"OPTIMUM DESIGN OF STRUCTURES"

BY

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SYNOPSIS.

The work presented in this thesis is to produce general computer programmes for the automatic optimum design of realistic, rigidly jointed multi-storey frames. The work carried out can be divided into three main parts. In the first part, a non-linear programming algorithm is proposed for the minimum weight design of structures in which joint displacements are included as design variables and the design problem is formulated by the matrix displacement method. An approximating programming method is employed to obtain its solution. This method is general and can be equally applied to rigidly jointed and pin jointed structures subject to deflexion and stress constraints. The move limits may be arranged, so that the number of iterations required to obtain the final design can be kept to a minimum.

In the second part, the theorems of structural variation are extended and shown to be applicable to rigidly jointed structures. These enable the prediction of the behaviour of a pin jointed structure, from the results of analysing another, more general, or parent rigidly jointed structure. It is shown that these theorems can be applied to select a more suitable shape of a structure in terms of minimum weight while satisfying stress and deflexion requirements.

In the third part, the theorems of structural variation are employed to evaluate the significance of each member in the behaviour of a rigidly jointed structure and the problem of minimum weight design is extended to include shape. A method is proposed for the design of rigidly jointed structures of optimum shape with stress and deflexion limitations. Finally, this method is applied to a number of frames some of which have architectural constraints.

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NOTATIONS.

a_{ij}	direction cosines of a bar (Chapter 1)
a,b,c,d,e,f	stiffness parameter
e_i	bound on displacement variable x_i
$\underline{f}$	matrix of member forces due to unit axial loads
f_c, f_{c_1}	constants (Chapter 3)
f_{ij}	force in member j due to unit loads at the ends of member i
$f(x)$	objective function (Chapter 1)
f_{ii}	matrix of axial forces for members due to first end unit loading for the variation of second moment of area of member i
$g_k(x)$	type I inequality constraints
$h_j(x)$	equality constraints
$\underline{k}$	member stiffness matrix
ℓ_p, ℓ_c	length of tensile and compressive member (Chapter 1)
ℓ_p, m_p	direction cosines
m_ℓ, m_u	move limits
$m_{f_{ii}}$	matrix of first end moments for members due to first end unit loading for the variation of second moment of area of member i
$m_{s_{ii}}$	matrix of second end moments for members due to first end loading for the variation of second moment of area of member i
p_i	axial force in member i
pv, pv2	constants (Chapter 3)
r_{α_i}	variation factor for area variable of member i
r_{β_i}	variation factor for second moment of area variable of member i
r	the penalty parameter
S_{ii}	matrix of shear force for members due to first end loading for the variation of second moment of member i
$S_\ell(x)$	type II inequality constraints
u_j	new design parameter unrestricted in sign (Chapter 1)

v_o	original volume before removing member i
v_i	volume of feasible structure on removal of member i
$\underline{x}$	vector of variables
x_i	i-th variable (Chapter 1), deflexion at node i (Chapter 2)
$\underline{x}_i$	vector of variables at i-th design point
$\underline{x}_{i+1}$	vector of variables at (i+1) the design point
x_i', x_i''	new positive variables for x_i
x_i^*	element of new deflexion vector
y	distance of extreme fibre from neutral axis of member
z_j^o	initial design parameter of element j
A	cross-sectional area
A_i	area of the adopted section for group i
$\underline{A}$	displacement transformation matrix, area variables (Chapter 2)
A_m^*	new area for member m
A_{11}, B_{11}, F_{11}	coefficients of area variables in the overall stiffness coefficient matrix
$A_{12}, B_{12}, F_{12}, C, T, e, f$	coefficients of second moment of area variables in overall stiffness coefficient matrix
A_{21}, A_{22}	coefficients of area variables in overall stress coefficient matrix
$\underline{B}$	load transformation matrix, matrix of coefficients of variables in the axial stresses (Chapter 2)
$\underline{B}_R$, $\underline{B}_S$	submatrices of overall stress coefficient matrix
B_{21}, B_{22}, d, e, f	coefficients of second moment of area variables in overall stress coefficient matrix
$\underline{B}_b, \underline{B}_r$	submatrices of $\underline{B}$
$\underline{C}$	unit load matrix
$\underline{C}(A)$	matrix of coefficients of area variables in bending stresses
E	Young's modulus of elasticity
$\underline{F}$	overall flexibility matrix, axial force matrix for members due to external loads (Chapter 5)
F_i	component of the external force at joint i

$\underline{F}_{bb}, \underline{F}_{br}, \underline{F}_{rb}, \underline{F}_{rr}$	submatrices of $\underline{F}$
F_{I_j}	axial force in member j after the variation of second moment of area at the first end of member i
$G(A,X)$	stiffness constraints
H	horizontal force
I_i	second moment of area of member i
I_i'	new second moment of area of member i
I_m^*	new second moment of area of member m
$\underline{K}$	overall stiffness matrix
$\underline{K}^*$	new overall stiffness matrix
$\underline{K}(A)$	stiffness coefficient matrix
$\underline{K}_{RR}, \underline{K}_{RS}, \underline{K}_{SR}, \underline{K}_{SS}$	submatrices of $\underline{K}$
$\underline{L}$	load matrix
L_i	length of member i
L	span of two storey frame (Chapter 5)
$\underline{M}_f$	matrix of first end moments for members due to external loads
$\underline{M}_s$	matrix of second end moments for members due to external loads
M_i	bending moment in member i
M_{f_i}	moment at the first end of member i
$M_{I_{f_{ij}}}$	moment at the first end of member j after variation of the second moment of area at the first end of member i
$M_{I_{f_{ii}}}$	moment at the first end of member i after variation of the second moment of area at the first end of member i
$M_{I_{s_{ij}}}$	moment at the second end of member j after variation of the second moment of area at the first end of member i
P,Q	local member axis
$\underline{P}$	vector of member forces
RHS	matrix of the right hand side of constraints in linearised problem
S_j	force in a bar (Chapter 1)
S_{AX}, S_{AY}	resultant of shear force at the first end of member on the reference axis

S_{BX}, S_{BY}	resultant of shear force at the second end of member on the reference axis
$S_{I_{ij}}$	shear force in member j after variation of second moment of area at the first end of member i
$\underline{S}$	matrix of shear force for members due to external loads
$\underline{U}$	vector of member displacement
V	structural volume (Chapter 1), vertical force (Chapter 3)
$\underline{V}$	vector of design variables
W	objective function
$\underline{X}$	joint displacement vector
$\underline{X}^*$	new joint displacement vector
$\underline{X}_b$	displacements corresponding $\underline{L}_b$
$\underline{X}_r$	displacements corresponding $\underline{L}_r$
α_i	proportional increase in area of member i
α	proportional increase of member areas
β_i	proportional increase of member second moment of area
γ_i	new displacement variable
δA	change in area
δI	change in second moment of area
δv_i	change in volume due to removal of member i
δ_{ji}	deflexion at joint j due to axial unit loading
ε	maximum permissible strain (Chapter 1)
γ_i	density of material for group i
$\gamma_{Ji}{}^*$	proportional increase of member areas and second moment of areas
$\gamma, \mu, \theta, \alpha, \beta$	constants (Chapter 5)
ν_j	proportional increase of member areas and second moment of areas
$\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$	constants (Chapter 5)
θ_j	deflexion at joint j after changing area of member i
π_i	force in member i
π_{ij}	force in member j while member i is changing

ρ_i	material density
σ_i	combined stress in member i
σ_t, σ_c	permissible tensile and compression stress (Chapter 1)
σ_ℓ^*	new stress in member ℓ
σ_b	bending stress
$\underline{\chi}$	matrix of displacements due to unit loads
χ_{ij}	displacement of node i due to unit loads at the ends of member j
ψ_{ij}	displacement of node i due to alterations in member j
ψ_J	new displacement at node J due to removal of member i
$\underline{\Delta}$	vector of permissible displacements
Δ_J	allowable deflexion at joint J
Δ_j	deflexion at joint j due to external loads
Δz_j	the change in design parameter z_j

CONTENTS

C H A P T E R 1

HISTORICAL REVIEW AND SCOPE OF THE PRESENT WORK.

1.1) INTRODUCTION.

Structural design can be considered as a decision making process in which a certain objective has to be achieved and at the same time some design requirements have to be satisfied. In general the first decision concerns the shape of the structure. The section properties of members are then determined to make the structure sustain the acting external loads safely.

This used to be accomplished using an iterative method in order to obtain a reasonable and feasible design. After World War II the development of a new branch of mathematics known as operational research made it possible to obtain the optimum design as opposed to a feasible design. In particular, availability of a section of this science called mathematical programming and computers has constituted a base for the rapid growth of the interest in the optimum structural design. In most of these works, the minimum material cost was taken as an objective. This was partly because the weight of structures can easily be expressed in terms of the design variables.

Formerly, much effort was spent on determining the optimum sectional properties of the members so that the weight of the structure could be reduced. Depending on the failure conditions chosen, both elastic and plastic theory were employed in the formulation of the design problem. Each formulation, obviously, had its own design requirements. However, it was later found possible to obtain much greater weight reduction with changes in the geometry than with changes only in member sectional properties. Hence, optimising the shape of a structure which is of major importance has attracted great interest in recent years.

It is now apparent that the choice of the optimisation procedure for solution of the design problem is equally as important as the selection of the method for its formulation. Hence, in this chapter, firstly the algorithms of mathematical programming which are utilised in the solution of the design problems are reviewed. Later, the published work on structural optimisation is reviewed considering the optimisation methods which have been used to solve the design problem.

1.2) HISTORICAL REVIEW OF OPTIMISATION METHODS.

In the last two decades, a large number of optimisation methods have been developed and they have been applied in every field of engineering. Practical applications have shown that the efficiency of these methods depends on the type of problem to be solved. Those which have a reasonable amount of experimental evidence, reliability and guarantee of convergence are obviously preferred and applied in structural design. The review of these optimisation techniques which are widely utilised in structural design will be given considering individual topics.

1.2.1) LINEAR PROGRAMMING.

A general linear programming problem may be stated as follows:-

Maximise $W = f(x_i)$ $\qquad i = 1, \ldots, n$

Subject to

$$h_j(x_i) = d_j \qquad j = 1, \ldots, p$$

$$g_k(x_i) \leqslant b_k \qquad k = p+1, \ldots, p+r$$

$$S_\ell(x_i) \geqslant e_\ell \qquad \ell = p+r+1, \ldots, p+r+t$$

$$x_i \geqslant 0$$

where W is the objective function, $h_j(x_i)$ are equality constants and p is their total number, $g_k(x_i)$ are known as type I inequality constraints and r is their total number. $S_\ell(x_i)$ are known as type II inequality constraints and t is their total number. n is the total number of variables.

The first publication on linear programming was by Kantorovich[3] in 1939. However, [illegible] after World War II; the simplex method by Dantzig[4] became available and provided great application of linear programming in every field. In the simplex method type I constraints may be transformed into equalities by adding slack variables $x_{n+1} \ldots x_{n+r}$ such as

$$\sum_{k=p+1}^{p+r} g_k(x_i) + x_{n+\nu} = b_k \qquad \nu = 1,\ldots,r$$

Type II inequalities can be converted into equalities by subtracting surplus variables $x_{n+r+1},\ldots,x_{n+r+t}$;

$$\sum_{\ell=p+r+1}^{p+r+t} S_\ell(x_i) - x_{n+r+\mu} = e_\ell \qquad \mu = 1,\ldots,t$$

The reason for these is to obtain a basic feasible solution in order to start simplex iteration. The detailed explanation of the method can be found in[4,5]. There is no obvious basic feasible solution in the case of equality constraints. This is overcome by introducing artificial variables. In the final simplex table these artificial variables should not appear as they violate the equality requirements and it is necessary to reduce them to zero during the simplex procedure. This can be done by the device suggested by Charnes[6] and known as "Charnes M" method. In minimising a problem a large positive price M is assigned to each of the artificial variables in order to force them to become non-basic. Once this happens they cannot become positive again because of the price M and the nature of the simplex method. It is convenient to write the terms involving M on a separate line. Considering this line as an objective function, normal simplex iterative procedure is carried out until all artificial variables are removed. There are three different stages at the end of the procedure:-

1) All artificial variables become non-basic and the optimality criterion is satisfied. This case indicates that the optimal solution has been obtained.

2) One or more artificial variables are basic variables with values of zero and the optimality criterion is satisfied. This condition also indicates that the optimal solution has been obtained.

3) One or more artificial variables cannot be made non-basic e.g. they have positive values. In this case there is no feasible solution to the original problem.

Another way of treating artificial variables has been developed by Dantzig and Orden[7] et al. at the RAND Corporation in 1954. The procedure consists of two phases. In phase I a new objective function is defined by assigning a price of +1 to the artificial variables and a price of zero to other variables. The simplex calculations are then carried out to remove artificial variables. After achieving this, in phase II the actual objective function is minimised without having artificial variables or having some of them at zero level. Phase I is terminated if one of three situations is reached:-

1) The value of the new objective function is zero and all the artificial variables are non-basic. In this case the basic feasible solution has been obtained.

2) The value of the new objective function is zero and one or more artificial variables are basic with a value of zero. This means that a degenerate basic feasible solution has been obtained.

3) The value of the new objective function is less than zero and one or more artificial variables are basic. This shows that there is no feasible solution to the original problem.

If conditions 1 or 2 are reached at the end of phase I,phase II proceeds as a normal simplex iteration.

The revised simplex method was also developed by Dantzig, Orchard-Hays et al.[8]. This method uses the same basic principle as the simplex method. It proceeds from one basic feasible solution to another in such a manner that at each iteration only one basic variable changes. As a result only the essential quantities are computed which saves quite a large amount of computer storage. If the original problem contains a high proportion of zeros, which in the case for many practical problems, the revised simplex method requires fewer arithmetic operations. This reduces the computing time and the rounding-off errors. For problems that require artificial variables to be added, a two phase technique is employed by the revised simplex method.

It was in 1947 John Von Neuman showed that every linear programming problem can be formulated in its dual form. Any given primal problem

$$\text{Maximise} \quad \underline{W} = \underline{C}^T \underline{X}$$

Subject to

$$\underline{A}\ \underline{X} \leqslant \underline{B}$$

$$\underline{X} \geqslant 0$$

has a dual of

$$\text{Minimise} \quad \underline{Y} = \underline{B}^T \underline{U}$$

Subject to

$$\underline{A}^T \underline{U} \geqslant \underline{C}$$

$$\underline{U} \geqslant 0$$

In the primal $\underline{A}$ is a $m \times n$ matrix where m is the number of constraints and n is the number of variables, $\underline{C}$ and $\underline{X}$ are n dimensional vectors while $\underline{B}$ is an m dimensional vector. In the dual $\underline{A}^T$ is an $n \times m$ matrix which is the transpose of $\underline{A}$ and $\underline{U}$ is an m dimensional solution vector.

In 1954 C.E.Lemke's[9] dual simplex method became available. This starts with an initial basic non-feasible solution and aims to obtain a feasible basic solution. The selection of basic and non-basic variables is the only difference between the dual and the standard simplex method. The dual simplex method may be preferred in problems which have more constraints than variables and may be an alternative to the use of artificial variables.

When using the simplex method, it is assumed the variables in programming problems can have continuous values between specific limits. In practice, however, variables can only have discrete values. This difficulty can be overcome by Integer programming such as that due to Gomory[10].

1.2.2) NON-LINEAR PROGRAMMING.

In the real world there are few problems which are entirely linear. Most structural design problems are indeed non-linear. Particularly, if elastic theory is used for the formulation, the design problem turns out to be non-linear. There is no general method for solving non-linear programming problems. None of them is superior under all conditions. The mathematical properties of linear programming problems have been studied in depth and efficient computer programs have been developed[7,8,9,11]. This makes the idea of approximating non-linear programming problems by a sequence of linear programming problems attractive. After a finite number of iterations such a procedure converges to an optimum solution. The simplex method is used for solving the linear programming problem during each iteration. An alternative method is to replace the constrained non-linear programming problem in an unconstrained form, and to solve this by one of the unconstrained algorithms. This also converges to an optimum solution after a number of iterations. There are also methods which approach the problem directly. These start from a feasible point and find a direction to move along to improve the objective function without violating the constraints. After moving a short distance this direction is redetermined and the procedure is repeated until the objective function cannot be improved. A review of non-linear programming methods is given by Zountedijk[12].

1.2.2.1) SEQUENCE OF LINEAR PROGRAMS (SLP).

There are two methods of linearisation. The first makes use of a Taylor series and takes the first order terms to approximate a non-linear function into a linear form. The second replaces each non-linear function by a series of linear segments.

In 1960 Kelley[13] used the first method of linearisation to devise an algorithm known as the cutting plane method. Any non-linear programming problem of the form:-

$$\text{Min. } W = f(x_i) \qquad i = 1,\dots,n$$

Subject to

$$h_j(x_i) = 0 \qquad j = 1,\dots,r$$

$$g_k(x_i) \leq 0 \qquad k = r+1,\dots,m$$

$$x_i \geq 0$$

can be linearised at any arbitrary point x_{io} to become

$$\text{Min } W = f(x_{io}) + \nabla f(x_{io})(x_i - x_{io})$$

Subject to

$$h_j(x_{io}) + \nabla h_j(x_{io})(x_i - x_{io}) = 0$$

$$g_k(x_{io}) + \nabla g_k(x_{io})(x_i - x_{io}) \leq 0$$

$$x_i \geq 0$$

where

$$\underline{\nabla f}(x_{io}) = \left[\frac{\partial f}{\partial x_{1,o}} \quad \frac{\partial f}{\partial x_{2,o}} \quad \dots \quad \frac{\partial f}{\partial x_{n,o}} \right]$$

is known as the gradient vector. The method proceeds as follows:-

1) The objective function and the constraints are linearised at any point x_{io} and the resulting linear programming problem is solved.

2) The results are substituted in the non-linear constraints and the most violated one is found.

3) This constraint is linearised at the optimum solution x_{ip} and then added to the linear programming problem to obtain a new optimum solution.

4) Steps 2 and 3 are repeated by adding one extra linearised constraint at a time until all the non-linear constraints are satisfied within the acceptable accuracy.

In 1961 P.Wolfe[14] suggested that by utilizing duality, convergence of this process may be accelerated.

Some undesirable points of the cutting plane method prevent its application to every non-linear programming problems. This is particularly so with non-convex practical problems. When the set of constraints is convex the procedure converges to a global optimum. If this is not so, the convergence cannot be guaranteed and the solution is not globally optimum. Another difficulty which arises with the non-convex problem is that of oscillations. The final difficulty is that if the optimum solution of the original non-linear problem is not a vertex of the feasible region, numerical accuracy becomes unacceptable because of round-off errors. However for structural design problem, the optimum point is often at a point of intersection of the constraint boundaries.

Griffin and Stewart[15] also used the Taylor series to linearise the non-linear problem. Their approach which is known as either the Move limit method or Approximating programming does not suffer the difficulties which arise in the cutting plane method. Complete re-linearisation is applied at each iteration and move limits are imposed on the variables which do not allow them to move very far. This method proceeds as follows:

1) The objective function and the constraints are linearised at any arbitrary point x_{io} and by applying move limits additional constraints are imposed.

$$\text{Min } W = f(x_{io}) + \nabla f(x_{io})(x_i - x_{io})$$

Subject to

$$h_j(x_{io}) + \nabla h_j(x_{io})(x_i - x_{io}) = 0$$

$$g_k(x_{io}) + \nabla g_k(x_{io})(x_i - x_{io}) \leq 0$$

$$(1-m_\ell).x_{io} \leq x_i \leq (1+m_u)x_{io}$$

where m_ℓ and m_u are move limits which are positive constants.

2) The result x_{ip} is taken as the optimum solution and relinearisation is utilized. The process is repeated until convergence is obtained.

In Approximating programming constraints do not have to be convex. There is no restriction on the initial design point which can be feasible or infeasible. The number of constraints which are linearised at each iteration does not increase. In the cutting plane method this is not so. Approximating programming method can easily be adopted for use by computer to facilitate calculations. In some problems the solution may oscillate. This is overcome by terminating the procedure when move limits have the value of predetermined limits. If functions are highly non-linear, linearisation may not be efficient and convergence cannot be guaranteed. However, Approximating programming is one of the powerful techniques and allows a large number of variables. Because of this advantage it has found wide application in structural design. The algorithm produced by H.O.Hartley and R.R.Hocking(16) also utilised the Taylor expansion.

The second method of linearisation is to approximate a function by a series of linear segments. This is known as piecewise linearisation and a wide explanation of the method is given by Hadley(17). One of the conditions in the use of the method is that the objective function and constraints must be separable. The degree of approximation can be improved by increasing the number of segments to be considered. However, this will also increase the number of variables involved in the problem. This is the main disadvantage of the method which limits the size of the problem to be solved.

1.2.2.2) SEQUENCE OF UNCONSTRAINED MINIMISATION TECHNIQUES [SUMT].

Sequence of unconstrained minimisation techniques have been widely applied to solve structural design problems in many recent research works(66,68,69). The reason for this is that the algorithms to minimise unconstrained problems are more powerful and better established than the methods for constrained optimisation. Further, formulation of the design problems by these methods is straightforward which simplifies the computer programming.

The basic idea of these methods is to transform the constrained non-linear problem into an unconstrained one by multiplying the constraints by a factor and adding them together to the objective function. Consider a constrained non-linear programming problem of the form:-

$$\text{Min } W = f(x_i) \qquad i = 1,\ldots,n$$

Subject to

$$g_j(x_i) \leqslant 0 \qquad j = 1,\ldots,m$$

which can be converted to an unconstrained problem of the form:-

$$P(x,r_k) = f(x_i) + \phi[g_1(x_i),\ g_2(x_i),\ldots,g_m(x_i),r_k]$$

where r_k is known as the responde factor and ϕ is some function of constraints which creates a penalty for violating them. For this reason this technique is also known as the penalty function method. n is the number of variables while m is the total number of constraints. $P(x,r_k)$ is the new unconstrained function which may be minimised by one of the unconstrained minimisation techniques.

The procedure starts with an arbitrary point x_0 and then the function $P(x,r_k)$ with a predetermined value of $r_k = r_1$ is minimised. This is continued until the point x_0 converges to the optimum x_p.

Courant[18] was first to use a type of penalty function method. However, the method introduced by Carroll[19] in 1961 has formed a basis for much of the recent work. He called this approach the created responde surface method. Theoretical foundation was established and developed by Fiacco and McCormick[20] in 1964 which made it possible to handle multivariable systems. The term "sequential unconstrained technique" is used by them for their approach.

There are two types of penalty functions. The first is known as the exterior penalty function method in which the new function $P(x_i,r_k)$ has the form:-

$$P(x_i,r_k) = f(x_i) + r_k \sum_{j=1}^{m} \phi[g_j(x_i)]$$

which is constructed by multiplying the penalty function by a respond factor r_k so that when r is increased ϕ changes proportionally. The method proceeds by selecting a value for r. P is then minimised. The minimum point is then substituted in all the constraints. If they are satisfied to an acceptable accuracy, the procedure is terminated; if not, then r is increased and $P(x_i,r_k)$ is minimised again. This is carried out until the constraints are satisfied to an acceptable accuracy. The algorithm can operate from infeasible initial design points as well as feasible initial design points.

The second method is known as the interior penalty function method and is commonly used in structural design. In this method, the new unconstrained function $P(x_i,r_k)$ has the form:-

$$P(x_i,r_k) = f(x_i) + r_k \sum_{j=1}^{m} [1/g_j(x_i)]$$

where the penalty term creates a barrier between the feasible and infeasible region. This is why they are also known as barrier-functions(21). The algorithm starts minimising the function $P(x,r_k)$ for a given feasible point with a determined value of r_k. The convergence criteria is then checked. If it is not satisfied, then r_k is reduced by a factor α, $r_k = \alpha r_k$ where $\alpha < 1$ and the new function $P(x,r_k)$ is minimised with this new value of r_k. The procedure is repeated until the convergence criteria is satisfied. Some disadvantages of the method have been given by Ramakrish and Campbell(22).

In the case where the design problem contains equality constraints, the interior penalty function suggested by Fiacco and McCormick(23) has found considerable application. It has the form:

$$P(x_i, r_k) = f(x_i) + r_k \sum_{j=1}^{m} \frac{1}{g_j(x_i)} + r_k^{-\frac{1}{2}} \sum_{\ell=1}^{p} [h_\ell(x_i)]^2$$

where $h_\ell(x_i)$ is the equality constraint and p is their total number. This method requires a quasifeasible initial design point to start with. After a finite number of iterations it converges to the optimum solution. However, practical applications have shown that difficulties may arise in obtaining the optimum solution when available unconstrained minimisation techniques are used. A detailed account of penalty function method is given by L.R.Fox[24].

The selection of an efficient method for unconstrained minimisation is extremely important. There is a wide choice of methods of search. A review of these methods is given by Spang[25] and also by J.Kowalik[26]. The procedures available for unconstrained functions can be divided into sequential and nonsequential methods. Nonsequential methods make use of random numbers. Grid search and random search are two of the methods which belong to this group. Sequential techniques converge to the optimum point after a number of successive iterations. They define a new point from the current one using the following equation:-

$$x_{i+1} = x_i + \alpha_i S_i$$

where x_i is the starting value for the i^{th} step, S_i is i^{th} direction vector, α_i i^{th} step length and x_{i+1} is the design vector corresponding to the minimum of the unconstrained function to be minimised along the current direction S_i. The sequential methods consist of two groups. The first group is known as the gradient method. They move along the negative direction of gradient vector at possible rate. As a result they require computation of first or higher order derivatives of the function, in addition to the values of the optimised function itself. Convergence is generally quick in these techniques. In the case where the computation of first or higher order derivatives becomes practically impossible or laborious, the second group of sequential methods can be used for unconstrained minimisation. This group is known as the direct search method. Because they do not require the evaluation of derivatives, they are preferred in complex or large problems.

Among the gradient methods the variable metric algorithm is a sophisticated technique. This was devised by Davidon(27) in 1959. It makes use of conjugate gradients. This method which was later modified and developed by Fletcher and Powell(28) is quite powerful and widely employed in conjunction with penalty function methods.

Alternatively, some of the direct search methods were proposed by Rosenbrock(29), Powell(30) and others. For problems of small dimensions all these methods have been found to work well. However, when they applied to problems of large dimensonality difficulties may arise. The method of Powell has been found to be superior to most of the direct search techniques currently available(26). The investigation which was carried out by Asaadi(31) has shown that the Variable Metric method of(28) is slightly more advantageous than the direct search method of Powell.

Both gradient and direct search methods need one dimensional search techniques for which Fibonacci and Golden-Section search can be employed which are found to be quite efficient(26).

1.2.2.3) BASIC NON-LINEAR PROGRAMMING APPROACHES [NLP].

These methods are in the class of direct search algorithms. They start at some feasible point and then find a direction to be moved along which the objective function can be improved while all the constraints are satisfied. This kind of direction is called useable-feasible direction. This travel is expressed in the form:-

$$x_{i+1} = x_i + \alpha_i S_i$$

where S_i is the direction to be moved and, α is the step size. The travel from x_i to x_{i+1} is achieved in two stages. In the first stage the direction vector S_i is computed. Since this vector has to be feasible and useable, it will satisfy the relationships:-

$$\underline{S}^T \cdot \underline{\nabla f}(x) \leqslant 0$$
$$\underline{S}^T \cdot \underline{\nabla g}_j(x) \leqslant 0$$

where $\underline{\nabla f}(x)$ and $\underline{\nabla g}_j(x)$ are the gradient of the objective function and active constraint respectively. There are many methods which make use of the above idea. Those which were successfully applied in structural optimisation problems are mentioned in this chapter.

Rosen's[32] gradient projection method obtains the new feasible direction by using the Kuhn-Tucker[33] conditions. It starts from a feasible point and moves in the direction of $-\underline{\nabla f}(x)$, until a constraint is encountered. Further movement cannot be made to improve the objective function without violating the constraints. The new direction is obtained by projecting $-\underline{\nabla f}(x)$ on to the constraint hyper-plane. The move along that direction will improve the objective function and remains within the feasible region. This direction is given by

$$\underline{S}_i = \underline{P} \cdot \underline{\nabla f}(x_i) / |\underline{P} \cdot \underline{\nabla f}(x_i)|$$

where P is a projection matrix and given by

$$\underline{P} = \underline{I} - \underline{N}_k (\underline{N}_k^T \underline{N}_k)^{-1} \underline{N}_k^T$$

in which $\underline{N}_k$ is the gradient vector of all active k constraints. Using this direction $\underline{S}_i$, the step size α_i is determined and the procedure continues until the projection P is zero.

The method is efficient if all the constraints are linear and becomes less efficient for non-linear constraints. Computational difficulties arise in inverting $(\underline{N}_k^T \underline{N}_k)$ and the development of a computer program for this purpose is not straightforward.

The idea of the best feasible direction was first suggested by Zountendijk[34] who starts from a feasible point. A step is taken until reaching a boundary of a constraint. The increase σ in the objective function is then maximized while none of the active constraints are violated. This is done over a smalll interval by linearising the objective function and constraints. The disadvantages of the method are given by Zountendijk[12].

Another feasible direction method is the generalized reduced gradient algorithm which was proposed by J.Abadie and J.Carpentier[35] in 1969. The method which is an extension of Wolfe's[36] algorithm can accommodate both non-linear objective function and constraints. It has been claimed in their paper[35] that this method can handle large non-linear programming problems in less computer time than other known methods.

1.3) HISTORICAL REVIEW OF STRUCTURAL OPTIMISATION.

A structural engineer aims at designing a structure which sustains the external loads safely while satisfying the imposed design requirements. This is carried out by first deciding the configuration (topology and geometry) of the structure and then determining member sizes for this fixed geometry. In the past, some work has been directed towards the optimisation of member sizes for fixed geometry where the objective was taken as the minimum material weight.

However, later, it was understood that the configuration of the structure could be a design parameter and that it is possible to reduce the cost of material by optimising its configuration as well as the member properties. The review of structural optimisation will be given according to this framework. A comprehensive review of recent developments is also given by Sheu and Prager[37], L.Schmit[38], K.I.Majid[39] and others.

1.3.1) OPTIMISATION OF FIXED SHAPE STRUCTURES.

The design problem of structures having fixed shape can be formulated by employing either the plastic theory or the elastic theory. The use of either forms the feature of the design problem. Structural design by plastic theory leads to a linear programming problem. This is due to the fact that simple plastic theory assumes a rigidly jointed structure statically determinate at collapse. The equations of static equilibrium are then used to evaluate the sectional properties. For this reason, deflection requirements are not included in the design problem. Because the elastic theory does not make such assumptions, the design problem by this theory turns out to be non-linear. Hence, the review of fixed shape optimisation will be carried out considering these topics.

1.3.1.1) STRUCTURAL DESIGN BY RIGID PLASTIC THEORY.

Plastic theory was developed and widely published in the early fifties by the Cambridge team of Baker, Heyman and Horne(40). It considers the state of failure and derives the sections required to sustain the working loads. Heyman(41) has described a method of inequalities to derive the minimum weight solution, but the method is cumbersome and becomes almost unworkable for complicated structures. Foulkes(42) has shown that plastic minimum weight design can be reduced to a linear programming problem, but his approach is suitable only for hand calculations. Livesley(43) was the first to produce a computer program to design frames. He used a modified form of steepest descent to solve the design problem. Because the plastic theory neglects the deflexions and applies equilibrium equations to the undeformed state of the structure, the deflexion constraints do not appear in the design problem.

M.F.Rubinstein and J.Karagozian(44) have used the weak beam-strong column model for tall frames so that the frame becomes a mechanism when hinges form at the points of maximum moments in the beams. The deflexion constraint was formulated in terms of energy stored in the beams. In this way they have presented a design approach for the preliminary design of tall frames.

Toakley[45] has formulated the design problem in a similar manner to Livesley and obtained its solution for a discrete set of sections. Three different techniques were employed. These are the search technique, Gomory's algorithm and the random-step method. It was found that convergence difficulty may arise in the use of Gomory's mixed integer algorithm. He[46] also described an efficient way of using the dual simplex algorithm in the minimum weight design of rigidly jointed structures by the rigid-plastic theory. In his later work[47] the optimum elastic-plastic design of rigidly jointed structures was introduced, in which the initial deformed shape was assumed. The actual deformations and the axial forces in the members were determined by employing the elastic-plastic analysis program which was developed by Jenning and Majid[48]. Using the new deformed shape the constraints were reformulated. This procedure continued until two successive designs were identical. However, it was stated that this approach does not guarantee that collapse due to instability does not occur before the required load factor is reached.

J.M.Davies[49] described an approximate minimum weight design method which is applicable to steel frames. It is known that in plastic design the conditions of equilibrium, mechanism and yield are required to be satisfied. However, an approximation can be carried out in the design problem by considering only the conditions of equilibrium and yield. The resulting problem is solved by residual bending moment distribution. It was claimed that this method required less computer time and storage as opposed to linear programming. The excess minimum weight over the true optimum was considered to be less than 2%.

1.3.1.2) STRUCTURAL DESIGN BY ELASTIC THEORY.

The design criteria which is commonly used in elastic structural design is that stresses in the members and deflexions at the joints of a structure should not exceed certain permissible values. These limitations may be imposed by appropriate specifications such as B.S.449. Hence, it becomes necessary to express the deflexions and stresses in the structure in terms of design variables. This can be carried out by employing either of two matrix methods of structural analysis.

In the case of the matrix displacement method the design problem becomes one of finding the sectional properties of the members so that three constraints are satisfied. These are the stiffness equalities, the deflexion and the stress inequalities. When the deflexions are expressed in terms of sectional properties by inverting the overall stiffness matrix of the structure, the first two constraints turn out to be the same.

