PH.D. THESIS

THE APPLICATION OF FINITE SUMMATION WAVEFORMS TO MULTIPLEX SIGNAL SYSTEMS

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SUMMARY

This thesis is concerned with telecommunications systems, and is an investigation into some aspects of multiplex signal systems. It is shown that a generalised theoretical approach to multiplex systems may be made, in terms of orthogonal functions. Some signal processing methods applicable to the interconversion of frequency-division-multiplex and time-division-multiplex systems are considered.

The properties of waveforms consisting of a selected group of harmonics are examined, and it is shown that these may be used as the basis of a time-division-multiplex system which requires only the minimum transmission bandwidth. The properties of the spectra of regular pulse sequences are derived, and it is shown that improved methods may be found for generating the set of carriers for a frequency-division-multiplex system.

ACKNOWLEDGEMENT

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CONTENTS

List of symbols and abbreviations.

T	periodic time = 1/frequency = 1/f = $\omega/2\pi$
τ	time displacement, also periodic time, where $\tau < T$
d	pulse duration
$f(t), g(t)$	general functions of time
$F(\omega), G(\omega)$	Fourier transforms of $f(t)$ and $g(t)$
$\delta(t)$	unit impulse function
$f^*(x)$	complex conjugate of $f(x)$
$f(x)_* g(x)$	convolution of $f(x)$ and $g(x)$
$\varphi(x)$	member of a set of orthogonal functions
$m(t)$	message, or a modulating or baseband signal
$D_n(t)$	periodic Dirichlet kernel, i.e. the summation of a set of harmonically related cosinewaves
$S_n(t)$	the summation of a set of odd-numbered, harmonically related sinewaves, with alternating polarity
prf	pulse-repetition-frequency
w.f.	weighting function
fdm	frequency-division-multiplex
tdm	time-division-multiplex
am	amplitude-modulated
pam	pulse-amplitude-modulated
dsb	double-sideband
ssb	single-sideband
LPF	lowpass filter
BPF	bandpass filter

CHAPTER 1

GENERALISED APPROACH TO LINEAR MULTIPLEXING

1.1 Introduction

The basic feature of a multiplex system is the simultaneous transmission of a number of signals along a common path. Each signal originates from a separate message source and all the messages have common characteristics in that they may all lie within the same bandwidth, and occur during arbitrary and possibly overlapping time intervals. Thus, any direct superposition would provide a composite from which the contributing messages could not be separated. The multiplex system must therefore change their form in some manner, such that from a linear combination of the modified messages any one of the composing signals may be separated and restored to the original form.

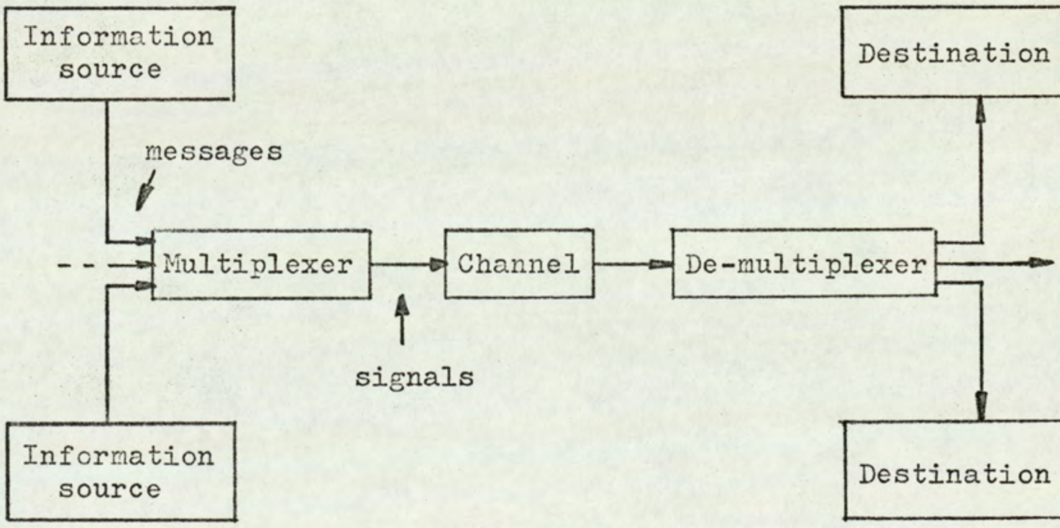


Figure 1

As shown in Figure 1, the sources may be said to produce information in the form of 'messages', which may not be combined directly until after conversion to 'signals' by the multiplexing process. The processes of multiplexing and de-multiplexing are assumed to be linear, and to operate in conjunction with a loss-free channel.

It is also assumed that the messages occur as variations in the amplitude of a voltage or current, and are physically representable as time varying waveforms. Then each is representable over some interval in Fourier series form, as the sum of a set of amplitude weighted harmonics. Thus, each message is a function of time, frequency, and phase, whilst the range within which these three parameters vary for any one message may be the same, or at least overlap, the range of any other message. As a function cannot be limited in both time and frequency, the assumption is made that there is some amplitude level below which a parameter has negligible value, and that the range limitation of a message is defined in this way.

Since overlapping ranges prohibit the direct superposition of messages, it is, perhaps, evident that all that is required is to translate each message to a different range, such that there is no overlapping. Techniques for achieving a translation in time, frequency, or phase, have long been known, and the two methods commonly used to realise a multiplex system were developed from this intuitive viewpoint.

Thus, frequency translation is obtained by forming the product of the message frequencies and a suitable carrier frequency; signal recovery is by bandpass filtering or an equivalent process, and message recovery by demodulation and filtering. Translation of the frequencies of a set of messages using carriers sufficiently different so as to produce non-overlapping sidebands is then the basis of frequency-division-multiplexing.

Time-division-multiplexing requires that the messages occur in pulse modulated form, or be first converted to that form, with sufficient interval between each pulse of a message to allow interpolation of the pulse waveforms of other messages.

Time translation may be achieved by delaying, or if the message is originally continuous, by different sampling instants. Time selective gating is used to separate a specific message, equivalent to time domain filtering.

It will be seen that, although the signals of a frequency-division-multiplex system continue to overlap in time, and the signals of a time-division-multiplex system share a common frequency spectrum, multiplexing is nevertheless effective, since the signal separation process discriminates only in the appropriate domain. The essential feature in either case is the translation of messages to some non-overlapping basis from which subsequent signal separation is possible.

1.2 Generalised approach of Zadeh and Miller

Although conventional frequency-division-multiplex and time-division-multiplex systems have been extensively developed and discussed, relatively little has appeared concerned with a generalised, or unifying, approach to multiplex systems. One such approach, however, was made in 1952 in a series of related papers by Zadeh and Miller, (references 1, 2, 3), who define a linear multiplex system as one in which the signal separation is effected by linear filters.

That is, if one source produces a signal s_1, and a second source a signal s_2, the operation performed by a filter F_1 is:

$$F_1(s_1 + s_2) = s_1$$

which must be satisfied for all values of s_1 and s_2. If all possible values of s_1 are in a set S_1, and all possible values of s_2 in a set S_2, it is shown that these must be linear sets. Furthermore, since the conditions that

$$F_1 \; s_1 = s_1, \text{ and } F_1 \; s_2 = 0$$

cannot be satisfied if S_1 and S_2 have any element in common, then the sets must be disjoint.

That these two properties of linearity and disjointness are the only necessary properties for a linear multiplex system is demonstrated by means of a geometric representation. Both these and subsequent authors employ the geometric model of signals in a vector space, first used in a communication theory context by Kotel'nikov, (reference 4).

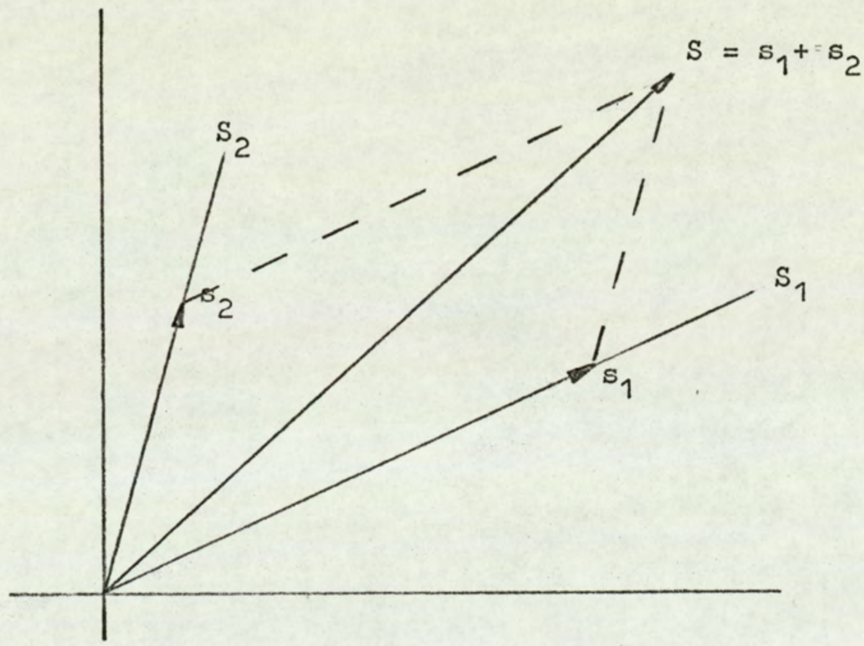


Figure 2

As shown in Figure 2, the linear disjoint subspaces S_1 and S_2 are represented by two straight lines intersecting at the origin, these being the locii of all possible values of the signals s_1, s_2. The composite signal which is the result of adding two signals s_1 and s_2 is shown as the vector sum, S. The operation of the filter F_1 is then represented by the projection of S on to S_1 along S_2. To obtain a transfer function for the filter, the transformation matrix is formed which transforms (or maps) S on to the subspace S_1. This matrix is constructed by combining terms from the matrix of coordinates of the defining vectors of the subspaces with terms from its inverse. If the signals have been defined as time domain functions, the transformation matrix gives the impulse response for F_1.

It is deduced from the representation of Figure 2 that the dimensions of S must be at least equal to the sum of the dimensions of S_1 and S_2 for signal recovery to be possible. This implies, for example, that the bandwidth of the common channel in a frequency-division- multiplex system may not be less than the sum of the bandwidths of the component messages.

Zadeh and Miller also describe an alternative purely analytical representation for a multiplex system. In the Fourier transform of a signal,

$$f(t) = \frac{1}{2\pi} \int F(\omega) \exp(j\omega t) \, d\omega$$

the terms under the integral are the set of complex exponentials exp (jωt), and a weighting function, F(ω).

A more general representation is

$$f(t) = \int F(\lambda) \, k(t,\lambda) \, d\lambda$$

where $k(t,\lambda)$ represents an arbitrary set of functions, or component signals, and $F(\lambda)$ remains a weighting function. Thus $k(t,\lambda)$ is used to define a λ - domain, of which the time and frequency domains are special cases. In the time domain $k(t,\lambda) = \delta(t - \lambda)$, with $\lambda =$ time, and in the frequency domain $k(t,\lambda) = \frac{1}{2\pi} \exp(\lambda t)$, with $\lambda = j\omega$. Multiplexing is achieved by allocating non-overlapping sets of values of λ to different messages.

An expression for the filter is deduced in terms of the unit impulse response, i.e. to an impulse $\delta(t - \tau)$, and is

$$h_1(t,\tau) = \int k(t,\lambda) \, k^{-1}(\lambda,\tau) \, d\lambda$$

where the inverse function $k^{-1}(\lambda, \tau)$ is obtained from

$$\int k(t,\lambda)\; k^{-1}(\lambda,\tau)\; d\lambda = \delta(t - \tau)$$

This impulse response is the continuous equivalent of the transformation matrix deduced from the geometrical representation.

The authors conclude that a multiplex system may be synthesised either by choosing a suitable set of component signals generated by the function $k(t, \lambda)$, and allocating non-overlapping sets of values to different messages or, with the geometrical approach, by choosing suitable linear and disjoint subspaces for allocation to different messages.

1.3 Use of orthogonal functions

In the discussion just considered, Zadeh and Miller point out that the chief difficulty of the λ-domain approach is that inverse functions are known for relatively few functions of the form $k(t, \lambda)$. It might appear, however, that there are further difficulties which have restricted practical exploitation of their analysis. Since an electrical signal has only the parameter domains of time, frequency, and phase, presumably a multiplex system must make use of one, or a combination of these three, which tends to limit the generality of the λ-domain. Furthermore, the requirement that a suitable set of functions $k(t,\lambda)$ or a suitable set of linear and disjoint subspaces be chosen is also perhaps too general.

Unless the synthesised system is to be purely theoretical, the functions chosen must be capable of being generated by means that are feasible from an engineering point of view, but otherwise there is little to guide the choice of what is 'suitable'. The filter response functions obtained will, in general, be lengthy expressions and, although these can, in theory, be realised by weighted tappings of delay lines, the difficulties encountered in engineering such filters may not be acceptable in practice.

There is, however, one class of functions which meet the general requirements, namely orthogonal functions. These are highly suitable for multiplexing purposes, and indeed are the basis of both time-division and frequency-division-multiplex systems. The λ- domain and geometric representations discussed previously might be regarded as the continuous and discrete versions of the same problem, and similarly orthogonal functions may appear in a continuous or discrete form.

Geometrically, orthogonal functions are representable as a set of mutually perpendicular vectors, such as the axes of a rectangular coordinate system. Algebraically, two functions $f_m(x)$, $f_n(x)$ are said to be orthogonal over the interval x_1, x_2 if

$$\int_{x_1}^{x_2} f_m(x)\, f_n^*(x)\, dx = \begin{cases} 0, & m \neq n \\ \text{a constant}, & m = n \end{cases}$$

where $f^*(x)$ denotes the complex conjugate of $f(x)$.
If the constant which the integral yields in the second case is unity, the functions are termed orthonormal.

The useful properties of these functions may be seen by considering that:

a) Orthogonal functions fulfil the conditions of forming the linear and disjoint subspaces required in the treament of Zadeh and Miller. If disjoint subspaces are interpreted as linearly independent systems of vectors, they are not necessarily orthogonal, but the converse is true (reference 5).

b) Realisation of the filtering process necessary to separate the multiplexed signals is greatly simplified. For if each message is transformed into a function which is orthogonal to that of any other message, it can be extracted from a composite signal by multiplication with an identical function, followed by integration. Both these processes can be approximated closely by established engineering techniques.

c) Orthogonal functions have been subject to considerable mathematical investigation, so that there is a wide range of results available as an aid to the interpretation of conventional systems, and the development of other systems. The analytical methods are well known and have been used widely in the general field of communication system analysis (e.g. reference 6).

This close relation between multiplexing and orthogonality has been appreciated for some time. Shannon pointed it out prior to 1951, (c.f. reference 7), whilst Zadeh and Miller remark, (reference 2), that frequency-division-multiplexing and time-division multiplexing are examples of orthogonal manifolds in signal space. Others have since extended the concept with a view to devising systems superior to those conventionally used, as seen in section 1.5. Noise theory and coding theory are particularly dependent on the use of orthogonal functions, and Filipowsky, (reference 8), has cited eighty-nine references relevant to the theory and application of these functions.

1.4 Orthogonality and multiplexing.

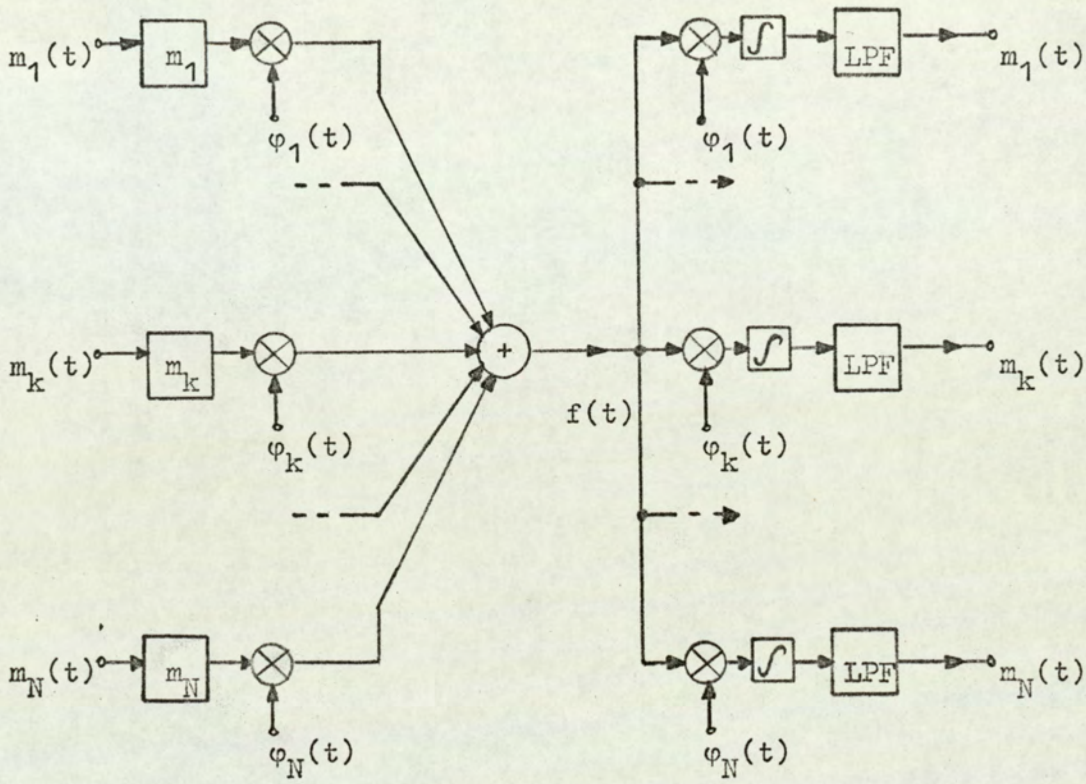


Figure 3

In the generalised multiplex system shown above, a message, $m_k(t)$, modulates one member, $\varphi_k(t)$, from a set of periodic orthogonal functions $\{\varphi_n(t)\}$. If the period of $m_k(t)$ is much greater than that of $\varphi_k(t)$ then $m_k(t)$ is approximately constant over the period of $\varphi_k(t)$, and the modulated signal is approximately $m_k\varphi_k(t)$. With N channels, the composite signal

$$f(t) = \sum_{k=1}^{N} m_k\varphi_k(t)$$

is transmitted along the common path. At the receiver, replicas of the orthogonal waveforms, $\varphi_j(t)$, are generated, and each separately multiplies the incoming signal f(t). The integral of each product is taken over an interval T, this being the interval with respect to which the $\{\varphi_n(t)\}$ are orthogonal. Then, from the definition of orthogonality,

$$\int_0^T m_k\varphi_k(t)\varphi_j(t)\,dt = \begin{cases} 0, & k \neq j \\ A.m_k, & k = j \end{cases}$$

where A is a constant depending on the choice of $\varphi_n(t)$. By further smoothing of the sequence of values of m_k thus obtained, the original message, $m_k(t)$, is recovered.

To examine this representation in more detail, and to demonstrate the synthesis of a multiplex system in terms of a manipulation with orthogonal functions, the various aspects will be considered separately, as follows.

1.4.a <u>Choice of a set of orthogonal waveforms.</u>

In the case of a conventional frequency-division-multiplex system the waveforms are, of course, a set of harmonic sinusoids. As is well known from the derivation of the Fourier series expansion,

$$\int_{-T/2}^{T/2} \cos(m\omega_o t)\cos(n\omega_o t)dt = \begin{cases} 0 , & m \neq n \\ T/2, & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_o t)\sin(n\omega_o t)dt = \begin{cases} 0 , & m \neq n \\ T/2, & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_o t)\cos(n\omega_o t)dt = 0 , \text{ all m and n}$$

where $\omega_o = 2\pi/T$. That is, the functions

$$\left\{1 , \cos\omega_o t , \cos2\omega_o t , .. \cos n\omega_o t ,.. \sin\omega_o t , \sin2\omega_o t ,..\sin n\omega_o t\right\}$$

form an orthogonal set over the interval $-T/2 < t < T/2$.

A set of complex exponentials $\left\{\exp(jn\omega_o t\right\}$ is also orthogonal over the same interval, but in practice this is equivalent to the use of a set of harmonic sines and cosines. Two general frequencies, $\exp(j\omega_1 t)$ and $\exp(j\omega_2 t)$, are not orthogonal over a finite interval unless ω_1 and ω_2 are harmonically related, that is, the ratio ω_1/ω_2 must be a rational number. This does not impose a severe restriction in practice since the integration time can be extended to reduce the error, and frequency-division-multiplex system carrier frequencies normally have a common harmonic base for convenience of carrier generation. Whilst the carrier frequencies may be spaced arbitrarily, provided only that the sidebands do not overlap, the system bandwidth is obviously minimised when the sets of sidebands are contiguous.

Conventionally, time-division-multiplexing is achieved by interlacing non-overlapping pulse-amplitude-modulated waveforms. In that case it is obvious from inspection that the instantaneous product of any two waveforms is zero, and hence that the integral of any product must be zero. In the limiting case, when the waveforms are periodic trains of unit impulse functions with period T,

$$f_1(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

$$f_1(t - \tau_1) = \sum_{-\infty}^{\infty} \delta(t - \tau_1 - nT)$$

it may be shown that the waveforms are orthogonal by considering that over a period T the product of $f_1(t - \tau_1)$ with some arbitrary function $\varphi(t)$ is

$$\int_0^T \delta(t - \tau_1)\, \varphi(t)\, dt = \varphi(\tau_1)$$

If $\varphi(t) = \delta(t - \tau_2)$ then $\varphi(t) = 0,\ t \neq \tau_2$, so that

$$\int_0^T \delta(t - \tau_1)\, \delta(t - \tau_2)\, dt = 0\ , \quad \tau_1 \neq \tau_2$$

The successive sets of pulses of an unmodulated pulse train generate identical amplitude-frequency spectra, which are nevertheless orthogonal in the frequency domain. This is seen by considering Parseval's theorem, which states that for two real functions, $f_1(t)$ and $f_2(t)$, with Fourier transforms $F_1(\omega)$ and $F_2(\omega)$,

$$\frac{1}{T}\int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\, F_2^*(\omega)\, d\omega$$

so that

$$\frac{1}{T}\int_{-\infty}^{\infty}\left|f(t)\right|^{2}\,dt \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^{2}\,d\omega$$

It follows that if two functions are orthogonal in the time domain they must also be orthogonal in the frequency domain,and vice-versa. This,of course,refers to the envelope functions in the complementary domain,not to the discrete components. Thus,the fact that frequency-division-multiplex signals overlap in time,and time-division-multiplex signals in frequency,is immaterial,since a signal separation process which operates on the discrete components in one domain will effectively operate on the envelope functions in the other domain.

Clearly,however,the choice of the set of orthogonal waveforms to be used in a multiplex system must be from periodic functions of time,so as to be compatible with a message set which also comprises functions of time. Periodic functions in the frequency domain are not realisable in a physical,i.e. bandlimited, system,and the only function which is periodic in both time and frequency domains is the idealised unit impulse train.

The orthogonality of the frequency-domain functions of an idealised pulse train may also be seen from Parseval's theorem,by using the version applicable to periodic functions;

$$\frac{1}{T}\int_{-T/2}^{T/2} f_1(t)\, f_2(t)\, dt \quad = \quad \sum_{n=-\infty}^{\infty} (c_1)_n\,(c_2)_{-n}$$

where $(c_1)_n$ and $(c_2)_n$ are the complex Fourier coefficients of $f_1(t)$ and $f_2(t)$ respectively.

The functions considered,with their Fourier series expansions,are,

$$f_1(t) \quad = \quad \delta(t - nT) \quad = \quad \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_o t}$$

$$f_2(t) \quad = \quad \delta(t - \tau - nT) \quad = \quad \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_o t}\, e^{-jn\omega_o \tau}$$

so that for $f_1(t)$,

$$(c_1)_n = \frac{1}{T}$$

and for $f_2(t)$,

$$(c_2)_{-n} = \frac{1}{T} e^{jn\omega_o\tau}$$

For orthogonality in the time domain

$$\int_{-T/2}^{T/2} f_1(t)\, f_2(t)\, dt = 0$$

so that

$$T \sum_{n=-\infty}^{\infty} (c_1)_n (c_2)_{-n}$$

must also be identically zero, that is,

$$T \sum_{n=-\infty}^{\infty} (c_1)_n (c_2)_{-n} = T \sum_{n=-\infty}^{\infty} \frac{1}{T^2} e^{jn\omega_o\tau} = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_o\tau} = 0 \; *$$

and this is so, since the latter summation represents the vector sum of an infinite number of constant-increment angular rotations of a fixed-length vector.

Non-overlapping pulse waveforms are not the only possible basis for a time-division-multiplex system. Ideally, it is only necessary that the waveforms should not interfere at the sampling points. For example, one may consider the spectrum of an isolated pulse, $p(t)$, having a time waveform which is identically zero outside the range $-\tau/2 < t < \tau/2$. If $p(t)$ is expanded as a Fourier series in this interval,

$$p(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_o t} \qquad \text{where } \omega_o = 2\pi/\tau$$

and the Fourier transform of a component term, $e^{jk\omega_o t}$, is

$$F_k(\omega) = \int_{-\tau/2}^{\tau/2} e^{jk\omega_o t} \, e^{-j\omega t} \, dt$$

$$= \frac{1}{\tau} \frac{\sin(\frac{1}{2}\omega\tau - k\pi)}{(\frac{1}{2}\omega\tau - k\pi)}$$

* $\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_o\tau} = \delta(\tau - nT) = 0$, since $\tau \neq T$

so that the Fourier transform of p(t) is

$$P(\omega) = \sum_{n=-\infty}^{\infty} \frac{c_n}{\tau} \frac{\sin(\frac{1}{2}\omega\tau - k\pi)}{(\frac{1}{2}\omega\tau - k\pi)}$$

Since the terms $\left\{e^{jn\omega_0 t}\right\}$ form an orthogonal set,it follows from Parseval's theorem that the terms

$$\left\{ \frac{\sin(\frac{1}{2}\omega\tau - k\pi)}{(\frac{1}{2}\omega\tau - k\pi)} \right\}$$

must also form an orthogonal set.

Thus a set of orthogonal functions over an infinite interval are formed by the set of (sin x)/x functions,in which the respective maxima occur at the zero crossings of the other functions,with all other zero crossings coincident. Being aperiodic, a set of these functions would not be usable as the orthogonal waveforms of a multiplexing system,but a set of periodic waveforms with zero crossing properties similar to those of the (sin x)/x function can be generated,as is described in a later chapter.

With two time-displaced but otherwise identical periodic waveforms,the autocorrelation function, $r_{11}(\tau)$,may be used to determine their orthogonality with respect to the relative displacement τ. Defined as

$$r_{11}(\tau) = \frac{1}{T}\int_0^T f_1(t)\, f_1(t+\tau)\, dt$$

where $f_1(t)$ has period T,it is evident that $r_{11}(0)$ yields the constant required as one condition of orthogonality,since $r_{11}(0)$ is the value of the mean power in the waveform $f_1(t)$.
If $r_{11}(\tau)$ becomes zero for some value of τ between 0 and T,the second condition of orthogonality is satisfied for that value of relative displacement.

For example,the autocorrelation function of a sinusoid

$$f_1(t) = A\cos(\omega_1 t + \varphi_1)$$

is

$$r_{11}(\tau) = \frac{A^2}{2}\cos\omega_1\tau$$

then since

$$r_{11}(\tau) = 0 \ , \quad \tau = \pi/2 \ \text{ or } \ \tau = 3\pi/2$$

it follows that

$\pm A \sin(\omega_1 t + \varphi_1)$ is orthogonal to $A \cos(\omega_1 t + \varphi_1)$.

Similarly, the crosscorrelation function may be used to investigate the orthogonality of two waveforms with different shape but common period. This case would arise, for example, when considering the effect of a rectangular sampling pulse train operating upon a distorted received signal. A direct indication will be provided of the crosstalk level when the time displacement deviates from the correct value.

1.4.b. Signal generation.

Having chosen a set of orthogonal waveforms, $\{\varphi_n(t)\}$, it is required to generate signals which are functions of both a message, $m_k(t)$, and a 'carrier', $\varphi_k(t)$. In general terms this can be regarded as a linear transformation from the message space to the signal space. As shown by Brillouin, for example, (reference 9, ch. 8), a general linear transformation is defined by the crosscorrelation function, in the form of

$$r(x) = \int_{-\infty}^{\infty} f(x + y)\, g(y)\, dy$$

where $f(x)$ is the function to be transformed and $g(y)$ is a transformation function.

If x and y are in the frequency domain, then $r(x)$ becomes

$$R(\omega) = \int_{-\infty}^{\infty} F(\omega + \omega')\, G(\omega')\, d\omega'$$

$$= \int_{-\infty}^{\infty} F(\omega')\, G(\omega' - \omega)\, d\omega'$$

The convolution of $F(\omega)$ and $G(\omega)$ is

$$F(\omega) * G(\omega) = \int_{-\infty}^{\infty} F(\omega')\, G(\omega - \omega')\, d\omega'$$

and since $R(\omega)$ may be written as

$$R(\omega) = \int_{-\infty}^{\infty} F(\omega') \, G\left[-(\omega - \omega')\right] \, d\omega'$$

then $R(\omega)$ is the convolution of $F(\omega)$ and $G(-\omega)$.

However

$$F(\omega)_* G(-\omega) = F(\omega)_* G(\omega)$$

if both $G(\omega)$ and $g(t)$ are real, which is the case for the waveforms used in practical multiplex systems.* Therefore, since the inverse Fourier transform is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[F(\omega)_* G(\omega)\right] e^{j\omega t} d\omega = 2\pi \left[f(t) \, g(t)\right]$$

a linear transformation, (i.e. translation), in the frequency domain is achieved by taking the product of the corresponding time domain functions. The product modulator is, of course, commonly used in signal processing.

If the transformed, or modulated, functions became an orthogonal set this would be convenient in one respect, since any cross-products caused by nonlinearity in the transmission path could be removed by integrating. However, this is not feasible since message recovery would require that the product be formed of two identical functions involving the message $m(t)$. But the latter is not available in the receiver until the recovery process is complete. What is required, rather, is that

$$\int_0^T m_k(t) \, \varphi_k(t) \, \varphi_j(t) \, dt = \begin{cases} 0, & k \neq j \\ A.m_k(t), & k = j \end{cases}$$

where $\varphi_k(t), \varphi_j(t)$ are from the set of orthogonal waveforms chosen. This will only be true if $m_k(t)$ is effectively a constant over the interval T. As this interval is the least value which contains an integral number of the periods of both $\varphi_k(t)$ and $\varphi_j(t)$, it follows that this must be much less than the period of the highest frequency component in $m_k(t)$ for a satisfactory approximation. If this condition cannot be realised, then a sample and hold circuit may be inserted before the modulator.

* i.e. $g(t)$ is assumed to be always representable as a real even function, so that $G(\omega)$ will also be real.

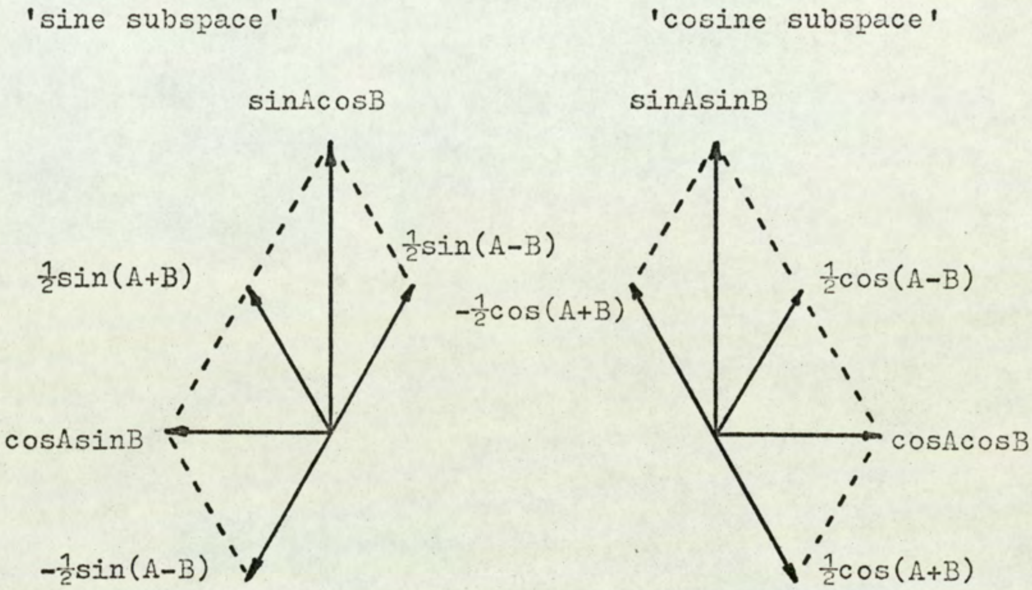


Figure 4

a real function of time, its Fourier transform is wholly imaginary, i.e. $j\pi\left[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right]$. Thus, when used as a function of frequency, it is reasonable to interpret sin x as an imaginary function.

Figure 4 shows vectors for the products of sinA, or cosA, with sinB, or cosB. One may consider, for example, the vector for the product sinAsinB, when sinB is a much slower oscillation than sinA. Then the amplitude of sinA varies in accordance with sinB, but the direction of sinA remains unchanged and is thus always orthogonal to cosA. But the product sinAsinB is a frequency translation and, in general, the frequency of sinAsinB would not be that of sinA, so that the vector for sinAsinB would be rotated from that of sinA. In that case, the sinAsinB vector could not be orthogonal to the cosA vector. In fact, however, the frequency translation implied by sinAsinB results in a pair of vectors, corresponding to the two sidebands, and the vector sum of this pair, which is the sinAsinB vector, is always orthogonal to the cosA vector.

Since these are simply relationships between in-phase and quadrature versions of signals,it might be said that this provides a 'physical' explanation for the occurrence of a pair of sidebands,as it is not necessary to invoke the usual trigonometric identities. As shown in Figure 4,the vectors for sinAsinB and cosAcosB are perpendicular,but both are in the 'real' cosine subspace,since the product sinAsinB implies a j^2 factor. Hence, the sidebands in this space are all cosine functions. The products of sines and cosines remain in the sine subspace,since the implied j factor occurs once only in each case,and hence the sidebands are all sine functions.

If s(t) is substituted for sinB,where s(t) is a finite-bandwidth modulating signal,and $\hat{s}(t)$ is substituted for cosB, where $\hat{s}(t)$ has the same components as s(t),but each component is in quadrature with the corresponding component of s(t),then this representation extends to include the concept of the analytic signal and the use of Hilbert transforms,(references 12 and 13), and the complex envelope representation for a bandpass signal, (reference 14,ch. 3).

As is well known,one method of generating a single-sideband signal is to form the functions

$$\cos \omega_m t \cos \omega_c t \pm \sin \omega_m t \sin \omega_c t$$

or

$$\sin \omega_m t \cos \omega_c t \pm \cos \omega_m t \sin \omega_c t$$

which yield a single sideband in terms of sines or cosines only. This is evident from the vector relationships of Figure 4,and it may also be seen that,whereas the recovery of the modulation component of a double-sideband signal requires only the translation back to the baseband of a single(equivalent) vector,the single-sideband signal is a function of two such vectors in quadrature, and recovery of the modulation component is less simply achieved.

The more commonly used method of obtaining a single sideband by bandpass filtering is also analytically representable in terms of a complex modulating signal,(reference 14,ch. 3), since this representation is generally applicable to bandpass signals. Finally, it is readily seen from the orthogonality of the sine and cosine functions that it is possible to multiplex two messages by modulating carriers of the same frequency, but in quadrature phase.

1.4.d. Signal separation and message recovery.

Assuming that the received signal is in the form

$$r(t) = \sum_{k=1}^{K} m_k \varphi_k(t)$$

then, as shown in Figure 3, r(t) is applied to a bank of multipliers and integrators so that

$$\int_0^{T_k} \left\{ \sum_{k=1}^{K} m_k \, \varphi_k(t) \right\} \varphi_j(t) \, dt = \begin{cases} A \, m_k , & k = j \\ 0 , & k \neq j \end{cases}$$

Thus, neglecting the effect of noise and spurious signals, the multiplication and integration effectively filters the received signal and recovers the message component directly. It is this property which is the whole justification for the use of orthogonal waveforms as carriers.

In the case where m_k is an approximation for $m_k(t)$ and the integration is approximated by a lowpass filter, the system becomes a product demodulator, i.e.

$$\cos\omega_c t \, (m_k(t)\cos\omega_c t) = m_k(t) \cos^2\omega_c t = \tfrac{1}{2}m_k(t) + \tfrac{1}{2}m_k(t)\cos 2\omega_c t$$

and, assuming that $\frac{1}{2}m_k(t)\cos 2\omega_c t$ is completely rejected by a lowpass filter, the message is recovered without distortion. This detection method is, of course, well known, and is referred to variously as coherent, synchronous, or homodyne detection. Whilst applicable to any form of amplitude-modulated signal, it is the only means of detection available for a single-sideband signal with no residual carrier component. The disadvantage of coherent detection is also well known, namely that the version of the carrier generated at the receiver must be exactly equal in frequency and phase to the transmitted carrier if the message is to be recovered without distortion.

In general, the above applies equally to both frequency-division-multiplex and time-division-multiplex systems, though in the latter case it may be noted that the orthogonal set used for recovery need not be identical to the set used for transmission. The transmitted waveforms may be of arbitrary shape provided they are non-interfering at specific instants, or over specific intervals. Signal separation is achieved by synchronised sampling switches; this is equivalent to multiplication by a unity amplitude rectangular pulse waveform, where the pulse duration may be equal to, or less than, the duration of the non-interfering zone of the transmitted signals. The harmonic components of the waveforms should have the same frequency values, although the magnitudes may differ. Since message recovery is achieved by integrating over the period of the fundamental, the other harmonic cross-products, which are of course an orthogonal set in themselves, are rejected.

In the case of a frequency-division-multiplex system where the message continuously modulates the carrier, the integration in the receiver may also be continuous, with the integrator output continuously discharged through an appropriate time constant. However, if the modulation is applied via a sample and hold circuit, or is a fixed amplitude data waveform, the integration must be performed over the exact period appropriate to the $\varphi_k(t)$. The integrator output is then sampled at the end of each period, and reset to zero. The ensuing samples are either lowpass filtered, if from an analogue message, or lengthened in a sample and hold circuit if the message is a data waveform.

This arrangement of a bank of multipliers and integrators is sometimes referred to as a correlation receiver, and is an optimum system for the recovery of signals with added white gaussian noise, (reference 6, ch. 4).

If a frequency-division-multiplex system transmits a double-sideband-plus-carrier signal, it is more economical to separate the signals by bandpass filtering and recover the message by envelope detection. That bandpass filtering may be regarded as an operation in terms of orthogonal functions is shown in the next subsection, whilst envelope detection is essentially the same as detection by multiplication and integration.

Rectification of the modulated carrier may be regarded as multiplication by a switching function with a period corresponding to that of the carrier. By integrating the product of the signal and the switching function harmonics over the period of the fundamental, (i.e. lowpass filtering), the message is recovered in the same way as by coherent detection.

1.4.e. Bandpass filtering.

Bandpass filters are, of course, an essential feature of any multiplex system, if only to exclude out-of-band noise, or restrict the message bandwidth. Since this section is concerned with the relation of orthogonal functions to multiplex systems, it is relevant to consider how bandpass filtering may be expressed in terms of such functions.

As previously mentioned, a rectangular coordinate system may be postulated for the combined message and signal space, and any vector in the space resolved into components on each of the axes. If the coordinate system is in terms of frequency, the whole space may be divided into subspaces representing different frequency bands. Any general vector in the space may have components in some, or all, of the subspaces; but the projection of the vector into any one subspace is equivalent to bandpass filtering over the range of frequencies contained in the subspace, and this is demonstrated in the following discussion.

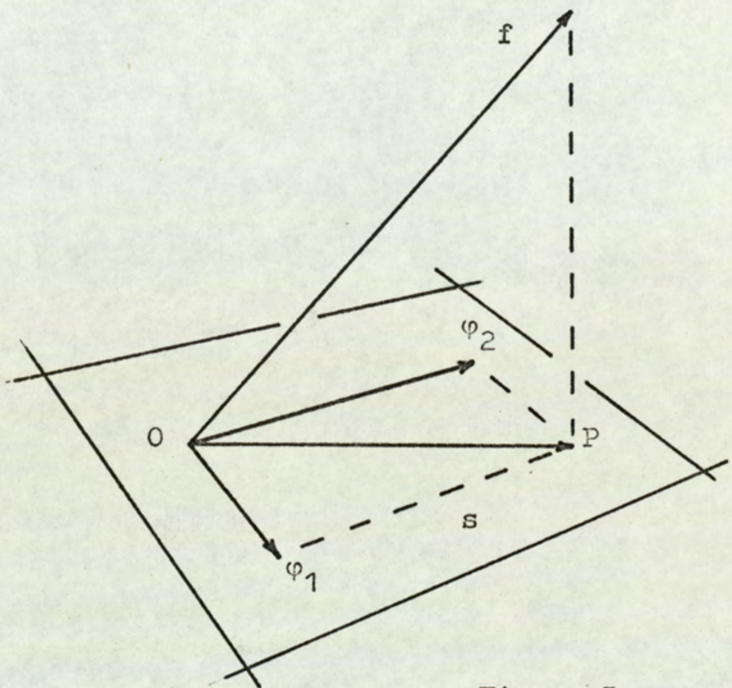


Figure 5

In Figure 5, the projection of the vector f into the subspace s is OP, which has components φ_1 and φ_2 in the coordinate system for that subspace. The projection of f in the direction of φ, where φ has unit length, is defined as

$$\text{proj}(f:\varphi) \quad = \quad (f.\varphi)\varphi$$

where $f.\varphi$ is the inner product of f and φ.

Following a method used by Davis, (reference 15, ch. 2), one may consider the projection of a vector f on to the pair of orthogonal vectors φ_1, φ_2, where

$$\varphi_1(t) = \sin k\omega t \qquad \varphi_2(t) = \cos k\omega t$$

Since

$$\int_0^{2\pi} (\varphi_1(t))^2 dt \quad = \int_0^{2\pi} (\varphi_2(t))^2 dt \quad = \quad \pi$$

the normalised versions of $\varphi_1(t)$ and $\varphi_2(t)$ are

$$\varphi_1(t) = \pi^{-\frac{1}{2}}\sin k\omega t \qquad \varphi_2(t) = \pi^{-\frac{1}{2}}\cos k\omega t$$

The projection of f on to φ_1 and φ_2 is

$$\text{proj}(f:\varphi_1,\varphi_2) = (f.\varphi_1)\varphi_1 + (f.\varphi_2)\varphi_2$$

which becomes

$$\left\{\int_0^{2\pi} f(t)\, \pi^{-\frac{1}{2}}\sin k\omega t\, dt\right\}\pi^{-\frac{1}{2}}\sin k\omega\tau + \left\{\int_0^{2\pi} f(t)\, \pi^{-\frac{1}{2}}\cos k\omega t\, dt\right\}\pi^{-\frac{1}{2}}\cos k\omega\tau$$

(where t and τ are used to distinguish between independent time variables)

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)\left\{\sin(k\omega t)\sin(k\omega\tau) + \cos(k\omega t)\cos(k\omega\tau)\right\} dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\left\{\cos k\omega(\tau - t)\right\} dt$$

which is recognisable as the convolution of f(t) with cos $k\omega t$, and has the alternative form

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau - t)\cos k\omega t\, dt$$

This represents the projection of f on to a subspace with only one frequency in it, defined by sin $k\omega t$ and cos $k\omega t$. If the subspace contains frequencies from zero to $n\omega$, it is defined by the set of orthonormal vectors

$$\left\{(2\pi)^{-\frac{1}{2}},\ \pi^{-\frac{1}{2}}\cos\omega t,\ \pi^{-\frac{1}{2}}\cos 2\omega t,\ ..\ \pi^{-\frac{1}{2}}\cos n\omega t,\ \pi^{-\frac{1}{2}}\sin\omega t,\ \pi^{-\frac{1}{2}}\sin 2\omega t,\ ..\ \pi^{-\frac{1}{2}}\sin n\omega t\right\}$$

Since each pair of sines and cosines of the same frequency yields the above convolution, the projection of f on to the whole subspace is

$$\text{proj}(f:s) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + \sum_{k=1}^{n} \frac{1}{\pi}\int_0^{2\pi} f(\tau-t)\cos k\omega t\,dt$$

The summation of a finite sequence of harmonic cosines has a closed form known as the periodic Dirichlet kernel, $D_n(t)$.

$$D_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{jk\omega t} = \frac{1}{2\pi}\left[1 + 2\cos\omega t + 2\cos 2\omega t + \ldots 2\cos n\omega t\right]$$

The closed form may be obtained by multiplying each side by $\sin\frac{1}{2}\omega t$ and using the relation

$$\sin A\cos B = \tfrac{1}{2}\sin(B + A) - \tfrac{1}{2}\sin(B - A)$$

Hence

$$D_n(t)\sin\tfrac{1}{2}\omega t = \frac{1}{2\pi}\left[\sin\tfrac{1}{2}\omega t + \left(\sin(3/2)\omega t - \sin\tfrac{1}{2}\omega t\right) \ldots \ldots + \left(\sin(2n+1)\tfrac{1}{2}\omega t - \sin(2n-1)\tfrac{1}{2}\omega t\right)\right]$$

$$= \frac{1}{2\pi}\sin(2n+1)\tfrac{1}{2}\omega t$$

so that

$$D_n(t) = \frac{\sin(n+\frac{1}{2})\omega t}{2\pi\sin\frac{1}{2}\omega t}$$

and

$$\text{proj}(f:s) = \int_0^{2\pi} f(\tau-t)\,\frac{\sin(n+\frac{1}{2})\omega t}{2\pi\sin\frac{1}{2}\omega t}\,dt$$

To interpret this result as filtering one may consider, as shown by Guillemin, (reference 16, ch. 15), that convolution of the periodic function f(t) over a single period with the periodic

Dirichlet kernel $D_n(t)$ is equivalent to the convolution of f(t) with the aperiodic Dirichlet kernel, $D_a(t)$, over the infinite interval, where

$$D_a(t) = \frac{n\omega}{\pi}\frac{\sin n\omega t}{n\omega t}$$

Since the convolution of two time domain functions is equivalent to the product of their corresponding functions in the frequency domain, then

$$F.T\left[\int_{-\infty}^{\infty} f(\tau-t)\, D_a(t)\, dt\right] = F.T\left[f(t)\right]\; F.T\left[D_a(t)\right]$$

where F.T(f(t)) denotes the Fourier transform of f(t), and similarly for $D_a(t)$.

The (sin x)/x form of $D_a(t)$ is recognisable as the Fourier transform of a rectangular function, that is

$$F.T\left[\frac{n\omega_o}{\pi}\;\frac{\sin n\omega_o t}{n\omega_o t}\right] = F.T\left[\frac{1}{\pi t}\sin n\omega_o t\right] = \begin{cases} 1 & , |\omega| < n\omega_o \\ 0 & , |\omega| > n\omega_o \end{cases}$$

Thus, the projection of f on to s is equivalent to filtering f(t) with a lowpass filter having a cutoff frequency $n\omega_o$.

To determine the subspace appropriate to bandpass filtering, it may be noted that that the impulse response of an ideal bandpass filter, (centre frequency ω_c, bandwidth $\omega_c \pm n\omega_o$), may be obtained from the product of cos $\omega_c t$ and the impulse response of an ideal lowpass filter with cutoff $n\omega_o$. Since $D_a(t)$ has the form of the required impulse response and can be replaced in the convolution integral by $D_n(t)$, the modified form of $D_n(t)$ is

$$\frac{1}{2\pi}\cos \omega_c t\ (\ 1 + 2\cos \omega_o t + 2\cos 2\omega_o t\ ..\ + 2\cos n\omega_o t)$$

$$= \frac{1}{2\pi}\left[\cos \omega_c t + \cos(\omega_c \pm \omega_o)t + \ \ldots\ \cos(\omega_c \pm n\omega_o t)\right]$$

By an inverse process to that used for obtaining the periodic Dirichlet kernel,it will be seen that the above is contained in a subspace defined by the coordinate vectors,

$$\left\{\frac{1}{2\pi}\cos\,\omega_c t,\ \pi^{-\frac{1}{2}}\cos(\omega_c \pm \omega_o)t\ ,\ \pi^{-\frac{1}{2}}\cos(\omega_c \pm 2\omega_o)t\ ,\ ..\pi^{-\frac{1}{2}}\cos(\omega_c \pm n\omega_o)t\ ,\ ...\pi^{-\frac{1}{2}}\sin(\omega_c \pm \omega_o)t\ ,..\pi^{-\frac{1}{2}}\sin(\omega_c \pm n\omega_o)t\right\}$$

Thus,the bandpass filtering operation may be interpreted as the convolution of the signal f(t) with a set of orthogonal functions defining the filter band.

1.5 Some methods for orthogonal waveform multiplexing

As the significance of orthogonal waveforms to multiplexing has become more widely appreciated, a number of schemes have been proposed where some advantage over conventional systems was obtained. Although none of these appears to have received widespread adoption, possibly due to difficulties in practical realisation, they nevertheless illustrate some of the approaches which are possible within the generalised framework previously described.

One of the first to be published was that by Harmuth in 1960, (reference 17). The application was to digital systems, and the choice of orthogonal waveforms was a set of harmonic sines and cosines, assumed time-limited to the period of the fundamental. Each digit position in a sequence of length n is allocated one member of the set, so that the transmitted signal is the sum of n sinusoids, each sinusoid being switched to zero amplitude if the digit is zero in the position to which it refers. Digit recovery is by a set of analogue multipliers and lowpass filters. Thus, the system resembles on-off keying of a carrier; but by using a number of different frequency carriers, the maximum switching rate is only required to be half the character rate rather than half the digit rate.

Bandwidth limitation is the main cause of crosstalk in such a system, but Harmuth considered that, with tolerable distortion of the demodulated pulses, and with a large number of digits per character, the bandwidth required approached the theoretical minimum of half the digit rate. Crosstalk was found to be worse between digits transmitted by the same waveform, and a worst-case figure of 10% was quoted. Reduction of required transmission path bandwidth is the advantage to be gained by use of this system, against which must be balanced the cost of the waveform generators required in the transmitter and receiver.

In 1962, Ballard described a system, (reference 18), in which the orthogonal waveforms used were based on the Legendre polynomials, $P_n(x)$. These polynomials appear as a standard example of an orthogonal function in most mathematical texts on the subject, and may be defined as:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

with, for example, $P_o(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Thus, as the order of the polynomial increases, it is successively representable as a constant, straight line, parabola, etc. For use in a multiplexing system, the polynomials are functions of time, $P_n(t)$, where x is related to t by $x = \frac{2}{T}(t - \frac{T}{2})$, $0 \leqslant t \leqslant T$.

In Ballard's system, each of the appropriate waveforms is synthesised over a period, T, and if a polarity reversal would occur at the end of the period, the next period of the waveform is inverted. It is shown that, in terms of a frequency $f = \frac{1}{2T}$, the spectral distribution of the polynomials is such that $P_1(t)$ has the maximum amplitude component at f, $P_2(t)$ at 2f, $P_3(t)$ at 3f, and so on. The other harmonics decrease asymptotically in the same way for all polynomials, with an initial slope of 12 db/octave.

Each of the waveforms is amplitude-modulated (as in a conventional frequency-division-multiplex system) and recovery is by use of a correlation receiver, i.e. multiplication and integration. It will be seen that this method resembles frequency-division-multiplex in that the signals overlap in time, but that the spectra of each signal also overlap, as in time-division-multiplex. Crosstalk depends on the highest-order harmonic which can be passed in the available bandwidth. The minimum bandwidth required depends on the highest-order polynomial used, since the spectrum peak for each waveform must be transmitted. With a five-channel system the author recommends a bandwidth of nine times the message bandwidth.

In a comparison made with the bandwidth requirements of standard telemetry systems, the bandwidth claimed for this system is that of ideal double-sideband frequency-division-multiplex; that is forty one-kiloHertz channels in an eighty kiloHertz band. The other advantages claimed for this system are those which normally apply to frequency-division-multiplex as opposed to time-division multiplex. In view of the sophisticated circuitry needed to realise this bandwidth saving, it might appear that this system is limited to special purpose applications.

An analysis by Karp and Higuchi in 1963, (reference 19), introduces another mathematically well-known set of orthogonal functions, the Hermite polynomials.* In the choice of pulse waveforms for a time-division-multiplex system, it is desirable to minimise the spectral spread for pulses of some fixed duration, and to use waveforms where most of the signal energy occurs near the centre of the waveform duration. Waveforms based on the gaussian function are known to comply with these requirements, and the Hermite polynomials, $H_n(x)$, are related to the gaussian function by

$$H_n(x) = \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left[\exp\left(-\frac{x^2}{2}\right)\right]$$

$H_o(t)$ is the well-known gaussian-shaped pulse, and higher orders of $H_n(t)$ are oscillatory waveforms, where the number of oscillations per period depends on the order of the polynomial. The authors investigated the effect of restricting these functions to short durations and narrow bandwidths, and concluded that considerable constraints were possible without significant loss

* The Hermite polynomials are orthogonal with respect to a weighting function $\exp(-x^2/2)$, and satisfy the conditions:- $\int_{-\infty}^{\infty}\exp(-x^2/2)\, H_m(x)\, H_n(x)\, dx = n!\sqrt{2\pi}$, $n = m$; $= 0$, $n \neq m$.

of orthogonality.

The relative merits of using a gaussian-shaped pulse in conventional pulse systems have been investigated by various authors but presumably the analysis by Karp and Higuchi is intended to apply to systems where either different messages, or different digits of a message, modulate the waveforms of different polynomials. It would appear that circuitry with a degree of complexity at least comparable to that used by Ballard would be required to generate and recover these waveforms, which again is a handicap to their practical exploitation.

A system whereby an increase in transmission-path bandwidth could be exchanged for the advantages to be gained by the use of all-digital circuitry was analysed by Titsworth in 1963 (reference 20). Messages in digital form are required, and each digit of a character sequence from a given message causes a code group to be passed to a logic network. If the digit is 0, the complement of the code group is passed. These code groups are required to be orthogonal, which has essentially the same meaning in digital system terminology as the more general definition considered earlier. With a set of n messages there are, thus, n orthogonal code groups applied in parallel to the logic network. This, in turn, produces a single output sequence for transmission to the receiver. It is shown that the optimum decision for the logic network is to output '+1' if less than half the inputs are 'ones' in a given digit interval, and '-1' if more than half are 'ones'.

Signal recovery is performed by correlation; that is the received pulse sequence is multiplied by the appropriate code group (as used for that channel at the transmitter) and integrated over the duration of the group. Sampling the integrator output at the end of each interval recovers the original digit. All these operations are, of course, carried out digitally.

Clearly, this system is based on the same general principles of orthogonal-waveform multiplexing that have been described earlier. Pseudonoise sequences are also suggested for use as code groups, as nearly orthogonal waveforms for which the cross-talk is small. The author points out that the peak-to-average signal power ratio is unity in the transmitted signal. Also, of course, a lower cost and increased reliability is obtained from the use of digital circuits throughout. Bandwidth increase depends on the length of the orthogonal code groups, since the frequency of the digital channel signals is multiplied by the number of digits in these groups (approximately 2^n for n channels). A practical realisation of this system has also been described recently, (reference 21).

In the approach used by Chang in 1966, (reference 22), the orthogonal waveforms are generated by the use of special bandpass filters. The messages to be multiplexed are required to be in pulse-modulated form, and all the pulse trains synchronised. From each channel the sampling-frequency fundamental and its sidebands are selected by a bandpass filter designed to have zero amplitude response at the limits of the sidebands, i.e. at $f_s \pm fm$, where f_m is the message bandwidth. This band is then translated, presumably by means of a single-sideband suppressed-carrier modulation technique, to become $f_{c1} \pm f_m$. A second channel is translated to become $f_{c2} \pm f_m$, where the separation between f_{c1} and f_{c2} is f_m, so that the sidebands of adjacent channels are overlapping.

Similarly, the remaining channels are translated to positions where the upper and lower sidebands of adjacent channels are overlapping. However, the phase response of the bandpass filters is designed to be such that the components of the upper sideband in one channel are always in quadrature with the components of the lower sideband in the next highest channel. In this way, the orthogonality of the different channel signals is preserved.

It is demonstrated by the author that bandpass filters with the required amplitude and phase response can be realised without undue difficulty. Signal recovery,by a correlation method,is dealt with in a subsequent paper,(reference 23). There it is shown that the receiver may use a bandpass filter at the input to each channel,which selects the band appropriate to that channel plus the pair of interfering sidebands. This band is multiplied by the band-centre frequency,and the resulting baseband signal is lowpass filtered and sampled. By correct choice of the phase of the multiplying frequency,and of the sampling instants,the components of the interfering sidebands are rejected.

The overall system therefore uses the same transmission path bandwidth as an ideal single-sideband frequency-division-multiplex system, and achieves this with the use of non-ideal filters. It is shown that for a reasonable tolerance on system parameters, reduction in system performance is not excessive.

In 1967, and subsequently in 1968, (references 8 and 24), Filipowsky described a set of orthogonal waveforms which can be formed from trigonometric products. That is, the product of a number of sines and cosines with harmonically-related frequencies forms a variety of different waveforms, some of which are orthogonal sets. The author provides a systematic classification procedure for these waveforms and their power spectra.

Although this study is limited to linear multiplex systems, some mention may be made of asynchronous time-division methods. In 1950, White (reference 25), discussed the possibility of transmitting pulse trains from independent message sources, where synchronism was not possible. This technique was said to be first used with low-information-rate navigation systems, such as Loran. Without synchronism the signals are semi-orthogonal, and the interference between channels depends on such factors as pulse duration, degree of correlation between different signals, spurious signals and noise, degree of redundancy in any coding system used, and hence transmission-path bandwidth. The author considered that by use of appropriate coding such a system might be acceptable.

In 1952, Pierce and Hopper (reference 7), suggested the use of random sampling of each of the message sources of a speech communication system. Each sample was converted to a digital code group, and in the authors' experimental system any overlapping from the pulses of other code groups caused the rejection of that sample. Subjective tests showed that speech quality was tolerable with up to eight interfering transmitters. In general, it would appear that such a system could only be applied to low-quality communication links.

A system which is nonlinear in the sense that the messages are not transmitted in a real-time scale was described by Flood and Urquhart-Pullen in 1964, (reference 26). The sampled values of any one message may be allowed to overlap since this is, of course, the means by which the continuous message is recovered. If a pulse-amplitude-modulated time-division-multiplex system consists of successive groups of pulses, each group consisting of samples of the same message, the transmission-path bandwidth need only be sufficient to prevent crosstalk between the different groups, rather than their constituent pulses. With guard pulses before and after each group, some overlapping of the groups is permissible. Thus, with sufficiently long groups, the bandwidth required approaches that of the message bandwidth.

As each message channel must be sampled continuously, the sample values for each group are obtained by storing a sequence, and transmitting them on a compressed time scale. The receiver store then reads out the samples on the real-time scale. Thus, the same amount of information is transmitted in unit time as with a wholly real-time system, but the transmission-path bandwidth required approaches that of an ideal single-sideband frequency-division-multiplex system.

1.6 Summary

In this chapter, an attempt has been made to show that conventional multiplexing systems can be described in general terms as an operation with orthogonal functions. A number of unconventional systems have also been seen to fit this description. The essential feature of any multiplex system is to transform the messages into signals which can be superimposed, reseparated, and transformed back into messages without distortion.

It was seen that this can be achieved by using a set of waveforms which are orthogonal functions of time, and modulating different members of the set by different messages. The signals are reseparated and messages recovered by a simple process of multiplication and integration. Theoretically, the process does not involve any distortion of the messages.

Conventional multiplex systems have developed in practice without recourse to this approach. However, the description is generally applicable, and (as a set of orthogonal waveforms can be realised in many different ways) a means for the synthesis and comparison of a variety of systems is thereby provided. Frequency-division and time-division-multiplex may be regarded as the two extreme cases. In the first, the orthogonal waveforms occupy an unlimited region of the time domain, but are delta functions in the frequency domain; whilst the reverse is true for ideal time-division.

However, a quadrature multiplexing system is an example of waveforms which are continuous in the time domain, but are orthogonal by virtue of a precise time (i.e. phase) displacement.

In a time-division-multiplex system, each orthogonal waveform is the same infinite set of time-continuous orthogonal waveforms, but the collective effect of the phase differences between any two sets results in periodic time-domain orthogonality.

In a pulse-amplitude-modulated system, each of the pulse spectral components is modulated by the message. Therefore, a time-division-multiplex system using pulse-amplitude-modulation comprises a multiply superimposed set of frequency-division systems. By filtering the appropriate modulated harmonic, a frequency-division-multiplex system can therefore be extracted from a pulse-amplitude-modulated time-division-multiplex system, (reference 27). It is thus seen that the two extreme cases are not disparate, but that each is a logical extension of the other.

A single-sideband frequency-division system with ideal filtering occupies the minimum possible bandwidth, namely $N f_b$ where N is the number of channels and f_b is the message bandwidth. However, the analogue multipliers and filters of a frequency-division sytem are more difficult to realise than the switching circuits of a time-division system. Whilst there are a number of factors which determine the practical usefulness of a particular multiplex system,(such as the performance in a non-ideal environment, the necessity or otherwise for synchronism between transmitter and receiver, and the transmitter efficiency), the factor which appears to have received the greatest attention is reduction of transmission bandwidth. Inevitably, however, this conflicts with another important factor, namely ease of realisation.

Some of the unconventional schemes proposed have aimed at reducing transmission bandwidth by using a set of orthogonal functions which are piece-wise continuous waveforms, having a spectrum which falls off more sharply than that of a rectangular pulse train. But the circuitry involved is always at least as complex as that of a conventional system with comparable bandwidth.

This difficulty would seem to apply to most of the sets of orthogonal polynomials. Ballard's system (which required elaborate circuits to realise) used the Legendre polynomials, $P_o(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3x^2}{2} - \frac{1}{2}$, ... which were converted to periodic functions by continuous repetition over a restricted range of the variable. Another well-known set are the Laguerre polynomials, the first three being $L_o(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = 1 - 2x + \frac{x^2}{2}$. A set which occurs in lowpass filter approximation theory are the Chebyshev polynomials, $C_o(x) = 1$, $C_1(x) = x$, $C_2(x) = 2x^2 - 1$, ... Bessel functions form orthogonal sets, and the Bessel polynomials are $B_o(x) = 1$, $B_1(x) = x + 1$, $B_2(x) = x^2 + 3x + 3$, ... It is evident that the same degree of difficulty would arise when attempting to convert any of these sets of polynomials into waveforms.

By way of contrast, another set of functions which would be easy to realise, but would hardly conserve bandwidth, is the Rademacher orthogonal sequence, (reference 28). This is defined by $R_k(x) = \text{sgn}(\sin 2^k \pi x)$, $0 \leqslant x \leqslant 1$, $k = 1, 2$ ... which yields a set of square waves successively doubling in frequency.

Thus, the property of orthogonality does not in itself imply that a set of functions would be useful in a practical multiplex system. As was pointed out by Gabor, (reference 12), sinusoids are the only functions which can be generated by constant circuit elements. Therefore, any sets of orthogonal waveforms which are neither linear combinations of sinusoids nor can be generated by simple switching will be less useful in practice.

In conclusion,it may be said that the orthogonal function approach provides a generally applicable mathematical model for linear multiplex systems. Both conventional,and other systems can be described in this way,and the various signal processing operations are seen to be related by this general treatment.

In theory,a multiplex system may be synthesised from any set of functions which have the orthogonality property. In practice,the choice is restricted by the need to generate the functions in a reliable and economical manner.

Conventional systems are based on functions which are either strictly orthogonal in the frequency domain,requiring precise bandwidth control,or in the time domain,requiring a wide bandwidth. Other systems must necessarily be hybrid,having some overlapping in both time and frequency. One such system is described in a subsequent chapter,where it is shown that time domain signal processing may be combined with a minimum transmission bandwidth.

CHAPTER 2

INTERCONVERSION OF TIME-DIVISION AND FREQUENCY-DIVISION MULTIPLEX SYSTEMS.

