calling for a good deal of Graph Theory, including results that are still to be found.

In addition, this analysis was too restrictive, mainly, as it could not deal with the case of coupled non-identical nonlinear oscillators, which can show more interesting results.

A numerical analysis was, therefore, to be carried out on larger arrays of coupled non-identical nonlinear oscillators.

Using order parameters meant to show whether, or not, synchronisation occurred, this analysis displayed some conditions on the coupling-strength for the studied array to reach synchrony, according to other parameters related to either the oscillators' internal dynamics, or the network's geometric structure.

The end of the analysis extended the study on other effects of self-entrainment, exploring the spatial behaviour of the array.

4.2 Opening to Future Research

The arrays studied in this paper do not mean to remain only theoretical: one might actually want to build them.

In this optic, one may want to choose optimal parameters for synchronisation, in accordance with the results found.

However, one must be careful, and remember that the choice of the parameters have a great impact, not only on the synchronisability of the array (which the previous analysis tried to display), but also on the actual cost of such a project.

Thus, one might want to keep in mind that the factors of increasing cost are:

- the number of oscillators;
- the coupling-strengths;
- the type of connectivity used: a sparse connectivity is much more difficult to build than a nearest-neighbour connectivity, which can directly use the conductivity properties of the lattice the oscillators are set on;
- the number of physical connections;
- the lengths of the physical connections;
- the nonlinearity of each oscillator: a nonlinear oscillator is more difficult to build than a linear oscillator;
- the total time required for the experiment to run.

Therefore, one may want to build a cost-function, to analyse the optimal solution to the following problems:

- Increasing the number of connections allows the critical coupling-strength to be smaller, which is cost-saving, but it is, itself, cost-increasing;
- Increasing the average distance between two oscillators coupled to one another allows the critical coupling-strength to be smaller, which is cost saving, but it is, itself, cost-increasing;
- Increasing the nonlinearity of each oscillator increases the speed of synchronisation, which is cost-saving, but it is, itself, cost-increasing;
- Using a sparse connectivity allows the critical coupling-strength to be smaller, which is cost-saving, but it is, itself, cost-increasing, compared to using a nearest-neighbour connectivity.

One should also keep in mind that there exists a minimal critical coupling-strength, which, from a certain point, cannot be decreased, not matter how many connections are added to the network. This statement may, first, seem to be a drawback, but it may prove to be quite useful in practice. It comes from the fact that an array using sparse connectivity is quite robust: from a certain point, adding oscillators or connections does not affect the critical coupling-strength. In turn, this means that, for a large enough array, with many enough connections, loosing a few oscillators or connections - which might happen in a physical array - does not affect the critical coupling-strength, and, thus, the synchronisability of the system...

Other aspects of the problem have not been tackled in this paper, and may be the object of further research.

In order to reduce the analysis to a simpler small array problem, one might, first, want to try and get a clusterisation of the large array, in order to get an equivalent system of a few coupled macro-oscillators, and, then, perform a simple analysis on this small number of macro-oscillators.

Instead of only looking at an array using either nearest-neighbour connectivity, or sparse connectivity, one might want to try to build an array using a combination of the two: for instance, one could consider the behaviour of an array using a strong nearest-neighbour connectivity - easier to build - and a weak sparse connectivity - easier to get synchronisation.

Considering the fact that it has been proved [18] that perfect synchronisation for short-ranged coupling is only possible in arrays of at least three dimensions, another idea would be to try and build three-dimensional arrays by superposing layers of two-dimensional arrays coupled to one another, and then get an array which would be easier and faster to synchronise, for smaller coupling-strengths, and using a nearest-neighbour connectivity, which would be cheaper.

A last issue one might want to consider, if the array is to be built in the MEMS field: the dynamics studied all through the paper may not be applicable when it comes to the small-world dynamics, and, therefore, one may need to analyse the problem using quantum physics...

Appendix A

Study of the Stable-Limit Cycle Behaviour of a Single Van der Pol Oscillator

There are plenty of ways of studying the stable-limit cycle behaviour of a single Van der Pol oscillator, among which is one using Liénard's Theorem.