An alternative formulation uses the matrix force method to find the same member properties while satisfying the compatibility constraints which are equalities, as well as the deflexion and the stress inequalities.

Whichever method is employed to formulate the design problem results in a non-linear programming problem. Different techniques were employed by various authors to obtain its solution. The review will be given according to this framework.

i) SOLUTION BY [SLP].

The first application of the linearisation technique to the structural design problem was doen by F.Moses(50). A three bar truss and a one storey frame each subjected to two distinct load conditions were solved by the cutting plane method and significant savings in the weight of the structure were achieved with only one iteration of linearisation.

Reinschmidt K.F. et al.[51] formulated the design problem by the matrix displacement method. Both stress and deflexion constraints were considered. The move limit method was adopted for solution of the problem. Various convergence aids have been employed such as move limits, constraint accumulation and second order corrections. Each of these have been applied in a number of ways. Adoptive move limits were utilised to prevent fluctuation. The method of constraint accumulation was found successful when the problem was strictly convex. It was stated that the best compromise of all would always remain dependent on the type of problem. In their later work[52] detailed explanation and the comparison between iterative design, which imposes no restriction on displacements and assumes that the best structure is a fully stressed one, and structural optimisation was given. They have also shown that the use of reciprocal areas as a design variable reduces linearisation errors because the stresses in the members are linearly related to their reciprocal area.

K.M.Romstad and C.K.Wang[53] also formulated the design problem in a similar manner to the previous work[51]. In order to obtain the objective function in a linear form, the weight of each discrete element in the system was expressed as a function of a single design parameter. Finite changes were made in all design parameters during the iterations to achieve the minimum weight solution. These changes could be positive or negative depending upon the relationship between the initial assumptions and this minimum weight solution. Due to the fact that linear programming does not allow negative solution vectors, it was necessary to make the following transformation to overcome this difficulty,

$$u_j = 1 + \Delta z_j / \overset{\circ}{z}_j$$

where $\overset{\circ}{z}_j$ is the initial design parameter for element j, Δz_j is the change in this design parameter and u_j is the new design parameter.

Stress constraints were expressed in terms of the forces in the members. In rigidly jointed structures in addition to the areas of the members their section modulus was also considered as a design parameter. Stress limiting criterion at any section was taken as

$$F_{new}/(F_{new})_{all} + M_{new}/(M_{new})_{all} \leq 1$$

in which F_{new} is the axial force in the new section, M_{new} is the bending moment in the new section, $(F_{new})_{all}$ and $(M_{new})_{all}$ are the allowable axial force and bending moment in the new section respectively. A computer program was described and various design examples were solved. It was found that on deflection limited trusses application of move limits was mandatory while they were only desirable in stress limited trusses.

G.G.Pope[54] used the feasible direction method due to Zountendijk to linearise the minimum weight design problem. The procedure only operated from feasible initial design points. Simple examples were described,one of which contained upper limits on the displacements.

G.G.Pope[55] has also reviewed the application of [SLP] to optimum structural design and described the use of the move limit method in the design of stressed-skin structures. The member cross sectional areas and skin thickness were considered as design variables. Finite element idealisation was used to represent structural configuration. Displacements were kept purely elastic and stresses were restricted against yielding which was calculated by the Von Mises criterion for the state of plane stress. The buckling of an unreinforced strip or skin between two spars of low torsional rigidity was also considered. A discussion of design problems which could be formulated in linear form was also given.

In his later work[56] the above method was applied to problems where two dimensional stress fields were involved. A fully stressed design and SLP were used to obtain the minimum weight of a plate containing a hole acted on by a uniform end load. It was shown that the minimum weight obtained by SLP was lighter than the one obtained by fully stressed design iterations. The influence of active displacement and stress constraints on convergence of the move limit method was also investigated. Such applications have proved the effectiveness of the method.

D.Johnson and D.M.Brotton(57) have formulated the design problem for redundant trusses utilizing the matrix force method. Three different types of variables were used. These were stress-area, stress-reciprocal area and force-area. A detailed explanation and comparison of these were given and superiority of force-area formulation was shown in a wide range of examples. Fixed value move limits have been employed and an 0.5% change of objective function in two successive cycles was considered accurate enough for convergence. Considerably larger trusses were designed without difficulty. It was noticed that most of the computer time was used by the linear programming portion of the program. Because of the small number of variables involved in the force method of analysis, the design method described was accepted as effective and economic. However Johnson and Brotton did consider deflection constraints.

K.Reinschmidt(58) proposed two methods for discrete structural optimisation. In the first, plastic theory was used which yielded a linear problem. In the second, their previous formulation(51) was utilised. Both integer linear problems were solved by implicit enumeration. The use of integer programming increased their computer time considerably. A good initial design point was obtained by rounding up a continuous linear programming solution. This local optimisation procedure was employed to search for the global optimum by generating random starting points.

G.Davies and H.S.Wang[59] made adjustment to the Cornell et al.[51] procedure using the principle of fully stressed design so that the method may converge to the optimum in a shorter time and in a very steady manner. Their program consisted of two steps at each iteration. At the initial design point the optimum solution was found by using the move limit method. The stresses and deflexions were obtained for the new design variables and checked. New design variables were then adjusted by interpolation and this point was employed to generate the next design cycle.

As an alternative Toakley[60] employed the peicewise linearisation technique to solve the minimum weight design problem of statically determinate pin-jointed frames subject to deflexion and stress limitations. The design problem was formulated by the unit load method. The reciprocal areas were considered as design variables. As a result deflexion constraints were in linear form and the objective function was strictly convex. Hence the optimum solution obtained by this procedure was the global one.

Majid and Anderson[61] formulated the design problem of statically indeterminate elastic structures by using the matrix force method. Both deflexion and stress constraints were considered. The piecewise linearisation technique was employed for the solution of the design problem. Due to the fact that the members in pin-jointed structures were subject to axial forces, the design variables considered were only the areas of the members and the forces in the redundant members. In sway frames, axial deformations were neglected. It was found that this procedure was only applicable to small bare frames. The reason for this was that piecewise linearisation required considerable computer storage and time to obtain an optimum solution.

ii) <u>SOLUTION BY [SUMT]</u>.

L.A.Schmit and R.L.Fox[62] introduced the integrated approach which combined the analysis and design process in order to achieve analysis for acceptable designs which reduce the weight. By employing heaviside penalty function the design problem was transformed into one of unconstrained minimisation. Steepest descent type procedure was utilized to obtain its solution.

In their later work[63] the method was applied to general three-dimensional trusses. Member buckling, joint displacement and member stress limitations were included. Some improvements were introduced to the design procedure. It was also shown that the integrated approach could be used in conjunction with matrix structural analysis.

D.Kavlie et al.[64] employed the penalty function method due to Fiacco and McCormick to transform the minimum weight design problem of structures into a sequence of an unconstrained problem. The variable metric method was used to obtain its local optimum. One of the disadvantages of the method was that during the unconstrained minimisation it was necessary to find the partial derivatives of the function with respect to all the free variables. This had to be carried out analytically. In the case of complex constraints this was time consuming and could easily have introduced errors. The method was applied to the optimisation of a corrugated transverse bulkhead of an oil tanker.

J.Moe[65] has discussed a way of reducing the amount of required redesign and computer time. The approximate behaviour model was employed to express member end forces in terms of member sizes in statically indetermined structures. This was carried out by utilizing the fact that variation of members sizes creates relatively slow changes in member forces. An application of the method to ship structure design was also explained.

D.Kavlie and J.Moe[66] have described the application of SUMT to the design of elastic grillages loaded laterally and in plane. Both deflexion and stress constraints were considered. A comparison of the variable metric method and Powell's direct search method was given. It was found that SUMT could be used for non-convex sets of design variables. It was also shown that the initial design point and initial response factor had decisive influence on the results. It was verified that a fully stressed design may not necessarily correspond to the minimum weight design.

K.M.Gisvold and J.Moe[67] have shown that buckling problems could be formulated by the energy method as an unconstrained non-linear programming problem. A direct search method was employed for its solution. The design procedure described was applied to the buckling problem of a stiffened plate subject to loads in its plane as well as lateral loads.

D.Kavlie-J.Moe[68] used SUMT for automated design and optimisation of statically indeterminate structures. It was demonstrated that infeasible initial design points could be employed with the extended penalty function technique. In the unidirectional searches a special type of polynomial approximation was applied which saved considerable computer time. It was shown that the way of constructing the member stiffness matrix was efficient particularly when the finite element method was used for the analysis. The choice of initial respond factor in case of several local optima was also investigated.

K.M.Gisvold and J.Moe[69] have modified SUMT by adding discretiazation penalty function in order to handle mixed integer problems. It was stated that although there was a little research which had been done in the area of mixed integer non-linear programming their algorithm has been applied relatively successfully to a number of different design problems. One of its disadvantageswas that optimalities could not be guaranteed.

R.M.Pickett et al.[70] have investigated the design of large structural systems where a reduced number of design variables were used. The actual design problem was replaced by using a small number of trial designs with a sufficiently small problem where the direct solution could be found with existing programming procedure. There was no need for these trial designs to be acceptable but they should be linearly independent. SUMT was used to generate the optimisation procedure which was solved by the deflected gradient method. The application of the method to a number of examples has shown that large reductions in computer effort were obtained.

B.M.E. De Silva and G.N.C.Grant[71] have made a comparison of penalty function formulations in the multi-bar truss optimisation. The heaviside and SUMT penalty function transformations were employed to convert the constrained problem into an unconstrained minimisation problem. These were solved using the methods of Rosenbrock[29], Powell[30] and Nelder-Mead[72]. The SUMT transformation was found superior to heaviside step function method. Among the non-gradient minimisation techniques utilized Powell's algorithm was found to be superior to others. As a result it was concluded that the SUMT/POWELL combined algorithm provided the best solutions.

iii) SOLUTION BY NLP.

Although in most of the recent work either SLP or SUMT was used to solve the minimum weight design problem of structures, there is a considerable amount of work which used one of the basic non-linear programming algorithms that treat constraints in their non-linear form.

L.A.Schmit and T.P.Kicher[73] applied the concept of structural design to the three bar truss to select the best material and configuration. The steepest descent algorithm was employed to obtain the optimum design. Later L.A.Schmit and W.M.Morrow[74] added buckling constraints to this problem.

L.A.Schmit and R.H.Mallet[75] considered material densities as design variables as well as member areas and inclination of the bars in three bar truss. In this way design parameter heirarchy was introduced. The design problem was solved by a method of alternate steps. As a result an automatic procedure was produced to select the material of the members and the geometric configuration to achieve a minimum weight design.

R.Razani[76] investigated the relationship between fully stressed design and minimum weight design. The Kuhn-Tucker[33] condition was used to verify the optimality of the fully stressed design. It was shown that fully stressed design was not always the one which had minimum weight.

T.P.Kicher[77] also examined the relationship between minimum weight design and the fully stressed design which was called simultaneous failure mode design. The problem was formulated using a simple elastic structural system and solution was obtained by Lagrange multiplier technique.

A.Gellatly and R.H.Gallegar[78] used the feasible direction method to solve the minimum weight design of trusses subject to stress and deflexion constraints. Matrix displacement analysis was employed to obtain the behaviour variables which were element stresses and nodal deflexions.

D.M.Brown and A.H.Ang[79] found that the gradient project method of Rosen could be adopted to solve the design problem of rigidly jointed structures. Design variables were taken as the second moment of areas of the various member groups which were related to weight and radius of gyration approximation. A computer program was described and a number of design examples were demonstrated. In the case of the non-convex design problem, the method leads to a local optimum.

F.Moses and S.Onoda[80] used the matrix displacement method to formulate the minimum weight design of elastic grillages made of straight orthogonal beams normally loaded. Beam section properties were related by an empirical relationship which reduced the design variables to the areas of each beam element. Only stress constraints were considered. Three algorithms were employed. These were the stress-ratio, the cutting plane and the useable-feasible gradient directions. A detailed comparison of these methods showed that the cutting plane method required fewer structural analysis cycles for convergence than others. In order to reduce the analysis cycle in the use of the useable-feasible method a technique was utilized which first found a fully-stressed design by the stress-ratio and then began moving in the useable-feasible vector direction. It was stated that the stress-ratio method could be useful to find a good initial design point if constraints were non-convex.

K.I.Majid and D.W.C.Elliott[81] proposed a general non-linear programming procedure to obtain the optimum design of a structure subject to deflexion limitations. The dynamic search method was employed which was an extension of the useable-feasible gradient direction method. This optimisation approach was utilized to produce design charts for fixed base pitched roof frames which could be used for design purposes. A method was also given which used an available discrete set of sections in the safe load tables.

G.N.Vanderplaats and F.Moses[82] described a general algorithm for structural design. Zountedijk's method of feasible direction was used in conjunction with the matrix force method. Modifications to the feasible direction method were introduced to improve the numerical stability of the problem and to deal with infeasible designs. It was only necessary to evaluate the analytic gradients of the constraints, which were active at a given stage, and the objective function in the design process. It was stated that the generality of approach prevented it from being the most efficient for the design of all structures. The procedure was applied to the elastic design of indeterminate trusses under multiple loading conditions where, in addition to stress and deflexion constraints, Euler buckling constraints were considered.

P.A.Seaburg and C.G.Salmon[83] investigated the minimum weight design of light gauge steel members. The design variables were taken as the thickness and the remaining cross-sectional dimensions of sections. The direct search and Rosen's gradient search methods were employed. The gradient search method was recommended because it took half as much time as the direct search.

M.Pappas and C.L.A.Rao[84] transformed the constrained design problem into a single unconstrained one by the penalty function method where penalty multipliers were used as functions of the design variables. The direct search method was employed to find its solution. The direct search method was improved and extended for use as an optimality check at points of direct search failure. It was mentioned that this optimum structural design approach required the solution of only one unconstrained problem and did not need an initial selection of the penalty multiplier which allowed infeasible points to be chosen as an initial design. It was concluded that the application of this algorithm to integrally stiffened cylindrical shells has verified its good convergence properties and computational efficiency.

Other optimisation methods have also been employed in the structural design. A.C.Palmer[85] applied dynamic programming to the optimal plastic design of continuous beams and rigidly jointed frames. On the other hand A.B.Templeman[86,87] formulated the structural design problem in such a manner that geometric programming could be used to obtain the minimum cost rapidly and simply. However, it should be pointed out that these techniques have a limited practical application.

1.3.2) SHAPE OPTIMISATION.

The first work on shape optimisation was published as early as 1904 by Michell[88], who used Maxwell's theory which states that for all the frames under a given system of applied forces, the member forces and lengths were related in the following relationship:-

$$\Sigma f_p \ell_p - \Sigma f_c \ell_c = C$$

where f_p is the tension in any tie bar of length ℓ_p and f_c is the thrust in any strut of length ℓ_c and C is constant which is a function of the applied forces and coordinates of their points of application independent of the form of the frame.

If σ_t and σ_c are the permissible stresses in tension and compression respectively, then the above equation becomes

$$\sigma_t \, v_t - \sigma_c \, v_c = C$$

it follows that

$$v = v_t + v_c = v_c \left(1 + \frac{\sigma_c}{\sigma_t}\right) + \frac{1}{\sigma_t} . C$$

$$= v_t \left(1 + \frac{\sigma_t}{\sigma_c}\right) + \frac{1}{\sigma_t} . C$$

where v_t is the volume of all the tension members and v_c is the volume of all the compression members. v is the total volume. It is concluded from this equation that the lightest structure is the one that has the least volume of compression members or the least volume of tension members.

Michell applied the principle of virtual work and stated that the volume of a frame is a minimum when the space occupied by it could be subject to an appropriate small deformation such that the axial strains along any member of the frame were equal to $\pm$ e where e was a small number and the sign was compatible with the member force. Furthermore, no other element in the space had a strain numerically greater than e. By using minimum strain energy approach it was shown that there was a unique geometry for the absolute minimum weight design under a given single load condition. L.C.Schmit[89] extended this theory to allow for alternative load cases. D.N.G.Ghista[109] applied Michell's theory to develop optimum structures for more than one load case. In all these works there was no control on deflexions. From the illustrated examples given by various authors, it can be concluded that this approach which employs variational theory of mathematics is extremely limited in practical use.

As an alternative method, P.Pederson[90] presented an approach for optimum geometry design of statically determinate trusses subject to stress and Euler buckling constraints in which joint coordinates were treated as independent design variables. Later the same idea was applied to statically indeterminate trusses[91] with displacement and stability constraints and also to space trusses[92]. The matrix displacement method was used to formulate the problem where length and direction cosines of members were expressed in terms of member end joint coordinates. The move limit method was employed to obtain the solution which proved to be very effective and flexible. It was concluded that with multiple loading systems, the optimal structure was statically indeterminate and not fully stressed. Trusses up to 40 joints have been optimised without difficulty. Although it was possible to extend the procedure to obtain the optimum geometry design of rigidly jointed frames, it was stated that the expressions of gradient derivation would be complicated.

G.N.Vanderplaats and F.Moses[93] also described a procedure for the design of elastic trusses for optimum geometry subject to stress and buckling constraints. The matrix force method was found suitable for formulating the problem. The objective function was expressed as

$$W = \sum_{i=1}^{n} \rho_i \, A_i \, \ell_i$$

where n was the total number of members which was constant during the process, ρ_i, A_i, ℓ_i are respectively the material density, the cross-sectional area and the length of member i. The length of the members were expressed in terms of member end coordinates as:-

$$\ell_i = \left[\sum_{r=1}^{2} (x_m - x_\ell)^2 \right]_i^{\frac{1}{2}}$$

in which m and ℓ subscripts are the ends of member i. The approach started with initial specified geometry and joints were moved until optimum geometry for the given structure was found. This was carried out first by obtaining the minimum weight design for initial geometry, then the objective function was minimised in the direction of steepest descent of coordinate design space. Member area design was updated to maintain optimality. The process was repeated until the weight could no longer be reduced. The number of design variables considered at any given stage was reduced by considering two separate but dependent design spaces. Area and coordinate linking was used to preserve symmetry and member grouping.

K.C.Fu[94] proposed the iterative search technique for optimising the configuration of trusses in which only the coordinates of unloaded joints were taken as design variables. The procedure started by choosing an initial truss geometry of a given topology. The coordinates of one of the unloaded joints were then selected as variables which were optimised while the other joint coordinates were kept constant. This was applied sequentially to the other unloaded joints and process was continued until no improvement could be achieved on the objective function. It was noticed that the examples solved showed good agreement with the Michell structure.

These works found the optimum geometry of a structure while its topology was kept unaltered. S.L.Lipson and K.M.Agrawall[95] optimised topology as well as geometry of indeterminate trusses subject to multiple load conditions. Independent variables were taken as joint coordinates and sectional areas which were selected from a discrete member spectrum. In the examples solved only the stress constraints were considered. During the design process those members which had zero areas and those joints which had zero coordinates were deleted and the relevant stress constraints were omitted automatically. The examples illustrated showed that a non-convex feasible space only increased the number of iterations but presented no difficulty. D.W.Alspaugh and K.Kunoo[96] also showed that considerable weight reduction can be achieved by allowing freedom in the number and the location of the joints in truss structures.

Another method of shape optimisation was first introduced by Dorn et al.[97] which made use of the concept of ground structures. The design space contained a set of admissible joints and all considered structures for a given problem will only select joints from this set. The ground structure was obtained by linking each admissible joint to others in the design space and the Minimum weight design of this ground structure was formulated in the following form:-

$$\text{Min. } W = \frac{\rho}{\sigma} \sum_{j=1}^{n} \ell_j |S_j|$$

Subject to

$$\sum_{j=1}^{n} a_{ij} S_j = F_i \qquad i = 1,2,\ldots,m$$

where n is the number of admissible bars, m is the number of admissible joints, S_j are forces in these bars which were taken as design variables, ρ,σ denotes the weight per unit volume, and the yield stress of the given material, a_{ij} is the direction cosines of the bar and F_i is the component of the external force at the joint. This linear programming problem was solved for S_j and bars with zero forces were removed. Removal of these members and unloaded joints made it possible to obtain a structure which had a new topology as well as a new geometry. However, it may be necessary to keep some of the zero force bars by assigning an arbitrary small cross-section to satisfy the rigidity requirements. Included in the examples were planar trusses under one loading condition, which were optimised and found to be statically determined and therefore fully stressed. Fleron[98] has also presented a similar method.

M.W.Dobbs and L.P.Felton[99] extended this approach to deal with multiple loading conditions. This made the design problem non-linear and the steepest descent alternate algorithm was utilized for its solution. They also made the approach iterative so that the process might be repeated until no further topological changes were possible. The method was proved successful and promising. However, it covered only stress constraints and gave no justification for the deletion of the members.

W.S.Lapay and G.G.Goble[100] compared the non-linear and linear formulation of the above approach and found that non-linearity made it possible to consider buckling constraints and was superior to the linear formulation.

K.I.Majid and D.W.Elliot [101] stated the theorems of structural variation which made it possible to predict exactly how the forces and deflexions throughout the strcture change when some or many of its members are either varied or totally removed. Later this was used[102] in conjunction with topological design of pin jointed structures. The matrix displacement method was used to formulate the design problem where stress and deflexion requirements were considered. A Ground structure was initially developed and then the members were removed until no further topological changes were possible. The manner by which the members were deleted was forecasted by using a benefit vector. The theorems of structural variation were employed to prepare this vector depending on whether stress or deflexion constraints were dominant. Members were arranged in such a way that the first member in the benefit vector may be removed with the largest reduction in the volume of the structure. The removal of members at each iteration yielded a new type of non-linear programming whose constraints and objective function were continuously changing. Self weight of members were also included as design variables and it was found that this changed the shape of the final design and speeded up the search for the optimum shape.

K.F.Reinschmidt and A.D.Russel(103) have formulated a design problem based on the equilibrium conditions and the stress constraints by temporarily neglecting the conditions of elastic compatibility. The examples considered show that the method made it possible to eliminate the surplus members and lead to better indeterminate truss configurations than did a stress ratio type algorithm.

On the other hand dynamic programming was applied to the shape optimisation of structures with reference to pinjointed structures by R.F.Goff(104) and also by Palmer(105).

1.4) CONCLUSIONS FROM PUBLISHED LITERATURE.

The work reviewed here shows that without any simplifying assumption the minimum weight design of structures turns out to be a non-linear programming problem. However, there are quite a large number of algorithms for solution of such problems; the move limit method and the penalty function method were widely applied and found effective in structural design. These methods do not restrict the designer to predict a feasible initial design point which provides flexibility to the procedures in which they were employed. Furthermore, their computer application is straightforward and does not generally introduce severe difficulty.

Although, it has been stated that the procedures developed can be extended to rigidly jointed frames, generally they were applied in the examples to pin jointed structures. In such structures the formulation of the design problem is simple due to the fact that members of the pin jointed structure are only subject to axial forces. Generally in these formulations the matrix force method was preferred and widely applied. However the matrix displacement method was also used in some of the structural design problems.

The way they were employed in the formulation of design problems required either the solution of simultaneous linear equations or sometimes the inversion of a matrix. The former was necessary with the use of the matrix force method to obtain the redundants from compatibility equations. The latter when used was necessary in the matrix displacement method to express the deflexions in terms of design variables which were generally considered as sectional properties of the members.

Some design procedures were also developed which were capable of handling complex structures. However these were only tested on special examples. It seems that most of the work on structural optimisation has been directed to pin jointed structures.

After having a considerable number of algorithms for the optimisation of fixed geometry, most of the recent work has been directed to the optimisation of structural configuration. This work can be divided into three groups. The first group uses Michell's theory which applies the calculus of variations to the design of structures for minimum volume. These algorithms have no control on deflexions and are extremely limited in practical use. The second group considers the joint coordinates as independent variables i.e. the joints of the initial geometry of the structure are allowed freedom. By optimising the weight of the structure, the joints having zero coordinates and the members having zero sectional properties are deleted during the procedure. This is continued until it becomes impossible to make further reduction in the weight of the structure. It is apparent that the number of design variables involved in this technique increases considerably which means an increase in computer time and core storage. The third group of works defines a ground structure which is determined by linking each admissible joint to others. Unloaded members and joints are then removed which are obtained as a result of the optimisation procedure. In this way at the end of each topological cycle a different structure with a new topology is extracted. This technique is found quite promising for practical applications. In particular when this technique is combined with the theorems of structural variation, it makes it possible to forecast the manner in which the numberous members should be removed. However, it is not yet possible to conclude which one of the last two techniques is superior.

1.5) THE SCOPE OF THE PRESENT WORK.

The structural optimisation procedure which is described in Chapter 2, formulates the design problem by the matrix displacement method and considers the displacements of joints in a structure as design variables. In this way it becomes possible to avoid either matrix inversion or the solution of simultaneous equations which were the case in some of the previous works as discussed in the last section. For the solution of the design problem approximating programming is employed which was widely applied in previous works and found to be effective. This automatic optimum design procedure is described with reference to general rigidly jointed structures where stiffness, stress and deflexion constraints are considered. The arrangement of the move limits are also given in Chapter 2. [SUMT] is employed to solve the design problem.

The procedure is computerized and explanation of the computer programming is given in Chapter 3 in detail.

Chapter 4 contains the examples solved by this procedure. The effect of the axial force, the arrangement of move limits, the nature of the initial design point and its effect to the design process are investigated. The results obtained are illustrated.

The theorems of structural variation are extended and proved in Chapter 5 to cover rigidly jointed structures. Interrelation of all structures is confirmed. It is shown that it is possible to predict the behaviour of a pin jointed structure from the results of the analysis of another rigidly jointed structure. These theorems are also employed to study the variation in member forces and joint deflexions of a structure when one or more of its members are varied or totally removed. The use of these theorems in the analysis and design of structures is explained,and illustrated examples are given to show their versatility.

In Chapter 6, the problem of minimum weight design is extended to include the shape of the structure. The theorems of structural variations are employed to prove that it is possible to evaluate the significance of each member in the behaviour of a structure. This makes it possible to calculate, in advance, the economy achieved by altering its topology. A design procedure is also described which makes use of the structural optimisation of Chapter 2. A number of design examples are solved and the results are illustrated.

CHAPTER 2

FORMULATION OF THE DESIGN PROBLEM BY THE MATRIX DISPLACEMENT METHOD.

2.1) INTRODUCTION.

There are basically two main matrix methods of structural analysis that are well established and can be employed to formulate the optimum design problem. These are the matrix force method and the matrix displacement method.

The matrix force method involves the concept of redundancies; consequently, it is not equally efficient for statically determinate and indeterminate structures. However, it involves the solution of a smaller number of equations, one per unknown redundant, than the displacement method. As a result it is widely used as a powerful tool in the design of structures where deflexions are part of the design criteria.

The matrix displacement method, however, expresses the internal member force in terms of the joint displacements. These unknown displacements are obtained by solving a set of joint equilibrium equations.

The design procedure is automised with the purpose of using the computer to set up the design problem and to carry out its solution. It is shown that the application of move limits is necessary to achieve convergence. These are arranged in such a way that the number of iterations required to obtain the optimum solution can be kept to a minimum.

2.2) THE MATRIX DISPLACEMENT METHOD.

The matrix displacement method which is employed in the formulation of the design problem, is probably the most efficient general method of structural analysis available. It does not involve the concept of redundancies and can easily be automised. It also requires a minimum amount of input data. This method will now be summarised.

In a member of rigidly jointed plane frame shown in

Figure 2.1, the member forces $\underline{P}_{AB} = \{P_{AB}\ S_{AB}\ M_{AB}\ M_{BA}\}$ are related to member distortions $\underline{U}_{AB} = \{u_{AB} v_{AB} \theta_{AB} \theta_{BA}\}$ by the equation

$$\underline{P}_{AB} = \underline{k}_{AB} \quad \underline{U}_{AB} \tag{2.1}$$

where the matrix $\underline{k}_{AB}$ is the stiffness matrix for the member and has the form

$$\underline{k}_{AB} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & d & d \\ 0 & d & e & f \\ 0 & d & f & e \end{bmatrix} \tag{2.2}$$

where

$$\begin{aligned} a &= EA/L, \quad b = 12EI/L^3, \quad d = -6EI/L^2 \\ e &= 4EI/L, \quad f = 0.5e \end{aligned} \tag{2.3}$$

in which A is the area, I is the second moment of area, E is the modulus of elasticity and L is the length of member AB. When the equation 2.1 is written down for all the other members of the plane frame and compounded together, the member stiffness matrix of the frame is obtained which consists of submatrices similar to 2.2.

$$\underline{P} = \underline{k}.\underline{U} \tag{2.4}$$

The member distortions are related to the joint displacements by the equation

$$\underline{U} = \underline{A}.\underline{X} \tag{2.5}$$

where the A matrix is known as the displacement transformation matrix. Its elements are constants referring to the cosines of the angles between the members and reference axes. Now, the member forces can be expressed in terms of joint displacements as:

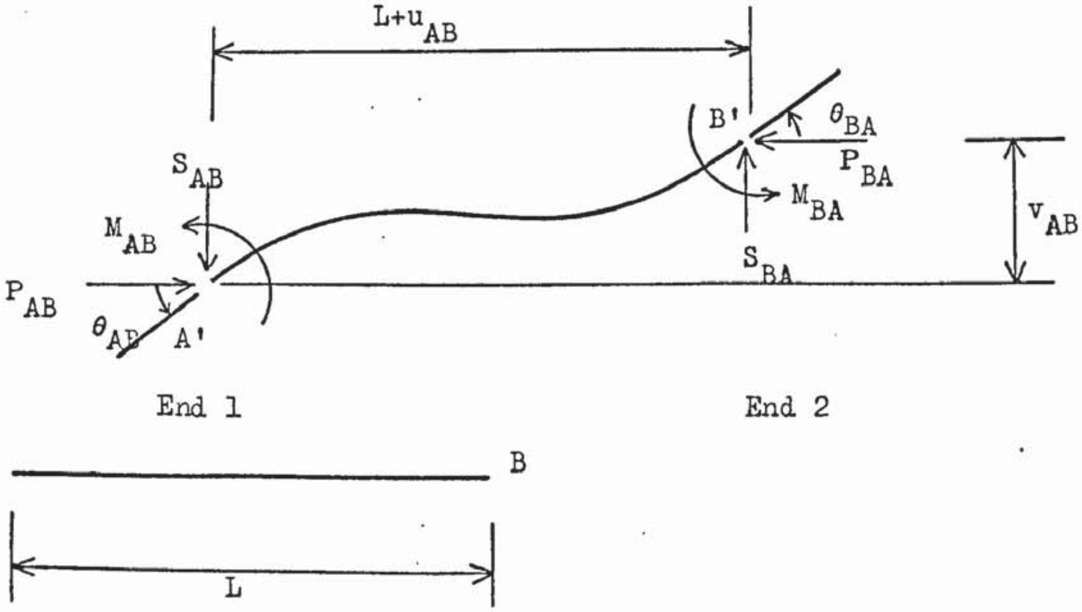
$$\underline{P} = \underline{k}.\underline{A}.\underline{X} \tag{2.6}$$

Using the principle of virtual work, we obtain

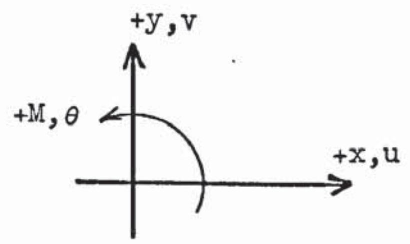
$$\underline{L} = \underline{A}^T.\underline{P} \tag{2.7}$$

which is

$$\underline{L} = \underline{A}^T.\underline{k}.\underline{A}.\underline{X} \tag{2.8}$$



a) End forces and displacements



b) Sign convention

FIGURE 2.1 DEFORMATION OF A MEMBER OF RIGIDLY JOINTED PLANE FRAME.

where L is the external load matrix and $\underline{A}^T.\underline{k}.\underline{A} = \underline{K}$ is known as the overall stiffness matrix of the structure. Thus

$$\underline{L} = \underline{K}.\underline{X} \qquad 2.9$$

2.3) THE AUTOMATIC CONSTRUCTION OF CONSTRAINTS.

There are a number of assumptions which the minimum weight design approach makes. The first one is that there is a continuous set of sections available from which to select. This arises from the nature of the mathematical programming and does not cause serious errors. In the case where a discrete set of sections is to be used, Integer or Dynamic programming can be employed. However, these methods complicate the design procedure and in some cases optimality cannot be guaranteed.

A second assumption is made by relating the area, the section modulus and the second moment of area of the section to each other. These are the variables in the design of rigid frames. It is desirable to use one of these variables to derive the objective function and the constraints. Although these sectional properties do not have any direct and linear relationship with each other, it is possible to obtain reasonable relationships for them. Templeman[36] has done this and concluded that for Universal beams

$$A = 0.78\, z^{\frac{2}{3}} \qquad z = 1.452A^{3/2}$$

$$\text{or} \qquad 2.10$$

$$A = 0.559\, I^{\frac{1}{2}} \qquad I = 3.20\, A^2$$

where A is the area, z is the section modulus and I is the second moment of area of a section.

The objective function which is the weight of the structure, can be expressed in terms of the areas of the members. If the members are grouped together for practical reasons, then the objective function becomes

$$W = \sum_{i=1}^{NG} \gamma_i\, \ell_i\, A_i \qquad 2.11$$

where NG is the total number of groups, ℓ_i is the total length of all the members in group i and A_i is the area of the adopted section for group i. γ_i is the density of the material for group i. In this case the objective function is in a linear form.

In the case where the second moment of area of a section is used as the design variable, then the objective function becomes non-linear and has the form of

$$W = 0.559 \sum_{j=1}^{NG} \gamma_i \, \ell_i \, I_i^{\frac{1}{2}}$$

where I_i is the second moment of area of the adopted section for group i. However, it becomes necessary to express the area and the section modulus in terms of the second moment of area of that section. The expression 2.10 can be used for these conversions.