2.1 Introduction.

Since there are two forms of multiplexing in common use, the necessity may arise of converting from one form to the other. This can be achieved by de-multiplexing, restoration to baseband, and re-multiplexing into the desired form. Whilst this method is, of course, effective, there exists the possibility of bypassing the intermediate conversion to baseband, so reducing the amount of equipment needed. For this to be possible, the fdm carrier frequencies and the tdm sampling frequency must have a common harmonic base. The standard telephone group of twelve 4-kHz channels has this property, and will be used as an example for other such systems.

The array of passive bandpass filters required for fdm systems represents a large part of the initial cost. Active filters using high gain feedback amplifiers are a possible alternative, though not necessarily more economical. Other active-circuit methods of obtaining a bandpass response are available, and some of these are considered in the latter part of this chapter.

2.2 Conversion from tdm to fdm.

It was mentioned, in the conclusion to chapter one, that a tdm system may be regarded as the superposition of a number of fdm systems. Each pulse of the tdm frame is a set of modulated harmonics. If a different modulated harmonic is bandpass filtered form each pulse of the frame, the sum of the bandpass filter outputs will be an fdm signal. A consecutive set of harmonics would, of course, be chosen, and either one, or both sidebands may be filtered.

The harmonics tend to have reduced amplitudes as the harmonic number increases. However, if, for example, the ratio of sampling period to pulse duration is greater than twice the highest harmonic number used, then the difference in amplitudes will be less than four db. In the standard telephone group, the twelve 4-kHz channels occur as the lower sidebands on a set of carriers

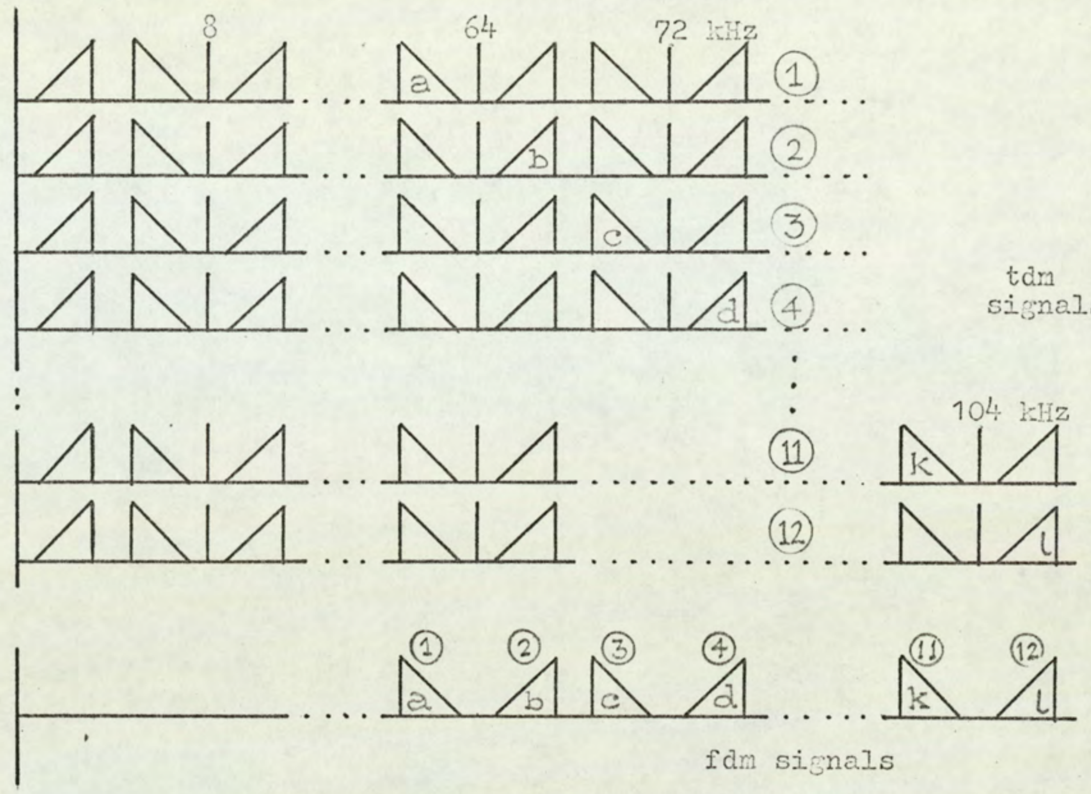


Figure 2.1

at 4 kHz intervals, from 64 kHz to 108 kHz. In the corresponding tdm system, each 4-kHz channel is sampled at 8 kHz, and the channel pulse trains are interpolated to form a multiplex signal with a pulse-repetition-frequency of 96 kHz.

It is evident that an fdm system of dsb signals on carriers at 8 kHz intervals can be bandpass filtered directly from the tdm signal. The essential features of such a system were described by Thrasher, and others, in 1964. (References 27 and 29.) The fdm signals were derived by direct sampling of the messages, rather than from an existing tdm signal. Amplitude losses in the bandpass filters were corrected by using the resonant transfer property, and message recovery was by bandpass filtering, re-sampling, and lowpass filtering.

As previously mentioned, it is not necessary to restrict this process to the formation of dsb signals. One method would be to convert the dsb signals to ssb in the translation to supergroup. Another possible method is shown in figure 2.1.

Each bandpass filter selects alternatively an upper and lower sideband from consecutive pairs of channels in the tdm signal. The fdm signal will therefore consist of a set of alternating upper and lower sidebands, occupying the same overall bandwidth as a normal ssb fdm signal. Signal recovery would follow the normal practice, that is, bandpass filtering of the wanted sideband, and multiplication by a carrier frequency appropriate to the position of the sideband.

Since any one multiplying carrier is applicable to a pair of sidebands, it is evident that only half the number of locally generated frequencies are required in the receiver, compared to the usual system. It is also evident that the bandpass filters must be effective in suppressing the adjacent sideband, since any demodulated component would be intelligible.

It would be desirable to gate the tdm signal, and separate the channel pulse trains, before filtering the required modulated harmonic. Otherwise there is a tendency for those harmonics to be suppressed which are not multiples of the tdm frequency, i.e. of 96 kHz. This, of course, applies whether a dsb or ssb system is to be formed.

In general, the method of converting from tdm to fdm by direct filtering would appear to be fully justified, assuming that the tdm frequencies, and frequency stability, are compatible with the required fdm signal. The method eliminates the need for lowpass filters, carrier frequency generators, and modulators. The number of bandpass filters required, and their performance, is no greater than would be needed when generating the fdm signal from baseband.

2.3 Conversion from fdm to tdm.

Although an fdm signal is inherently contained within the framework of a tdm signal, the reverse is not true. Even if the modulated carriers of the fdm signal had the same amplitude and phase relationships that would occur if they were part of a tdm signal, it would still be necessary to add a large number of appropriately phased and amplitude weighted components to approximate the discrete-time tdm signal. In general, fdm system carriers are generated with arbitrary phase relationships, which will, in any case, be modified by the transmission path characteristics. It would, therefore, appear to be necessary to bandpass filter the fdm signal and recover the separate channel signals as the first step.

Initially, one may consider the case of an isolated dsb signal, when the envelope is a direct representation of the modulating message. It is evident that if the signal is half-wave rectified, the resulting waveform, consisting of amplitude modulated half-sinusoids, is one form of a pulse-amplitude-modulated signal. The baseline pulse duration would be half the carrier period, so that to obtain a set of pam signals with the same pulse duration throughout, it would be necessary to sample the dsb-am waveform. Clearly, the sampling instants should be synchronised to the peak values of the carrier waveform. As it was postulated that a specific harmonic relationship exists between fdm and tdm signals, it is possible to sample at the frequency of the common harmonic base, and remain synchronised.

If the common base is 8 kHz, the pulse train resulting from the sampled dsb-am waveform will be at the required sampling frequency for a message bandwidth of 4 kHz. However, the carrier frequencies of the standard telephone group are the 16th to the 27th harmonics of 4 kHz. Thus, only half the carrier frequencies are integer multiples of 8 kHz. Assuming that the sampling instants are synchronised to the positive-going peaks of the carrier waveforms which are integer multiples of 8 kHz, then sampling those carriers which are only integer multiples of 4 kHz at a rate of 8 kHz will result in samples which occur alternately at a positive peak and a negative peak. Therefore, to generate the required 8 kHz pulse train each negative sample must be inverted. (The effect of inverting the polarity of alternate samples is discussed further in chapter three.)

To obtain the spectrum of the sampled signal, one may consider the carrier, $\cos \omega_c t$, to be modulated by a message $m(t)$. The modulated signal is, in general, given by,

$$f_1(t) = (1 + m(t)) \cos \omega_c t$$

The sampling process may be represented by taking the product of $f_1(t)$ and a unity-amplitude rectangular pulse train, $f_2(t)$, having a fundamental frequency ω_s.

$$f_2(t) = c_o + 2 \sum_{n=1}^{\infty} c_n \cos n\omega_s t$$

where

$$c_n = \frac{d}{T} \frac{\sin n\pi d/T}{n\pi d/T}, \quad d = \text{pulse duration}, \quad T = 2\pi/\omega_s .$$

Since the sampling frequency and the carrier frequency are harmonically related, one may write $\omega_c = k\omega_s$, where k is some integer. The sampled signal is, therefore,

$$f_1(t)\, f_2(t) = f_s(t) = \Big[(1 + m(t)) \cos k\omega_s t\Big]\Big[c_o + 2 \sum_{n=1}^{\infty} c_n \cos n\omega_s t\Big]$$

$$= \Big[c_o(1 + m(t)) \cos k\omega_s t\Big]$$

$$+ \Big[(1 + m(t)) \Big\{ 2 \sum_{n=1}^{\infty} c_n \cos n\omega_s t \cos k\omega_s t \Big\}\Big]$$

The product $\left\{ 2 \sum_{n=1}^{\infty} c_n \cos n\omega_s t . \cos k\omega_s t \right\}$ is

$$\left\{ \sum_{n=1}^{\infty} c_n \Big[\cos(k + n)\omega_s t + \cos(k - n)\omega_s t\Big] \right\}$$

which expands to

$$\left\{ c_k + \Big[\sum_{p=1}^{k-1} (c_{k-p} + c_{k+p}) \cos (p\omega_s t)\Big] + \Big[c_{2k} \cos (k\omega_s t)\Big] + \Big[\sum_{n=1}^{\infty} (c_n + c_{2k+n}) \cos (k+n)\omega_s t\Big] \right\}$$

In practice, the sampled signal will be band-limited. Provided the bandwidth extends to at least $k\omega_s$, and the pulse duration is such that c_n is effectively a constant over this bandwidth, the expression may be approximated to

$$c_o + \sum_{p=1}^{p \geqslant k} c_p \cos p\omega_s t$$

The sampled signal may then be written as

$$\begin{aligned} f_s(t) \simeq & \left[c_o(1 + m(t))\cos k\omega_s t\right] \\ & + \left[c_o + 2\sum_{p=1}^{p \geqslant k} c_p \cos p\omega_s t\right] \\ & + \left[c_o m(t) + 2\, m(t) \sum_{p=1}^{p \geqslant k} c_p \cos p\omega_s t\right] \end{aligned}$$

and it will be seen that lowpass filtering recovers the message in the form $c_o m(t)$.

The effect of sampling at instants other than those of the peak values of the carrier waveform may be seen by considering a carrier with arbitrary phase, φ.

Assuming, (for brevity), a suppressed carrier signal, in the form

$$m(t)\cos(k\omega_s t - \varphi)$$

the sampled signal will be

$$f_s(t) = \left[c_o m(t)\cos(k\omega_s t - \varphi)\right] + \left[2m(t)\sum_{n=1}^{\infty} c_n \cos(k\omega_s t - \varphi)\cos n\omega_s t\right]$$

The second bracket expands to

$$\begin{aligned} & m(t)\sum_{n=1}^{\infty} c_n \left[(\cos k\omega_s t \cos\varphi + \sin k\omega_s t \sin\varphi)\cos n\omega_s t\right] \\ = \; & m(t)\sum_{n=1}^{\infty} \frac{c_n}{2}\Big\{\cos\varphi\left[\cos(k+n)\omega_s t + \cos(k-n)\omega_s t\right] \\ & + \sin\varphi\left[\sin(k+n)\omega_s t + \sin(k-n)\omega_s t\right]\Big\} \end{aligned}$$

It will be seen that if $\varphi = \pi/2$, the expression has sine terms only, and that no value of n can result in a dc term. Therefore, the message could not be recovered by lowpass filtering. For intermediate values of φ, between 0 and $\pi/2$, the message component is proportionately reduced. This effect is, of course, the same as that obtained when attempting to realise coherent detection with an out-of-phase local oscillator.

Considering next a single-sideband signal, it will be seen that a pam waveform may again be obtained by direct sampling. This follows since a single-sideband signal may be expressed as

$$f(t) = m(t) \cos \omega_c t \pm \hat{m}(t) \sin \omega_c t$$

where $\hat{m}(t)$ is the quadrature version of m(t).

Hence, if the signal is sampled at the peaks of the cosine carrier, the sample will be centred on a zero crossing of the sine carrier. If ω_c is sufficiently large compared to the highest frequency component in m(t), the sample of the sine carrier will have a negligible dc value. The message, m(t), may then be recovered with negligible distortion arising from the quadrature component, $\hat{m}(t)$.

Thus, the sampled version of an upper-sideband signal is,

$$\begin{aligned} f_s(t) &= \left[c_o m(t)\cos k\omega_s t + 2m(t)\sum_{n=1}^{\infty} c_n \cos k\omega_s t \cos n\omega_s t\right] \\ &\quad + \left[c_o \hat{m}(t)\sin k\omega_s t + 2\hat{m}(t)\sum_{n=1}^{\infty} c_n \sin k\omega_s t \cos n\omega_s t\right] \\ &= c_o\left[m(t)\cos k\omega_s t + c_o\hat{m}(t)\sin k\omega_s t\right] \\ &\quad + m(t)\sum_{n=1}^{\infty} c_n\left[\cos(k+n)\omega_s t + \cos(k-n)\omega_s t\right] \\ &\quad + \hat{m}(t)\sum_{n=1}^{\infty} c_n\left[\sin(k+n)\omega_s t + \sin(k-n)\omega_s t\right] \end{aligned}$$

and since $\hat{m}(t)$ is not multiplied by a dc term, only m(t) will be recovered by lowpass filtering with a cutoff frequency of $\omega_s/2$.

If the sampling instants are located at the peaks of the sine carrier,

$$f_s(t) = c_o\left[m(t)\cos(k\omega_s t) + \hat{m}(t)\sin(k\omega_s t)\right] + m(t)\sum_{n=1}^{\infty} c_n\left[\sin(k+n)\omega_s t - \sin(k-n)\omega_s t\right] + \hat{m}(t)\sum_{n=1}^{\infty} c_n\left[\cos(k-n)\omega_s t - \cos(k+n)\omega_s t\right]$$

in which case, $\hat{m}(t)$ is recovered by lowpass filtering. Whilst this would be acceptable for speech messages, it would not be so for data waveforms, since the quadrature version of a waveform is generally markedly different in shape from the original waveform.

The process of converting directly from fdm to tdm requires, therefore, that the fdm signal be first bandpass filtered into the constituent channels, and that each channel be then sampled by a correctly synchronised switch. The synchronisation is the same as that which occurs in normal coherent detection. A residual carrier, or a stable local oscillator, is normally used to provide the timing information.

The pam signals obtained by sampling could not be inserted directly into a tdm frame, since it cannot be assumed that the carrier peaks will occur at instants corresponding to the position of the channel pulse in the tdm frame. The pam signals would, therefore, require to be passed to a holding circuit, and resampled at the correct instant.

The total circuitry involved in this conversion system is substantially the same as would be required in converting to baseband, and remodulating. The differences lie in the demodulation of the fdm signals being replaced by switching circuits operating at a subharmonic of the carrier frequency, and in the elimination of passive lowpass filters. It would appear, therefore, that direct conversion from fdm to tdm, by this method, is less advantageous than direct conversion from tdm to fdm.

2.4 Multiple path system for lowpass to bandpass transformation.

Bandpass filtering is an essential feature of the multiplex conversion systems which have been described. The use of passive elements, i.e. inductance, capacitance, and crystals, is well proven. However such filters would represent a large part of the bulk and initial cost of the system, and the merits of any alternative are worth consideration.

One possible method is based on a set of modulators and lowpass filters. This does not necessarily dispense completely with the need for passive filters, but it is generally the case that lowpass filters with a given response are easier and cheaper to realise than bandpass filters with the same (translated) response.

The process depends upon the generation of a multiple set of sidebands with phase relationships such that all but the wanted components are self-cancelling. To consider first the simpler case of sinusoidal modulation, the effect upon a signal of the network shown in figure 2.2 is derived. In the complete system a number of such networks are placed in parallel.

The input signal, V_i, to the network shown in figure 2.2.a. is assumed to have a bandwidth extending above and below ω_s, the frequency of the modulating signal. From the diagram,

$$V_1 = V_i \cos \omega_s(t-\tau) \quad , \quad V_o = V_2 \cos \omega_s(t-\tau)$$

Two frequencies, ω_x , ω_y , are defined from the input signal, such that

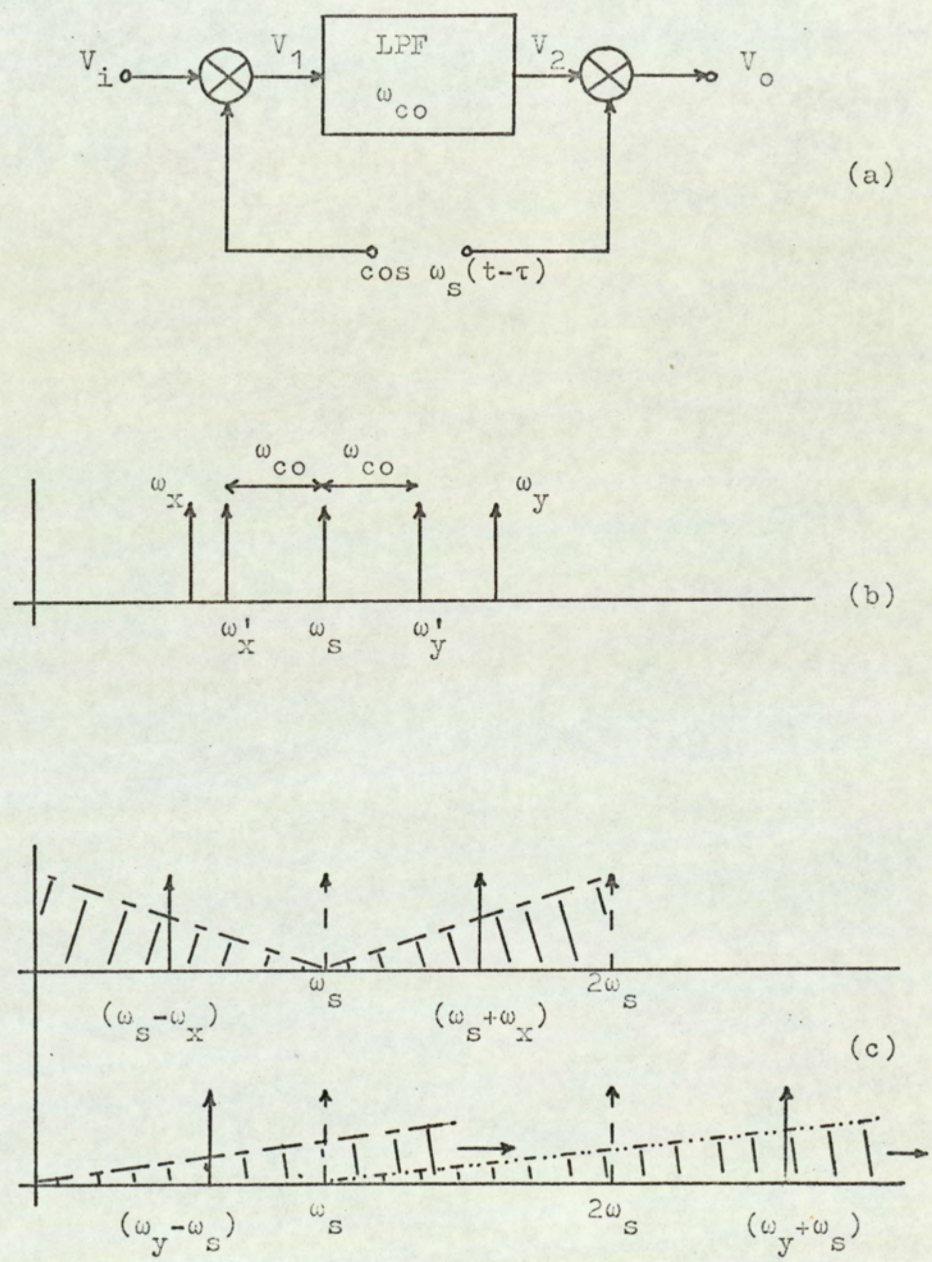
$$\omega_x \leqslant \omega_s \leqslant \omega_y$$

and two other frequencies, ω'_x , ω'_y , such that

$$\omega'_x = \omega_s - \omega_{co} \quad , \quad \omega'_y = \omega_s + \omega_{co}$$

as shown in figure 2.2.b. The cutoff frequency of the lowpass filter is defined as ω_{co}.

The requirement is that the band of frequencies from ω'_x to ω'_y be filtered from the input spectrum.



V i
V 1
LPF
ω co
V 2
V o
(a)
cos ω s (t-τ)
ω co
ω co
ω x
ω y
(b)
ω' x
ω s
ω' y
ω s
2ω s
(ω s -ω x)
(ω s +ω x)
(c)
(ω y -ω s)
ω s
2ω s
(ω y +ω s)

Figure 2.2

The input, in terms of the two general frequencies ω_x and ω_y, is,

$$V_i = \cos \omega_x t + \cos \omega_y t$$

hence, the output from the first modulator is

$$V_1 = V_i \cos \omega_s(t-\tau)$$

$$= \frac{1}{2}\left\{\cos\left[(\omega_s + \omega_x)t - \omega_s\tau\right] + \cos\left[(\omega_s - \omega_x)t - \omega_s\tau\right] + \cos\left[(\omega_y + \omega_s)t - \omega_s\tau\right] + \cos\left[(\omega_y - \omega_s)t + \omega_s\tau\right]\right\}$$

i.e. V_1 consists of overlapping sets of sidebands, which may extend from zero frequency upwards, as shown in figure 2.2.c. The assumption is made that

$$\omega_{co} < \omega_s$$

so that

$$\omega_{co} < (\omega_s + \omega_x) \text{, and } \omega_{co} < (\omega_y + \omega_s)$$

The highest frequency component that can appear in V_2, the output from the lowpass filter, will be ω_{co}. Since

$$\omega'_x = \omega_s - \omega_{co} \text{, then } \omega_{co} = \omega_s - \omega'_x$$

$$\omega'_y = \omega_s + \omega_{co} \text{, then } \omega_{co} = \omega'_y - \omega_s$$

so that, in terms of the highest frequency components,

$$V_2 = \frac{1}{2}\left\{\cos\left[(\omega_s - \omega'_x)t - \varphi' - \omega_s\tau\right] + \cos\left[(\omega'_y - \omega_s)t - \varphi' + \omega_s\tau\right]\right\}$$

where φ' is the phase shift introduced by the lowpass filter.

Frequencies intermediate between ω_x' and ω_y' appear as lower frequency components. These will also be passed by the lowpass filter, and will have proportionately reduced values of φ. With a non-ideal lowpass filter, ω_{co} is here taken to be the highest frequency passed which retains significant amplitude.

The output of the second modulator is :-

$$\begin{aligned} V_o &= V_2 \cos \omega_s(t-\tau) \\ &= \frac{1}{4}\Big\{\cos\left[(2\omega_s - \omega_x')t - \varphi' - 2\omega_s\tau\right] \\ &\quad + \cos(\omega_x't + \varphi') + \cos(\omega_y't - \varphi') \\ &\quad + \cos\left[(2\omega_s - \omega_y')t + \varphi' - 2\omega_s\tau\right]\Big\} \end{aligned}$$

In the term $\cos\left[(2\omega_s-\omega_x')t -\varphi'-2\omega_s\tau\right]$, ω_x' can take any value between 0 and ω_s, so that this term may represent any frequency between ω_s and $2\omega_s$, and will overlap the desired component ω_y'. In the term $\cos\left[(2\omega_s-\omega_y')t +\varphi'-2\omega_s\tau\right]$, ω_y' can take any value between ω_s and $2\omega_s$, so that this term may represent any frequency from 0 to ω_s, and will overlap the desired component ω_x'.

It will be noted, however, that with respect to the desired components, ω_x' and ω_y', the two unwanted components have experienced phase shifts of $2\omega_s\tau$. This effect may be used to eliminate the unwanted components.

Thus, if the modulating signal is redefined as

$$\cos\left[\omega_s\left[t - (n-1)\tau\right]\right] \quad , \quad n = 1,2, .. N,$$

where $N\tau$ has the value derived later, then if N parallel networks, of the form shown in figure 2.2.a., have phase increments to their modulating signals as just indicated, summation of the N outputs will cause the unwanted components to be cancelled.

The first unwanted component becomes

$$\cos\left[(2\omega_s - \omega_x')t - \varphi' - 2\omega_s(n-1)\tau\right]$$

$$= \cos\left[\theta t - \varphi' - 4\pi(n-1)\tau/T_s\right]$$

$$= \text{Re.}\; e^{j\left[\theta t - \varphi' - 4\pi(n-1)\tau/T_s\right]} \quad \text{, where } T_s = 2\pi/\omega_s \;.$$

The summation of these terms over the N paths is

$$\text{Re.}\; e^{j(\theta t - \varphi')} \sum_{n=1}^{N} e^{-j4\pi(n-1)\tau/T_s}$$

and the geometric series formed by the summation has the closed form

$$\frac{1 - e^{-j4\pi N\tau/T_s}}{1 - e^{-j4\pi\tau/T_s}}$$

$= 0$, provided $\tau = k(T_s/2N)$, where k is any integer not a multiple of N.

Similarly, the second unwanted component will sum to zero, and the overall output will be

$$\sum_N V_o = \frac{N}{4}\left[\cos(\omega_x' t + \varphi') + \cos(\omega_y' t - \varphi')\right]$$

which provides the required bandpass filtering of the input spectrum. It may be noted that there are no band-limiting restrictions on the input.

If the number of paths, N, is two, the phase displacement between the modulating signals should be $T_s/4$, i.e. they should be in quadrature. The phase difference between the unwanted components in the two paths becomes π at the output, leading to mutual cancellation. Two-path systems, based on this principle, have been discussed by Weaver, (reference 30), and by Paris, (reference 31).

In the system described by Weaver, the baseband message is translated to a single-sideband signal. The message modulates quadrature carriers at the baseband centre frequency. This results in overlapping sidebands which are passed through lowpass filters with a cutoff frequency equal to half the bandwidth of the baseband. (The baseband should not extend to zero frequency). The filter outputs represent the difference frequencies between the baseband components and the centre frequency. These are then used to modulate a second pair of quadrature carriers at the appropriate frequencies to translate the message to the required range. Summing the modulator outputs produces an upper sideband, and subtracting, a lower sideband. The filter requirements for isolating the sidebands are thus less critical than with the usual method.

The system described by Paris, (the 'Rixon Bandshift Modulator'), is intended primarily as a variable centre-frequency bandpass filter. The input signal is first modulated by a variable -frequency oscillator, then wide bandpass filtered to allow fixed-frequency working for the quadrature path oscillators. The final modulator frequency is chosen to translate the narrowband signal to the lowpass range.

Systems of this nature require careful balancing of the signal paths for effective operation. However, it was indicated by Paris that balanced modulators with a carrier leakage of less than 50 db could be achieved without great difficulty. Also, that by matching the lowpass filter components to within 1%, the distortion could also be kept to within 1% of the maximum output.

Modulation by a single sinusoid represents only a special case, since the modulating signal may be any periodic function. A general analysis for such systems was provided by Franks and Sandberg in 1960, (reference 32). If the modulating functions occupy an infinite bandwidth, as with pulse modulation, it was shown that the input spectrum must be bandlimited to $N\omega_s/2$. Also, that the modulating signals should be displaced by equal increments over their fundamental period.

With rectangular-pulse modulation, the modulators are, of course, sampling switches. Because of the multiple pulse harmonics, the system response is that of a comb filter, i.e. a set of passbands centred on harmonics of the sampling frequency. If a single passband only is required, then provided ω_s sufficiently exceeds ω_{co}, the unwanted response peaks can be eliminated by a simple bandpass filter at the output.

The N-path method thus provides a means for realising narrow bandpass systems without the use of inductors, and is particularly useful at low frequencies. It is also potentially useful for constructing bandpass systems in integrated-circuit form, (reference 33). From the point of view of fdm-tdm interconversion, however, the advantage over passive filters is minimal.

The centre frequency of the N-path filter is determined only by the modulating frequency. It might appear, therefore, that by changing the centre frequency in appropriate steps, an fdm signal could be separated and appear directly in pam tdm form. However, this is feasible only at low stepping rates. A similar problem occurs with spectrum analysers having a cro display. The input spectrum is hetrodyned against a sweep frequency oscillator and applied to a narrow-bandpass filter. If the sweep rate is too high, the filter response is attenuated and distorted.

To attempt to obtain clear-cut pulses from the N-path filter would be to require that a narrow-bandwidth source should provide wideband signals. The limitation lies in the minimum interval for which the N-path filter must dwell on the centre frequency for the output to reach the steady state level. A bandpass filter can be treated in terms of an equivalent lowpass filter, (reference 14, ch.3), and as discussed by Schwartz, (reference 34, ch.2), the minimum response time is the reciprocal of the lowpass filter bandwidth.

With the standard telephone group, the N-path filter would, therefore, have to remain at one centre frequency for at least 0.5 ms, which clearly rules out the possibility of multiplexing the twelve channels at a 96kHz rate by stepping the centre frequency. In fact, each channel would require a separate N-path filter, which would not represent any saving over the use of passive filters. Thus, it would appear that although the N-path filter has useful properties, its use would not provide any direct improvement in an fdm-tdm interconversion system.

2.5 Digital filtering.

The increasing use of pcm, (pulse-code-modulation), suggests that some mention be made of the possibility of using 'digital' methods for multiplex interconversion. It is evident that multiplexed pcm signals convert directly to and from pam tdm. The conversion of fdm signals to and from pcm would normally require an intermediate conversion to pam.

Depending on the context, the term 'digital filtering' appears in the literature with two different meanings. The first implies an operation upon a signal waveform in quantised binary form, so as to modify the signal bandwidth for transmission purposes. This can be achieved with conventional filters, but the necessary linear phase characteristic is difficult to realise. However, a linear characteristic can be relatively easily obtained from filters which are based on summing the weighted outputs of a tapped delay line. With binary signals, the delay line is readily achieved by means of a shift register. The binary waveform can also be modulated in various ways, using only logic gates. A recent description of these techniques has been given by Gerwen, (reference 35).

It will be seen that since these are operations upon the binary-coded waveform, rather than upon the signal which has been coded, they are not applicable to the fdm conversion problem. The other meaning which attaches to the term 'digital filtering' is that of a purely numerical operation, carried out either by a digital computer, or by equivalent circuitry.

Thus, for fdm conversion, the fdm waveform would be sampled directly by an analogue-to-digital converter at the computer input. Sequential separation of the channels, and demodulation,would be performed by an appropriate algorithm,so that the computer output would be multiplex pcm signals,which could be converted directly to pam tdm,if required. Alternatively, the computer input would be a multiplex pcm signal,which,after processing and digital-to-analogue conversion,would appear as the sampled version of an fdm signal. After smoothing,the samples would be transmitted as a normal fdm signal. However,conversion to fdm would be more simply achieved by the methods described at the beginning of this chapter.

It is evident that,for speech transmission,these operations must be in real time. The maximum acceptable delay would be about 10 ms. Furthermore,it is unlikely that a computer would be available solely for signal processing,so that the time would have to be shared with other operations,such as traffic routing.

Taking the standard telephone group as an example, the fdm signal has a 48kHz bandwidth,from 60 to 108 kHz. The minimum sampling rate is given by

$$f_s = 2 f_2/m$$

where f_2 is the highest frequency in the bandwidth,and m is the largest integer not greater than f_2/(bandwidth).

$$\text{Hence,} \quad f_s = (2.108)/2 = 108\text{kHz.}$$

Assuming that conversion to eight-bit binary provided the minimum acceptable computation error,the binary signal would emerge at 864 kbits/sec. Although high,this conversion rate is feasible. (At the time of writing,the highest speed analogue-to-digital converter which is commercially available is capable of 200,000 conversions per second with a 15 bit resolution,(the EECO 1202,USA), although the cost is comparable to that of a small general-purpose computer.)

The required input/output rate is well within the capacity of current general-purpose computers,and the required digital-to-analogue conversion rate is not excessive. The peripheral equipment would not,therefore,set any inherent limitation to real time operation.

The restriction,if any,would be in the speed at which the computer program could be performed. This will obviously depend on such factors as the characteristics of the computer and the computer code,as well as the complexity and method of the computation. The problem has been discussed by Salzer, (reference 36),in relation to the use of on-line computers in control systems. A considerable body of literature also exists on computer techniques for obtaining the Fourier transforms of arbitrary functions,and for realising various non-linear operations, such as adaptive filtering. The 'Fast Fourier Transform' permits real time operation at low frequencies,as described,for example, by Shively,(reference 37).

For real time operation,filtering is best carried out by a process comparable to the analogue method of summing the weighted outputs of a tapped delay line. The filter may be recursive,i.e. the output for a given input sample is a function of both the previous input values and of the previous output values; or non-recursive,when the output is a function only of the previous input values.

Generally,a recursive filter is used when a continuous analogue filter is to be simulated. There will,therefore,be a delay to allow accumulation of the required number of output weightings. A comprehensive general survey of digital filters has been given by Kaiser,(reference 38),and another detailed treatment is that by Robinson,(reference 39). The real time realisation of a lowpass filter with a cutoff of 0.1 Hz has also been described recently,(reference 40).

So far as fdm conversion is concerned,the direct conversion would involve multiplication by the carrier frequency of the desired channel,followed by lowpass filtering. The computational procedure would have to include a means for aligning the phases of the multiplying and carrier frequencies,and for variations of computer clock frequency with respect to the signal frequency. Unless parallel arithmetic units were provided, the computation would have to be repeated for each channel of the multiplex signal, and all computations completed in the interval between successive samples of the signal.

It was pointed out by Brillouin, (reference 9, ch. 7 & 9), that with the (sin x)/x response of an ideal lowpass filter, the summation of at least 10^n terms is required for an error of less than 10^{-n}. In practice, if the digital filter is based on a realisable continuous filter, fewer terms may give the desired accuracy. For example, the computational equivalent to a 10-tap filter might approximate the desired response sufficiently closely. With a 100 kHz sampling rate, this implies a delay of 100 microsec.

Because of the variety of ways in which the filter algorithm might be constructed, a specific processing time for arithmetical operations cannot realistically be postulated. However, it can reasonably be assumed that there will be at least one multiplication per filter tap. Taking a figure of 20 microsec per multiplication, which would apply to a general-purpose computer such as the PDP-9, the 10-tap filter would require 200 microsec for multiplications. To this must be added the time for transfers, addition, and other operations.

This applies to only one channel, so that with serial processing the total time would then be increased by the number of channels. It is evident that an input rate of one new sample every 10 microsec cannot be accommodated. Even if each channel had the use of a separate arithmetic unit, the computation time would still have to be reduced by at least an order of magnitude. Neither does this allow for sharing the computer time with other operations.

Generally, it would appear that real time signal processing on current general-purpose computers is limited to frequencies of not more than a few hundred Hertz. Whilst this includes many useful applications, the possibility of processing at even low communications frequencies would appear to need the development of computers operating at several hundred megaHertz, or the use of special purpose circuitry.

2.6 Conclusions.

It has been seen that the structure of a pam tdm signal is such that there is a simple means for converting directly from the tdm signal to an equivalent ssb fdm signal. All that is required is a set of bandpass filters, so that this method offers considerable saving over the conventional approach of demodulating to baseband, and remodulating.

Fdm can also be converted directly to tdm by filtering, followed by correctly synchronised sampling. This has less advantage over the conventional method, since the circuitry involved is substantially the same in each case.

Because bandpass filters are a necessary feature in either case, some active-circuit methods for realising a bandpass response have been examined. It was seen that the N-path filter makes possible the realisation of narrow bandpass filters without the use of inductors. However, the circuitry involved is such that no clear advantage would be gained by the use of such a system. Direct processing by digital computation was considered, but was found to require excessively high computing speeds.

The use of the two conventional forms of multiplexing, and consequently the need for interconversion, arises from the particular advantage that each has to offer. Thus, fdm allows the narrowest possible transmission bandwidth, whilst tdm is convenient for terminal operations.

This suggests that it would be useful if a multiplex system could combine these properties, i.e. have minimum bandwidth for transmission, and be recovered at the receiver by sampling. It would also represent a useful saving if the number of bandpass filters could be reduced from that needed for a conventional system. A system which meets these requirements will be described in the next chapter.

CHAPTER 3

A MINIMUM BANDWIDTH TIME-DIVISION-MULTIPLEX SYSTEM USING FINITE SUMMATION IN THE FREQUENCY DOMAIN.

3.1. Introduction.

Two of the criteria by which a particular multiplex system may be judged are the transmission bandwidth needed, and the degree of complexity required in a practical realisation. With the signal processing techniques which are presently available, a time-division-multiplex system using switching circuits to generate and recover the signals is more easily realised than a frequency-division-multiplex system using analogue multipliers and sharp cutoff bandpass filters.

However, the pulse waveforms of the tdm (i.e. time-division-multiplex) system require a greater bandwidth for faithful reproduction than the smooth continuous waveforms of an fdm (i.e. frequency-division-multiplex) system. An ideal single-sideband fdm system occupies the minimum possible bandwidth for a linear real-time system, but realisation of a close approximation is relatively difficult, even when the synchronisation problem is eased by the use of single-sideband-plus-carrier. Double-sideband-plus-carrier fdm systems are much simpler to achieve, but precise bandpass filtering is still necessary for bandwidth conservation.

Some of the fdm schemes described in chapter one were able to dispense with passive bandpass filters, but only at the expense of circuitry at least as complex as that of a conventional single-sideband system. It would appear to be the case with fdm that nothing comparable in simplicity to time-domain filtering by simple switching is achievable. Therefore a more fruitful approach might be to seek ways of band-limiting a tdm system.

Such a band limited system would have waveforms corresponding to the sum of the first few harmonics of the original pulse waveforms. It was seen in chapter one that the summation of a finite and complete set of harmonic frequencies has a simple closed form function, namely the periodic Dirichlet kernel. This chapter is devoted to the consideration of waveforms obtained from various finite summations in the frequency domain and their application to practical tdm systems.

3.2. Band limited tdm signals.

Since a rectangular pulse waveform is analytically representable as the interference pattern of an infinite summation of harmonically-related continuous sinusoids, truncations of the spectrum may be expected to lead to functions with less discontinuities, and eventually to smooth continuous waveforms. The effect of limiting the spectrum of the tdm signal is, therefore, to spread the pulses in time so that eventually the distorted pulses occupy overlapping time intervals and their orthogonal property is lost. A reduction in overall signal amplitude will be a further undesirable consequence.

The mutual interference caused by pulse overlapping is termed the crosstalk. The crosstalk ratio may be defined as the ratio of the mean amplitude of the band-limited signal from a particular channel over a specific finite time interval, (such as the duration of the original rectangular pulse), to the mean amplitude of the sum of overlapping components from the other channels. Any non-ideal tdm system will have a finite value for this ratio which will depend, to some extent, on the system bandwidth.

However, bandwidth is by no means the only factor which determines crosstalk, since such system parameters as the initial pulse shape and duration, and the attenuation and phase characteristics of the transmission path are also involved. It has been shown, (Reference 41), that there is no simple relationship between crosstalk and bandwidth which is useful in the general case. If the messages in each channel occupy the same bandwidth, the theoretical minimum bandwidth for the multiplex signal is the product of the number of channels and the channel bandwidth. For speech transmission, where the crosstalk must be low, the system bandwidth must be many times greater than the theoretical minimum. For telemetry systems the crosstalk specification may be reduced, and, for example, a figure of six times the theoretical minimum bandwidth has been quoted as one giving satisfactory results, (Reference 42).

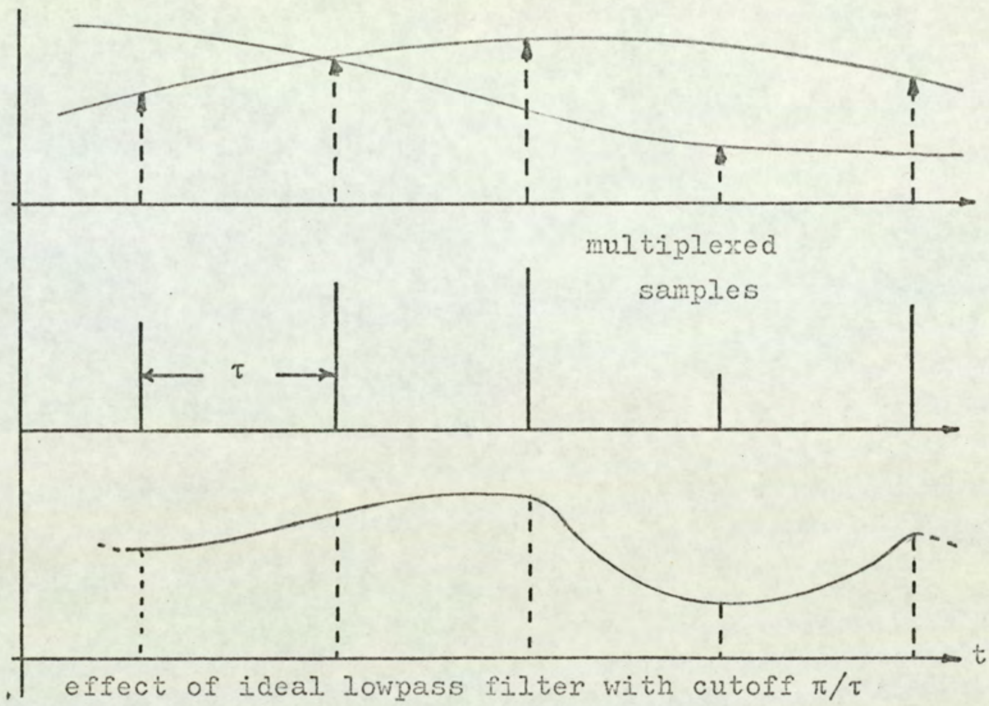


Figure 3.1

The waveform of a tdm signal having the theoretical minimum bandwidth may be envisaged by considering the tdm signal to be the sampled version of some arbitrary waveform. This waveform is sampled at the multiplex signal pulse-recurrence-frequency, that is, at Nf_s, when there are N channels each sampled at a frequency f_s. If the arbitrary waveform is recovered by passing the tdm signal through an ideal lowpass filter with cutoff at $Nf_s/2$, then this waveform, (shown in figure 3.1), is uniquely defined, having a bandwidth from zero to $Nf_s/2$, and containing without distortion all the amplitude values necessary to reconstruct the pam channel signals by subsequent resampling.

However, only non-ideal filters are available, which will inevitably introduce crosstalk. In the case of multiplexed binary signals, such as pcm, the crosstalk may be acceptable. Thus, practical filters with gradual cutoffs starting at half the bit rate may be used, whilst still retaining sufficient amplitude difference between mark and space to reconstruct the original waveform with negligible error.

Despite the crosstalk introduced by practical lowpass filters, narrowband tdm systems for speech transmisson have been described, which incorporate some device for reducing the crosstalk. In 1949 Boothroyd and Creamer described a system whereby the received signal was corrected by feeding back a proportion of a particular channel pulse to the other channels in such a way as to cancel the unwanted components introduced by a non-ideal lowpass filter. (reference 43).

The authors implied that a bandwidth approaching the theoretical minimum was obtainable, although this point was not developed analytically. Schwartz, (reference 34, ch. 4), has described a system using less elaborate corrector circuits which uses a bandwidth of about twice the theoretical minimum. However, as has been pointed out, (reference 44), it is difficult in a practical system to maintain the initial corrector adjustments so that the crosstalk is always kept below the level required for high quality speech telephony.

3.3.a. Ideal lowpass filter response.

The response of an ideal lowpass filter and various approximations to the ideal response are discussed in most standard texts, notably by Schwartz, (reference 34, ch. 4), and by Guillemin, (reference 16, ch. 16).

Since the derivation of this response is pertinent to the material which follows, a summary of the procedure is included. First, the frequency domain characteristic of an ideal lowpass filter with cutoff ω_{co} is defined as

$$G(\omega) = e^{j\omega t'}, \ |\omega| < \omega_{co}; \qquad G(\omega) = 0, \ |\omega| > \omega_{co}$$

i.e. the filter is considered to have a unity amplitude response and a linear phase characteristic, $e^{j\omega t'}$, throughout the passband.

The time domain representation, ie. impulse response, is obtained from the Fourier transform of $G(\omega)$,

$$g(t) = \frac{1}{2\pi} \int_{-\omega_{co}}^{\omega_{co}} e^{j\omega(t-t')} \, d\omega = \frac{1}{\pi(t-t')} \sin \omega_{co}(t-t')$$

$$= \frac{\omega_{co}}{\pi} \frac{\sin \omega_{co}(t-t')}{\omega_{co}(t-t')}$$

The response, $r(t)$, of the filter to a signal, $s(t)$, is then obtained from the convolution of $s(t)$ and $g(t)$,

$$r(t) = \int_{-\infty}^{\infty} s(t) \, g(\tau-t) \, dt$$

If the input signal is represented as an idealised sampled signal, in which the sample values are the values of impulse functions spaced at intervals T, that is

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT) \, \delta(t-nT)$$

and if $\omega_{co} = \frac{1}{2}(2\pi/T)$, then the filter response may be obtained directly,

$$r(t) = \frac{\omega_{co}}{\pi} \sum_{n=-\infty}^{\infty} s(nT) \frac{\sin \omega_{co}(t-t'-nT)}{\omega_{co}(t-t'-nT)}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} s(nT) \frac{\sin \left[(\pi/T)(t-t'-nT)\right]}{(\pi/T)(t-t'-nT)}$$

which indicates that each successive sample, $s(nT)$, generates a separate $(\sin x)/x$ waveform. Thus, the discrete input is converted into the time continuous waveform resulting from the superposition of an infinite set of $(\sin x)/x$ waveforms. Before and after the peak value occurring at the instant of the generating impulse, each $(\sin x)/x$ waveform has zero crossings equally spaced at intervals T. Since it is at these instants that a fresh impulse appears, the amplitude of the overall waveform is always linearly proportional to the signal value at the sampling instants.

From the sampling theorem, the sample values of a signal in the time domain are the Fourier coefficients of the same signal when it is expressed as a Fourier series in the frequency domain. This Fourier series is the transform of the original time-domain signal, so that the sample values also define the original signal for the whole interval between samples. Thus, the ideal filter response is an undistorted replica of the original signal in the sense of being the best mean-square approximation between sampling instants.

So far, the ideally sampled version of a single signal only has been considered. As was previously mentioned, a multiplexed signal, consisting of interleaved channels, each an ideally sampled signal, may also be passed through an ideal lowpass filter without distortion, as was indicated in Figure 3.1.

If the sample values of the multiplex signal are separated by intervals τ so that the multiplex frequency is $2\pi/\tau$, and if the filter cutoff frequency is $\frac{1}{2}(2\pi/\tau)$, then as with a single channel the $(\sin x)/x$ waveforms have zero crossings at intervals τ, and no crosstalk occurs. The overall waveform is arbitrary in the sense that it is formed from the arbitrarily related instantaneous values of the set of messages, but it is a continuously defined function bandlimited to ω_{co}. With ideal sampling in the receiver it is, of course, only the values of the waveform at the sampling instants which are of interest in the subsequent de-multiplexing.

Thus, with an ideal lowpass filter, and an ideally sampled multiplex signal, the latter could be restricted to the theoretical minimum bandwidth without producing crosstalk. It is clearly necessary that the separate channels should also be recovered by ideal sampling, since the $(\sin x)/x$ waveforms are non-interfering only at the zero crossings. However, an ideal filter is not physically realisable, and even if an acceptable approximation were to be used, the ideally sampled signal could itself only be approximated by the use of finite duration pulses.

The response of an ideal lowpass filter to a rectangular pulse may be deduced from the step response. This is derived from the integral of the impulse response, and takes the form,

$$r(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\omega_{co}(t-t')} \frac{\sin \omega_{co}(t-t')}{\omega_{co}(t-t')} \, d(\omega_{co}(t-t'))$$

$$= \frac{1}{2} + \frac{1}{\pi} \mathrm{Si} \left[\omega_{co}(t-t')\right]$$

and the response of the filter to a rectangular pulse of duration d may be obtained from the difference between unit steps at $(t-t'-d/2)$ and at $(t-t'+d/2)$, which is

$$r(t) = \frac{1}{\pi} \left\{ \mathrm{Si} \left[\omega_{co}(t-t'-d/2)\right] - \mathrm{Si} \left[\omega_{co}(t-t'+d/2)\right] \right\}$$

The shape of this function depends on the ratio of filter cutoff frequency and pulse duration (e.g. Reference 34, p45) but it is not necessary to consider this aspect in detail. The oscillatory tails of the step response undergo maxima and minima around a mean value at the instants when the corresponding $(\sin x)/x$ function has zero crossings. It would therefore be impossible to use a cutoff frequency of half the sampling rate without experiencing crosstalk.

This may also be seen by considering the rectangular pulse to be formed from a set of adjacent impulses. Each impulse generates a $(\sin x)/x$ waveform and it is not possible to define uniformly spaced intervals at which these waveforms sum to zero. This indicates, too, a further drawback to the use of an ideal lowpass filter for obtaining the theoretical minimum bandwidth, namely, that the cutoff frequency must be exactly half the sampling frequency, since deviation either way results in a shift in the timing of the zero crossings. Thus, it is seen that even if the use of an ideal lowpass filter were feasible, the response to any physical pulse waveform would inevitably be accompanied by crosstalk.

To summarize the previous remarks, the theoretical minimum bandwidth is attainable with an idealised pulse-amplitude-modulated multiplex system because the impulse response of the ideal lowpass filter is a (sin x)/x function. This function has regularly spaced zero crossings at intervals corresponding to the intervals between the multiplexed samples, provided that the filter cutoff frequency is exactly half the multiplex frequency. Hence, in the response of the ideal lowpass filter there is no crosstalk at the sampling instants, and subsequent re-sampling by a periodic impulse train at the channel sampling frequency will allow the original signal to be recovered without distortion.

3.3.b. Synthesis of the ideal lowpass filter response.

Since it is the appropriately spaced zeros of the (sin x)/x function which are of major significance, it is relevant to consider what other functions would satisfy this criterion. First, one may exclude any function which is identically zero for a finite time interval, since this function would require an infinite bandwidth. The desired function must be bandlimited, and subsequent to a peak value at $t = 0$, should be identically zero at all instants $t = nT$. Hence, the function must be oscillatory with zero crossings at $t = nT$. It would be desirable for the amplitude of the oscillations to decay rapidly, and to have the lowest possible gradient at the zero crossings, so as to minimise crosstalk when resampling with finite duration pulses.

Bennett and Davey (Reference 45) have described the application of Nyquist's theorems on intersymbol interference in band-restricted systems. The theorem on vestigial symmetry shows that an ideal lowpass filter modified by a skew symmetrical rounding of the rectangular response still retains the original zero crossings. The modified transfer function may be written as

$$g(t) = \frac{\sin \omega t}{\omega t} + g_1(t)$$

and, since $g_1(t)$ is of the form $(\sin \omega t).f(t)$, the original zeros are retained although additional zeros may appear. Other criteria show how zeros may be created at instants halfway between the original zeros, either additionally, or as an alternative. In general, however, an extension of the lowpass filter bandwidth is required, so that these methods could not be used for a system devised to operate with the minimum possible bandwidth.

Rather than attempt to realise a useful approximation to an ideal lowpass filter, however, one might instead attempt to synthesise the filter response directly. Obviously, it is not necessary to first generate a wideband sampled signal, and filter it to obtain a narrowband version, if the latter can be achieved by some other means.

Thus, one may consider the ideally-sampled signal represented by a periodic unit impulse train, i.e. the sampled function is a unity dc level;

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nt) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_o t}$$

(where $\omega_o = 2\pi/T$), whose Fourier transform is

$$S(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} 2\pi \, \delta(\omega - n\omega_o) = \omega_o \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_o)$$

Passing this signal through an ideal lowpass filter will cut off the spectrum above some frequency ω_{co}. If ω_{co} is such that the impulse at $p\omega_o$ is included, but the impulse at $(p+1)\omega_o$ is excluded, then

$$S(\omega) = \omega_o \sum_{n=-p}^{p} \delta(\omega - n\omega_o)$$

so that the time domain signal is

$$s_p(t) = \frac{1}{T} \sum_{n=-p}^{p} e^{jn\omega_o t}$$

This finite summation in the time domain is recognisable as the periodic Dirichlet kernel, $D_n(t)$, which was derived in section 1.4.e. of the first chapter. Thus the equivalent closed form is

$$s_p(t) = \frac{1}{T} \frac{\sin\left[(2p+1)\omega_o t/2\right]}{\sin(\omega_o t/2)} = D_n(t)$$

Noting that $D_n(t)$ is no more than the sum of a set of cosine harmonics, there are obviously a variety of ways in practice of generating such a function. If the harmonics are all modulated by the same message, then

$$s_p(t) = \frac{1}{T} \sum_{n=-p}^{p} e^{jn(\omega_o \pm \omega_m)t}$$

which is essentially the same result that would be obtained by passing an ideally-sampled message through an ideal lowpass filter. However, the objective is to generate a bandlimited tdm signal, and the possibility of so doing by using waveforms which are finite summations of harmonically related frequencies is considered in the remainder of this chapter.

3.4.a. Waveform properties of $D_n(t)$.

The various forms in which the periodic Dirichlet kernel, $D_n(t)$, may be written are,*

$$\text{(a)} \quad D_n(t) = \frac{1}{T} \sum_{n=-m}^{m} e^{jn\omega_o t} \qquad (\omega_o = 2\pi/T)$$

$$\text{(b)} \quad D_n(t) = \frac{2}{T}\left(\tfrac{1}{2} + \cos \omega_o t + \cos 2\omega_o t + \ldots \cos m\omega_o t \right)$$

$$\text{(c)} \quad D_n(t) = \frac{\sin \frac{1}{2}(2m+1)\omega_o t}{2 \sin \frac{1}{2}\omega_o t}$$

It is seen from (b) that $D_n(t)$ is a periodic function, with period T. Also, it may be seen from (c) that the numerator of this function,

$$\sin \tfrac{1}{2}(2m+1)\omega_o t$$

has (2m+1) zero crossings in the interval T, whilst the denominator,

$$\sin \tfrac{1}{2}\omega_o t$$

has only one zero crossing in this same interval.

When both numerator and denominator are zero, then

$$\mathrm{Lt}_{\frac{1}{2}\omega_o t \to 0} \quad \frac{\sin (2m+1)\frac{1}{2}\omega_o t}{2 \sin \frac{1}{2}\omega_o t}$$

$$= \quad \mathrm{Lt}_{\frac{1}{2}\omega_o t \to 0} \quad \frac{(2m+1) \cos (2m+1)\frac{1}{2}\omega_o t}{2 \cos \frac{1}{2}\omega_o t}$$

$$= \quad \tfrac{1}{2}(2m+1)$$

Therefore $D_n(t)$ has a peak value of $\frac{1}{2}(2m+1)$ at t=0, followed by oscillations which have their zero crossings at intervals of T/(2m + 1). The waveforms obtained from evaluating $D_n(t)$ for a range of values of m are shown in figure 3.2.

*(a) and (b) are consistent with the derivation of $D_n(t)$ from a truncation of the spectrum of a periodic train of unit impulses. The more general form of (c) results from summing the series in (b), but omitting the coefficient 2/T.

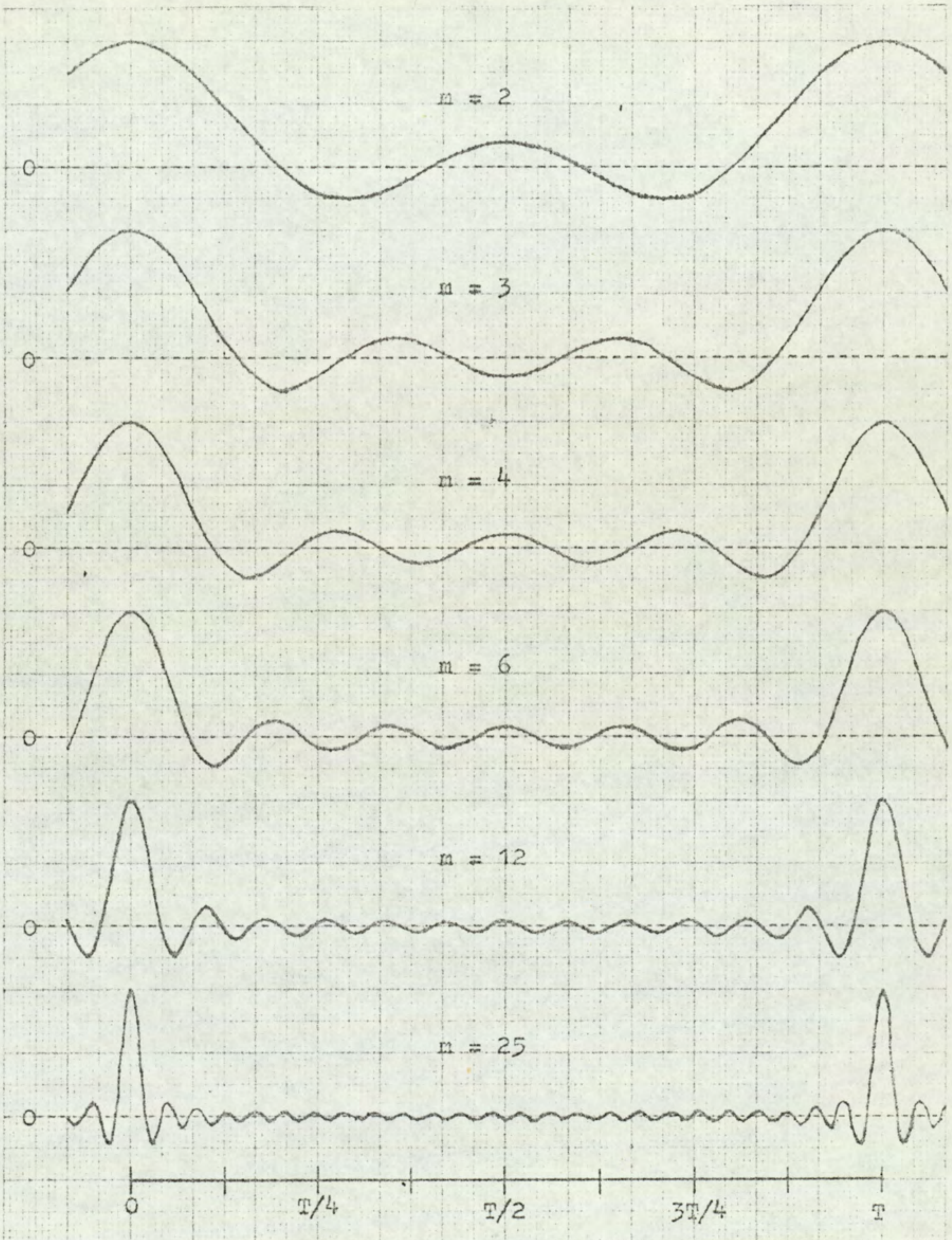


Figure 3.2

It is evident from (a) that as m approaches infinity, $D_n(t)$ approaches a periodic impulse train. Another limiting form for $D_n(t)$, which is less relevant from the multiplexing system point of view, is obtained by maintaining a finite value for m, but allowing ω_o to approach zero.

In that case

$$D_n(t)_{\omega_o \to 0} \rightarrow \frac{1}{2\pi} \int_{-m\omega_o}^{m\omega_o} e^{j\omega t}\, d\omega$$

$$= \frac{m\omega_o}{\pi} \frac{\sin m\omega_o t}{m\omega_o t}$$

which is the appropriate time-domain expression for the implied rectangular frequency-domain function.

If $D_n(t)$ is modulated by a frequency ω_a, and $\omega_a \leqslant \omega_o/2$, the resultant function is

$$\frac{2}{T}\left[\tfrac{1}{2}\cos \omega_a t + \cos(\omega_o \pm \omega_a)t + \cos(2\omega_o \pm \omega_a)t \ldots + \cos(m\omega_o \pm \omega_a)t\right]$$

so that the bandwidth occupancy extends from ω_a to $m\omega_o + \omega_a$. If ω_a represents the highest frequency of a message bandwidth extending from zero to ω_a, then the modulated version of $D_n(t)$ occupies a bandwidth which is (2m+1) times the message bandwidth.

It will be seen from the waveforms of figure 3.2, that if a second set of cosine harmonics are generated, having the same value for m as the first, but with the origin displaced by an integer multiple of T/(2m+1), then the peak of one waveform occurs at a zero crossing of the other, and the remaining zero crossings coincide. Furthermore, it will be seen that (2m+1) waveforms can be superimposed in this way.

If this composite waveform is sampled by infinitely-short duration pulses at instants corresponding to the peak values of the constituent waveforms, then clearly these peak values are recovered without crosstalk. Modulation of the $D_n(t)$ 'carriers' does not affect the timing of the zero crossings with respect to the unmodulated waveforms. The modulated function is

$$f(t) = \left[\frac{\sin\,(2m+1)\tfrac{1}{2}\omega_o t}{2\,\sin\tfrac{1}{2}\omega_o t}\right]\left[A\,\sin\,(\omega_a t + \varphi_a)\right]$$

and the right-hand bracket will not eliminate any of the zeros of the left-hand bracket.

Since, in practice, finite duration sampling pulses must be used, the recovered messages will suffer from crosstalk. Assuming, for the moment, that this crosstalk can be maintained at a tolerably low level, it will be seen that (2m+1) messages can be multiplexed using a bandwidth of (2m+1) times the message bandwidth, (which is assumed to be the same for all channels). Thus, the $D_n(t)$ function may be used as the basis of a tdm system which occupies the theoretical minimum bandwidth.

3.4.b. Crosstalk ratio using $D_n(t)$.

Before assessing the crosstalk incurred by the use of finite duration sampling pulses, it is necessary to determine an appropriate definition for the crosstalk ratio. In general, this is a signal-to-noise ratio, where the noise is that caused by the overlapping of other channels within the duration of the pulse sampling the wanted channel. Specifically, however, it is necessary to decide whether the instantaneous, rms, or mean value of the signal and noise components should be used.

Generally, the message waveform is recovered from the sample values by linear amplification and lowpass filtering. A holding circuit to prolong the sample to the duration of the sampling period is commonly interposed, and with an ideal sample and hold circuit, an instantaneously sampled value will be held. In that case, the system crosstalk would be zero. However, a practical sample and hold circuit has finite rise time sampling pulses, and the holding capacitor also has a finite charging time.

Whether or not a holding circuit is used, an instantaneous value of the sampled waveform is not realisable. Furthermore, the recovered message will appear, initially at least, as a voltage waveform, rather than a power waveform. Hence the rms value is not appropriate. The output of the lowpass filter will, in fact, be a waveform which is proportional not only to the mean value of the sequence of sampling pulses, but also the mean value of each pulse over the pulse duration.

The crosstalk ratio may therefore be evaluated by integrating a particular channel waveform between the limits set by the duration of the sampling pulse. The crosstalk contributed by the other channels may be obtained by integrating their relatively displaced waveforms over the same limits, or over limits appropriate to their relative time displacement. A rectangular sampling pulse is assumed.

Initially the crosstalk ratio for $m = 2$ is determined, and the appropriate waveform is shown in figure 3.3.a. Since $(2m + 1) = 5$ waveforms can be multiplexed, the superposition of the relevant portions of the five waveforms is shown in figure 3.3.b

In determining the crosstalk ratios, the unmodulated waveforms only are considered. This is based on the assumption that in a multiplexed system, each modulated channel will approximate to the same mean value over some interval. Thus the crosstalk ratios obtained will be applicable to the average values over this interval for modulated signals.

In figure 3.3.b., sampling pulses are superimposed for specimen pulse durations. These are defined in terms of the percentage ratio of the pulse duration to the interval between consecutive sampling points in the multiplexed signal. This is $(2m + 1)$ times greater than ratio of the sampling pulse duration to the period of the fundamental component of the waveform.