Theorem 11 (Liénard's Theorem) *Consider the following system:*

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)y. \end{cases} \tag{A.1}$$

Suppose $f(x)$ and $g(x)$ satisfy the following conditions:

- *$f(x)$ and $g(x)$ are continuously differentiable for all x;*
- *$g(x)$ is an odd function;*
- *$g(x) > 0$ for $x > 0$;*
- *$f(x)$ is an even function;*
- *The odd function $F(x) = \int_0^x f(u)\mathrm{d}u$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.*

Then, the system (A.1) has a unique, stable limit cycle surrounding the origin in the phase plane.

Applying this result to the single Van der Pol oscillator case, one gets as follows.

Consider the following Van der Pol oscillator:

$$\ddot{x} + (\beta + \alpha x^2)\dot{x} + \omega^2 x \;=\; 0, \tag{A.2}$$

where α and ω are strictly positive constants.

Defining the following functions:

$$\begin{aligned} f(x) &= \beta + \alpha x^2, \\ g(x) &= \omega^2 x, \end{aligned}$$

one gets that the system (A.2) is equivalent to the system (A.1).

Therefore, one can notice that:

- $f(x)$ and $g(x)$ are continuously differentiable for all x;
- $g(x)$ is an odd function;
- $g(x) > 0$ for $x > 0$;
- $f(x)$ is an even function;
- $F(x) = (\beta + \frac{\alpha}{3}x^2)x$, which has a zero if, and only if, $\beta < 0$; in that case, $a = \sqrt{-3\frac{\beta}{\alpha}}$, and $F(x)$ is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

One can, therefore, conclude that the Van der Pol oscillator has a unique, stable limit-cycle surrounding the origin in the phase plane, if, and only if, $\beta < 0$.

Bibliography

[1] J. T. Ariaratnam and S. H. Strogatz. Phase Diagram for the Winfree Model of Coupled Nonlinear Oscillators. *Physical Review Letters*, 86(19):4278–4281, 2001.

[2] M. Bennett, M. F. Schatz, H. Rockwood, and K. Wiesenfeld. Huygens's clocks. *The Royal Society in London A*, 458:563–579, 2002.

[3] H. Daido. Lower Critical Dimension for Populations of Oscillators with Randomly Distributed Frequencies: A Renormalization-Group Analysis. *Physical Review Letters*, 61(2):231–234, 1988.

[4] H. Gould and J. Tobochnik. Normal Modes and Waves. Lecture notes available on the World-Wide Web, 2000.

[5] G. P. Imponente. Complex Dynamics of the Biological Rythms: Heart and Gallbladder Cases. Presentation slides available on the World-Wide Web, 2003. `http://www.pv.infn.it/ frontier/2003/talks/Posters/Imponente2.pdf`.

[6] A. Jadbabaie, N. Motee, and M. Barahona. On the Stability of the Kuramoto Model of Coupled Nonlinear Oscillators. Article available on the World-Wide Web, 2004. `http://www.seas.upenn.edu/ jadbabai/papers/kuramoto_final.pdf`.

[7] J. Jouffroy. A Relaxed Criterion for Contraction Theory: Application to an Underwater Vehicle Observer. Article available on the World-Wide Web, 2003. `http://www.cesos.ntnu.no/ jouffroy/Publications/Jouffroy-2003av.pdf`.

[8] Y. Kuramoto. *Chemical Oscillations, Waves, and Turbulence.* Springer Verlag, Berlin edition, 1984.

[9] C. Lighty. Algebraic Graph Theory. Lecture notes available on the World-Wide Web. `http://www.math.mcgill.ca/ vetta/CS760.dir/chrisl.pdf`.

[10] P. J. Martinez, L. M. Floria, F. Falo, and J. J. Mazo. Intrinsically Localized Chaos in Discrete Nonlinear Extended Systems. *Europhysics Letters*, 45(4):444–449, 1999.

[11] Misc. Égalité de Courant-Fischer. Lecture notes available on the World-Wide Web. `http://www.bibmath.net/exercices/mps/quadratique.pdf`.