2.3.1) <u>THE STIFFNESS CONSTRAINTS</u>.

In the design of a structure by the matrix displacement method, the structure must be sufficiently stiff to carry the externally applied loads while the stress and deflexion requirements are observed. It is, therefore, necessary to include the stiffness constraints in the design problem which are in the form

$$\underline{K}.\underline{X} = \underline{L}$$

The overall stiffness matrix k is constructed by carrying out the triple multiplication of $\underline{A}^T\underline{k}\underline{A}$. The contribution of a single member linking the joints R and S, to this matrix is

$$\begin{array}{cc} & \begin{array}{cc}\text{at joint R} & \text{at joint S}\end{array} \\ \begin{array}{c}\text{at joint R} \\ \\ \underline{K} = \\ \\ \text{at joint S}\end{array} & \left[\begin{array}{ccc:ccc} A & B & -C & -A & -B & -C \\ B & F & -T & -B & -F & -T \\ -C & -T & e & C & T & f \\ \hdashline -A & -B & C & A & B & C \\ -B & -F & T & B & F & T \\ -C & -T & f & C & T & e \end{array}\right] \end{array}$$

i.e.

$$\underline{K} = \begin{bmatrix} \underline{K}_{RR} & \underline{K}_{RS} \\ \underline{K}_{SR} & \underline{K}_{SS} \end{bmatrix} \qquad 2.12$$

where

$$\begin{aligned} A &= a\ell_p^2 + bm_p^2 \\ B &= (a-b)\,\ell_p m_p \qquad 2.13 \\ C &= -\,dm_p \\ T &= d\,\ell_p \\ F &= am_p^2 + b\ell_p^2 \end{aligned}$$

in which ℓ_p, m_p are the direction cosines of the member.

It can be seen that the above expressions contain the area and the second moment of areas of sections which are the variables of the programming problem. Hence, it becomes necessary to separate the elements of expression 2.13. In this way, the submatrices $\underline{K}_{RR}$, $\underline{K}_{RS}$, $\underline{K}_{SR}$, $\underline{K}_{ss}$ will have the form:

$$\underline{K}_{RR} = \left[\begin{array}{c:c:c:c:c} A_{11}.A & A_{12}.I & B_{11}.A & B_{12}.I & -C.I \\ B_{11}.A & B_{12}.I & F_{11}.A & F_{12}.I & -T.I \\ 0 & -C.\,I & 0 & -T.\,I & e.I \end{array}\right]$$

$$\underline{K}_{RS} = \left[\begin{array}{c:c:c:c:c} -A_{11}.A & -A_{12}.I & -B_{11}.A & -B_{12}.I & -C.I \\ -B_{11}.A & -B_{12}.I & -F_{11}.A & -F_{12}.I & -T.I \\ 0 & C.\,I & 0 & T.\,I & f.I \end{array}\right] \qquad 2.14$$

$$\underline{K}_{SR} = \begin{bmatrix} -A_{11}.A & \vdots & -A_{12}.I & \vdots & -B_{11}.A & \vdots & -B_{12}.I & \vdots & C.I. \\ -B_{11}.A & \vdots & -B_{12}.I & \vdots & -F_{11}.A & \vdots & -F_{12}.I & \vdots & T.I. \\ 0 & \vdots & -C.\ I & \vdots & 0 & \vdots & -T.\ I & \vdots & f.I \end{bmatrix}$$

2.14 contd

$$\underline{K}_{SS} = \begin{bmatrix} A_{11}.A & \vdots & A_{12}.I & \vdots & B_{11}.A & \vdots & B_{12}.I & \vdots & C.I \\ B_{11}.A & \vdots & B_{12}.I & \vdots & F_{11}.A & \vdots & F_{12}.I & \vdots & T.I \\ 0 & \vdots & C.\ I & \vdots & 0 & \vdots & T.\ I & \vdots & e.I \end{bmatrix}$$

where

$$A_{11} = \frac{E}{L}\,\ell_p^2 \quad , \quad B_{11} = \frac{E}{L}\,\ell_p m_p \quad , \quad F_{11} = \frac{E}{L}\,m_p^2$$

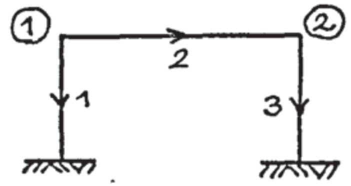
are the coefficients of the area variables A and

$$A_{12} = \frac{12E}{L^3}\,m_p^2 \quad , \quad B_{12} = \frac{12E}{L^3}\,\ell_p m_p \quad , \quad C = -\frac{6E}{L^2}\,m_p$$

$$T = -\frac{6E}{L^2}\,\ell_p \quad , \quad e = \frac{4E}{L} \quad , \quad f = \frac{2E}{L}$$

are the coefficients of the second moment of area I. The relationship 2.10 can now be used to express the second moments of area of the sections in terms of the areas. In this way, the first and third columns of the submatrices 2.14 contain the coefficients of the first order terms of areas, the second, fourth and fifth columns contain the coefficients of the second order terms of areas.

In the analysis of structures, there are three rows and columns corresponding to each joint of a structure in the overall stiffness matrix. This matrix can be constructed firstly by finding the members connected to each joint and then adding their contributions together at the location corresponding to that joint. This is carried out for every joint in the structure. In the design of structures this cannot be done due to the fact that the areas of the members are the design variables. As a result it becomes necessary to keep the contribution of each member separate. At each joint these contribution matrices are subsequently located to the overall stiffness coefficient matrix. As an example, in Figure 2.2 a layout of the overall stiffness coefficients matrix is given for a portal



		JOINT 1										JOINT 2									
		x_1				y_1				θ_1		x_2				y_2				θ_2	
		MEM. 1		MEM. 2		MEM. 1		MEM. 2		MEM. 1	MEM. 2	MEM. 2		MEM. 3		MEM. 2		MEM. 3		MEM 2	MEM 3
JOINT 1	x_1	A_{11}	A_{12}	A_{11}	A_{12}	B_{11}	B_{12}	B_{11}	B_{12}	$-C$	$-C$	$-A_{11}$	$-A_{12}$	0	0	$-B_{11}$	$-B_{12}$	0	0	$-C$	0
	y_1	B_{11}	B_{12}	B_{11}	B_{12}	F_{11}	F_{12}	F_{11}	F_{12}	$-T$	$-T$	$-B_{11}$	$-B_{12}$	0	0	$-F_{11}$	$-F_{12}$	0	0	$-T$	0
	θ_1	0	$-C$	0	$-C$	0	$-T$	0	$-T$	e	e	0	C	0	0	0	T	0	0	f	0
JOINT 2	x_2	0	0	$-A_{11}$	$-A_{12}$	0	0	$-B_{11}$	$-B_{12}$	0	C	A_{11}	A_{12}	A_{11}	A_{12}	B_{11}	B_{12}	B_{11}	B_{12}	C	C
	y_2	0	0	$-B_{11}$	$-B_{12}$	0	0	$-F_{11}$	$-F_{12}$	0	T	B_{11}	B_{12}	B_{11}	B_{12}	F_{11}	F_{12}	F_{11}	F_{12}	T	T
	θ_2	0	0	0	$-C$	0	0	0	$-T$	0	f	0	C	0	C	0	T	0	T	e	e

FIGURE 2.2. THE LAYOUT OF THE OVERALL STIFFNESS COEFFICIENT MATRIX

frame. Each joint of the frame is numbered from 1 upwards. At each joint the overall stiffness coefficients matrix has three rows and give columns for each member connecting to this joint. It is obvious that generally, this matrix has 3*N number of rows where N is the total number of joints in the structure and $5*\left(\sum_{i=1}^{N} M_i\right)$ number of columns where M_i is the total number of members connected to joint i. If members are grouped for practical reasons, then M_i will be defined as the total number of different member groups at joint i.

In case there are hinges in the structure the order of the overall stiffness coefficients matrix will be increased. There will be an additional row and column corresponding to each of the hinges. It is common to locate these rows and columns after the row corresponding to the last joint. For this reason, the hinge number is given as 3*N+1 so that the row belonging to that hinge is specified by the hinge number. Since the overall stiffness coefficient matrix is not symmetric, the column number of that hinge is different from its row number and is obtained from $5.\left(\sum_{i=1}^{N} M_i\right) + 1$. As a result, if there are p hinges in the frame, the order of the overall stiffness coefficient matrix will be $\left[(3N + P)*(5(\sum_{i=1}^{N} M_i) + P)\right]$.

2.3.2) <u>THE STRESS CONSTRAINTS</u>.

The design of a feasible structure requires the satisfaction of stress constraints, such that stresses in the members should not exceed specified limits. In rigid frames, the critical stress at the outer fibres of a member is obtained by combining the axial and the bending stress.

$$\sigma_i = \frac{P_i}{A_i} \pm \frac{M_i}{z_i} \qquad 2.15$$

where σ_i is the combined stress, P_i is the axial force, M_i is the bending moment at the end of the member i, A_i and z_i are the area and the section modulus of member i. The combined stress at any point of the section must satisfy the following inequality

$$-\sigma_c \leqslant \sigma_i \leqslant \sigma_t \qquad 2.15(a)$$

where σ_c is the permissible compressive stress and σ_t is the permissible tensile stress. In the inequality 2.15(a), tension is taken as positive and compression as negative. Substituting σ_i from 2.15 into 2.15(a), it follows that:

$$-\sigma_c \leqslant \frac{P}{A} \pm \frac{M}{z} \leqslant \sigma_t$$

That is

$$\frac{P}{A} \pm \frac{M}{z} \leqslant \sigma_t$$

$$\frac{P}{A} \pm \frac{M}{z} \geqslant -\sigma_c$$

By multiplying both sides of the second inequality by -1, it follows:

$$\begin{aligned} \frac{P}{A} + \frac{M}{z} &\leqslant \sigma_t \\ \frac{P}{A} - \frac{M}{z} &\leqslant \sigma_t \\ -\frac{P}{A} + \frac{M}{z} &\leqslant \sigma_c \\ -\frac{P}{A} - \frac{M}{z} &\leqslant \sigma_c \end{aligned} \qquad 2.16$$

Therefore, it is necessary to consider four independent stress constraints for a section with two axis of symmetry to cover all the possible stresses which can occur at the outer fibres of a member.

The axial force and the bending moment of the member are obtained from the equation

$$\underline{P} = \underline{kAX}$$

For a member linking joints R and S, the product matrix kA has the form

$$\underline{k}.\underline{A} = \begin{array}{c} \begin{array}{cc} \text{at joint R} & \text{at joint S} \end{array} \\ \left[\begin{array}{ccccc:cccc} -a\ell_p & -am_p & 0 & \cdots & & \cdots & a\ell_p & am_p & 0 \\ bm_p & -b\ell_p & d & \cdots & & \cdots & -bm_p & b\ell_p & d \\ dm_p & -d\ell_p & e & \cdots & & \cdots & -dm_p & d\ell_p & f \\ dm_p & -d\ell_p & f & \cdots & & \cdots & -dm_p & d\ell_p & e \end{array}\right] \end{array}$$

Once again a,b,c,d,e,f contain the design variables such as the areas and the second moment of area. It is only possible to compute their coefficients in the member force matrix

$$\underline{k}\,\underline{A} = [\underline{B}_R \quad \underline{B}_S]$$

where

$$\underline{B}_R = \begin{array}{c} \text{At the first end of the member} \\ \begin{bmatrix} -A_{21}.A & -A_{22}.A & 0 \\ B_{21}.I & -B_{22}.I & -d.I \\ -D_{21}.I & D_{22}.I & e.I \\ -D_{21}.I & D_{22}.I & f.I \end{bmatrix} \end{array} \text{ and } \underline{B}_S = \begin{array}{c} \text{At the second end of the member} \\ \begin{bmatrix} A_{21}.A & A_{22}.A & 0 \\ -B_{21}.I & B_{22}.I & -dI \\ D_{21}.I & -D_{22}.I & fI \\ D_{21}.I & -D_{22}.I & eI \end{bmatrix} \end{array}$$

in which

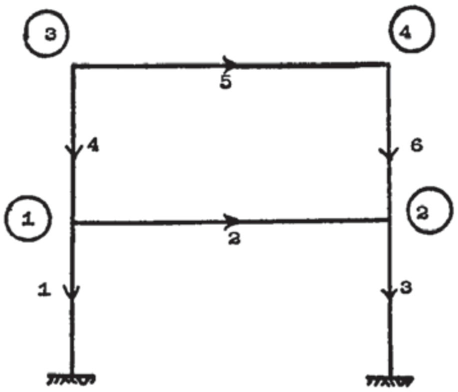
$$A_{21} = \frac{E}{L}\,\ell_p \qquad , \quad A_{22} = \frac{E}{L}\,m_p$$

are the coefficients of the area A

$$B_{21} = \frac{12E}{L^3}\,m_p \qquad B_{22} = \frac{12E}{L^3}\,\ell_p$$

$$D_{21} = \frac{6E}{L^2}\,m_p \qquad D^{22} = \frac{6E}{L^2}\,\ell_p$$

are the coefficients of the second moment of area I. The first row of submatrices $\underline{B}_R$ and $\underline{B}_S$ contains the first order term of area and is used to calculate the axial stress. Consequently, the axial stress will be independent of the section areas and only dependent on the function of joint displacements. The bending moments at the first and the second ends of the member are calculated by using the third and fourth rows of $\underline{B}_R$ and $\underline{B}_S$. The elements of these rows are the coefficients of the second moment of areas. Further, in calculating the bending stresses these elements are divided by the section modulus. It is, once again, necessary to employ the relationship 2.10 to express the second moment of areas and the section modulus in terms of the areas. In this way, the elements of these rows become the coefficients of variables in the bending stresses and they contain the 0.5 order of areas. The overall matrix which contains the coefficients of the variables in the stress constraints can be constructed automatically. This is shown in Figure 2.3. For a structure consisting of m members



JOINTS

$$\begin{array}{ccc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} P_1 \\ P_1 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = & \begin{bmatrix} B_{R_1} & 0 & 0 & 0 \\ B_{R_2} & B_{S_2} & 0 & 0 \\ 0 & B_{S_3} & 0 & 0 \\ B_{S_4} & 0 & B_{R_4} & 0 \\ 0 & 0 & B_{R_5} & B_{S_5} \\ 0 & B_{S_6} & 0 & B_{S_6} \end{bmatrix} & \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \end{array}$$

where $\underline{\delta}_i = \{x_i \quad y_i \quad \theta_i\}$

and

$\underline{P}_i = \{P_{AB} S_{AB} \; M_{AB} \; M_{BA}\}$

<u>Figure 2.3</u> The layout of the overall stress coefficient matrix for two storey frame.

and n joints, this matrix has the order [4*M, 3*N].

2.3.3) THE DEFLEXION CONSTRAINTS.

It is necessary for a safe structure to have deflexions which are not more than some given specified values. Since the joint displacements are introduced as the design variables, the deflexion requirements are reduced to upper bound constraints.

$$\underline{X} \leqslant \underline{\Delta} \qquad 2.17$$

where $\underline{\Delta}$ is a set of allowable deflexions. It can be seen that the displacements of joints can be either positive or negative, while the mathematical programming operates only with non-negative variables. There are two ways in which the problem can be converted to this form. The first way is to write each joint displacement as the difference of two non-negative variables, such as:

$$x_i = x_i' - x_i''$$

where x_i is the horizontal displacement of joint i and x_i', x_i'' are the new non-negative variables. Depending on the magnitude of x_i' and x_i'', x_i can have any sign. It is shown by Hadley[5] that in the solution of the problem, only one of these new variables can appear. That is either $x_i'' = 0$ and $x_i = x_i'$ or $x_i' = 0$ and $x_i = - x_i''$ or $x_i' = x_i'' = 0$ and $x_i = 0$. This device was utilized in the design examples but found unsuccessful because it doubled the number of displacement variables. The device can also give rise to the appearance of both x_i' and x_i'' c.f.(Hadley). Indeed in one trial this actually happened.* Another way of overcoming the non-negativity restriction which does not involve introducing an extra variable, is to substitute

$$x_i = y_i - e_i \qquad 2.18$$

where y_i is a new non-negative variable and e_i is constant. If y_i is not in the solution x_i becomes equal to $-e_i$ which is the selected most negative value x_i can possibly take. If at joint i the displacement

*Notice that the deflexion variables x_i' and x_i'' do not appear is limited to Δ_i then the deflexion constraint for x_i becomes

$$x_i = y_i - e_i \leq \Delta_i$$

$$y_i \leq \Delta_i + e_i \qquad 2.19$$

In the case where the most negative and positive values x_i can take, are equal, then 2.19 becomes:

$$x_i = y_i - \Delta_i \leq \Delta_i$$

$$y_i \leq 2\Delta_i \qquad 2.20$$

In some structures, the deflexion at a joint is known to be always negative, that is

$$x_i = y_i - e_i \leq 0$$

$$y_i \leq e_i \qquad 2.21$$

Each joint of a plane frame introduces three variables to the design problem. These are the horizontal displacement, the vertical displacement and the rotation of that joint. So the bounds on these variables can be placed into three categories. Those for the horizontal and vertical displacement variables can be obtained from specifications. For example, B.S.449 considers a structure safe when midspan deflexion of its beams does not exceed $\ell/360$, where ℓ is the length of a beam. The horizontal deflexion of its columns is restricted by the same code to not more than h/325 where h is the height of a column. In case there is no restriction on one category of displacements, then the bound is taken sufficiently large to include all the possible values that a particular displacement can take. For example it is not usual to put an upper bound on rotations. In this case the value of e = 0.08 radians is convenient, because linear structural theories are only applicable for small deflexions with $|\theta| \leq 0.08$ radians.

2.4) THE APPROXIMATING PROGRAMMING.

As shown in the previous sections, the formulation of the design problem by the matrix displacement method reduces to a non-linear programming problem. The approximating programming which is described in Chapter 1, is found quite effective for structural design problems by many of the previous research workers[51,53,55,57,91]. This method employs the first two terms of Taylor's series to linearise the non-linear function. It is known that by this series a function of several variables $f(\overset{\circ}{x}_i)$ can be expanded at the point x_i^1 so that the value of $f(x_i^1)$ is obtained

$$f(x_i^1) = f(\overset{\circ}{x}_i) + \sum_{i=1}^{n} \frac{\partial f(x_i^1)}{\partial \overset{\circ}{x}_i} (x_i^1 - \overset{\circ}{x}_i) \qquad i = 1,2,\ldots n$$

or in matrix form

$$f(\underline{x}^1) = f(\underline{x}^o) + \underline{\nabla} f(\underline{x}^o)\ (\underline{x}^1 - \underline{x}^o) \qquad 2.22$$

where $\underline{\nabla} f(x^o)$ is the row vector $\left[\frac{\partial f}{\partial x_1^o} \ \frac{\partial f}{\partial x_2^o} \ \ldots \ \frac{\partial f}{\partial x_n^o}\right]$ in which n is the number of variables. By applying this to the non-linear problem consisting of n variables, m constraints which is in the form:

$$\text{Min} \quad W = W(x)$$

Subject to

$$h_k(x) = 0 \qquad k = 1,\ldots \ell$$

$$g_j(x) \leqslant 0 \qquad j = \ell+1,\ldots m$$

can be transferred to the linear programming problem of the form

$$\text{Min } W = W(\underline{x}^o) + \nabla W(\underline{x}^o)\ [\underline{x}^1 - \underline{x}^o]$$

Subject to

$$h_k(\underline{x}^o) + \nabla h_k(\underline{x}^o)[\underline{x}^1 - \underline{x}^o] = 0$$

$$g_j(\underline{x}^o) + \nabla g_j(\underline{x}^o)[\underline{x}^1 - \underline{x}^o] \leqslant 0$$

where the value of every function is known at $\underline{x}^o$ and $\underline{x}^1$ is the unknown variable. Once this linear programming problem is solved, the process can be repeated with $\underline{x}^1$ replacing $\underline{x}^0$ to obtain a new solution $\underline{x}^{(2)}$ until the value of the objective function remains unaltered in two successive iterations.

2.4.1) THE DERIVATIVES OF CONSTRAINTS.

Since the approximating programming requires the gradient vectors of the constraints, it is necessary to compute the derivatives of these with respect to the design variables.

There are two ways of computing the derivatives of a function, when a computer is used to carry out the calculations. The first one utilises finite differencing. The basic idea of this is very simple. It approximates the derivative of the function G(x)

$$\frac{\partial G}{\partial x_k} \simeq \frac{G(X^k) - G(X)}{\Delta x_k}$$

where

$$\underline{X}^k = (x_1\ x_2\ \ldots, x_k + \Delta x_k, \ldots, x_n)$$

and Δx_k is some small change in x_k. The errors which are introduced in this way are large and the value of Δx_k requires adjustment. As a result it may be used where the explicit differentiation is not available or very complex. The second way is the exact computation of derivatives. This may be carried out in the following manner

$$y = x^m$$

$$\frac{dy}{dx} = m.x^{m-1} \qquad 2.23$$

It is known that in the design problem the relationship between the area, the section modulus and the second moment of area is in the form of

$$I = p.A^r$$

$$z = S.A^t$$

where p,r,s,t are constants and their values may change depending on the type of beam used in the structure. Since the constraints have only area variables which are high ordered, it is possible to compute the exact derivatives for them.

The design variables vector has the form:

$$V = \{v_1\ v_2\ \dots\ v_m, v_{m+1}\ \dots\ v_{m+3n}\} \qquad 2.24$$

where the first m variables represent the areas of the groups, the rest is the displacements of joints and n is the number of joints in the structure.

In matrix form:

$$\underline{V} = \{\underline{A}\quad \underline{X}\}$$

where the submatrix $\underline{A} = \{A_1\ \dots\ A_m\}$ contains the areas and $\underline{X} = \{x_1\ y_1\ \theta_1\ \dots\ x_n\ y_n\ \theta_n\}$ contains the displacements of joints.

The stiffness constraints are functions of the areas of the members and the displacements of joints. They have the form:

$$\underline{G}(A,X) = \underline{K}(A).\underline{X} - \underline{L} = 0$$

where $\underline{G}(A,X)$ represents the stiffness constraints, $\underline{K}(A)$ refers to the overall stiffness coefficient matrix and $\underline{L}$ is the external load matrix. The gradient vector of this is

$$\nabla G = \left\{ \frac{\partial G}{\partial v_1}\ \frac{\partial G}{\partial v_2}\ \dots\ \frac{\partial G}{\partial v_{m+3n}} \right\}$$

which is

$$\nabla G = \left\{ \frac{\partial G}{\partial A_1}\ \frac{\partial G}{\partial A_2}\ \dots\ \frac{\partial G}{\partial A_m}\ \frac{\partial G}{\partial x_1}\ \dots\ \frac{\partial G}{\partial \theta_n} \right\}$$

It can be seen that the overall stiffness coefficient matrix is a function of areas only. The derivatives of the stiffness constraints with respect to areas will be

$$\frac{\partial G}{\partial A_i} = \frac{\partial K(A)}{\partial A_i}.\underline{X}$$

and the derivatives of the stiffness constraints with respect to displacement variables will be

$$\frac{\partial G}{\partial x_i} = \underline{K}(A)$$

The stress constraints consist of the combination of the axial and bending stresses. It is known that in rigidly jointed plane frames the axial stresses are only functions of the joint displacements. On the other hand, the bending stresses are functions of area variables as well as joint displacements.

$$\underline{\sigma}(A,X) = [\underline{B} + \underline{C}(A)]\,\underline{X} - \underline{\sigma}_p \leqslant 0$$

where $\underline{\sigma}(A,X)$ represents the stress constraints, σ_p is the permissible stress, matrix $\underline{B}$ contains the coefficient of variables in the axial stresses and matrix $\underline{C}$ contains the coefficients of variables of bending stresses. The matrices $\underline{B}$ and $\underline{C}$ are obtained by collecting the elements of the rows in the product matrix $\underline{kA}$ which correspond to the axial forces and bending moments respectively. The gradient vector of stress constraints is

$$\nabla\sigma(A,X) = \left[\frac{\partial\sigma}{\partial A_1} \;\cdots\; \frac{\partial\sigma}{\partial A_m} \;\; \frac{\partial\sigma}{\partial x_1} \;\cdots\; \frac{\partial\sigma}{\partial\theta_n}\right]$$

where the derivatives of $\underline{\sigma}(A,X)$ with respect to areas are

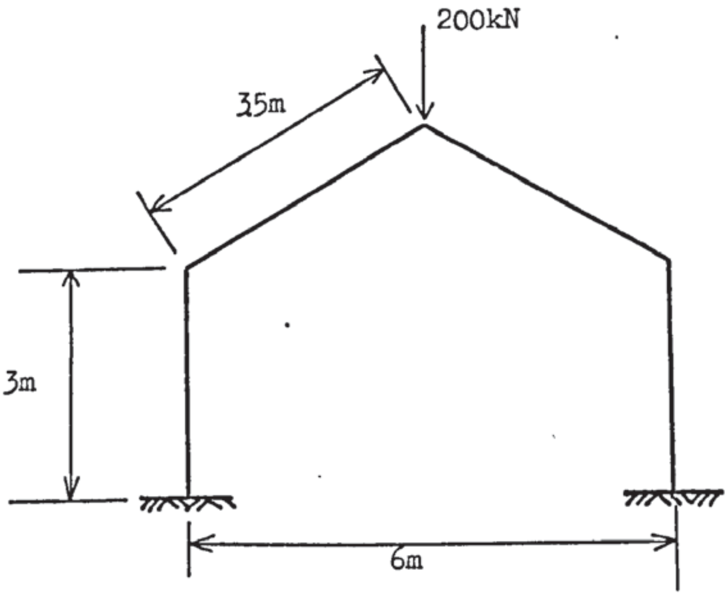
$$\frac{\partial\sigma}{\partial A_i} = \frac{\partial C(A)}{\partial A_i}\,.\underline{X}$$

and with respect to displacement variables are

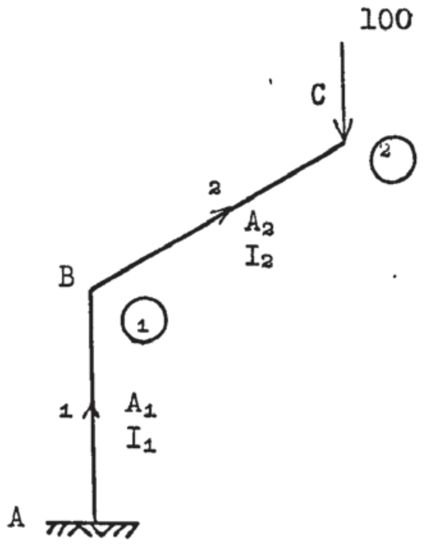
$$\frac{\partial\sigma}{\partial x_i} = \underline{B} + \underline{C}(A).$$

2.4.2) <u>THE LINEARISATION AND MOVE LIMITS</u>.

The linearisation of the non-linear constraints was carried out in the manner described in Section 2.4. As an example, the process was applied to the stiffness constraints of the pitched roof frame shown in Figure 2.4. The columns are 3m high and the span is 6m. The angle of pitch is 30°. It was required to design the frame so that the stresses were limited to 0.10804 kN/mm^2 in the columns and 0.2034 kN/mm^2 in the rafters. The horizontal deflexion at the eaves were limited to 4.225 mm and the vertical deflexion at C to 8.23 mm. The columns were required to be manufactured out



a) STRUCTURE AND LOADING



b) MEMBER NUMBERING

FIGURE 2.4: A PITCHED ROOF FRAME

of the same section while the inclined members were to have the same section but might be different from the columns.

This frame was designed by the method described in this chapter and optimum areas for the columns and rafters were found to be 1×10^4 mm^2 and 0.5×10^4 mm^2 respectively.

The first stiffness constraint of this pitched roof frame takes the form:

$$
\begin{aligned}
h_1(x) = {} & 0.029x_1^2x_3 + 44.35x_2x_3 + 0.0045x_2^2x_3 + 25.607x_2x_4 \\
& - 0.00788x_2^2x_4 + 4.416x_1^2x_5 - 1.622x_2^2x_5 - 25.607x_2x_6 \\
& + 0.00788x_2^2x_6 - 0.0563x_1^2 + 1.0564x_2 + 0.00823x_2^2 = 0
\end{aligned}
$$

where x_1, x_2 are the areas of the columns and the rafters respectively, x_3, x_4, x_5 are the non-negative new variables for the displacements of joint 1 and x_6 is the non-negative variable for the vertical displacement of joint 2. The gradient vector is

$$
\begin{aligned}
\nabla h_1(x) = \Big[& (0.058x_1x_3 + 8.832x_1x_5 - 0.01126x_1) \\
& (44.35x_3 + 0.009x_2x_3 + 25.607x_4 - 0.01576x_2x_4 - 3.244x_2x_5 \\
& -25.607x_6 + 0.01576x_2x_6 + 1.0564 + 0.01646x_2) \\
& (0.029x_1^2 + 44.35x_2 + 0.0045x_2^2)(25.607x_2 - 0.00788x_2^2) \\
& (4.416x_1^2 - 1.622x_2^2)\ (-25.607x_2 + 0.00788x_2^2) \Big]
\end{aligned}
$$

Choosing the initial design point as

$$\underline{X}^{o} = \{110. \quad 60. \quad 0.1158 \quad 0.0368 \quad 0.01064 \quad 0.216\}$$

the linearised constraint will have the form of

$$h_\ell(x) = h(x^o) + \underline{\nabla h}(x^o)\ [\underline{x} - \underline{x}^o] = 0$$

which is

$$-1.4025 + [-1.31 \quad 0.75 \quad 3028.1 \quad 1508.05 \quad 47594.4 \quad -1508.05] \begin{bmatrix} x_1 & - & 110 \\ x_2 & - & 60 \\ x_3 & - & 0.1158 \\ x_4 & - & 0.0368 \\ x_5 & - & 0.01064 \\ x_6 & - & 0.216 \end{bmatrix} = 0$$

Simplyfying this becomes:

$$h_\ell(x) = 1.31x_1 - 0.75x_2 + 3028.1x_3 - 1508.05x_4 - 47595.4x_5 + 1508.05x_6 - 489.11 = 0$$

In this manner the non-linear programming problem is transferred to a linear programming one. It is obvious that this linearisation will introduce errors in the solution of the problem. These may be controlled by imposing some bounds on the design variables which are known as move limits. These can be arranged arbitrarily, but usually they can be based on a certain percentage of the present values of the design variables. In the design procedure presented in this chapter they are arranged in the following way:

$$(1-m)\, x_j^{(o)} \leq x_j^{(1)} \leq (1+m)\, x_j^{(o)} \qquad j = 1,\ldots,n$$

where n is the total number of the design variables, $x_j^{(o)}$ is the present value of the design variable, $x_j^{(1)}$ is the value which will be found at the end of the next iteration. m is the preselected percentage. Experience obtained from the examples considered shows that 90% move limits can be chosen and then reduced by 10% at each iteration. This method provides large move limits during the first iterations and small move limits during the last iterations which satisfies the requirements of:

i) If the initial design point is chosen far from the true optimum, then it is necessary to employ large move limits in order to reduce the number of iterations to reach the optimum point.

ii) Tight move limits are required to obtain the convergence in case the optimum is not fully stressed. In case the convergence is not obtained when the value of m becomes 0.10, then the iterations are continued with these particular values of the move limits. It is also found that it is only necessary to put move limits on the area variables. As explained in Section 2.3.3, the substitution which is carried out for the displacement variables to satisfy the non-negativity restriction introduces enough bounds to control the linearisation errors. This saves 3n number of constraints, where n is the number of joints in the structure, which have to be added to the design problem.

After arranging the move limits the linear problem can be solved by the simplex method. The presence of equality constraints makes it necessary to add artificial variables to each of them. These have to be eliminated during the simplex iterations. There are two versions of the simplex method which deal with artificial variables. These are Charnes M method and two-phase method which are described in Chapter 1. Both canbe employed in the design procedure. It has been shown previously that the two phase method is better than Charnes M method particularly when a computer is used to carry out the calculations. If the artificial variables can not be eliminated during the simplex iterations and they appear in the final solution, then the linear problem has no feasible solution. This does not necessarily mean that a feasible solution is not available for the programming problem. There are a number of reasons for obtaining the nonfeasible solution. One reason is that if the value of the move limits are selected too small, then the design problem will be bounded too tightly. It then becomes impossible to find a feasible solution for the design problem within these bounds imposed on the area variables. The relaxation of these move limits therefore improves the possibility of obtaining a feasible solution. Another reason for obtaining a non-feasible solution is that the starting point may be chosen far from the optimum point. Then the linearisation errors become very large and a feasible solution cannot be found. This can also be overcome by trying a different initial design point. Finally, due to the accumulation of the linearisation errors involved at each iteration, the displacement variables violate the stiffness constraints excessively. This may happen after a number of iterations. In this case, it becomes necessary to solve the stiffness equations with the present values of areas and to adjust the displacements. These new values are then used to carry out the next iteration.

2.5) THE DESIGN PROCEDURE.