Inspection of figure 3.3.b. shows that the waveforms are symmetrical about the centre of the sampling pulse, so that

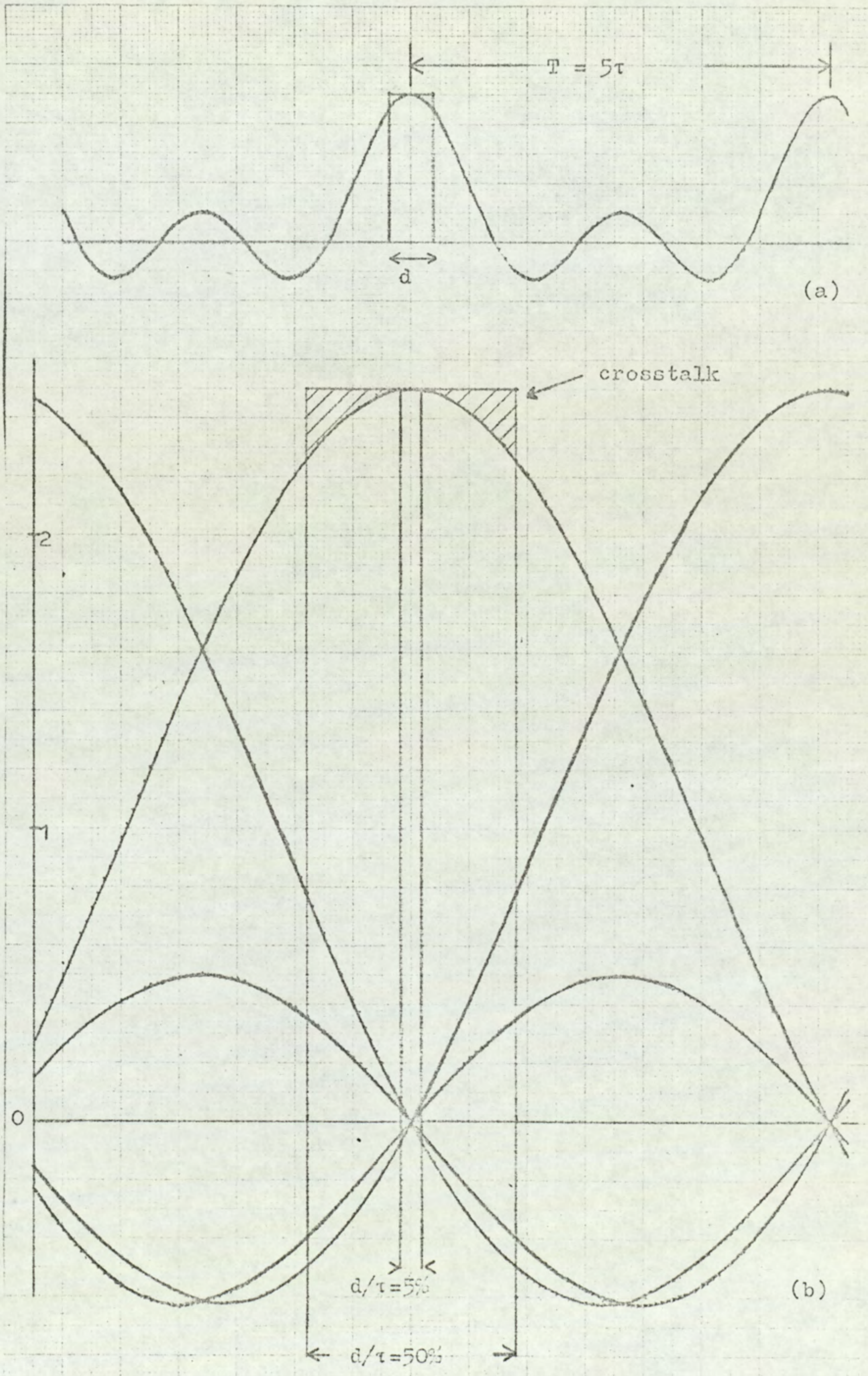


figure 3.3

it is only necessary to obtain the crosstalk ratio for either half of the pulse. The diagram shows that, whereas two of the interfering waveforms have positive areas over the half pulse duration, the other two have negative areas. Thus, in terms of the resultant mean value, the interfering components are largely self cancelling.

When the five waveforms are added, a specific envelope must, of course, be produced. The resultant is that which occurs from adding sets of equal-amplitude sinusoids with equal phase displacements over a complete period. That is, the sinusoids are mutually cancelling, and the resultant is a constant which is the sum of the dc terms. If, in fact, all the waveforms were modulated by the same constant value there would be no crosstalk, since each pulse samples the same dc value. The crosstalk components would add to the sample of the wanted channel to produce a flat - topped pulse. However, on the assumption that it is only the average value which is the same for all waveforms, the crosstalk ratio is interpreted as the ratio of the area of the sampled primary channel to the algebraic sum of the areas of the other components in the same sample.

Since the envelope of the superimposed waveforms is a constant, the envelope of the crosstalk components is obtained by subtracting the primary channel waveform from this constant. As the constant is the sum of the dc terms, it is equal to the peak value of any one of the waveforms. Hence, the envelope of the crosstalk components is simply an inverted version of any one of the channel waveforms, where what was the peak now touches the time axis.

Referring again to figure 3.3.b., it follows that the crosstalk ratio is simply the ratio of the area of the primary channel waveform beneath the rectangular sampling pulse to the residual area of the rectangular pulse waveform which lies outside the channel waveform. This latter is easily obtained by subtracting the computed area of the sampled primary channel from the area of the rectangular sampling pulse.

When generating the waveforms of $D_n(t)$, it is, of course, unnecessary to include the normalising constant 2/T which appears in the theoretical derivation. That is, the waveforms are formed simply by adding

$$\left[\tfrac{1}{2} + \cos \omega_o t + \cos 2\omega_o t + \ldots \cos m\omega_o t\right]$$

As was mentioned, the crosstalk ratio need only be determined over half the pulse duration since the waveforms are symmetrical. Therefore, for a sampling pulse of duration d, the sampled area of the primary channel is,

$$A_s = \int_0^{d/2} \left[\tfrac{1}{2} + \cos \omega_o t + \cos 2\omega_o t \ldots + \cos m\omega_o t\right] dt$$

$$= \frac{1}{\omega_o} \left[\omega_o d/4 + \sin \omega_o d/2 + \frac{1}{2} \sin 2\,\omega_o d/2 \ldots + \frac{1}{m}\sin m\omega_o d/2\right]$$

$$\frac{1}{\omega_o}\left[\tfrac{1}{2}\pi d/T + \sin \pi d/T + \frac{1}{2} \sin 2\pi d/T \ldots + \frac{1}{m} \sin m\pi d/T\right]$$

where $\omega_o = 2\pi/T$

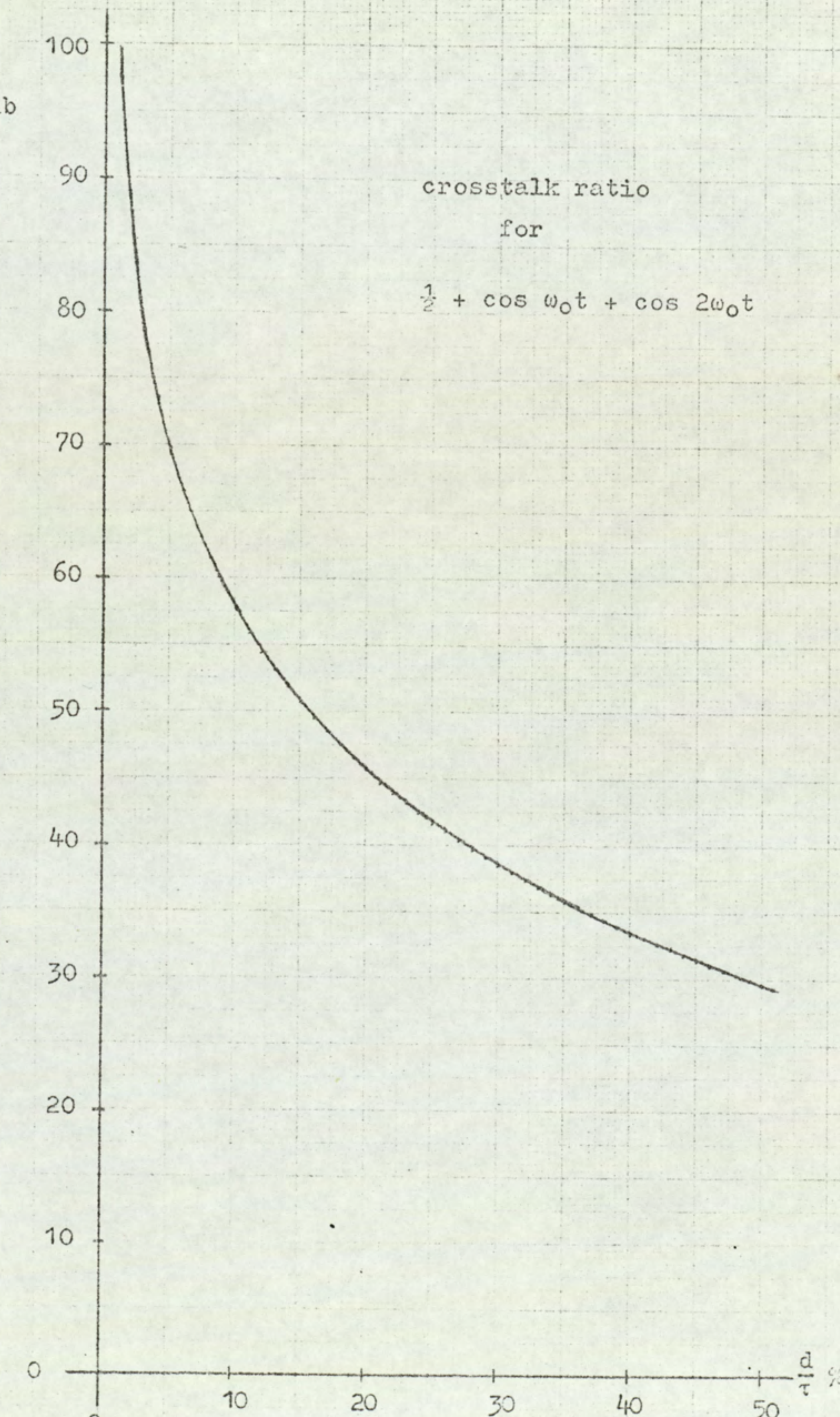
The area of the rectangular sampling pulse is,

$$A_p = \frac{1}{\omega_o} \cdot (\pi d/T) \cdot (\text{peak value of } D_n(t))$$

$$= \frac{1}{\omega_o} \cdot (\pi d/T) \cdot \left(\frac{2m + 1}{2}\right)$$

so that the crosstalk ratio is

$$R = \frac{A_s}{A_p - A_s} = 20 \log \left(\frac{A_s}{A_p - A_s}\right) \text{ db}$$



d = sampling pulse duration
τ = channel time slot

figure 3.4

Figure 3.4 shows the calculated crosstalk ratio for a five-channel system. It may be seen that the ratio achieves 60 db for a sampling pulse duration of about 8% of the channel time slot. If, for example, the fundamental component of the waveform is 8 kHz, the channel time slot is 125/5 = 25 microsec. Hence, the required pulse duration is 2 microsec. for a 60 db crosstalk ratio, which does not represent an impracticably short value.

As the number of harmonics composing the waveform is increased to allow more channels to be multiplexed, additional pairs of waveform crossovers occur at the sampling points, (as may be envisaged from figure 3.2). However, for a given ratio of sampling pulse duration to channel time slot, the resulting decrease in crosstalk ratio is negligible, as may be seen from the values tabulated below.

CROSSTALK RATIO (db)

d/τ %	m = 2	m = 3	m = 6	m = 12
1	97.6	97.4	97.3	97.3
5	69.7	69.5	69.4	69.3
10	57.6	57.4	57.3	57.3
15	50.6	50.4	50.3	50.2
20	45.6	45.4	45.3	45.2
25	41.7	41.5	41.4	41.3

These calculated crosstalk ratios are, of course, for a system with no errors. In practice, the zero crossings would be adversely affected by phase and amplitude errors in the harmonic generator, and by the characteristics of the transmission path. The latter must have a flat amplitude response and a linear phase response over the transmitted signal bandwidth. Timing errors in the receiver will also affect the crosstalk ratio to a greater extent than for conventional tdm.

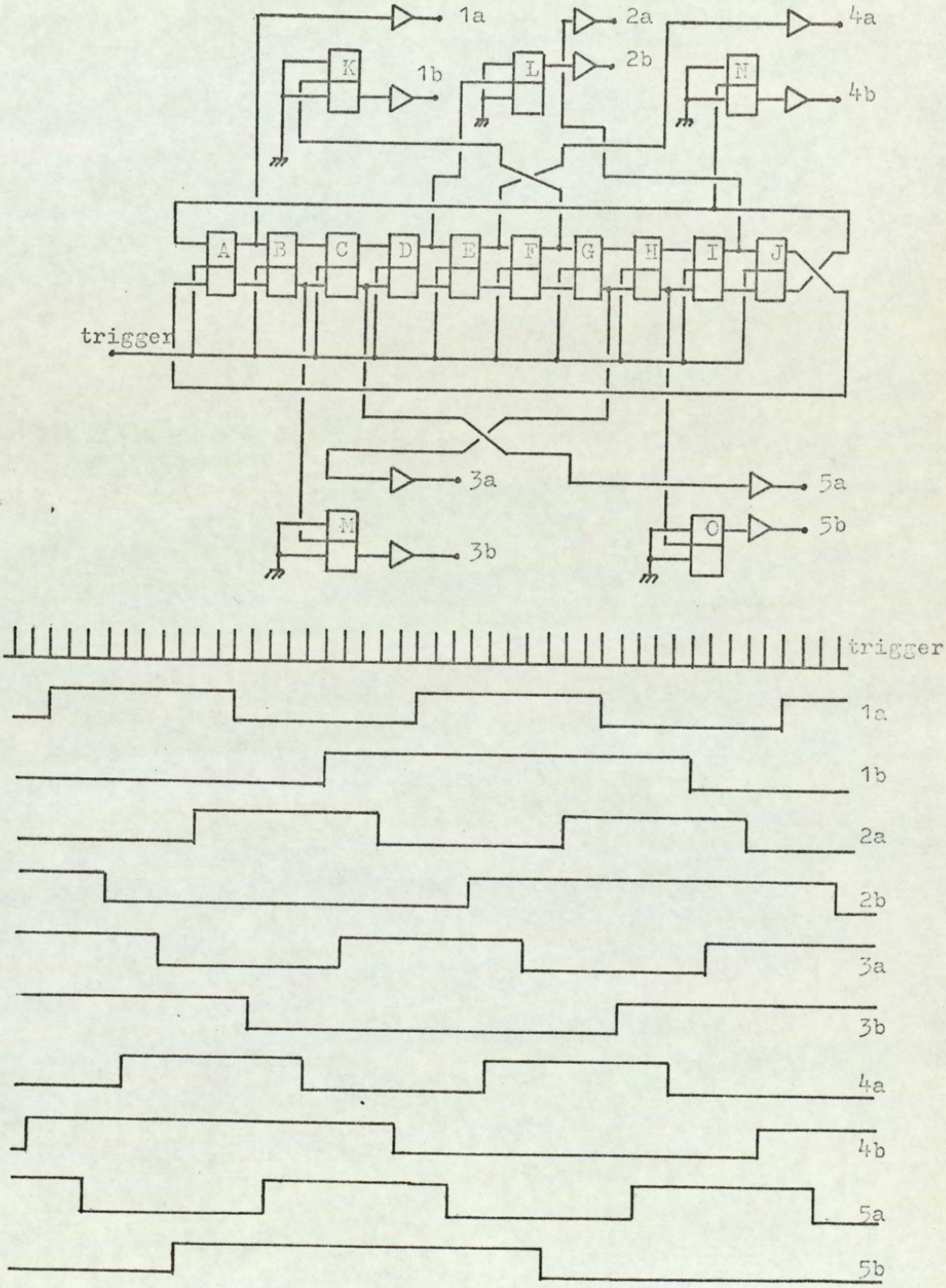
The effect of these errors is not examined for the $D_n(t)$ waveform, since, as discussed below, there are other practical disadvantages to the use of this waveform.

3.4.c. System realisation with $D_n(t)$.

In principle, the realisation of this system is straightforward. First, a set of harmonic cosines are generated and summed. The summation waveform is then modulated by a message. Further sets of cosine harmonics are generated, with phase shifts appropriate to their position in the multiplex frame, and each set modulated by a different message. The group of modulated waveforms are then summed and transmitted as the multiplex signal.

An obvious way to generate a set of harmonics is to filter the required components from a pulse train with appropriate prf. However, these filtered harmonics will be phased as sine waves, that is, each harmonic has a positive-going zero crossing at the instant when the leading edge of the pulse occurs. This could be corrected by phase-shifting networks, which would, however, represent an additional source of error. On the basis that digital circuits should be used where possible, it would be preferable to generate a set of harmonically-related square waves. These square waves must have their mid-points aligned with respect to the lowest frequency component.

It is a relatively simple matter, using flip-flops, to generate two square waves in quadrature. Each output, '1' and '0', is used to trigger a further pair of flip-flops. However, when a number of count-downs have to be performed to obtain the required harmonics, and to generate sets of square waves with arbitrary phase displacement for the different channels, the process becomes impracticable.



waveform generator for five-channel multiplexer

figure 3.5

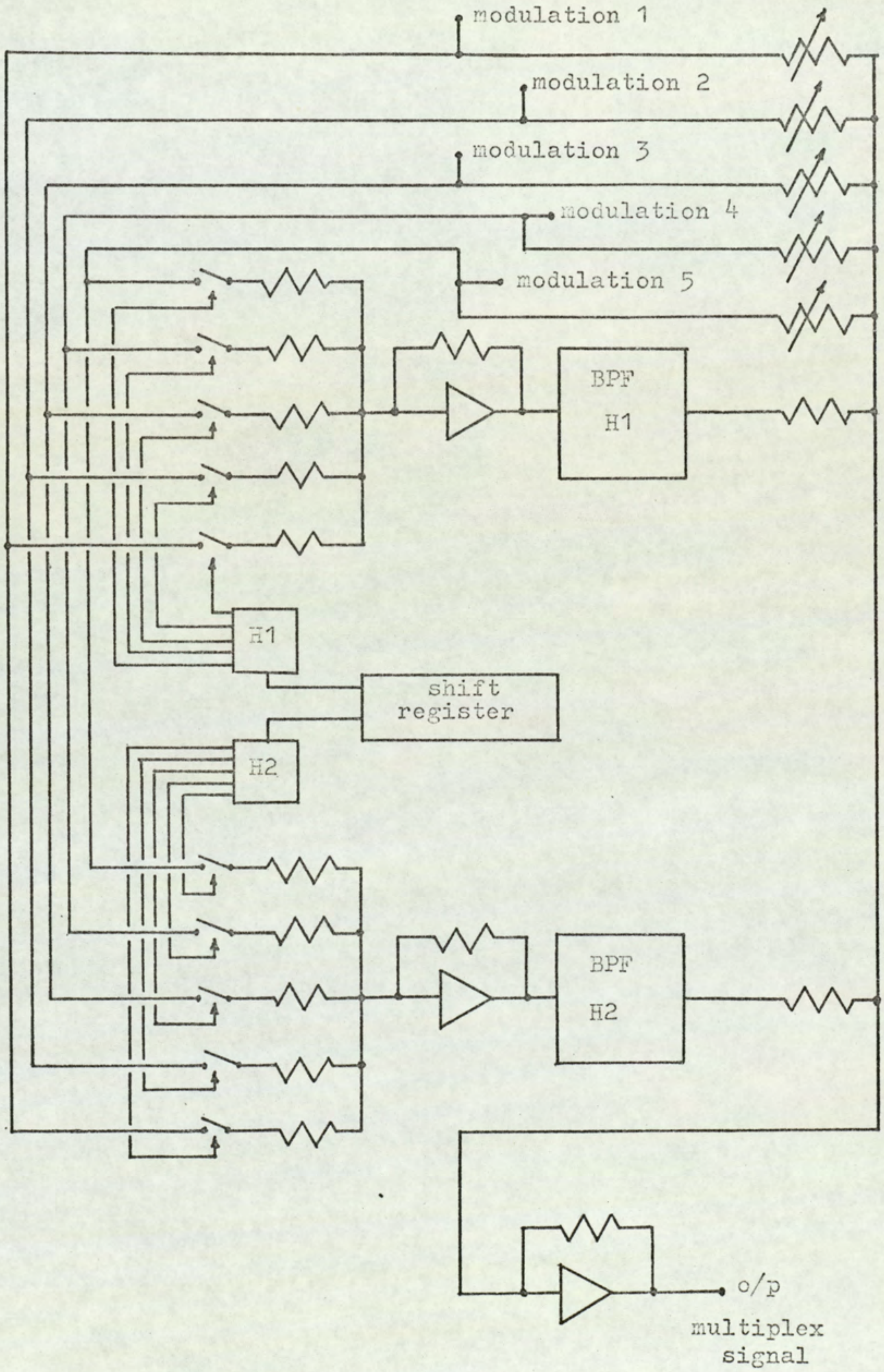
In an experimental system, having $m = 2$, it was found that a 'twisted ring' feedback shift register could provide a relatively simple solution. The waveform at each stage is a square wave displaced from the previous stage by one trigger period, and the square-wave period is twice the number of stages times the trigger period. The set of square waves at $2\omega_o$ can be taken off directly, and the set at ω_o by use of further 'divide by two' stages. The arrangement is shown in figure 3.5.

The output square waves were then used to drive sampling switches to which the modulating signals were applied. The fundamental of each modulated square wave was next extracted by a bandpass filter. As all the waveforms are eventually superimposed, it is only necessary to provide one filter per harmonic of the basic 'carrier' waveform. That is, the system requires only m filters for $(2m + 1)$ channels. All the modulated square waves for a particular harmonic were therefore applied to a summing amplifier, and passed through a common filter.

In the experimental system, the filters used were bandpass-coupled transformers, and, for convenience, standard IF transformers were modified for this purpose. The harmonic frequencies used were 20 kHz and 40 kHz, which represented a compromise between the frequency limitations of the summing amplifiers and the bandpass transformers.

The outputs from the two filters were then combined in a further summing amplifier. Also added at this stage were the components contributed by the dc terms. The output of this summing amplifier was then the required multiplexed signal. The overall system is shown in figure 3.6.a. and 3.6.b.

The experimental arrangement sufficed to demonstrate some of the difficulties which might be encountered in a fully-engineered version. Although waveform generation is relatively easy for $m = 2$, the circuitry required to generate cosine-related harmonics would become increasingly cumbersome as m was increased. Difficulty was also encountered in applying the dc term, or the



five-channel multiplexer

figure 3.6.a.

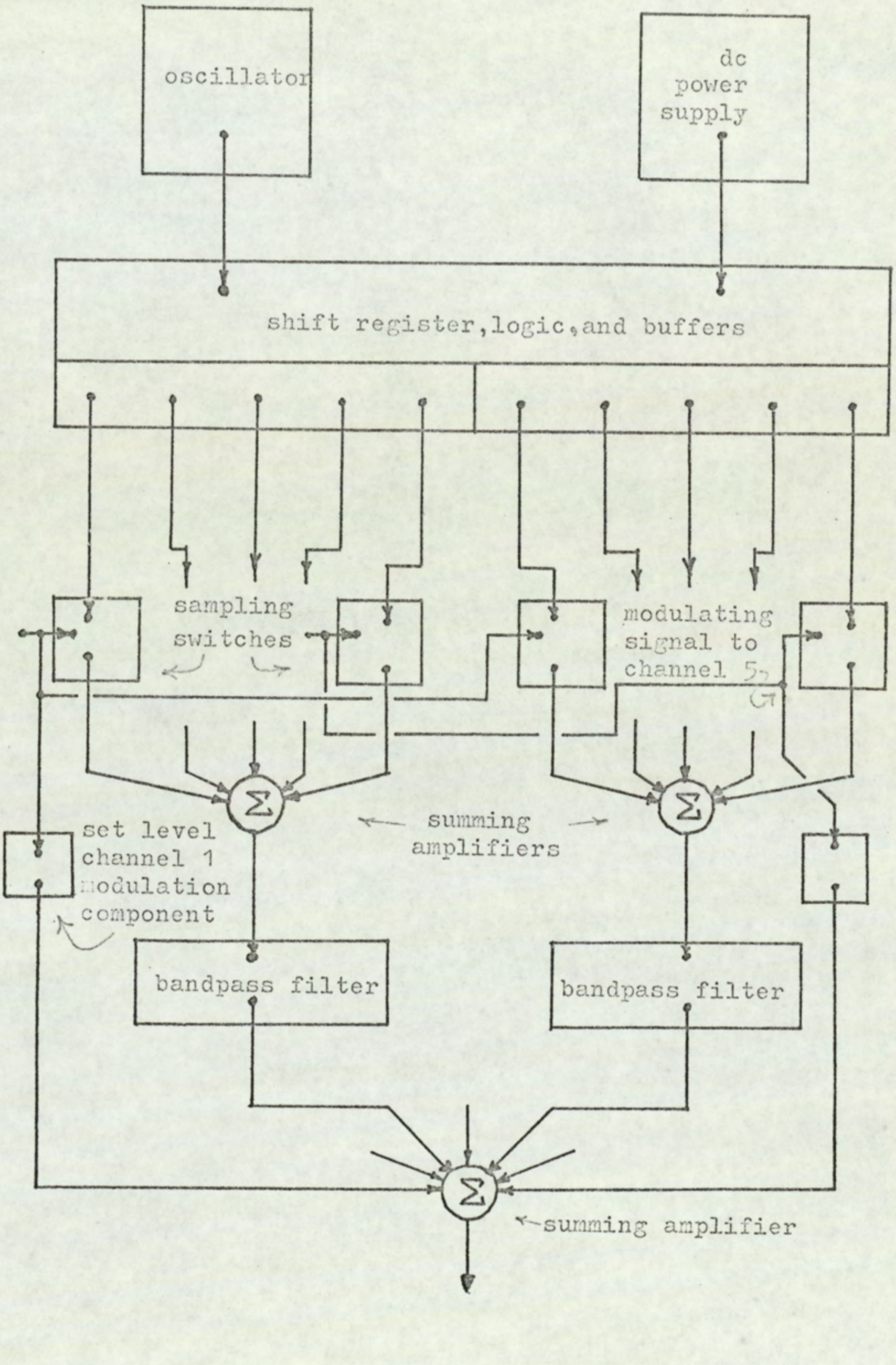

oscillator
dc power supply
shift register, logic, and buffers
sampling switches
modulating signal to channel 5
set level channel 1 modulation component
summing amplifiers
bandpass filter
bandpass filter
summing amplifier

Figure 3.6.b.

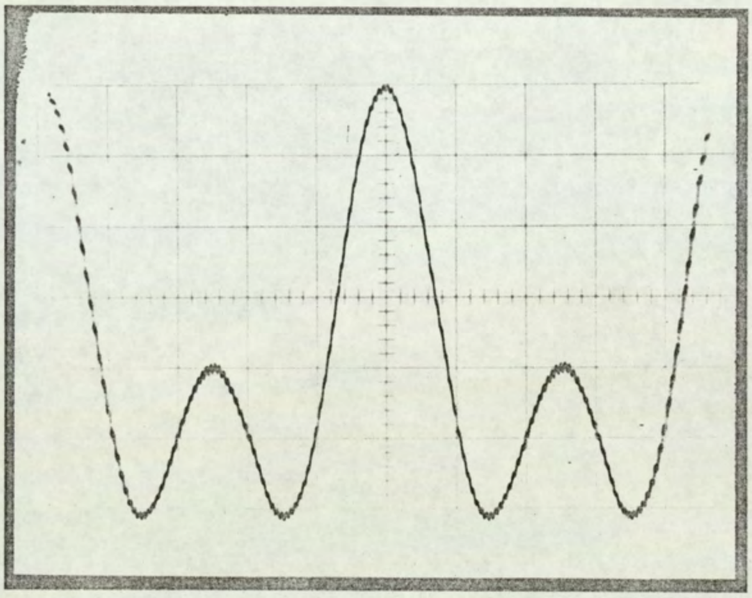


figure 3.7

modulation component corresponding to the dc term, in a sufficiently precise and stable manner. There were also the problems of drift in amplifier gain and phase, potentiometer setting drift, and noise pickup, common to the realisation of any system.

Figure 3.7 is an oscilloscope photograph of the waveform obtained by summing $\cos(2\pi.20.10^3t)$ and $\cos(2\pi.40.10^3t)$. It is evident that there was an appreciable amount of noise pickup caused by the proximity of the pulse circuitry and the high impedance level analogue circuitry. It was not possible to obtain adequate selectivity from the bandpass transformers, which was one reason why square wave inputs were preferred.

In figure 3.8 the same waveform is shown with approximately 50% modulation, and in figure 3.9 with approximately 100% modulation. It will be observed that the zero crossings are diffused. Apart from lack of precision in generating the waveforms, this also arose from synchronisation drift within the oscilloscope.

figure
3 .8

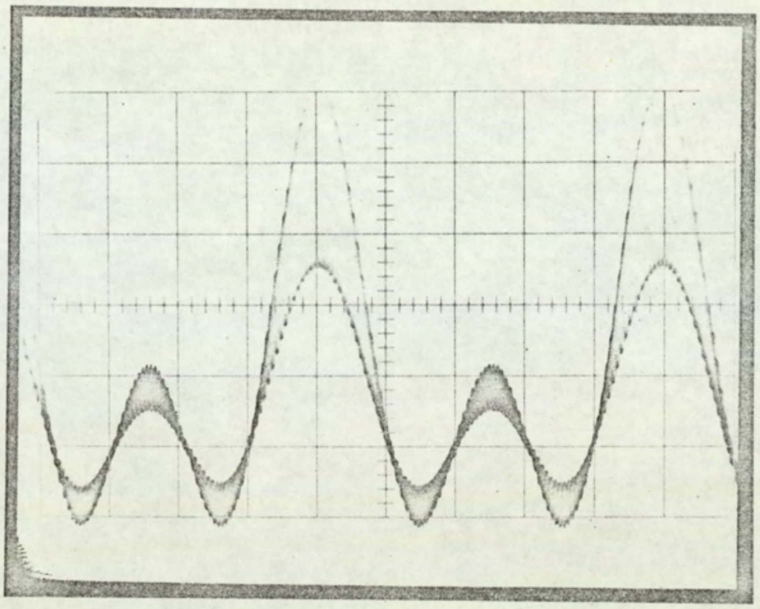
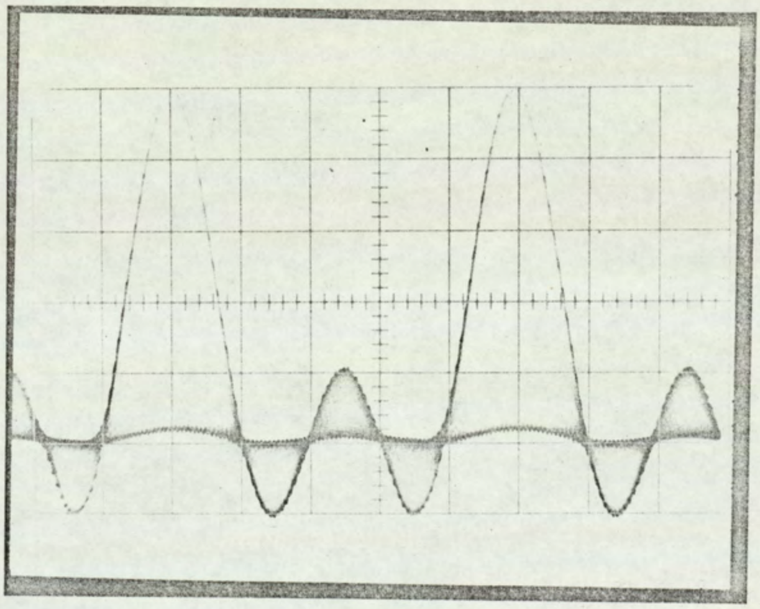


figure
3 .9



The inherent disadvantages of this system are considered to be the amount of logic circuitry needed to generate the cosine-related harmonics, and the need to add a precise dc level to each waveform to locate correctly the zero crossings. The latter could be avoided by generating the $D_n(t)$ function directly from the closed form using analogue dividers, but since a divider would be required for each channel, this would not represent an economical solution. In the next sub-sections the possibility of using other summation waveforms is considered.

3.5. Waveforms generated by finite summations of sinusoids.

There are, of course, an infinite number of possibilities when adding sines and cosines with different amplitude, frequency, and phase. The waveforms examined are those of finite summations with a closed form, which would be practicable to generate. A published set of series provided a guide in this respect. (reference 46). The particular property which was sought was that the waveform should have zero crossings similar to those of the (sin mx)/(sin x) function.

Summations of cosines.

(1) $$\sum_{n=1}^{m} \cos nx = \frac{\sin \frac{1}{2}(2m+1)x}{2 \sin \frac{1}{2}x} - \frac{1}{2}$$ (figure 3 .10.a)

This function has already been considered, as, with the exception of the dc component, it is the same as the $D_n(t)$ function. The effect of omitting the dc level is seen in figure 3 .10.a.

$$\sum_{n=1}^{m} (-1)^n \cos nx = (-1)^m \frac{\cos \frac{1}{2}(2m+1)x}{2 \cos \frac{1}{2}x} - \frac{1}{2}$$

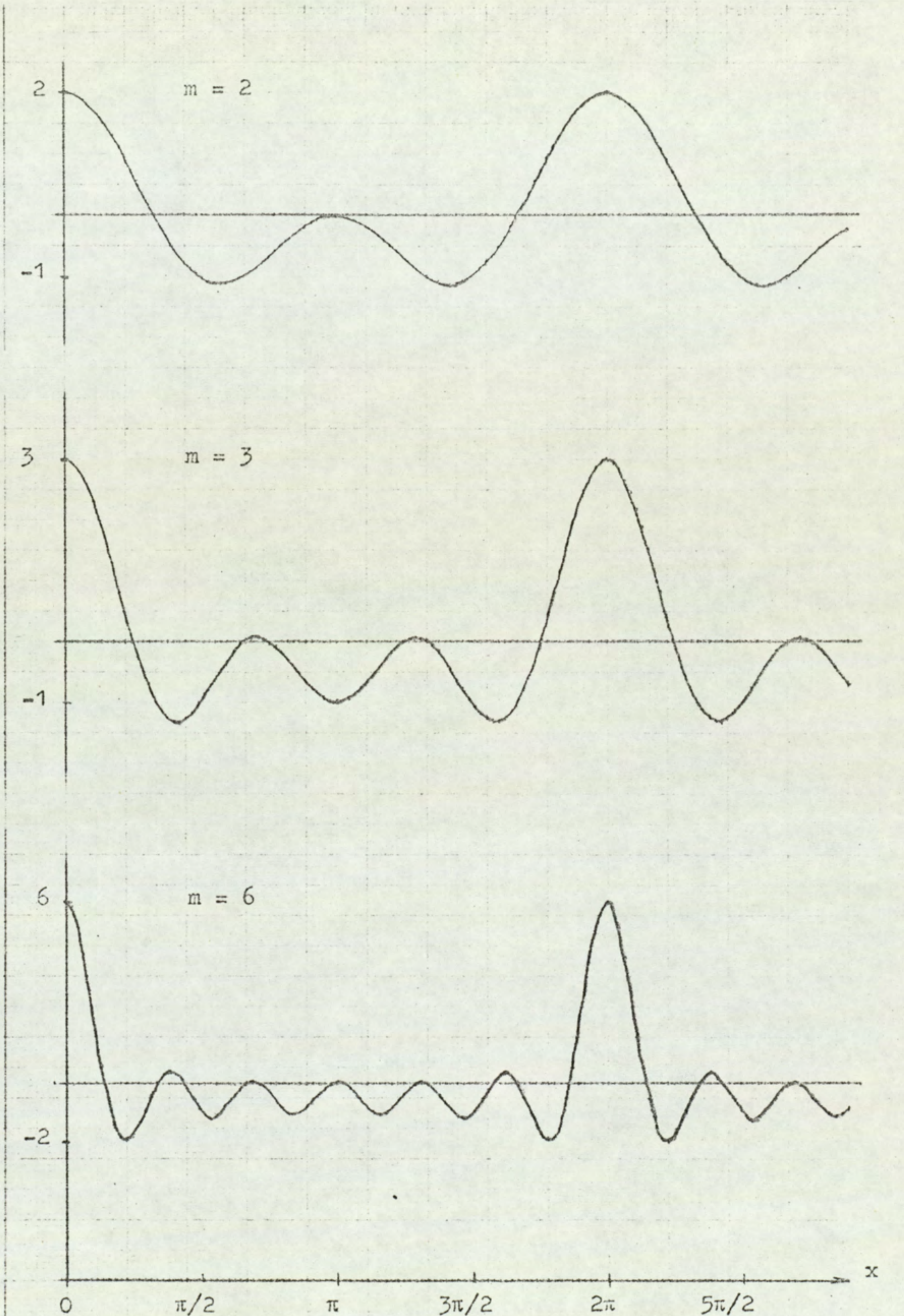
This waveform is the same as the one above, but is displaced by half the period of the fundamental component.

(2) $$\sum_{n=1}^{m-1} n \cos nx = \frac{m \sin \frac{1}{2}(2m-1)x}{2 \sin \frac{1}{2}x} - \frac{1-\cos mx}{4 \sin^2(\frac{1}{2}x)}$$

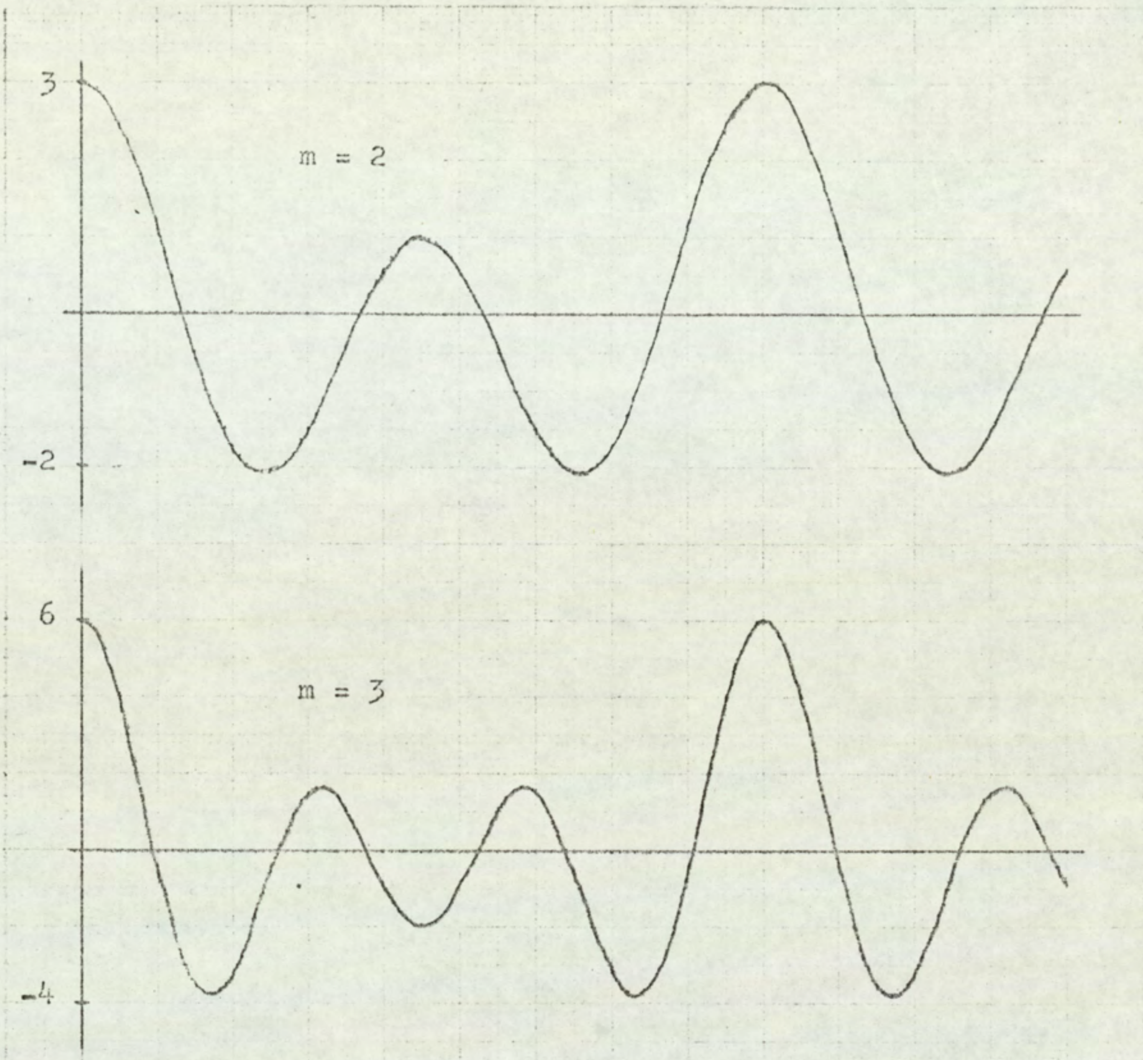
(figure 3 .10.b)

Inspection of the waveform shows that the zero crossings are unsuitable.

$$\sum_{n=1}^{m} (-1)^n n \cos nx$$ The closed form for this sum is lengthy, but the waveform is the same as shown in figure 3.10.b, displaced by half the fundamental period.



$\sum^{m}\cos nx$ figure 3.10.a.



$\sum^{m} n \cos nx$

figure 3.10.b.

(3) $$\sum_{n=1}^{m} \cos(2n-1)x = \frac{\sin 2mx}{2 \sin x}$$

The waveform is the same as (8), but displaced by 3/4 of the fundamental period.

(4) $$\sum_{n=1}^{m} (-1)^n \cos(2n-1)x = \frac{1-(-1)^m \cos 2mx}{2 \cos x}$$

The waveform is the same as (7), but displaced by 1/4 of the fundamental period.

(5) $$\sum_{n=1}^{m} (2n-1)\cos(2n-1)x$$ (figure 3.10.c)

This summation is not tabulated in reference 46, but it will be seen that the zero spacings are unsuitable.

Summations of sines.

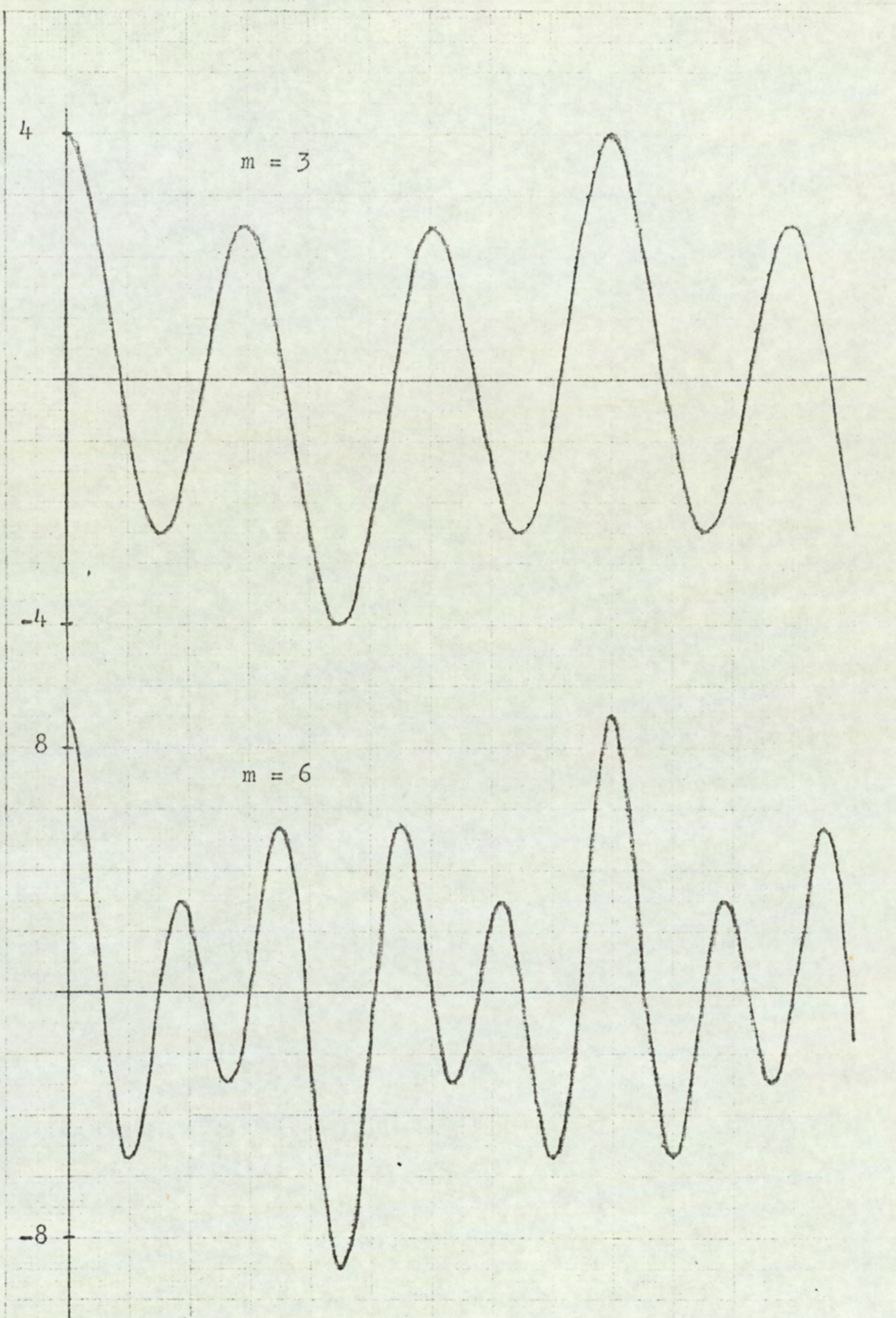
Since these will form odd functions, they might appear to be unsuitable for a multiplexing basis. This is not necessarily so, as discussed later. All that is necessary is that the spacing of the zero crossings should be suitable.

(6) $$\sum_{n=1}^{m} \sin nx = \frac{\cos \frac{1}{2}x - \cos(2m+1)\frac{1}{2}x}{2 \cdot \sin \frac{1}{2}x}$$ (figure 3.10.d)

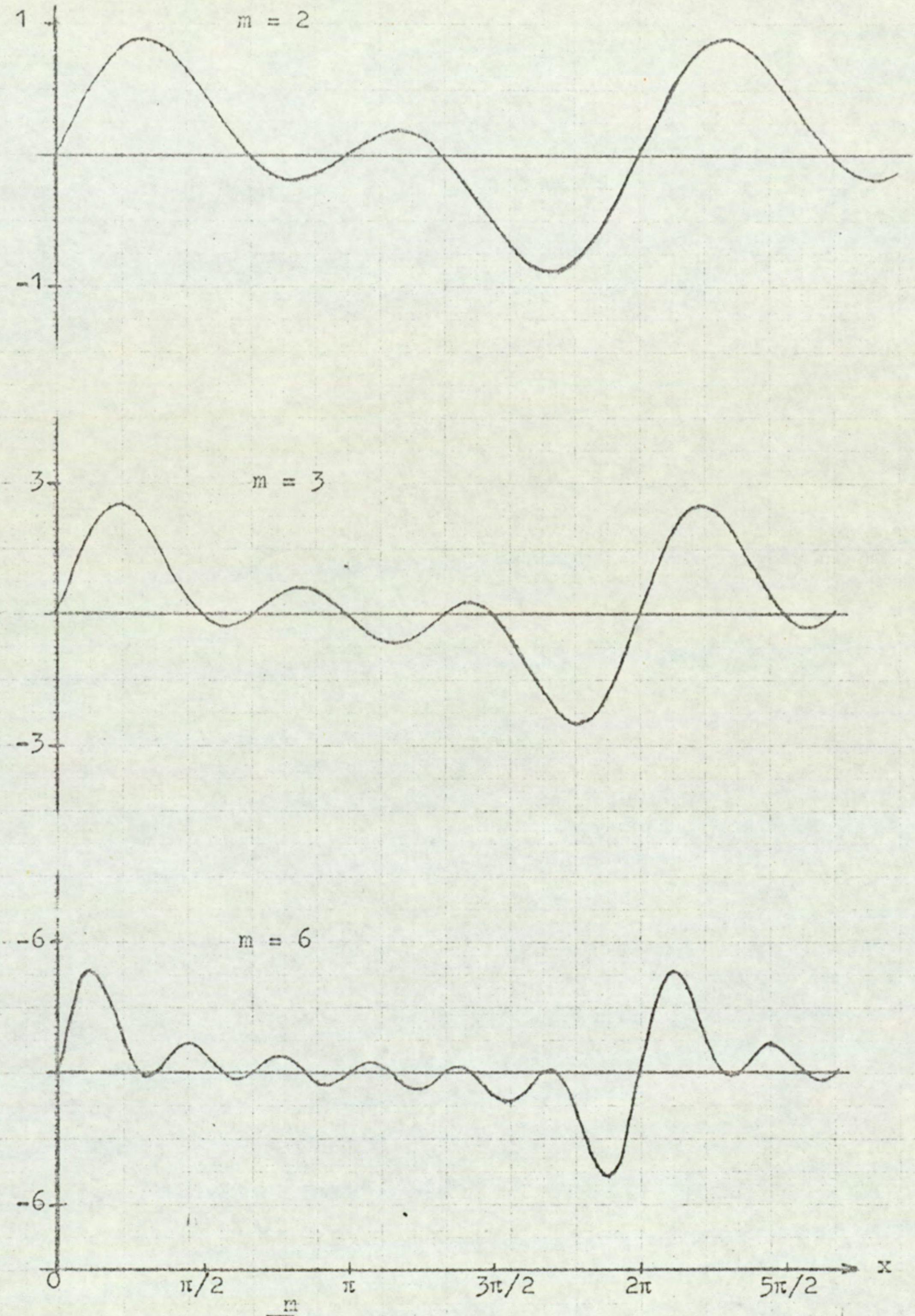
Spacing of the zero crossings is unsuitable.

(7) $$\sum_{n=1}^{m} \sin(2n-1)x = \frac{\sin^2 mx}{\sin x}$$ (figure 3.10.e)

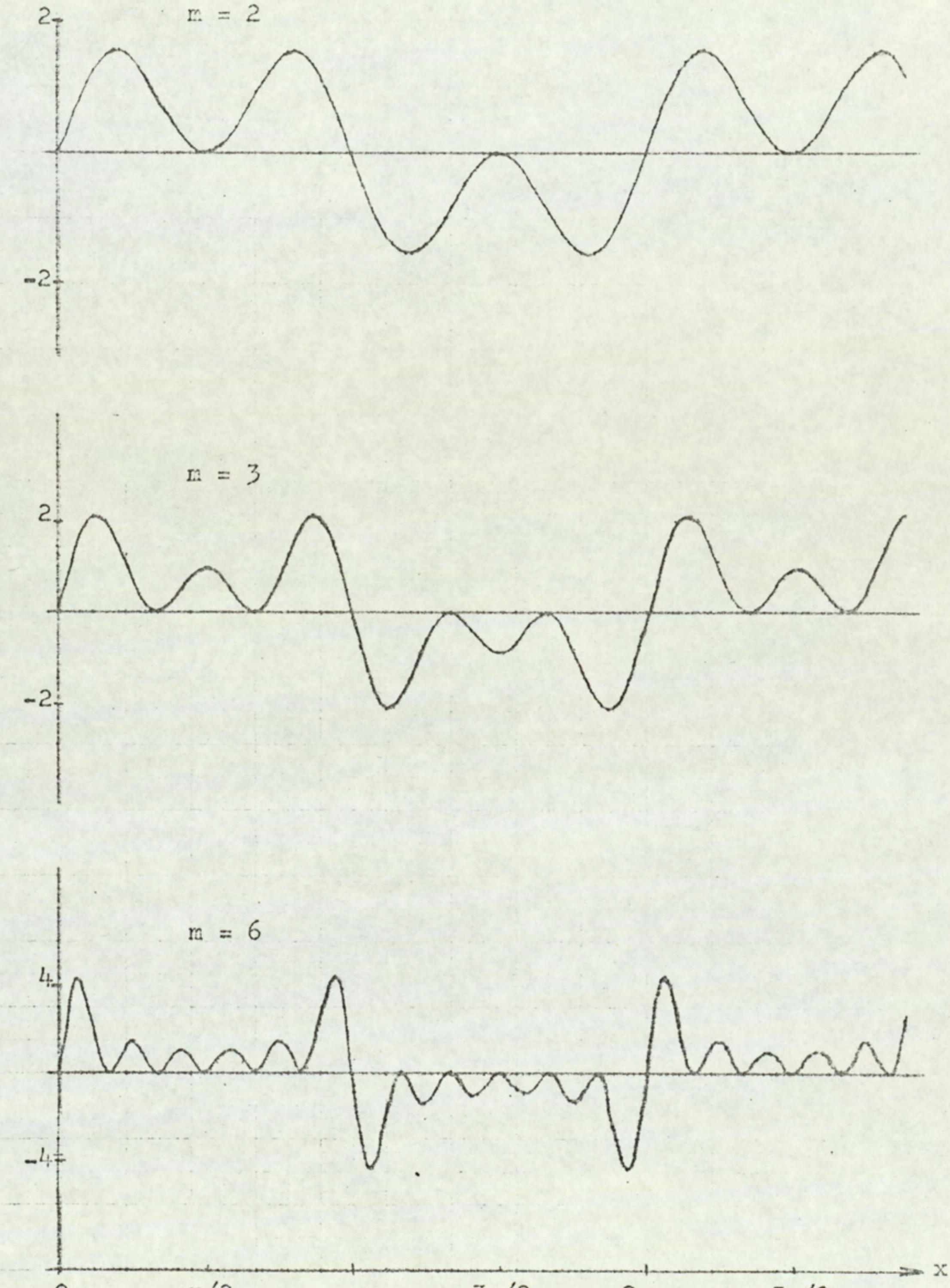
As all the zero crossings are equally spaced, the waveform is unsuitable for multiplexing.



$\sum^{m}(2n-1)\cos(2n-1)x$ figure 3.10.c.



$\sum^{m} \sin nx$ figure 3.10.d.



$\sum^{m} \sin(2n-1)x$ figure 3.10.e

(8) $\sum_{n=1}^{m} (-1)^n \sin(2n-1)x = (-1)^m \dfrac{\sin 2mx}{2\cos x}$ (figure 3.10.f)

The zero crossing spacings of this waveform are suitable for multiplexing 2m channels, provided both negative and positive peaks are used.

(9) $\sum_{n=1}^{m-1} n \sin nx = \dfrac{\sin mx}{4\sin^2(\frac{1}{2}x)} - \dfrac{m \cos\frac{1}{2}(2m-1)x}{2\sin\frac{1}{2}x}$ (figure 3.10.g)

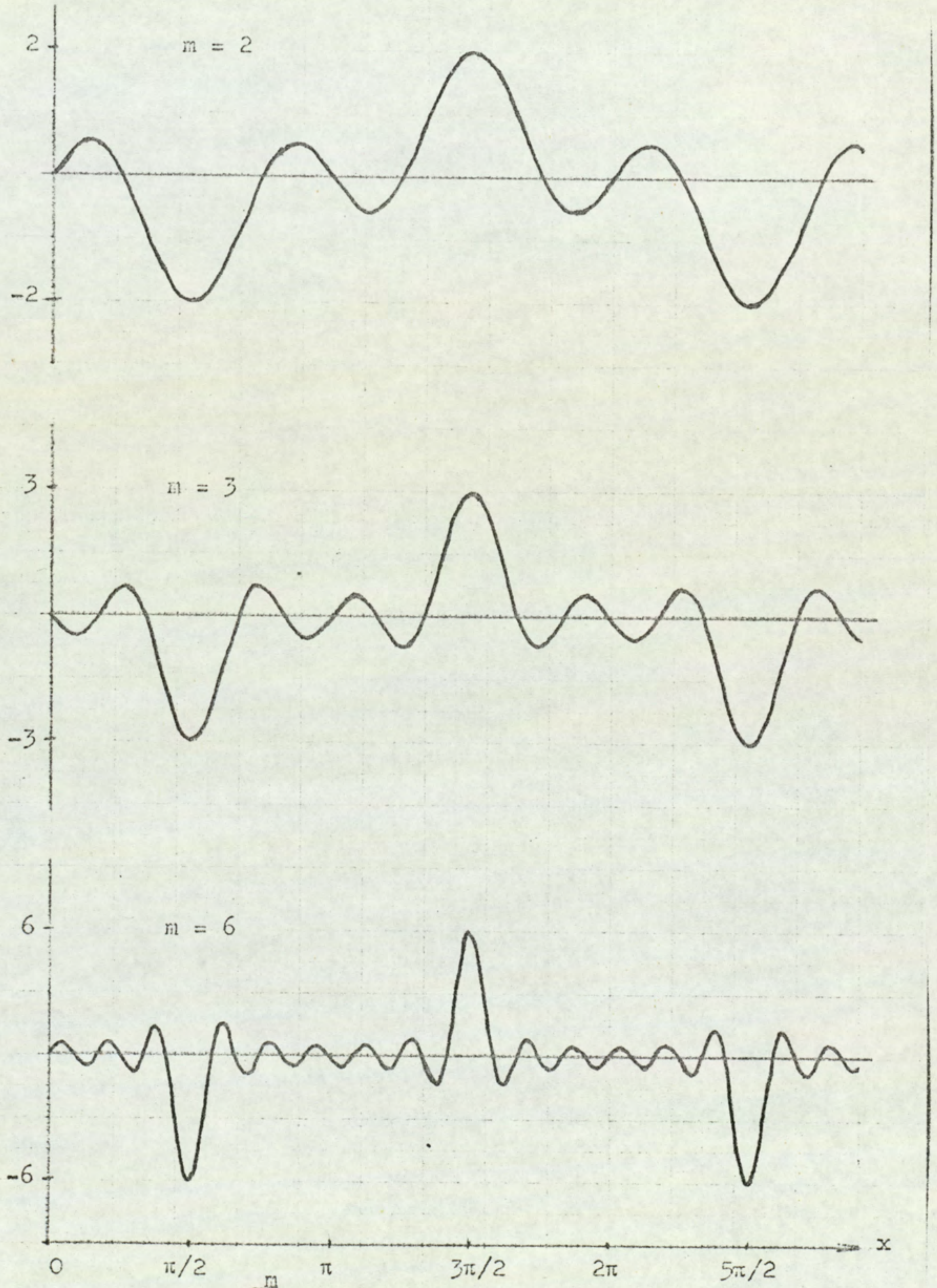
Spacing of the zero crossings is unsuitable.

$\sum_{n=1}^{m} (-1)^n n \sin nx$ The closed form is lengthy, but the waveform is the same as in figure 3.10.g, but displaced by half the fundamental period.

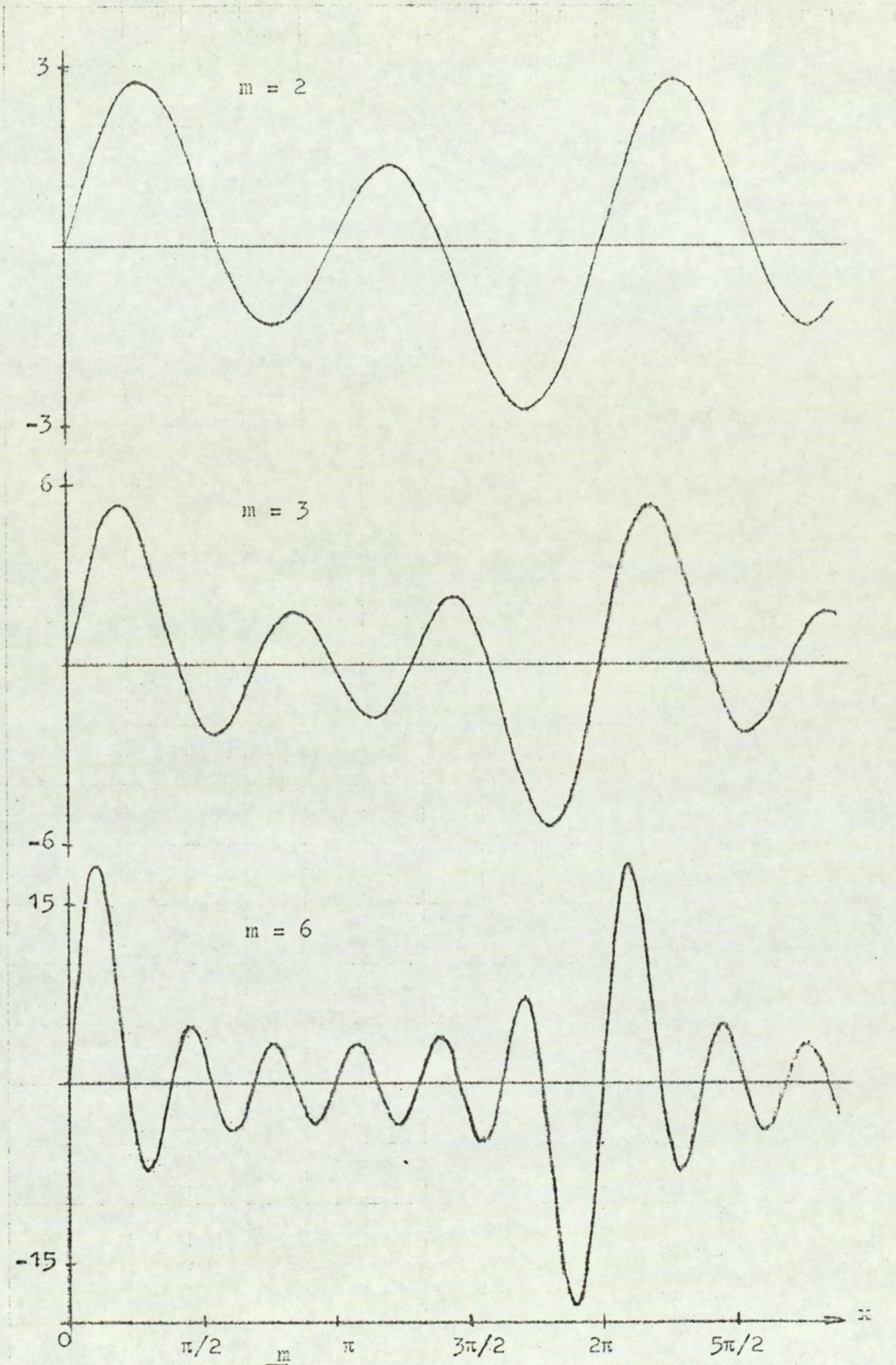
(10) $\sum_{n=1}^{m} (2n-1)\sin(2n-1)x$; $\sum_{n=1}^{m} (-1)^n(2n-1)\sin(2n-1)x$

The closed forms for these summations are lengthy, but the waveforms are shown in figures 3.10.h, and 3.10.i, where it will be seen that neither are suitable for multiplexing.