[12] Misc. Formes Quadratiques sur un Espace Vectoriel Euclidien. Applications Géométriques. Lecture notes available on the World-Wide Web. `http://www.reunion.iufm.fr/dep/mathematiques/DocFC/LAgregI/AlgTourn/FormesQD.pdf`.

[13] Misc. Formes Sesqi-Linaires et Produit Scalaire. Lecture notes available on the World-Wide Web. `http://irrotationnels.epfl.ch/documentation/pdf/AlgLin_SJ/AlgebreLineaireChapitre11.pdf`.

[14] Misc. Dérivation. Lecture notes available on the World-Wide Web, 2000. `http://www.prepasciences.com/maths/sup/Chapitre_06_-_Derivation.pdf`.

[15] E. Niebur, H. G. Schuster, D. M. Kammen, and C. Koch. Oscillator-Phase Coupling for Different Two-Dimensional Network Connectivities. *Physical Review A*, 44(10):6895–6904, 1991.

[16] S. Pati. The Third Smallest Eigenvalue of the Laplacian Matrix. *The Electronic Journal of Linear Algebra*, 8:128–139, 2001.

[17] J. D. Rosenhouse. Algebraic Graph Theory. Lecture notes available on the World-Wide Web. `http://www.math.ksu.edu/ jasonr/book7.pdf`.

[18] H. Sakaguchi, S. Shinomoto, and Y. Kuramoto. Local and Global Self-Entrainments in Oscillator Lattices. *Progress of Theoretical Physics*, 77(5):1005–1010, 1987.

[19] K. Satoh. Computer Experiment on the Cooperative Behavior of a Network of Interacting Nonlinear Oscillators. *Journal of the Physical Society in Japan*, 58(6):2010–2021, 1989.

[20] J. J. E. Slotine and W. Wang. A Study of Synchronization and Group Cooperation Using Partial Contraction Theory. *Block Island Workshop on Cooperative Control*, Kumar V. Editor, Springer Verlag, 2003.

[21] J. J. E. Slotine, W. Wang, and K. El Rifai. Contraction Analysis of Synchronization in Networks of Nonlinearly Coupled Oscillators. Sixteenth International Symposium on Mathematical Theory of Networks and Systems, Belgium, 2004.

[22] D. A. Spielman. Applied Extremal Combinatorics - Lecture 2. Lecture notes available on the World-Wide Web, 1998. `http://www-math.mit.edu/ spielman/AEC/lect2.ps`.

[23] S. H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.* Reading, MA, Perseus Books, Cambridge MA edition, 1994.

[24] S. H. Strogatz. Exploring Complex Networks. *Nature*, 408:268–276, 2001.

[25] S. H. Strogatz. *Sync: The Emerging Science of Spontaneous Order.* Hyperion, New York edition, 2003.

[26] S. H. Strogatz, R. E. Mirollo, and P. C. Matthews. Coupled Nonlinear Oscillators Below the Synchronization Threshold: Relaxation by Generalized Landau Damping. *Physical Review Letters*, 68(18):2730–2733, 1992.

[27] S. H. Strogatz and I. Stewart. Coupled Oscillators and Biological Synchronization. *Scientific American*, 269(6):102–109, 1993.

[28] S. H. Strogatz and D. J. Watts. Collective Dynamics of 'Small-World' Networks. *Nature*, 393:440–442, 1998.

[29] W. Wang and J. J. E. Slotine. Fast Computation with Neural Oscillators. Submitted to *Physical Review E*, 2003.

[30] W. Wang and J. J. E. Slotine. On Partial Contraction Analysis for Coupled Nonlinear Oscillators. Submitted to *Biological Cybernetics*, 2003.

[31] W. Wang and J. J. E. Slotine. Adaptive Synchronization in Coupled Dynamics Networks. Submitted to 43rd IEEE Conference on Decision and Control, 2004.

[32] W. Wang and J. J. E. Slotine. Where To Go and How To Go: a Theoretical Study of Different Leader Roles. Submitted, 2004.

[33] A. T. Winfree. Biological Rhythms and the Behavior of Populations of Coupled Oscillators. *Journal of Theoretical Biology*, 16(1):15–42, 1967.

[34] A. T. Winfree. *The Geometry of Biological Time.* Springer, New York edition, 1980.