The design procedure which will be described in this section consists of three stages. The first stage is to set up the design problem. The second stage is to linearise the non-linear problem at some chosen initial design point and put the move limits on the area variables. The third stage is to solve this linear programming problem by the simplex method. Using the results thus obtained in the last step, the second and third steps are repeated until the required convergence is obtained. The flow diagram of the procedure is given in Figure 2.5. The data which the procedure requires is small and requires the selection of an initial design point. In most optimum structural design algorithms, a feasible initial design point is necessary to start with. It is not always easy to predict such a point. As shown in the flow diagram, the procedure can operate from both a feasible and infeasible initial design point which gives flexibility to the design procedure. The only restriction which this procedure needs for the initial design point is that it has to satisfy the stiffness equalities. This can easily be carried out by solving the stiffness equalities with the selected values of area variables to find the displacement variables. Then these values of areas and displacement variables are used as an initial design point. As explained in the last section the solution of the stiffness equalities may be required for adjustment of the displacement variables. Hence, it becomes necessary to carry out the construction of the overall stiffness matrix of the structure. This does not introduce any difficulty. Since the overall stiffness coefficients matrix contains the contributions of each member connected to each joint, the only necessary computation is to multiply these by the selected values of the areas and add them together at each joint to obtain the elements of the overall stiffness matrix at each joint. The linearisation process is carried out as explained

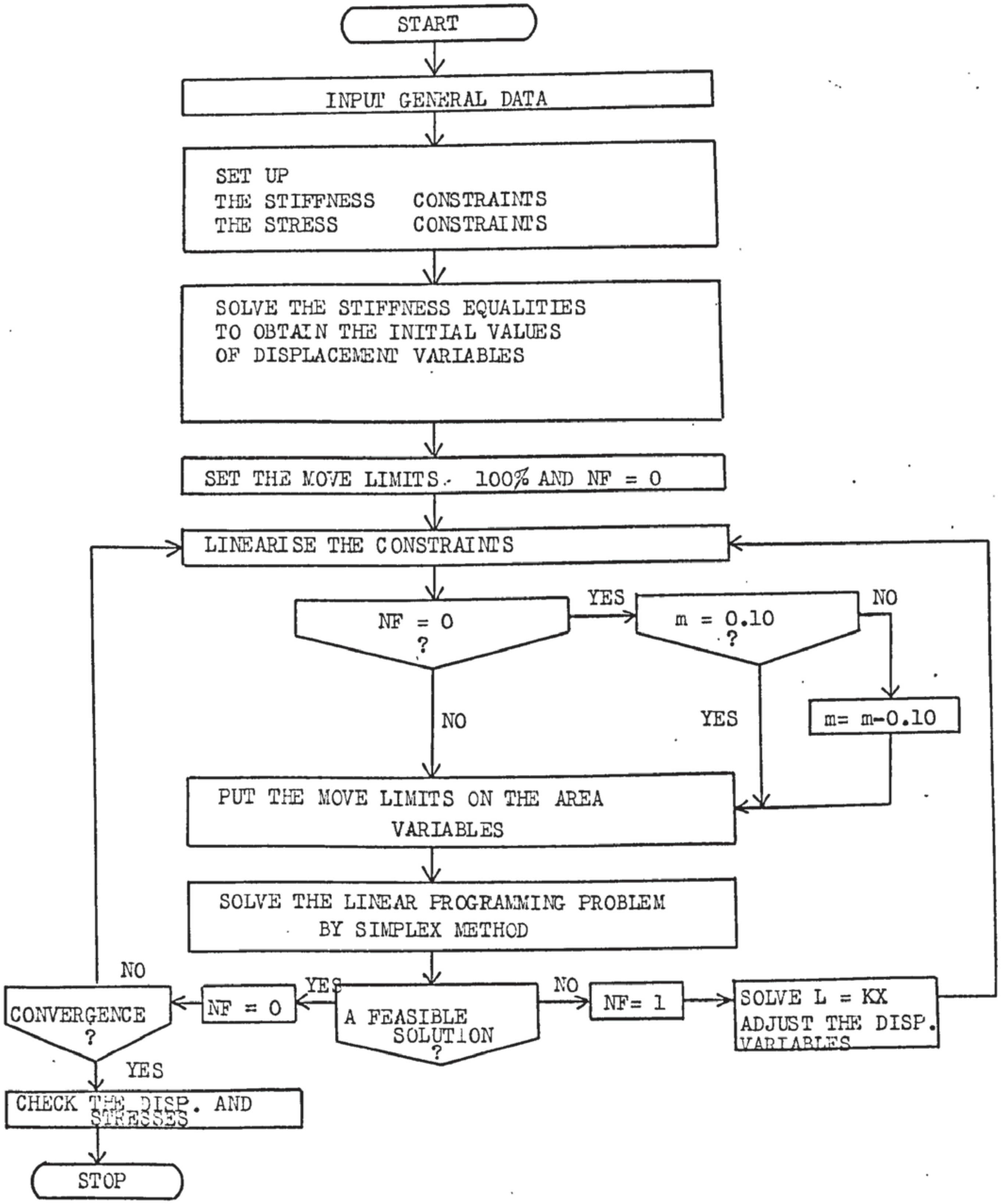


FIGURE 2.5: THE FLOW-DIAGRAM OF THE DESIGN PROCEDURE

in the previous sections.

The convergence limit utilised is that the change of the objective function on two successive cycles will be less than ε of its current value, where ε is a selected small constant.

Thus: $$[W(x^{i+1}) - W(x^{i})] / W(x^{i}) \leq \varepsilon$$

where $W(x^{i})$ is the present value of the objective function, $W(x^{i+1})$ is the value obtained in the next iteration. It is found reasonable to take the value of ε as 0.1%.

2.6) SOLUTION OF THE DESIGN PROBLEM BY [SUMT].

In most of the recent research work, the non-linear structural design problems are either solved by approximating programming or by the penalty function method. In the last section approximating programming is employed for solution of the design problem and found to be effective. The penalty function method can also be used to solve the design problem. The basic idea of this method, described in Chapter 1, is to convert the constrained problem, with its objective function, equality and inequality constraints, into a problem in which some new function is minimised without regard for constraints. The solution of the original constrained minimisation problem is then obtained through a sequence of unconstrained minimisations. There are different types of penalty functions. The one, which is developed by Fiacco and McCormick[23] has found great application for solving structural design problems and it has the form:

$$P(x_k,r) = W(x_k) + r \sum_{i=1}^{m} \frac{1}{g_i(x_k)} + r^{\frac{1}{2}} \sum_{j=1}^{\ell} h_j^2(x_k)$$

$$k = 1,2,\ldots n$$

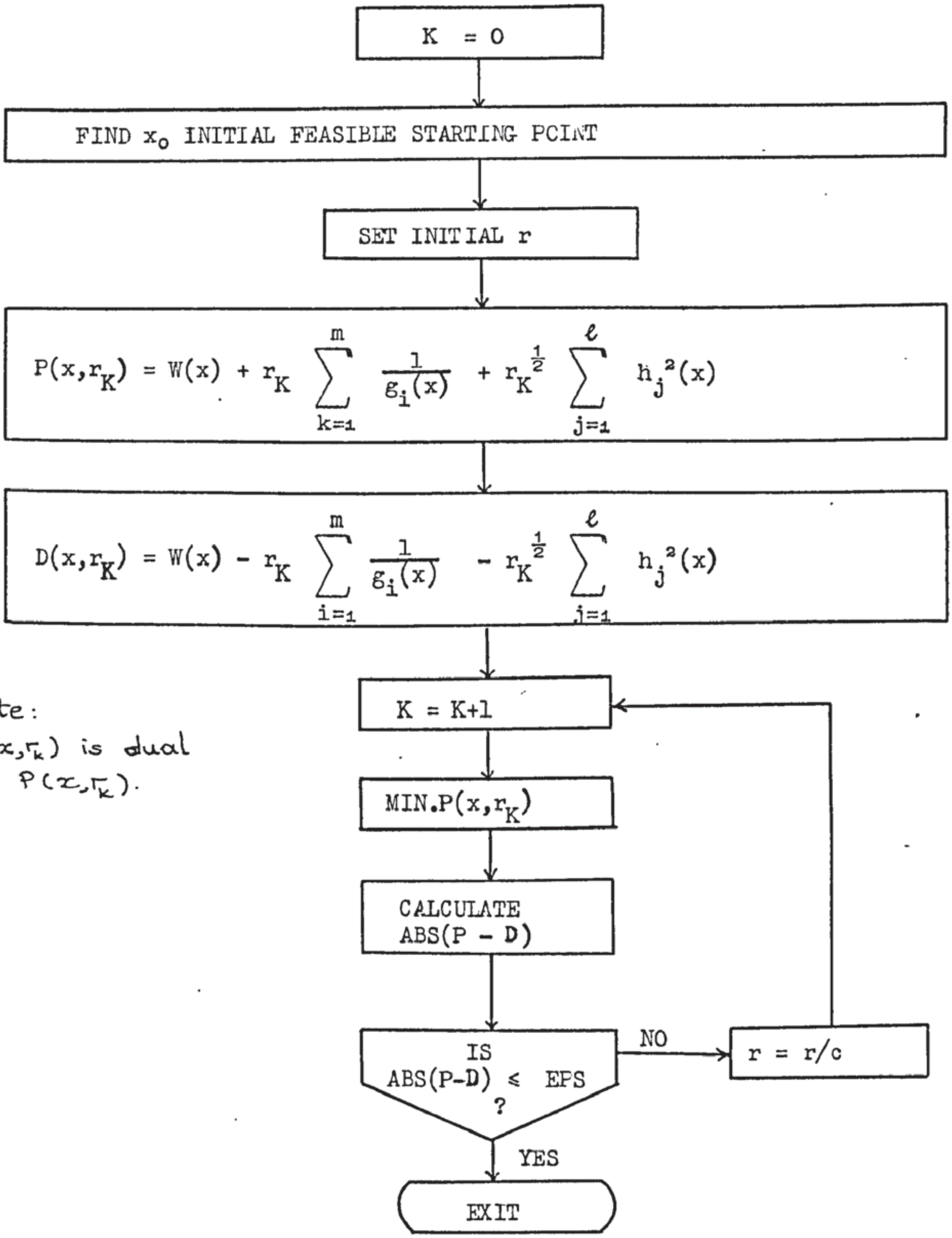
where $P(x,r)$ is the unconstrained function to be minimised, $W(x_k)$ is the objective function, $g_i(x)$ are the inequality constraints, m is the number of inequality constraints, $h_j(x)$ is the equality constraints and ℓ is the number of the equality constraints, n is the number of variables and r is the penalty parameter. The flow diagram of the method is given in Figure 2.6 which is also used by Asaadi[31].

Generally the unconstrained minimisation methods proceed by choosing a starting point x_i and calculating

$$x_{i+1} = x_i + \alpha_i S_i$$

where x_{i+1} is the new point, α_i is the step length and S_i is the direction vector to move along. There is a considerable amount of algorithms which minimise a multivariable function. These can be divided into two groups. The first group uses the gradient vector which requires computation of first or higher order derivatives of the function. The second group is called the direct search technique. They do not require the evaluation of function derivatives. The gradient methods are found[26] quick and effective. Among these the variable metric method developed by Fletcher-Powell[28] is suggested to be superior to others[31]. This technique requires a one dimensional minimisation to find the α^* which minimises $P(x_i+\alpha_i S_i, r)$. The cubic interpolation is the one which is utilised for this purpose by Fletcher-Powell in their variable Metric Method.

This procedure, shown in Figure 2.6, was employed to solve the proposed non-linear design problem. Difficulty arose in the cubic interpolation technique which failed to find the α^*. The reason for this was thought to be the differences among the numerical values of the design variables. That is, while the areas have values such as 10^2, the deflexions are 10^{-2}. Although the variables were scaled in order to make them of similar order and constraints were normalised[106], the difficulty of convergence was not overcome.



Note:
$D(x,r_k)$ is dual of $P(x,r_k)$.

FIGURE 2.6 A FLOW-DIAGRAM OF PENALTY FUNCTION METHOD.

It is therefore concluded that further research is required before the penalty function method can be applied to the optimum design of structures where the displacements of joints are taken as design variables as well as sectional properties of the members. Research is particularly needed to find out a suitable method for scaling the variables.

C H A P T E R   3

COMPUTER PROGRAMMING.

3.1) INTRODUCTION.

A computer program for the optimum design of rigidly jointed plane frames which makes use of the design procedure described in the previous chapter, was written in Fortran and run on the ICL 1905 computer at the University of Aston. The program consists of a master segment calling a number of subroutines, the functions of which are illustrated by the flow diagram shown in Figure 3.1.

The input data is divided into two parts. The first part is the data concerning the structural properties. The second is the data which is relevant to mathematical programming i.e. the initial design point which consists of the selected areas of groups in the structures. The format of the data for a simple frame is shown in Figure 3.2. The first input card contains the total number of joints, the total number of members, the total number of hinges, if there are any, the total number of supports and the number of groups in the structure. In addition to joints, supports are also numbered by considering the first support number as N+1 where N is the last joint number. The first support number represented by KSF is required as a datum in order to identify the members which are supported. NB is the maximum number of members connecting to a joint in the structure.

The relationship between the section modulus z, the second moment of area I, and the area of the section A is also entered.

$$I = fc.A^{pv}$$

$$z = fcl.A^{pv2}$$

where fc, fcl, pv, pv2 are constants. This is followed by the specification of the members which is the joint number of first and second ends, the hinge number of first and second ends and the group numbers of the members. The material properties of the

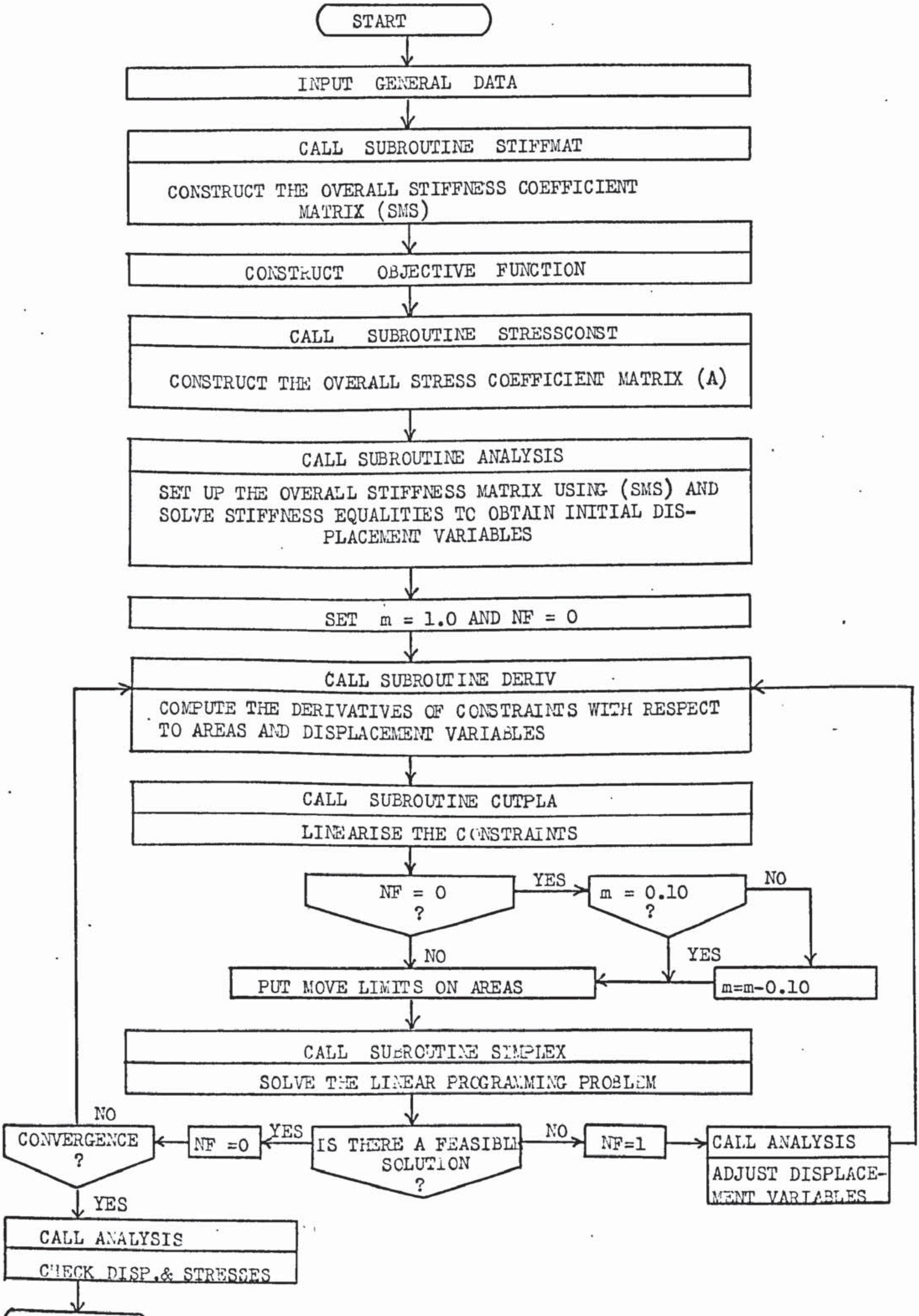


FIGURE 3.1 - MASTER PROGRAM.

member are specified by stating its modulus of elasticity E and permissible stresses in compression and tension, which are read by the arrays PSTRC, PSTRT respectively.

The joint data contains the coordinates of each joint in the structure and the bounds imposed on the displacements of joints. The external loads acting on each joint in turn, in x,y and θ direction are also given in the form of a row vector. This load vector is of order 3.NOJ where NOJ is the total number of joints in the structure. This is followed by the preselected area variables corresponding to each of the groups in the structure.

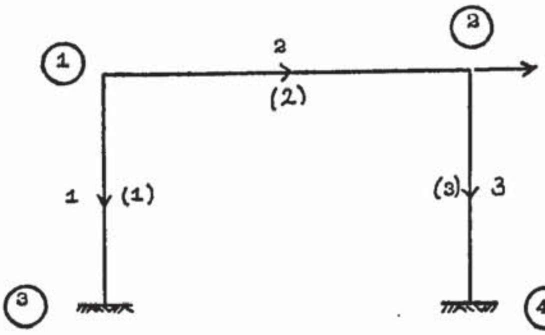
After entering the data, the program starts by computing the following at each joint:-

i) The total number of members connecting to this joint,

ii) The member number of these members,

iii) The group number of these members,

Joint number of first and second ends of each member are also checked at each joint against the joint number to determine whether that particular joint is the first end of the member. If so, then the member number is multiplied by -1. It can be seen that some of the members connected to a joint may belong to the same group. As a result their contributions to the overall stiffness coefficient matrix can be added together at the rows and columns corresponding to that particular joint. Hence, it becomes necessary to find the total number of different groups at each joint. This is carried out by checking the group numbers of members connected to that joint and storing the different ones in the array NG(J). The Jth elenent of this array gives the number of different groups at joint J. As shown in Chapter 2, the contribution of each group is stored separately at each joint in the overall stiffness coefficient matrix. The total number of "different groups of each joint" for the structure is obtained by adding the elements of the array NG together for all



( ) : represents the group numbers of members

a) STRUCTURAL CONFIGURATION

NOJ NOM NOH NOS NOG KSF NB

b) RELATIONSHIPS OF THE SECTIONAL PROPERTIES

FC FC1 PV PV2

c) MEMBER PROPERTIES

MEMBER NO	ME1	ME2	HN1	HN2	GRP	PSTRC	PSTRT	E

d) JOINT DATA

d1) COORDINATES OF JOINTS INCLUDING SUPPORTS

JOINT NO	XJ	YJ

d2) BOUNDS ON THE DISPLACEMENTS AND LOAD VECTOR

JOINT NO	UBX	UBY	UBT	VL(Ho)	VL(ver.)	VL(Mom)

e) INITIAL VALUES OF AREAS VARIABLES

GROUP NO	XA

Figure 3.2. FORMAT OF THE DATA.

the joints and storing in the array NI. The Jth element of this array gives the total number of "different groups of each joint" up to the joint J. Then the program proceeds to compute the size of the overall stiffness coefficient matrix and other arrays which are necessary for each subroutine. When convergence criteria are satisfied, the displacements of joints and stresses in the members are checked and printed out with the optimum values of areas for the selected groups. Then the program is terminated.

3.2) CONSTRUCTION OF THE STIFFNESS CONSTRAINTS.

Subroutine STIFFMAT consists of three nested loops as shown in Figure 3.3. The outer loop takes each joint of the structure in turn. In the second loop the total number of different member groups at this joint are cycled. In the final loop each of the members connected to that joint are taken and checked as to whether the group number of the member coincides with the group number of the second loop. If so, the contribution of it to the submatrices $\underline{K}_{RR}$, $\underline{K}_{RS}$, $\underline{K}_{SR}$ and $\underline{K}_{SS}$ is computed. Then the program proceeds to determine whether the joint taken in the first loop is the first end of the member. If so, the array NI explained in the previous section is used to determine the addresses of the submatrix $\underline{K}_{RR}$, and this matrix is inserted in the corresponding rows and columns of the overall stiffness coefficient matrix. The addresses of the submatrix $\underline{K}_{RS}$ are found by using the array NIG. This array contains the number of different group numbers at each joint. This is used to compute the number of groups before the member group at the joint corresponding to the second end of the member. This number is employed to find the addresses of the submatrix $\underline{K}_{RS}$ and this matrix is inserted in the corresponding rows and columns of the overall stiffness coefficient matrix. In the case when the second end of the member is supported, the member will have

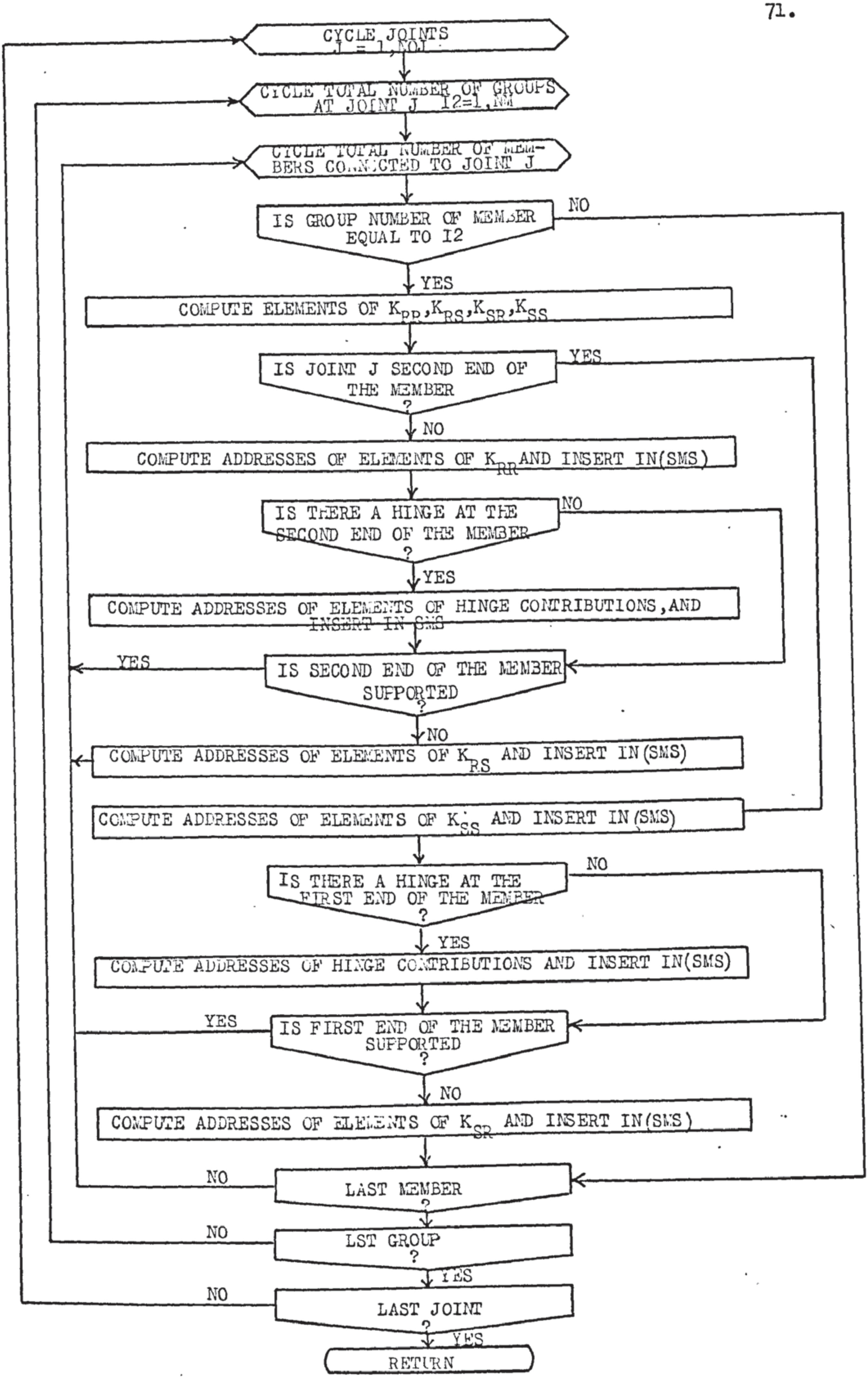


FIGURE 3.3: SUBROUTINE STIFFMATT

no contribution from that end. If the joint taken in the first loop is the second end of the member, then the same procedure is applied to compute the addresses of the submatrices $\underline{K}_{SS}$ and $\underline{K}_{SR}$.

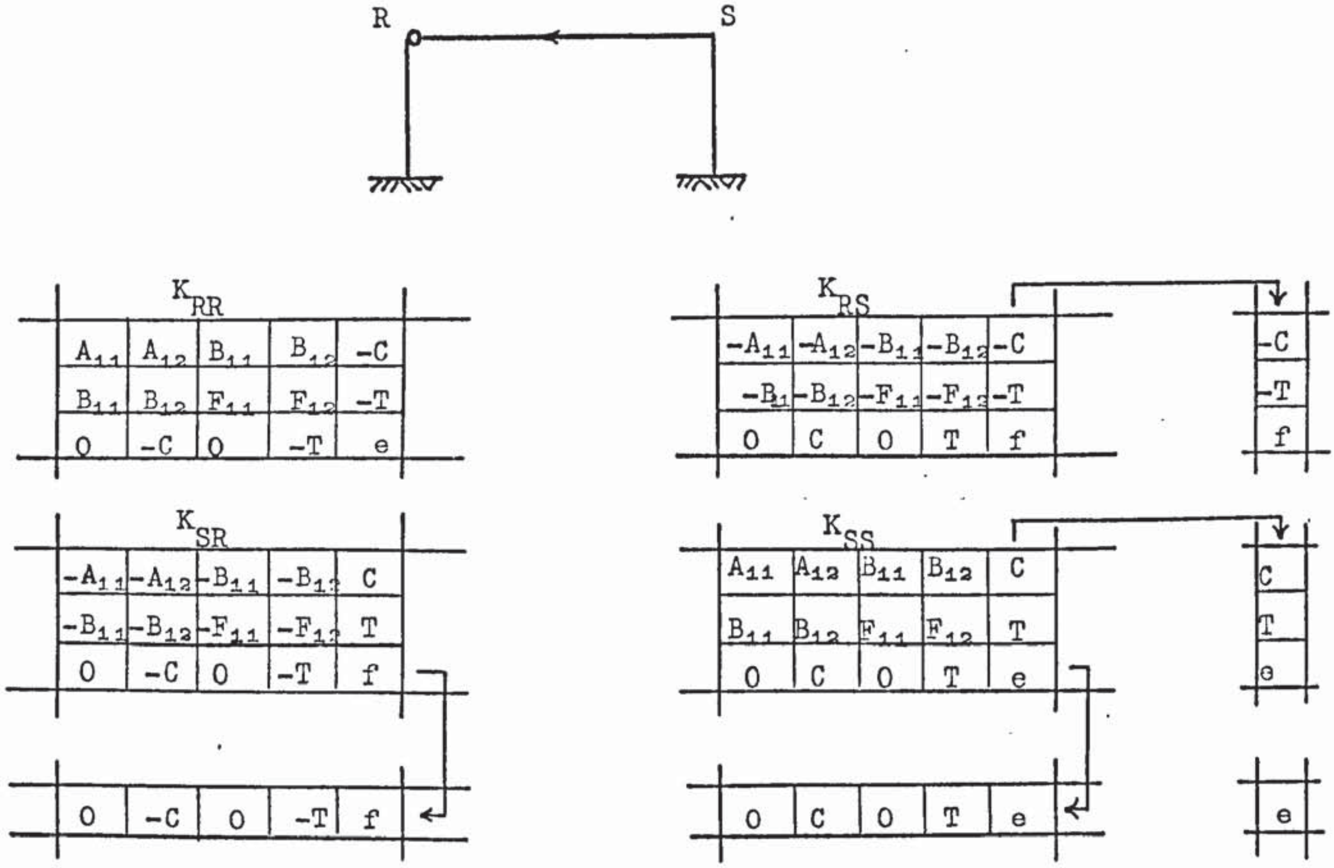
In the case when there is a hinge at the end of the member, the additional contributions are placed in the overall stiffness coefficient matrix. This is shown in Figure 3.4. As explained in Chapter 2, these contributions are stored in the overall stiffness coefficient matrix after the contributions of the members connected to the last joint in the structure. The row number of the first hinge contribution is identical to the hinge number, and the column number is given by

$$HC = 5.NI(NOJ) + 1$$

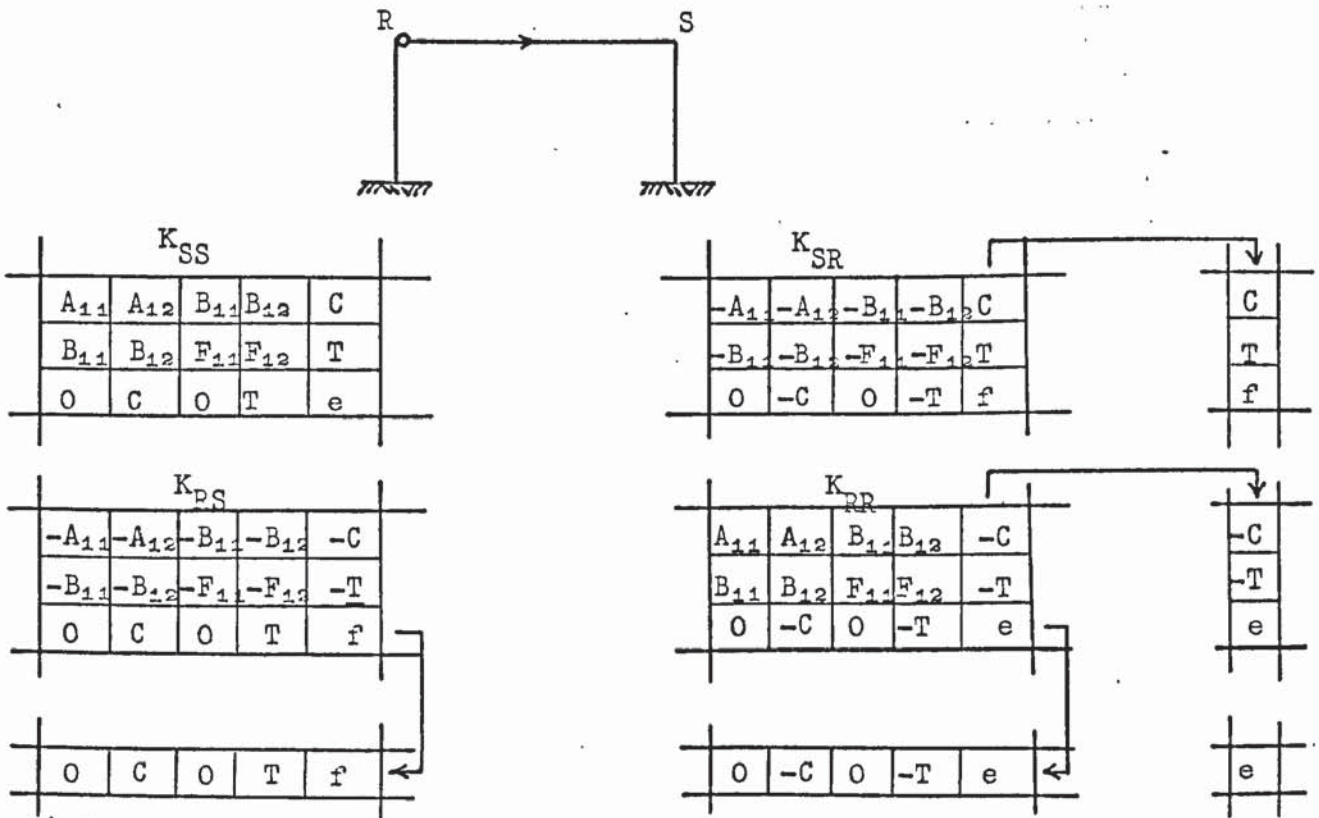
where NI(NOJ) gives the summation of the "different number of groups at each joint" for all the joints, HC is the column number of the first hinge contribution and NOJ is the number of joints in the structure.

3.3) CONSTRUCTION OF THE STRESS CONSTRAINTS.