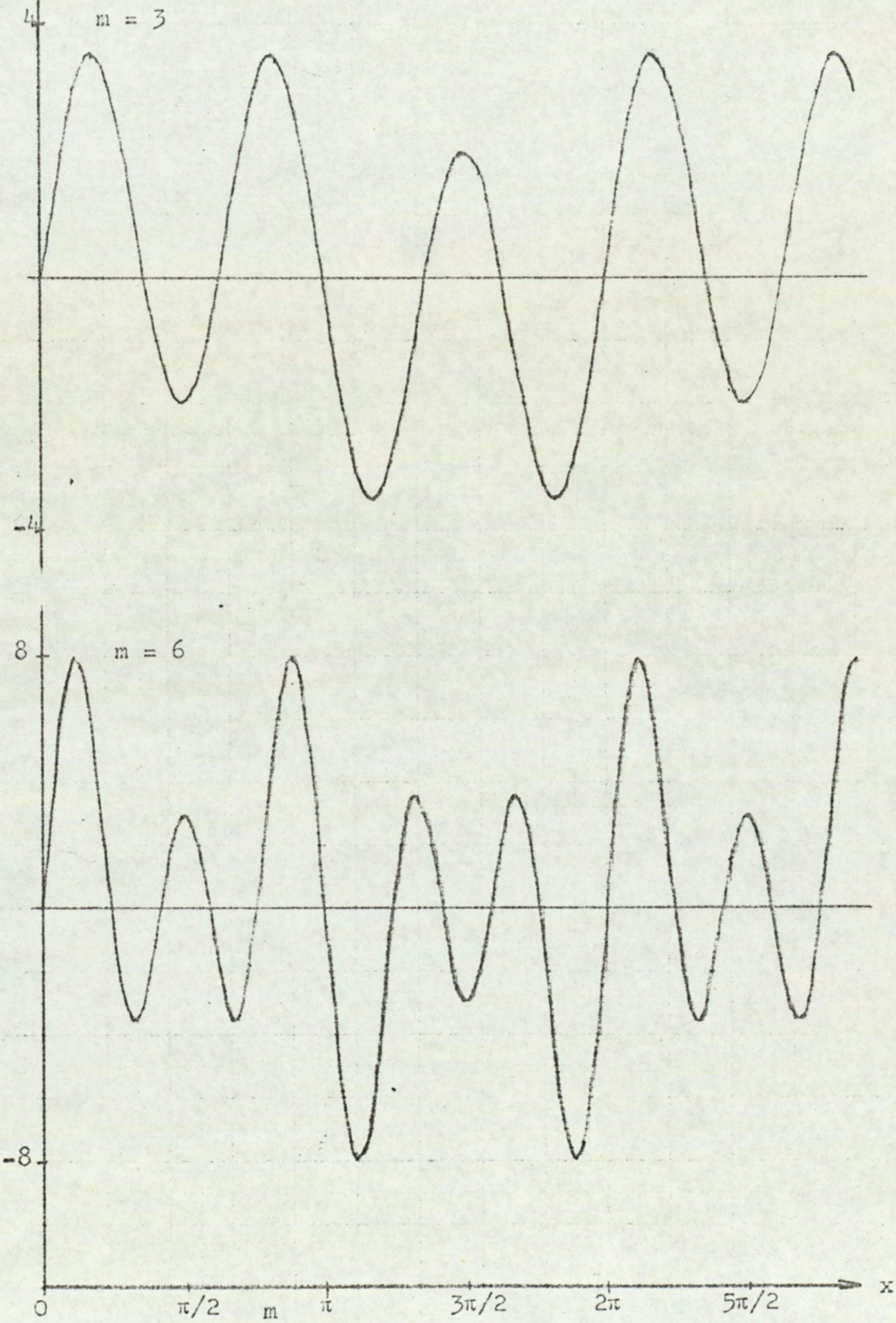
Form these various summations, the only one which appears to have useful properties is that given by (8). This waveform is therefore considered in detail in the next section.



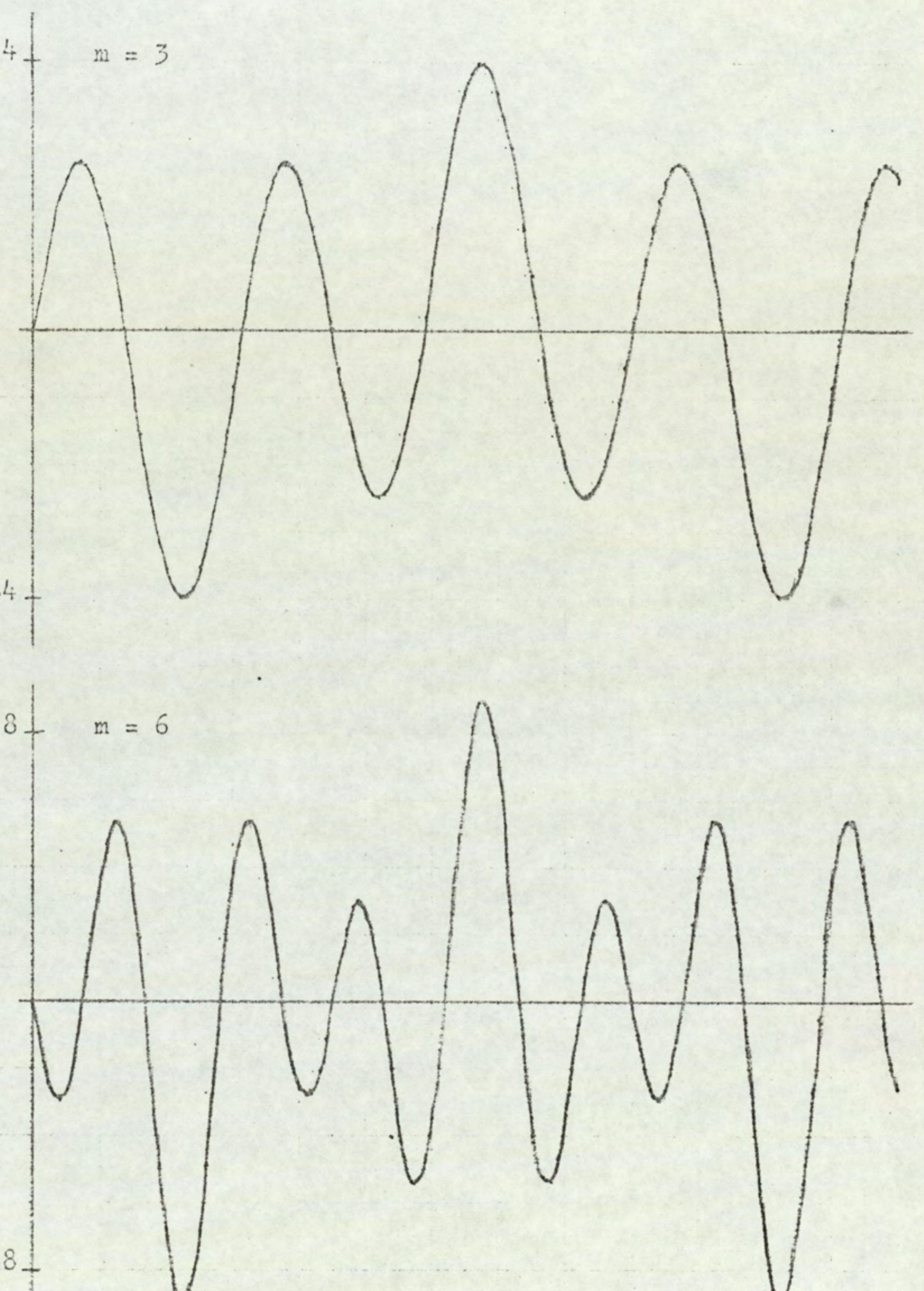
$\sum^{m}(-1)^{n}\sin(2n-1)x$ figure 3.10.f



$\sum^{m} n \sin nx$ figure 3.10.g



$\sum^{m}(2n-1)\sin(2n-1)x$ figure 3.10.h.



$\sum^{m}(-1)^{n}(2n-1)\sin(2n-1)x$ figure 3.10.i.

3.6.a. Waveform properties of $S_n(t)$.

The finite summation which is to be considered will be referred to as $S_n(t)$, where

$$S_n(t) = \left\{-\sin\omega_o t + \sin3\omega_o t - \sin5\omega_o t \ldots(-1)^m \sin(2m-1)\omega_o t\right\}$$

$$= \sum_{n=1}^{m}(-1)^n \sin(2n-1)\omega_o t$$

$$= (-1)^m \frac{\sin2m\omega_o t}{2\cos\omega_o t}$$

The closed form is evidently similar to that of the periodic Dirichlet kernel, $D_n(t)$, since

$$D_n(t) = \frac{\sin(2m+1)\tfrac{1}{2}\omega_o t}{2\sin\tfrac{1}{2}\omega_o t}$$

However, the numerator of the closed form of $S_n(t)$ has 4m zeros in the interval T, (where $T = 2\pi/\omega_o$), and the denominator has two zeros in this interval. Therefore, there are two limiting cases per period, when the function attains a peak value.

These peak values may be obtained from

$$\mathrm{Lt}_{\,t\rightarrow T/4} \; (-1)^m \frac{\sin2m\omega_o t}{2\cos\omega_o t}$$

$$= \mathrm{Lt}_{\,t\rightarrow T/4} \; (-1)^m \frac{2m\omega_o \cos2m\omega_o t}{-2\omega_o \sin\omega_o t}$$

$$= (-1)^m \frac{m \cos2m(\pi/2)}{(-1)} = -(-1)^m . m . (-1)^m = -m$$

and from

$$\text{Lt}_{t \to 3T/4} \; (-1)^m \frac{\sin 2m\omega_o t}{2\cos\omega_o t}$$

$$= (-1)^m \frac{m \cos 2m(3\pi/2)}{1} \qquad = (-1)^m . m . (-1)^m = m$$

Since there are two peaks per period, the fundamental component of the summation waveform would have to be equal to, (or greater than), the highest frequency in the message bandwidth. The receiver must sample the modulated waveform twice in each period of the fundamental, once at the negative peak, and once at the positive peak. This is to fulfil the condition of sampling at not less than the minimum Nyquist rate. There would be no particular difficulty in arranging for alternate samples to be inverted in polarity before the message was recovered by filtering.

2m waveforms can be multiplexed with this system, as opposed to (2m+1) when using the $D_n(t)$ function, since there is no dc term to take the bandwidth to zero frequency. The multiplex signal still occupies only the theoretical minimum bandwidth. The relative phasing of the waveforms for a four-channel system, (i.e. m = 2), is shown in figure 3.11.a., and for twelve channels in figure 3.11.b.

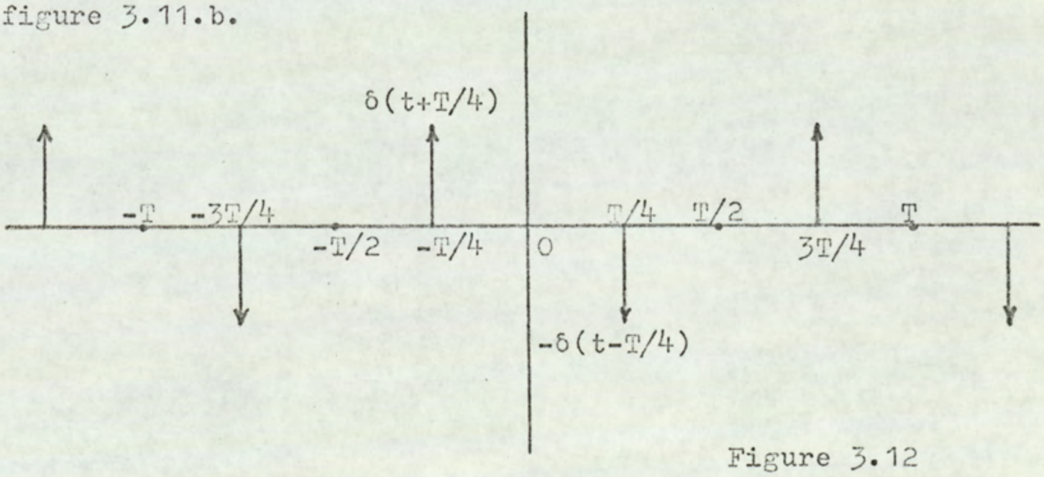
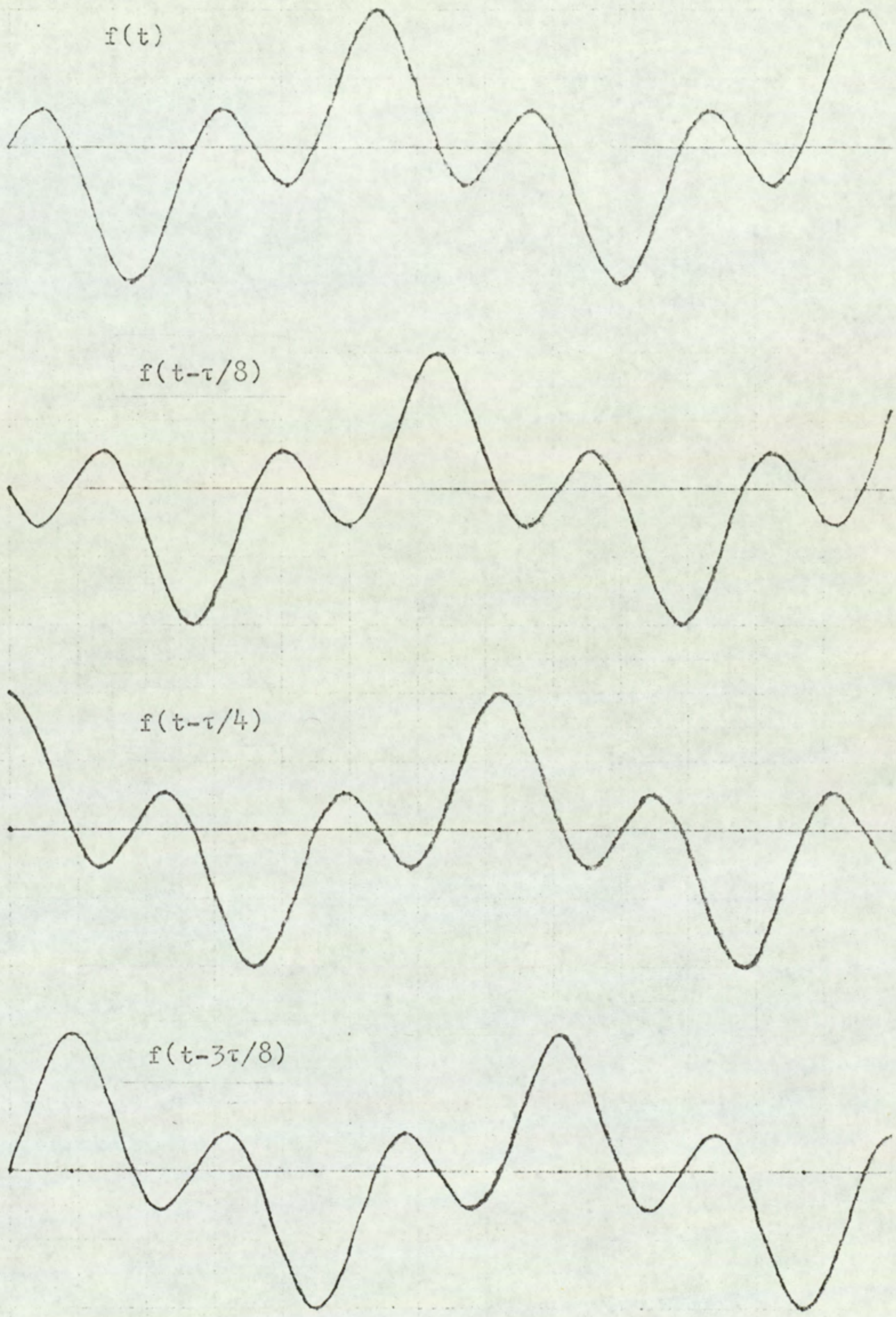


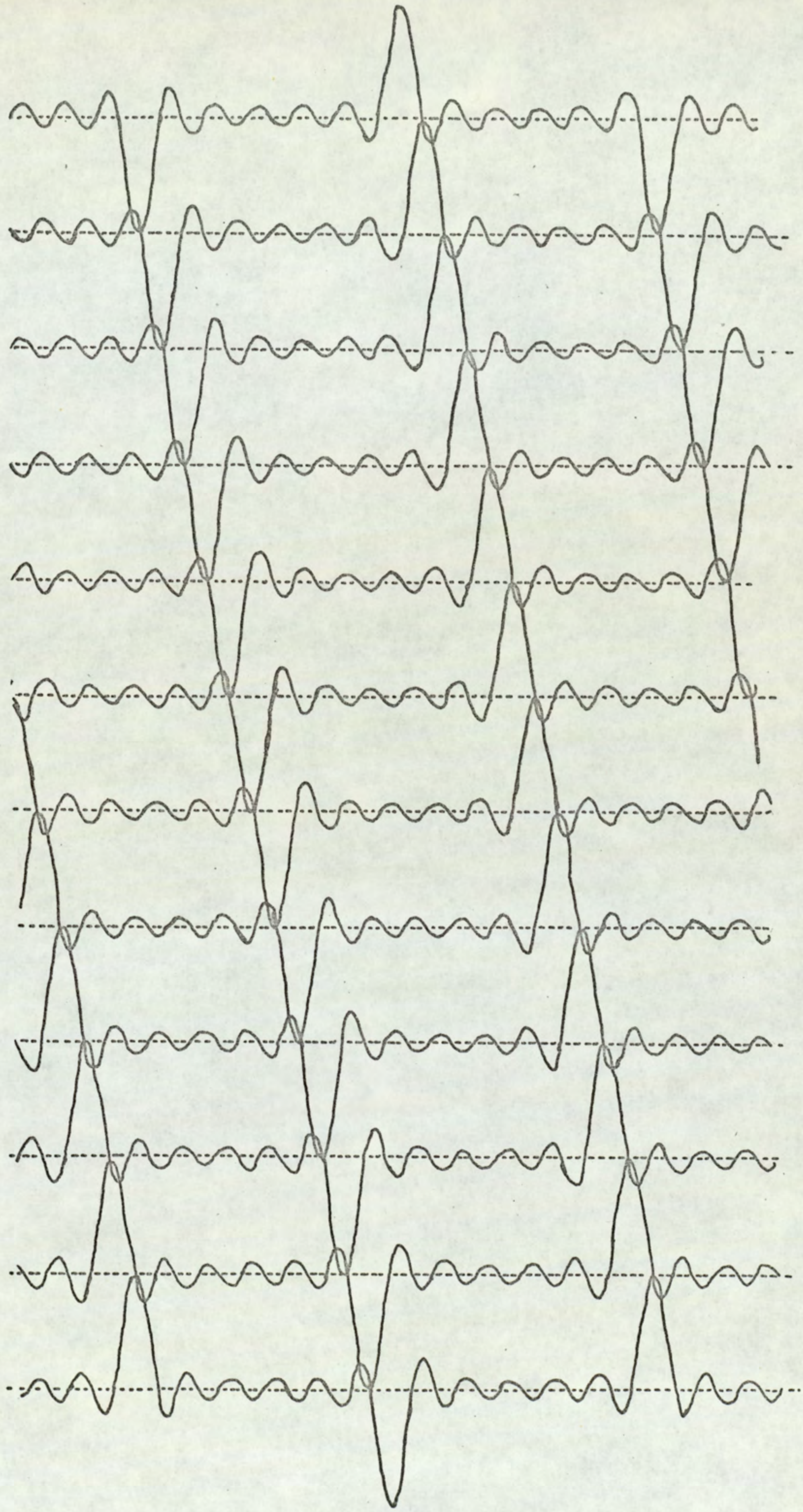
Figure 3.12

From inspection, it would appear that the limiting form of $S_n(t)$ as m approaches infinity is a train of alternate polarity impulse functions, as shown in figure 3.12. That this is



$f(t) = -\sin(\omega t) + \sin(3\omega t)$

figure 3.11.a.



Relative phasing for m = 6

Figure 3.11.b.

so may be seen by considering that the function in figure 3.12 is,

$$f(t) = \sum_{n=-\infty}^{\infty}\delta(t + T/4 - nT) - \sum_{n=-\infty}^{\infty}\delta(t - T/4 - nT)$$

then since

$$\sum_{n=-\infty}^{\infty}\delta(t \pm T/4 - nT) = \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty}\cos(n\omega_o(t \pm T/4))$$

$$(\omega_o = 2\pi/T)$$

the relation

$$\cos(A + B) - \cos(A - B) = -2\sin A\sin B$$

may be used to obtain

$$f(t) = -\frac{2}{T}\sum_{n=1}^{\infty}\left[\cos(n\omega_o t + n\pi/2) - \cos(n\omega_o t - n\pi/2)\right]$$

$$= -\frac{4}{T}\sum_{n=1}^{\infty}\sin n\omega_o t \sin n\pi/2$$

$$= \frac{4}{T}\left[-\sin\omega_o t + \sin 3\omega_o t - \sin 5\omega_o t \ \ldots\right]$$

$$= \frac{4}{T}\sum_{n=1}^{\infty}(-1)^n \sin(2n-1)\omega_o t$$

which, apart from the constant $4/T$, is the same as $S_n(t)$.

The envelope formed by the superposition of a set of $S_n(t)$ functions does not reduce in the simple manner which applies to a set of $D_n(t)$ functions. The phase difference between adjacent fundamentals, ω_o, in a set of multiplexed $S_n(t)$ functions is $\pi/2m$. The superposition of the fundamentals therefore results in,

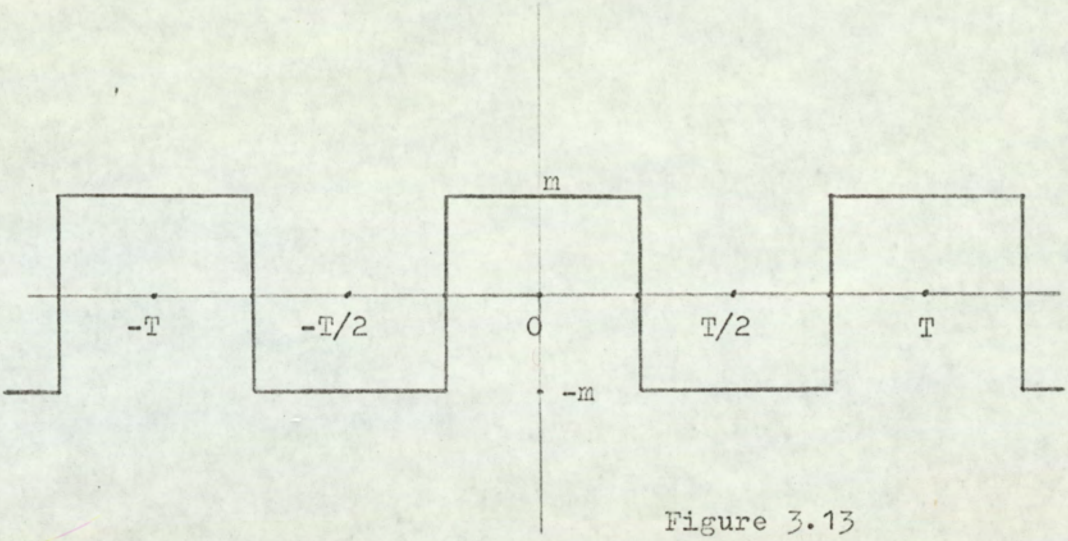


Figure 3.13

$$f_1(t) = \sum_{k=0}^{2m-1} -\sin(\omega_o t - k\pi/2m)$$

$$= \sum_{k=0}^{2m-1} \left\{ \frac{e^{-j(\omega_o t - k\pi/2m)}}{2j} - \frac{e^{j(\omega_o t - k\pi/2m)}}{2j} \right\}$$

$$= \frac{e^{-j\omega_o t}}{2j} \sum_{k=0}^{2m-1} e^{jk\pi/2m} - \frac{e^{j\omega_o t}}{2j} \sum_{k=0}^{2m-1} e^{-jk\pi/2m}$$

then, since

$$\sum_{k=0}^{2m-1} e^{jk\pi/2m} = \frac{1 - e^{j\pi}}{1 - e^{j\pi/2m}} = \frac{2}{1 - e^{j\pi/2m}} = \frac{je^{-j\pi/4m}}{\sin(\pi/4m)},$$

$$\sum_{k=0}^{2m-1} e^{-jk\pi/2m} = \frac{1 - e^{-j\pi}}{1 - e^{-j\pi/2m}} = \frac{2}{1 - e^{-j\pi/2m}} = \frac{-je^{j\pi/4m}}{\sin(\pi/4m)},$$

therefore,

$$f_1(t) = \frac{e^{-j\omega_o t}}{2j} \frac{je^{-j\pi/4m}}{\sin(\pi/4m)} + \frac{e^{j\omega_o t}}{2j} \frac{je^{j\pi/4m}}{\sin(\pi/4m)}$$

$$= \frac{\cos(\omega_o t + \pi/4m)}{\sin(\pi/4m)}$$

As $m \to \infty$, $\pi/4m \to 0$, and $\sin(\pi/4m) \longrightarrow \pi/4m$
so that

$$f_1(t) \underset{m \to \infty}{\longrightarrow} \frac{4m}{\pi} \cos \omega_o t$$

The summation of the third harmonics is

$$f_3(t) = \sum_{k=0}^{2m-1} \sin(3\omega_o t - 3k\pi/2m)$$

and in a similar manner this reduces to

$$f_3(t) = - \frac{\cos 3(\omega_o t + \pi/4m)}{\sin(3\pi/4m)}$$

so that

$$f_3(t) \underset{m \to \infty}{\longrightarrow} - \frac{4m}{\pi} \frac{1}{3} \cos 3\omega_o t$$

Therefore the summation of all the harmonic components is

$$\sum f(t) \underset{m \to \infty}{\longrightarrow} \frac{4m}{\pi} \left[\cos \omega_o t - \frac{1}{3} \cos 3\omega_o t + \frac{1}{5} \cos 5\omega_o t \quad \ldots \right]$$

$$= -\frac{4m}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n-1)} \cos(2n - 1)\omega_o t$$

which is recognisable as the square wave function shown in figure 3.13. That the limiting form for the envelope is a square wave may also be seen by considering that the superposition of an infinite set of impulse functions of the form shown in figure 3.12 would also have a square wave envelope as the displacement between adjacent functions approaches zero.

3.6.b Crosstalk ratio using $S_n(t)$.

As before, the crosstalk ratio will be defined as the ratio of the area of a pulse sampling the wanted channel at a peak value, to the mean area of the other waveforms which are intercepted by the same pulse. The same assumption is made, that the results for the unmodulated waveforms will be applicable to modulated waveforms when the messages have the same average value.

Since the envelope of a finite set of $S_n(t)$ functions is not the simple function which occurs when summing the $D_n(t)$ functions, the derivation of the crosstalk ratio is a more lengthy process.

Initially, one may consider the area, A, under a rectangular pulse which samples the $S_n(t)$ function. The pulse has a duration d, and is centred at some arbitrary instant t_q.

$$A = \int_{t_q-\frac{1}{2}d}^{t_q+\frac{1}{2}d} \left[-\sin\omega_o t + \sin 3\omega_o t \;\ldots\; (-1)^m \sin(2m-1)\omega_o t \right] dt$$

$$= \frac{1}{\omega_o} \left[\cos\omega_o t - \frac{1}{3}\cos 3\omega_o t \;\ldots\; - \frac{(-1)^m}{(2m-1)} \cos(2m-1)\omega_o t \right]_{t_q-\frac{1}{2}d}^{t_q+\frac{1}{2}d}$$

$$= \frac{1}{\omega_o} \Bigg\{ \left[\cos\omega_o(t_q + \tfrac{1}{2}d) - \cos\omega_o(t_q - \tfrac{1}{2}d) \right]$$

$$-\frac{1}{3}\left[\cos 3\omega_o(t_q + \tfrac{1}{2}d) - \cos 3\omega_o(t_q - \tfrac{1}{2}d) \right]$$

$$\vdots$$

$$- \frac{(-1)^m}{(2m-1)} \left[\cos(2m-1)\omega_o(t_q+\tfrac{1}{2}d) - \cos(2m-1)\omega_o(t_q-\tfrac{1}{2}d) \right] \Bigg\} ;$$

using $\cos(A + B) - \cos(A - B) = -2\sin A\sin B$,

$$A = \frac{1}{\omega_o}\left\{\left[-2\sin\omega_o t_q \sin\omega_o \tfrac{1}{2}d\right] - \frac{1}{3}\left[-2\sin 3\omega_o t_q \sin 3\omega_o \tfrac{1}{2}d\right] \cdots - \frac{(-1)^m}{(2m-1)}\left[-2\sin(2m-1)\omega_o t_q \sin(2m-1)\omega_o \tfrac{1}{2}d\right]\right\}$$

Since $\omega_o = 2\pi/T$, therefore

$$A = -\frac{2}{\omega_o}\left\{\sin\omega_o t_q \sin\pi d/T - \frac{1}{3}\sin 3\omega_o t_q \sin 3\pi d/T \cdots - \frac{(-1)^m}{(2m-1)}\sin(2m-1)\omega_o t_q \sin(2m-1)\pi d/T\right\} \quad \ldots(1)$$

The first peak of the wanted channel, (which will be negative-going), occurs at $t_q = T/4$. Therefore, the area of the signal sample will be obtained by substituting $(2\pi/T)(T/4)$ for $\omega_o t_q$ in (1), which gives

$$A_s = -\frac{2}{\omega_o}\left\{\sin(\pi/2)\sin(\pi d/T) - \frac{1}{3}\sin(3\pi/2)\sin 3(\pi d/T) \cdots - \frac{(-1)^m}{(2m-1)}\sin(2m-1)\pi/2\,\sin(2m-1)(\pi d/T)\right\}$$

$$= -\frac{2}{\omega_o}\left\{\sin\pi d/T + \frac{1}{3}\sin 3\pi d/T \,..\, + \frac{1}{(2m-1)}\sin(2m-1)\pi d/T\right\}.$$

If the successive sampling instants,(i.e. zero crossings),are denoted t_1, t_2, ... t_{2m}, and the integrals over the interval d,centred at t_1, t_2, ... t_{2m}, are denoted A_1, A_2, ... A_{2m}, then as t_1 = T/4,the sampled signal is A_1, and the sampled noise,(i.e. the crosstalk),is caused by $(A_2 + A_3 + \dots A_{2m})$.

To envisage the form taken by the sum of the crosstalk components,reference may be made to figure 3.11.a,showing the staggered waveforms for a four-channel system,and to figure 3.11.b,showing the staggered waveforms for a twelve-channel system.

It will be observed that the channel waveforms are odd functions about T/2. Hence the integral of the sample centred at T/2,which is the (m+1)th sample,has zero value. Furthermore,the algebraic sum of the integrals of any two samples which are equidistant about T/2 will also have zero value. That is,

$$A_m + A_{m+2} = 0$$

$$A_{m-1} + A_{m+3} = 0$$

$$\vdots$$

$$A_2 + A_{2m} = 0$$

Inspection of the waveforms shows that the crosstalk in the first channel,sampled at $(T/4 \pm \frac{1}{2}d)$,consists of components which are symmetrically distributed about T/2. This also applies to the last,i.e. (2m)th,channel. In an error-free system the crosstalk in these two channels is therefore zero. In the second channel the only components which are not cancelled are those from the adjacent channels,and similarly for the (2m-1)th channel. In the third channel an additional pair of crosstalk components appears,and so on.

In general, if the magnitude of the total crosstalk occurring in a sample at t_q is denoted c_q, then it will be seen that

$$c_1 = c_{2m} = 0$$
$$c_2 = c_{2m-1} = 2A_2$$
$$c_3 = c_{2m-2} = 2(A_2+A_3)$$
$$\vdots$$
$$c_m = c_{m+1} = 2(A_2+A_3 \cdot \cdot +A_m)$$

within the first half-period, and

$$c_1 = c_{2m} = 0$$
$$c_2 = c_{2m-1} = 2A_{2m}$$
$$c_3 = c_{2m-2} = 2(A_{2m}+A_{2m-1})$$
$$\vdots$$
$$c_m = c_{m+1} = 2(A_{2m}+A_{2m-1} \cdot\cdot +A_{m+2})$$

within the second half-period.

The crosstalk magnitudes are the same in each half-period, but the polarities are, of course, inverted for the second set of samples.

Thus, the crosstalk in the (p)th channel is

$$c_p = 2\sum_{q=2}^{p} A_q$$

$$\text{where } A_q = \int_{t_q-\frac{1}{2}d}^{t_q+\frac{1}{2}d} \left\{ \sum_{n=1}^{m} (-1)^n \sin(2n-1)\omega_o t \right\} dt$$

$$\text{and where } t_q = T/4 + (q-1)T/4m = \frac{T}{4m}(m + q - 1)$$

The integral in A_q is the same as that previously derived for (1), so that

$$c_p = -\frac{4}{\omega_o} \sum_{q=2}^{p} \Big\{ \sin\omega_o t_q \sin\pi d/T - \tfrac{1}{3}\sin 3\omega_o t_q \sin 3\pi d/T$$
$$\quad . \ . \ - \frac{(-1)}{(2m-1)} \sin(2m-1)\omega_o t_q \sin(2m-1)\pi d/T \Big\}$$

The summation of the fundamentals, i.e. the first term in the bracket, is

$$-\frac{4}{\omega_o}\sin\pi d/T \Big[\sin(2\pi/T)\,\frac{T}{4m}(m + 2 - 1)$$
$$+ \sin(2\pi/T)\,\frac{T}{4m}(m + 3 - 1)$$
$$\cdot$$
$$\cdot$$
$$+ \sin(2\pi/T)\,\frac{T}{4m}(m + p - 1) \Big]$$

$$= -\frac{4}{\omega_o}\sin\pi d/T \Big[\sin(\pi/2m)(m+1) + \sin(\pi/2m)(m+2)$$
$$. \ . \ + \sin(\pi/2m)(m + p - 1) \Big]$$

Using the relation $\sin(A + B) = \sin A\cos B + \cos A\sin B$, the expression becomes

$$-\frac{4}{\omega_o}\sin\pi d/T \Big\{ \big[\sin(\pi/2)\cos(\pi/2m) + \cos(\pi/2)\sin(\pi/2m)\big]$$
$$+ \big[\sin(\pi/2)\cos 2(\pi/2m) + \cos(\pi/2)\sin 2(\pi/2m)\big]$$
$$\cdot$$
$$\cdot$$
$$+ \big[\sin(\pi/2)\cos(p-1)(\pi/2m) + \cos(\pi/2)\sin(p-1)(\pi/2m)\big] \Big\}$$

$$= -\frac{4}{\omega_o}\sin\pi d/T \Big\{ \cos(\pi/2m) + \cos 2(\pi/2m) \ldots + \cos(p-1)(\pi/2m) \Big\} \ ;$$

since $$\sum_{n=1}^{m} \cos nx = \left\{ \frac{\sin(m + \frac{1}{2})x}{2\sin\frac{1}{2}x} - \frac{1}{2} \right\} ,$$

the previous expression becomes

$$- \frac{4}{\omega_o} \sin\pi d/T \left\{ \frac{\sin(p - 1 + \frac{1}{2})(\pi/2m)}{2 \sin(\pi/4m)} - \frac{1}{2} \right\}$$

$$= - \frac{2}{\omega_o} \sin\pi d/T \left\{ \frac{\sin(2p-1)(\pi/4m) - \sin(\pi/4m)}{\sin(\pi/4m)} \right\} .$$

Using the relationship $\sin A - \sin B = 2 \cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$, the expression may be written as

$$- \frac{2}{\omega_o} \frac{\sin(\pi d/T)}{\sin(\pi/4m)} \left\{ 2 \cos \frac{2p(\pi/4m)}{2} \sin \frac{2(p-1)(\pi/4m)}{2} \right\}$$

$$= - \frac{4}{\omega_o} \sin\pi d/T \left\{ \frac{\cos(p\pi/4m) \sin(p-1)(\pi/4m)}{\sin(\pi/4m)} \right\}$$

with $2 \leqslant p \leqslant m$

Similarly, the summation corresponding to the third harmonics will be

$$- \frac{4}{3\omega_o} \sin3\pi d/T \left\{ \frac{\cos(3p\pi/4m) \sin3(p-1)(\pi/4m)}{\sin3(\pi/4m)} \right\} ,$$

Hence, the sum of the crosstalk components in the (p)th channel is

$$c_p = -\frac{4}{\omega_o}\left\{\left[\sin(\pi d/T)\ \frac{\sin(p-1)(\pi/4m)\cos(p\pi/4m)}{\sin(\pi/4m)}\right] + \frac{1}{3}\left[\sin 3(\pi d/T)\ \frac{\sin 3(p-1)(\pi/4m)\ \cos 3(p\pi/4m)}{\sin 3(\pi/4m)}\right]\right.$$

$$\vdots$$

$$\left. + \frac{1}{(2m-1)}\left[\sin(2m-1)(\pi d/T)\ \frac{\sin(2m-1)(p-1)(\pi/4m)\cos(2m-1)(p\pi/4m)}{\sin(2m-1)(\pi/4m)}\right]\right\}$$

The crosstalk ratio, R, is A_s/c_p, so that

$$R = \frac{-(2/\omega_o)\left\{\sin(\pi d/T) + (1/3)\sin 3(\pi d/T) \ldots + (1/(2m-1))\sin(2m-1)(\pi d/T)\right\}}{-(4/\omega_o)\left\{\sin(\pi d/T)\ \dfrac{\sin(p-1)(\pi/4m)\cos(\pi/4m)}{\sin(\pi/4m)} \ldots + \dfrac{1}{(2m-1)}\sin(2m-1)(\pi d/T)\dfrac{\sin(2m-1)(p-1)(\pi/4m)\cos(2m-1)(p\pi/4m)}{\sin(2m-1)(\pi/4m)}\right\}}$$

$$= \frac{\sin x + \frac{1}{3}\sin 3x \ . \ . \ . \ \frac{1}{(2m-1)}\sin(2m-1)x}{2\left\{\sin x\left[\dfrac{\sin(p-1)y\ \cos py}{\sin y}\right] \ldots + \dfrac{\sin(2m-1)x}{(2m-1)}\left[\dfrac{\sin(2m-1)(p-1)y\ \cos(2m-1)py}{\sin(2m-1)y}\right]\right\}} \qquad \ldots\ldots\ldots\ldots(2)$$

where $x = \pi d/T$ and $y = \pi/4m$.

The factors in terms of x may be regarded as arising from the integration over a specific interval d. The factors in terms of y may be regarded as weighting functions, (in both sign and magnitude), which vary according to the instant upon which d is centred.

As was indicated earlier, there is zero crosstalk in the first and last channel, regardless of the value of m . This may be seen by observing that with respect to these sampling instants, one channel is separated by T/4, and will have zero mean over the sampling interval, whilst the remaining channels are in two equal groups which contribute components of equal and opposite polarity.

The second, and last but one, channels will experience crosstalk only from their adjacent channels. The third, and last but two, will experience crosstalk only from the adjacent pair on each side, and so on.

For example, if $m = 2$, then from equation number one on page eighty-nine, the area of the crosstalk components will be given by

$$A = -\frac{2}{\omega_o}\left[\sin\omega_o t_q \sin\pi d/T - \frac{1}{3}\sin 3\omega_o t_q \sin 3\pi d/T\right]$$

With $m = 2$, the sampling instants for negative polarity will occur at T/4, 3T/8, T/2, and 5T/8, where T is the period of the fundamental, ω_o. The total crosstalk in the first channel will be the sum of the integrals at 3T/8, T/2, and 5T/8.

If $t_q = 3T/8$,

$$A_2 = -\frac{2}{\omega_o}\left[\sin\frac{3\pi}{4}\sin\pi d/T - \frac{1}{3}\sin\frac{9\pi}{4}\sin 3\pi d/T\right]$$
$$= -\frac{2}{\omega_o}\left[\sin\frac{\pi}{4}\sin\pi d/T - \frac{1}{3}\sin\frac{\pi}{4}\sin 3\pi d/T\right]$$

If $t_q = T/2$,

$$A_3 = -\frac{2}{\omega_o}\left[\sin\pi \sin\pi d/T - \frac{1}{3}\sin 3\pi \sin 3\pi d/T\right] = 0$$

If $t_q = 5T/8$,

$$A_4 = -\frac{2}{\omega_o}\left[\sin\frac{5\pi}{4}\sin\pi d/T - \frac{1}{3}\sin\frac{15\pi}{4}\sin 3\pi d/T\right]$$
$$= -\frac{2}{\omega_o}\left[-\sin\frac{\pi}{4}\sin\pi d/T + \frac{1}{3}\sin\frac{\pi}{4}\sin 3\pi d/T\right] = -A_2$$

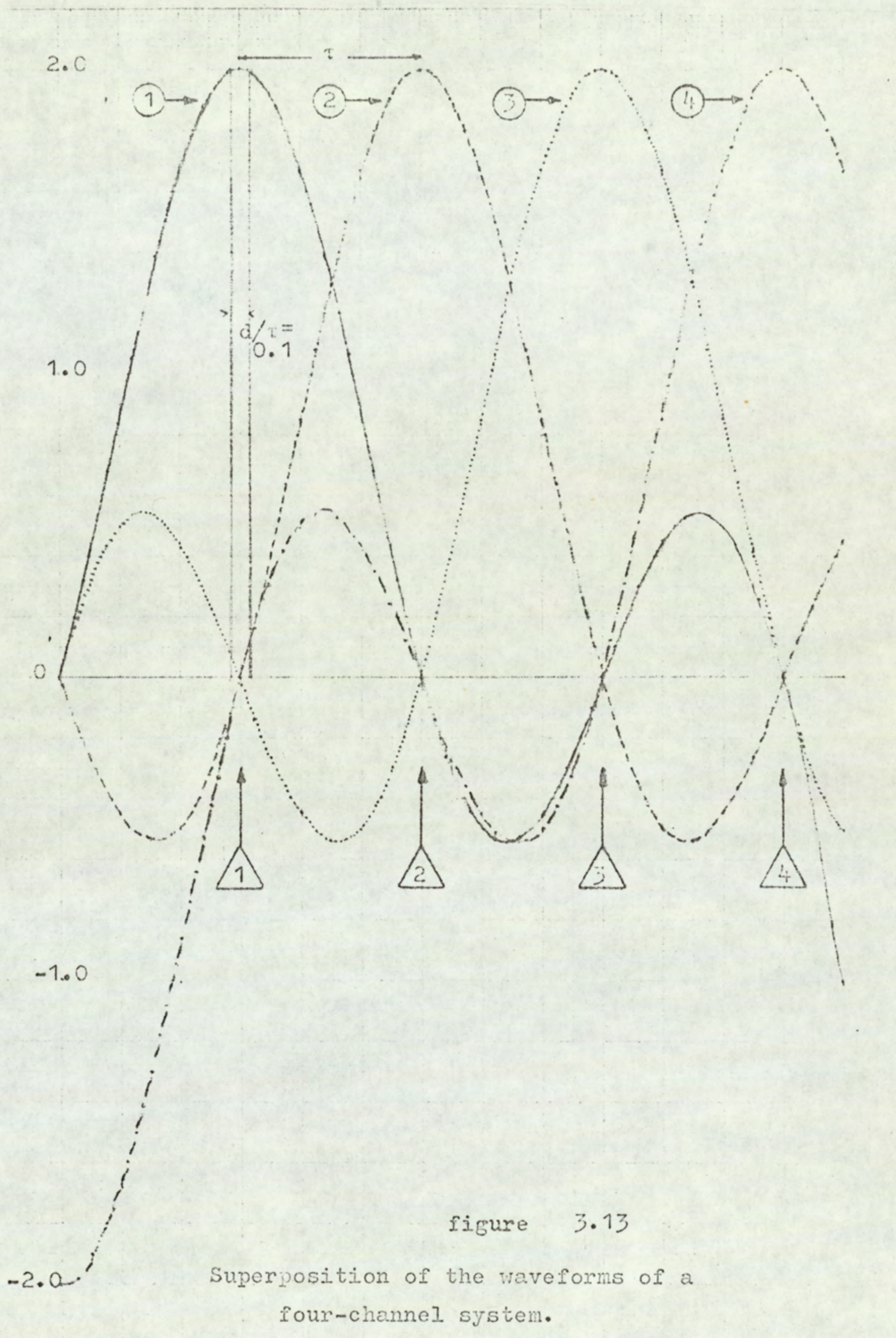


figure 3.13

Superposition of the waveforms of a four-channel system.

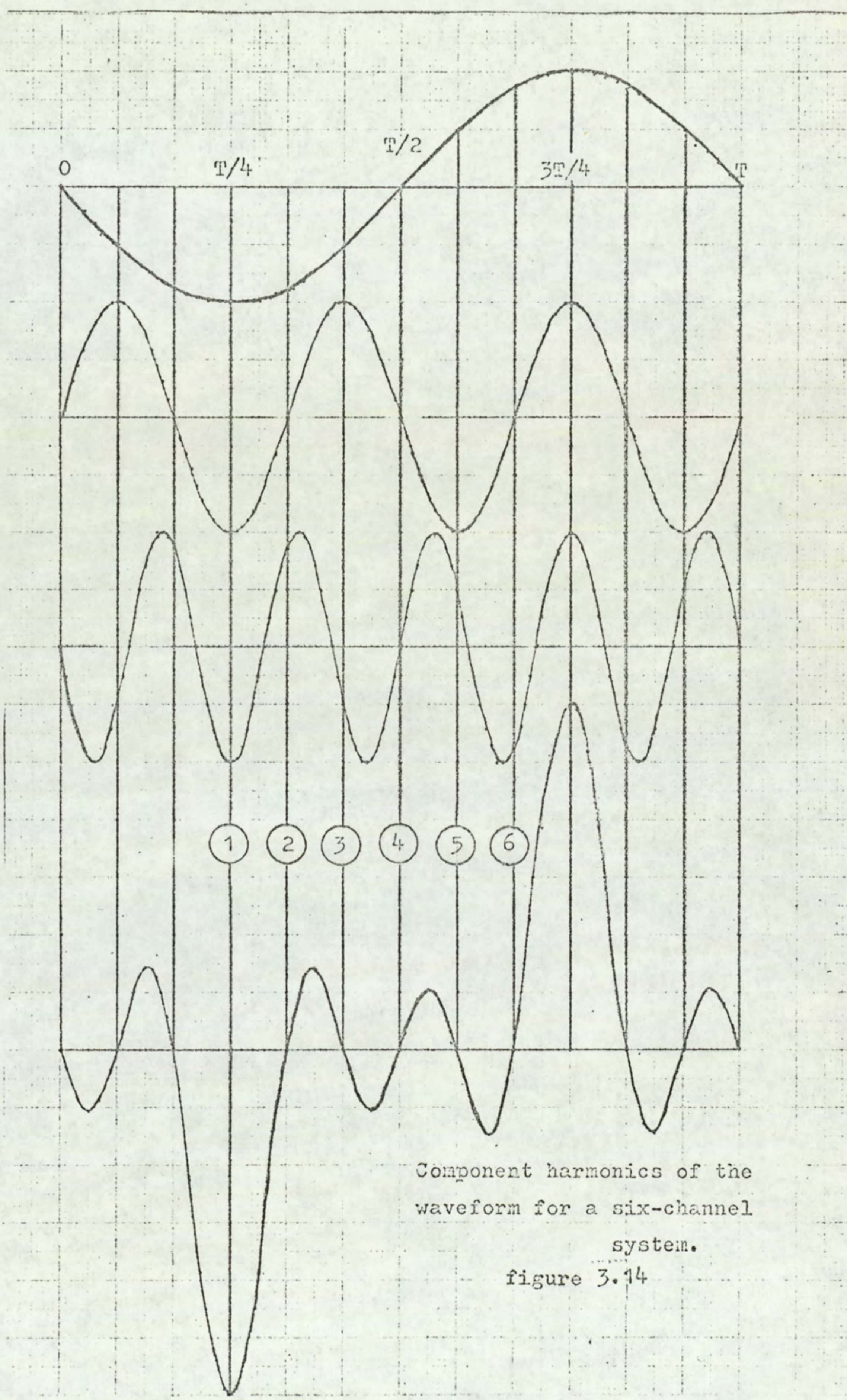
Hence, the sum of the crosstalk components is zero in the first channel, and the same result is obtained when evaluating the crosstalk in the fourth channel.

Figure 3.13 shows the superimposed waveforms of a four-channel system. It will be seen that at sampling point (1), the contributions of waveforms (2) and (4) are equal and opposite during the sampling interval, whilst the mean value of waveform (3) is zero over this interval. At sampling point (4), the cancelling waveforms are (1) and (3), whist waveform (2) has zero mean. At sampling point (2), waveform (4) has zero mean, but waveforms (1) and (3) will contribute a crosstalk component. Similarly, at sampling point (3), waveforms (2) and (4) contribute the crosstalk.

The constituent waveforms for a six-channel system are shown in figure 3.14. At sampling point (4), the composing waveforms all have odd symmetry, so that integration about this point must have zero resultant. It may also be noted that the contributions from any one harmonic from points (2) to (6) will sum to zero, so that their superposition at point (1) cannot give anycrosstalk component. This applies if the waveform shown is the first or the last in the multiplex frame.

Although the number of crosstalk components progressively increases from the second to the (m)th channel, and decreases from the (m+1)th to the (2m-1)th, the crosstalk in channels (m) and (m+1) does not represent the worst case. The magnitude of the crosstalk component from another channel decreases as the relative displacement of that channel increases. Thus, although the second, and last but one, channels experience crosstalk only from adjacent channels, the crosstalk amplitude is greatest, and these channels have the worst crosstalk ratio.

The difference between channels is not, however, great. Using the general expression for the crosstalk ratio, (equation 2, page 94), a range of values was computed, and is shown in Table 1. The crosstalk ratio in each channel for systems with 4, 6, 8, 10, and 12 channels is tabulated against d/τ, the ratio of sampling pulse duration to the appropriate time slot for that number of



Component harmonics of the waveform for a six-channel system.

figure 3.14

Table 1

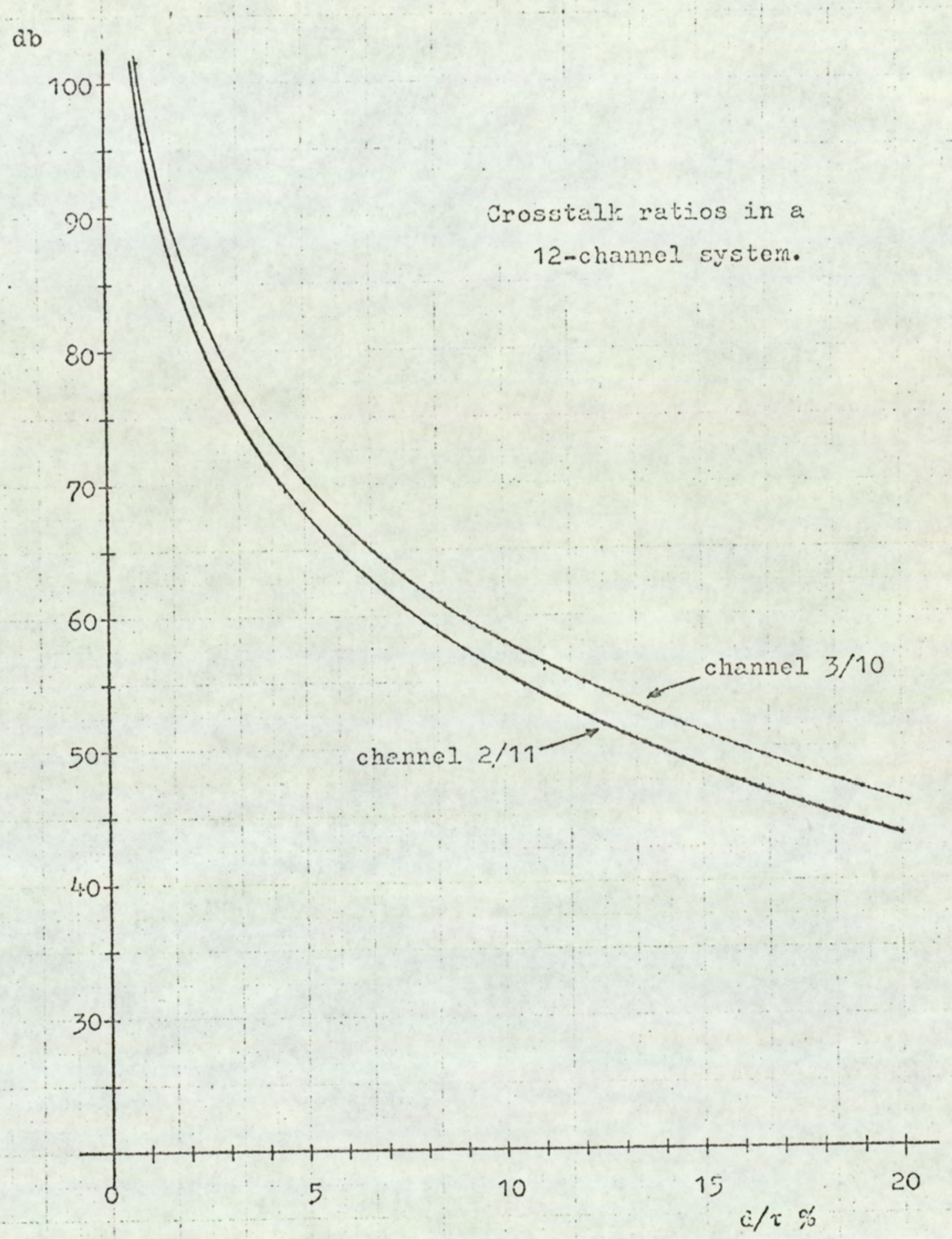
CROSSTALK RATIO (db)

m →	2	3		4			5				d/τ
channel→	2/3	2/5	3/4	2/7	3/6	4/5	2/9	3/8	4/7	5/6	%
	109	108	110	108	110	109	108	110	109	109	0.5
	96.8	96.0	97.9	95.8	98.0	97.2	95.7	98.0	97.1	97.4	1.0
	89.7	89.0	90.8	88.8	91.0	90.2	88.7	91.0	90.0	90.4	1.5
	84.7	84.1	85.8	83.8	86.0	85.2	83.7	86.0	85.0	85.4	2.0
	80.8	80.1	81.9	79.9	82.1	81.3	79.8	82.1	81.2	81.5	2.5
	77.7	76.9	78.8	76.7	78.9	78.2	76.6	79.0	78.0	78.3	3.0
	75.0	74.2	76.1	74.0	76.2	75.5	73.9	76.3	75.3	75.7	3.5
	72.7	71.9	73.8	71.7	73.9	73.2	71.6	74.0	73.0	73.3	4.0
	70.6	69.9	71.7	69.7	71.9	71.1	69.6	71.9	70.9	71.3	4.5
	68.8	68.1	69.9	67.8	70.0	69.3	67.8	70.1	69.1	69.5	5.0
	67.1	66.4	68.3	66.2	68.4	67.6	66.1	68.4	67.5	67.8	5.5
	65.6	64.9	66.7	64.7	66.9	66.1	64.6	66.9	65.9	66.3	6.0
	64.2	63.5	65.3	63.3	65.5	64.7	63.2	65.5	64.6	64.9	6.5
	62.9	62.2	64.1	62.0	64.2	63.4	61.9	64.2	63.3	63.6	7.0
	61.7	61.0	62.9	60.8	63.0	62.2	60.7	63.0	62.1	62.4	7.5
	60.6	59.9	61.7	59.7	61.9	61.1	59.6	61.9	60.9	61.3	8.0
	59.6	58.8	60.7	58.6	60.8	60.1	58.5	60.9	59.9	60.2	8.5
	58.6	57.8	59.7	57.6	59.8	59.1	57.5	59.9	58.9	59.3	9.0
	57.6	56.9	58.8	56.7	58.9	58.1	56.6	58.9	58.0	58.3	9.5
	56.7	56.0	57.9	55.8	58.0	57.2	55.7	58.0	57.1	57.4	10.0
	55.9	55.2	57.0	55.0	57.2	56.4	54.9	57.2	56.2	56.6	10.5
	55.1	54.4	56.2	54.1	56.3	55.6	54.1	56.4	55.4	55.8	11.0
	54.3	53.6	55.4	53.4	55.6	54.8	53.3	55.6	54.6	55.0	11.5
	53.6	52.8	54.7	52.6	54.8	54.1	52.5	54.9	53.9	54.3	12.0
	52.9	52.1	54.0	51.9	54.1	53.4	51.8	54.2	53.2	53.5	12.5
	52.2	51.4	53.3	51.2	53.4	52.7	51.1	53.5	52.5	52.9	13.0
	51.5	50.8	52.6	50.6	52.8	52.0	50.5	52.8	51.8	52.2	13.5
	50.9	50.2	52.0	49.9	52.1	51.4	49.9	52.2	51.2	51.6	14.0
	50.3	49.5	51.4	49.3	51.5	50.8	49.2	51.6	50.6	51.0	14.5
	49.7	49.0	50.8	48.8	50.9	50.2	48.7	51.0	50.0	50.4	15.0
	49.1	48.4	50.2	48.2	50.4	49.6	48.1	50.4	49.4	49.8	15.5
	48.6	47.8	49.7	47.6	49.8	49.1	47.5	49.9	48.9	49.2	16.0
	48.0	47.3	49.2	47.1	49.3	48.5	47.0	49.3	48.4	48.7	16.5
	47.5	46.8	48.6	46.6	48.8	48.0	46.5	48.8	47.8	48.2	17.0
	47.0	46.3	48.1	46.1	48.3	47.5	46.0	48.3	47.3	47.7	17.5
	46.5	45.8	47.6	45.6	47.8	47.0	45.5	47.8	46.8	47.2	18.0
	46.0	45.3	47.2	45.1	47.3	46.5	45.0	47.3	46.4	46.7	18.5
	45.6	44.8	46.7	44.6	46.8	46.1	44.5	46.9	45.9	46.3	19.0
	45.1	44.4	46.2	44.2	46.4	45.6	44.1	46.4	45.5	45.8	19.5
	44.7	44.0	45.8	43.7	45.9	45.2	43.7	46.0	45.0	45.4	20.0

Table 1 (cont.)

CROSSTALK RATIO (db)

	m = 6					d/τ
channel→	2/11	3/10	4/9	5/8	6/7	%
	108	110	109	110	109	0.5
	95.7	98.1	97.0	97.5	97.3	1.0
	88.6	91.0	89.9	90.4	90.2	1.5
	83.6	86.0	85.0	85.4	85.2	2.0
	79.7	82.1	81.1	81.6	81.4	2.5
	76.6	79.0	77.9	78.4	78.2	3.0
	73.9	76.3	75.2	75.7	75.5	3.5
	71.6	74.0	72.9	73.4	73.2	4.0
	69.5	71.9	70.9	71.3	71.1	4.5
	67.7	70.1	69.0	69.5	69.3	5.0
	66.0	68.4	67.4	67.9	67.7	5.5
	64.5	66.9	65.9	66.4	66.1	6.0
	63.1	65.5	64.5	65.0	64.8	6.5
	61.9	64.2	63.2	63.7	63.5	7.0
	60.7	63.0	62.0	62.5	62.3	7.5
	59.5	61.9	60.9	61.4	61.1	8.0
	58.5	60.9	59.8	60.3	60.1	8.5
	57.5	59.9	58.8	59.3	59.1	9.0
	56.6	58.9	57.9	58.4	58.2	9.5
	55.7	58.0	57.0	57.5	57.3	10.0
	54.8	57.2	56.1	56.6	56.4	10.5
	54.0	56.4	55.3	55.8	55.6	11.0
	53.2	55.6	54.6	55.0	54.8	11.5
	52.5	54.9	53.8	54.3	54.1	12.0
	51.8	54.2	53.1	53.6	53.4	12.5
	51.1	53.5	52.4	52.9	52.7	13.0
	50.4	52.8	51.8	52.3	52.1	13.5
	49.8	52.2	51.1	51.6	51.4	14.0
	49.2	51.6	50.5	51.0	50.8	14.5
	48.6	51.0	49.9	50.4	50.2	15.0
	48.0	50.4	49.4	49.9	49.6	15.5
	47.5	49.9	48.8	49.3	49.1	16.0
	47.0	49.3	48.3	48.8	48.6	16.5
	46.4	48.8	47.8	48.2	48.0	17.0
	45.9	48.3	47.3	47.7	47.5	17.5
	45.4	47.8	46.8	47.2	47.0	18.0
	45.0	47.3	46.3	46.8	46.6	18.5
	44.5	46.9	45.8	46.3	46.1	19.0
	44.1	46.4	45.4	45.9	45.7	19.5
	43.6	46.0	44.9	45.3	45.2	20.0



d = sampling pulse duration

τ = channel time slot

Figure 3.15

channels. The first and (2m)th channels are not included, since, as was mentioned, these have zero crosstalk.

It will be seen that the difference between the worst and best cases is always less than 3db. These two cases, for a twelve-channel system, are plotted in figure 3.15. If a value of d/τ = 7.5 % is chosen, so that the crosstalk ratio is always greater than 60db, the required sampling pulse duration for the twelve channel system is 0.8 microsec. Although short, this would be within the capability of existing techniques.

The crosstalk ratios which have been derived are, of course, for a system in which the waveforms are correctly formed, and the sampling switch is correctly synchronised. The effect of errors will be considered in the next sub-section.

3.6.c. Effect of tolerance upon system performance.

In any practical system there will be some deviation from the ideal conditions under which the waveforms are realised with absolute precision, and the sampling pulses are centred precisely upon the zero-crossings. Therefore it is necessary to determine what tolerance, if any, is permissible, before the crosstalk ratio deteriorates by an unacceptable amount.

The effect of timing error in the sampling pulses is considered first. With a correctly located pulse, it was seen that the total crosstalk resulted from components of opposite polarity, and almost equal magnitude, which occurred at each side of the pulse centre-line. With a displaced pulse, this balance is disturbed. Some compensation still occurs, but the crosstalk resultant will have a greater magnitude. Furthermore, the magnitude of the sample from the wanted channel will be reduced as the sampling instant is displaced from the peak amplitude.

Consideration of the waveform patterns shows that if the sampling pulse is displaced by half the channel time slot, the wanted channel and the adjacent channel contribute components of equal magnitude and polarity. Although the crosstalk from the other channels may be largely self-cancelling, the resulting crosstalk ratio cannot be greater than unity. Hence, the permissible displacement will be small.

The crosstalk ratio may be evaluated from equation 1, page 89, by substituting appropriate values for t_q, the arbitrary sampling instant. For an offset pulse, this may be put in the form,

$$t_q = t_s + kd$$

where t_s is the instant of a zero-crossing, d is the pulse duration, and k is a multiplying factor less than unity.
Equation 1 then becomes:-

$$\begin{aligned} A = -\frac{2}{\omega_o}\Big\{ & \sin\pi d/T \left[\sin 2\pi t_s/T \cos 2\pi kd/T + \cos 2\pi t_s/T \sin 2\pi kd/T\right] \\ & - \frac{1}{3}\sin 3\pi d/T \left[\sin 3(2\pi t_s/T)\cos 3(2\pi kd/T) + \cos 3(2\pi t_s/T)\sin 3(2\pi kd/T)\right] \\ & \quad \vdots \\ & - \frac{(-1)^m}{(2m-1)}\sin(2m-1)\pi d/T \left[\sin(2m-1)(2\pi t_s/T)\cos(2m-1)(2\pi kd/T) \right. \\ & \left. \quad + \cos(2m-1)(2\pi t_s/T)\sin(2m-1)(2\pi kd/T)\right] \Big\} \end{aligned}$$

which may be expanded as:-

$$\begin{aligned} -\frac{1}{\omega_o}\Big\{ & \sin(2\pi t_s/T)\left[\sin(1+2k)(\pi d/T) + \sin(1-2k)(\pi d/T)\right] \\ & + \cos(2\pi t_s/T)\left[\cos(1-2k)(\pi d/T) - \cos(1+2k)(\pi d/T)\right] \\ & - \frac{1}{3}\sin(6\pi t_s/T)\left[\sin(1+2k)(3\pi d/T) + \sin(1-2k)(3\pi d/T)\right] \\ & - \frac{1}{3}\cos(6\pi t_s/T)\left[\cos(1-2k)(3\pi d/T) - \cos(1+2k)(3\pi d/T)\right] \\ & \quad \vdots \\ & - \frac{(-1)^m}{(2m-1)}\sin(2m-1)(2\pi t_s/T)\left[\sin(1+2k)(2m-1)(\pi d/T) + \sin(1-2k)(2m-1)(\pi d/T)\right] \\ & - \frac{(-1)^m}{(2m-1)}\cos(2m-1)(2\pi t_s/T)\left[\cos(1-2k)(2m-1)(\pi d/T) - \cos(1+2k)(2m-1)(\pi d/T)\right] \Big\} \end{aligned}$$

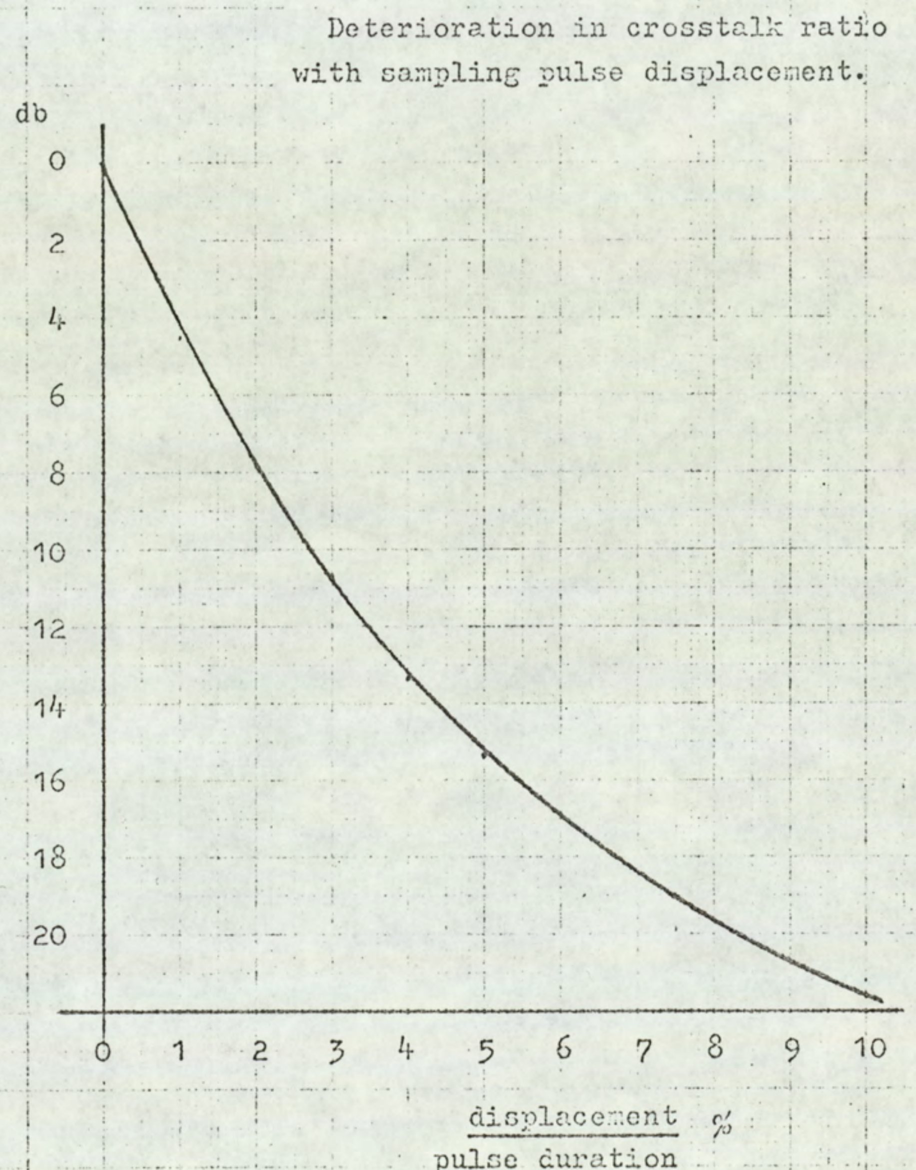
Because the waveforms are odd functions about T/2, each interfering waveform contributes a different component. The expression for the crosstalk ratio cannot be simplified to the form used for correctly located sampling pulses. The total crosstalk differs in successive channels, although pairs of channels at equal displacements either side of the frame centre have the same crosstalk ratio. Whether the pulse is advanced, or delayed, is immaterial, since this merely changes the polarity of the crosstalk resultant.

Initially, the crosstalk ratio was evaluated for a displacement of half the pulse duration, i.e. the leading edge of the pulse aligned with the zero-crossing. The ratio of pulse duration to channel time slot was taken as 10%, which produced the following results,

		crosstalk ratio
m = 2	channel 1 (or 4)	22.8db
	channel 2 (or 3)	29.5db
m = 3	channel 1 (or 6)	23.0db
	channel 2 (or 5)	27.8db
	channel 3 (or 4)	33.6db
m = 6	channel 1 (or 12)	22.7db

Since a 10% ratio of pulse duration to time slot results in crosstalk ratios of about 57db with correctly timed sampling pulses, displacement by d/2 produces a considerable deterioration. It will be seen that the first and last channels of a given system, which previously had zero crosstalk, now represent the worst case. As with correct timing, the difference between systems having differing numbers of channels is small.

The crosstalk ratio in the first channel of a four-channel system may, therefore, be taken as representative of the worst case values for higher values of m. Figure 3.16 shows the result of evaluating the crosstalk ratio in channel 1 of a four-channel system for a range of sampling pulse displacement. The curve indicates the decrease in crosstalk ratio, (from the value for zero displacement), against fractional displacements of a pulse duration.



Deterioration in crosstalk ratio
with sampling pulse displacement.
db
0
2
4
6
8
10
12
14
16
18
20
0
1
2
3
4
5
6
7
8
9
10
displacement / pulse duration %

Figure 3.16

It is evident that the system is highly sensitive to small errors in the timing of the sampling pulse. If, say, a value of d/τ is chosen from Table 1 to give a 4db margin for errors, a displacement of 1% of the pulse duration will absorb this margin. Clearly, there are physical limitations to the smallest value of d/τ that may be used. Furthermore, the absolute value of the permissible displacement would decrease as the pulse duration was reduced.

The effect of a high frequency jitter in the timing of the sampling pulses would be less important. Since the crosstalk polarity depends on the direction in which the pulse is displaced, then provided the jitter frequency is higher than the message bandwidth, the crosstalk would be that for the mean displacement.

Errors in realisation of the waveforms may also be treated as timing errors. The zero-crossings are dependent upon the algebraic sum of the component harmonic sinusoids becoming zero. Thus, if the harmonics of the waveform for any one channel do not all have the same peak value, the zero-crossings will be displaced. This also applies, of course, if the harmonics are not in the correct phase relationship.

If each of the harmonic components is realised with an amplitude tolerance of $\pm x\%$, then in the worst case the errors sum to displace the waveform by $x\%$ above or below the zero axis. The effect upon a zero-crossing will depend upon the slope of the waveform through that point. The worst case, i.e. the least slope, occurs at $T/2$, as may be seen from figure 3.13, or figure 3.14. The two waveform peaks immediately adjacent to this point are equal and opposite in value, and occur at approximately half the channel time slot. Hence, a displacement of the waveform at this point by $x\%$ will displace the zero-crossing by $x/2$ %, (assuming a straight-line approximation for small displacements). At other sampling instants, the displacement would be less.

The percentage error in amplitude of the waveform harmonics must, therefore, be not more than twice the permitted timing error. If, for example, a pulse duration were chosen so that $d/\tau = 5\%$, and the timing error was not to exceed 1% of the pulse duration,then the ratio of maximum deviation to channel time slot would be 0.05%. The waveform amplitude tolerance would then be 0.1% .

Errors in the phase,or relative alignment,of the harmonic components of a waveform would be equivalent to an amplitude error. The maximum error would occur when each of the harmonics was passing through a point of maximum gradient,which is again the case for $t = T/2$. If the channel time slot,τ,is taken as the reference interval,i.e. a displacement between two harmonics of τ represents a displacement of 100%,then for small values in the region of $T/2$ an incremental change in phase of x% will produce an incremental change in amplitude also of x % . Thus the phase is subject to the same constraints as the amplitude.

Since all three forms of error will be present simultaneously,it is clear that the permitted tolerances become very small. For example,if a value of $d/\tau = 5\%$ is chosen for for a four-channel system,giving a nominal crosstalk ratio of 68db,then if the crosstalk ratio is not to be reduced below 60db by the three sources of error,each must be realised within a tolerance of $\pm$ 0.05 % .

In practice the situation would be eased slightly, since the timing error,and the phase-amplitude error,apply to different worst cases. However,the saving is small,and it would appear that the necessity for precise realisation restricts this system to applications where a high crosstalk ratio is not required.

3.6.d. Modulation, and message recovery.

Modulation of the $S_n(t)$ function requires that each harmonic present in the waveform be identically amplitude modulated to produce double sidebands on each harmonic. In order to use the minimum possible bandwidth, the fundamental component of the waveform is chosen to be the same as the highest frequency component, ω_{max}, in the message bandwidth.

Since there are m harmonics, the signal band is from zero to $2\omega_{max}$. Each of the 2m channels is modulated in the same way, having the same, overlapping, bandwidth, so that the overall bandwidth occupied is $2m\omega_{max}$.

If the message in the (q)th channel is denoted $m_q(t)$, the modulated signal is

$$f'_q(t) = m_q(t) \sum_{n=1}^{m} (-1)^n \sin\left[(2n-1)\left[\omega_o t - (q-1)\pi/2m\right]\right]$$

where $\omega_o = \omega_{max}$.

This implies that the lower sideband on the fundamental overlaps the message band, so that a balanced modulator must be used. Alternatively, the message may modulate a suitably high frequency, from which the baseband can be separated by a lowpass filter, followed by a further translation of the modulated signal to the required position.

If a balanced modulator is used, a simple means of modulating the waveform is to chop the message with a square-wave switching function. A symmetrical square-wave, of frequency ω_o, with amplitude values of ± 1, may be written as

$$f(t) = \frac{4}{\pi}\left[\sin\omega_o t + \frac{1}{3}\sin 3\omega_o t + \frac{1}{5}\sin 5\omega_o t \ \ldots\right]$$

which contains the necessary frequency components to form the $S_n(t)$ waveform. The modulated harmonics may be separated by bandpass filters, and each component normalised to a reference amplitude, with a sign change for alternate components.

In practice, it would be necessary to follow the usual procedure of bandlimiting the messages to slightly less than ω_o so as to provide a transition region for the filter cutoff. This would also prevent the modulated signals from extending to dc, which would avoid some transmission difficulties.

The same modulation methods apply, whether the message be an analogue signal, or a data signal. When modulating by a binary waveform, the highest frequency component that need be retained is half the bit rate. The binary waveform may be put into nonreturn-to-zero form, and used to modulate the waveform directly via the switching modulator, as is sometimes done in data transmission by amplitude modulated carrier. (Reference 45, ch.3).

Whilst 100% modulation can be used if it is required to maximise the efficiency of the transmitted signal, it may be more convenient to use less than 100%, and retain the approximate square-wave shape of the multiplex signal. This reduces the ratio of peak-to-mean signal power, but is mainly useful for providing synchronisation information at the receiver, as discussed later.

Message recovery, by sampling, requires, generally, that alternate samples be inverted in polarity. This may be regarded as the product of the multiplex signal with a correctly synchronised pulse train, which has unity amplitude pulses of alternating polarity. This pulse train is the difference between two other pulse trains, which if their frequency is $\omega_o (=2\pi/T)$, are displaced by T/2. Thus, the alternating polarity pulse train is

$$p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} - \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o (t-T/2)}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} (1 - e^{-jn\omega_o T/2})$$

Since $(1-e^{-jn\omega_o T/2}) = (1-e^{-jn\pi}) = \begin{cases} 0, & n=0, \text{ or } n \text{ even} \\ 2, & n \text{ odd} \end{cases}$

therefore,

$$p(t) = 2\sum_{n=\mp(1,3,5,..)} c_n e^{jn\omega_o t} = 4\sum_{n=1,3,5,..} c_n \cos n\omega_o t$$

The product of this pulse train with the first channel signal is

$$p(t)\, f_1(t) = 4 \sum_{n \text{ odd}} c_n (\cos n\omega_o t) \left[m_1(t) \sum_{q=1}^{m} (-1)^q \sin(2q-1)\omega_o t \right]$$

The product of the (p)th harmonic of the sampling pulse train with the (q)th harmonic of the channel waveform is

$$4\, c_p\, m_1(t)\, (-1)^q \left[\cos p\omega_o t\, \sin(2q-1)\omega_o t \right]$$

Writing this as $k_1 \cos(a\omega_o t)\sin(b\omega_o t)$,then following the discussion in chapter 1,

$$\int_{-T/2}^{T/2} k_1 \cos(a\omega_o t) \sin(b\omega_o t)\, dt = 0 \qquad (\omega_o = 2\pi/T)$$

That is ,the functions are orthogonal,so there would be zero resultant after lowpass filtering of these samples. To produce an output,the sampling pulse train must be synchronised,i.e. have an appropriate phase shift with respect to the channel waveform. If the pulse train has a displacement φ,then

$$k_1 \cos(a(\omega_o t-\varphi))\sin(b\omega_o t) =$$

$$k_1 \sin(b\omega_o t) \left[\cos(a\omega_o t)\cos(a\varphi) + \sin(a\omega_o t)\sin(a\varphi) \right]$$

$$= k_2 \left\{ \left[\sin(a+b)\omega_o t - \sin(a-b)\omega_o t \right] \cos(a\varphi) + \left[\cos(a-b)\omega_o t - \cos(a+b)\omega_o t \right] \sin(a\varphi) \right\}$$

where $k_2 = 2c_p m_1(t)(-1)^q$

It is evident that the only term which does not become zero after integration is $\cos(a-b)\omega_o t\, \sin(a\varphi)$, when a=b. It also follows that this term is maximum when $\sin(a\varphi) = \pm 1$, i.e. $\varphi = \pm \pi/2$. As was seen, a takes only odd values.

When all the harmonics in the channel waveform are taken into account, the result of the integration is

$$k_3\left[-\sin\varphi + \sin 3\varphi - \sin 5\varphi \;.\;.\;.(-1)^m \sin(2m-1)\varphi\right]$$

$$= -mk_3 \text{ if } \varphi = \pi/2, \text{ or } +mk_3 \text{ if } \varphi = -\pi/2$$

where $k_3 = Tc_1m_1(t)$, and the pulse duration is assumed to be such that the Fourier series coefficients are approximately equal to at least the (2m-1)th harmonic.

As is obvious, the negative sampling pulse should, therefore, be aligned with the negative peak of the waveform to recover the message without reversal of polarity.

As was seen, if $\varphi = 0$, the sampled signal is identically zero, and this is also the case if $\varphi = \pm\pi$. If $\varphi = \pi/2m$, the reult of the integration is

$$(m=2),\quad k_3(-\sin(\pi/4) + \sin(3\pi/4)) = 0$$

$$(m=3),\quad k_3(-\sin(\pi/6) + \sin(\pi/2) - \sin(5\pi/6)) = 0$$

$$(m=4),\quad k_3(-\sin(\pi/8) + \sin(3\pi/8) - \sin(5\pi/8) + \sin(7\pi/8)) = 0$$

The components will always sum to zero, regardless of the value of m, since the expression is the same as that for forming the zero crossings of the waveform.

Hence, it will be seen that the sampling pulse train and the channel waveform are orthogonal at sampling instants spaced by intervals of $\pi/2m$ from the origin, excluding the points at $\pi/2$ and $3\pi/2$. Since the channel waveforms are displaced by intervals of $\pi/2m$, they are orthogonal to each other at these instants. In general, the preceding merely confirms what is evident from an inspection of the waveforms.

3.6.e. Synchronisation.

As has been seen, a small error in the timing of the sampling pulses results in a large deterioration in crosstalk ratio. In order to obtain the necessary precise synchronisation, it would be preferable to base the timing on the transmitted waveform, thus avoiding errors due to variable transmission delay.

If the channel waveforms were modulated by less than 100%, one method would be to make use of the approximate square-wave envelope. A zero-crossing detector would indicate the transition between positive-going and negative-going halves of the multiplex waveform. However, this would be unlikely to provide a sufficiently precise reference, since the exact point of this transition will depend upon the relative amplitudes of the adjacent waveform peaks. The shape of the envelope at this point is determined mainly by the first and last channels of the frame, which will, in general, have different instantaneous amplitudes.

This could be overcome by allotting the first and last channels to an arbitrary fixed level tone, or by modulating both channels with the same message. This would, of course, be wasteful of bandwidth, and an alternative is to filter one of the harmonic carrier components. The filter output will be the resultant of the 2m harmonics of that frequency present in the waveform.

Provided the sidebands do not extend to the carrier, the amplitude of the filtered component will be independent of the modulation, and the phase will be constant with respect to the multiplex signal. The required clock frequency to drive the sampling circuits is $4m\omega_o$, whilst the highest carrier frequency is $(2m-1)\omega_o$. In general, the clock frequency may be obtained by filtering both the fundamental and the highest frequency carrier, and multiplying these to obtain $2m\omega_o$, which could then be frequency doubled.

One other requirement is to lock each sampling pulse to a particular channel, to prevent slipping. The waveform property by which each frame is alternately reversed in polarity may be used for this purpose. A zero-crossing detector will define the start of each frame.

3.6.f. System realisation with $S_n(t)$.

The general considerations which apply to a practical realisation of this system are the same as those for the system using $D_n(t)$, the periodic Dirichlet kernel. However, there are two significant advantages to be gained when generating the waveforms for the $S_n(t)$ function.

First, the harmonics are sine related, rather than cosine related as with $D_n(t)$. This means that the harmonics can be filtered from pulse trains which have their leading edges aligned. This latter condition is, of course, the one which prevails when performing the usual digital operations, such as gating and counting. The necessity of aligning the centres of the pulses, so as to obtain cosine related harmonics, is removed.

Second, the $D_n(t)$ function required the addition of a precise dc level to locate correctly the zero-crossings. This does not apply to the $S_n(t)$ function, and since precise ac circuits are more easily realised than precise dc circuits, the system is that much easier to realise.

As was mentioned earlier, the harmonics necessary to form one channel waveform can be obtained by bandpass filtering from a square-wave at the fundamental frequency. Hence, a set of square-waves must be generated with appropriate displacements to act as a source for the different channel waveforms. This could be simply achieved by the use of a feedback shift register, such as was shown in figure 3.5.

If the period of the clock frequency is τ, and the shift register has 2m stages, the output square-waves have a period T, $= 4,\tau$, and each square-wave is delayed by $\tau, = T/4m$, which is the required condition. For example, if a twelve-channel system were required, with message bandwidths of 4kHz, then the shift register would require 12 stages, and a clock frequency of 96kHz.

Each square-wave could then be used to drive a balanced modulator, to which the message was applied. The modulated square-wave would then be passed to a set of bandpass filters, there being one bandpass filter for each harmonic to be extracted.

It may be noted that only one bandpass filter per harmonic is required for the whole system,i.e. a total of (2m-1),rather than one per harmonic for each channel. This follows since the square-waves can be superimposed before filtering.

Each bandpass filter would require to be combined with an amplifier to restore the amplitude of the (unmodulated) harmonic to that of the fundamental. It would,in any case,be necessary to provide each filter with a summing amplifier for the purpose of combining the square-waves. Because there are 2m square waves superimposed in this amplifier,the signal level may be high,but this does not represent a serious disadvantage.

The modulated harmonics from the output of each filter may then be combined in a further summing amplifier,the output of which will be the multiplex signal. As the square waves require precise bandpass filtering in order to properly define the waveforms,the multiplex signal is precisely bandlimited,and no further bandlimiting is necessary before transmission,or remodulation.

The receiver may follow standard practice for time-division-multiplex systems. Assuming that synchronisation was achieved by filtering two harmonics from the multiplex waveform, as previously suggested,then these filters,plus the modulator, frequency-doubler,and pulse shaper would also be required. For channel synchronisation,a zero-crossing detector with associated logic circuits would also have to be included.

The most severe problem would be that of maintaining sufficient timing stability,and some degree of thermal control would probably be necessary. Also,passive delay elements would be preferable to monostables. A general block diagram for the system is shown in figure 3.17.

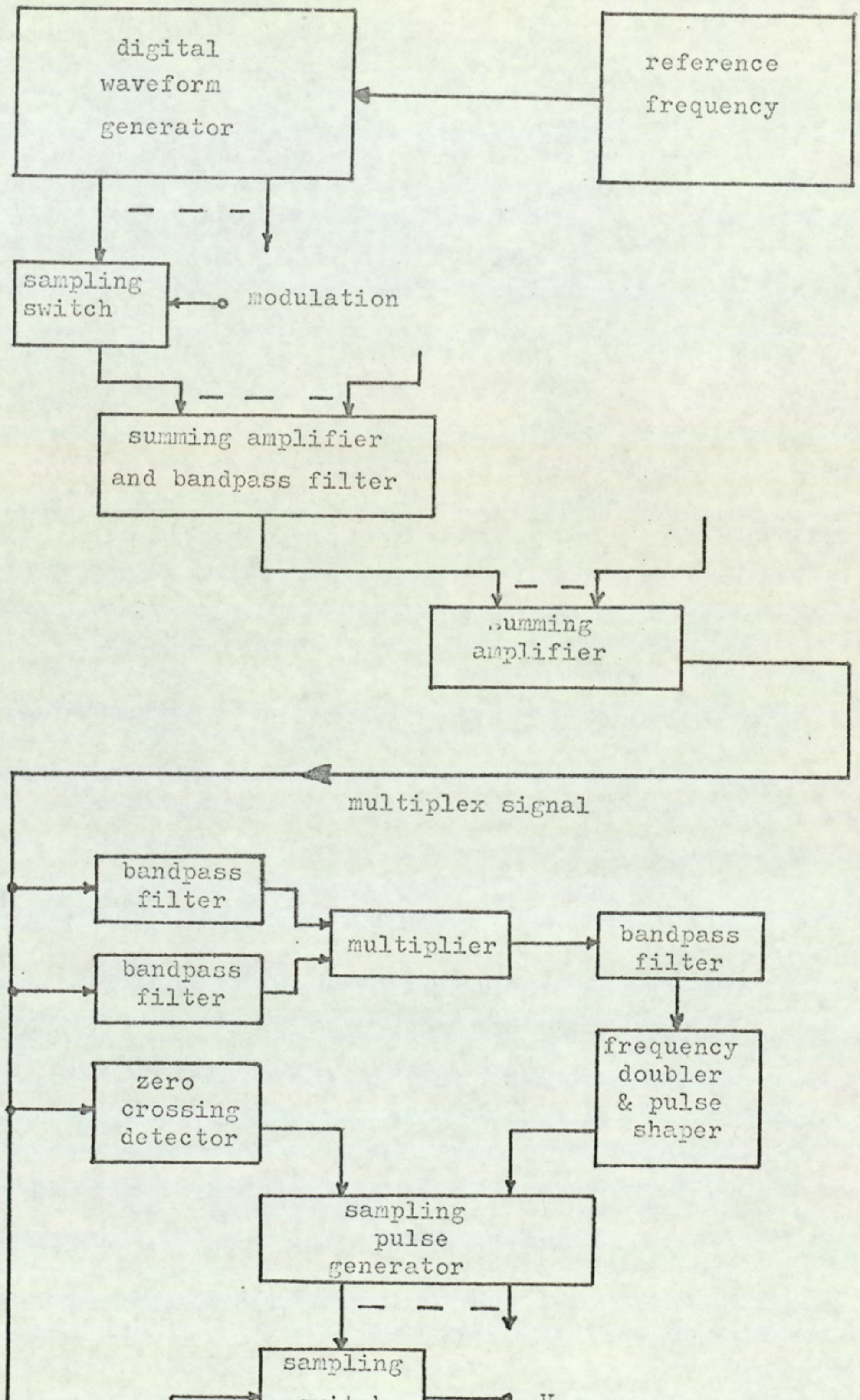

digital waveform generator
reference frequency
sampling switch
modulation
summing amplifier and bandpass filter
summing amplifier
multiplex signal
bandpass filter
bandpass filter
multiplier
bandpass filter
zero crossing detector
frequency doubler & pulse shaper
sampling pulse generator
sampling switch
V0

Figure 3.17

3.6.g. Experimental system.

Because of the need for precise realisation of the waveforms, it was not anticipated that in the limited time available it would be possible to construct a full-scale system which would allow useful measurements. The experimental work was confined to generating the waveform for one channel of a four-channel system, from which it was hoped to check the theoretical crosstalk ratios previously derived.

An existing piece of apparatus was adapted for this purpose. This was a 12-channel integrated circuit multiplexer, with associated driving stages and logic. The multiplexer, which incorporated FET sampling switches, had all outputs internally connected to a common point. However, the switches were bilateral, so that a dc level could be applied to the common point, to appear as pulses at the different switch 'input' points.

A divide-by-eight mode was selected, and the first four switches were used to produce a sequence of four pulses, separated by intervals of 1/8 of the fundamental period. The same summing amplifiers that were used to obtain the waveforms of the $D_n(t)$ system were used, so that a fundamental frequency of 20kHz was retained, but the second bandpass filter was adjusted to a centre frequency of 60kHz.

A pulse train at 20kHz was generated via the multiplexer, and applied through operational amplifiers to the bandpass filters. The outputs from each filter were then adjusted to have the same amplitude and phase, by correcting the amplifier gain and filter response. The logic driving the multiplexer switches allowed only for a maximum pulse duration equal to the clock pulse train period. This was less satisfactory than a square wave, since the harmonic amplitudes were low, and even harmonics were present. However, it proved possible to generate a waveform which approximated that of the ideal four-channel waveform, and the oscilloscope photograph of this waveform is shown in figure 3.18.

By applying an ac component in addition to the dc level at the multiplexer common point, it was possible to examine

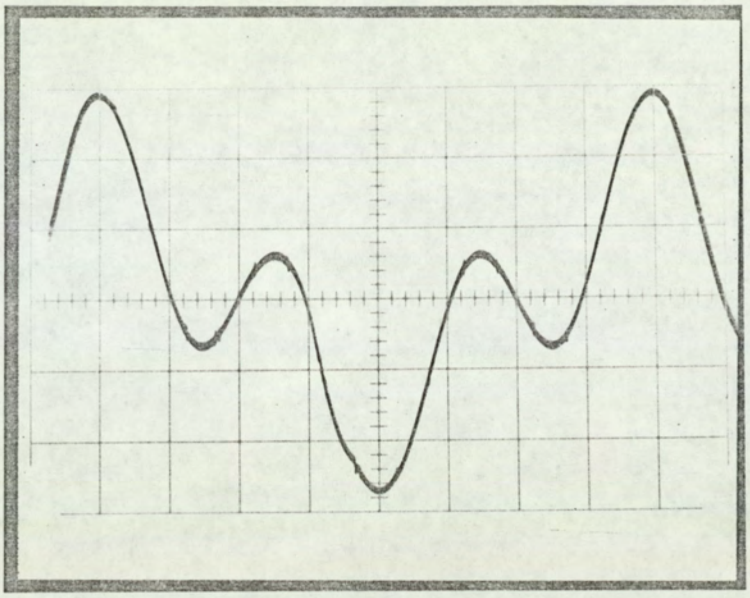


Figure 3.18

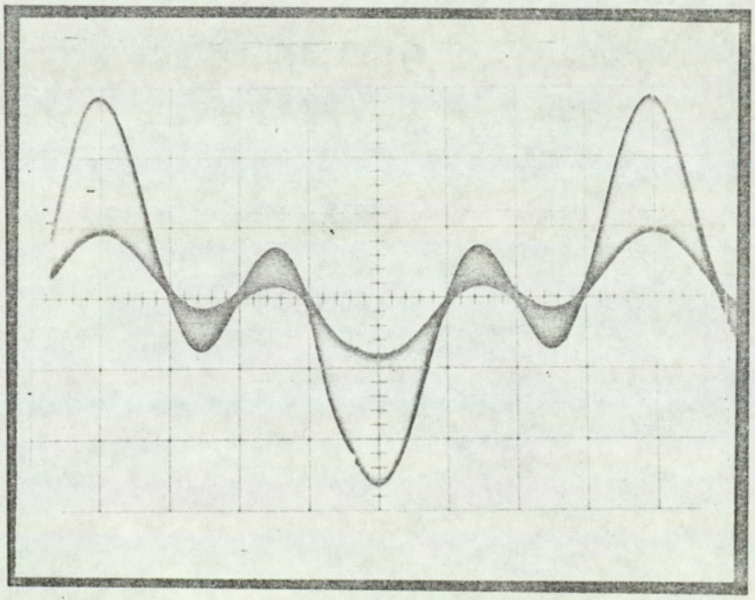


Figure 3.19

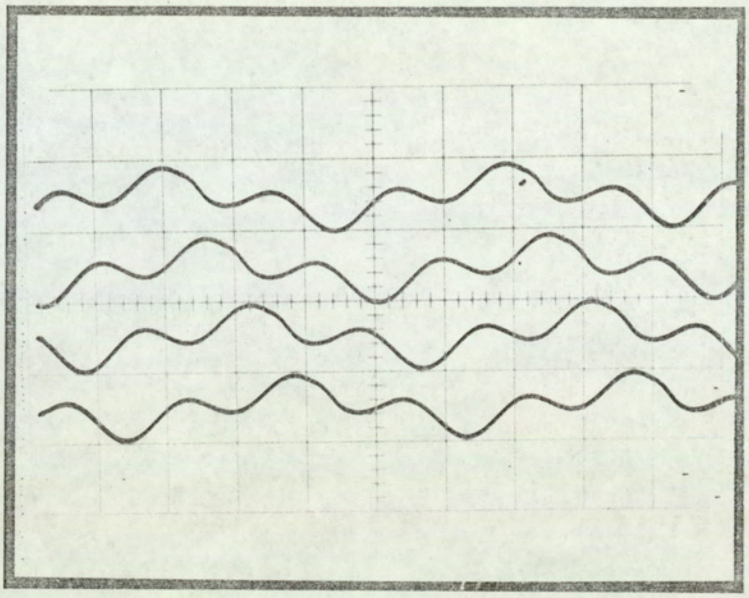


Figure 3.20

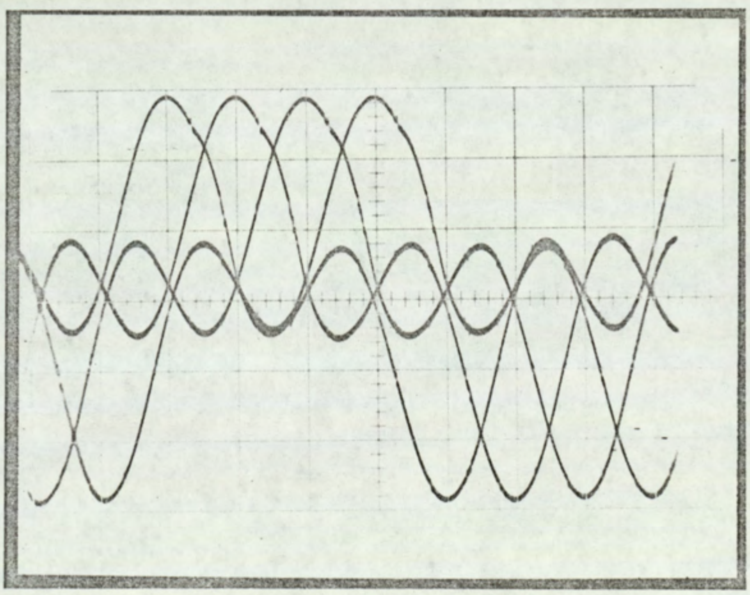


Figure 3.21

the appearance of the modulated waveform,and this is shown in figure 3.19. The diffused zero-crossings indicate that the basic waveform had not been precisely realised.

To check the timing of the four-pulse source,each pulse was applied in turn to the filters,and the resulting waveforms superimposed on one photograph. This is shown in figure 3.20, and on a larger scale,in figure 3.21. Although the displacement between waveforms appears to be close to the correct value,the failure to obtain overlapping zero-crossings again shows that the basic waveform was in error.

In order to assess the crosstalk,a separate sampling switch was used. This was driven from the delayed output of a pulse generator,the external excitation for which was provided by one of the multiplexer output pulses. This allowed the channel waveform to be sampled at the main peak and zero-crossings in quick succession. The appropriate sampling positions are indicated in figure 3.22. The sampler output was then lowpass filtered,and applied to a digital voltmeter.

By comparing the reading for the main peak to the algebraic sum of the readings at the three zero-crossings,an approximate measure of the crosstalk ratio was obtained. The range over which this could be measured was restricted by various practical difficulties. For example,the response of the sampling switch became distorted at pulse durations less than about 10% of the fundamental period. The digital voltmeter readings of the low amplitude sample values were also obscured by erratic drifting,caused mainly by thermal effects in the integrated circuit operational amplifiers. The delay circuit in the pulse generator also suffered from jitter.

These effects,and others,meant that the measurements had to be restricted to larger pulse durations,i.e. greater than 10% of the fundamental period. Since the channel spacing in a four-channel system is 12½% of the fundamental period,these pulse durations could not be used in practice. However they provided a check on the extrapolated theoretical values.

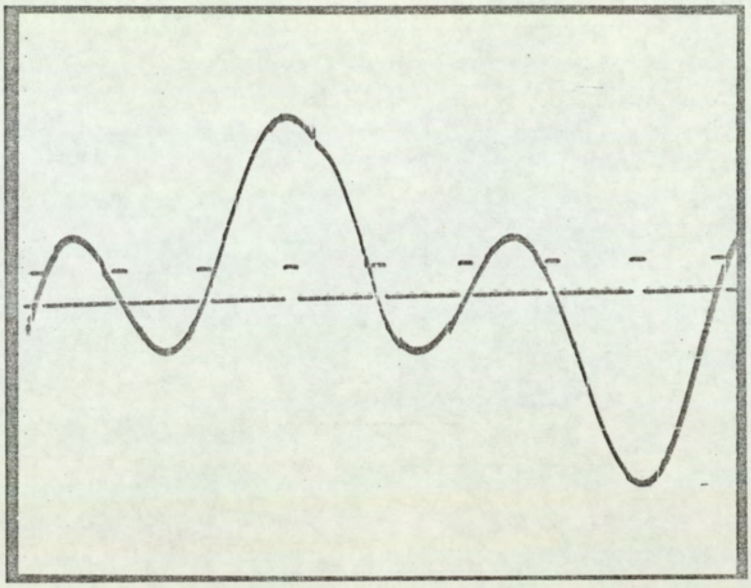


Figure 3.22

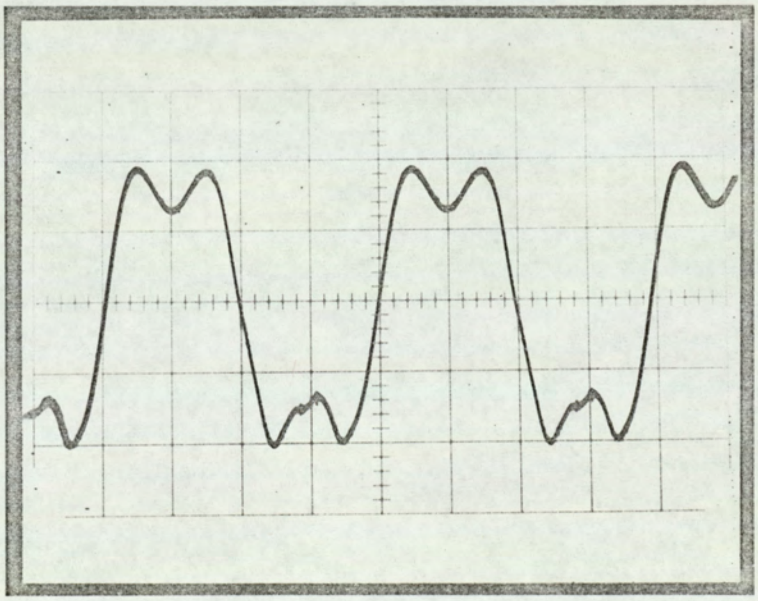


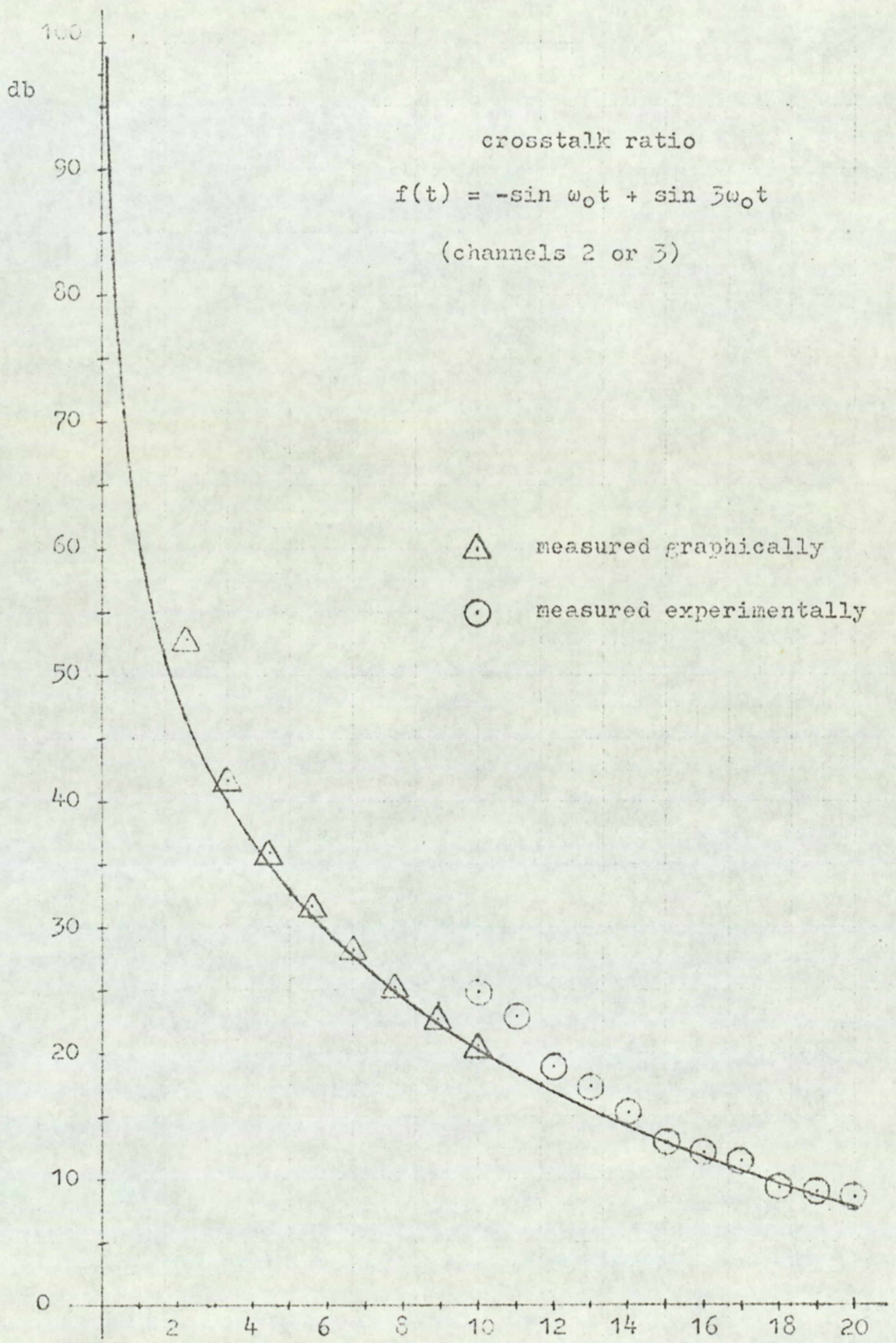
Figure 3.24

The measurement procedure was, first, to sample the waveform at T/2. Theoretically, this sample should have zero mean value, and the sampling pulse was aligned to produce the lowest reading, which was treated as a residual error for correcting the other readings. The driving pulse to the sampler was then derived from a different member of the multiplexer output four-pulse sequence. This shifted the sample point to another zero-crossing, or to the waveform peak. A comparison of the peak reading with the sum of the two readings from the zero-crossings at 3T/8 and 5T/8 gave the crosstalk ratio which would occur when four such readings were superimposed.

The results are shown in figure 3.23, against the theoretical curve drawn from the results of Table 1 (in section 3.6.b) for channel 2/3 of a four-channel system. It will be seen that the measured results are in close agreement at the lower end of the curve, but appeared to be superior to the theoretical values as the duration was decreased. It was not clearly established why this happened, but was assumed to be due to amplifier drift causing too low a value of crosstalk to be read.

To obtain a further check on the theoretical curve, the waveform was plotted to a large scale on millimetre graph paper, and the area under different sampling pulse durations was obtained by counting squares. This confirmed the theoretical values down to a ratio of d/T of approximately 4%, though below this figure it appeared that the crosstalk was again being underestimated.

Finally, the four pulses from the multiplexer were applied simultaneously to the summing amplifiers and bandpass filters, to obtain a qualitative picture of the four-channel multiplex waveform envelope. This is shown in figure 3.24. The amplifiers suffered from overloading with four pulses present, as is shown by the distorted waveform. However the approximate square-wave appearance of the envelope is demonstrated. The position at which the envelope is sampled may be seen from figure 3.25, where



d = sampling pulse duration
$T = 2\pi/\omega_0$

figure 3.23

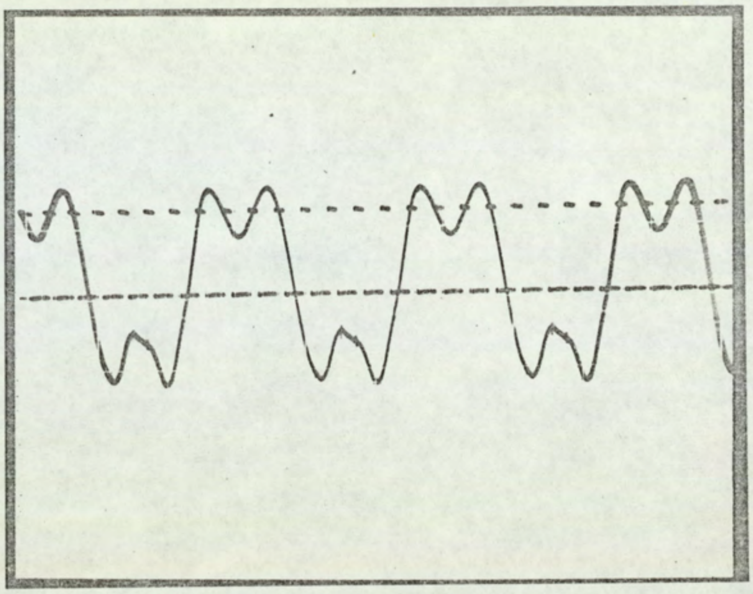


Figure 3.25

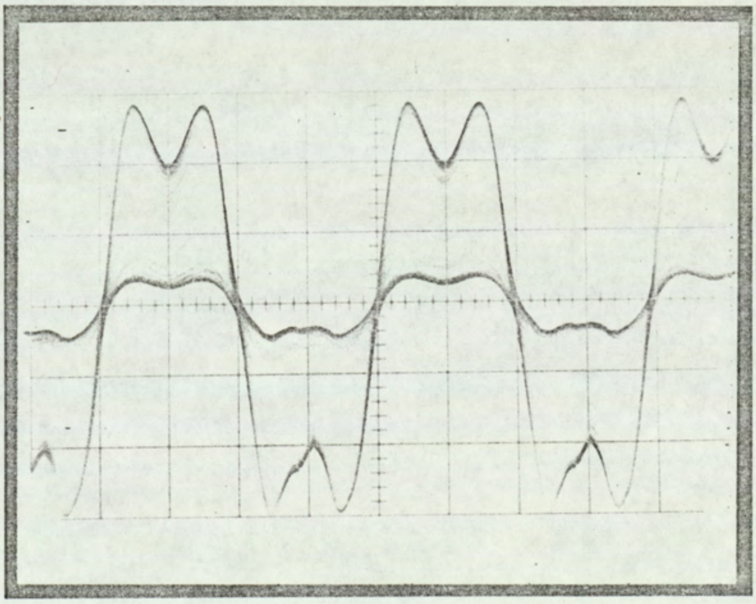


Figure 3.26

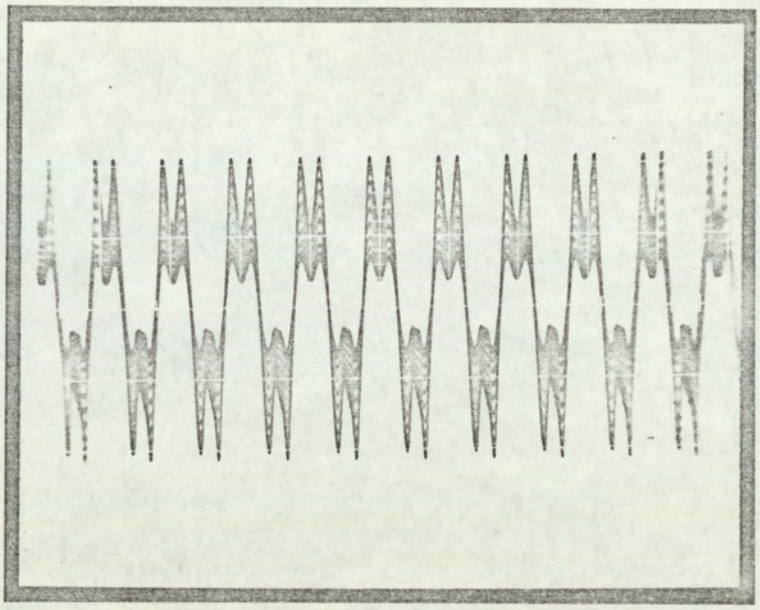


Figure 3.27

a synchronised pulse train at the multiplexing frequency is superimposed on the waveform envelope.

In figure 3.26, the four channels are modulated to the same depth and by the same frequency to give some idea of the appearance of the transmitted multiplex signal waveform. This is also shown with a different time scale in figure 3.27, above, from which it is evident that there would be little difficulty in obtaining frame synchronisation by the method previously described. The arrangement used for the experimental measurements is shown in figure 3.28.

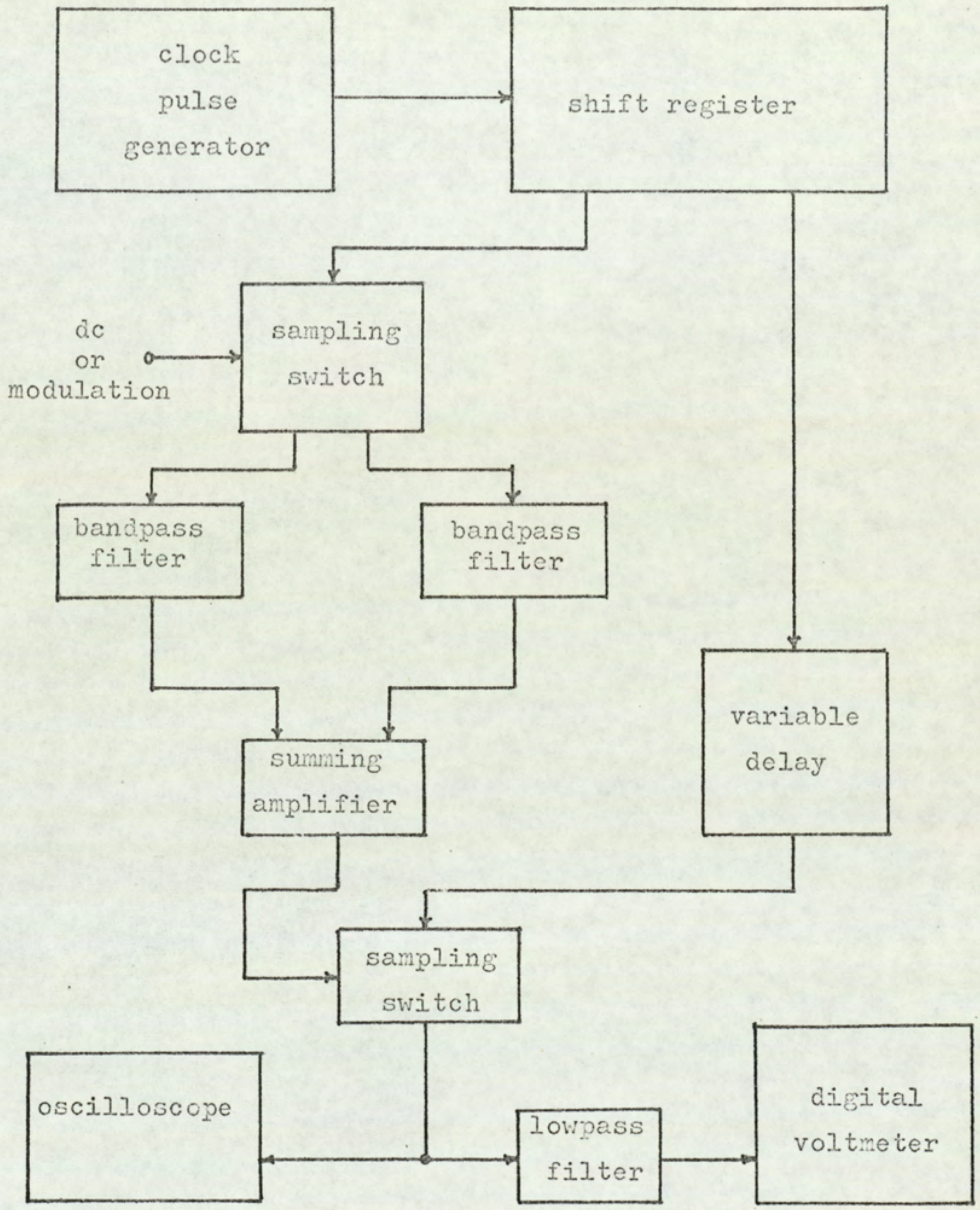

clock
pulse
generator
shift register
dc
or
modulation
sampling
switch
bandpass
filter
bandpass
filter
variable
delay
summing
amplifier
sampling
switch
oscilloscope
lowpass
filter
digital
voltmeter

Figure 3.28

3.7. Conclusions.

In this chapter, the possibility has been investigated of transmitting a time-division-multiplex signal in a bandwidth no greater than that of single-sideband frequency-division-multiplex. Initially, the spectrum of a bandwidth truncated unit impulse train was examined, and was seen to yield a function with a simple closed form, known as the periodic Dirichlet kernel.

The possibility of using this function as the basis of a time-division-multiplex system was then considered, and was seen to be feasible, although subject to considerable practical difficulties.

Since it was the zero-crossing properties of this waveform which were of prime significance, a number of other waveforms, which would result from a simple summation, were plotted. It was seen that the waveform produced by summing alternate polarity odd sine harmonics had potentially useful properties.

This waveform, referred to as $S_n(t)$, was examined in detail, and it was found that, with precise realisation, the crosstalk ratios were such as to allow the system to be used for speech transmission. However, the severe deterioration caused by small errors in realisation would make the system difficult to use in practice.

The properties of the waveform were such that modulation, message recovery, and synchronisation could, in principle, be achieved without great complication. Experimental measurements of the crosstalk ratio for lower values confirmed those predicted by theory.

The need for precise realisation handicaps the application of this system to speech transmission, but the lower crosstalk ratios permissible for data transmission would allow a system having wider tolerances to be used. The additional complexity compared to a conventional time-division-multiplex system may be balanced against the saving in transmission bandwidth.

The transmitter requires only one bandpass filter per two channels. If the need for very precise timing is removed, the receiver is no more complex than would be the case for a conventional time-division-multiplex system.

No special precautions would be required to achieve a crosstalk ratio capable of handling data transmission. A signal-to-noise ratio of 14.4 db to allow an error rate of 10^{-4} has been quoted for bipolar baseband data transmission.(Reference 45,ch.11). One useful feature of this system is that in-band noise which is added during transmission will be translated outside the passband by the alternate polarity reversal of the receiver sampling pulses.

A restrictive feature is that the transmission path must not introduce any significant phase or amplitude distortion which cannot be compensated,or the zero-crossings will be displaced. This means that the system would be more useful for radio than for line transmission.

Thus,although this system has been examined more with a view to establishing the theoretical properties,than to devising a practically useful system,it would appear that the method is worth consideration when bandwidth conservation is an important criterion.

CHAPTER 4

SPECTRUM SHAPING BY FINITE SUMMATION IN THE TIME DOMAIN.

4.1. Introduction.

In the previous chapter, the property of interest was the characteristic shape of the time domain function produced by summing a finite number of regularly spaced, discrete, frequency domain components. Because of the duality of the time and frequency domains, it may be anticipated that there will be an equivalent process involving the summation of regularly spaced, discrete, time domain components.

A finite summation of such time-discrete components may be regarded as the generation of periodically repeating groups of contiguous pulses, as shown, for example, in figure 4.3. A continuous pulse train would be equivalent to an infinite summation. The pulses may be idealised unit impulses, rectangular pulses, or of any arbitrary shape.

In this chapter, only finite-duration, rectangular pulses are considered. The summation will result in a specific spectral distribution, and it is the envelope of this distribution which is of interest. As is well known, pseudorandom, (or pseudonoise), pulse groups have useful properties in communication systems, and these sequences and their spectra have been the subject of considerable attention, (e.g. reference 47, ch.3). As will be seen, regular pulse groups not only have useful properties, but lend themselves to a simple analytical treatment.

Since this thesis is concerned with multiplex systems, the application to be described is related to multiplexing. A short report on this application has been published, and is included as an appendix to this chapter.

4.2. Principle of spectrum shaping.

The finite summation of a set of cosine harmonics was seen, in the previous chapter, to have a closed form function known as the periodic Dirichlet kernel. As might be expected, a similar function occurs when dealing with a finite summation of pulses.

Before considering these summations analytically, however, a simple qualitative understanding of the principle involved may be obtained with the help of figure 4.1. This shows, in figure 4.1.a., a portion from three continuous pulse trains, each being displaced by the same amount, and, as an example, the relative displacement is shown as T/12, i.e. 30° . Each pulse train will have a (sin x)/x spectral distribution, the modulus of which is shown as the heavy line in figure 4.1.c.

If these three pulse trains are superimposed, the spectral distribution of the composite will differ from that of any one pulse. In this case, the superposition results in a multilevel pulse train, but a similar effect occurs if each pulse remains separate in the composite.

With respect to the fundamental harmonic component of the first pulse train, the fundamentals of the second and third pulse trains are delayed by 30°, and 60°, respectively. Relative to the second harmonic of the first pulse train, the second harmonics of the other two are delayed by 60°, and 120°, respectively, and so on. As the harmonic number increases, the phase difference between corresponding harmonics is increased in proportion.

Figure 4.1.b. is an attempt to show this effect in a three-dimensional diagram. Taking the (sin x)/x spectral envelope of the first pulse train as reference, the envelopes of the other two are rotated in the 'phase plane'. The locii of the tips of the frequency vectors may be regarded as helices.

If a number of sinusoids of the same frequency, but different phase, are added, the result is another sinusoid of the same frequency. The amplitude and phase of the resultant are, of course, obtained from the vector sum of the components.

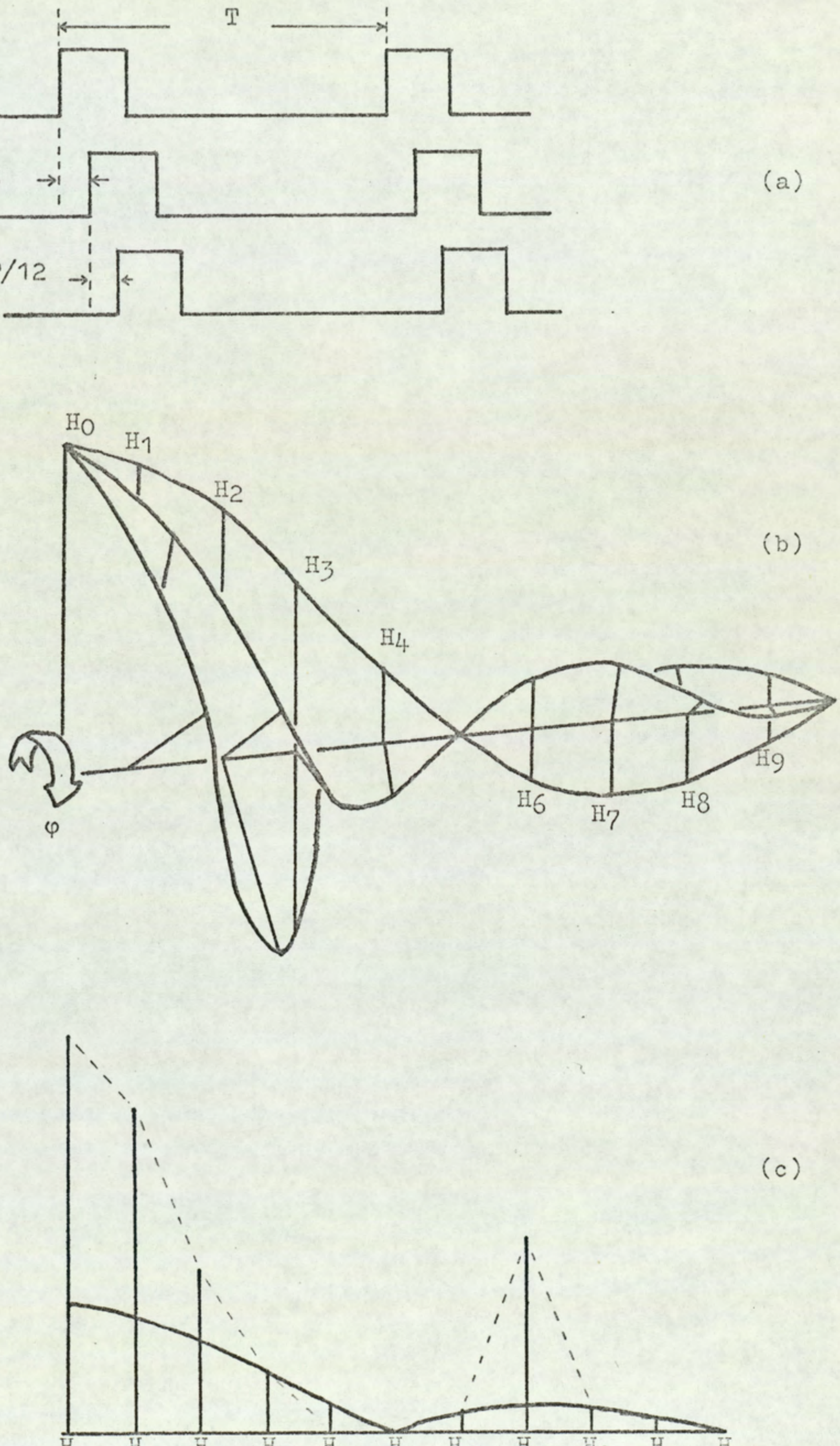

T
T/12
T/12
(a)
H_0
H_1
H_2
H_3
H_4
H_6
H_7
H_8
H_9
φ
(b)
(c)
H_0 H_1 H_2 H_3 H_4 H_5 H_6 H_7 H_8 H_9 H_{10}

Figure 4.1

Therefore, the spectral envelope of the pulse train obtained by superimposing the three pulse trains in figure 4.1.a. will be that resulting from the vector sum of corresponding harmonics. This is shown as the dotted envelope in figure 4.1.c. Generally, it is the amplitude of a harmonic component, rather than the phase, which is of importance, and figure 4.1.c. shows the modulus of the amplitudes in the envelope of the resultant.

It follows that, with an appropriate choice of pulse spacing, and number of pulses, the addition of a group of displaced, but otherwise identical, pulse trains will permit a predetermined shaping of the spectral envelope of any one pulse train.

The spectrum zeros are determined by the pulse duration, and cannot be removed, but additional zeros may be created by vector-summing to zero. At other points the envelope may be given an accentuated, or a diminished amplitude.

4.3. Fourier series of pulse train by superposition.

To derive an expression for the spectral envelope of superimposed pulse trains, one may first consider the simple case of a continuous pulse train formed by the superposition of a number of other pulse trains.

Thus, in figure 4.2. the Fourier series expansion of waveform 1 is

$$f(t) = \frac{Ad}{\tau} \sum_{n} \frac{\sin n\pi d/\tau}{n\pi d/\tau} e^{jn2\pi t/\tau}$$

Let waveform 1 be due to the superposition of p separate waveforms, 2a , 2b , 2c , . . . 2p .

For 2a, $$f(t) = \frac{Ad}{p\tau} \sum_{n} \frac{\sin n\pi d/p\tau}{n\pi d/p\tau} e^{jn2\pi t/p\tau}$$

for 2b, $$f(t) = \frac{Ad}{p\tau} \sum_{n} \frac{\sin n\pi d/p\tau}{n\pi d/p\tau} e^{jn2\pi(t-\tau)/p\tau}$$

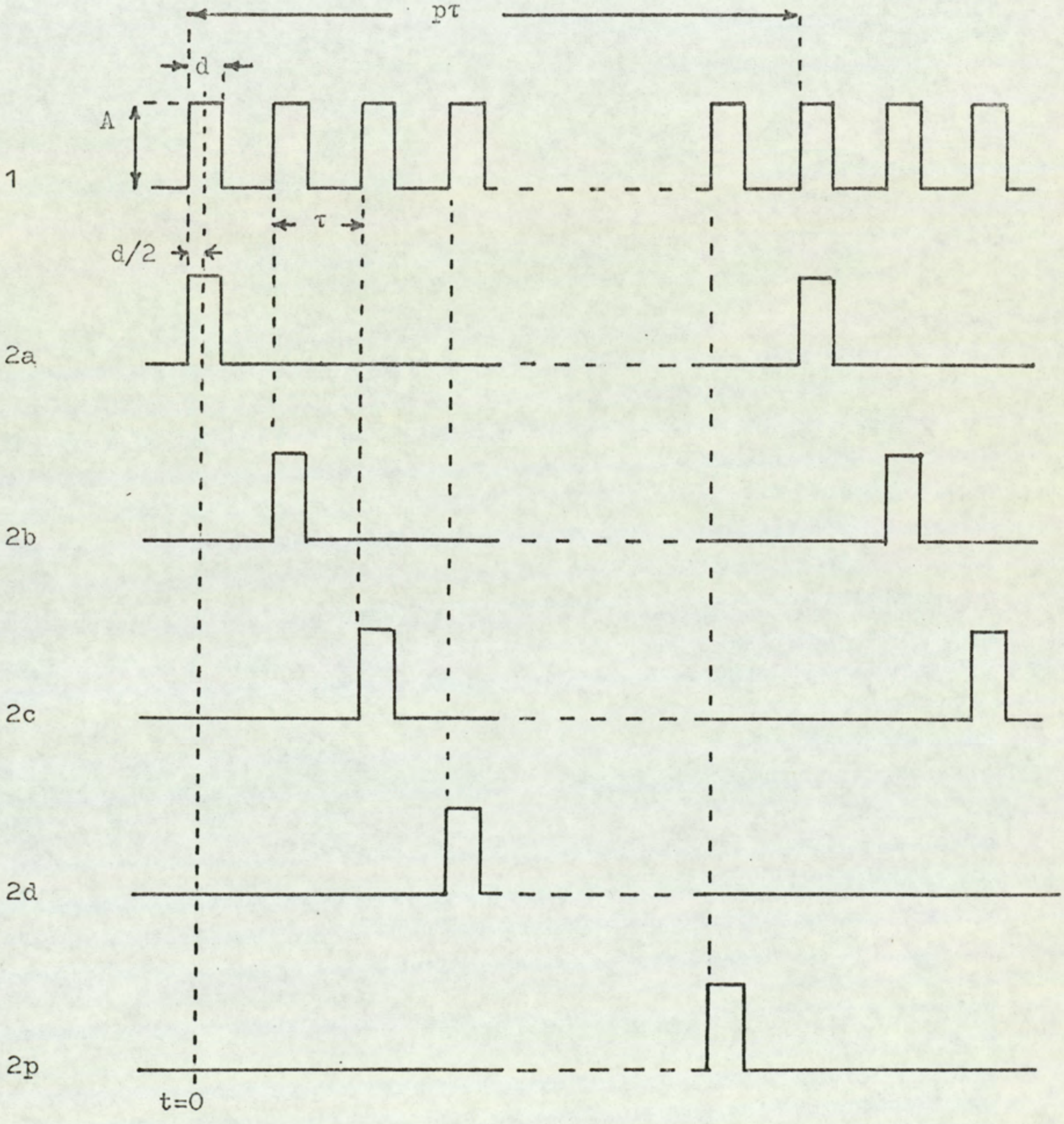


Figure 4.2

and for 2p , $f(t) = \frac{Ad}{p\tau} \sum_n \frac{\sin n\pi d/p\tau}{n\pi d/p\tau} e^{jn2\pi(t-(p-1))/p\tau}$

The summation of the waveforms, $\left[2a + 2b + \ldots 2p\right]$ is

$$\frac{Ad}{p\tau} \sum_n \frac{\sin n\pi d/p\tau}{n\pi d/p\tau} \left[1 + e^{-jn2\pi\tau/p\tau} + e^{-jn2\pi\, 2\tau/p\tau} \ldots \ldots e^{-jn2\pi(p-1)\tau/p\tau}\right] e^{jn2\pi t/p\tau}$$

in which the time displacement terms

$$\left[1 + e^{-jn2\pi/p} + e^{-jn4\pi/p} \ldots + e^{-jn2\pi(p-1)/p}\right]$$

sum to $\frac{1 - e^{-jn2\pi}}{1 - e^{-jn2\pi/p}} = \frac{\sin n\pi}{\sin n\pi/p} e^{-jn\pi(1-\frac{1}{p})}$

$\frac{\sin n\pi}{\sin n\pi/p} = 0$, unless $n = qp$, where q is an integer,

substituting $n = qp$, $\frac{\sin n\pi}{\sin n\pi/p} = \frac{\sin p(q\pi)}{\sin (q\pi)}$

$$\mathrm{Lt}_{x \to q\pi} \frac{\sin px}{\sin x} = \mathrm{Lt}_{x \to q\pi} \frac{p \cos px}{\cos x} = \frac{p \cos pq\pi}{\cos q\pi}$$

$$\frac{p \cos pq\pi}{\cos q\pi} = \begin{cases} +p & q \text{ even or } q \text{ odd, and } p \text{ odd} \\ -p & q \text{ odd, and } p \text{ even} \end{cases}$$

substituting $n = qp$ into $e^{-jn\pi(1-\frac{1}{p})}$ gives $e^{-jq(p-1)\pi}$

$e^{-jq(p-1)\pi} = \cos q(p-1)\pi$ since q and p are integers, and

$$\cos q(p-1)\pi = \begin{cases} +1 & q \text{ even or } q \text{ odd, and } p \text{ odd} \\ -1 & q \text{ odd, and } p \text{ even} \end{cases}$$

Therefore, $\frac{\sin n\pi}{\sin n\pi/p}$ and $e^{-jn\pi(1-\frac{1}{p})}$ take the same sign, and

$$\frac{\sin n\pi}{\sin n\pi/p}\ e^{-jn\pi(1-\frac{1}{p})} = \begin{cases} 0, & n \neq qp \\ +p, & n = qp \end{cases}$$

Hence, the summation of the waveforms (2a + 2b + . . 2p) is

$$\frac{Ad}{p\tau} \sum_{n=qp} \frac{\sin n\pi d/p\tau}{n\pi d/p\tau}\ p\ e^{jn2\pi t/p\tau}$$

$$= \frac{Ad}{\tau} \sum_{q} \frac{\sin q\pi d/\tau}{q\pi d/\tau}\ e^{jq2\pi t/\tau}$$

which is identical to the Fourier series expansion of waveform 1. This result is to be expected, since the superposition of p pulse trains regularly spaced over the period is indistinguishable from a pulse train at p times the original frequency. However, if the first m waveforms only are summed, i.e. 2a + 2b +. . 2m, then

$$\sum_{m} f(t) = \frac{Ad}{p\tau} \sum_{n} \frac{\sin n\pi d/p\tau}{n\pi d/p\tau} \left[1 + e^{-jn2\pi/p} + e^{-jn4\pi/p} \ldots + e^{-jn2\pi(m-1)/p} \right] e^{jn2\pi t/p\tau}$$

and the time displacement terms sum to

$$\frac{\sin n\pi m/p}{\sin n\pi/p}\ e^{-jn\pi(m-1)/p}$$

Each of the waveforms 2a, 2b, . . 2m, has a period = $p\tau$

Substituting $T = p\tau$ into the expression for the summation,

$$\sum_{m} f(t) = \frac{Ad}{T} \sum_{n} \frac{\sin n\pi d/T}{n\pi d/T}\ \frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T}\ e^{-jn\pi(m-1)\tau/T}\ e^{jn2\pi t/T}$$

Thus, the Fourier series of the pulse sequence formed by taking the partial summation is

$$f(t) = \frac{Ad}{T} \sum_{n} \frac{\sin n\pi d/T}{n\pi d/T} e^{jn\omega_o t} \left[\frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T} e^{-jn\omega_o(m-1)\tau/2} \right]$$

$$(\text{where } \omega_o = 2\pi/T)$$

which is the Fourier series for one of the constituent waveforms, but modified by the weighting function

$$\frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

and with the time origin of the harmonics displaced to the centre of the pulse sequence.