Subroutine stressconst constructs the overall stress coefficient matrix. The flow diagram of it is given in Figure 3.5. Each joint is taken in turn and members connected to that joint are checked to find out whether the joint is at the first end. If so, their first end contributions to submatrix $\underline{B}_R$ is determined, the addresses of the elements of this matrix are computed and inserted in the overall stress coefficient matrix. If the joint is the second end of the member, then the addresses of submatrix $\underline{B}_S$ are found and inserted in the $\underline{A}$ matrix. The hinge contributions come from each hinge at either end of the member are placed after the last column corresponding to the last joint in the overall stress coefficient matrix. Hence, the column number of the first hinge contribution is the same as the hinge number which is given as



a) The hinge contributions of a member with a hinge at its second end



b) The hinge contributions of a member with a hinge at its first end

FIGURE 3.4 THE LAYOUT OF HINGE CONTRIBUTIONS NUMERICALLY R > S

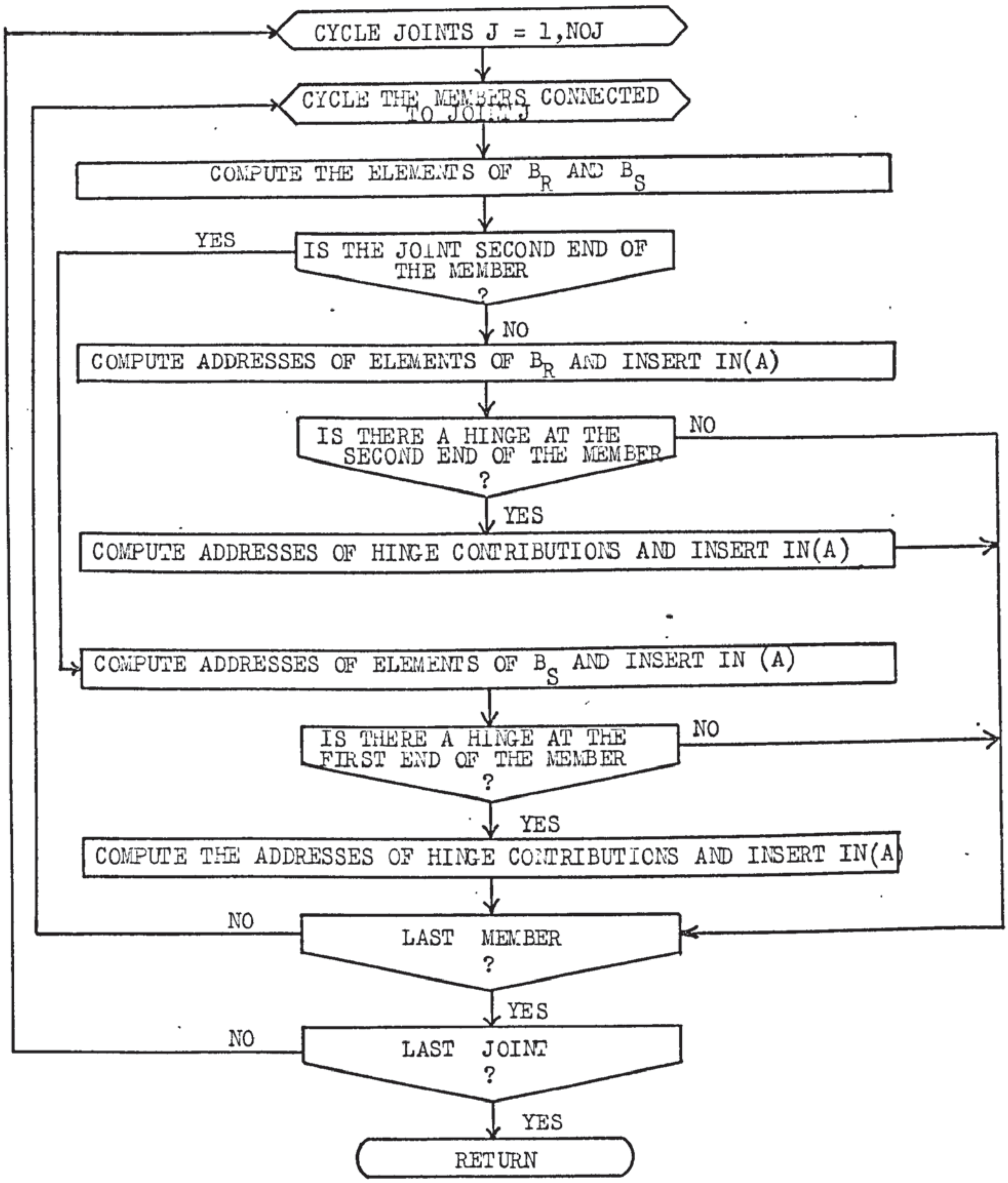


FIGURE 3.5: SUBROUTINE STRESSCONST

3.NOJ + 1.

3.4) THE SOLUTION OF THE STIFFNESS CONSTRAINTS.

As shown in Chapter 2, the solution of stiffness equations may be necessary to adjust the displacement variables. It may also be necessary to carry out the solution of the stiffness equations to check the displacements of the joints and the stresses in the members when convergence is obtained. Subroutine ANALYSIS is written for this purpose. When it is called in the master program before convergence is obtained, it only solves the stiffness equations for displacements. The stresses in the members are not computed, since they are not needed during the iterations. When it is called after convergence is obtained and a check is required for displacements and stresses, it solves for the stresses in the members as well as the displacements of joints.

As a result, subroutine ANLAYSIS constructs the overall stiffness matrix by using the overall stiffness coefficient matrix which is set up by subroutine stiffmatt. It is known that this matrix contains the contributions of each group connected to each joint and it is also known which column of this matrix contains the coefficient of the first order and the second order terms of the area variable. The only necessary computation to obtain the overall stiffness matrix of the structure is to multiply these elements of the overall stiffness coefficient matrix by the preselected area variables. This is carried out for all the joints and the elements are placed in the columns and rows of the overall stiffness matrix corresponding to these joints. The hinge contributions are also computed and placed in the corresponding rows and columns of the overall stiffness matrix. A flow diagram is given in Figure 3.6.

The methods for the solution of sets of linear

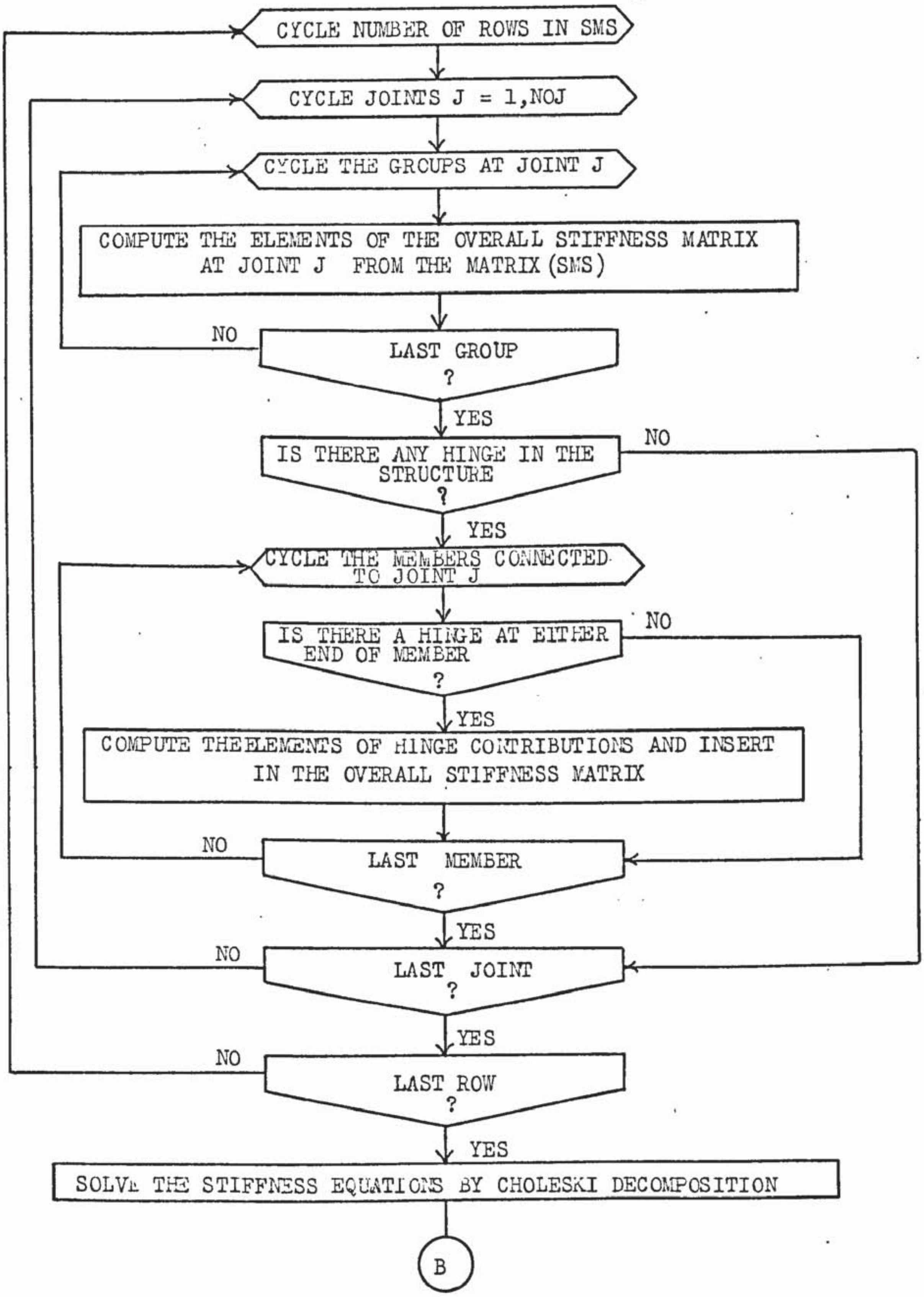


FIGURE 3.6: SUBROUTINE ANALYSIS

equations can be divided into two categories. These are direct and indirect methods. In direct methods, the results are obtained by carrying out a single set of operations on the equations. In the indirect methods the solution is obtained by a series of successive approximations. In these methods, the rate of convergence may be slow or the results may not converge at all. For these reasons the direct methods are preferred in the solution of sets of linear equations. Choleski's triangular decomposition method is one of the direct methods which is particularly suitable for the solution of structural equations. It factorizes any square symmetric matrix $\underline{K}$ in the form

$$\underline{K} = \underline{A}.\underline{A}^T$$

where $\underline{A}$ is the lower triangular matrix. As explained in(107), no term of $\underline{K}$ is used more than once. Therefore the procedure requires no additional storage, the locations in $\underline{K}$ can be overwritten by the elements of $\underline{A}$ as each one is obtained. A flow diagram is also given in(107). This Choleski decomposition is employed in subroutine Analysis.

In Figure 3.7 a flow diagram is given for the computation of the stresses in the members. Each member is taken in turn.

The axial and bending stresses coefficients, which are computed by subroutine stressconst are multiplied by the area adopted for the group of that member and the displacements which are known by the solution of stiffness equations. These axial and bending stresses due to each end displacement are added together to obtain the combined stresses for the members.

3.5) COMPUTATIONS OF GRADIENT VECTORS.

The derivatives of constraints with respect to design variables are computed by subroutine DERIV and stored in matrix DK.

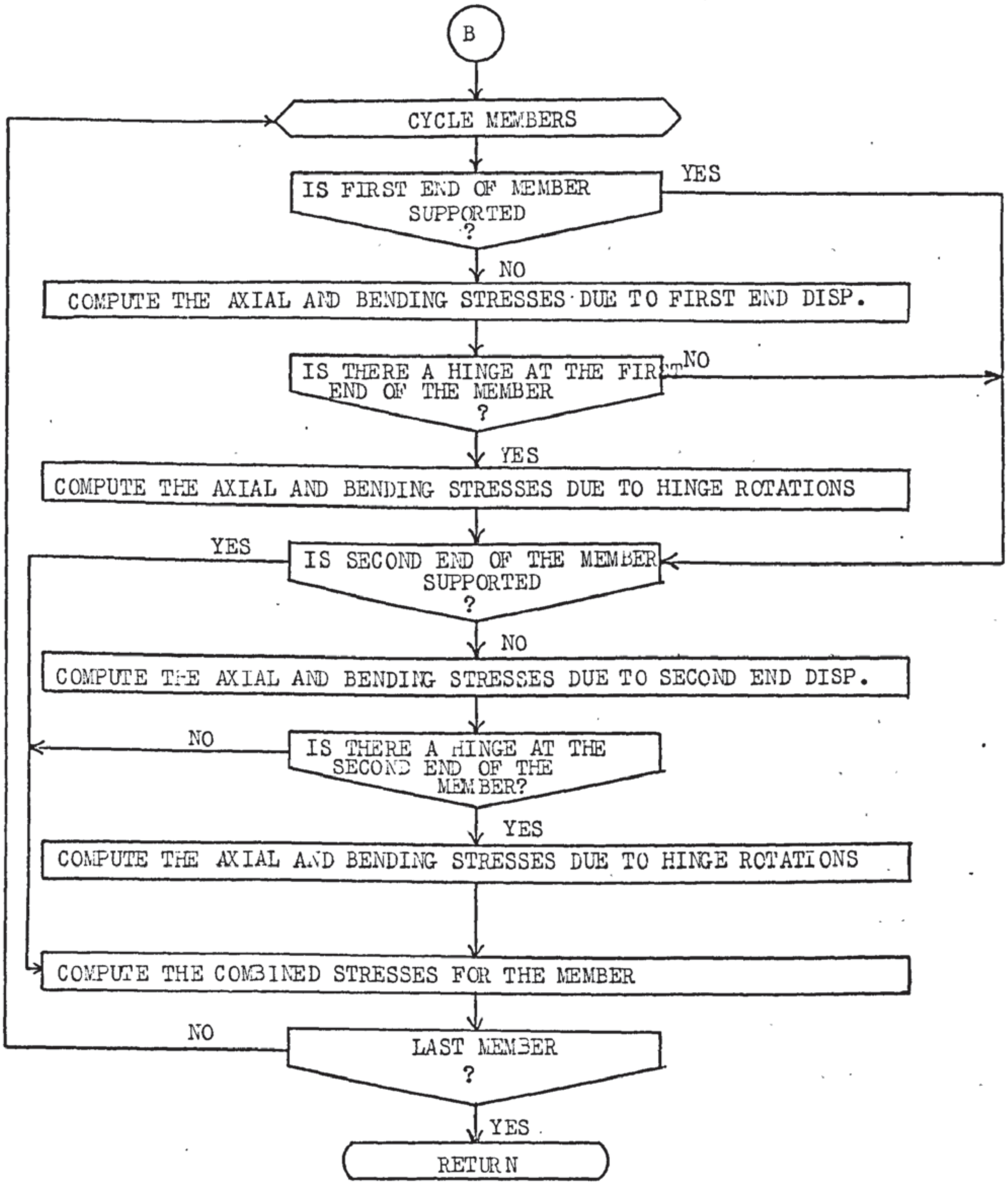


FIGURE 3.7: SUBROUTINE ANALYSIS.

The order of this matrix is [NR+8.NOM, NOG+3.NOJ] where NR is the number of stiffness constraints, NOM is the total number of members, NOG is the total number of groups and NOJ is the total number of joints in the structure. The first NOG columns of DK contain the derivatives of the constraints with respect to area variables. The rest, contain the derivatives of the constraints with respect to displacement variables. Subroutine DERIV consists of three parts.

In the first two parts the derivatives of stiffness equalities with respect to areas and displacement variables are computed. The flow diagram is given in Figure 3.8. The stiffness coefficient matrix is transferred to this subroutine. It cycles the group numbers and at each joint determines whether there is an element corresponding to this group in the overall stiffness matrix. If there is, then the derivatives of stiffness equalities with respect to areas are computed at this joint. As shown in Chapter 2, the contribution matrices have five columns for each member. It is also known that first and third columns contain the areas and the remainder contain the second moment of areas. So the derivatives of the elements at each column are computed by adding together first order terms and then second order terms. By carrying this out for all the joints, the derivative of the stiffness constraint is computed with respect to the area available adopted for that group. It is shown in Chapter 2 that the derivatives of stiffness constraints with respect to displacement variables are equal to the element of the overall stiffness matrix corresponding to that particular displacement. Therefore the necessary computation is to find the elements in the overall stiffness coefficient matrix corresponding to that displacement and multiply them by the current values of areas. By carrying this out for each joint, the derivatives are computed and inserted in the derivative matrix DK.

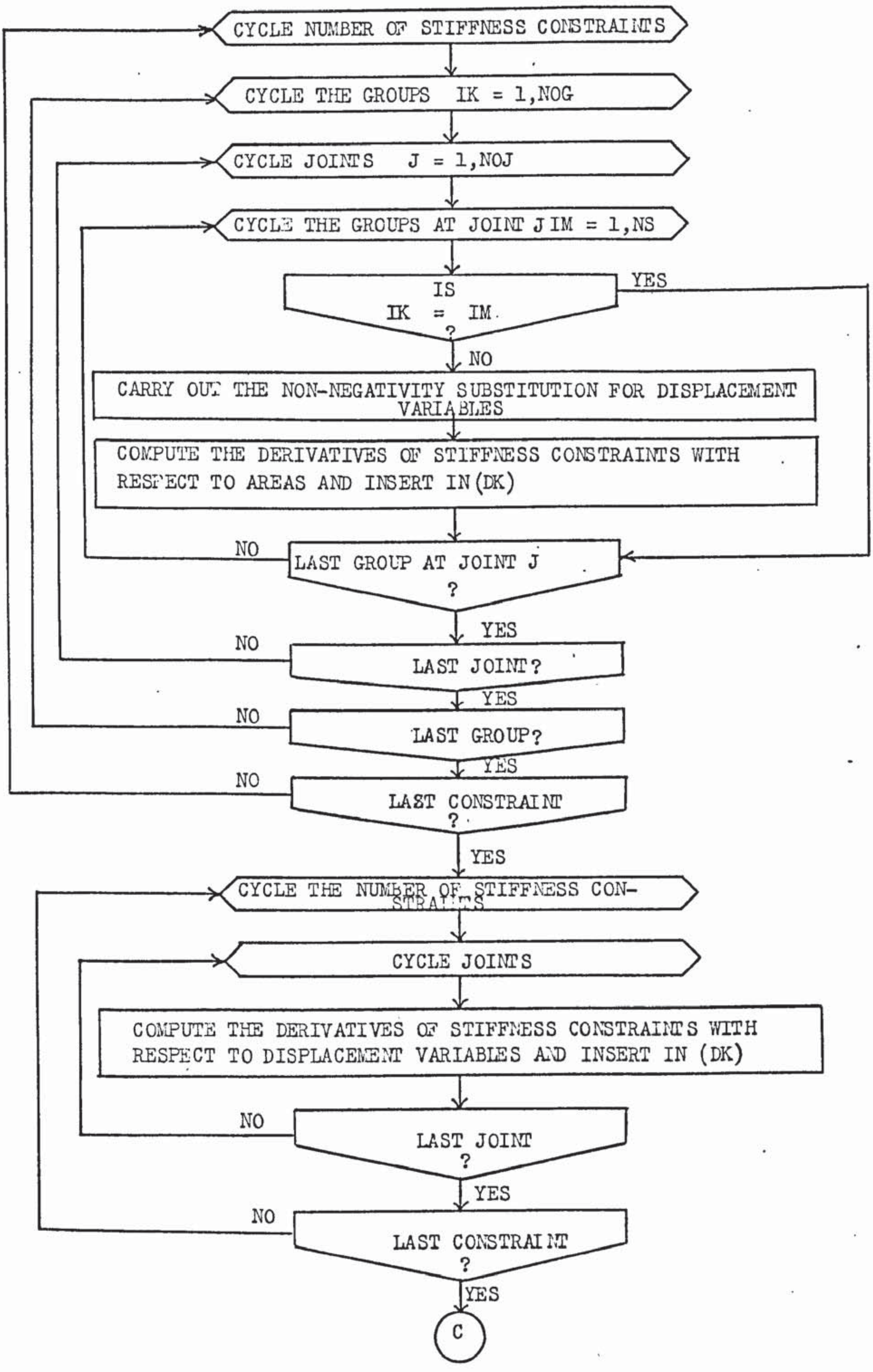


FIGURE 3.8: SUBROUTINE DERIV

In the final part, the derivatives of the stress constraints are computed with respect to displacement and area variables. A flow diagram is given in Figure 3.9. As shown in Chapter 2 each member contributes two submatrices corresponding to each end of the members to the overall stress coefficient matrix. Hence, the derivatives of the stress constraints of each member with respect to displacement variables are computed in two parts. The derivatives with respect to the displacement variables belonging to the joint which is the first end of the member are computed using the matrix B_R and the derivatives with respect to the displacement variables belonging to the joint which is the second end of the member are computed by using the matrix B_S. At the same time the derivatives with respect to areas are also computed for each end and these are added together and inserted at the corresponding places in matrix DK.

3.6) THE LINEARISATION OF THE CONSTRAINTS.

The linearisation of the constraints is carried out by subroutine CUTPLA. It can be seen from the flow diagram given in Figure 3.10 that, firstly, the values of stiffness constraints are computed at the current design points. Then using the gradient vectors constructed by subroutine DERIV, the right hand side of the constraints in the linearised problem are computed. It is shown in Chapter 2 that the stiffness equality constraint

$$h_i(\underline{x}) = 0$$

can be linearised in the form

$$h_i(\underline{x}_o) + \nabla h_i(\underline{x}_o)(\underline{x}-\underline{x}_o) = 0 \qquad 3.1$$

where the $\underline{x}_o$ is the current design point and $\underline{x}$ is the design point to be found at the end of the next iteration and so at the current design point $\underline{x}_o$, the values of $h(\underline{x}_o)$ and $\nabla h(\underline{x}_o)$ are known. Equation 3.1 can be simplified to

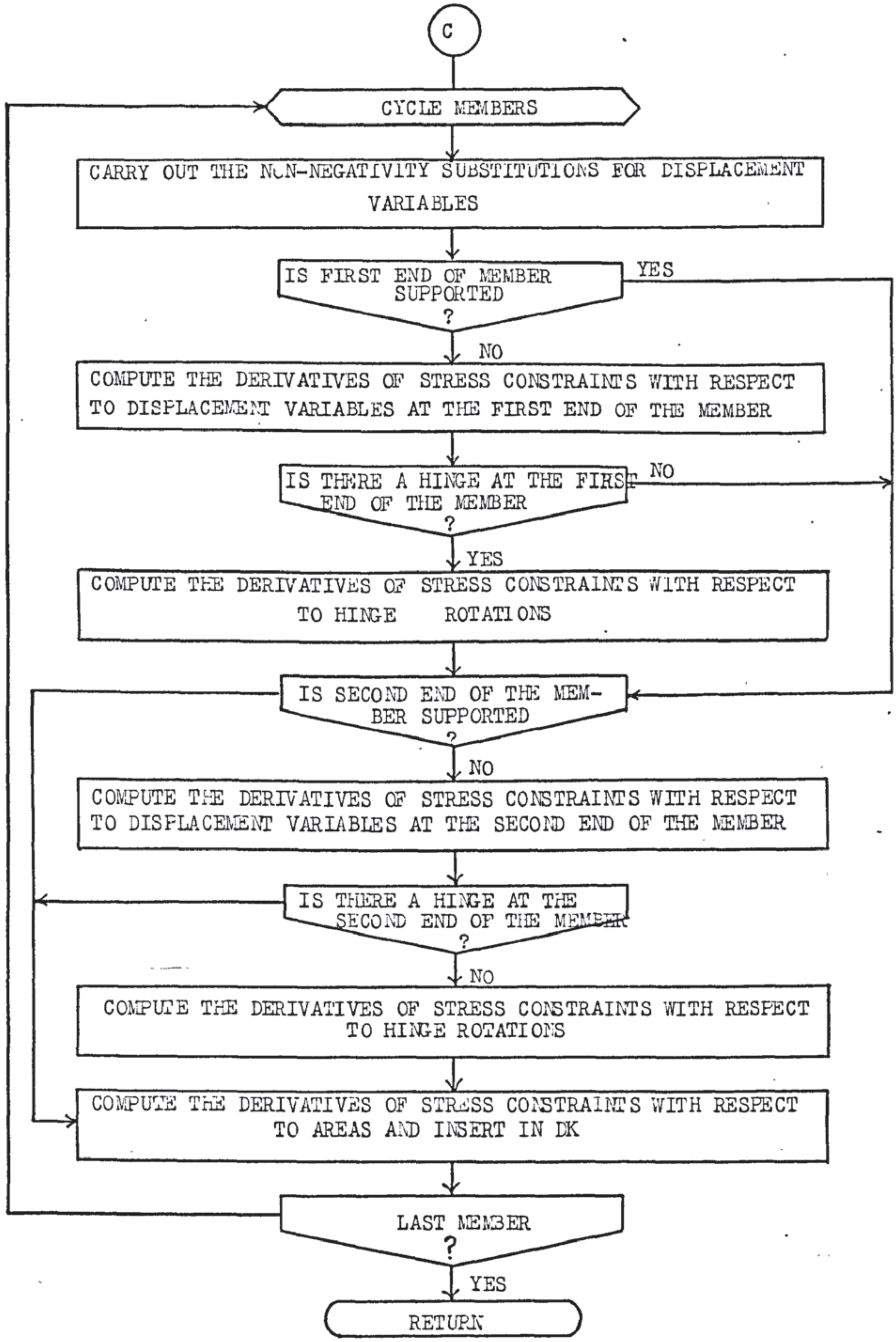


FIGURE 3.9: SUBROUTINE DERIV.

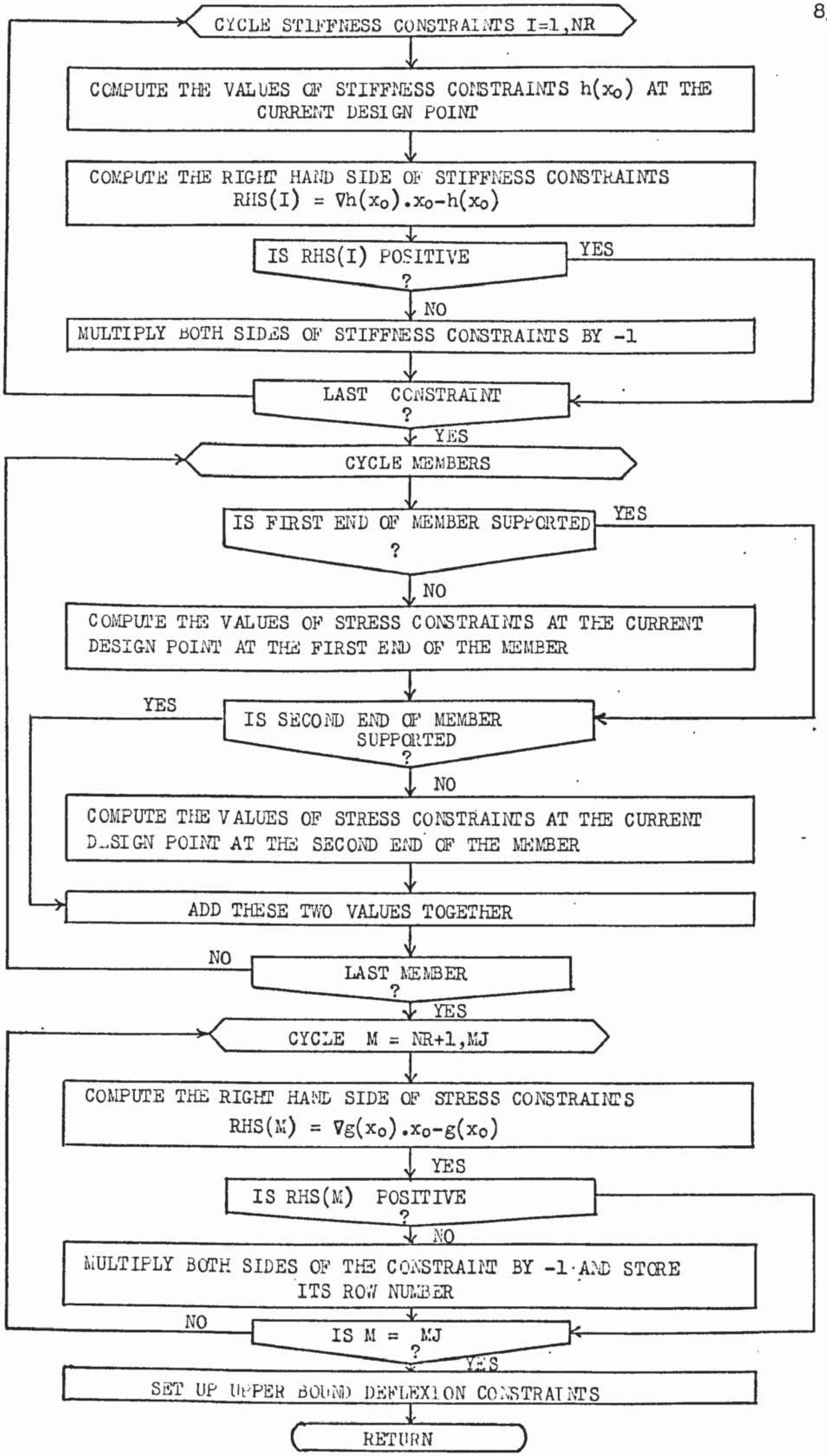


FIGURE 3.10: SUBROUTINE CUTPLA

$$\nabla h_i(\underline{x}_o).\underline{x} = \nabla h_i(\underline{x}_o)\underline{x}_o - h_i(\underline{x}_o)$$

where the matrix $\nabla h_i(\underline{x}_o)$ in the left hand side contains the coefficients of the design variables in the linear programming problem and the right hand side of i^{th} stiffness constraint becomes

$$RHS_i = \nabla h_i(\underline{x}_o).\underline{x}_o - h_i(\underline{x}_o)$$

which can be positive or negative. It is found advantageous to have the right hand side of the constraint positive. Hence if RHS_i is found to be negative both sides of the stiffness equality is multiplied by -1 to ensure that each RHS_i is positive.

The value of stress constraints at the current design point can be computed in two parts and added together. This is because of the fact that if an end of the member is supported, then the stress constraint will have no term corresponding to the supported end. Using these values of stress constraints and gradient vectors, the right hand side can be computed. If the sign turns out to be negative, then it is necessary to multiply both sides of the constraint by -1. This changes the constraint from type I to type II where type I represents $\leq$ sort of inequalities and type II represents $\geq$ sort of inequalities. It is also necessary to store the row number of this constraint because it will need an artificial variable to be added.

In this way the linear programming problem is set up and the coefficients of variables are stored in the matrix DK. The size of it is increased to [KB, NV] where

$$KB = NR + 8.NOM+3.NOJ +2.NOG$$

$$NV = NOG+3.NOJ$$

in which NR is the number of stiffness constraints, NOM is the total number of members, NOJ is the total number of members, NOJ is the total number of joints and NOG is the total number of groups in the structure. Hence, it can be seen that the first NR rows of the matrix DK contain the coefficients of variables in the linearised stiffness constraints, the rows from NR+1 to NR+8NOM contain the coefficients of variables in the linearised stress constraints. The coefficient of variables in the deflexion constraints are placed in the rows from NR+8.NOM to NR+8.NOM+3.NOJ. The rest contains the constraints due to move limits. The same order is followed in the matrix RHS.

3.7) SUBROUTINE SIMPLEX.

Two different simplex subroutines are employed to solve the linear programming problem. Firstly, a program is written which uses the Charnes M method as described in(6). This is used within the design procedure, applied to a number of relatively small examples and found successful. But, when it is utilized in the design of relatively large frames, it is noticed that difficulties arise in finding feasible solutions. It is thought that this happens due to the round-off errors during the simplex iterations. The round-off errors become important when some of the numbers are very large and some are quite small in the right hand side matrix. It can easily be seen that this is the case in most design problems. The right hand side of the linearized stiffness constraints are of the order of 10^4, the right hand side of deflexion constraints for rotations are of the order of 10^{-2}. In such problems after a number of iterations a small round-off error in a critical element can cause an error in selecting the maximum element of the objective function row which results in an incorrect selection of the variable entering the solution. In the simplex method what is known as check column is constructed by adding the elements of the columns of the table, row for row. As shown in(108), if the same simplex operation is applied to this column as is used in other columns, the elements of the check column will equal the row sums at each iteration. This is found to be a very powerful check on the calculations. When some elements of the right hand side matrix are very large and some are small, the entries to the check column will be dominated by the elements of the right hand side matrix. After a number of iterations, the check column becomes almost useless, as small round-off errors in the columns containing the large numbers make it impossible to detect large errors in the columns containing the small numbers.

Although it is possible to control these round-off errors by scaling the problem, it is found practical to use the simplex subroutine written by ICL[11] in the design of relatively large frames. This routine uses the two-phase revised simplex algorithm with multiple column selection. As will be shown in the next chapter, this routine is found to be effective and successful from the results of the examples solved.

C H A P T E R 4

DESIGN EXAMPLES

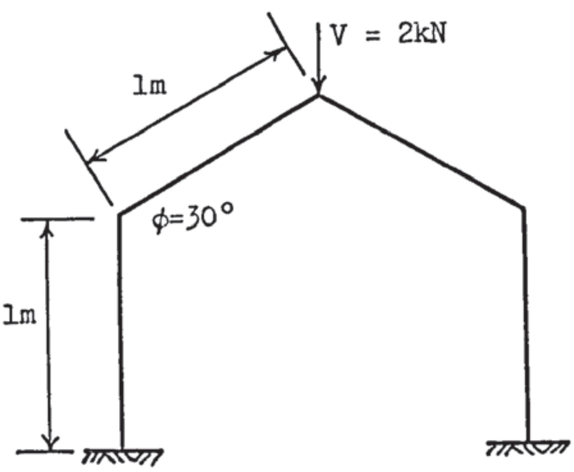
4.1) HAND EXAMPLES.

The optimum design method described in Chapter 2 was applied to a number of examples and the numerical results were illustrated.

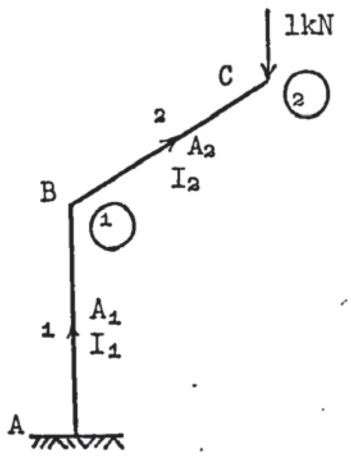
Firstly, two pitched roof frames were designed by hand to investigate the effect of the axial force and the initial design point. In the first pitched roof frame, which has also been solved graphically by K.I.Majid[1], the effect of axial force was neglected with the purpose of simplifying the design problem. The second moment of areas of columns and rafters were taken as design variables. This gave rise to a non-linear objective function. However, it was shown that the approximation of this non-linear objective function in linear form, in this case, did not introduce any error to the answer. The design problem was also formulated in terms of the areas of the columns and rafters. Two cases were compared and it was found that in the case where the effect of axial force is neglected, the use of the second moment of areas as design variables leads to a less non-linear design problem. As a result the linearisation errors involved were small and the application of move limits were not necessary. In the second pitched roof frame, the design problem was considered in the general form and the areas of the columns and the rafters were taken as design variables. Different initial design points were used and their effect on the optimum answer was investigated.

4.1.1) THE EFFECT OF AXIAL FORCE.

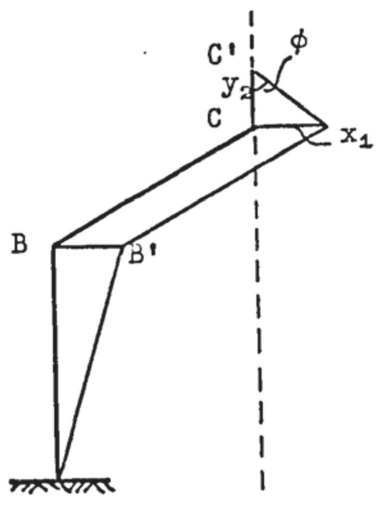
The design of the symmetrical pitched roof frame, shown in Figure 4.1, subject to a vertical load of 2kN acting at the apex C was considered. The columns were of the same section with second moment of area I_1 while rafters have second moment of area I_2. The modulus of elasticity was 207 kN/mm². The vertical deflexion of the apex was restricted to 4.8 mm while the horizontal deflexions at B



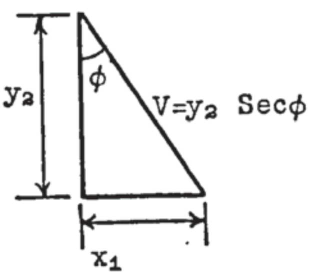
(a) DIMENSIONS AND LOADING OF FRAME



(b) NUMBERING OF MEMBERS AND JOINTS



(c) IDEALIZED DISPLACEMENT OF FRAME



(d) VECTOR DIAGRAM FOR DISPLACEMENTS

FIGURE 4.1: SYMMETRICAL PITCHED ROOF FRAME.

and D were restricted to 2.78 mm. The allowable combined stress for the members was 0.15 kN/mm^2.

Due to the symmetry it was only necessary to consider one half of the frame. Neglecting the axial stiffness EA/L of the members, the joint displacement vector was given by

$$X = \{x_1 \quad \theta_2 \; y_2\}$$

From the geometry of the frame it was possible to express the horizontal deflexion of B in terms of y_2, shown in Figure 4.1, as:

$$x_1 = y_2 \tan\phi$$

where ϕ is the angle of pitch. Hence the joint displacement vector was reduced to $X = \{\theta_1 \; y_2\}$. It was known that the vertical deflexion at the apex was always downwards. Hence, the deflexion constraint for it was to be

$$y_2 \leqslant 0.$$

In order to satisfy the non-negativity restriction for displacement variables, the expressions

$$y_2 = y_2 - 4.8$$

$$\theta_1 = y_1 - 0.01$$

were substituted in the stiffness and stress constraints. Hence the design problem had the form

$$\text{Min } W = I_1^{\frac{1}{2}} + I_2^{\frac{1}{2}}$$

Subject to

$$\begin{array}{lcl}
0.828I_1\gamma_1+0.828I_2\gamma_2+0.7166\times10^{-3}I_1\gamma_2-1.4354\times10^{-3}I_2\gamma_2-0.01172I_1-0.0014I_2 & = & 0 \\
0.7166\times10^{-3}I_1\gamma_1-1.4354\times10^{-3}I_2\gamma_2+0.0827\times10^{-5}I_1\gamma_2+0.3308\times10^{-5}I_2\gamma_2-0.1113\times10^{-3}I_1- 0.0153\times10^{-3}I_2 & = & -1.0 \\
0.68238I_2^{\frac{1}{4}}\gamma_1-2.3644\times10^{-3}I_2^{\frac{1}{4}}\gamma_2+4.5253\times10^{-3}I_2^{\frac{1}{4}} & \leq & 0.15 \\
-0.68238I_2^{\frac{1}{4}}\gamma_1+2.3644\times10^{-3}I_2^{\frac{1}{4}}\gamma_2-4.5253\times10^{-3}I_2^{\frac{1}{4}} & \leq & 0.15 \\
1.3647I_2^{\frac{1}{4}}\gamma_1-2.3644\times10^{-3}I_2^{\frac{1}{4}}\gamma_2-2.2985\times10^{-3}I_2^{\frac{1}{4}} & \leq & 0.15 \\
-1.3647I_2^{\frac{1}{4}}\gamma_1+2.3644\times10^{-3}I_2^{\frac{1}{4}}\gamma_2+2.2985\times10^{3}I_2^{\frac{1}{4}} & \leq & 0.15 \\
0.68238I_1^{\frac{1}{4}}\gamma_1+1.181\times10^{-3}I_2^{\frac{1}{4}}\gamma_2-12.492\times10^{-3}I_2^{\frac{1}{4}} & \leq & 0.15 \\
-0.68238I_2^{\frac{1}{4}}\gamma_1-1.181\times10^{-3}I_2^{\frac{1}{4}}\gamma_2+12.492\times10^{-3}I_2^{\frac{1}{4}} & \leq & 0.15 \\
\gamma_1 & \leq & 0.02 \\
\gamma_2 & \leq & 4.8
\end{array} \qquad \ldots 4.1$$

It was found that the stress constraints at C were dominant. Hence in the design problem only this constraint was considered and the problem was linearised at the point

$$\{I_1\ I_2\ \gamma_1\ \gamma_2\} = \{\ 4.0\times10^4 \quad 6.0\times10^4 \quad 0.0065 \quad -0.4\} \qquad 4.2$$

At the end of the first iteration the result was found to be:

$$\{I_1\ I_2\ \gamma_1\ \gamma_2\} = \{\ 5.373\times10^4 \quad 5.4667\times10^4 \quad 0.008 \quad 0.0\} \qquad 4.3$$

The weight function W being 1.08×10^4 mm^2. The minimum weight found by graphical design was 1.05×10^4 mm^3. The frame considered was analysed with $I_1 = 5.373 \times 10^4$ mm^4, $I_2 = 5.466 \times 10^4$ mm^4 and the values of γ's were found to be $\gamma_1 = 0.008$, $\gamma_2 = 0.0$ which agree very well with that shown in equation 4.3. It was shown that after one iteration reasonable convergence was obtained without application of move limits.

The same design problem was also linearised at the infeasible initial design point which was far from the optimum. When the change in the value of the objective function in two successive iterations was 1%, the procedure was terminated. The results obtained without application of move limits are shown in the following table

ITERATION NO	$I_1 \times 10^4 mm^4$	$I_2 \times 10^4 mm^4$	y_1 (rad)	y_2 (mm)	$W \times 10^4\ mm^3$
INITIAL DESIGN POINT	2.25	3.00	0.004	-5.1	0.525
1	3.273	4.748	0.0061	0.0	0.8021
2	4.35	6.368	0.0077	0.0	1.072
3	5.201	5.619	0.0074	0.0	1.082

TABLE 4.1

It can be seen that the minimum weight of the structure found in Table 4.1 agrees very well with the result shown in 4.3. It is clear that it is not always possible to predict the severe stress constraint. For this reason the design problem 4.1 was again solved by using the same initial design point but with all the stress constraints present.

ITERATION NO	$I_1 \times 10^4 mm^4$	$I_2 \times 10^4 mm^4$	y_1 (rad)	y_2 (mm)	$W \times 10^4\ mm^3$
INITIAL DESIGN POINT	2.25	3.00	0.004	-5.1	0.525
1	5.1766	5.946	0.0062	0.0	1.1123
2	7.0309	4.2484	0.0085	0.4	1.1279

TABLE 4.2

It can be seen that the final result obtained in Table 4.2 was different from that shown in Table 4.1. This is due to the fact that the presence of all the stress constraints generates more vertices in the feasible region which may increase the possibility of having a local optimum. It is known that, in this case, approximating programming tends to converge to one of these local optima.

In the case, where the areas of the columns and rafters are taken as design variables instead of the second moment of areas in the design of the pitched roof frame considered, the design problem 4.1 becomes:

$$\text{Min } W = A_1 + A_2$$

Subject to

$$2.6496A_1^2y_1+2.6496A_2^2y_1+2.293\times10^{-3}A_1^2y_2-4.59\times10^{-3}A_2^2y_2-0.0375A_1^2-4.464\times10^{-3}A_2^2 = 0$$

$$2.293\times10^{-3}A_1^2y_1-4.59\times10^{-3}A_2^2y_1+0.265\times10^{-5}A_1^2y_2+1.058\times10^{-5}A_2^2y_2-0.356\times10^{-3}A_1^2-0.049\times10^{-3}A_2^2 = -1.0$$

$$0.91239A_2^{\frac{1}{2}}y_1-3.1614\times10^{-3}A_2^{\frac{1}{2}}y_2+6.051\times10^{-3}A_2^{\frac{1}{2}} \leq 0.15$$

$$-0.91239A_2^{\frac{1}{2}}y_1+3.1614\times10^{-3}A_2^{\frac{1}{2}}y_2-6.051\times10^{-3}A_2^{\frac{1}{2}} \leq 0.15$$

$$1.8248A_2^{\frac{1}{2}}y_1-3.1614\times10^{-3}A_2^{\frac{1}{2}}y_2-3.073\times10^{-3}A_2^{\frac{1}{2}} \leq 0.15$$

$$-1.8248A_2^{\frac{1}{2}}y_1+3.1614\times10^{-3}A_2^{\frac{1}{2}}+3.073\times10^{3}A_2^{\frac{1}{2}} \leq 0.15$$

$$0.91239A_1^{\frac{1}{2}}y_1+1.5793\times10^{-3}A_1^{\frac{1}{2}}y_2-16.705\times10^{-3}A_2^{\frac{1}{2}} \leq 0.15$$

$$-0.91239A_1^{\frac{1}{2}}y_1-1.5793\times10^{-3}A_1^{\frac{1}{2}}y_2+16.705\times10^{-3}A_1^{\frac{1}{2}} \leq 0.15$$

$$y_1 \leq 0.02$$

$$\ldots\ 4.4 \qquad y_2 \leq 4.8$$

where A_1 and A_2 are the areas of the columns and rafters respectively. It can be seen by comparing the design problems 4.1 and 4.4 that, the use of areas as design variables increases the non-linearity of the stiffness constraints and reduces the non-linearity of the stress constraints. The initial design point was selected as:

$$\{A_1\ A_2\ y_1\ y_2\} = \{1.0\times10^2\ \ 1.0\times10^2\ \ 0.007\ \ 0.0\}$$

The result obtained in the first iteration was

$$\{A_1\ A_2\ y_1\ y_2\} = \{0.6422\times10^2\ \ 1.9922\times10^2\ \ 0.0\ \ 0.0\}$$

when this was used to generate the next iteration no feasible solution was obtained. This is due to the fact that the increase of the non-linearity of the stiffness constraints generates large linearisation errors and the displacement variables obtained by the simplex method violates the stiffness constraints excessively. Hence, it became necessary to adjust the displacement variables. This was carried out by solving the stiffness constraints with $A_1 = 0.6422\times10^2$ mm^2, $A_2 = 1.9922\times10^2$ mm^2 and the adjusted values of y's were found. The new design point became

$\{A_1\ A_2\ y_1\ \ y_2\} = \{\ 0.6422\text{x}10^2\ \ 1.9922\text{x}10^2\ \ \ 0.0016\ \ \ldots\ 0.9\}$

which was employed for the second iteration. The feasible solution was found to be

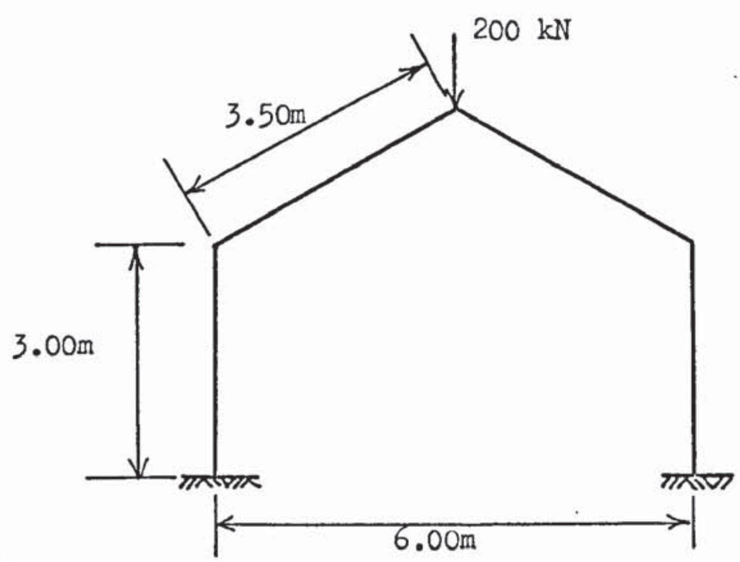
$\{A_1\ A_2\ y_1\ y_2\} = \{\ 1.2648 \times 10^2\ \ \ 0.0\ \ \ 0.0078\ \ \ 0.0\}$

Although a different initial design point was employed, convergence was not obtained. It was found that the application of move limits was necessary to obtain convergence. Later, this matter was investigated in more detail in an example in which the effect of axial force was not neglected. As a result it may be concluded that in design problems where the effects of axial forces can be ignored, the use of the second moment of areas as design variables reduces the non-linearity of the stiffness contraints and therefore yields better results. However, in general it is not possible to ignore the axial stiffness of the members. This is taken into account in the next example.

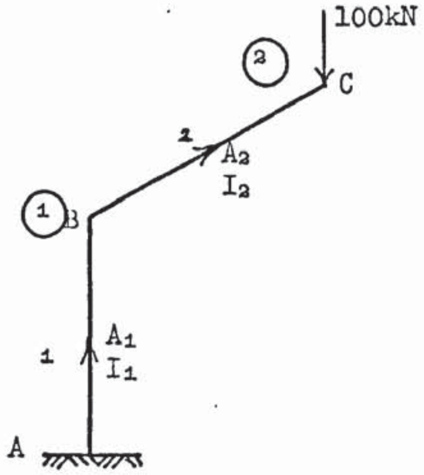
4.1.2) THE EFFECT OF THE INITIAL DESIGN POINT.

The design of the symmetrical pitched roof frame shown in Figure 4.2 was considered. The columns are of the same section with area A_1 and the rafters have area A_2. The modulus of elasticity is 207.0 kN/mm^2. Because the axial forces in the members are significant, it is required to limit the combined bending and direct compressive stresses to 0.10804 kN/mm^2 in member AB and to 0.2034 kN/mm^2 in member BC. The outward horizontal deflection at the eaves is to be limited to 4.225 mm and the downward deflection at C to 8.23 mm.

Again, due to symmetry, only half of the structure is considered. This is shown in Figure 4.2(b) where the joints and the members are numbered. Because the axial forces in the members are significant, the vertical deflexion y_1 at joint 1 cannot be suppressed. The joint deflexion vector is

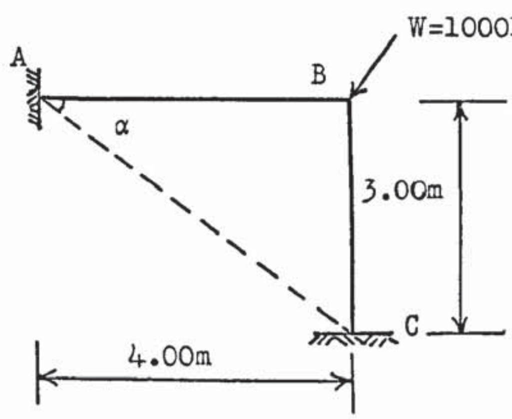


(a) DIMENSIONS AND LOADING OF FRAME

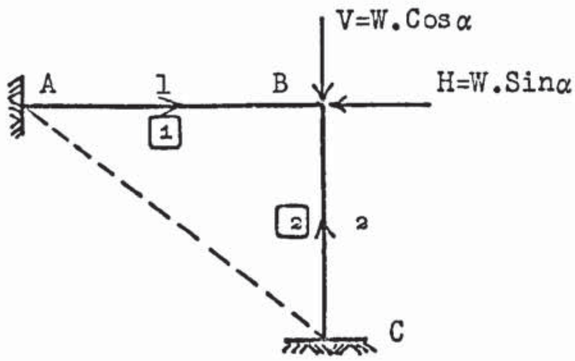


(b) NUMBERING OF MEMBERS AND JOINTS

FIGURE 4.2: SYMMETRICAL PITCHED ROOF FRAME



(a) FRAME AND LOADING



(b) APPLIED LOAD VECTOR AND NUMBERING OF MEMBERS

FIGURE 4.3: TRIANGULATED FRAME.

$$\underline{X} = \{x_1\ y_1\ \theta_1\ y_2\}$$

The corresponding load vector is

$$\underline{L} = \{0 \quad 0 \quad 0 \quad -100\}$$

It is known that x_1 and y_2 are always negative. Hence the deflexion constraints for these are

$$\left.\begin{aligned} x_1 &\leq 0 \\ y_2 &\leq 0 \end{aligned}\right\} \qquad 4.5$$

Substituting $x_1 = \gamma_1 - 4.225$ and $y_2 = \gamma_2 - 8.23$ to satisfy the non-negativity restrictions, the expressions 4.5 become

$$\gamma_1 \leq 4.225$$

$$\gamma_2 \leq 8.23$$

The vertical deflexion y_1 at joint 1 is also negative and a reasonable bound on this can be calculated by assigning the smallest universal section available to the column. This has an area of 3230 mm^2. The axial load in the column is 100 kN, thus from:

$$P = EAy_1/L$$

$$y_1 = \frac{100\times3000}{207\times3230}$$

$$y_1 = 0.448 \text{ mm} \simeq 0.5 \text{ mm}$$

The substitution $y_1 = \gamma_3 - 0.5$ gives the vertical deflexion constraint at joint 1 as

$$\gamma_3 \leq 0.5$$

Finally the rotation θ_1 at joint 1 may be positive or negative. Assuming that $-0.01 \leq \theta \leq 0.01$, the substitution $\theta_1 = \gamma_4 - 0.01$ gives the constraint for this joint rotation as

$$\gamma_4 \leq 0.02$$

The design problem is formulated as explained in Chapter 2 and the necessary substitutions for the displacements as shown above are carried out in the constraints. Due to the fact that the number of variables and constraints was too great to carry out the calculations by hand, a computer was used for solution of each linearised problem, ICL's[11] simplex routine was employed for this purpose. Firstly, an infeasible initial design point was used, the move limits were not utilized and the results obtained at each iteration are shown in Table 4.3.

ITERATION NO	$A_1 \times 10^4 mm^2$	$A_2 \times 10^4 mm^2$	y_1(mm)	y_2(mm)	y_3(rad)	y_4(mm)	$W \times 10^8 mm^3$
INITIAL DESIGN POINT	0.50	0.25	-12.778	0.2102	0.01433	-23.069	0.23750
1	0.8673	0.3500	4.225	0.423	0.0122	7.119	0.38271
2	1.1721	1.3559	3.192	0.360	0.0071	8.23	0.82622
3	1.8334	2.0425	4.225	0.455	0.013	6.955	1.26495
4	2.2665	3.1596	4.225	0.275	0.0074	8.189	1.78583

TABLE 4.3

It was noticed that the value of the objective function increased continuously and convergence was not obtained. The adjustment of displacement variables was carried out at each iteration to reduce the linearisation errors. The results are shown in Table 4.4.

ITERATION NO	$A_1 \times 10^4 mm^2$	$A_2 \times 10^4 mm^2$	y_1(mm)	y_2(mm)	y_3(rad)	y_4(mm)	$W \times 10^8 mm^3$
INITIAL DESIGN POINT	0.50	0.25	-12.778	0.2102	0.0143	-23.069	0.23750
1	0.8673	0.35	- 3.288	0.333	0.01245	- 6.122	0.38271
2	1.7356	0.4427	0.945	0.417	0.0114	1.41	0.67564
3	3.486	0.4978	4.225	0.50	0.01036	7.03	1.22005

TABLE 4.4

It can be seen that the results are improved but convergence is still not obtained. It is obvious that the adjustment of displacement variables is not enough to control the linearisation errors. It is, therefore, necessary to apply some bounds on the design variables so that their change during the iterations can be restricted. Firstly, the move limits are chosen as an upper bounds. That is:

$$x_j^{(1)} \leqslant (1.0 \mp m)\, x_j^{(0)} \qquad j = 1,\ldots n \qquad 4.6$$

where n is the number of design variables, $x_j^{(0)}$ is the current value of the design variable and $x_j^{(1)}$ is the value of the design variable to be obtained in the next iteration. m is the selected percentage known as a move limit. In the expression 4.6 the minus sign is used for feasible design points and the plus sign is used for infeasible design points. There are two methods to apply move limits. The first is to put these upper bounds on both area and displacement variables. This method was not found successful, because in the design problem the displacement variables were already bounded enough by the substitution carried out to satisfy the non-negativity restriction. Further restriction prevented obtaining a feasible solution. The second method is to put these upper bounds on the area variables only. This was applied to the example considered where the adjustment of displacement variables was also carried out at each iteration and results are shown in Table 4.5.

ITERATION NO	$A_1 \times 10^4 mm^2$	$A_2 \times 10^4 mm^2$	y_1(mm)	y_2(mm)	y_3 (rad)	y_4(mm)	$W \times 10^8 mm^3$
INITIAL DESIGN POINT	0.50	0.2500	-12.778	0.2102	0.01433	-23.063	0.23750
1	0.8423	0.3500	- 3.288	0.333	0.01245	- 6.122	0.38271
2	1.0108	0.4900	1.78	0.366	0.01033	3.230	0.48375

TABLE 4.5

Move limits of 100% were employed in the first iteration, different move limits of 20% and 40% were applied to areas A_1 and A_2 respectively in the second iteration. The pitched roof frame considered was analysed using the areas obtained at the second iteration and it was found that the deflexions of the joints and stresses in the members were at their allowable bounds.

A feasible initial design point was also employed and it was found that the adjustment of displacement variables has to be carried out when it is necessary. This is shown in Table 4.6.

ITERATION NO	$A_1 \times 10^4 mm^2$	$A_2 \times 10^4 mm^2$	y_1(mm)	y_2(mm)	y_3(rad)	y_4(mm)	$W \times 10^8 mm^3$
INITIAL DESIGN POINT	3.00	2.00	4.019	0.452	0.01	7.615	1.6500
1	1.50	0.75	3.663	0.428	0.01006	6.872	0.71250
2	1.05	0.525	2.038	0.382	0.01049	3.719	0.49875
3	0.9975	0.4987	0.215	0.356	0.0109	0.375	0.47381

TABLE 4.6

In the first iteration different move limits of 50% and 62.5% were used on A_1 and A_2, then equal move limits employed in the next two iterations which were 30% and 5% respectively. In this example, the optimum result was found without carrying out the adjustment of displacement variables at each iteration. It was noticed that this was not always the case. When the feasible point

$$\{A_1\ A_2\ y_1\ y_2\ y_3\ y_4\ y_5\} = \{1.50\times10^4\ \ 0.75\times10^4\ \ 2.419\ \ 0.402\ \ 0.01044\ \ 4.488\}$$

was taken as an initial design point and 30% move limits were employed on areas, in the first iteration the new design point was found to be

$$\{A_1\ A_2\ y_1\ y_2\ y_3\ y_4\ y_5\} = \{1.05\times10^4\ \ 0.525\times10^4\ \ 1.282\ \ 0.374\ \ 0.01074\ \ 2.328\}$$

Although a number of different move limits were employed and the above design point was used in the next iteration a feasible solution has not been found. Then the adjustment of displacement variables were applied and 4% move limits were utilized on both area variables, the result was found to be

$$\{A_1\ A_2\ y_1\ y_2\ y_3\ y_4\} = \{1.008\times10^4\ \ 0.504\times10^4\ 0.678\ \ 0.356\ \ 0.0109\ \ 1,095\}$$

which is an optimum. The results obtained from the application of a number of different feasible initial design points indicates that the solution of stiffness equalities to adjust the displacement variables is carried out only when it becomes necessary. As a result it may be concluded that feasible and infeasible initial design points may be employed, but the use of the latter may require a greater number of adjustments of displacement variables which is carried out during the iterations to obtain the optimum answer.

4.2) TRIANGULAR FRAME.

The computer program described in the previous chapter was used to design the simple triangular frame of Figure 4.3(a). As explained in that chapter, for the design of this and the next example the first simplex routine was employed in the design procedure which utilises the Charnes M method. The frame is fixed at A and C and rigidly jointed at B, and the dimensions are shown in the figure. A force of 1000 kN acts at right angle to the line joining A and C which is resolved into its components as shown in Figure 4.3(b). The vertical deflexion of point B was limited to 1.436 mm and the horizontal deflexion was restricted to 2.793 mm. The combined stresses in members AB and BC were limited to 0.176 kN/mm² and 0.116 kN/mm² respectively. The elasticity modulus was 207 kN/mm².

There are a total of 24 constraints and 5 design variables in the design problem. Since it is only necessary to apply move limits on areas, there are consequently 2 constraints. The rest consist of 3 stiffness, 16 stress and 3 deflexion constraints.

As seen from the examples solved by hand, the move limits can be chosen quite arbitrarily. In previous research work, a number of different ways were employed. For example, fixed value move limits were found satisfactory by Johnson and Brotton[57] to obtain convergence in the examples which they considered. These were arranged by selecting 50% for the first 3 cycles, 25% for the next cycles and 10% for the final cycles. On the other hand Reinschmidt and others[51] have found adaptive move limits practical to obtain convergence. It still remains impossible to give a general rule for their application. This is due to the fact that the selection of move limits depends on the behaviour of the programming problem. If a problem behaves poorly, then tighter move limits are necessary, while if convergence is direct and orderly, the move limits may decelerate convergence. Hence adoptive move limits may give better convergence. However, both methods of move limit application were used in the design examples considered. In the present example, an initial value was selected for the move limits and this was changed at each iteration. In the following example fixed value move limits were employed. In these two examples, the adjustment of displacement variables was avoided, instead different move limits were tried, to overcome obtainining non-feasible solution. The examples solved by hand have shown that, in the case where a feasible design point is used to start the iteration, tighter move limits may prohibit the linear program from finding a feasible solution. In such cases these limits are relaxed. Hence the program was written in such a way that when no feasible solution was obtained in the use of a feasible initial design point, the simplex routine was repeated a number of times by increasing the value of the move limit each time by 10%. If a feasible solution was unobtainable, then the adjustment of displacement variables was carried out and the new design point was used to continue the iteration with the adjusted values of the move limits.

ITERATION NO	$A_1 \times 10^4 mm^2$	$A_2 \times 10^4 mm^2$	$y_1(mm)$	$y_2(mm)$	$y_3(rad)$.	$W \times 10^8 mm^3$
INITIAL DESIGN POINT	1.50	2.5	2.1	0.987	0.0102	1.3500
1	0.75	1.25	1.717	0.756	0.0104	0.67500
2	0.45	0.75	0.877	0.248	0.0107	0.40500
3	0.3887	0.8105	0.047	0.0	0.0111	0.39862
4	0.4015	0.7980	0.009	0.0	0.0111	0.40001

TABLE 4.7

Table 4.7 shows the results obtained during the iterations Move limits of 50% were used to start with,which was then increased to 60% at the second iteration. It was then found that a feasible solution was not available with the move limits having a value of 70% and so the simplex method was applied 4 times, each one having increased move limit constraints. In the fourth application, with the move limits at 100%, a feasible solution was obtained as shown in the third iteration. The minimum weight of the structure was found to be $W = 0.400012 \times 10^8$ mm^3 with areas $A_1 = 0.4015 \times 10^4$ mm^2 and $A_2 = 0.798 \times 10^4$ mm^2. The structure was analysed with these areas and the deflexion and stresses were found to be at their allowable bounds. This verified the optimality of the result.

4.3) THE PORTAL FRAME.

The design of the fixed base rectangular portal frame shown in Figure 4.4 which is subject to a horizontal load 1 kN acting at B is now considered. The modulus of elasticity is 207 kN/mm^2 and it is required to make the columns out of the same section while the beam may have a different section. The maximum horizontal sway at B is restricted to 4 mm and the bending stress in the members should not exceed 0.15 kN/mm^2.

In the example considered a feasible initial design point and fixed value move limits were employed. For the first three iterations 75% move limits were used. It was then found that a feasible solution cannot be obtained with this particular value of move limits. They have to be relaxed. The simplex method was repeated twice by increasing the value of move limits to 85% at each iteration and the feasible solution was obtained with the move limits having values of 95%. The results are shown in Tablo 4.8.

ITERATION NO	$A_1 \times 10^2$ mm^2	$A_2 \times 10^2$ mm^2	y_1 (mm)	y_2 (mm)	y_3 (rad)	y_4 (mm)	y_5 (mm)	y_6 (rad)	$W \times 10^5$ mm^3
INITIAL DESIGN POINT	5.00	3.00	4.497	0.503	0.0095	4.489	0.497	0.0095	13.000
1	3.75	2.10	4.76	0.504	0.0092	4.750	0.496	0.0093	9.600
2	2.812	1.47	5.324	0.505	0.0086	5.309	0.495	0.0086	7.090
3	2.109	1.029	6.412	0.506	0.0074	6.392	0.494	0.0074	5.249
4	2.0039	0.9261	7.493	0.507	0.0061	7.467	0.493	0.0061	4.930
5	2.0041	0.9261	7.404	0.506	0.0060	7.378	0.494	0.0061	4.930

TABLE 4.8

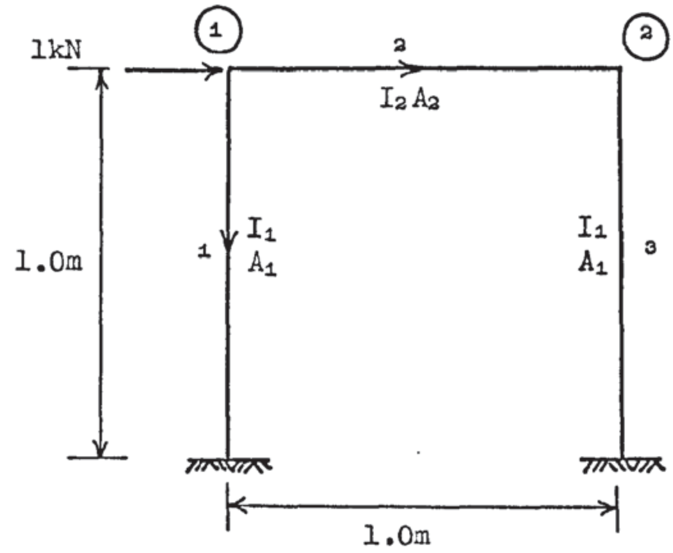
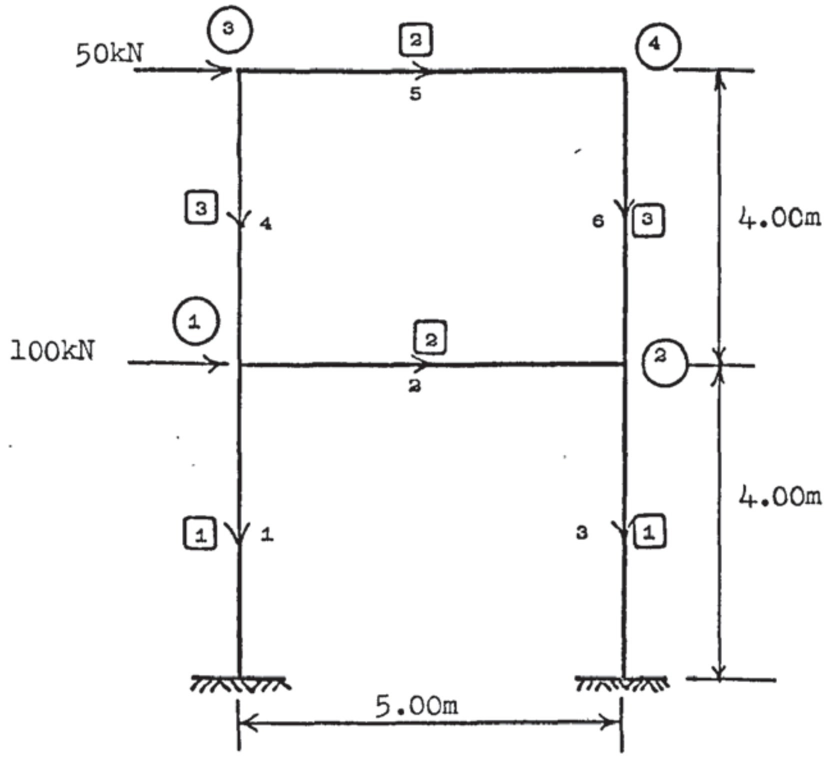


FIGURE 4.4 A PORTAL FRAME



Numbers inside ooxes indicate area groupings
the remainder are member numbers.

FIGURE 4.5 TWO STOREY FRAME I.

As seen from Table 4.8 convergence was obtained in the fifth iteration and the minimum weight of the frame was found to be 4.93×10^5 mm^3. The columns have the optimum area of 2.0×10^2mm^2 and the beam has the optimum area of 0.926×10^2 mm^2. The same frame was designed by K.I.Majid(1) where the axial stiffness of the members were neglected and the minimum weight of the frame was obtained as 4.463×10^5 mm^3. In that case, the optimum areas for columns and beam were found to be $A_1 = 1.85 \times 10^2$ mm^2, $A_2 = 0.94 \times 10^2$ mm^{2}respectively. It can be seen that when the axial stiffness of the members are not neglected the member forces in the columns will increase while the axial force in the beam will remain approximately the same. Hence, the area required for the columns will increase and the area required for the beam will remain very similar. The results shown in Table 4.8 verify this.