4.4. Properties of the weighting function.

Expression (1), above, will be designated the weighting function, and abbreviated to w.f. It will be recalled that in chapter 3, sec. 3.5, the finite summation of different sets of sinusoids were tabulated and plotted. The summation of a finite set of cosine harmonics was

$$\sum_{n=1}^{m} \cos n\theta = \frac{\sin(2m+1)(\theta/2)}{2\sin(\theta/2)} - \frac{1}{2}$$

and this function, with different constants, appears when considering the finite sum of exponential harmonics. As was seen in chapter 3, sec. 3.3.b, the function is then known as the periodic

Dirichlet kernel, $D_n(t)$.

$$D_n(t) = \frac{1}{T} \sum_{n=-m}^{m} e^{jn\omega_0 t} = \frac{1}{T} \frac{\sin[(2m+1)(\omega_0 t/2)]}{\sin(\omega_0 t/2)}$$

Also, in chapter 3, section 3.5, the finite sum of a set of alternate polarity odd sine harmonics emerged as a function with useful properties, and was there designated $S_n(t)$.

$$S_n(t) = \sum_{n=1}^{m} (-1)^n \sin(2n-1)\theta = (-1)^m \left[\frac{\sin(2m\theta)}{2\cos\theta}\right]$$

and, apart from a constant phase displacement, this function is the same as the finite summation of odd cosine harmonics,

$$\sum_{n=1}^{m} \cos(2n-1)\theta = \frac{\sin(2m\theta)}{2\sin\theta}$$

Writing the weighting function as

$$\text{w.f.} = \frac{\sin m\theta}{\sin\theta} \quad (\text{where } \theta = n\pi\tau/T)$$

the similarity to the above finite summations is evident. Although the weighting function is not itself expressible as a simple finite summation of first - order sinusoids, it will be seen that if m is even, the envelope varies in the same manner as $S_n(t)$, and if m is odd, the variation is similar, in some respects, to that of $D_n(t)$. As the variable is $\theta/2$ in the latter function, the weighting function has two peak values in each period, compared to one with $D_n(t)$.

It follows that the $(\sin x)/x$ pattern of the basic pulse-train harmonic amplitudes undergoes a cyclic modification from the weighting function. Of course, the pulses need not necessarily have a (sin x)/x distribution, since the Fourier coefficients do not enter into the derivation of the weighting function.

This particular function has been discussed previously in the literature, (reference 48), where the authors use the term 'repetition function', and describe a graphical method for using this function to determine the harmonic amplitudes of regular pulse sequences.

When dealing with the finite summation expressions, it was the zero-crossing behaviour which was of primary interest. With a weighting function, however, the envelope amplitude over the whole range is of interest, and some limiting cases will be determined, as follows.

(1) If m = 1, then w.f. = 1, and the pulse sequence spectrum is obviously that for a regular pulse train, of period T, and pulse duration d.

(2) If $m = \frac{T}{\tau}$, then $\text{w.f.} = \frac{\sin n\pi}{\sin n\pi/m}$

$$\frac{\sin n\pi}{\sin n\pi/m} = \begin{cases} \pm m, & n = km, \text{ (where k is an integer)} \\ 0, & n \neq km \end{cases}$$

$$\text{Lt}_{\theta \to \pi} \frac{\sin km\theta}{\sin k\theta} = \text{Lt}_{\theta \to \pi} \frac{km \cos km\theta}{k \cos k\theta} = \begin{cases} -m, & \text{k odd and m even} \\ +m, & \text{otherwise} \end{cases}$$

for $m = T/\tau$, the pulse sequence spectrum is, therefore,

$$f(t) = \frac{Ad}{T} \sum_{n=km} \frac{\sin n\pi d/T}{n\pi d/T} (\pm m)\, e^{jn\omega_o(t-(m-1)\frac{\tau}{2})}$$

$$= \frac{Ad}{m\tau} \sum_{km} \frac{\sin k\pi d/T}{k\pi d/T} (\pm m)\, e^{-jk(m-1)\pi}\, e^{jk2\pi t/\tau}$$

since

$$e^{-jk(m-1)\pi} = \begin{cases} -1 \text{ , k odd and m even} \\ +1 \text{ , otherwise} \end{cases}$$

it follows that m will always occur with a positive sign in f(t), so that

$$f(t) = \frac{Ad}{\tau} \sum_{k} \frac{\sin k\pi d/\tau}{k\pi d/\tau} \; e^{jk2\pi t/\tau}$$

which is,of course,the spectrum of a regular pulse train,period τ, pulse duration d, as shown in figure 4.3.a.

Although this treatment is limited to pulse sequences having periods which are integer multiples of τ,(the spacing between pulses), the choice of m,(the number of pulses),and of T, (the sequence period),is otherwise arbitrary. For example,one might choose $m = 3$ and $T = 7\tau$,as in figure 4.3.b,so that $m\tau/T = 3/7$.

A special case,which is of interest,occurs when $m\tau/T = \frac{1}{2}$, i.e. a 'half-length' pulse sequence,as in figure 4.3.c. Thus,

(3) if $m = T/2\tau$, then w.f. $= \dfrac{\sin n\pi/2}{\sin n\pi/2m}$

$$\frac{\sin n\pi/2}{\sin n\pi/2m} = \begin{cases} \pm m, & n = 2km \;(= kT/\tau) \\ 0, & n \text{ even, but } n \neq 2km \\ \dfrac{\pm 1}{\sin n\pi/2m}, & n \text{ odd} \end{cases}$$

(where k is any integer)

from which it is seen that even-numbered harmonics are eliminated from the pulse sequence spectrum,except when these are multiples of T/τ.

As an extension to this case,one may consider the bipolar sequence in figure 4.3.d,where groups of m pulses alternate in polarity. Since the negative-going group is displaced from the positive-going group by half the period,the bipolar waveform is

$$f(t) = \sum_{n} c_n \, e^{jn\omega_o t} \, \frac{\sin n\pi/2}{\sin n\pi/2m} \, e^{-jn\omega_o(m-1)\tau/2} \, (1 - e^{-jn\omega_o T/2})$$

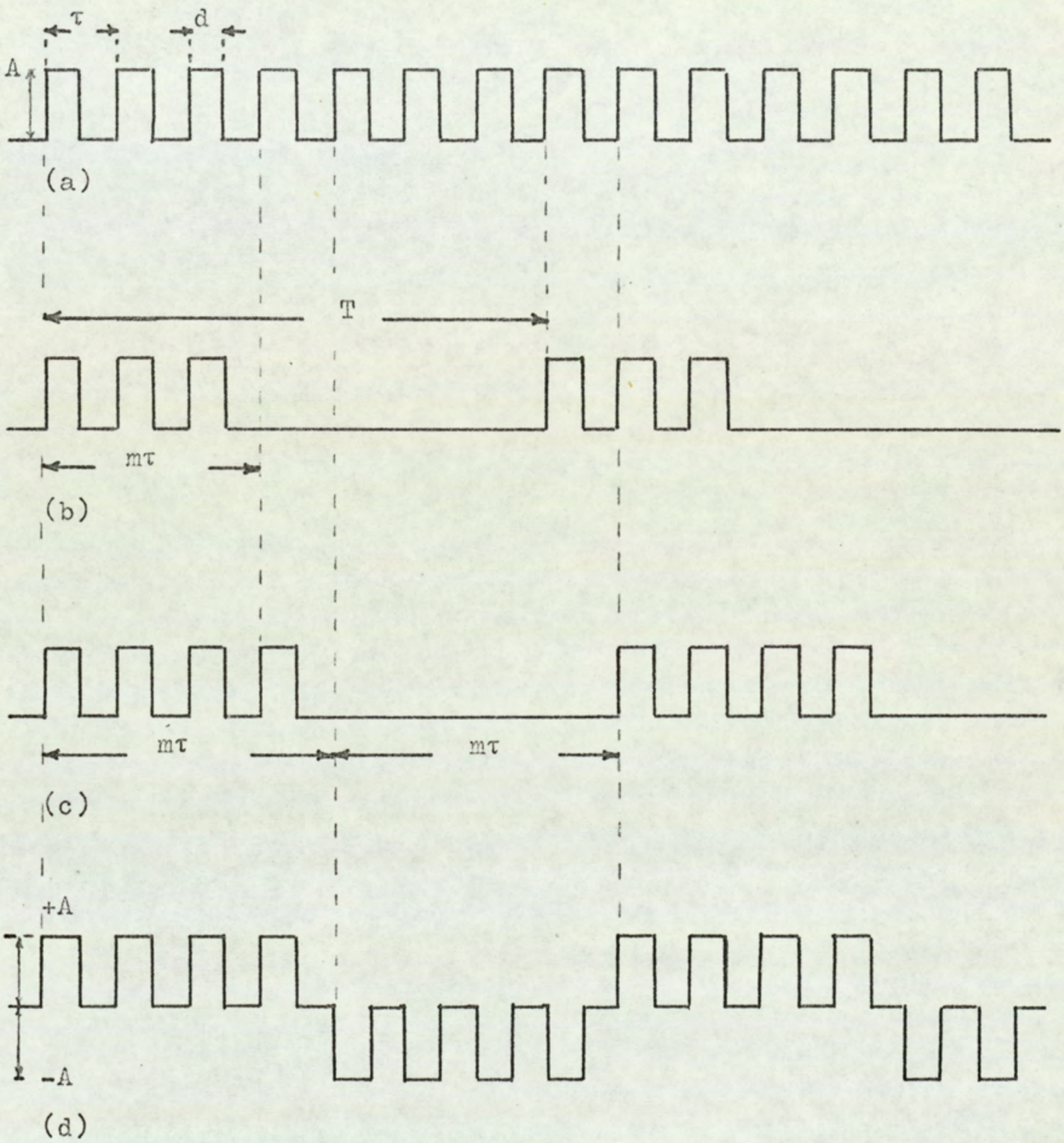


Figure 4.3

and since

$$(1 - e^{-jn\omega_o T/2}) = (1 - e^{-jn\pi}) = \begin{cases} 2 , & n \text{ odd} \\ 0 , & n \text{ even} \end{cases}$$

the half-length bipolar sequence retains only odd harmonics, as with the half-length unipolar sequence, but has doubled amplitudes. Also, the even harmonics at multiples of T/τ will disappear in the bipolar sequence.

(4) Range of values taken by the weighting function.

It may be seen from inspection that $\frac{\sin m\theta}{\sin \theta}$ is an even function about π; and that it is also an even function about $\pi/2$ if m is odd, and an odd function about $\pi/2$ if m is even. Hence, the principal values occur in the range $0 \leqslant \theta \leqslant \pi$.

$$\text{If } \theta = 0 \text{, then } \frac{\sin m\theta}{\sin \theta} = +m$$

$$\text{If } \theta = \pi \text{, then } \frac{\sin m\theta}{\sin \theta} = \begin{cases} +m & \text{(if m is odd)} \\ -m & \text{(if m is even)} \end{cases}$$

There are evidently (m-1) zero-crossings, and (m-2) turning points in the range $0 < \theta < \pi$, and this may also be seen by using a standard expansion, (reference 46),

$$\frac{\sin m\theta}{\sin \theta} = (2\cos\theta)^{m-1} + \sum_{k=1}^{k \leqslant \frac{m-1}{2}} (-1)^k \left\{ (m-k-1)(m-k-2)\ldots \ldots(m-2k)\frac{(2\cos\theta)^{m-2k-1}}{k!} \right\}$$

$$= a_o + a_1\cos\theta + a_2\cos^2\theta + \ldots\ldots a_{m-1}\cos^{m-1}\theta$$

Since the cosine power series is of order (m-1), there are (m-1) roots of the expression, i.e. the function has (m-1) zeros in the range $0 < \theta < \pi$. Since the differential of the series is of order (m-2), there are (m-2) turning points in the range $0 < \theta < \pi$,

i.e. $f'(\cos\theta) = (-\sin\theta) \left[a_{m-1}(m-1)\cos^{m-2}\theta + a_{m-2}(m-2)\cos^{m-3}\theta + \ldots \right]$

which has roots for (m-2) values of $\cos\theta$, and also for $\theta = 0$ and π.

When $\theta = 0$ or π, the modulus of the weighting function was seen to be

$$\left|\frac{\sin m\theta}{\sin \theta}\right| = m$$

To establish that the weighting function does not exceed this value in the range $0 < \theta < \pi$, one may consider that,

(1): the maximum value of $|\sin m\theta|$ is unity, when $m\theta = \pi/2$;

(2): the minimum value of $|\sin \theta|$, (where $\theta = n\pi\tau/T \neq 0$), for a given value of τ/T, occurs when $n = 1$, and when $n = \frac{T}{\tau} - 1$, since n takes only integer values.

If $n = 1$, $\sin \theta = \sin(\pi\tau/T)$.

If $n = \frac{T}{\tau} - 1$, then
$$\begin{aligned}\sin\theta &= \sin(\tfrac{T}{\tau} - 1)(\pi\tfrac{\tau}{T}) = \sin(\pi - \pi\tfrac{\tau}{T}) \\ &= \sin\pi \cos\pi\tfrac{\tau}{T} - \cos\pi \sin\pi\tfrac{\tau}{T} \\ &= \sin(\pi\tau/T)\end{aligned}$$

Therefore,

$$\left|\frac{\sin m\theta}{\sin \theta}\right|_{max} = \frac{1}{\sin(\pi\tau/T)}$$

For the pulse sequence to exist, the minimum value of $\frac{T}{\tau} = 2$, in which case,

$$\frac{1}{\sin(\pi\tau/T)} = \frac{1}{\sin(\pi/2)} = 1$$

As $\frac{T}{\tau} \to \infty$, $\frac{1}{\sin(\pi\tau/T)} \to \frac{1}{\pi\tau/T} = \frac{T}{\pi\tau}$

so that the greatest value which can be taken by $\left|\frac{\sin m\theta}{\sin \theta}\right|_{max} = \frac{T}{\pi\tau}$

If $m\theta = \pi/2$, and $n = 1$,

then $m\theta = mn\pi\tau/T = m\pi\tau/T = \pi/2$, and $m = \frac{T}{2\tau}$,

but since $\frac{T}{\pi\tau} < \frac{T}{2\tau}$, therefore $\left|\frac{\sin m\theta}{\sin \theta}\right|_{max} < m$.

If $m\theta \neq \pi/2$, but $\theta = \theta_{min} = \pi\tau/T$,

then $\frac{\sin m\theta}{\sin \theta} = \frac{\sin m\pi\tau/T}{\sin \pi\tau/T} < \frac{1}{\sin \pi\tau/T}$, (since $m\pi\tau/T < \pi/2$).

If $m\theta = \pi/2$, but $\theta = n\pi\tau/T \neq \theta_{min}$,

$$\text{then} \quad \frac{\sin m\theta}{\sin \theta} = \frac{1}{\sin n\pi\tau/T} < \frac{1}{\sin \pi\tau/T}$$

(since $n\pi\tau/T > \pi\tau/T$)

Therefore, in the range $0 < \theta < \pi$, the greatest value of the weighting function is always less than m . In this range, the greatest values will occur for those harmonic numbers which are immediately adjacent to the peak values at $n = 0$, and $n = T/\tau$, i.e. at $n = 1$, and $n = \frac{T}{\tau} - 1$.

The weighting function was evaluated over the range $0 \leq \theta \leq 2\pi$, for several values of m, and the results are shown in figures 4.4, 4.5, and 4.6. The significant parameter is $m\tau/T$, the ratio of the actual number of pulses in the sequence, m, to the maximum possible number of pulses, T/τ. Clearly, as this ratio approaches unity, the weighting function is such that harmonics which are not multiples of T/τ are increasingly suppressed.

The effect of the weighting function may be envisaged by postulating that a typical curve from figure 4.4, 4.5, or 4.6, be superimposed on the spectral envelope of the basic pulse train. The peaks at $\theta = 0, \pi, 2\pi, 3\pi, \ldots$ coincide with harmonic numbers $n = 0, T/\tau, 2T/\tau, 3T/\tau, \ldots$ The spectrum of the pulse sequence is obtained from the product of the weighting function and the basic envelope, at each harmonic number. A practical application for this effect will be considered in the remainder of this chapter.

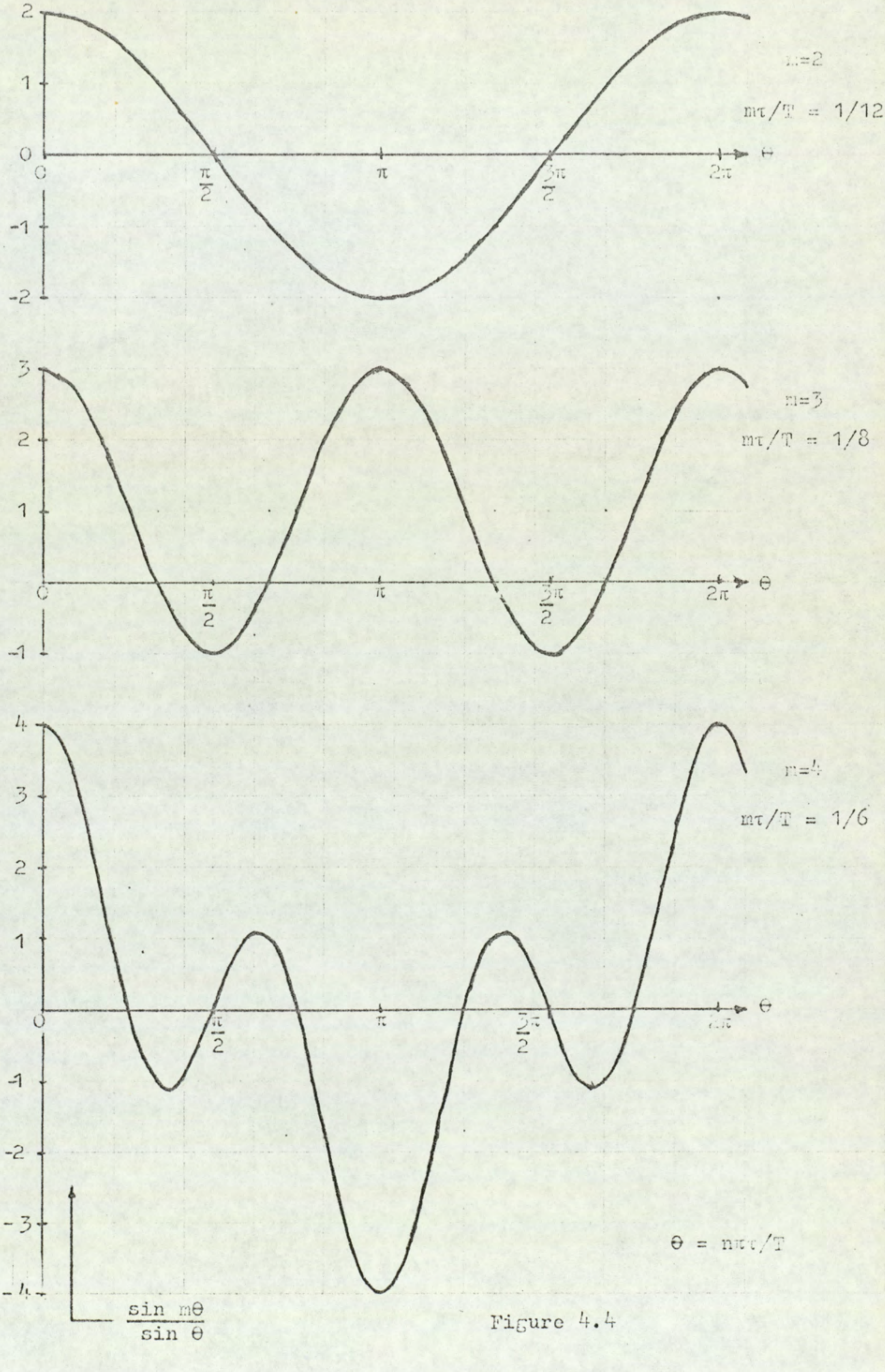
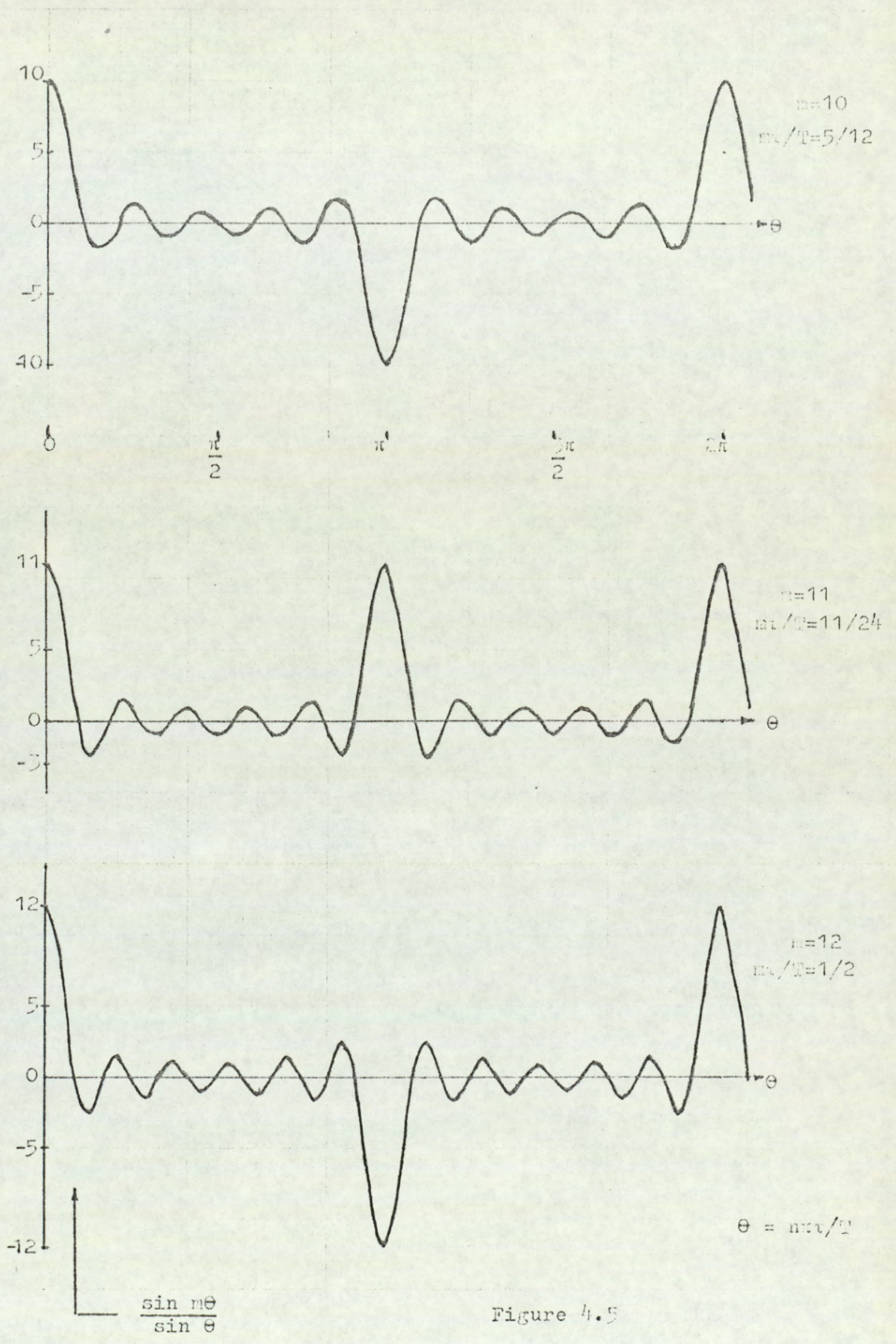


Figure 4.4



10
5
0
-5
-10
m=10
mτ/T=5/12
θ
0
π/2
π
3π/2
2π
11
5
0
-5
m=11
mτ/T=11/24
θ
12
5
0
-5
-12
m=12
mτ/T=1/2
θ
θ = nπτ/T
sin mθ / sin θ

Figure 4.5

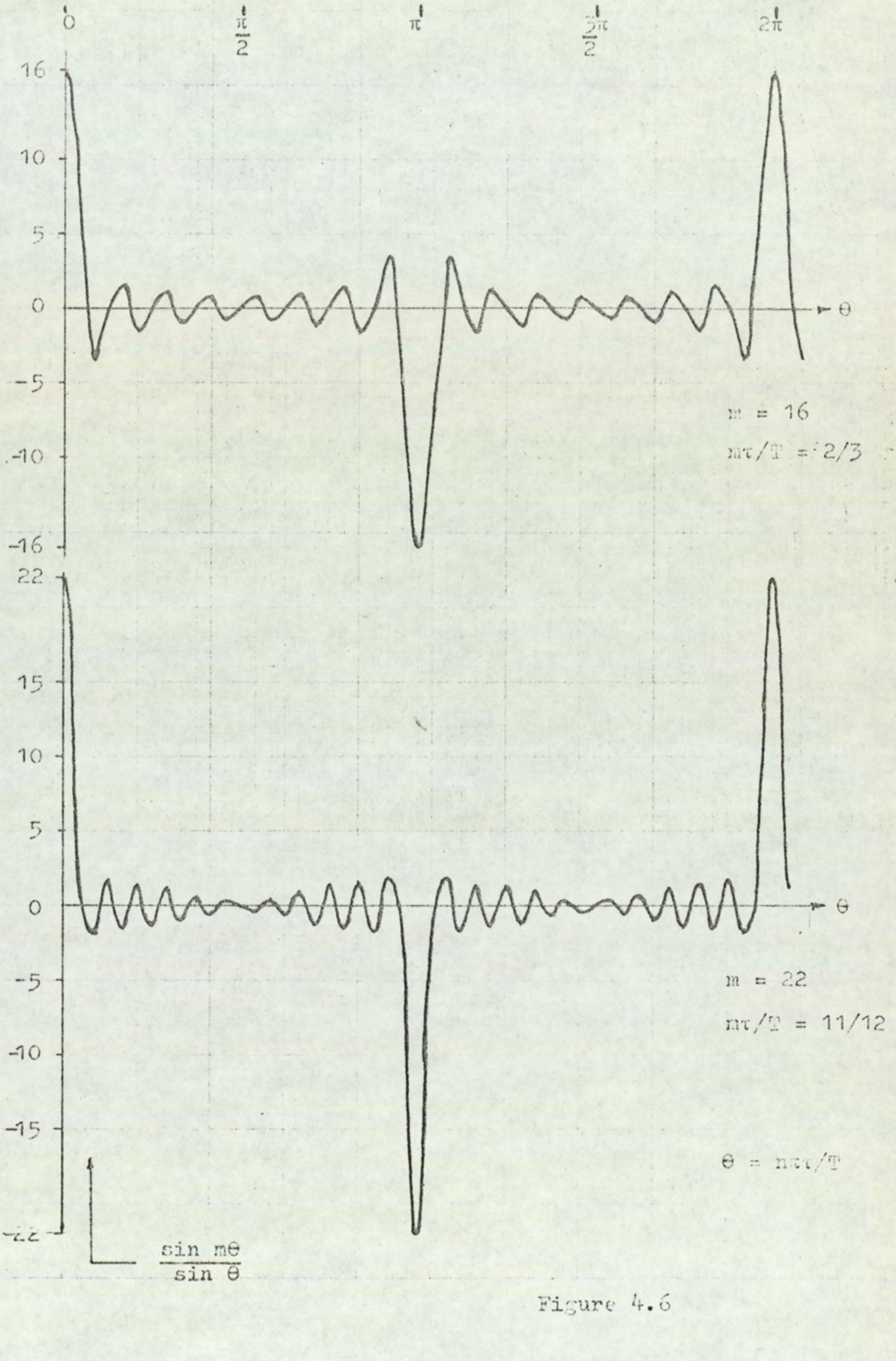

0
π/2
π
3π/2
2π
16
10
5
0
-5
-10
-16
θ
m = 16
mτ/T = 2/3
22
15
10
5
0
-5
-10
-15
-22
θ
m = 22
mτ/T = 11/12
θ = nπτ/T
sin mθ / sin θ

Figure 4.6

4.5. Application of pulse sequences to carrier frequency generation.

A frequency-division-multiplex system requires a set of carrier frequencies which are, in most cases, adjacent harmonics of some base frequency. It is evident that these harmonics may be obtained from a pulse-train at the base frequency. Since this base frequency may be low, frequency division from a stable, higher frequency, source is often necessary. Such a system, using digital frequency dividers, and transistor pulse generators, has been described in reference 49.

Although this method is an improvement upon earlier systems, using regenerative frequency division and saturating inductors, there remain some practical difficulties. If the required harmonics are of a high order, the pulse duration must be restricted so as to generate these harmonics with significant amplitude. This means that the duty ratio will be low, so that the pulse generator must operate with a high peak value of voltage, or current, in order to generate the harmonics at usable levels.

Greater efficiency could be obtained with a more continuous loading of the output stage. This can be achieved by replacing each recurrent, narrow pulse by a group, or sequence, of pulses. The technique has been discussed in reference 50, where the proposed method involved the use of pseudorandom pulse sequences.

Such sequences can be generated with feedback shift registers and digital circuit elements, and their properties have been discussed by many authors, (e.g. reference 47). Of course, pseudorandom sequences are periodically repeating groups of finite duration pulses. Consequently, they cannot have the uniform amplitude spectra associated with 'white' noise. For the purpose of generating a set of carriers, it is a uniform amplitude property which would be valuable, rather than minimum out-of-phase autocorrelation.

Pseudorandom sequences of rectangular pulses may, in fact, have a spectral distribution which differs only slightly from the (sin x)/x distribution of a regular, rectangular pulse train. Various methods have been described for determining the spectra of pseudorandom, or other irregular periodic pulse sequences, (reference

51, and reference 52), but, in general, these require a relatively lengthy analysis.

The regular pulse sequences described earlier in this chapter can be generated in the same way as pseudorandom sequences, and, as was seen, the spectrum of a sequence can be predicted from a simple analytical form. Thus, it would appear to be not inappropriate to consider the generation of a set of fdm carrier frequencies by the use of regular pulse sequences.

To generate these sequences requires that a regular pulse train be available as a source frequency. Then, either shift registers, or counters, combined with other digital circuit elements may be used to form the pattern. For this application, shift registers require slightly more circuitry, so that the circuits to be described all make use of counters.

For example, if the source frequency has a prf of $2\pi/\tau$, then a counter dividing by a ratio T/τ will produce a square wave at a frequency of $2\pi/T$. This square wave may be used to gate the source, i.e. input waveform, to yield the sequence shown in figure 4.7.a. The number of pulses, m, is there equal to $T/2\tau$, i.e. a half-length sequence. For other values of m, gates with three, or more, inputs may be used, or cascaded two-input gates. An example of a circuit for generating a two-pulse sequence is shown in figure 4.7.b.

By these methods, the input pulse train, with harmonics at $n(2\pi)/\tau$, is converted into a periodic signal having harmonics at $n(2\pi)/T$. The amplitudes of these harmonics will differ from those of a regular pulse train with frequency $2\pi/T$, in a manner depending on the value of the weighting function,

$$\frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T}$$

and, hence, on the value of m.

The effect upon the basic pulse spectrum is shown in figure 4.8. These spectra were obtained by applying the appropriate pulse train to a sweep-frequency spectrum analyser, and recording the output on an X-Y plotter, as indicated in figure 4.9.

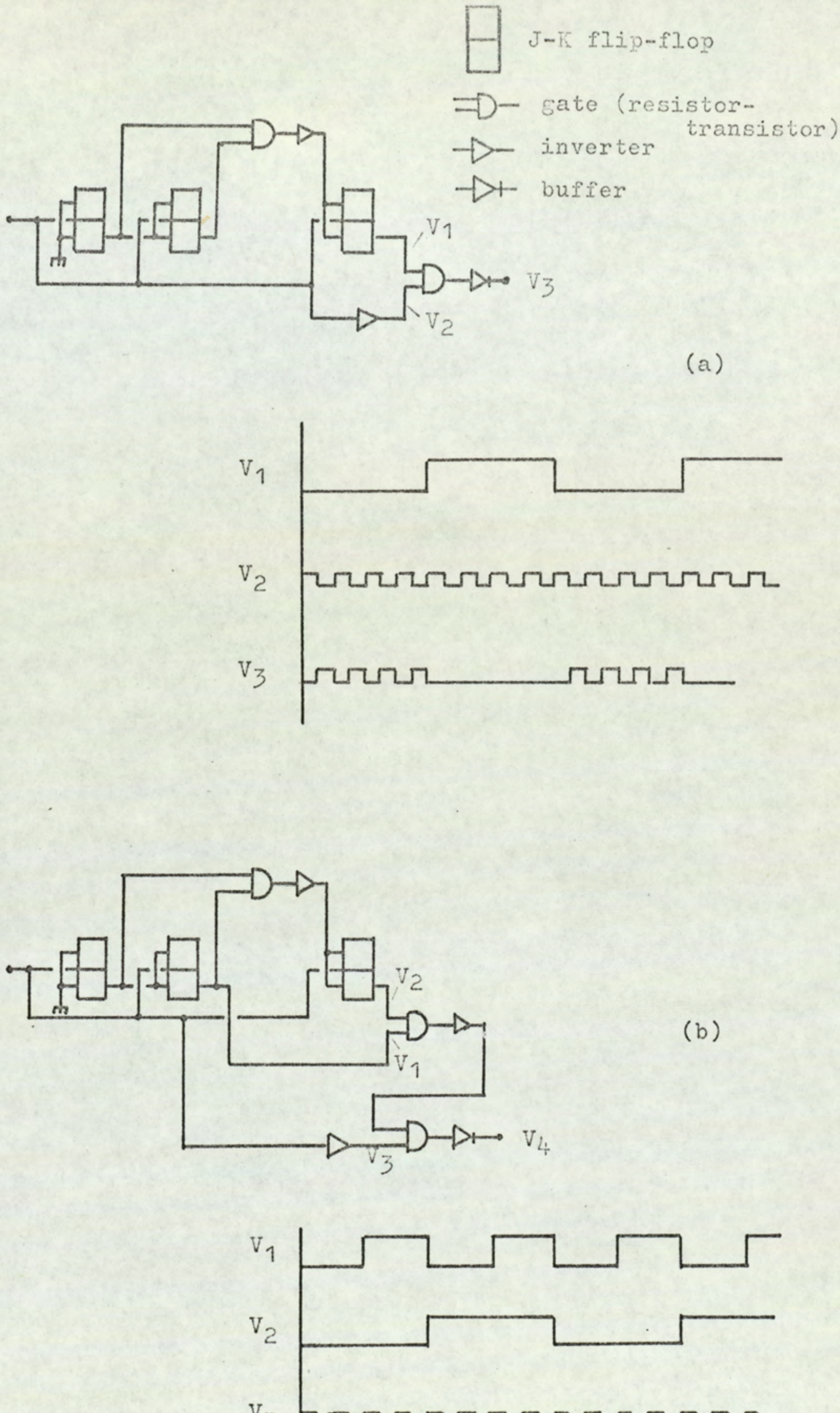

J-K flip-flop
gate (resistor-transistor)
inverter
buffer
V1
V2
V3
(a)
V1
V2
V3
V2
V1
V3
V4
(b)
V1
V2
V3
V4

Figure 4.7

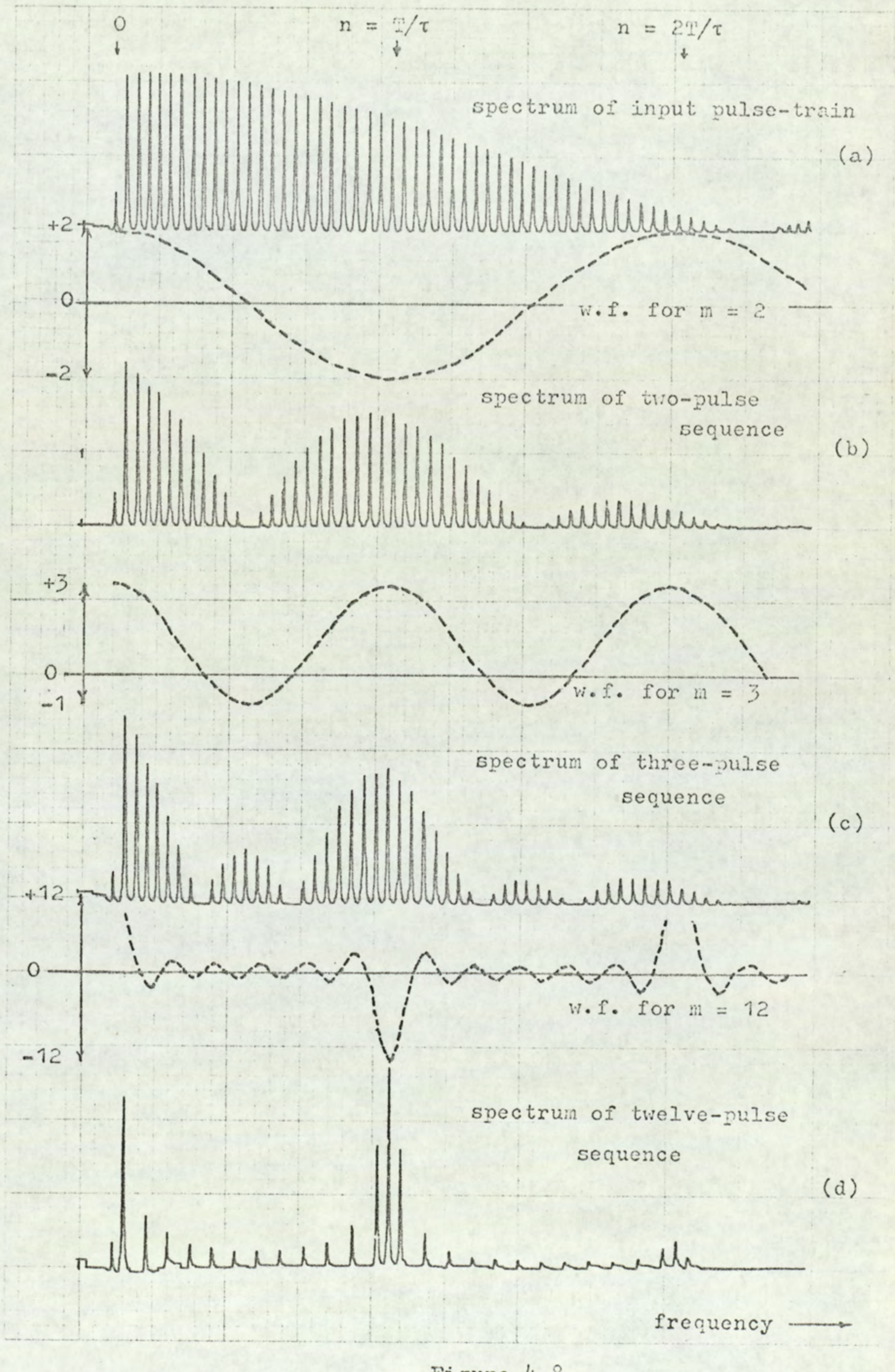
Figure 4.8.a. shows the spectrum, up to approximately the first zero, of the source pulse train. Because the spectrum analyser displayed only the magnitude of each harmonic, the sign reversals, where applicable, are not shown.

Figure 4.8.b. shows the spectrum of a two-pulse sequence obtained from a counter dividing by 24. Above the spectrum is displayed the weighting function appropriate to $m = 2$, and it will be seen that the spectrum in figure 4.8.b. is the result of multiplying the spectrum in figure 4.8.a. by this weighting function.

In figure 4.8.c. the same counter was used to generate a three-pulse sequence, and the weighting function for $m = 3$ is seen to produce the spectrum as shown.

Finally, the counter was used to form a twelve-pulse sequence, (a half-length sequence, since $T/\tau = 24$). The spectrum in figure 4.8.d. is seen to be the basic spectrum of figure 4.8.a. modified by the weighting function for $m = 12$.

The formation of simple, unipolar, sequences does not exhaust the possibilities for carrier -frequency generation. However, in order to relate this application to an actual system, the various elaborations which are possible will be included in the subsequent section on the experimental work.



0
n = T/τ
n = 2T/τ
spectrum of input pulse-train
(a)
+2
0
w.f. for m = 2
-2
spectrum of two-pulse sequence
(b)
+3
0
-1
w.f. for m = 3
spectrum of three-pulse sequence
(c)
+12
0
w.f. for m = 12
-12
spectrum of twelve-pulse sequence
(d)
frequency

Figure 4.8

4.6. Experimental work.

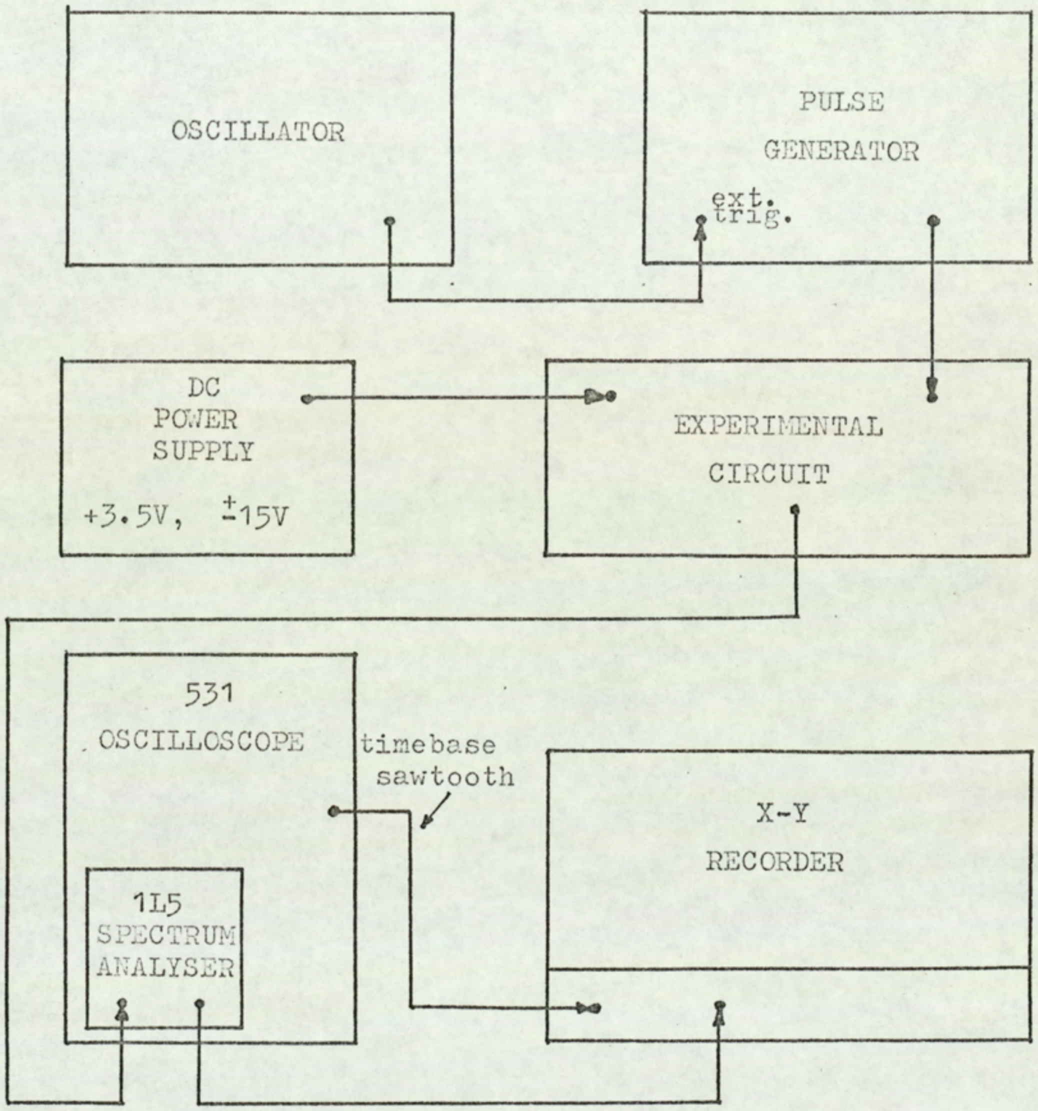
One important example of a frequency-division-multiplex system is the carrier-telephony standard group, which uses a set of twelve carrier frequencies at 4kHz intervals, from 64kHz to 108kHz. The generation of the carriers for this system, using various methods involving regular pulse sequences, was the object of the experimental work. The spectra shown for each experimental system are direct reproductions of a spectrum analyser output.

The arrangement of the apparatus is shown in figure 4.9. A stable laboratory oscillator was used as the frequency source, which was converted to a rectangular pulse-train by means of a laboratory pulse generator. This pulse-train was applied to an experimental sequence generator, constructed from integrated-circuit digital elements, such as flip-flops, gates, and inverters.

The integrated-circuit digital elements were from the MOTOROLA MC700P MRTL range. High-speed switching was not required, and the only criterion was that this range happened to be the lowest-priced available at the time.

The pulse sequence was then applied to a spectrum analyser. This was a TEKTRONIX, Type 1L5, plug-in unit, which was used in conjunction with a TEKTRONIX, Type 531, oscilloscope main frame. The 1L5 spectrum analyser incorporates a variable-frequency oscillator, which is driven from the oscilloscope time-base. The oscillator output is hetrodyned with the input signal, and applied to a variable-bandwidth bandpass filter. The filter output is displayed on the oscilloscope screen, and is also available as a varying dc level to drive an external recorder.

An X-Y recorder was used to plot the spectra, the x-axis being driven by the oscilloscope time-base. To minimise errors in the analyser and recorder response, the sweep-rate was set to the slowest obtainable, using both pre-set and variable controls at their lowest setting. This gave a sweep-rate of approximately 1 cm/sec. Also, the spectrum analyser resolution, i.e. bandwidth, was set to 200Hz, which allowed the recorder to make a gradual transition into each spectral line.



Apparatus for recording experimental spectra.

Figure 4.9

As previously mentioned, the recorded spectra show only the modulus of each component, and any phase reversals are not displayed. Either a linear, or a logarithmic response could be selected, but the spectra shown in this chapter are all to a linear scale. The amplitude scale is, in general, arbitrary, and intended to show only the relative magnitude of each spectral component.

The source frequency must be an integer multiple of 4kHz, in order to generate a sequence at a 4kHz rate. The value chosen for this multiple, which is T/τ, depends on the relative weighting which may be required. The value of m determines the characteristic shape of the weighting function, but the first peak will always occur at zero frequency, and the second peak at T/τ times the base frequency. Thus, the choice of T/τ determines the relative stretching, or compression, of the weighting function with respect to the basic pulse-train spectrum.

It is convenient if the source frequency corresponds to one of the desired carrier frequencies. As discussed later, it is also convenient if the source frequency has multiple factors. In the experimental systems, 96kHz was chosen, since this fulfilled the first two conditions, and would also be available if the communications terminal handled both fdm and tdm systems.

4.6.a. Carrier generation with 4kHz separation.

When using a normal pulse-train to generate the harmonics, the bandpass filters must discriminate against components at 4kHz intervals. The simplest approach, using a pulse sequence, does not alter the harmonic interval, but will increase the amplitudes of the desired harmonics.

The 96kHz input is divided by 24, which generates a set of harmonics of 4kHz. The desired carrier frequencies are provided by the 16th to the 27th harmonics of the pulse sequence, the 24th harmonic being the same as the input frequency. Therefore, the weighting function takes a value of m for the 96kHz component, since $n\tau/T = 24.1/24$, and

$$\frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T} = \frac{\sin m\pi}{\sin \pi} = m$$

The weighting function has a modulus of unity for integer multiples of the harmonic number given by

$$(n\pi m\tau/T) \quad = \quad \pi - (n\pi\tau/T)$$

$$\text{i.e.} \quad n = T/(m + 1)\tau$$

and is zero for integer multiples of harmonic numbers given by

$$n = T/m\tau$$

The specific value of m which is used must be such that all the desired harmonics are present,and,preferably,each should receive the greatest possible weighting. Clearly,it is desirable to avoid a value such that the weighting would become less than unity for any harmonic within the desired range.

Substituting $m = 2$ into the weighting function gives a minimum in-range value of unity for the 16th harmonic,(64kHz). If $m = 3$, the weighting function is unity for the 18th harmonic, but is zero for the 16th. Higher values of m result in the attenuation,or elimination,of other harmonics in the desired range. Hence,a two-pulse sequence is the only useful choice.

It is assumed,of course,that the desired harmonics are present in the basic pulse-train spectrum. However,the latter is that of a 4kHz pulse-train obtained by taking every 24th pulse of a 96kHz pulse-train,without change in the pulse duration. It is evident that the duty ratio of the 4kHz pulse-train will be low, so that the first spectrum zero will be at a relatively high harmonic number.

The 96kHz source frequency may be generated as a square wave. In that case,the pulse duration would be 5.2 microseconds, and the 4kHz basic pulse-train,with a period of 250 microseconds, would have a duty ratio,(d/T),of 1/48. Hence,the first spectrum zero would occur at 192kHz. Assuming a rectangular pulse waveform, with amplitude T/d,the 16th harmonic would have a relative amplitude of 0.83, and the 27th, an amplitude of 0.56. In the two-pulse sequence, where the weighting function reduces to

$$\frac{\sin 2\theta}{\sin \theta} = 2\cos\theta = 2\cos(n\pi/24) ,$$

the weighted amplitude of the 16th harmonic becomes (1.0)(0.83) = 0.83, and of the 27th harmonic, (1.85)(0.56) = 1.04 . The maximum amplitude is attained by the 22nd harmonic, which is increased from 0.69 to 1.33. The ratio of the highest and lowest amplitude components in the desired range is, therefore, 1.33/0.83 = 1.6 = 4.1db.

In reference 49, the author recommends that the pulse duration should be restricted to $T/2n_{max}$, where n_{max} is the highest harmonic number in the required range. With a 4kHz pulse train, this leads to a pulse duration of 4.6 microseconds, and a ratio d/T = 1/54. The ratio of the highest and lowest amplitude components (i.e. the 16th and 27th harmonics) is 0.86/0.64 = 1.34 = 2.6db. With a two-pulse sequence, having pulse durations of 4.6 microseconds, the ratio of the highest and lowest amplitude components, (i.e. the 22nd and 16th harmonics), becomes 1.45/0.86 = 1.69 = 4.5db.

However, there does not appear to be any significant advantage in minimising the harmonic amplitude spread. It is convenient to generate the 96kHz source as a square-wave, and the amplitude spread is not such as to justify further pulse-shaping circuitry. What is significant, is that, when generating the harmonics by means of a normal pulse-train, the highest-order harmonic in the range will have the lowest amplitude. The pulse generator must, therefore, be designed to provide this component at some usable level. With a two-pulse sequence, none of the harmonics will occur with lower amplitudes than in a normal pulse-train, and the highest-order harmonic is approximately doubled in amplitude. Thus, the pulse generator output stage need now operate at only a quarter of the power level previously required.

Either process makes use of digital frequency-division, and, assuming that 96kHz, or some other suitable frequency, can be generated, the additional circuitry required is merely a 3-input gate.

harmonic number	$\frac{\sin mn\pi\tau/T}{\sin n\pi\tau/T}$	d/T = 1/48: $\frac{\sin n\pi d/T}{n\pi d/T}$	d/T = 1/48: weighted spectrum	d/T = 1/54: $\frac{\sin n\pi d/T}{n\pi d/T}$	d/T = 1/54: weighted spectrum
16	1.00	0.83	0.83	0.86	0.86
17	1.22	0.81	0.99	0.85	1.04
18	1.41	0.79	1.11	0.83	1.17
19	1.59	0.76	1.21	0.81	1.29
20	1.73	0.73	1.26	0.79	1.37
21	1.85	0.71	1.31	0.77	1.42
22	1.93	0.69	1.33	0.75	1.45
23	1.98	0.66	1.31	0.72	1.43
24	2.00	0.64	1.28	0.71	1.42
25	1.98	0.61	1.21	0.69	1.37
26	1.93	0.58	1.12	0.66	1.27
27	1.85	0.56	1.04	0.64	1.18

Calculated values for two-pulse sequences, having the d/T ratios previously mentioned, are tabulated above.

To provide an experimental verification, the circuit shown in figure 4.7.b was used to generate a two-pulse sequence from a 96kHz pulse-train, which produced the spectra shown in figure 4.10. For comparison, three spectra are shown. First, that of the source waveform, the 96kHz pulse-train from which the carrier frequencies are to be generated. Second, that of the two-pulse sequence at 4kHz, obtained by digital operations upon the source waveform. Third, the spectrum of a 4kHz pulse-train, which would be the harmonic source using the method of reference 49.

The vertical scale is arbitrary, i.e. chosen merely to fit the particular format. A correction factor of 0.89 may be used to compare the recorded amplitudes with the theoretical values, above. The results for the two-pulse sequence spectrum then become:-

n	16	17	18	19	20	21	22	23	24	25	26	27
A_m	0.95	1.12	1.32	1.40	1.57	1.61	1.65	1.62	1.64	1.51	1.46	1.32
A_c	0.85	1.00	1.17	1.25	1.40	1.43	1.47	1.44	1.46	1.34	1.30	1.17
A_t	0.86	1.04	1.17	1.29	1.37	1.42	1.45	1.43	1.42	1.37	1.27	1.18
e	-1.2	-3.8	0	-3.1	+2.2	+0.7	+1.4	+0.7	+2.8	-2.2	+2.4	-0.8

n = harmonic number, A_m = measured value, A_c = corrected value, A_t = theoretical value, e = percentage error.

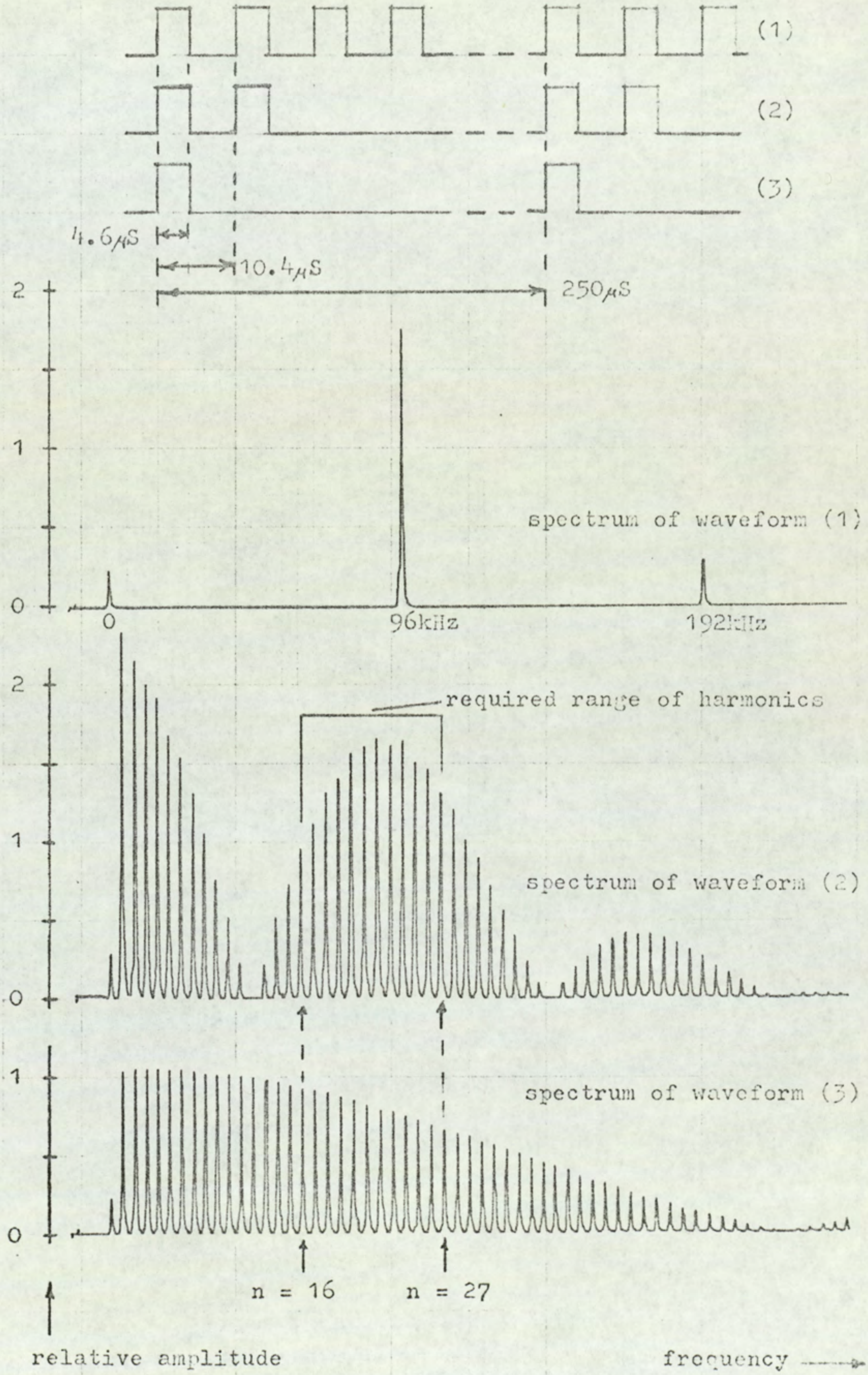

(1)
(2)
(3)
4.6μS
10.4μS
250μS
2
1
0
0
96kHz
192kHz
spectrum of waveform (1)
2
1
0
required range of harmonics
spectrum of waveform (2)
1
0
spectrum of waveform (3)
n = 16
n = 27
relative amplitude
frequency

Figure 4.10

The specified error for the spectrum analyser, and X-Y recorder, depends on the sweep rate. Under the best conditions, however, the error in each is not less than 5%. The recorded values are, therefore, within the limits of the possible maximum experimental error, and may be said to agree with the theoretical values.

4.6.b. Carrier generation with 8kHz separation.

The use of double pulse sequences is a simple adaptation, and could be applied to existing systems to reduce the output stage power levels. Another important advantage might be gained, however, if the filtering problem could be reduced.

When the carriers are generated as a set of 4kHz harmonics, the filters are required to isolate single frequencies from adjacent components of almost the same amplitude, at 4kHz separation. This implies highly selective filters. As a basis for comparison, the approximate Q-factor of a single tuned circuit may be determined, which would reduce the adjacent component by 60db.

Using the standard relationship,

$$Q \simeq (s f_o) / (2 f') ,$$

[where

s = factor by which the amplitude response at fo exceeds that at some other frequency, f. ($s \gg 1$).

f_o = centre-frequency of tuned circuit.

f' = difference between f_o and f.]

shows that the Q required to isolate 64kHz is approximately 8.10^3, and to isolate 108kHz is approximately $13.5\ 10^3$. To achieve such values in practice requires more elaborate methods than a single tuned circuit, and would absorb a large proportion of the cost of a complete system.

If the harmonic separation were greater, the filter specification would be less stringent, and allow some saving. With a separation of 8kHz, the required Q is obviously halved. It is, in fact, a relatively simple matter to generate the required set of frequencies with an 8kHz separation, regardless of whether the pulse sequence technique is used, or not.

The set of frequencies in the standard fdm range may be divided into two groups of six. First, the even harmonics of 4kHz, namely 64, 72, . . 104 kHz, which are also the eighth to the thirteenth harmonics of 8kHz. Second, the odd harmonics of 4kHz, from the seventeenth to the twenty-seventh, namely 68, 76, . . 108kHz.

Clearly, these harmonics can be generated separately, using an 8kHz pulse-train, and a 4kHz square-wave. Pulse sequences at these base frequencies may then be used to produce the harmonics at a greater level. The counter which divides by 24 to give 4kHz from 96kHz must, in any case, include a stage dividing by 12, which may be used to gate a sequence at 8kHz.

A two-pulse sequence at 8kHz will produce the required range of harmonics at levels which are at least equal to, and generally greater than, those of a normal pulse train, as has been seen. The 4kHz sequence from the divide-by-24 stage is now required to produce odd harmonics only, and as was seen previously, (page 123), this is achieved by using a half-length sequence, with $m = T/2\tau = 12$.

The harmonic levels in the half-length sequence will have greater amplitudes, in the required range, than for the two-pulse sequence. There is, however, one disadvantage. In addition to the odd harmonics of 4kHz, a component at T/τ = 96kHz will appear, and will receive the maximum accentuation. Not only is this component redundant, since it is postulated that 96kHz is available as the source frequency, but so far as the harmonics at 92 and 100kHz are concerned, the filtering problem is made worse.

That is, the filters concerned would have to reject a component at 4kHz separation, having a significantly greater amplitude than the desired component. There is, however, a simple remedy, since, as was seen previously, (page 123-4), a half-length bipolar sequence eliminates the redundant harmonic at the source frequency. Furthermore, the level of the remaining harmonics is doubled, compared to a unipolar sequence.

Thus, at the expense of the extra circuitry needed to form a two-level sequence, (i.e. +1, 0, -1), the output levels may be considerably increased, compared to the conventional method. Also, there is now a uniform separation of 8kHz between the spectral components, compared to 4kHz when using the conventional method. The theoretical amplitudes of a set of harmonic amplitudes are tabulated below.

4kHz pulse-train. d/T = 1/48.

harmonic number	frequency kHz	amplitude $(\sin n\pi/48)/(n\pi/48)$	amplitude ratio w.r.t. 108kHz	db.
16	64	0.83	1.48	3.4
17	68	0.81	1.45	3.2
18	72	0.79	1.41	3.0
19	76	0.76	1.36	2.6
20	80	0.73	1.30	2.3
21	84	0.71	1.27	2.1
22	88	0.69	1.23	1.8
23	92	0.66	1.18	1.4
24	96	0.64	1.14	1.2
25	100	0.61	1.09	0.7
26	104	0.58	1.04	0.3
27	108	0.56	1.00	0

8kHz 2-pulse sequence. d/T = 1/24.

harmonic number	frequency kHz	$\left\lvert\frac{\sin n\pi/24}{n\pi/24}\right\rvert$	w.f. = $\lvert 2\cos(n\pi/12)\rvert$	amplitude	db. w.r.t. 0.56
8	64	0.83	1.00	0.83	3.4
9	72	0.78	1.41	1.12	6.0
10	80	0.73	1.73	1.26	7.0
11	88	0.69	1.93	1.33	7.5
12	96	0.64	2.00	1.28	7.2
13	104	0.58	1.93	1.12	6.0

4kHz 12-pulse sequence. d/T = 1/48.

harmonic number	freq. kHz	$\frac{\sin n\pi/48}{n\pi/48}$	$\left\lvert\frac{1}{\sin(n\pi/24)}\right\rvert$	$\lvert amp^{dc}\rvert$	db. w.r.t. 0.56	$\lvert amp^{dc}\rvert$ bipolar	db. w.r.t. 0.56
17	68	0.81	1.27	1.03	5.3	2.06	11.3
19	76	0.76	1.64	1.25	7.0	2.50	13.0
21	84	0.71	2.63	1.87	10.5	3.74	16.5
23	92	0.66	7.69	5.07	19.1	10.14	25.2
24	96	0.64	12.00	7.68	22.7	-	-
25	100	0.61	7.69	4.69	18.5	9.38	24.5
27	108	0.56	2.63	1.47	8.4	2.94	14.4

The first table shows the amplitudes of the relevant harmonics from a 4kHz rectangular pulse-train, amplitude T/d, with d/t = 1/48. The ratio, in db, of each harmonic amplitude with respect to the lowest amplitude component, (i.e. 108kHz), is indicated.

The second table shows the amplitudes of the harmonics in an 8kHz 2-pulse sequence, and the amplitude ratios with respect to the lowest amplitude component in the 4kHz pulse-train of the first table. The modulus of the numerical values is shown in each case.

The third table shows the amplitudes of the harmonics in a 4kHz 12-pulse sequence. The ratio of each amplitude to that of the 108kHz component in the spectrum of the 4kHz pulse-train is again calculated. Since $m = T/2\tau$, and $T/\tau = 24$, the weighting function is

$$\frac{\sin nm\pi\tau/T}{\sin n\pi\tau/T} = \frac{\sin n\pi/2}{\sin n\pi/24} = \frac{\pm 1}{\sin n\pi/24},$$

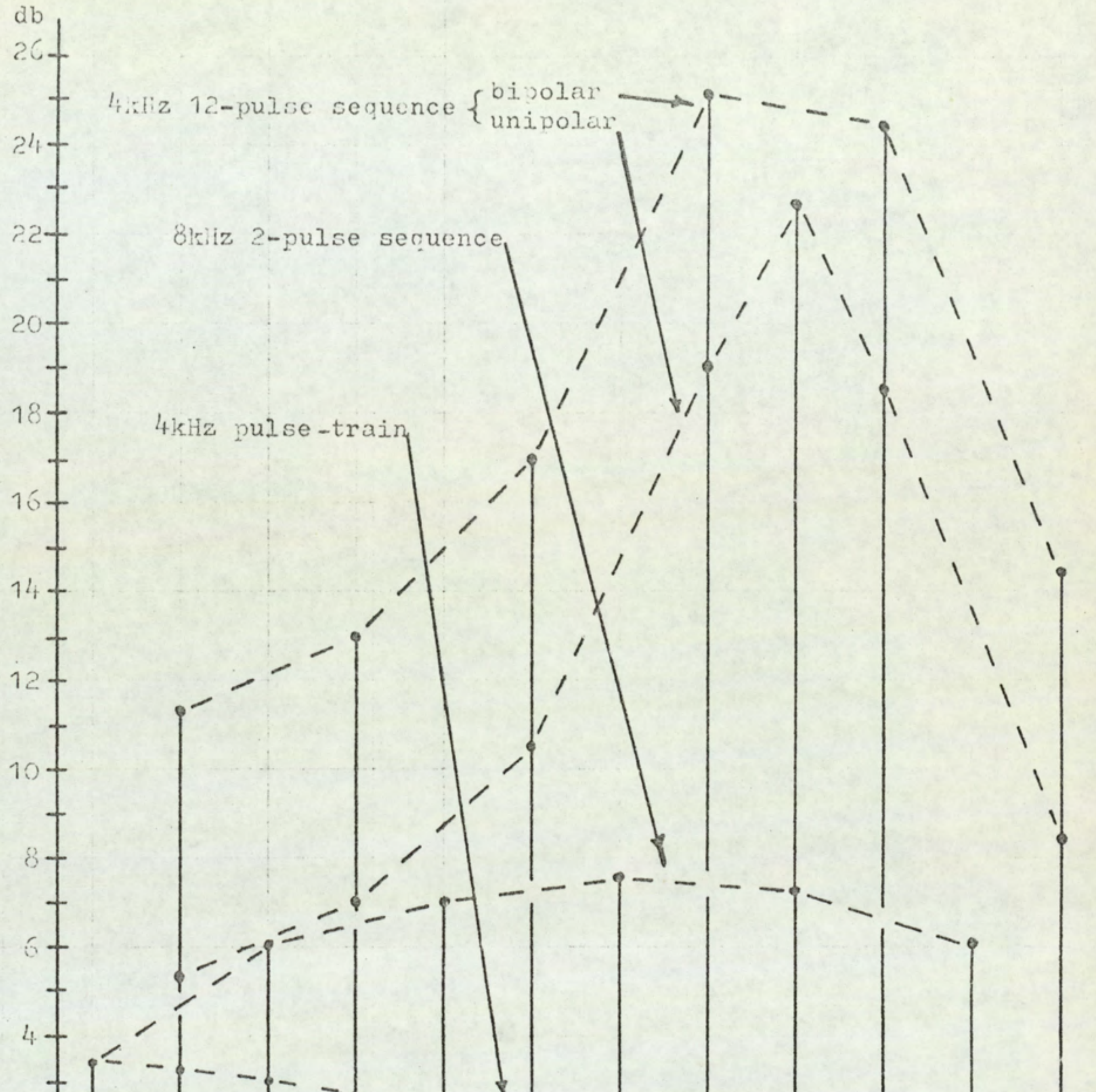
but, again, the modulus only is shown. The amplitudes and ratios of the bipolar sequence are included, although the amplitudes are, of course, simply twice those of the unipolar sequence, with the exception of 96kHz, which becomes zero.

These tabulated ratios are displayed graphically in figure 4.11, from which the increased harmonic amplitudes and frequency separation may be envisaged, compared to the use of a normal 4kHz pulse-train.

Experimentally obtained spectra for these types of sequence are shown in figures 4.12, and 4.13. The first circuit shown, at the top of figure 4.12, is based on a counter dividing by 24. The first two flip-flops are cross-coupled to provide a count of three, whilst the remaining three flip-flops each divide by two. The outputs from the stages divide successively by 3, 6, and 12, and are combined as logic functions by means of a 3-input gate.

This provides a pulse of duration 2τ for every 12 pulses of the source, or clock, frequency. The logical combination of this pulse and the source frequency, by means of a 2-input gate, provides the 8kHz 2-pulse sequence. The 4kHz 12-pulse sequence is obtained by logical combination of the source frequency and the square-wave output of the divide-by-24 stage.

Figure 4.12.a shows the first and second harmonic of the 96kHz source frequency. To display the effect of the weighting



db
26
24
22
20
18
16
14
12
10
8
6
4
2
0
4kHz 12-pulse sequence
bipolar
unipolar
8kHz 2-pulse sequence
4kHz pulse-train
64
68
72
76
80
84
88
92
96
100
104
108
harmonic amplitude db
harmonic frequency, kHz

Figure 4.11

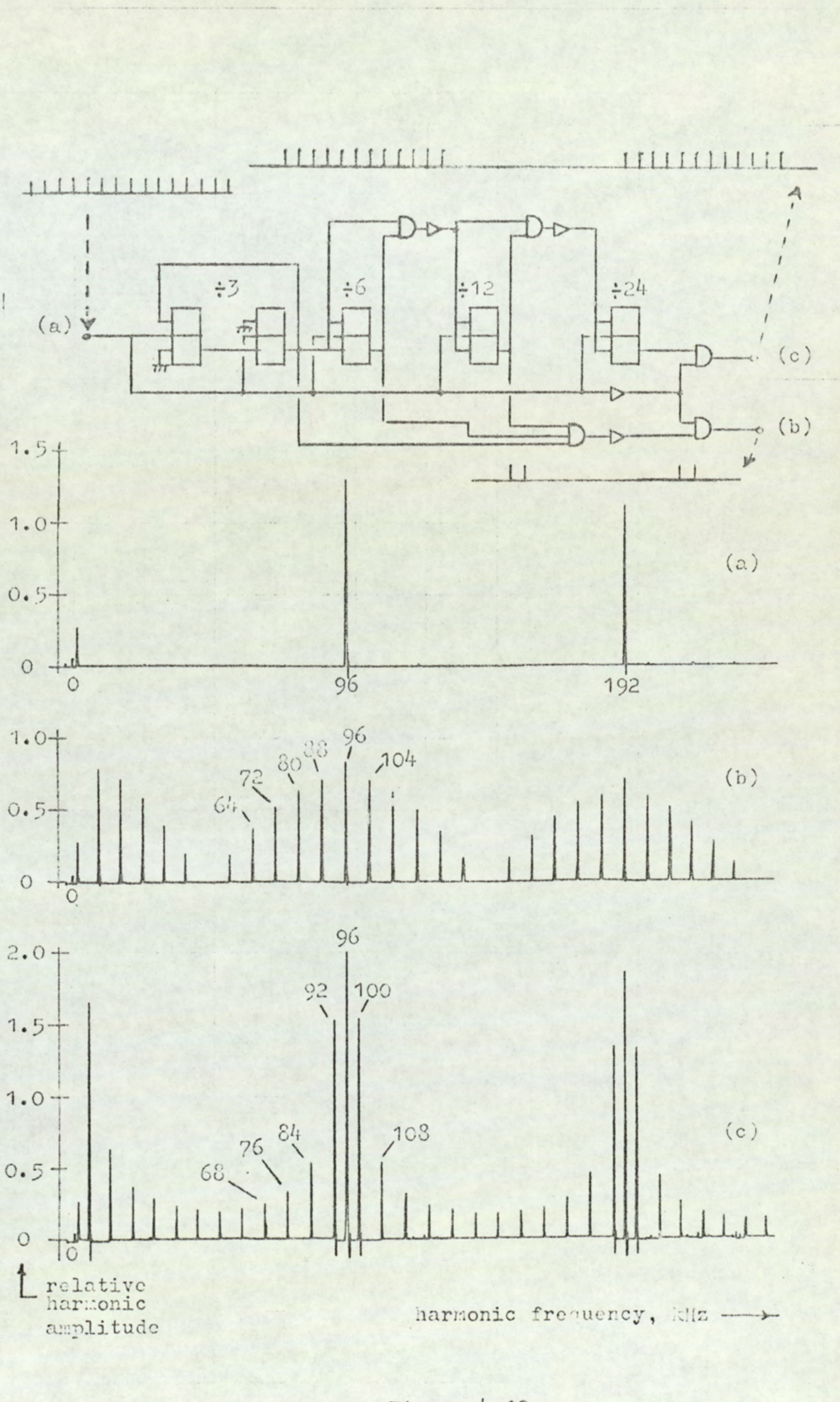

÷3
÷6
÷12
÷24
(a)
(b)
(c)
1.5
1.0
0.5
0
0
96
192
(a)
1.0
0.5
0
0
64
72
80
88
96
104
(b)
2.0
1.5
1.0
0.5
0
0
68
76
84
92
96
100
108
(c)
relative
harmonic
amplitude
harmonic frequency, kHz

Figure 4.12

function more clearly, a relatively short pulse duration was used, with $d/\tau = 1/6$, (where τ is the period of the source frequency). This reduced the amplitude spread of the basic pulse-train harmonics over the range of interest.

Figure 4.12.b. shows the effect of the weighting function upon the spectrum of the basic 8kHz pulse-train. Zeros occur for n = 6,18,30,etc., and maxima for n = 12,24,36,etc. Figure 4.12.c. shows the spectrum of the 12-pulse unipolar sequence, and the more accentuated peaking of the weighting function for m = 12 is evident. (c.f. figure 4.5).

As mentioned earlier, the recorded spectra do not show any reversals of sign. In fact, the harmonics alternate in polarity, with the exception of the two harmonics adjacent to each peak, which take the same sign. The peaks alternate in polarity, and the zeros of the weighting function, for this case, fall halfway between each spectral line, thus eliminating the even harmonics of 4kHz, (except multiples of T/τ).

A comparison of the theoretical and measured values is tabulated below.

8kHz 2-pulse sequence. $d/T = 1/72$

harmonic no.	8	9	10	11	12	13
frequency kHz	64	72	80	88	96	104
$(\sin n\pi d/T)(n\pi d/T)$	.980	.974	.968	.962	.955	.947
w.f. $= 2\cos n\pi/12$	1.00	1.41	1.73	1.93	2.00	1.93
theoretical amp^{de}	0.98	1.37	1.67	1.86	1.91	1.83
scaled amplitude	0.38	0.53	0.65	0.72	0.74	0.71
measured amp^{de}	0.38	0.52	0.63	0.72	0.83	0.71
% error	0	-1.9	-3.1	0	+12.2	0

4kHz 12-pulse sequence. $d/T = 1/144$

harmonic no.	17	19	21	23	24	25	27
frequency kHz	68	76	84	92	96	100	108
$(\sin n\pi d/T)(n\pi d/T)$	.977	.972	.965	.958	.955	.951	.943
w.f. $= 1/(\sin n\pi/24)$	1.27	1.64	2.63	7.69	12.0	7.69	2.63
theoretical amp^{de}	1.24	1.59	2.54	7.37	11.5	7.31	2.48
scaled amplitude	0.26	0.33	0.53	1.53	2.39	1.52	0.51
measured amp^{de}	0.27	0.34	0.54	1.53	2.01	1.55	0.54
% error	+3.8	+3.0	+1.9	0	-11.7	+2.0	+5.9

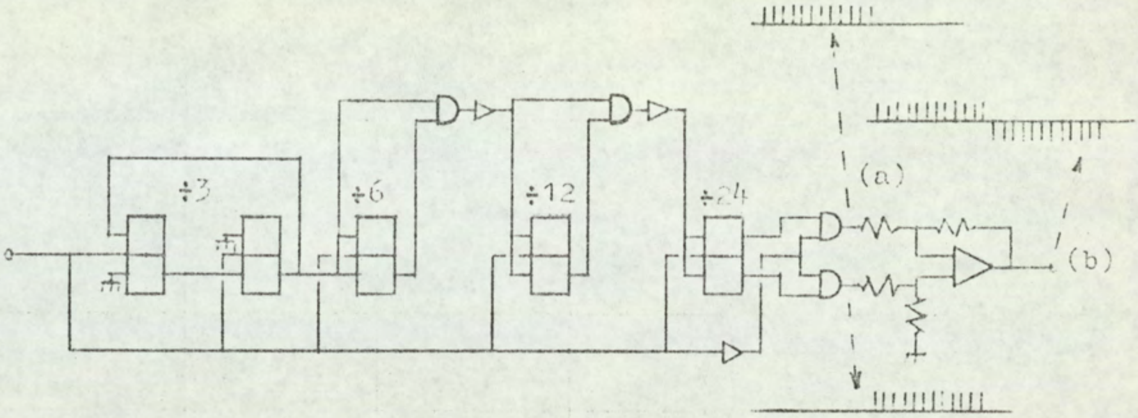
It will be seen that the errors are small, with the exception of the 96kHz component. This may be attributed to direct pick-up of the source frequency at the spectrum analyser input.

Figure 4.13 shows the circuit used to generate a bipolar sequence,and the spectrum which was obtained from that circuit. As for the unipolar 12-pulse sequence,the circuit is based on a divide-by-24 counter,but the complemetary outputs from the final flip-flop are now used. Alternate half-period sequences are gated to the two input points of a differential amplifier. A commercial operational amplifier module was used,the Analog Devices, Type 111. One input point provides an inverted signal at the common output point,and the other,a non-inverted signal. The feedback resistors were adjusted to match the amplitudes of the positive-going and negative-going half-sequences.

Figure 4.13.a. shows the spectrum of a unipolar half-period sequence for comparison. It will be seen that in the spectrum of the bipolar sequence,figure 4.13.b.,the amplitudes are doubled,and the 96kHz component is rejected,but that otherwise the spectra are identical. Elimination of the 96kHz component depends on accurate matching of the amplitudes and pulse shapes in the bipolar sequence. The residual component exhibited in the experimentally recorded spectrum could have been reduced by more elaborate matching techniques.

The theoretical and measured values for the 4kHz 12-pulse bipolar sequence,d/T = 1/48,are tabulated below.

frequency kHz	68	76	84	92	100	108
theoretical amplitude	2.06	2.50	3.74	10.14	9.38	2.94
scaled amplitude	0.40	0.48	0.72	1.95	1.80	0.56
measured amplitude	0.38	0.47	0.73	2.01	1.82	0.52
error %	-5.0	-2.1	+1.4	+3.1	+1.1	-7.1


÷3
÷6
÷12
÷24
(a)
(b)

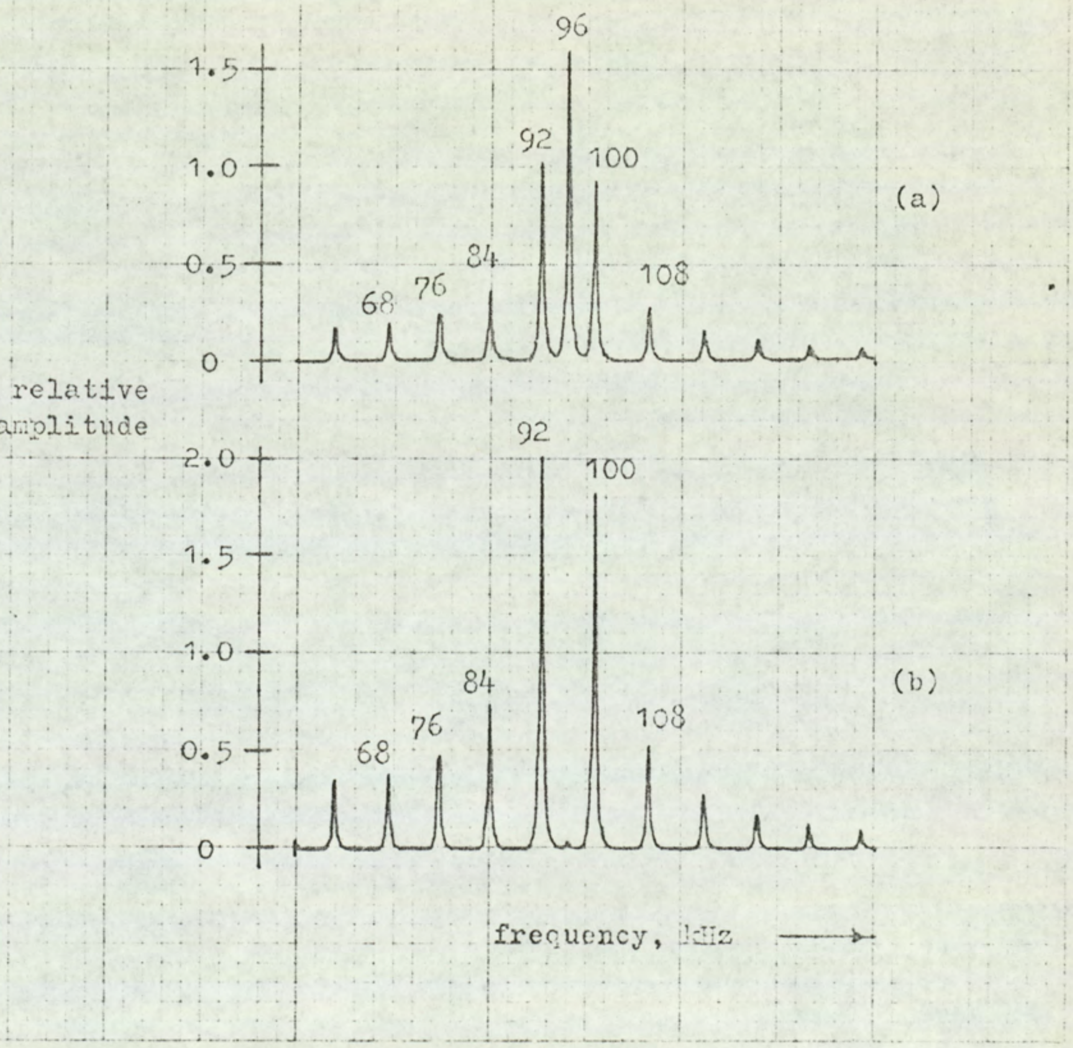

relative
amplitude
1.5
1.0
0.5
0
96
92
100
84
76
68
108
(a)
2.0
1.5
1.0
0.5
0
92
100
84
76
68
108
(b)
frequency, kHz

Figure 4.13

4.6.c. Quadrature sequences.

Although the advantages of all-digital circuitry are lost, the introduction of analogue circuits, i.e. the differential amplifier, allows greater flexibility in systems designed to isolate particular spectral lines. In an ideal carrier-frequency generator, all the carriers would be available as pure sinusoids at separate output points. Inevitably, discrete-level pulse sequences generate multiple harmonics, but the use of multi-level waveforms improves the possibility of their selective attenuation.

The objective was restricted to the production of a spectrum in which a desired harmonic was so accentuated, that a relatively simple bandpass filter might be used to extract the sinusoidal carrier. It has been seen that a bipolar half-length sequence has a spectrum in which those pairs of harmonics are emphasised which are immediately adjacent to the harmonics which are multiples of the source frequency. Generally, the overriding effect of the Fourier-series coefficient ensures that harmonic numbers greater than a few times that of the source frequency will yield insignificant amplitudes.

The significant harmonics will be those immediately adjacent to the source frequency, i.e. that pair for which $n = (T/\tau \pm 1)$. If one or other of these two harmonics can be suppressed, the resulting spectrum will have the desired properties.

Since the weighting function is symmetrical about T/τ, neither harmonic can be have zero amplitude in the pulse sequences described so far. However, if two sequences are formed, in one of which the unwanted harmonic is antiphase, then summing these sequences will cause the unwanted harmonic to be cancelled. There are various possible ways of achieving this phase difference, but it is clearly desirable that the circuitry should not become over-elaborate.

One relatively simple method is to generate two half-length bipolar sequences which have a quadrature phase relationship. A pulse sequence may be regarded as having arisen from the product of a rectangular gating waveform, and the source-frequency waveform. The fundamental and higher harmonics of the source-frequency waveform will, therefore, each take a set of side frequencies, formed by the sum and difference with each harmonic of the gating waveform.

Denoting the fundamental of the source-frequency waveform as $\cos\omega_s t$, and the fundamental of the gating waveform as $\cos\omega_g t$, there will be a pair of frequencies,

$$\cos(\omega_s + \omega_g)t \;+\; \cos(\omega_s - \omega_g)t\ ,$$

occuring at those points in the sequence spectrum where $n = (T/\tau \pm 1)$. However, if the fundamentals are denoted as $\sin\omega_s t$ and $\sin\omega_g t$, the phasing becomes such that those same frequencies are expressed as

$$\cos(\omega_s - \omega_g)t \;-\; \cos(\omega_s + \omega_g)t\ .$$

Consequently, if a waveform in which the fundamentals have a cosine phasing is added to a waveform in which they have a sine phasing, the term $\cos(\omega_s + \omega_g)t$ is cancelled. If the waveforms are subtracted, $\cos(\omega_s - \omega_g)t$ disappears.

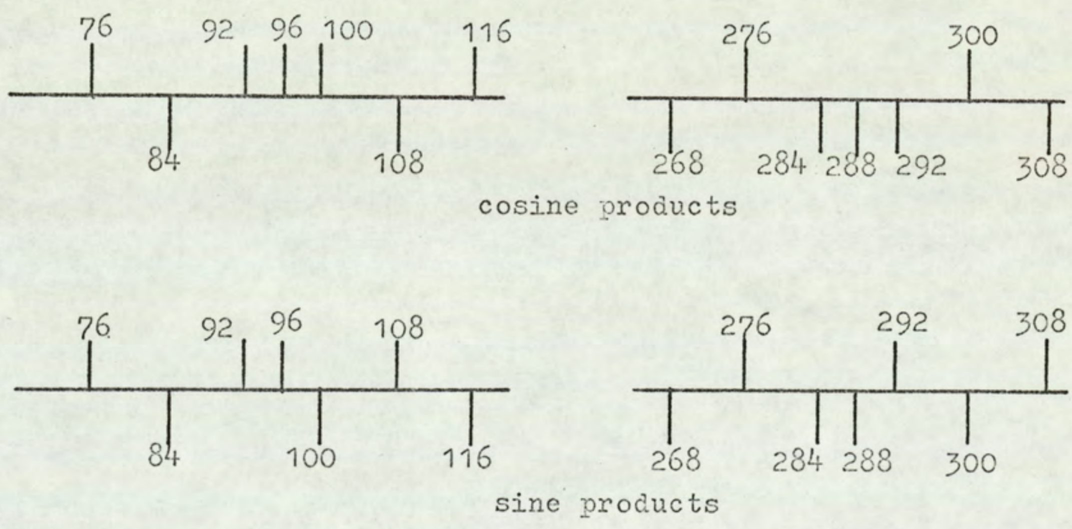


Figure 4.14

If both the gating waveform and the source-frequency waveform are square waves, with prf 4kHz and 96kHz respectively, then the polarity of some of the sum and difference terms is indicated in figure 4.14. The upper set shows components adjacent to the first and third harmonics of 96kHz when the fundamentals of the two waveforms are cosine related. With square waves, of course, only odd-numbered harmonics exist, and these are of alternating polarity. The lower set is the case when the fundamentals are sine related.

It will be seen that the lower side-frequencies on 96kHz will reinforce when the two spectra are added,but that the upper side-frequencies will cancel. In the same way,the lower side-frequencies on higher order harmonics of 96kHz are mutually reinforcing,but the upper side-frequencies are mutually cancelling.

When the lower side-frequencies fold back from zero frequency,the cosine products retain the same polarity,but the sine products take the opposite polarity. Consequently,the folded back terms are mutually cancelling when the spectra are added. Of course,the reverse situatuon applies when the spectra are subtracted.

However,using this approach,the derivation of analytical expressions for the spectra involves the product of infinite series. A simpler approach is possible by using the pulse sequence relationships.

The general expression for the spectrum of a pulse sequence,namely

$$f(t) = \frac{Ad}{T}\sum_{n}\frac{\sin n\pi d/T}{n\pi d/T}\, e^{jn\omega_o t}\left[\frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T}\, e^{-jn\omega_o(m-1)\tau/2}\right],$$

was derived by postulating a sequence in which the time origin occured at the centre of the first pulse in the sequence.

Considering a sequence in terms of the product of a source-frequency square wave,and a gating square wave,then if the fundamentals are sine related,the waveform generated will be in the form shown in figure 4.15.a. The sequence is the product of the two upper waveforms. The only modification required to the general expression is to introduce a delay term, $e^{-jn\omega_o d/2}$.

However,for a sequence in which the fundamentals are cosine related,the waveforms will be as in figure 4.15.b. Each of the two upper waveforms has a quarter-period shift with respect to the corresponding waveforms in figure 4.15.a. The product waveform,i.e. the quadrature sequence,will have a split pulse at each end of the sequence.

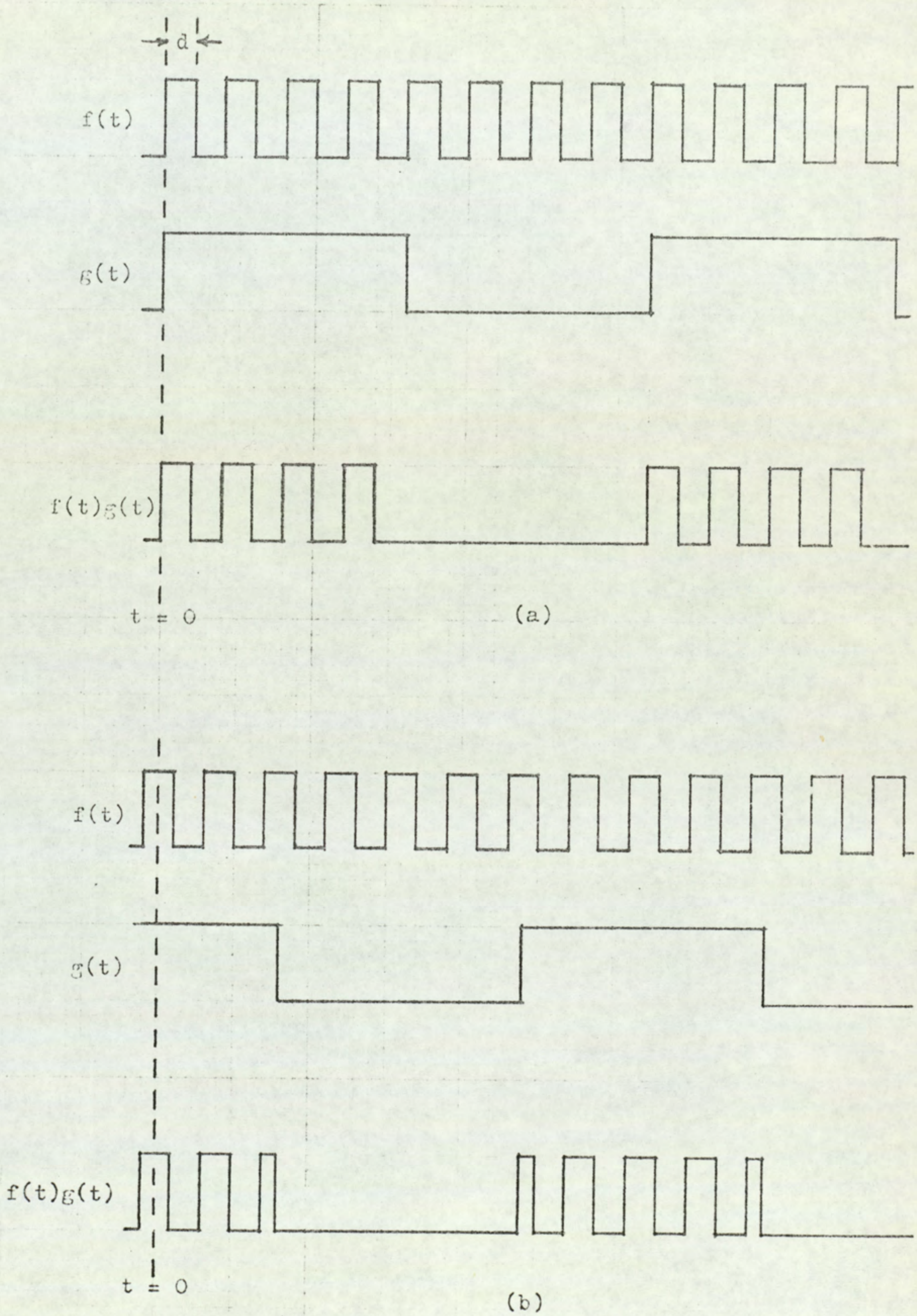


Figure 4.15

To obtain an expression for the spectrum of such a waveform, one may consider this 'split-pulse' sequence, (with a shifted origin), to be decomposed into two other sequences, having half the original pulse durations, as shown in figure 4.16.a.

Waveform a_1, which is a half-length sequence, may be expressed as

$$f(t) = \frac{Ad}{2T} \sum_n \frac{\sin n\pi d/2T}{n\pi d/2T} e^{jn\omega_o t} \left[\frac{\sin n\pi/2}{\sin n\pi\tau/T} e^{-jn\omega_o(m-1)\tau/2} \right]$$

Waveform a_2 differs only in being delayed by a factor

$$e^{-jn\omega_o 3\tau/4} = e^{-jn\pi 3\tau/2T}$$

Summing the expressions for the two waveforms will, therefore, introduce the term

$$(1 + e^{-jn\pi 3\tau/2T}) = 2\cos(n\pi 3\tau/4T)\, e^{-jn\omega_o 3\tau/8}$$

The expression for the recomposed pulse sequence is, therefore,

$$f(t) = \frac{Ad}{2T} \sum_n \frac{\sin n\pi d/2T}{n\pi d/2T} e^{jn\omega_o t} \left[\frac{\sin n\pi/2}{\sin n\pi\tau/T} 2\cos(n\pi 3\tau/4T) e^{-jn\omega_o(4m-1)\frac{\tau}{8}} \right]$$

If the time origin is at the leading edge of the first pulse in the sequence, the phase term becomes

$$e^{-jn\omega_o(4m-1)\tau/8}\, e^{-jn\omega_o\tau/8} = e^{-jn\omega_o m\tau/2}$$

The normal half-length pulse sequence may also be decomposed into two half-pulse-duration sequences, as in figure 4.16.b. Since these two sequences are displaced by $\tau/4$, their addition yields a term

$$(1 + e^{-jn\omega_o\tau/4}) = 2\cos(n\pi\tau/4T)\, e^{-jn\omega_o\tau/8}$$

and the expression for the recomposed pulse sequence becomes

$$f(t) = \frac{Ad}{2T} \sum_n \frac{\sin n\pi d/2T}{n\pi d/2T} e^{jn\omega_o t} \left[\frac{\sin n\pi/2}{\sin n\pi\tau/T} 2\cos(n\pi\tau/4T)\, e^{-jn\omega_o(4m-3)\frac{\tau}{8}} \right]$$

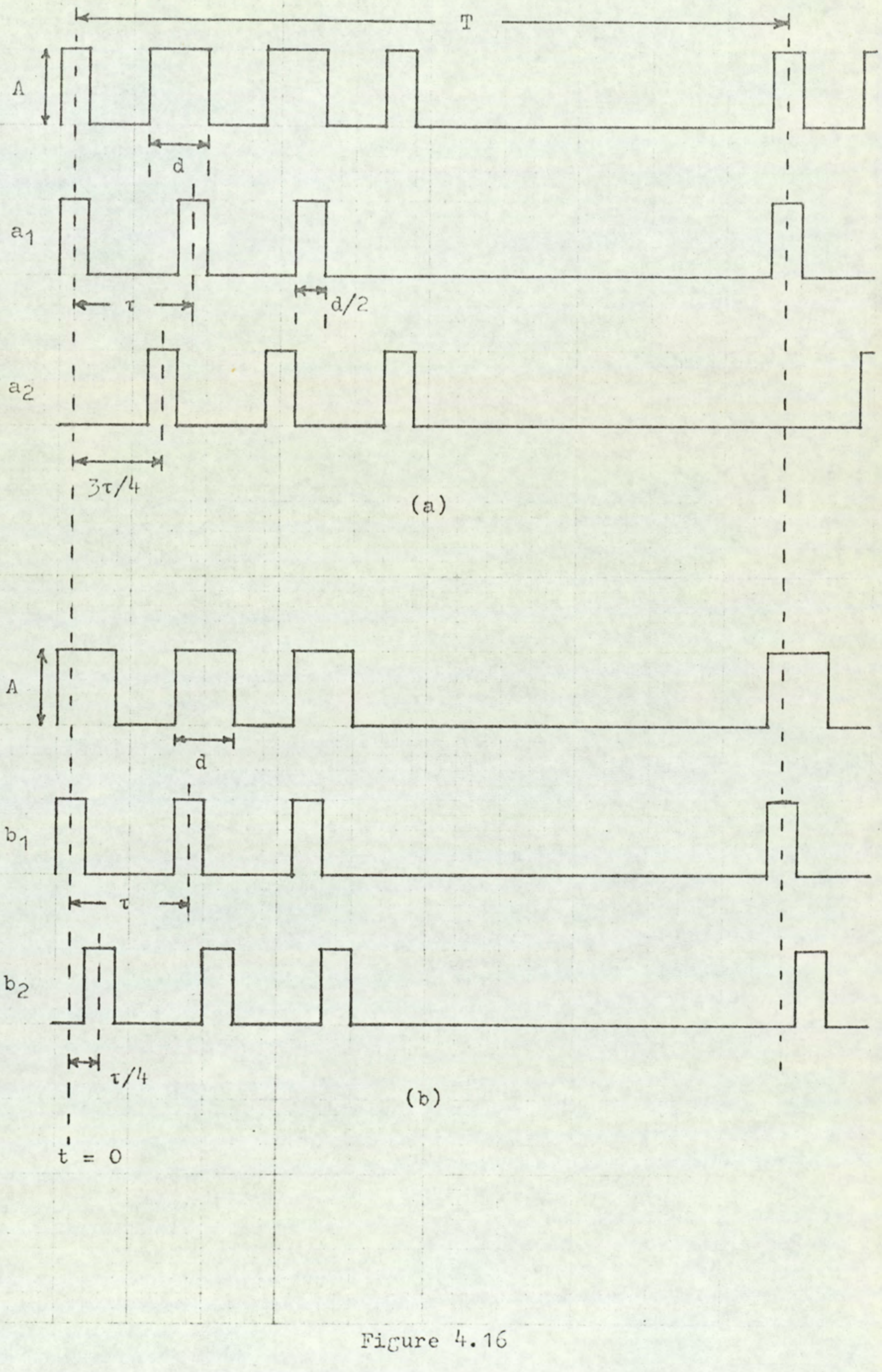


Figure 4.16

[This will, of course, reduce to an expression in terms of d, the actual pulse duration. The product

$$\left[\sin(n\pi d/2T)\right]\left[2\cos(n\pi\tau/4T)\right]$$

may be written as

$$2\sin(n\pi d/2T)\cos(n\pi d/2T) \quad = \quad \sin(n\pi d/T)$$

(since $d = \tau/2$), which gives the original expression for the sequence.] If the time origin is at the leading edge of the first pulse in the sequence, the phase term becomes

$$e^{-jn\omega_o(4m-3)\tau/8}\, e^{-jn\omega_o\tau/8} \quad = \quad e^{-jn\omega_o(2m-1)\tau/4}$$

The spectral envelope of the split-pulse sequence differs from that of a normal sequence. For example, the envelope of a split-pulse sequence is plotted in figure 4.17, for $m = 6$, and $T/\tau = 12$. Figure 4.17.a. is that part of the weighting function given by

$$2\cos n\pi 3\tau/4T \quad = \quad 2\cos n\pi 3/48$$

Figure 4.17.b. is the remaining part of the weighting function,

$$\frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T} = \frac{\sin n\pi/2}{\sin n\pi/12}$$

The latter is, of course, the weighting function for the normal pulse sequence.

Figure 4.17.c. shows the product of these two terms, which is the overall weighting function. It will be seen that the weighting tends to attenuate those components in the region between 0 and T/τ, $3T/\tau$ and $5T/\tau$, etc., and to accentuate those components in the region between T/τ and $3T/\tau$, $5T/\tau$ and $7T/\tau$, etc. Those components at, or near, $4T/\tau$, $8T/\tau$, etc., are, however, accentuated.

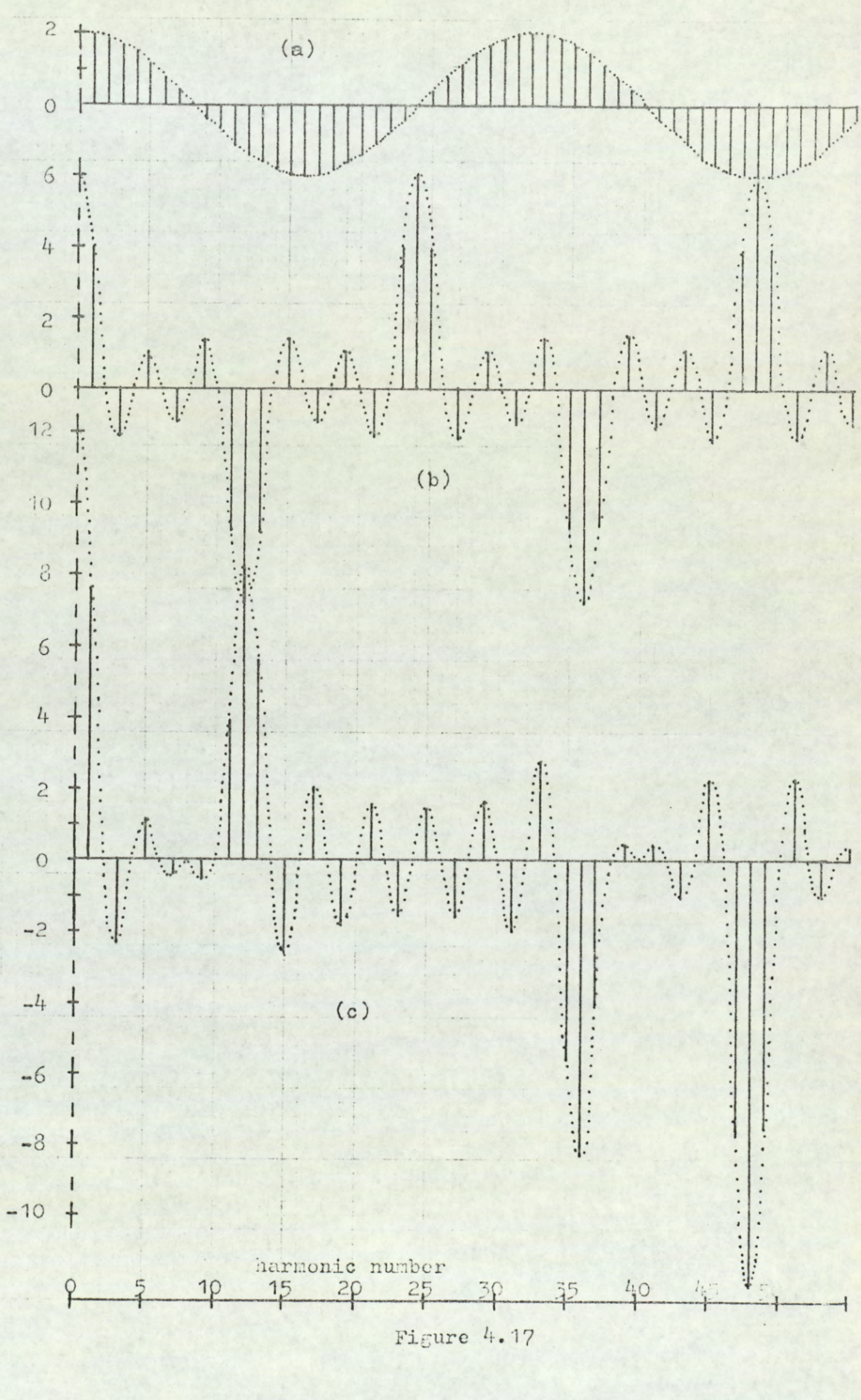

2
0
(a)
6
4
2
0
12
(b)
10
8
6
4
2
0
-2
(c)
-4
-6
-8
-10
harmonic number
0
5
10
15
20
25
30
35
40
45
50

Figure 4.17

Of course, the weighting function operates in conjunction with the normal Fourier Series coefficient. In the case of the split-pulse sequence, the effective pulse duty-ratio is 1/48, so that the harmonic amplitudes at $4T/\tau$, $8T/\tau$, etc., are, in fact, zero. This is demonstrated in the two experimentally recorded spectra, shown in figure 4.18.

The upper spectrum, figure 4.18.a., is that of a normal 6-pulse half-length sequence. Since the effective duty-ratio is 1/24, there are spectrum zeros at n = 24, 48, 72, etc., and the spectrum is accentuated in the region of n = 12, 36, 60, etc.

The lower spectrum, figure 4.18.b., is for the corresponding split-pulse sequence. The components are generally weighted in accordance with figure 4.17.c., but zeros occur at n = 48, 96, etc.

As with a normal sequence, a bipolar sequence with split pulses may be formed, in which the spectral amplitudes are doubled, and the components at multiples of T/τ are eliminated.

If two bipolar sequences are aligned so that the leading edge of the first pulse of the normal sequence coincides with the centre of the split pulse sequence, the combination of these two sequences may be expressed as

$$f(t) = \frac{Ad}{T}\sum_n \frac{\sin n\pi d/2T}{n\pi d/2T}\,\frac{\sin n\pi/2}{\sin n\pi\tau/T}\left\{\cos(n\pi 3\tau/4T)e^{jn\omega_o t} \pm \cos(n\pi\tau/4T)e^{jn\omega_o\left[t-(2m-1)\tau/4\right]}\right\}$$

(The appearance of the combined waveforms is indicated in figure 4.19) The significant feature in this expression is the phase difference between the terms in the curly bracket.

If, for example, $T/\tau = 12$, (so that m = 6), the two harmonics of primary interest are those at $n=(T/\tau)\pm 1$, i.e. at n = 11 and n = 13.

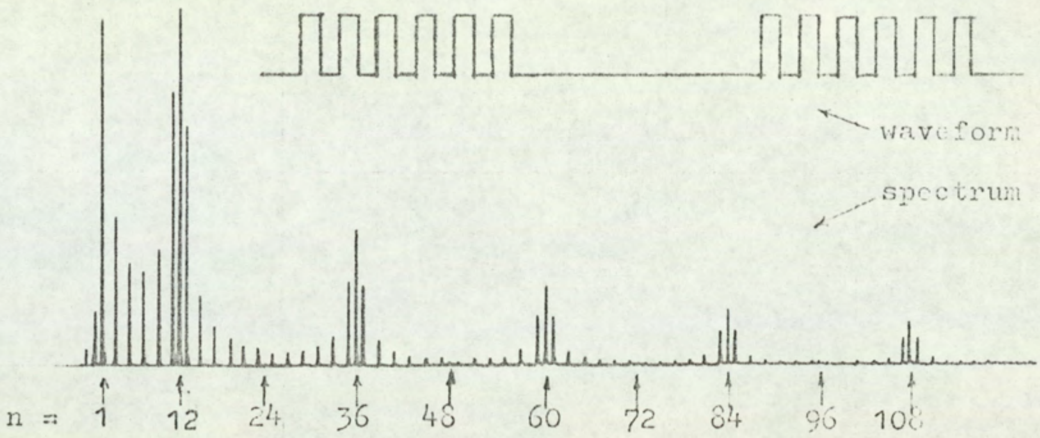
If n = 11

$\cos n\pi 3\tau/4T = \cos(11\pi/16) = \cos 123.75^o = -0.5554$

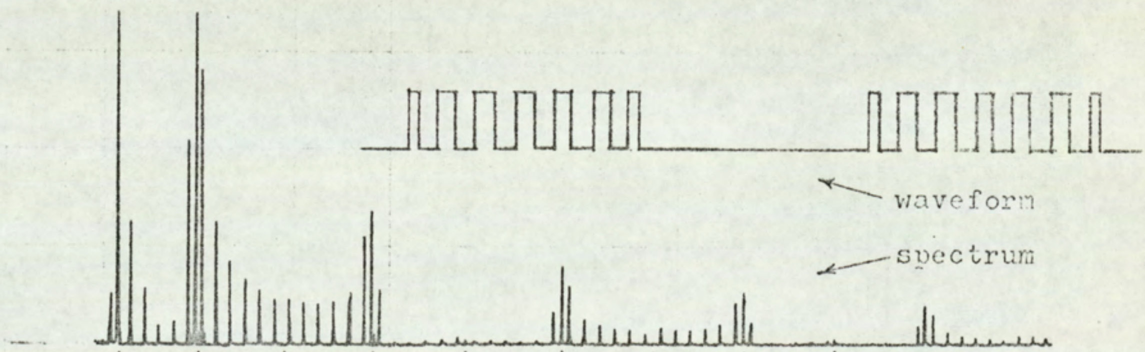
$\cos n\pi\tau/4T = \cos(11\pi/48) = \cos 41.25^o = 0.7490$

The phase difference is

$-n\omega_o(2m-1)\tau/4 = -121\pi/24 = -187.5^o$



(a)



(b)

Figure 4.18

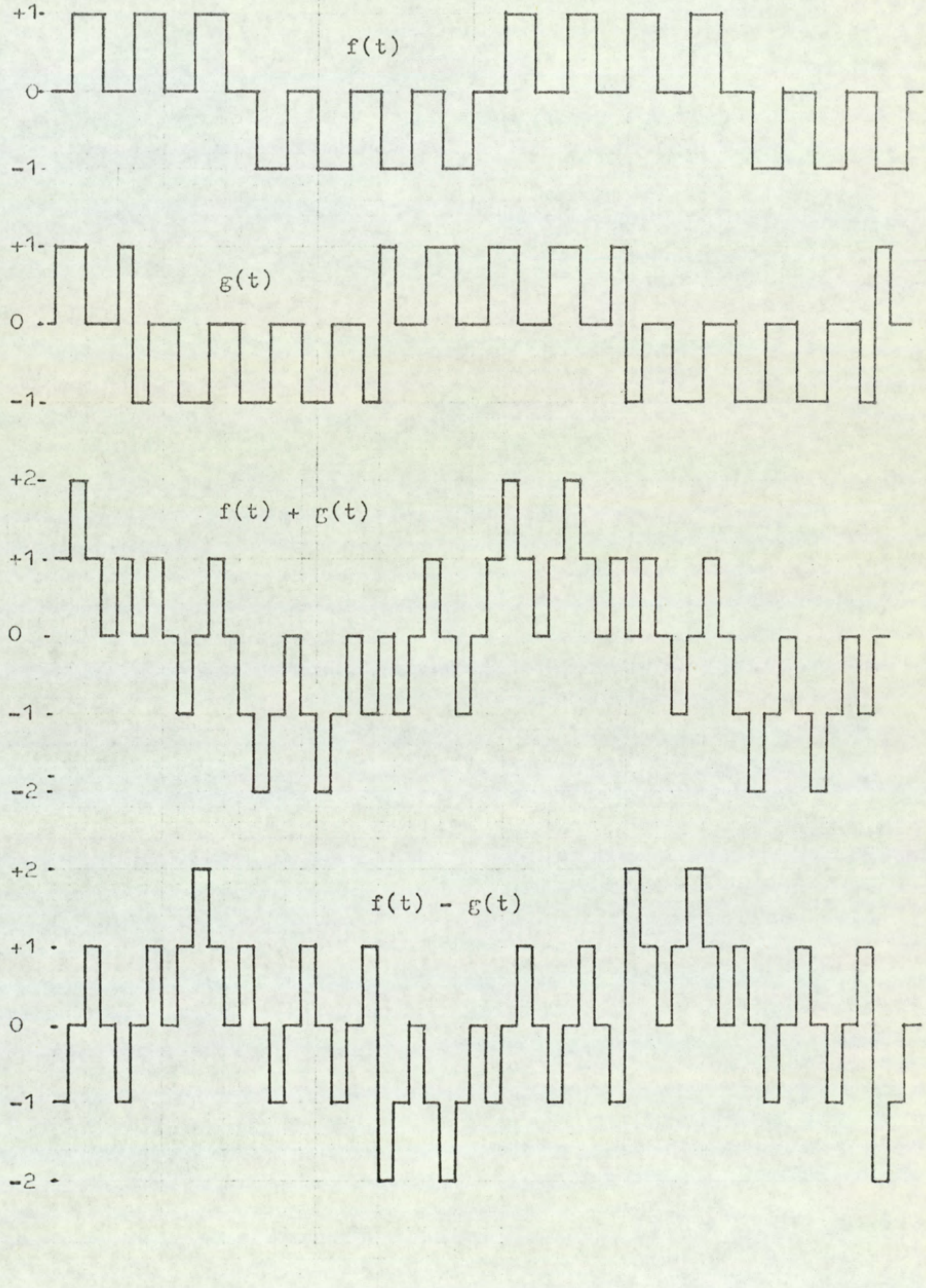


Figure 4.19

If $n = 13$

$\cos n\pi 3\tau/4T = \cos(13\pi/16) = \cos 146.25^{\circ} = -0.8315$

$\cos n\pi\tau/4T = \cos(13\pi/48) = \cos 48.75^{\circ} = 0.6593$

$-n\omega_o(2m-1)\tau/4 = -143\pi/24 = 352.5^{\circ}$

For comparison, the vector components of the four adjacent harmonics are also evaluated.

$n = 7$

$\cos 7\pi/16 = -0.1950$, $\cos 7\pi/48 = 0.8969$

phase difference $= -77\pi/24 = -217.5^{\circ}$

$n = 9$

$\cos 9\pi/16 = -0.1950$, $\cos 9\pi/48 = 0.8315$

phase difference $= -99\pi/24 = -22.5^{\circ}$

$n = 15$

$\cos 15\pi/16 = -0.9808$, $\cos 15\pi/48 = 0.5556$

phase difference $= -165\pi/24 = -157.5^{\circ}$

$n = 17$

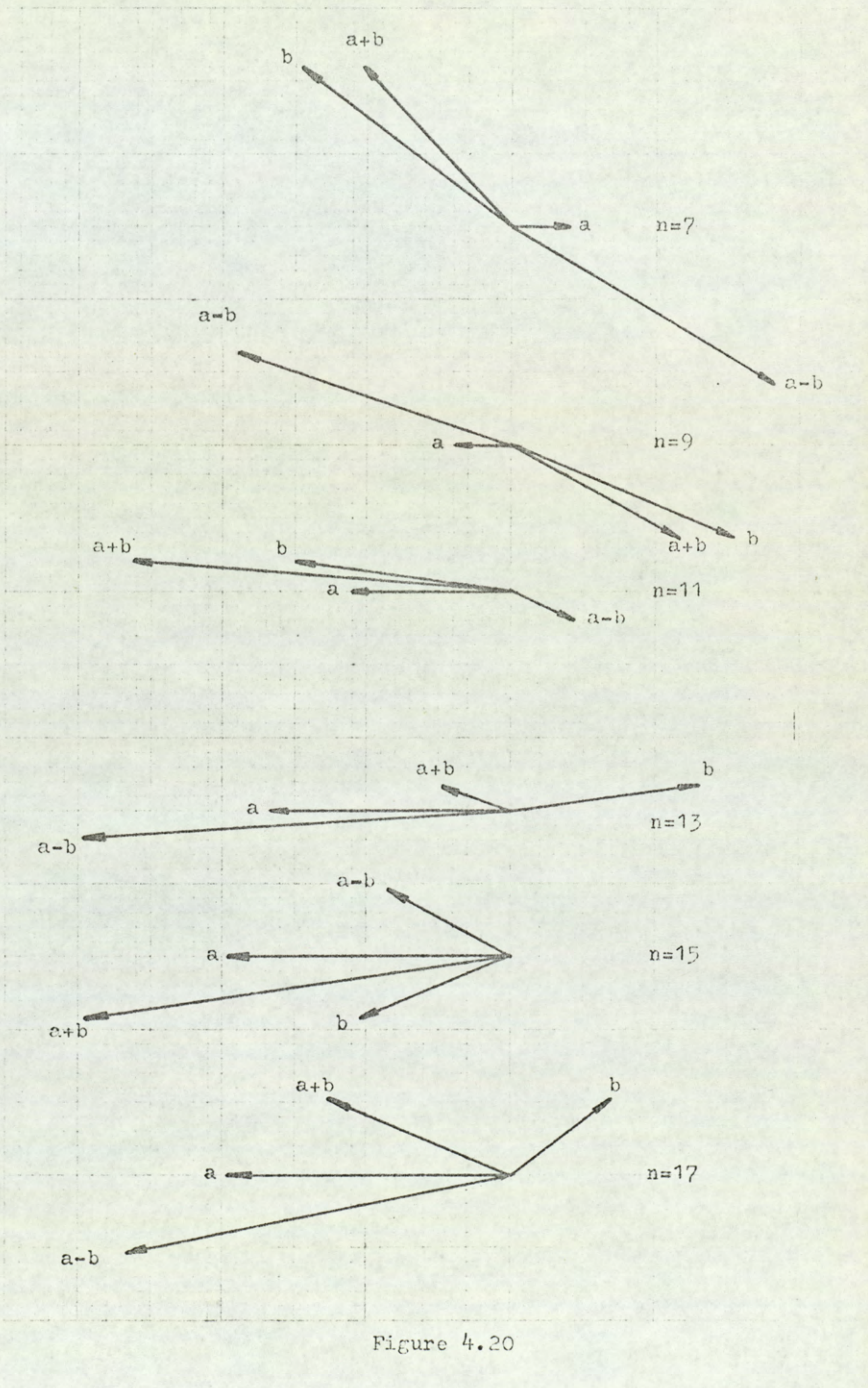
$\cos 17\pi/16 = -0.9808$, $\cos 17\pi/48 = 0.4423$

phase difference $= -187\pi/24 = -322.5^{\circ}$

These vector relationships are plotted in figure 4.20. The sum and difference vector is also shown for each case. It will be observed that, relative to the vector corresponding to the split-pulse sequence, a, (amplitude = $\cos n\pi 3\tau/4T$), the vector for the normal sequence, b, (amplitude = $\cos n\pi\tau/4T$), experiences a rotation of $11\pi/12$ between harmonics. (Even -numbered harmonics being non-existent).

At the harmonic number corresponding to $(T/\tau)-1$, i.e. $n = 11$, the vectors are closest to the in-phase condition. Hence, the vector sum is maximised, and the vector difference minimised. At the harmonic number corresponding to $(T/\tau)+1$, i.e. $n = 13$, the vectors are closest to the antiphase condition. Hence, the vector sum is minimised, and the vector difference maximised.

As the harmonic number departs from T/τ, the difference in amplitude between the sum-vector and the difference vector



a+b
b
a
n=7
a-b
a-b
a
n=9
a+b
b
a+b
b
a
n=11
a-b
a+b
b
a
n=13
a-b
a-b
a
n=15
a+b
b
a+b
b
a
n=17
a-b

Figure 4.20

becomes less marked. However, these harmonics will be increasingly attenuated by the half-length sequence weighting function.

Thus, the addition, or subtraction, of the two phase-displaced sequences produces the desired result, which was to attenuate one harmonic from the predominant pair in a bipolar half-length sequence.

The general circuit which was used to generate these waveforms experimentally is shown in figure 4.21. A dividing ratio of 24 is obtained with the arrangement shown, but the same principle was used for other dividing ratios.

A driving frequency of 384kHz was applied to flip-flop F1. This provided two complementary square-waves at a prf of 192kHz, which were used to drive the pair of flip-flops, F2 and F3. This produced two square waves at a prf of 96kHz which were in quadrature, i.e. had a relative displacement of a quarter-period.

Flip-flops F4 and F5 were cross-connected to give a stage dividing by three, whilst flip-flops F6, F7, and F8 each provided a stage dividing by two. Thus, the overall division from the input of F4 to the output of F8 was by a factor of twenty-four.

The complementary 4kHz square-wave outputs of F8 were combined with the reference-phase 96kHz square-wave by means of the double NAND gates, G1 and G2. This produced two half-length unipolar sequences, displaced by half the period of the 4kHz square-wave. One sequence was applied to the non-inverting input of the differential amplifier A1, whilst the other sequence was applied to the inverting input. The output of A1 was, therefore, a half-length bipolar sequence.

Using the appropriate logical combination, the double AND gate G3, and the flip-flop F9, were used to generate complementary 4kHz square-waves, which were displaced by a quarter-period from those at the output of F8. The quadrature version of the 96kHz square-wave was then gated by these 4kHz square-waves, using the two double NAND gates G4 and G5. A bipolar quadrature sequence was then obtained by means of the differential amplifier A2.

The two bipolar sequences were applied to a pair of operational amplifiers, A3 and A4. The output of A3 was the sum of

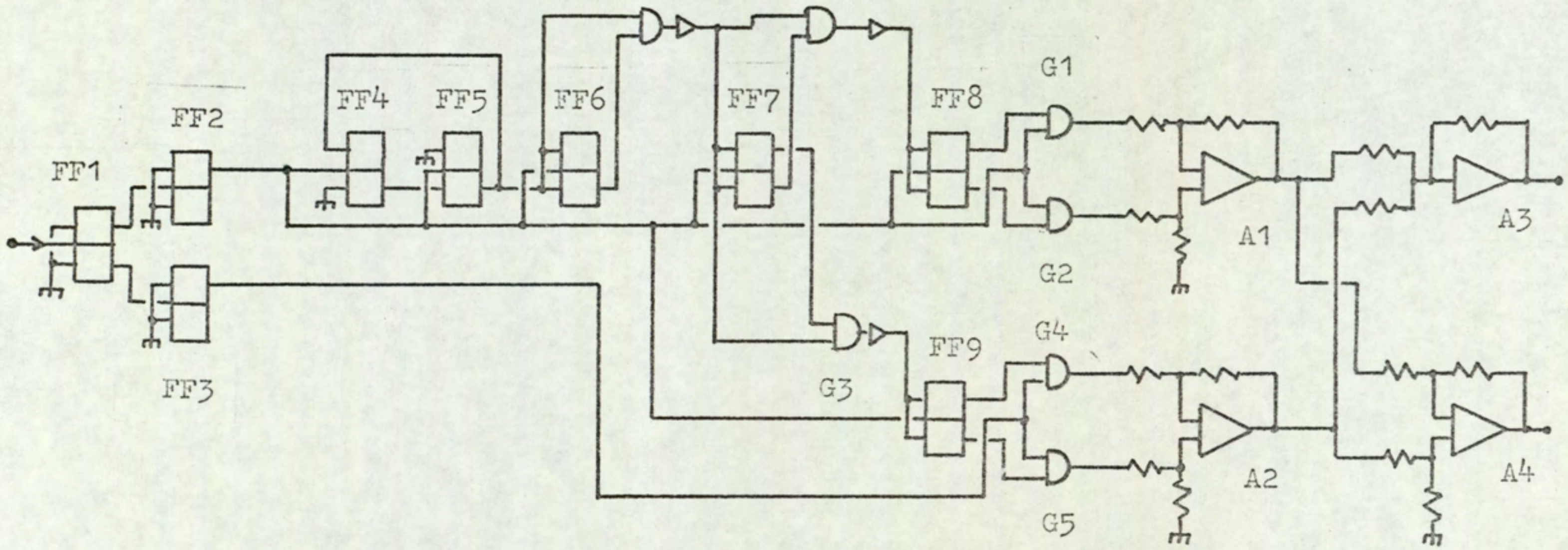


Figure 4.21

the two bipolar waveforms, and the output of A4 was their difference. The general appearance of the output waveforms was as indicated in figure 4.19, but subject to some rounding of the pulse edges caused by the limited amplifier bandwidth.

As with the circuits previously described, the logic circuits were assembled from MOTOROLA integrated-circuit elements, whilst the amplifiers were modular elements, ANALOG DEVICES Type111.

Again, the multi-level waveforms were applied directly to the input of the sweep-frequency spectrum analyser, the output of which operated an X-Y recorder to produce the experimental spectra. When used to generate pure carriers, the outputs would , of course, be applied to bandpass filters. It would be feasible to use the amplifiers, A1- A4, as active filters, in addition to their function as summing and difference amplifiers. This would allow further simplification to the final stage of passive bandpass filtering.

In the experimental circuit, the gain-setting resistor chain across the amplifiers was made adjustable, to permit balancing of the waveform amplitudes. It was found that a slight unbalance improved the suppression of the main unwanted harmonic.

This was attributed to the lowpass characteristic of the amplifiers. It has been seen that the amplitude of any particular harmonic in a combined pulse sequence is determined by the net effect of the Fourier Series coefficient, the bipolar half-length sequence weighting function, and a weighting function depending on the form and relative displacement of the composing sequences. For a complete description, the effect of the system response must also be included.