At this point it is worth while saying that for relatively small frames in the case where the initial design point is chosen as a feasible point, then it may be possible to obtain the optimum solution without carrying out the adjustment of the displacement variables, but by just repeating the simplex tables with different values of move limits. These move limits are only applied as upper bound constraints. Later, when relatively large frames were designed, this way of move limit arrangement was found impractical. Due to the reasons explained in Chapter 3, it became necessary to employ the simplex routine written by ICL and repetition of the simplex routine with different move limits was found to be time consuming especially for large design problems which require more than 100 simplex iterations to obtain the optimum solution. It was also found that when the linear programming problem has no feasible solution due to tight move limits, it may be possible to obtain a feasible solution by adjusting the displacement variables and carrying out the linearisation using this new design point and repeating the simplex routine with tight move limits. It is obvious that the solution of stiffness equations will take less computing time than carrying out the simplex routine. Hence in the following relatively large examples, analysis routine was employed when adjustment of displacements became necessary. Firstly move limits were applied as upper bound constraints. It was found that convergence was not obtained and fluctuation occurred. As a result it became necessary to employ lower bounds to prevent the fluctuation and move limits were arranged in the manner described in Chapter 2.

4.4) TWO STOREY FRAME I.

The two storey frame shown in Figure 4.5 has a total of 4 joints and 6 members. The members of the structure are grouped into 3 area groups. The first storey columns belong to group 1, second storey columns belong to group 3 and the beams belong to group 2. This frame was designed to satisfy the deflexion and stress limitations under the horizontal loading shown in Figure 4.5. The horizontal sway of the frame was limited to 5 mm, the stresses in members 1 and 3 were limited to 0.0227 kN/mm^2, the stresses in members 2 and 5 were limited to 0.0267 kN/mm^2 and the stresses in members 4 and 6 were limited to 0.01 kN/mm^2. The bounds on the vertical deflexion of joints were taken as 0.5 mm and those for the rotations were taken as 0.01 radians.

The design problem of the frame considered consists of 15 variables and 75 constraints 12 of which are stiffness constraints, 48 of which are stress constraints, 12 of which are deflexion constraints and the last 3 are due to move limits which are only upper bounds on area variables. Convergence rate was taken as 0.2% and the results for the areas obtained at each iteration are shown in Table 4.9.

ITERATION NO	$A_1 \times 10^4$ mm^2	$A_2 \times 10^4$ mm^2	$A_3 \times 10^4$ mm^2	$W \times 10^8$ mm^3
INITIAL DESIGN POINT	5.000	2.5000	3.7500	9.50000
1	3.7161	1.8374	2.7647	7.02206
2	3.9917	1.9796	2.9829	7.55936
3	3.9962	1.9992	2.9895	7.58784
4	4.0056	2.0000	2.9892	7.60397

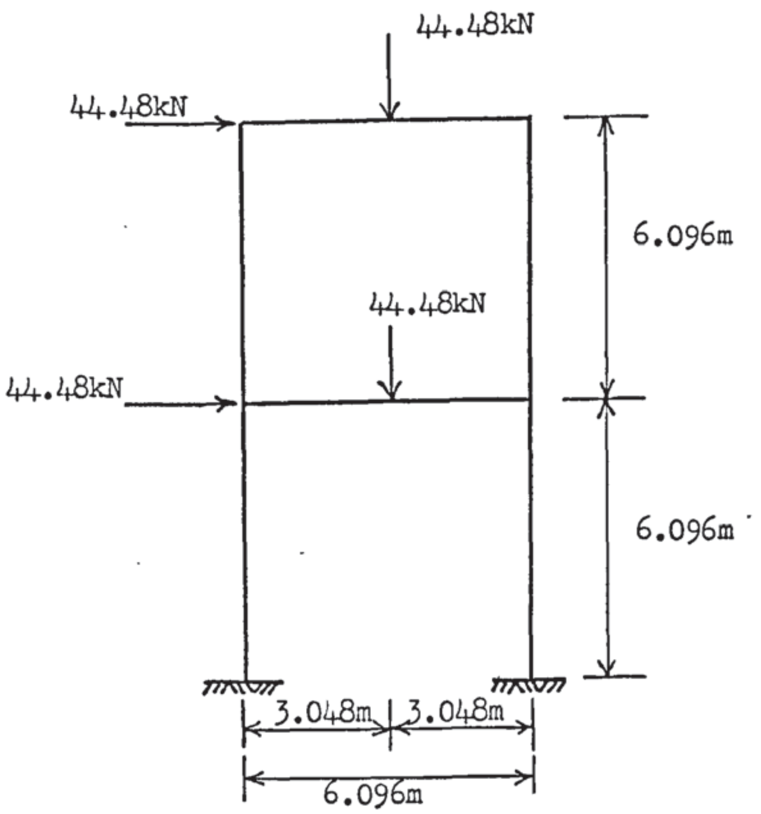
TABLE 4.9

The value of the weight function corresponding to this design was found to be 7.60397×10^8 mm^3 and was achieved after 4 iterations of the optimisation procedure, where the ICL's simplex routine was used to solve each linearised problem.

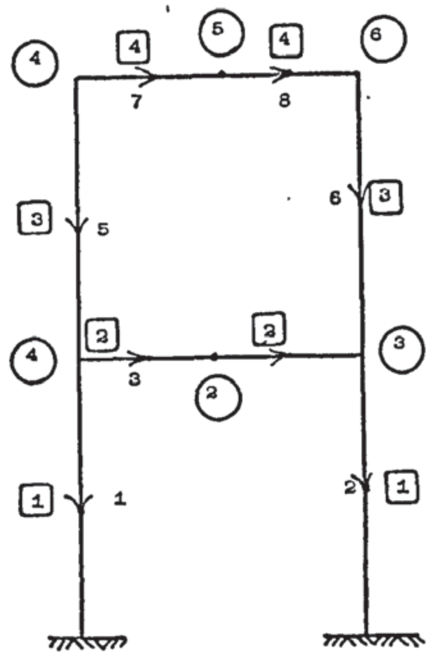
4.5) TWO STOREY FRAME II.

The frame of Figure 4.6 was originally designed for minimum weight by Toakley[45] using the rigid-plastic theory where a discrete set of sections was considered to be available. The frame has 8 members grouped together, with some members made out of the same section, so that there are 4 groups. The member, and group numbering is indicated in the figure. The frame is subject to equal loads of 44.48 kN acting at joints 1,2,4 and 5 shown in the figure. The limitations for the deflexion of joints were taken from B.S.449 which are that the midspan deflexion of a beam should not exceed the value of L/360 where L is the length of the beam and horizontal deflexion of a column should not be more than h/325 where h is the height of the column. Hence, the sway of the frame at joints 1,2,3 was limited to 18.757 mm, at joints 4,5,6 limited to 37.5138 mm and the vertical deflexion of joints were limited to 16.93 mm. The combined stresses in the members were limited to 0.15 kN/mm^2. Elasticity modulus of the material was 207 kN/mm^2.

Altogether, there are 18 stiffness constraints, 64



a) DIMENSIONS AND LOADING OF THE FRAME



b) NUMBERING OF MEMBERS AND GROUPS

FIGURE 4.6: TWO STOREY FRAME II

stress constraints and 18 deflexion constraints in the design problem. In addition to these, firstly upper bound move limits were imposed on the area variables.

The initial design point was chosen to be an infeasible point and equal values were taken for the group areas. As shown in Figure 4.7, the value of the minimum weight function fluctuated and convergence was not obtained after 22 iterations. Different initial design points were utilized and adoptive move limits were employed, but the convergence difficulty was not overcome. Comparing the sizes of the previous problem and this one, it can be seen that the problem considered here has more constraints and more displacement variables. When the size of the design problem increases, the errors involved in linearisation become large and the upper bound move limits are not enough to control the changes on the design variables during the optimisation process. Hence it becomes necessary to impose lower bounds on the area variables. This was done by adding 4 more lower bound constraints to the design problem which were of type II which require artificial variables. As shown in Figure 4.8, by means of lower and upper bound move limits convergence was obtained after 11 iterations. Two more iterations were carried out and no change was seen in the value of the minimum weight function which verifies the convergence. The convergence rate was taken as 0.1% and the values of areas obtained at each iteration are shown in Table 4.10. The final volume was obtained as 3.49634×10^8 mm^3. It is obvious that this result will be different to the one obtained by Toakley, due to the fact that a continuous set of sections was assumed available for selection in the design procedure used in this thesis, whereas a discrete set of sections was assumed to be available in the algorithm employed by Toakley. Hence, if a discrete set of sections are to be used, then it becomes necessary to round-off the derived

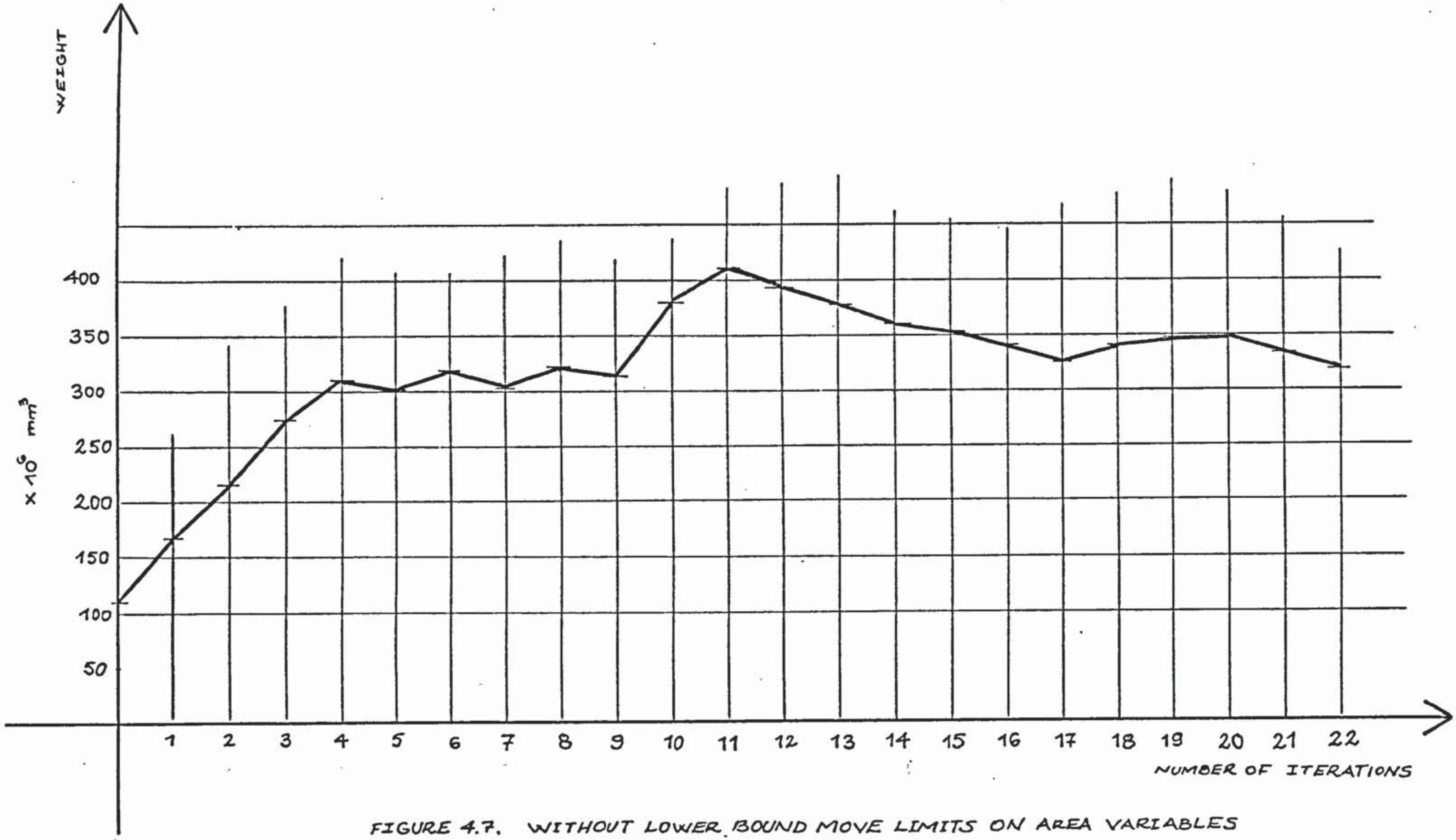


FIGURE 4.7. WITHOUT LOWER BOUND MOVE LIMITS ON AREA VARIABLES

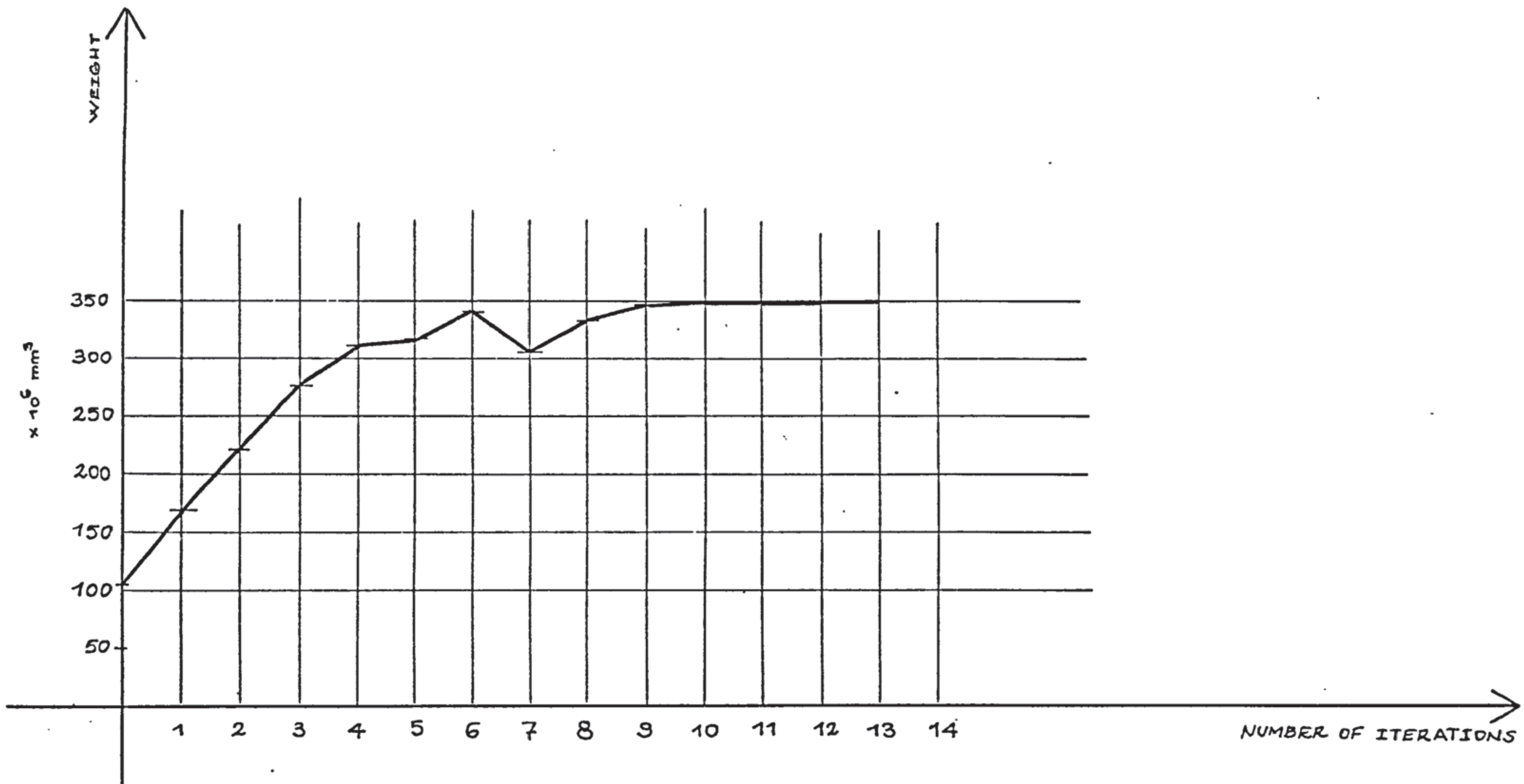


FIGURE 4.8. WITH LOWER BOUND MOVE LIMITS ON AREA VARIABLES

ITERATION NO	AREAS × 10^3 mm^2				WEIGHT × 10^8 mm^3
	A_1	A_2	A_3	A_4	
INITIAL DESIGN	3.000	3.000	3.000	3.000	1.09728
1	4.621	4.650	4.752	4.354	1.69170
2	6.483	7.341	5.606	5.148	2.23516
3	7.806	12.687	6.048	5.194	2.77916
4	9.871	8.387	7.185	8.829	3.12904
5	9.646	13.419	7.594	3.532	3.13524
6	10.914	8.619	9.937	5.298	3.39047
7	9.363	12.067	5.962	7.417	3.05621
8	9.196	15.687	7.751	5.192	3.33894
9	11.035	12.549	7.792	6.230	3.44023
10	9.932	13.804	8.394	6.853	3.49360
11	10.925	12.857	7.555	7.538	3.49634

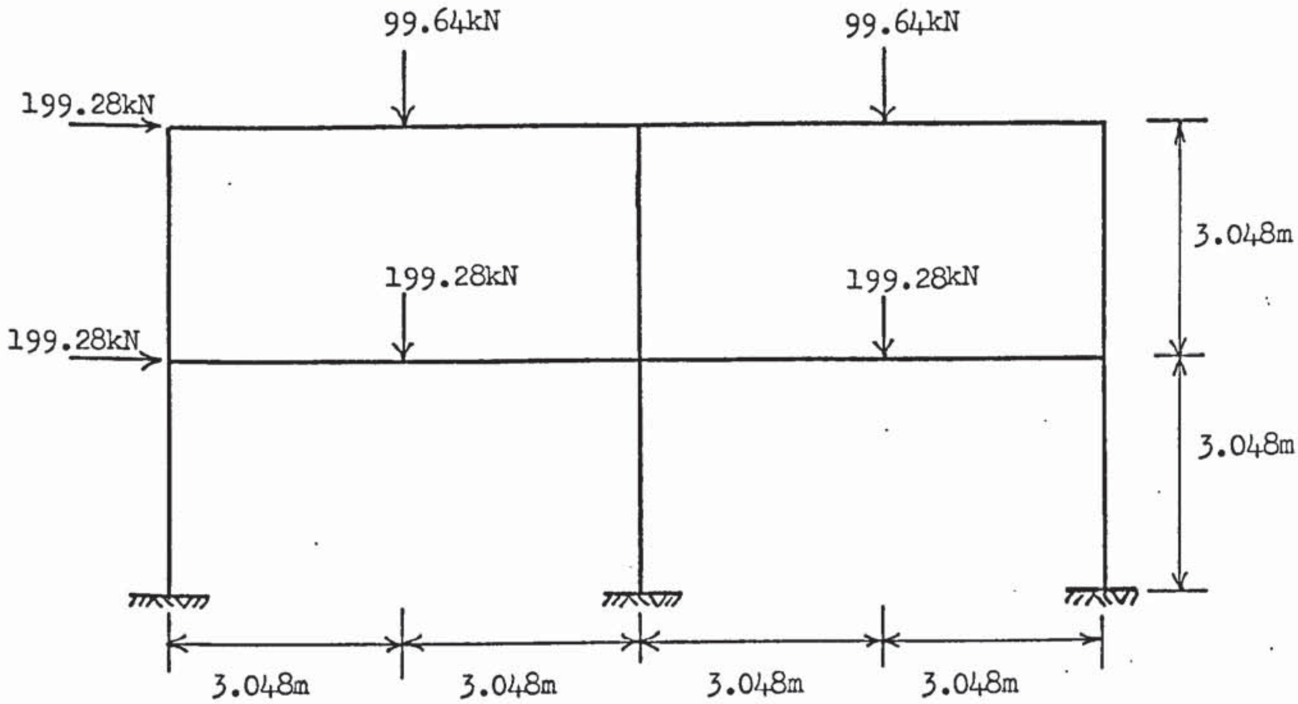
TABLE 4.10 TWO STOREY FRAME II DESIGN.

sections shown in Table 4.9 to available sections by using the universal beam and column tables given in B.S.4:1972. This was carried out and the final volume was obtained to be 2701.8 kg as compared to 1678.3 kg obtained by Toakley. It is known that to round-off the derived sections to available ones does not necessarily yield an optimum structure. On the other hand, the design by rigid-plastic theory does not impose any limitation on the deflexions due to the fact that it assumes that the structure does not deflect until collapse. This is one of the weaknesses of the rigid plastic-theory. When the frame considered was analysed with the final are as shown in Table 4.9, it was found that the deflexions were at their limits while the members were not fully-stressed. This showed the deflexion constraints are dominant in the design problem and it is apparent that the optimum result obtained without considering them will yield an unsafe solution.

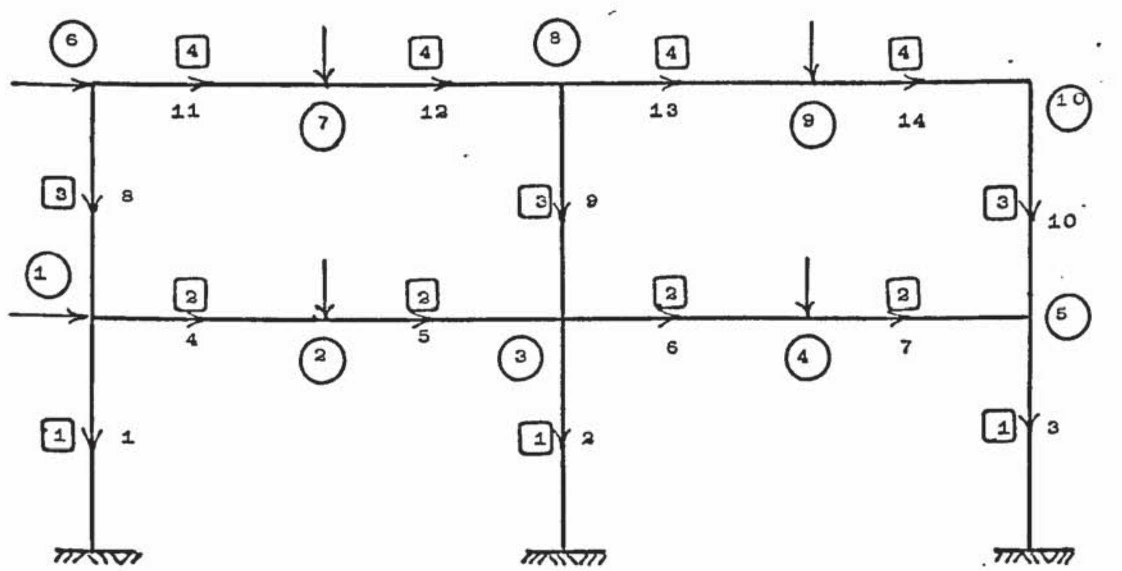
4.6) TWO STOREY, TWO BAY FRAME.

The structure shown in Figure 4.9 is a practical frame commonly used in structural engineering. This structure was also originally designed by Toakley[45] using the rigid-plastic theory where a discrete set of sections was considered to be available. The dimensions and the loading of the frame are shown in the figure. The horizontal deflexions of the joints at the first storey were not to exceed 9.378 mm, and the horizontal deflexions of the joints at the second storey were not to exceed 18.756 mm. Vertical deflexions of the joints were limited to 16.93 mm. These limitations were imposed as previously with reference to B.S.449.

Young's modulus for the material was 207 kN/mm^2 and the combined axial and bending stresses were limited to 0.15 kN/mm^2. The frame has 10 joints and 14 members which were collected together



a) DIMENSIONS AND LOADING OF FRAME



b) NUMBERING OF MEMBERS AND GROUPS

FIGURE 4.9. TWO STOREY,TWO BAY FRAME.

ITERATION NO	AREAS $\times 10^3$ mm^2				WEIGHT $\times 10^8 mm^3$
	A_1	A_2	A_3	A_4	
INITIAL DESIGN	6.000	5.000	5.000	5.000	2.07264
1	8.926	7.688	6.694	6.890	3.20576
2	11.831	10.832	7.694	8.262	4.11318
3	13.428	13.215	7.850	8.725	4.62081
4	13.698	13.978	7.833	8.775	4.74284
5	13.717	14.032	7.833	8.776	4.75134
FINAL DESIGN	13.717	14.033	7.833	8.775	4.75138

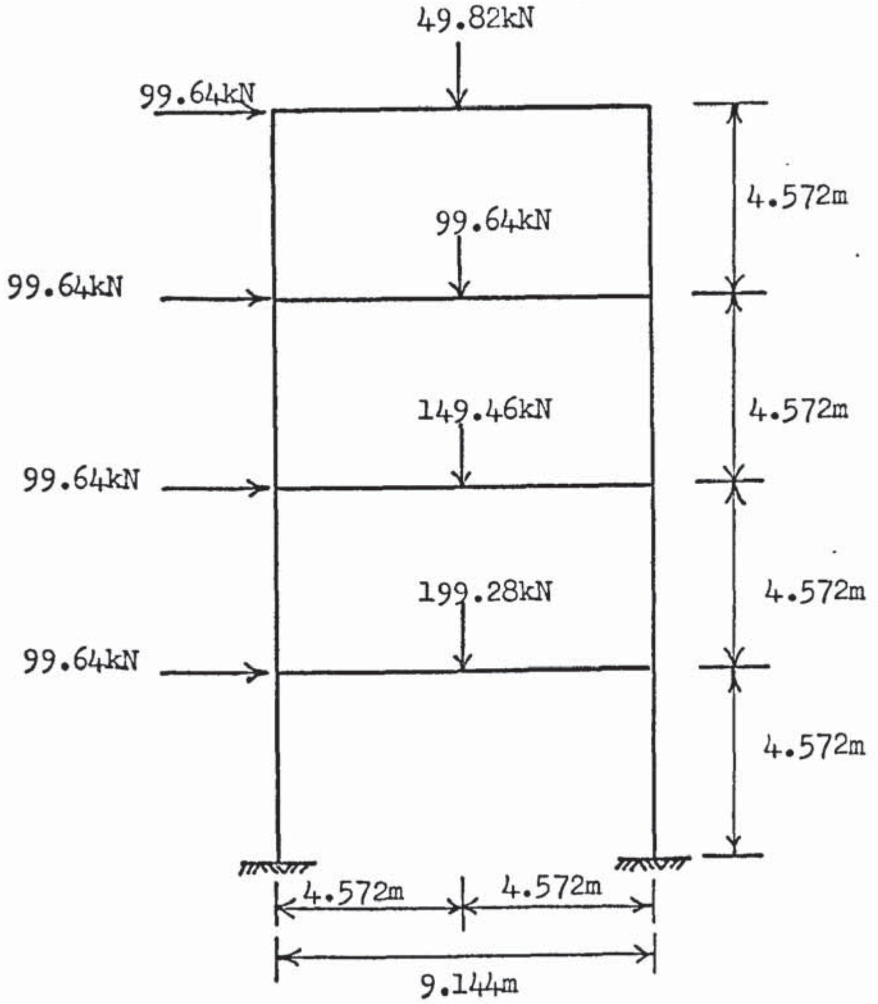
TABLE 4.11 TWO STOREY TWO BAY FRAME DESIGN.

into 4 area groups as shown in the figure. The design problem was therefore one of 34 variables 4 of which were areas, the rest were displacement variables and there were altogether 176 constraints. It was found that to obtain the optimum solution of each linearised problem an average of 140 simplex iterations were required and convergence was obtained after 6 iterations of the optimisation procedure. The areas for the groups obtained during the iterations are tabulated in Table 4.11. The minimum value of the weight function corresponding to this design was found to be 4.75138×10^8 mm^3. The analysis of the frame at the optimum point has shown that the stress constraints dominate the design problem, and that the combined stresses in members 2,5,9 and 12 were at their limits while the deflexions of the joints had values less than the bounds imposed.

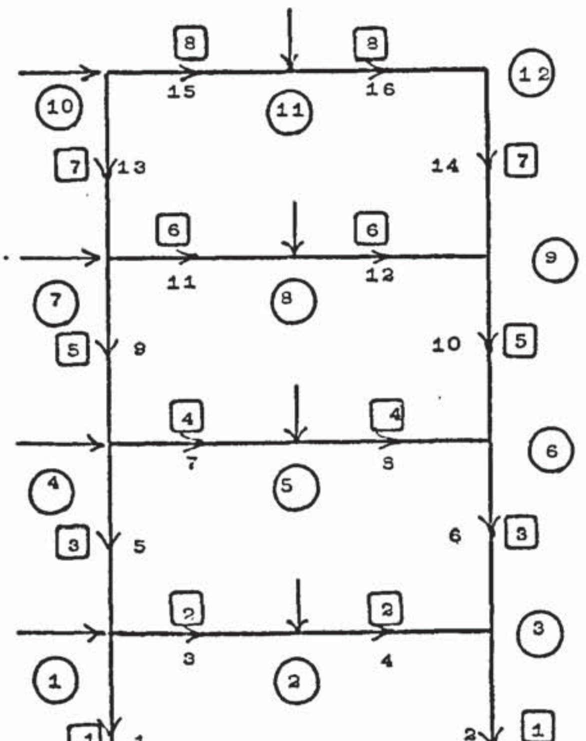
4.7) FOUR STOREY FRAME.

The final example was the design of the frame shown in Figure 4.10 which was also originally considered by Toakley[45]. The structure has 12 joints and 16 members grouped together, with some members made out of the same section, so that there are 8 groups. The member, and group numbering is indicated in the figure. Dimensions of the frame and the loads acting at the joints are also shown in the figure. The limitations on deflexions and stresses were taken from B.S.449. Hence, the horizontal deflexions of joints 1,2,3 should not exceed 14.068 mm, the horizontal deflexions of joints 4,5,6 should not be more than 28.135 mm, the horizontal deflexions of joints 7,8,9 should be less than 42.203 mm and the horizontal deflexions of joints 10,11,12 were restricted to 56.27 mm. The vertical deflexions of the joints were limited to 25.4 mm. The combined axial and bending stresses were restricted to 0.165 kN/mm^2. Elastic modulus of the material was taken as 207 kN/mm^2.

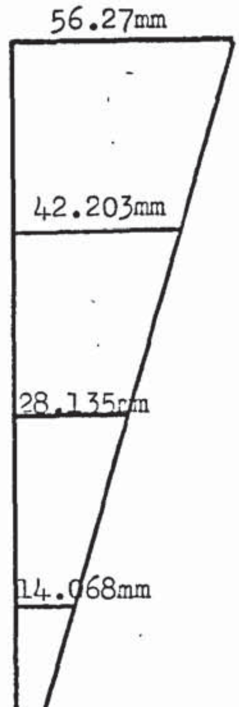
Altogether there were 44 variables (8 of which were



a) DIMENSIONS AND LOADING OF FRAME



b) NUMBERING OF MEMBERS AND GROUPS



c) BOUNDS ON HORIZONTAL DEFLEXIONS OF STOREYS

FIGURE 4.10; FOUR STOREY FRAME.

ITERATION NO	AREAS × 10^3 mm^2								WEIGHT × 10^8 mm^3
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	
INITIAL DESIGN	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	7.3152
1	14.965	14.620	13.511	15.068	12.085	13.706	7.624	6.677	8.9847
2	19.363	20.880	15.776	20.952	11.892	17.873	7.6275	8.867	11.2684
3	22.347	23.414	16.033	24.035	17.221	15.0733	12.478	7.816	12.6572
4	23.468	21.646	16.921	24.500	11.133	22.669	4.991	11.295	12.4985
5	20.137	32.469	17.939	18.933	16.789	11.423	7.487	5.647	11.9628
6	22.720	19.482	17.999	25.937	10.159	15.992	10.036	7.906	11.8902
7	24.266	25.326	17.196	18.684	13.207	20.789	9.996	10.273	12.7780
8	23.497	21.559	18.027	22.421	15.849	23.902	11.395	8.222	13.3023
9	24.535	23.715	16.224	24.663	14.264	21.512	13.194	9.045	13.4558
10	23.472	21.629	17.847	24.153	15.690	22.675	11.875	9.949	13.4685
11	22.835	23.793	16.484	26.568	14.121	20.408	11.928	10.944	13.4493
FINAL DESIGN	23.460	21.675	18.132	23.912	15.534	22.448	10.736	11.219	13.4523

TABLE 4.12: FOUR STOREY FRAME DESIGN.

area groups, the rest were displacement variables) and 216 constraints which required a core of 60K in the ICL 1905E computer. It was found that 12 iterations were required to reach the optimum solution. In the solution of each linarised problem, an average of 160 simplex iterations were needed which required considerable computer time. An initial design point was chosen to be infeasible. In the last three design examples the initial design point was selected in such a way that areas had values equal to each other. This introduced no difficulty in obtaining convergence which was found to be practical and provided flexibility to the design procedure. The frame was analysed with the optimum areas and it was found that the horizontal deflections of joints at the fourth storey were at their limits and members 2 and 4 were fully-stressed.

Using a tolerance on the objective function of 0.02% the final areas obtained are tabulated in Table 4.12. The final volume obtained was 13.45232×10^8 mm^3.

4.8) CONCLUSIONS.

The procedure for optimum elastic design of structures described in Chapter 2 was applied to the design or rigid frames where the criterion defining the optimum design was minimum weight and the results obtained are shown in this chapter. The application of the method is general. It approaches the design problem in a fundamental manner. It does not have the nature of iterative analysis. However, due to the linearisation error involved in the approximating programming, analysis is carried out when and if it becomes necessary. The examples solved have shown that the number of iterations required to obtain the optimum design is relatively small.