The amplifiers used in the experimental arrangement had a flat amplitude response up to approximately 150kHz. However, the amplitude at any particular harmonic number is partly determined by higher frequency components. Attenuation of the latter means that the system will approach the simple case of a single carrier and a single modulating term.

For the components at $n = (T/\tau) \pm 1$, this implies that the vectors will more nearly approach the in-phase or antiphase condition. Thus, a differential adjustment of the sequence amplitudes will have a more pronounced effect on this pair of harmonic amplitudes.

Reducing the amplitude of the main unwanted harmonic may result in an increase in amplitude of the subsidiary unwanted harmonics. This would be acceptable, provided that the increase in amplitude was linearly proportional to the frequency difference from the wanted harmonic. In the experimental spectra shown subsequently, some adjustment was made to minimise the amplitude of the main unwanted harmonic. However, a detailed optimisation in this respect, which would have involved shaping the amplifier response, was not attempted.

The means by which spectra having a predominant harmonic at any one carrier frequency from the standard frequency-division-multiplex group are next described.

4.6.d. Choice of source frequency.

The steps involved in generating spectra which emphasise a wanted harmonic consist of:-

(1) Division of a source frequency, f_s, by some factor, r, to produce a square-wave having a prf, f_o, = f_s/r.
(2) Formation of a half-length bipolar sequence, in which the predominant harmonics, (within the range of interest), will be at $f_s \pm (f_s/r)$.
(3) Generation of a quadrature version of the sequence in (2), followed by their combination to suppress one of the predominant harmonics.

Having previously discussed steps (2) and (3), some further comment is required on step(1).

It is evident that if only one source frequency, f_s, is available, (which would be the most economical method), then it must be possible to divide f_s by some integer, r, to produce a pair of frequencies, $f_s \pm (f_s/r)$, corresponding to two of the desired carrier frequencies; and then to divide f_s by some other integer, r', to produce another pair of wanted frequencies at $f_s \pm (f_s/r')$, etc.

Since the side-frequencies fall symmetrically about f_s, the most efficient choice would be to place f_s at the centre of the desired range. As the standard fdm group of carriers lie at 4kHz intervals from 64kHz to 108kHz, the centre frequency is 86kHz. If this were chosen as the source frequency, the first pair of side-frequencies would require to be at 84 and 88kHz. These could be obtained if 86 were divided by 43. The next pair would be 80 and 96kHz, i.e. having a 6kHz separation from 86kHz. However, this makes the dividing ratio 86/6, which is non-integer. The other frequencies in the range would also involve non-integer ratios, so that 86kHz is not a useful choice.

The source frequency may be chosen to correspond with one of the carrier frequencies, say, 84 or 88kHz. The first pair of side-frequencies would be separated by 4kHz from f_s, and both 84 and 88 are divisible by 4. However, the next pair require division by 8, so that 88kHz must be chosen. This conflicts with the requirement for the next division, which must be by 12.

Thus, if the chosen frequency corresponds to one of the carriers, it should, ideally, be divisible by 4, 8, 12, 16, etc., the number of factors depending on how far f_s is offset from the centre.

frequency of harmonic

source freq.	64	68	72	76	80	84	88	92	96	100	104	108
64	1	16	8		4				2			
68	17	1	17									
72	9	18	1	18	9	6			3			2
76			19	1	19							
80	5		10	20	1	20	10		5	4		
84			7		21	1	21		7			
88					11	22	1	22	11			
92							23	1	23			
96	3		4		6	8	12	24	1	24	12	8
100					5				25	1	25	
104									13	26	1	26
108			3						9		27	1

dividing ratios

The above table shows the appropriate dividing factor, (where one exists), to produce a desired carrier frequency from each possible choice of source frequency. It will be seen that the most useful source frequency is 96kHz. Only two carriers cannot be generated directly, 68kHz and 76kHz. These could be obtained by dividing 72kHz by 18. In practice, the 72kHz source frequency would, in any case, be available as the filtered output from the divide-by-4 stage operating with the 96kHz source frequency. As was previously mentioned, a source frequency of 96kHz might be available if the signal processing terminal included a standard tdm system.

The results from the experimental carrier generators are next considered.

4.6.e. Generation of 92kHz and 100kHz.

This pair of frequencies was generated by using the circuit of figure 4.21. Dividing the 96kHz source frequency by 24 produced a gating square-wave at 4kHz. In-phase and quadrature sequences were combined in analogue amplifiers, as previously discussed, and applied to the 1L5 spectrum analyser. The recorded analyser output is shown in figure 4.22. For greater clarity, the frequency scale was expanded to show only those harmonics within the range of interest.

Figure 4.22.a. shows the spectrum in which the component at 92kHz was emphasised. The closest harmonics with significant amplitudes are those at 76, 84, and 108kHz. The component at 100kHz, which is of comparable magnitude to the 92kHz component in a half-length bipolar sequence, has been largely suppressed.

An approximate theoretical value for the ratio of the amplitude of an unwanted component to that of the wanted component may be obtained from the ratio of

$$\frac{\cos(p\pi 3\tau/4T)+\cos(p\pi\tau/4T)e^{-jp\omega_0(2m-1)\tau/4}}{\sin(p\pi\tau/T)} \quad \text{to}$$

$$\frac{\cos(q\pi 3\tau/4T)+\cos(q\pi\tau/4T)e^{-jq\omega_0(2m-1)\tau/4}}{\sin(q\pi\tau/T)}$$

where p is the harmonic number of the unwanted component, and q is the harmonic number of the wanted component.

This is the ratio obtained by taking only the vector

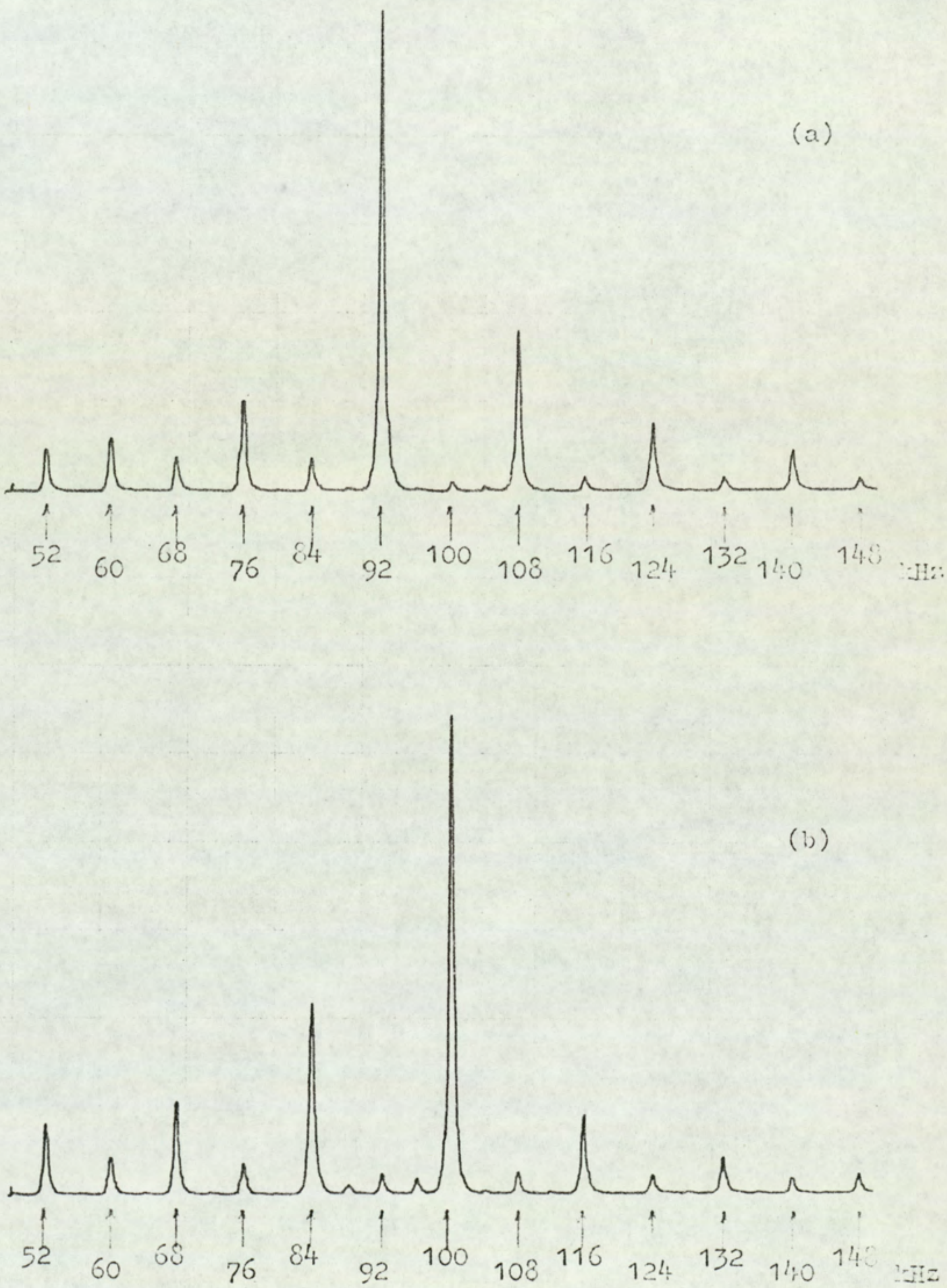

(a)
52
60
68
76
84
92
100
108
116
124
132
140
148
kHz
(b)
52
60
68
76
84
92
100
108
116
124
132
140
148
kHz

Figure 4.22

resultants,(which arise from the combination process),but with each resultant modified by the appropriate value of the bipolar half-length weighting function.

For example,the ratio of the magnitudes of the components at 84kHz and 92kHz will be determined.

Substituting $T/\tau = 24$, $m = 12$, and,(for the 92kHz component),$n=23$,

$\cos n\pi 3\tau/4T = \cos 23\pi/32 = -0.634$

$\cos n\pi\tau/4T = \cos 23\pi/96 = 0.730$

$n\omega_o(2m-1)\tau/4 = 23^2\,\pi/48$, equivalent to a phase lag of 183.75°

The vector summation is

$$\left[(0.634)^2 + (0.730)^2 + (2)(0.634)(0.730)(\cos 3.75^\circ)\right]^{\frac{1}{2}} = 1.36$$

The effect of the bipolar half-length weighting function is found by dividing the vector resultant by $\sin n\pi\tau/T$.

$\sin n\pi\tau/T = \sin 23\pi/24 = 0.130$

$(1.36)/(0.130) = 10.47$

The 84kHz component has $n = 21$.

$\cos n\pi 3\tau/4T = \cos 21\pi/32 = -0.475$

$\cos n\pi\tau/4T = \cos 21\pi/96 = 0.773$

$n\omega_o(2m-1)\tau/4 = 161\pi/16$,equivalent to a phase lag of 11.25°

The vector summation is

$$\left[(0.475)^2 + (0.773)^2 - (2)(0.475)(0.773)(\cos 11.25)\right]^{\frac{1}{2}} = 0.33$$

$\sin n\pi\tau/T = \sin \pi/8 = 0.383$

$(0.33)/(0.383) = 0.86$

Before taking the ratio of these two amplitudes,a correction may be applied for the effect of the Fourier Series coefficient. This will be in terms of a duty ratio of d/2T,so that the first zero of the (sin x)/x function appears at a harmonic number of 96. The normalised coefficient for $n = 23$ is 0.908,and for $n = 21$ is 0.922. The component at $n = 21$ should,therefore,be increased by a factor $(0.922)/(0.908) = 1.02$

$(0.86)(1.02) = 0.88$

The theoretical ratio of the amplitudes at 84kHz and 92kHz is therefore $(0.88)/(10.47) = 0.084$ (= 21.5 db)

This value is not corrected for the amplifier response,or for the effect of adjusting the sequence amplitudes to minimise the component at 100kHz.

In figure 4.22.a.,the measured ratio of the components at 84kHz and 92kHz is 0.066.

The tabulated amplitude ratios for the significant unwanted harmonics are

frequency kHz	calculated ratio	measured ratio
76	0.17	0.18
84	0.084	0.066
100	0.076	0.018
108	0.39	0.33

It is evident that the arbitrary adjustment did not greatly alter the amplitude ratios at 76kHz and 108kHz.

As a basis for comparison,one may consider the Q-factor necessary to separate 92kHz from a component of equal magnitude at,say,96kHz,(as would be required when using the spectrum of a 4kHz pulse-train). Assuming that the harmonics have the same amplitude,and that a 60db rejection is required,then using the relationship mentioned earlier,namely

$$Q \simeq (s)(f_o)/(2)(f')$$

the required value for 92kHz is

$$Q \simeq (10^3)(92)/(2)(4) = 11500$$

In the spectrum of figure 4.22.a.,the component at 84kHz is 0.066 of the amplitude of the component at 92kHz,so that for an overall attenuation of 60db, $s = (0.066)(10^3)$, and

$$Q \simeq (0.066)(10^3)(92)/(2)(8) = 380$$

To reject the component at 108kHz by 60db requires that

$$Q \simeq (0.33)(10^3)(92)/(2)(16) = 949$$

which,although high,is an improvement upon 11500.

Figure 4.22.b. shows the spectrum obtained by taking the difference of the in-phase and quadrature sequences. In this case,the component at 100kHz is emphasised. It will be seen that

neither the components at 96kHz or at 92kHz have been fully suppressed in this experimental spectrum. This may be attributed to differential drift in the amplifiers after the initial setting-up. However, the significant interfering harmonic is at 84kHz, for which the calculated amplitude ratio is 0.3, and the measured amplitude ratio is 0.4.

In terms of the experimentally obtained amplitudes, the required factor is

$$Q \simeq (0.4)(10^3)(100)/(2)(16) = 1250$$

which may be compared with the required Q, when using a 4kHz pulse spectrum, of 12500.

4.6.f. Generation of 88kHz and 104kHz.

To generate this pair of frequencies, the only modification required to the circuit of figure 4.21 was to omit one of the divide-by-2 stages, FF6, and the associated NAND gate. FF7 was driven by FF5 to obtain a final count of 12.

In-phase and quadrature sequences were generated, as before, and their sum and difference taken to emphasise the components at 88kHz and 104kHz. Since T/τ took a value of 12, a spectrum of odd 8kHz harmonics was obtained, i.e. the harmonics were separated by 16kHz. The experimentally recorded spectra are shown in figure 4.23.

It will be seen that the major interfering component is at 72kHz in both cases. In the spectrum which emphasises 88kHz, figure 4.23.a., the measured amplitude ratio of the 72kHz component to the 88kHz component is 0.18. The calculated ratio is found from:-

88kHz, $n = 11$, $T/\tau = 12$, $m = 6$

$\cos n\pi 3\tau/4T = \cos 11\pi/16 = -0.5556$

$\cos n\pi\tau/4T = \cos 11\pi/48 = 0.7518$

phase diff. $= -121\pi/24 = -187.5^\circ$

resultant $= \left[(0.56)^2 + (0.75)^2 + (2)(0.56)(0.75)(0.99)\right]^{\frac{1}{2}} = 1.31$

$\sin n\pi\tau/T = \sin 15^\circ = 0.259$

$(1.31)/(0.259) = 5.05$

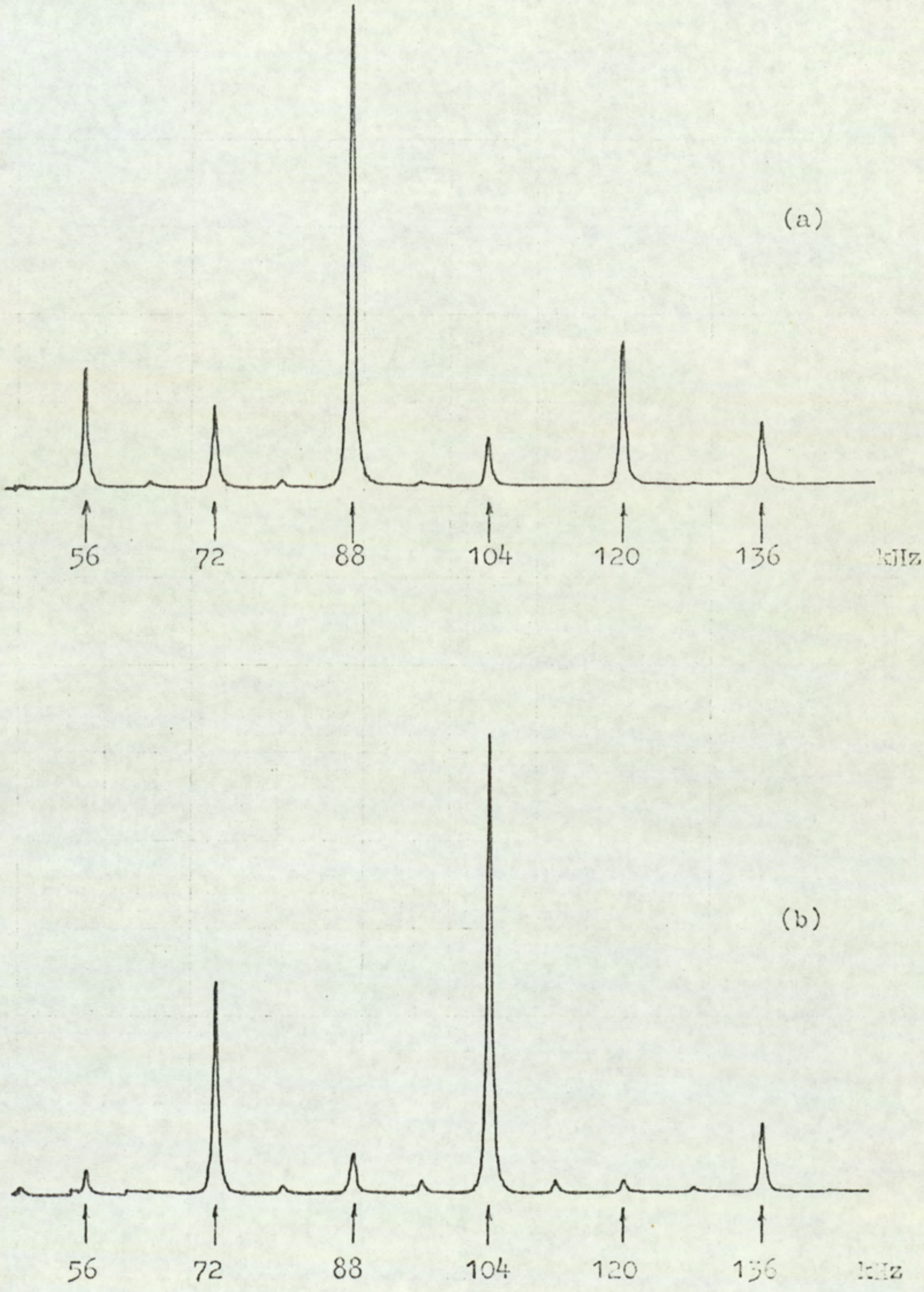


Figure 4.23

72kHz , $n = 9$

$\cos n\pi 3\tau/4T = \cos 9\pi/16 = -0.1950$

$\cos n\pi\tau/4T = \cos 9\pi/48 = 0.8315$

phase diff. $= -33\pi/8 = -22.5^{\circ}$

resultant $= \left[(0.195)^2 + (0.83)^2 - (2)(0.195)(0.83)(0.92)\right]^{\frac{1}{2}} = 0.65$

$\sin n\pi\tau/T = \sin 45^{\circ} = 0.707$

$(0.65)/(0.707) = 0.92$

Correction for Fourier coefficient $= (0.942)/(0.915) = 1.03$

Amplitude ratio $= (1.03)(0.92)/(5.05) = 0.19$

which is in good agreement with the measured value of 0.18.

Comparing on the basis of Q-factors, the value when using a 4kHz pulse spectrum is

$$Q \simeq (10^3)(88)/(2)(4) = 11000$$

and for the spectrum of figure 4.23.a. is

$$Q \simeq (0.18)(10^3)(88)/(2)(16) = 495$$

Figure 4.23.b. shows the spectra obtained by taking the difference of the in-phase and quadrature sequences. In this case, the component at 104kHz is emphasised.

The significant unwanted component is, again, at 72kHz, and the measured amplitude ratio is 0.46. The calculated value is 0.27. It was found that the levels at 88kHz and at 96kHz could not be reduced below those shown in figure 4.23.b., but the cause of the error was not determined.

The required Q using a 4kHz pulse spectrum is

$$Q \simeq (10^3)(104)/(2)(4) = 13000$$

and for the spectrum of figure 4.23.b., is

$$Q \simeq (0.46)(10^3)(104)/(2)(32) = 748$$

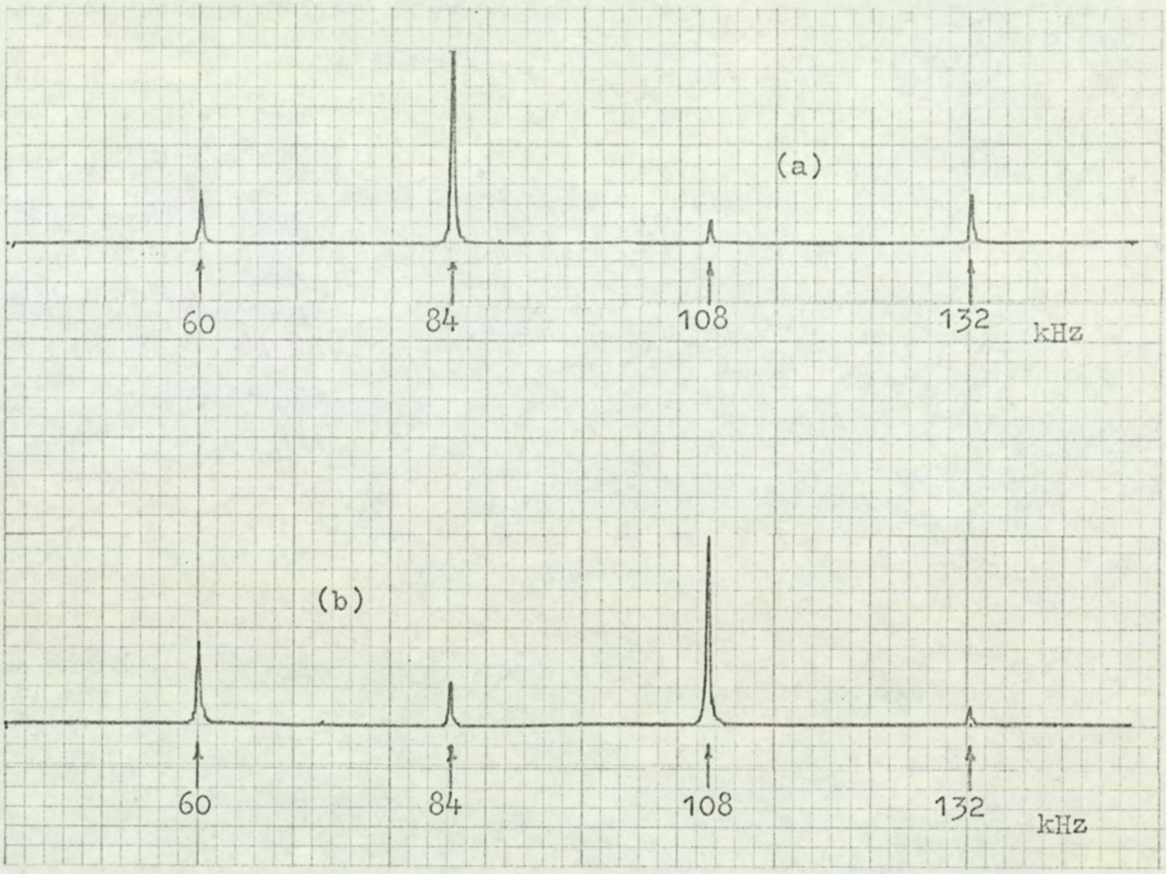


Figure 4.24

4.6.g. Generation of 84kHz and 108kHz.

This pair of frequencies occurs at (96 ± 12)kHz, so that the dividing ratio is 96/12 = 8. The circuit of figure 4.21 was used, by omitting the divide-by-3 stage. Bipolar 4-pulse sequences were formed, and combined, as before. As the spectral components are now at odd multiples of 12kHz, it is evident that the filtering problem is considerably eased.

In the experimental spectrum of figure 4.24.a., there are unwanted components at 60kHz and 108kHz. The former is, of course, outside the fdm carrier range. However, if it is desired to reject this component by 60db, then using the measured amplitude ratio of 0.25, the required Q is

$$(0.25)(10^3)(84)/(2)(24) = 440$$

compared to the required Q when using a 4kHz pulse spectrum of

$$(10^3)(84)/(2)(4) = 10500$$

The unwanted component at 108kHz, which has a lower amplitude than the 60kHz component, will be rejected by the same filter.

The experimental spectrum accentuating the 108kHz component, shown in figure 4.24.b., has the significant interfering component at 84kHz. The amplitude ratio is 0.23, and the filter Q-factor for 60db rejection is

$$(0.23)(10^3)(108)/(2)(24) \quad = \quad 520$$

compared to the required Q when using a 4kHz pulse spectrum of

$$(10^3)(108)/(2)(4) \quad = \quad 13500$$

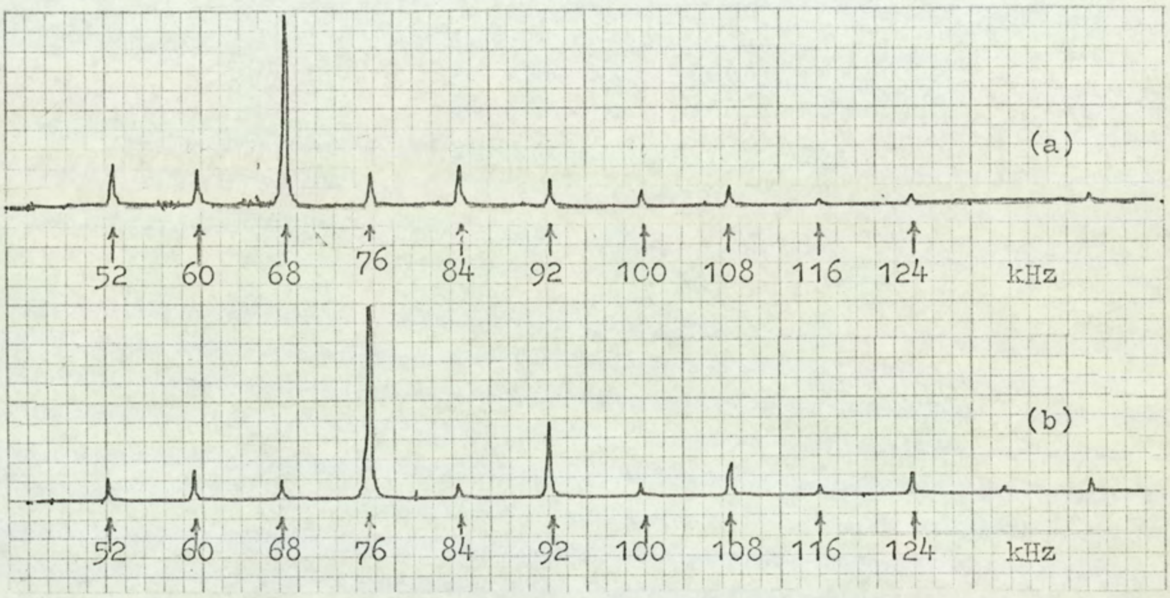


Figure 4.25

4.6.h. Generation of 68kHz and 76kHz.

As previously discussed, it is not possible to generate these two components directly, with a source frequency of 96kHz. One possibility is to use a source of 72kHz, and to divide this by 18. Formation of 9-pulse bipolar in-phase and quadrature sequences would then allow these two components to be separately accentuated.

However, it was found convenient to produce the experimental spectra by different methods, so as to make use of existing circuitry. The 68kHz component was obtained by using a source of 64kHz, and a counter dividing by 16. The 76kHz component was obtained with a source of 80kHz, and a counter dividing by 20.

Bipolar in-phase and quadrature sequences were generated and combined, in the same way that was used to generate the previous spectra. In both cases, the spectral components were at odd multiples of 4kHz, as would be the case if a source of 72kHz had been used.

The experimental spectrum in figure 4.25.a. shows significant interfering components at 52, 60, 76, and 84kHz. The amplitude ratios are approximately 0.2, and a filter which rejects all these components by at least 60db requires a Q of

$$(0.2)(10^3)(68)/(2)(8) = 850$$

compared to a 4kHz pulse spectrum Q of

$$(10^3)(68)/(2)(4) = 8500$$

An experimental spectrum which emphasised 76kHz is shown in figure 4.25.b. The major interfering component is at 92kHz, with an amplitude ratio of 0.35. To reject this by 60db requires a Q of

$$(0.35)(10^3)(76)/(2)(16) = 831$$

compared to a 4kHz pulse spectrum Q of

$$(10^3)(76)/(2)(4) = 9500$$

4.6.i. <u>Generation of 80kHz, 72kHz, and 64kHz.</u>

These remaining frequencies in the fdm range require that 96 be divided by six, four, and three respectively. Because the spectral components occur at relatively large separations, it is not necessary to use the combined sequence method.

If an appropriate bipolar sequence is formed in each case, the unwanted frequency in the predominant pair of harmonics will lie outside the fdm range, and at such a separation that rejection can be achieved with moderate values of Q.

The generating circuitry is, of course, simplified. The spectrum which emphasises 80kHz, shown in figure 4.26.a., was obtained with a 96kHz square-wave driving a divide-by-3 stage, FF4 and FF5 in figure 4.21. This was followed by a divide-by-2 flip-flop, driving a pair of NAND gates and a summing amplifier, as FF8 and A1 in figure 4.21.

As will be seen from the experimental spectrum, the components occur at a separation of 32kHz. The 112kHz component, with a measured amplitude ratio of 0.65, may be rejected by a Q of $(0.65)(10^3)(80)/(2)(32) = 813$, compared to a 4kHz pulse spectrum Q of 10000.

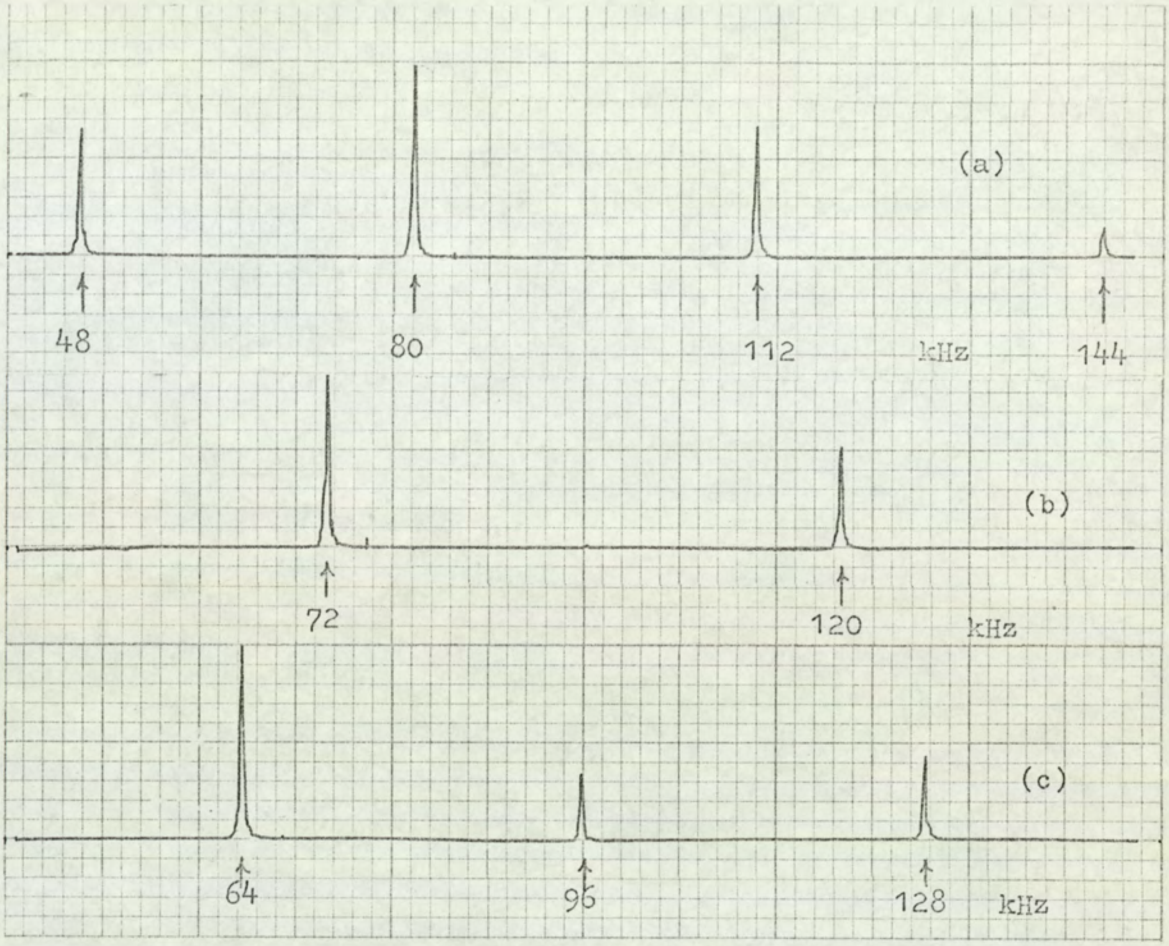


Figure 4.26

To produce a spectrum accentuating the 72kHz component, the 96kHz source frequency was divided by four, and a 2-pulse bipolar sequence was generated. The harmonics occured at odd multiples of 24kHz, i.e. with a separation of 48kHz. The circuit consisted simply of a pair of flip-flops, connected as FF7 and FF8 in figure 4.21, together with the gates, G1 and G2, and the amplifier, A1. The experimental spectrum is shown in figure 4.26.b., and the unwanted component at 120kHz is rejected by a Q of

$$(0.5)(10^3)(72)/(2)(48) = 375$$

compared to the required 4kHz pulse spectrum Q of 9000

To generate a spectrum in which 64kHz occurs as a predominant side-frequency on 96kHz, the counter must divide by 3. This precludes the use of half-length sequences, unless the centre pulse is split. The added complication is unnecessary, since a sequence with, say, two positive-going pulses, followed by a single negative-going pulse, will reduce the amplitude of the 96kHz component. Together with the harmonic separation of 32kHz, the resulting spectrum is then such that only a moderate value of Q is required to separate the 64kHz component.

The circuit used to generate the experimental spectrum consisted of a divide-by-3 stage, FF4 and FF5 in figure 4.21, driving the two gates, G1 and G2, and the amplifier A1. The output waveform from FF5 is a rectangular pulse-train, with a duty ratio of 2/3. The recorded spectrum is shown in figure 4.26.c., and the component at 96kHz may be rejected by a Q of

$$(0.35)(10^3)(64)/(2)(32) = 350$$

compared to the required Q when using a 4kHz pulse-train of 8000.

This completes the required number of carrier frequencies for the standard carrier-telephony fdm system. An overall representation of the experimental spectra which were generated by the methods described is shown in figure 4.27. For comparison, the recorded spectrum of a 4kHz pulse-train is shown at the base of the diagram.

The factors by which the filter selectivity may be reduced, when using these methods, are tabulated below. The ratio is that of Q_1/Q_2, where

Q_1 = Q-factor required for the rejection, by at least 60db, of the unwanted components in the experimental spectrum.

Q_2 = Q-factor required for the rejection, by at least 60db, of the unwanted components in the spectrum of a 4kHz pulse-train, (assuming negligible variation in amplitude of the spectral components).

frequency, kHz	64	68	72	76	80	84	88	92	100	104	108
$(Q_1/Q_2).10^2$	4.4	10.0	4.2	8.7	8.1	4.2	4.5	8.2	10.0	5.7	3.8

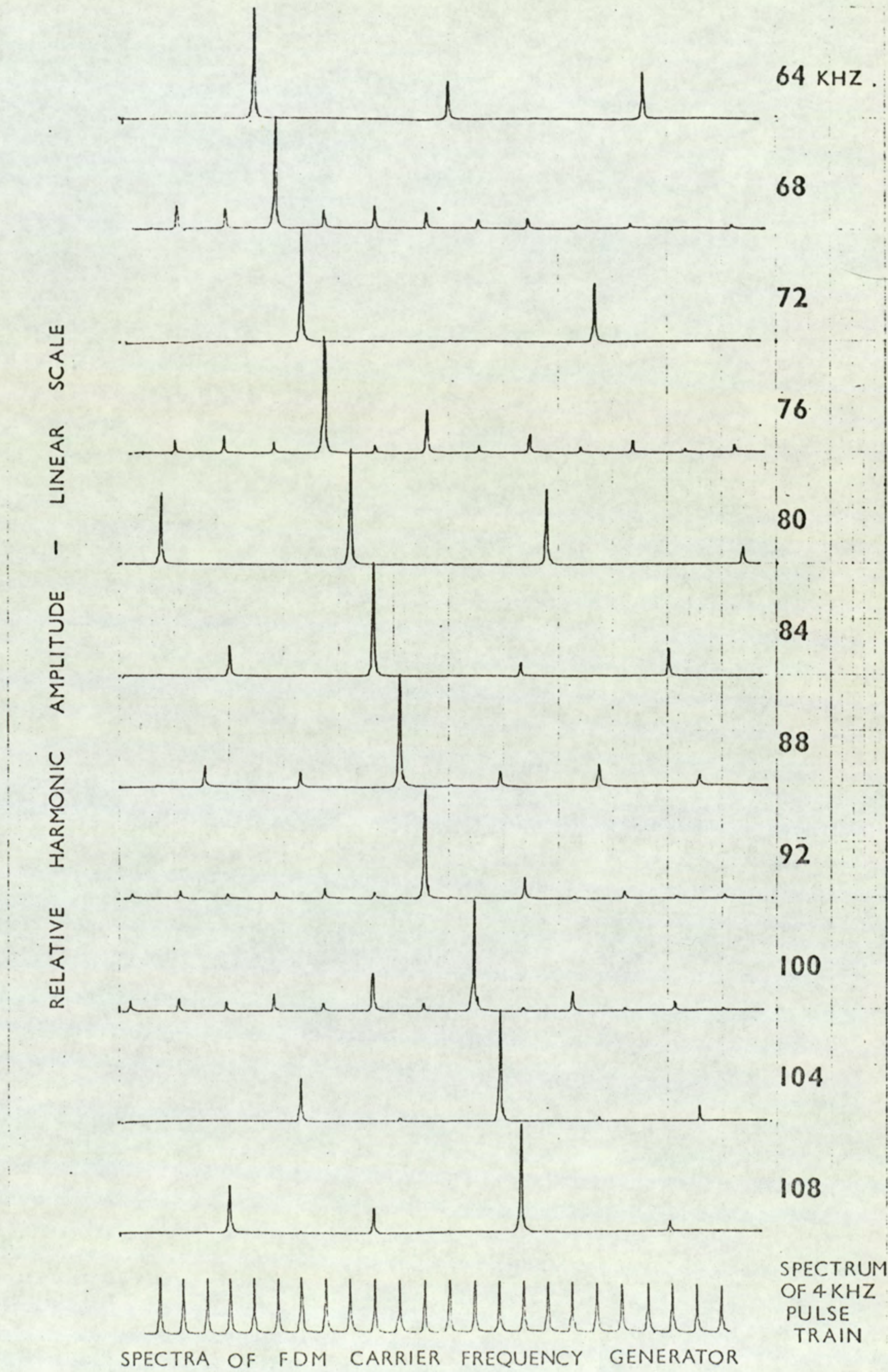

RELATIVE HARMONIC AMPLITUDE - LINEAR SCALE
64 KHZ .
68
72
76
80
84
88
92
100
104
108
SPECTRUM OF 4 KHZ PULSE TRAIN
SPECTRA OF FDM CARRIER FREQUENCY GENERATOR

Figure 4.27

4.7. Conclusions.

In this chapter, the spectral properties of simple regular pulse sequences have been examined. It was seen that these properties could be expressed in a relatively simple analytical form, which indicated ways of shaping the spectrum of a basic continuous pulse-train.

The pulse sequences were such that their practical realisation required only the simpler forms of digital circuitry. However, the combination of unipolar pulse sequences to produce multi-level waveforms was found to be a useful elaboration. The extra circuitry could be realised with standard integrated-circuit amplifiers.

The combination of two pulse sequences having a quadrature phase relationship was considered. This permitted an approach similar to that used for the generation of single-sideband signals by the phasing method.

The generation of a set of harmonically related carrier frequencies, for an fdm system, is conventionally achieved by bandpass filtering the appropriate harmonics of a continuous pulse-train. With pulse sequences, spectra may be generated in which the wanted harmonic is predominant. This makes possible the use of simpler and more economical bandpass filters.

The basis of a system for generating the carriers in the carrier-telephony standard group was investigated experimentally. A set of spectra were obtained, in each of which a wanted harmonic was emphasised. Comparisons on the basis of a theoretical filter Q-factor showed that there was an improvement of at least an order of magnitude over the conventional method.

Although the experimental work demonstrated that this was a feasible application, further investigation would be necessary to optimise the system. The cost of the extra circuitry and power consumption would have to be compared with the saving achieved by simplifying the bandpass filters. The effect of shaping the amplifier response, and of combining sequences with different amplitudes, also requires further investigation.

It is evident that, if a sufficient number of different pulse sequences were generated and combined, it would be possible to synthesise waveforms having spectra which conformed more closely to the ideal of a single frequency. However, the elaboration might be such as to exclude any practical application.

During the course of the experimental work, it became evident that a different approach might be possible, involving the use of either, (a) a chopped sinusoid, or, (b) a sampled sinusoid. For example, if the 96kHz source frequency to the digital circuits was a pure sinusoid, and if the logic gates which form the pulse sequences were replaced by analogue gates, i.e. sampling switches, the spectra would consist only of the sum and difference frequencies formed between 96kHz and the harmonics of the 4kHz gating waveform.

Conversely, if the 4kHz waveform was a piece-wise pure sinusoid, and was sampled by the pulse sequence, the spectra would be less influenced by the higher order harmonics of 4kHz. Although there was insufficient time to investigate the relative merits of these methods, it is felt that they would provide a useful topic for further investigation.

The relative simplicity of the analytical expressions for pulse sequences suggests a possible application to the analysis of pseudorandom and other irregular pulse trains. Although various analyses have been published, these generally involve a relatively lengthy treatment. Non-uniform sampling is another technique which might be more easily analysed by the use of superimposed pulse sequences. Another application might be to the analysis of naturally occuring, apparently random, spectra. However, an investigation of these latter possibilities would have involved a departure from the theme of multiplex systems, so that their validity remains conjectural.

Appendix to chapter 4

Report published in Electronics Letters, 1968, 4, pp. 338-339

CARRIER-FREQUENCY GENERATION USING REGULAR PULSE SEQUENCES

The formation of simple pulse sequences, by the periodic gating of pulse trains, is considered as a method for generating sets of harmonically related frequencies at increased levels and with some suppression of unwanted harmonics. The method may be used to generate the carrier frequencies for multichannel transmission systems.

The carrier frequencies for a frequency-division-multiplex transmission system are usually adjacent harmonics of a base frequency, and are derived from a periodic train of pulses at the base frequency. Since high-order harmonics are required, pulses of short duration and high peak power must be used. An alternative method[1] has been proposed using pseudorandom pulse sequences. This permits the generation of harmonics at given amplitudes with a smaller peak-pulse power. In addition, selected components may be either increased or reduced.

Regular pulse sequences may also be used to produce similar results. Generation of such pulse patterns is straightforward, since digital circuits are used to divide a given input frequency, and gate a train of pulses at the input frequency into sequences with the desired number of pulses.

If the input to the system is a train of rectangular pulses of period τ and pulse duration d, and if the output pulse train is a sequence of m pulses, which recur with period T, the Fourier-series expansion of the output may be expressed as

$$f(t)=\frac{Ad}{T}\sum_{n=-\infty}^{+\infty}\frac{\sin(n\pi d/T)}{n\pi d/T}\,\frac{\sin(n\pi m\tau/T)}{\sin(n\pi\tau/T)}\exp(jn2\pi t/T)$$

where the pulse amplitude is A, and the time origin is at the centre of the pulse sequence.

The spectrum is therefore that of a pulse train of period T, amplitude A and pulse duration d, but the Fourier coefficients are modified by the periodic weighting function

$$\frac{\sin n\pi m\tau/T}{\sin n\pi\tau/T}$$

This function attains a maximum value of $\pm m$ whenever the harmonic number is a multiple of the ratio T/τ, and oscillates over intermediate values in a manner determined by the value of m. The amplitudes of the harmonics of the sequence frequency therefore experience a selective increase or decrease.

In a standard carrier-telephony group, the twelve channel carriers are at 4kHz intervals, from 64kHz to 108kHz. Since these are the 16th to 27th harmonics of 4kHz, they may be generated from a 4kHz pulse train. To generate these same harmonics with a regular pulse sequence, a source frequency must be available which is a multiple of 4kHz and which corresponds to one of the desired harmonics at or near the middle of the range. If, for example, 96kHz is chosen, a square wave at this frequency can be divided by 24 to produce a grating frequency at 4kHz, which then gates the 96kHz input pulse train to produce the desired sequence. For a system which is required to produce all the required range of harmonics at a single output point, a 2-pulse sequence is optimum.

Figs. 1*a* and *b* show the spectrum of a 4kHz pulse train with a pulse duration of $5\cdot2\,\mu$s, and the spectrum of a 2-pulse sequence having the same pulse duration, with an interval of $5\cdot2\,\mu$s between the two pulses of the sequence. In the former, the ratio of maximum to minimum amplitude is 3·6dB, and in the latter it is 4·1dB. The amplitude of the harmonic at 64kHz is unchanged, but the amplitudes of higher harmonics are increased by the weighting function $2\cos(n\pi/24)$. Thus the amplitude at 96kHz is doubled. Only that part of the spectrum containing the desired frequencies is shown.

To ease the filtering problem, the group of twelve carrier frequencies may be generated in subgroups using several base frequencies, i.e. a pulse train at 16kHz, and square waves at 8kHz and 4kHz. The same effect may be achieved with pulse sequences, and the wanted harmonics may be produced with greater amplitudes.

If the number of pulses in the sequence is $T/2\tau$, the even harmonics of the base frequency are eliminated, except for those components which are multiples of T/τ. Dividing the 96kHz input by 24 and forming a 12-pulse sequence, as shown in Fig. 1*c*, will therefore generate only the odd harmonics of 4kHz. A spectrum with only the even harmonics of 4kHz is generated if the 96kHz input is divided by twelve to produce an 8kHz 2-pulse sequence.

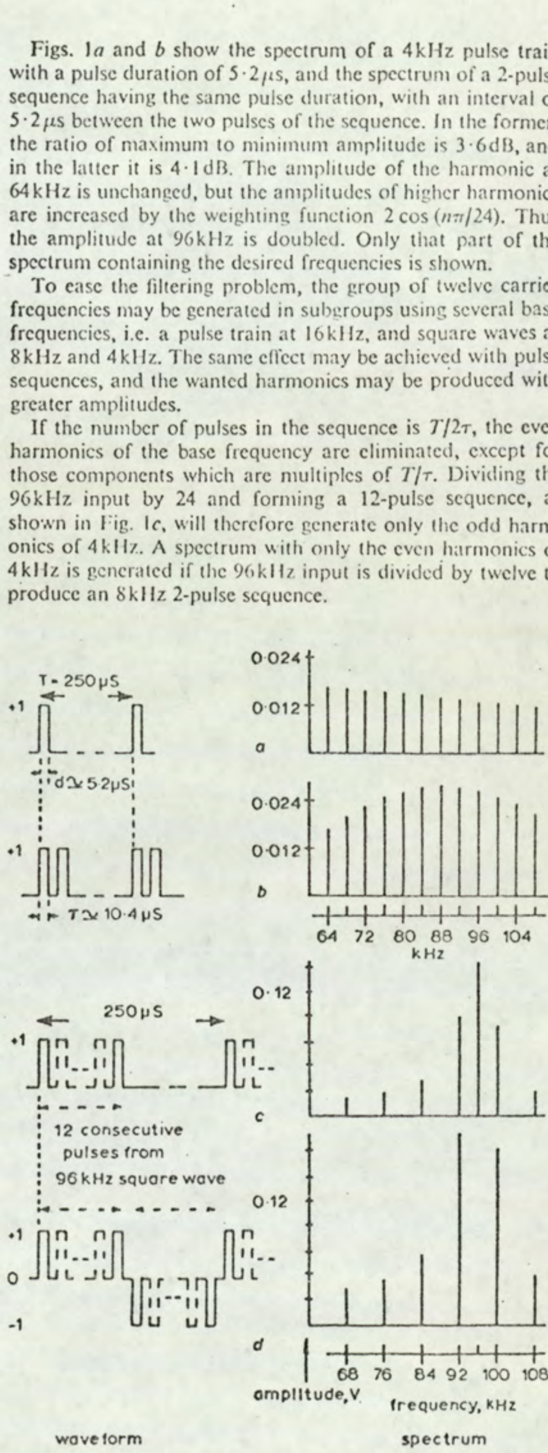


Fig. 1 *Generation of a group of frequencies using unipolar and bipolar sequences*

By this means, the group of twelve carrier frequencies may be separated into two groups of six, which have, with the exception of the components at $(T/\tau) \pm 1$ in the 12-pulse sequence, a separation between components of 8kHz. The component at T/τ, corresponding to the input frequency, is redundant, and may be eliminated by forming a bipolar sequence, as shown in Fig. 1*d*. A uniform spacing of 8kHz is thereby obtained for both sets of harmonics.

Introducing multilevel sequences into the system allows further operations of a 'filtering' nature to be performed. Ideally the output of the carrier generating system should be 12 pure sinusoids at the appropriate frequency from 12 separate output points. An approach to this may be made by considering that, for 'half-length' pulse sequences, i.e. where $m = T/2\tau$, the harmonics at $n = T/\tau$ and $n = (T/\tau) \pm 1$ are more heavily weighted than the other harmonics in the required range. The frequencies corresponding to a given harmonic number depend, of course, on the value of the dividing ratio T/τ. Thus, any pair of frequencies corresponding to $(T/\tau) \pm 1$, with predominant weighting over the remainder of the in-range spectrum, may be generated from the division of the source frequency by an even integer of the appropriate value and the formation of a half-length bipolar sequence.

Having generated a spectrum in which a pair of harmonics predominates, one will be the desired output frequency requiring further isolation. An approximation to a quadrature version of a given pulse sequence may be generated without difficulty using digital divider circuits, provided that the input frequency to the system is increased by a factor of four, e.g. to 384kHz. The combination of a bipolar sequence and its quadrature version by analogue addition, or subtraction, results in the partial suppression of one of the predominant harmonics.

The spectrum has the desired output frequency as a major component, although the fundamental and some low-order harmonics also retain large amplitudes. Separation between harmonics is, of course, not less than twice the base frequency, or sequence frequency, which is used for a given output frequency. Spectra obtained by this method are shown in Fig. 2, although the precise amount by which unwanted harmonics are suppressed depends on the accuracy with which positive- and negative-pulse amplitudes are balanced.

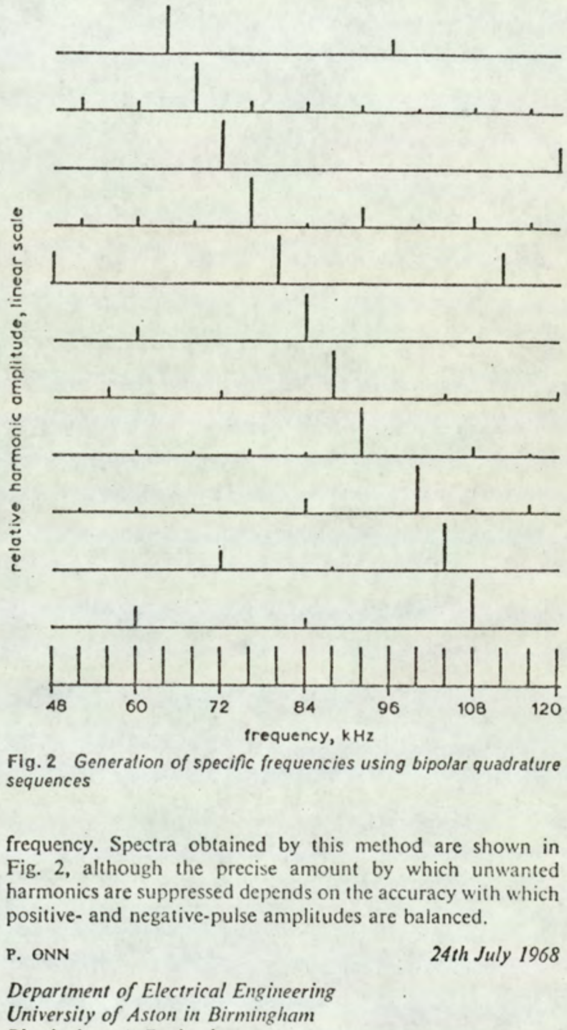


Fig. 2 *Generation of specific frequencies using bipolar quadrature sequences*

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CHAPTER 5

CONCLUSIONS

This thesis has been concerned with multiplex signal systems,and,relevant to that general theme,four topics have been discussed. Each of the previous chapters presented a self-contained aspect,so that conclusions were drawn when appropriate. In this final chapter,which is a survey and interpretation of the results as a whole,some previous material will,therefore,re-appear.

The first chapter was an attempt to show that a unifying representation of linear multiplexing systems is provided by regarding such systems as an operation with orthogonal functions. That these functions may be used as a basis for multiplexing is not,in itself,a new concept. The earliest reference which was found was to some unpublished work of Shannon,(reference 7),prior to 1951. Since then,various authors have discussed this topic, but in such a way that the use of orthogonal functions was presented as an alternative to conventional methods of multiplexing.

The intention in chapter one was to show that the most commonly used methods of frequency-division-multiplexing and time-division-multiplexing,together with a number of unconventional methods which emerged from a literature survey,are inherently dependent upon the use of orthogonal functions. It is true that this applies only to linear multiplexing;however no mention was found in the literature of any strictly non-linear system,in the sense of using a non-linear filter for signal recovery.

This approach is less general than that given by Zadeh and Miller,(reference 1). However,the latter may be regarded as a completely general mathematical description of multiplexing, which requires some restriction before being applied to practical systems. The orthogonal function approach does not call for any specialised mathematical technique,and was found to provide an apt theoretical model for the various operations involved in a practical multiplex system.

As signals are classically conceived of as functions of time,the convenient definition of orthogonality was seen to be that which involved the integral of the product of two such functions. The processes of modulation,demodulation,and lowpass filtering,were seen to be conformable to that general description. The transmitted signal is a function of both the message and an orthogonal carrier,and it is the latter which makes possible separation at the receiver when a number of messages are multiplexed.

The complementary definition of orthogonality,which implies mutual perpendicularity in a geometrical sense,was seen to be applicable when using the signal-vector space methods introduced by Kotel'nikov,and by Shannon. This representation was found to be convenient for discussing the related topics of single-sideband generation,and bandpass filtering.

Some unconventional multiplexing methods which have been described in the literature were examined,and were found to conform with the orthogonal function model. However,it appears to be the case that none of these methods has supplanted those conventionally used. On the one hand,single-sideband fdm will provide a system which uses the minimum possible bandwidth. Although the necessity for precise bandpass filtering,and for coherent demodulation,mean that the system is relatively expensive to realise,the necessary techniques are well established. On the other hand,when sufficient transmission bandwidth is available,a tdm system using pulse-amplitude-modulation,or pulse-code-modulation,is more economically realised.

The other methods which were examined,and various possible schemes which could be suggested from the general orthogonal function approach,fall into the intermediate region. The required transmission bandwidth is generally greater than the minimum possible,and realisation is generally more difficult than for conventional tdm.

It is concluded from this,that the generalised approach in terms of orthogonal functions is useful for the interpretation and comparison of different multiplexing methods. However,the mere knowledge that,theoretically,any set of functions having the property of orthogonality could be used as the basis of a multiplexing system,does not,in itself,suggest any clearly superior techniques to those conventionally used.

The topics discussed in the second chapter had formed the original subject for investigation. A method for converting directly from a 12-channel tdm signal,with a prf of 8kHz,to a 12-channel standard carrier-telephony fdm signal,had been suggested in the literature,and ways were sought of developing this theme.

It appeared that a possible improvement might be to bandpass filter alternate upper and lower sidebands from the tdm signal harmonics. This would reduce by half the number of carrier frequency generators required at the receiver. However,this method could equally well be used when generating a ssb fdm signal by conventional means.

The direct conversion from fdm to tdm was found to be less easily realised. Although it was demonstrated that a tdm signal could be obtained by direct sampling of a dsb or ssb fdm signal,the channels had first to be separated by bandpass filtering. The required circuitry was such that no clear advantage emerged over the usual methods.

Systems have been described in the literature for the realisation of a bandpass function with variable centre-frequency. These involve the use of parallel networks of modulators and fixed cutoff-frequency lowpass filters. An analysis,in terms of sinusoidal modulation,was performed,as an adjunct to the generalised analysis of Franks and Sandberg,(reference 32). However, there are inherent limitations to the rate at which the centre-frequency can be changed,which preclude the use of a stepped-centre-frequency,or commutating,bandpass filter for fdm to tdm conversion. The possibility of performing this conversion by real-time digital computation was briefly considered,but was found to require excessively high computation speeds.

That there should be two forms of multiplexing in common use arises from the minimum transmission bandwidth required by ssb fdm,but the superior performance of tdm for terminal operations. This suggested that a tdm system which could be transmitted in the minimum bandwidth would remove the need for system interconversion.

The possibility of such a system was considered in chapter three. If an ideal lowpass filter were used to exclude all frequencies greater than half the prf of a pam tdm signal, the resultant would occupy the minimum possible bandwidth. With practical filters and sampling pulses, this would lead to waveform distortion and crosstalk. However, an approach was developed based on the fact that truncation of the spectral range of a pulse-train gives rise to a waveform composed from the summation of a finite set of harmonics. This compares with the infinite summation required for the original pulse-train.

A finite summation of cosine harmonics was seen to reduce to a form known as the periodic Dirichlet kernel. Each period of the waveform was observed to contain a peak, followed by regularly spaced zero-crossings. If m harmonics were summed, there were 2m zero crossings. Thus, (2m+1) waveforms could be superimposed, each displaced in such a manner that the peak of any one waveform coincided with the zero-crossings of other waveforms.

Modulating the waveforms did not affect the positions of these zero-crossings. If the fundamental component of a waveform had a frequency twice that of the message bandwidth, then sampling the waveform at the peaks would provide a normal pam signal. The summation had to include a dc term to locate the zero-crossings correctly, so that the bandwidth of a modulated waveform became (2m+1) times the modulating signal bandwidth. This was the same for all waveforms, so that the multiplex signal occupied a bandwidth given by the product of the number of channels and the modulating bandwidth, i.e. the minimum possible.

With any practical sampling pulse, a portion of the waveforms each side of the zero-crossings would also be sampled, causing crosstalk. In the superimposed waveforms, however, some mutual cancellation occured. It was found that a sampling-pulse duration of 8% of the interval between successive peaks of the multiplex signal, i.e. the channel time slot, permitted a crosstalk ratio of 60db.

The experimental realisation of these waveforms was found to involve some difficulty, because of the need to provide a precise dc component, and to generate the harmonics with a cosine phasing. The set of phase-shifted harmonics could be generated economically by digital circuits, and considerable simplification would be possible if the harmonics were sine phased.

For this reason, a number of different summations of sinusoids were examined. These were all standard expressions, having an analytical closed form. Each waveform was evaluated over a range of parameters by means of a computer programme, and the results plotted for comparison. It was found that only one of these had useful properties, the waveform obtained by summing odd-numbered sine harmonics with alternating polarity.

However, this waveform being easier to realise than the periodic Dirichlet kernel, the properties were examined in detail. The waveform was observed to contain two peaks of opposite polarity in each period. The zero-crossings were such that, with m composing harmonics, 2m waveforms could be multiplexed. Hence, if the fundamental harmonic had a frequency corresponding to the message bandwidth, the multiplex signal again occupied the minimum possible bandwidth.

A normal pam signal could be obtained by sampling the waveform at the peaks, and reversing the polarity of alternate samples. An expression was derived for the crosstalk, which showed that the sampling pulse duration again required to be of the order of 8% of the channel time slot for crosstalk ratios of 60db. The waveforms were such that synchronising information could be extracted at the receiver without allocating a channel for that purpose.

It was established, however, that the crosstalk was critically dependent upon precise timing of the sampling pulses, and precise realisation of the harmonic amplitudes and phases. The tolerance on each of these might require to be as low as $\pm$ 0.05 % in order to obtain a crosstalk ratio of 60db.

In order to check the theoretically derived crosstalk, an experimental system was assembled, which generated the waveform for a four-channel system. The errors were such that the crosstalk ratio could only be measured for lower values, and a graphical check was made at the centre of the range. The experimental results showed close agreement with the theoretical values.

It was concluded that a tdm system based on these principles was not entirely impracticable, and might have some useful properties in practice. Bandwidth restriction does not depend, directly, on precise filtering. If digital circuits were to be used for generating the carrier waveforms, bandpass filters would be necessary to separate the harmonics, but only one filter per two channels would be required. Realisation would be less expensive, in that respect, than a ssb fdm system.

The receiver does not entail any greater elaboration than is normally required for the recovery of a tdm signal, and the transmitted waveform contains the necessary information for synchronisation. However, although precise timing of the sampling pulses might be obtained without great difficulty, the need for precise realisation of the waveforms is a major restriction.

Not only must the position of the zero-crossings be located with great precision when transmitted, but the transmission path characteristic must be such as to produce negligible distortion over the multiplex signal bandwidth. This limits the maximum number of channels which could be multiplexed.

These restrictions indicate that the system is unlikely to be useful for speech transmission. However, considerably lower crosstalk ratios are permissible for binary data transmission. It is current practice to lowpass filter a pcm signal with a cutoff of half the prf, but the transition band may extend to beyond the prf. Using finite summations of harmonics as carriers, the transmission bandwidth is precisely fixed at half the prf. It is considered, therefore, that a further development of this method might have useful applications to data transmission.

The latter investigation had been based on the properties of time-domain waveforms generated by summing a small number of discrete frequency-domain components,i.e. a set of harmonically related frequencies. The dual to this is the frequency-domain waveform,or spectral envelope,generated by summing a small number of discrete time-domain components. The latter might be interpreted as meaning a periodic and regular sequence of pulses.

This approach was investigated in the fourth chapter, where expressions were derived for simple regular sequences. These showed that the normal Fourier series coefficient became modified by a periodic weighting function. The waveforms generated by these weighting functions were comparable to the time-domain waveforms encountered in chapter three.

The properties of the weighting function were examined, and expressions obtained for the spectral envelopes of simple unipolar and bipolar sequences. These indicated ways in which a particular harmonic might be increased in amplitude with respect to adjacent harmonics.

A particular application was seen to be the generation of a set of harmonically related carrier-frequencies,such as those used in the standard carrier-telephony fdm system. Conventionally, these are generated from a 4kHz pulse train. This implies short pulse durations,and hence,high peak pulse power,which can,however, be reduced by the use of pulse sequences. More importantly,the 4kHz separation of the harmonics requires highly selective filters, and this requirement can also be eased by the use of pulse sequences.

It was found that if the method was extended to include the combination of two phase-displaced sequences,the accentuation of a wanted harmonic could be particularly marked. Hitherto,realisation of the pulse sequences had required only simple digital circuits. Combination of sequences required the use of analogue circuits,but it was considered that the advantages to be gained justified this elaboration.

As an experimental exercise,circuits were assembled to generate spectra which emphasised each of the standard group carrier frequencies in turn.

The pulse sequences were applied to a sweep-frequency spectrum analyser,and the resulting spectra were recorded. Measurements of the spectral amplitudes showed that the results were generally in accordance with the theoretically predicted values. The degree of isolation of a particular harmonic was such that the theoretical Q-factor of the final bandpass filter was an order of magnitude less than that required by the conventional method.

It was concluded that this would represent a useful practical application. However,there might be ways of improving upon this system,which need not make direct use of pulse sequences. This particular application was one related to the general theme of multiplexing,but it is considered that the analytical expressions which have been developed are potentially useful in other fields.

In conclusion,it may be remarked that the topics discussed in this thesis all invite further development. Although the orthogonal function approach did not immediately indicate ways of improving multiplex systems,the number of functions which were examined was by no means exhaustive. The dual nature of the waveforms obtained by finite summations in the time and frequency-domain indicates that further useful properties might readily be found.

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