Although, in this thesis, rigidly jointed plane structures were considered, the method may be extended to deal with rigidly jointed space frames. Since the displacements of joints were taken as design variables, it may be possible to use the finite element method to formulate the design problem, which will eventually make it possible to deal with the optimum design of more complex structures.

The investigation has shown that when the design problem is formulated by the matrix displacement method in the proposed way, both the stiffness and stress constraints satisfy the properties of convex functions. Furthermore, because both the objective function and deflexion constraints are always linear, they are also convex. It is known that, in minimisation problems, if the objective function and the set of constraints are convex, then a local optima is also global[2]. Hence, it can be concluded that the solutions of the non-linear design problems posed in this chapter are global. This can also be verified by commencing with the solution of a particular problem from several widely different initial design points. However, it is obvious that this requires a considerable amount of computer time.

C H A P T E R 5

STRUCTURAL VARIATION FOR RIGIDLY JOINTED STRUCTURES

5.1) INTRODUCTION.

There are a number of cases where it becomes necessary to know the manner in which the member forces and the joint deflections throughout a structure change when one or more of its members are varied or removed. This becomes particularly important in the design of structures with variable topology. Here "topology" is defined as the number and the distribution of joints and the manner in which they are linked together. It has been shown in previous research works(93,99,102) that economy in the material cost of the structure can be achieved by altering its topology. In this way, it may be possible to select that topology which makes the material cost minimal, while both stress and deflection requirements are satisfied.

It is apparent that when the topology of a structure changes, it becomes necessary to analyse each of these structures by using either the matrix force or the matrix displacement method. When there are several changes to be made to a structure, these methods involve the repeated construction and solution of a large number of simultaneous equations. It is clear that this is rather cumbersome and time consuming. These repeated analyses can be avoided by deriving an explicit relationship which can be utilised for the above purposes. K.I.Majid and D.Elliot(101) have established what are known as the theorems of structural variations, which make it possible to predict the behaviour of one structure from that of another.

The work of Maxwell, Mohr and Muller-Breslau clearly indicates that there is a heirarchy of structures in which the analysis of complex structures can be obtained from the analysis of more elementary structures. The reverse process which gives rise to the theorems of structural variations, predicts the exact behaviour of elementary structures from more general structures. Once the analysis of complex structures has been carried out, the proposed theorems can be applied to analyse a multitude of derived structures, without further application of either of the basic methods of analysis. Further, these theorems make it possible to predict the behaviour of a structure from the results of another which has the same shape but with different material and cross-sectional properties. Before entering into the details of these theorems, it is convenient to state their abilities.

5.2) THE THEOREMS OF STRUCTURAL VARIATIONS.

i) The first theorem predicts the forces throughout a resulting new structure when the cross-sectional properties of one or more members of the parent structure are varied independently or totally removed.

ii) The second theorem predicts the deflexions throughout a resulting new structure when the cross-sectional properties of one or more members of the parent structure are varied independently or totally removed.

iii) The third theorem predicts the forces and deflexions throughout a structure when the cross-sectional properties of all the members are varied proportionally.

5.2.1) ASSUMPTIONS.

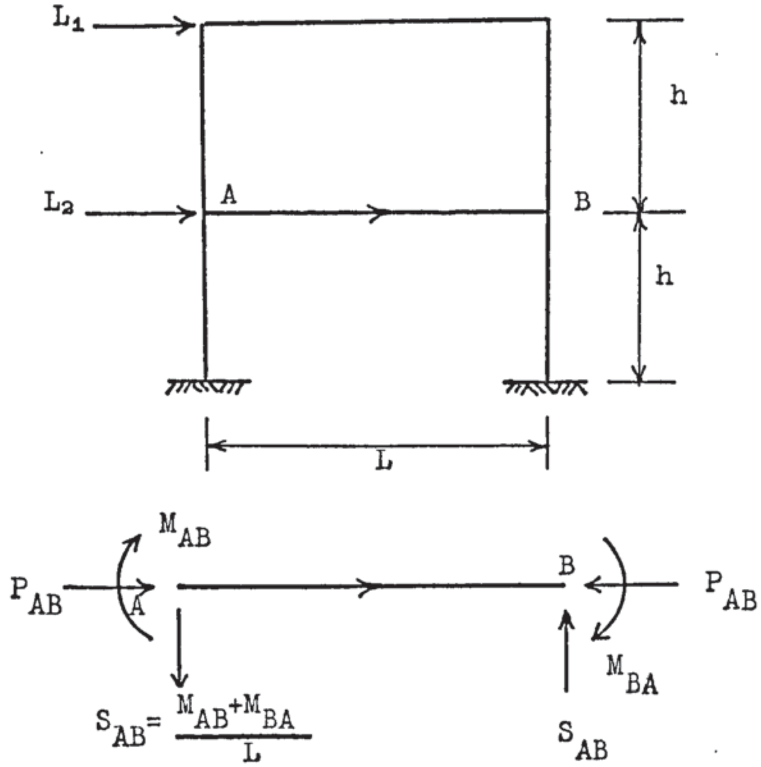
There are two assumptions which are made in conjunction with the structural variation. The first is that the stress-strain relationship of the material of the structure is linear elastic and obeys Hooke's law. The second is that the load deflexion relationship is also linear and that the principle of superposition is valid. However, recent work by Majid and Celik has shown that none of these assumptions are necessary.

The above theorems have been established with reference to linear plastic pin jointed structures where it is sufficient to consider the areas of the members to be basic variables, due to the fact that members and joints are only subjected to direct loads. It is known that in rigidly jointed structures the members and the joints are considered not only to sustain the direct forces but also moments. Hence it becomes necessary to include the second moment of areas of the members as further basic variables. In this chapter, these theorems are extended to cover rigidly jointed plane structures where the basic variables are the second moment of areas and the areas of the members. However, it is also possible to extend the theorems so that they can be used for rigidly jointed space structures.

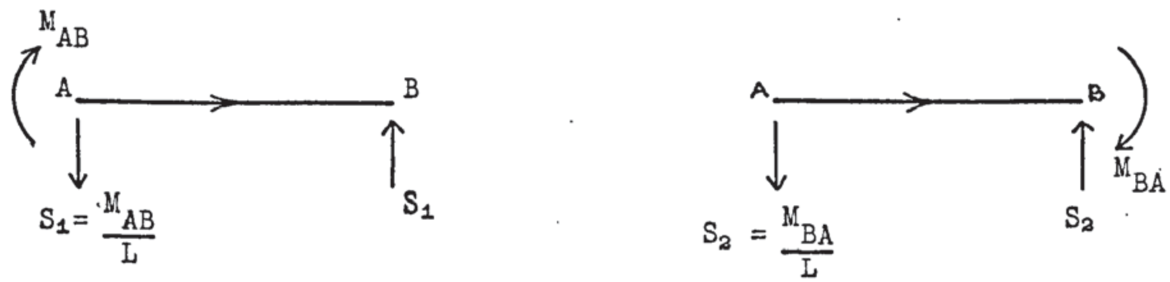
5.2.2) THE UNIT LOAD MATRIX.

As stated above due to the fact that members and joints of pin jointed structures are only subjected to direct loads, it is possible to predict the influence of the variation in the area of any member on the rest of a structure by first applying a pair of equal and opposite unit loads which act axially at the ends of that member. Then the results of the analysis of the structure can be used to study the effect of that member on the rest of the structure. Since there are two basic variables in rigidly jointed structures, which are the second moment of areas and the areas of the members, it becomes necessary to study the influence of the variation in each variable of any member separately.

A member whose first and second ends are connected to a structure at joints A and B respectively is shown in Figure 5.1(a) which is subjected at its ends to bending moments M_{AB} and M_{BA}, shear forces S_{AB} and axial force P_{AB}. It is known that the moments and shear forces are transferred by the bending stiffness of the member while the axial force is transferred by its axial stiffness. Since the moments of each end are not independent of each other, it becomes necessary to consider them separately as shown in Figure 5.1(b),(c).

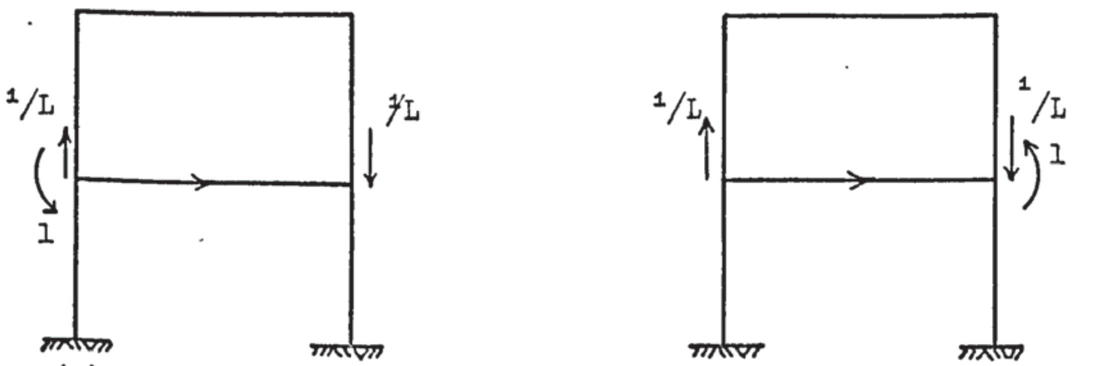


(a) FORCES IN MEMBER AB



(b) (c)

FIGURE 5.1 A RIGIDLY JOINTED STRUCTURE AND LOADING



(a) FIRST END UNIT LOADING (b) SECOND END UNIT LOADING

FIGURE 5.2 UNIT LOADINGS REQUIRED FOR THE VARIATION OF THE SECOND MOMENT OF AREA OF MEMBER AB

Hence, the influence of the variation in the second moment of area of a member will be studied in two steps. The first step requires an analysis of the structure under a unit moment and unit forces acting at the ends of the member. These will be in the opposite direction of the moments and forces shown in Figure 5.1(b). This analysis enables the study of the influence of the variation in the second moment of area at the first end of a member. The second step requires an analysis of the structure under the unit moment and forces acting at the ends of the member, in the opposite direction to those shown in Figure 5.1(c). This enables us to investigate the influence of the variation of the second moment of area on the second end of that member and also on the rest of the structure. Hence to study the independent effect of the second moment of areas of every member requires an analysis of the structure under two load cases as shown in Figure 5.2(a)(b). For convenience, these forces are resolved so that their components may act parallel to the overall reference axes X,Y of the structure.

A member whose first and second ends are connected to a structure at joints A and B respectively as shown in Figure 5.3(a) is now considered. The components of the external forces S_A and S_B can be expressed in terms of the direction cosines of the member as:

$$\begin{bmatrix} S_{AX} \\ S_{AY} \\ M_{AB} \\ S_{BX} \\ S_{BY} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} -m_p & 0 & 0 \\ \ell_p & 0 & 0 \\ 0 & 0 & 1 \\ 0 & m_p & 0 \\ 0 & -\ell_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_A \\ S_B \\ M_A \end{bmatrix} \qquad 5.1$$

For the loading of the second step as shown in Figure 5.3(b)

$$\begin{bmatrix} S_{AX} \\ S_{AY} \\ M_{AB} \\ S_{BX} \\ S_{BY} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} -m_p & 0 & 0 \\ \ell_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & m_p & 0 \\ 0 & -\ell_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_A \\ S_B \\ M_B \end{bmatrix} \qquad 5.2$$

where S_X and S_Y are the components of S in X and Y directions respectively and ℓ_p and m_p are the direction cosines of the p axis of the member.

For $S_A = 1/L$, $S_B = 1/L$, $M_A = 1$ and $M_B = 1$ the first step unit load matrix 5.1 reduces to

$$\left\{ S_{AX} S_{AY} M_{AB} S_{BX} S_{BY} M_{BA} \right\} = \left\{ -\frac{1}{L} m_p \ \frac{1}{L} \ell_p \ \ 1 \ \ \frac{1}{L} m_p \ -\frac{1}{L} \ell_p \ \ 0 \right\} \qquad 5.3$$

and the second step unit load matrix 5.2 reduces to

$$\left\{ S_{AX} S_{AY} M_{AB} S_{BX} S_{BY} M_{BA} \right\} = \left\{ -\frac{1}{L} m_p \ \frac{1}{L} \ell_p \ \ 0 \ \ \frac{1}{L} m_p \ -\frac{1}{L} \ell_p \ \ 1 \right\} \qquad 5.4$$

An analysis of the structure under these two load cases enables us to study the effect of the second moment of areas of the member on the rest of the structure. If the effects of the area of that member are required to be studied, then it is necessary[101] to carry out further analysis of the structure under a pair of equal and opposite unit loads acting axially at the ends of a member as shown in Figure 5.3(c). The components of the external forces $P_A = 1$ and $P_B = -1$ can be expressed in terms of the direction cosines of the members as:

$$\left\{ S_{AX} S_{AY} M_{AB} S_{BX} S_{BY} M_{BA} \right\} = \left\{ -\ell_p \ -m_p \ \ 0 \ \ \ell_p \ \ m_p \ \ 0 \right\} \qquad 5.5$$

Consequently, to study the influence of the variation in the second moment of area and the area of the member requires an analysis of the structure under three load cases which are given by the matrices 5.3, 5.4 and 5.5. These components of the unit external loads at the ends of the other members are similar and once these are collected

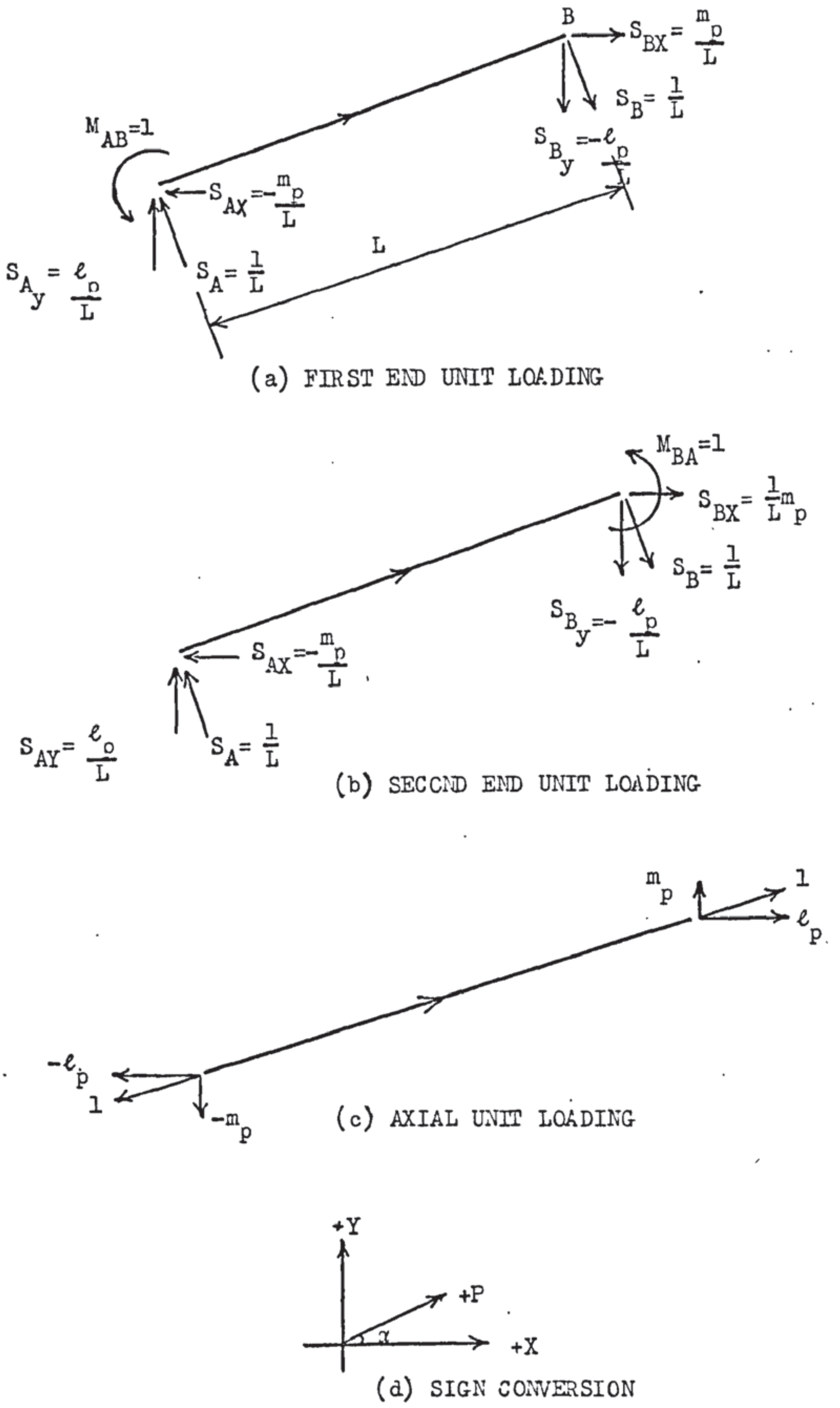


FIGURE 5.3 COMPONENTS OF UNIT LOADINGS ACTING AT THE END OF A MEMBER

together, they can be expressed as a matrix C which is known as the unit load matrix.

A part of matrix C, for a member u which is connected to joint r at its first end, joint s at its second end and a member f which is similarly connected to t and p, is of the from

$$
\underline{C} = \left[\begin{array}{ccc|ccc}
\vdots & & & & & \\
-\frac{1}{L_u}m_{p_u} & -\frac{1}{L_u}m_{p_u} & -\ell_{p_u} & 0 & 0 & 0 \\
\frac{1}{L_u}\ell_{p_u} & \frac{1}{L_u}\ell_{p_n} & -m_{p_u} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -\frac{1}{L_f}m_{p_f} & -\frac{1}{L_f}m_{p_f} & -\ell_{p_f} \\
0 & 0 & 0 & \frac{1}{L_f}\ell_{p_f} & \frac{1}{L_f}\ell_{p_f} & -m_{p_f} \\
0 & 0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{L_u}m_{p_u} & \frac{1}{L_u}m_{p_u} & \ell_{p_u} & 0 & 0 & 0 \\
-\frac{1}{L_u}\ell_{p_u} & -\frac{1}{L_u}\ell_{p_u} & m_{p_u} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \frac{1}{L_f}m_{p_f} & \frac{1}{L_f}m_{p_f} & \ell_{p_f} \\
0 & 0 & 0 & -\frac{1}{L_f}m_{p_f} & -\frac{1}{L_f}\ell_{p_f} & m_{p_f} \\
0 & 0 & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right]
\begin{array}{l}
\\ \\ \text{at r} \\ \\ \\ \\ \text{at p} \\ \\ \\ \\ \text{at s} \\ \\ \\ \\ \text{at t} \\ \\ \\
\end{array}
$$

for member u for member p

While analysing the structure under the actual external loads $\underline{L}$, it is now possible at the same time to evaluate the joint displacement $\underline{\chi}$ and the member forces $\underline{f}$ due to the unit load matrix. Combining $\underline{L}$ with $\underline{C}$, and also $\underline{X}$ with $\underline{\chi}$ the overall stiffness equation become

$$[\underline{X} \quad \underline{\chi}] = \underline{K}^{-1}[\underline{L} \quad \underline{C}]$$

and the member forces

$$[\underline{P} \quad \underline{p}] = \underline{kA}[\underline{X} \quad \underline{\chi}]$$

5.2.3) VARIATION OF FORCES WITH THOSE OF MEMBER AREAS AND SECOND MOMENT OF AREAS.

The first theorem of structural variation predicts the member forces throughout a structure when the second moment of area and area of one member are varied or when they are totally removed, thus resulting in a completely new structure which has a new topology. It is known[101] that it is necessary to change the area and the second moment of area of a varying member independetly and not simultaneously. Further, as explained in the previous section, the variation of the second moment of area of a member is carried out in two steps. In the first step, the new member forces in the other members of the structure are computed when the second moment of area varies at the first end of that member. In the second step, these forces are used to compute the final forces in the members of the structure when the second moment of area varies at the second end of that member. During these steps the area of the member is kept unaltered.

Firstly an expression will be derived for the new forces in any member, when the second moment of area of a given member is being varied.

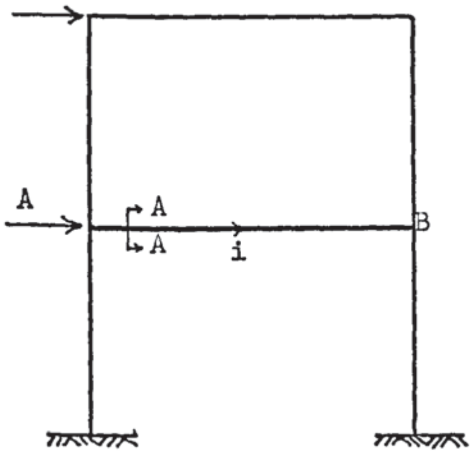
Consider a member i of a structure shown in Figure 5.4(a). The second moment of area of the member i is being varied from I_i to I_i' by an amount δI_i. When δI_i is positive the second moment of area increases and defining β as $\delta I_i/I_i$, it follows that β is also positive. On the other hand, if the second moment of area decreases, δI_i is negative and so is β. In the case where the second moment of area of member i decreases, then the remaining second moment of area is

$$I_i' = I_i - \delta I_i$$

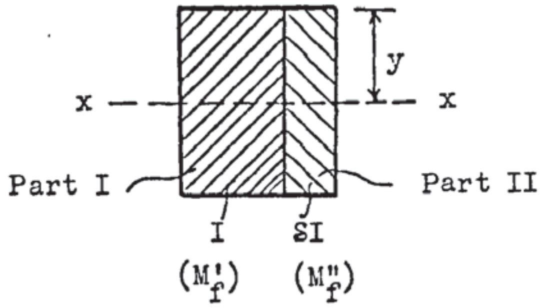
$$\beta = -\delta I_i/I_i \qquad 5.6$$

and hence $$I' = (1+\beta)I_i$$

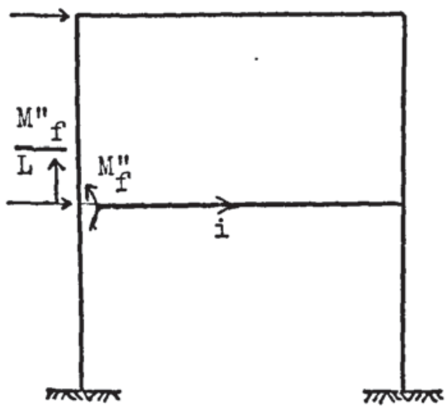
The structure shown in Figure 5.4(a) is subjected to a



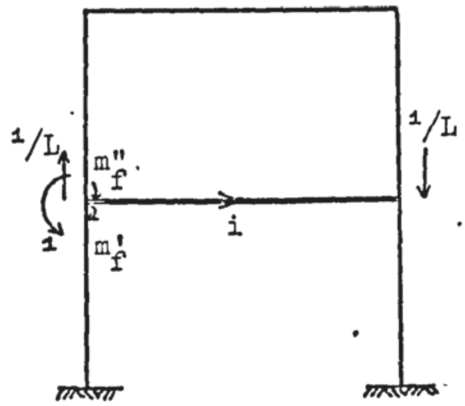
(a) STRUCTURE AND LOADING



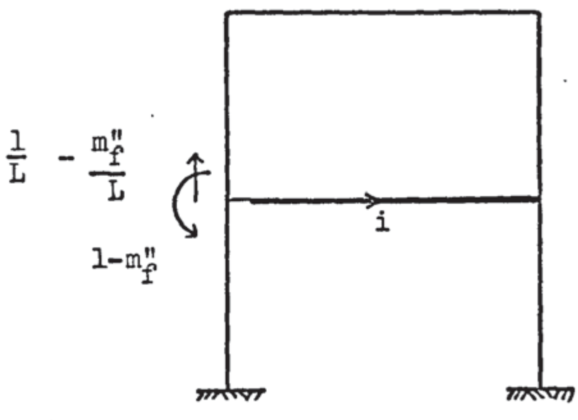
(b) SECTION A - A



(d) PART II OF THE SECTION AT JOINT A REPLACED BY MOMENT M''_f AND SHEAR M''_f/L



(d) FIRST END UNIT LOADING



(c) PART II OF THE SECTION AT JOINT A REPLACED BY MOMENT m''_f AND SHEAR FORCE m''_f/L

FIGURE 5.4 A GENERAL RIGIDLY JOINTED STRUCTURE.

general set of external loads

$$\underline{L} = \{L_1 \ \ L_2 \ \ \ldots \ \ L_d\}$$

where d is the total number of degrees of freedom. There are N members in the structure and the resulting member forces in these are:

$$\underline{P} = [\underline{F} \ \ \underline{S} \ \ \underline{M}_f \ \ \underline{M}_s]$$

where

$$\underline{F} = \{F_1 \ \ F_2 \ \ \ldots \ F_N\}$$
$$\underline{S} = \{S_1 \ \ S_2 \ \ \ldots \ S_N\}$$
$$\underline{M}_f = \{M_{f_1} \ M_{f_2} \ \ \ldots \ M_{f_N}\}$$
$$\underline{M}_s = \{M_{s_1} \ M_{s_2} \ \ \ldots \ M_{s_N}\}$$

in which $\underline{F}$ is the axial force matrix for the members, $\underline{S}$ is the shear force matrix, $\underline{M}_f$ is the moments at their first ends and $\underline{M}_s$ is the moments at their second ends.

Considering member i, which is connected to joints A and B, let us take the second moment of area of the first end in two parts I_i' and δI_i as shown in Figure 5.4(b). It should be pointed out that the member has to be divided into two parts perpendicular to x-x axis of its cross-section. An expression will be derived to find the new member forces in the other members of the structure due to this change in the second moment of area of member i while its area remains unaltered. Since the moments are transferred by the bending stiffness of the member, the corresponding moments at the first end of the member due to this change would be M_{f_i}' and M_{f_i}''. This is possible provided that:

$$M_{f_i} = M_{f_i}' + M_{f_i}''$$

and 5.7

$$\frac{M_{f_i}}{I_i} y = \frac{M_{f_i}'}{I_i'} y = \frac{M_{f_i}''}{\delta I_i} y = \sigma_b$$

where σ_b is the bending stress in member i as well as in each part of that member. It follows from equation 5.6 and 5.7 that

$$M''_{f_i} = -\beta\, M_{f_i} \qquad 5.8$$

$$M'_{f_i} = (1+\beta)M_{f_i}$$

The part II at the first end of the member with the second moment of area δI_i, the moment M''_{f_i} and the shear S_i can be removed without altering the member forces elsewhere in the structure, by applying their equal and opposite forces acting at joint A. This is shown in Figure 5.4(c). In the case where the second moment of area of the member is totally removed at the first end, this means that a hinge is inserted at the first end of the member which does not transfer shear and moment, i.e. it can be called shearless hinge.

The original structure is analysed under the load case shown in Figure 5.4(d) which is a unit moment at joint A and opposite loads of 1/L acting at joints A and B. The resulting member forces due to these are given as

$$\underline{p}_i = \left[\underline{f}_{ii}\ \underline{s}_{ii}\ \underline{m}_{f_{ii}}\ \underline{m}_{s_{ii}}\right]$$

$$\underline{f}_{ii} = \left\{f_{i_1}\ f_{i_2}\ \ldots\ f_{i_N}\right\}$$

$$\underline{s}_{ii} = \left\{s_{i_1}\ s_{i_2}\ \ldots\ s_{i_N}\right\}$$

$$\underline{m}_{f_{ii}} = \left\{m_{f_{i_1}}\ m_{f_{i_2}}\ \ldots\ m_{f_{i_N}}\right\}$$

$$\underline{m}_{s_{ii}} = \left\{m_{s_{i_1}}\ m_{s_{i_2}}\ \ldots\ m_{s_{i_N}}\right\}$$

where $m_{f_{i_2}}$ is the first end moment of member 2 due to unit loads acting at the ends of member i. Since, the second moment of area of member i is taken in two parts I'_i and δI_i, the moments $m'_{f_{ii}}$ and $m''_{f_{ii}}$ are given as

$$m''_{f_{ii}} = -\beta m_{f_{ii}} \qquad 5.9$$

$$m'_{f_{ii}} = (1+\beta)m_{f_{ii}}$$

Similarly the second moment of area of part II at joint A can be removed by compensating the moment and shear at that joint. Hence the total external moment acting at joint A becomes $1-m''_{f_{ii}}$ as shown in Figure 5.4(e).

The magnitude of the unit loads can always be increased by a factor r_{β_i} and the resulting member forces become

$$r_{\beta_i}\,\underline{f}_{ii} = \left\{r_{\beta_i}\,f_{i_1} \quad r_{\beta_i}\,f_{i_2} \quad \cdots \quad r_{\beta_i}\,f_{i_N}\right\}$$

$$r_{\beta_i}\,\underline{s}_{ii} = \left\{r_{\beta_i}\,s_{i_1} \quad r_{\beta_i}\,s_{i_2} \quad \cdots \quad r_{\beta_i}\,s_{i_N}\right\} \qquad 5.10$$

$$r_{\beta_i}\,\underline{m}_{f_{ii}} = \left\{r_{\beta_i}\,m_{f_{i_1}} \quad r_{\beta_i}\,m_{f_{i_2}} \quad \cdots \quad r_{\beta_i}\,m_{f_{i_N}}\right\}$$

$$r_{\beta_i}\,\underline{m}_{s_{ii}} = \left\{r_{\beta_i}\,m_{s_{i_1}} \quad r_{\beta_i}\,m_{s_{i_2}} \quad \cdots \quad r_{\beta_i}\,m_{s_{i_N}}\right\}$$

Then the removal of the second moment of area of part II will therefore require compensation of $r_{\beta_i}\,m''_{f_i}$ and the net externally applied moment at A becomes $r_{\beta_i} - r_{\beta_i}\,m''_{f_{ii}}$. It follows that under the actual external loads L, if δI_i is removed without compensation

$$r_{\beta_i} - r_{\beta_i} m''_{f_{ii}} - M''_{f_i} = 0 \qquad 5.11$$

Substituting the values of $m''_{f_{ii}}$ and M''_{f_i} from the equations 5.8 and 5.9, it follows that

$$r_{\beta_i} = \frac{-\beta\, M_{f_i}}{(1+\beta\, m_{f_{ii}})} \qquad 5.12$$

where r_{β_i} is known as the variation factor for member i.

The final moment in member i which has the remaining moment of areas I'_i is the sum of the moment due to the actual external loads M'_{f_i} and the change in this moment which was shown to be $r_{\beta_i}\ m'_{f_{ii}}$ and is therefore given by

$$M_{I_{f_i}} = M'_{f_i} + r_{\beta_i}\ m'_{f_{ii}} \qquad 5.13$$

Using equations 5.8, 5.9, 5.12 and 5.13, it follows that

$$M_{I_{f_i}} = \frac{(1+\beta)\ M_{f_i}}{(1+\beta)m_{f_{ii}})} \qquad 5.14$$

The forces in any other member j are also found by the superposition of the member forces due to external loads and the member forces given by matrices of 5.10. Hence the final member forces in member j due to the variation of the second moment of area of member i at the first end is given by

$$\begin{aligned} \underline{F}_{I_j} &= \underline{F}_j + r_{\beta_i}\ \underline{f}_{ji} \\ \underline{S}_{I_j} &= \underline{S}_j + r_{\beta_i}\ \underline{s}_{ji} \\ \underline{M}_{If_j} &= \underline{M}_{f_j} + r_{\beta_i}\ \underline{m}_{f_{ji}} \\ \underline{M}_{I_{s_j}} &= \underline{M}_{s_j} + r_{\beta_i}\ \underline{m}_{s_{ji}} \end{aligned} \qquad 5.15$$

or in matrix form:

$$\underline{P}_{I_j} = \underline{P}_j + r_{\beta_i}\ \underline{p}_j \qquad 5.16$$

or

$$\underline{P}_{I_j} - \underline{P}_j = r_{\beta_i}\ \underline{p}_j$$

The first theorem of structural variation is given by equation 5.15 for rigidly jointed plane structures, which states that when the second moment of area at the first end of a member i in a structure is changed by an amount δI_i, the change in the member forces in another member j is given by the product of the variation factor for member i and the force in member j produced by unit loads acting on the structure at the ends of i.

In the case where the second moment of area of member i is changed by an amount δI_i, the computation of the new member forces is carried out in two steps as mentioned previously. In the first step the second moment of area of member i is changed by an amount δI_i and by means of equations 5.14 and 5.15 the new member forces in the other members and in the second end of the member i are computed. Then the first theorem of structural variation is applied to the second end of the member i.e. the second moment of area is also changed by an amount δI_i at the second end of member i.

In the case where the second moment of area of member i is completely removed, then $\beta_i = -1$ and equation 5.12 gives the removal factor r_{β_i} as

$$r_{\beta_i} = \frac{M_{f_i}}{1 - m_{f_{ii}}} \qquad 5.17$$

It is clear that the total removal of the second moment of area will remove the bending stiffness of the member. Hence the member will only be able to transfer the axial forces by its area, i.e. the removal of the second moment of area of a member is equivalent to inserting two real hinges at each of its ends. It is interesting to see that if this is carried out for all the members of the structure a rigidly jointed structure will change into a pin jointed structure. Hence as will be shown by examples, by means of the first theorem of structural variation it becomes possible to find out the forces in the members of a pin jointed structure from the results of the analysis of a rigidly jointed structure.

It has been shown[101] that in pin jointed structures, it is possible to forecast the forces throughout when one or more of its members are varied or totally removed. If p_i and p_j are the forces in the members i and j of a pin jointed structure under the external loads and π_i and π_j are these forces when the area of member i is altered, then

$$\pi_i = (1+\alpha_i)\, p_i \,/(1+\alpha_i\, f_{ii}) \qquad 5.18$$

$$\pi_j = p_j + r_{\alpha_i}\, f_{ji} \qquad 5.19$$

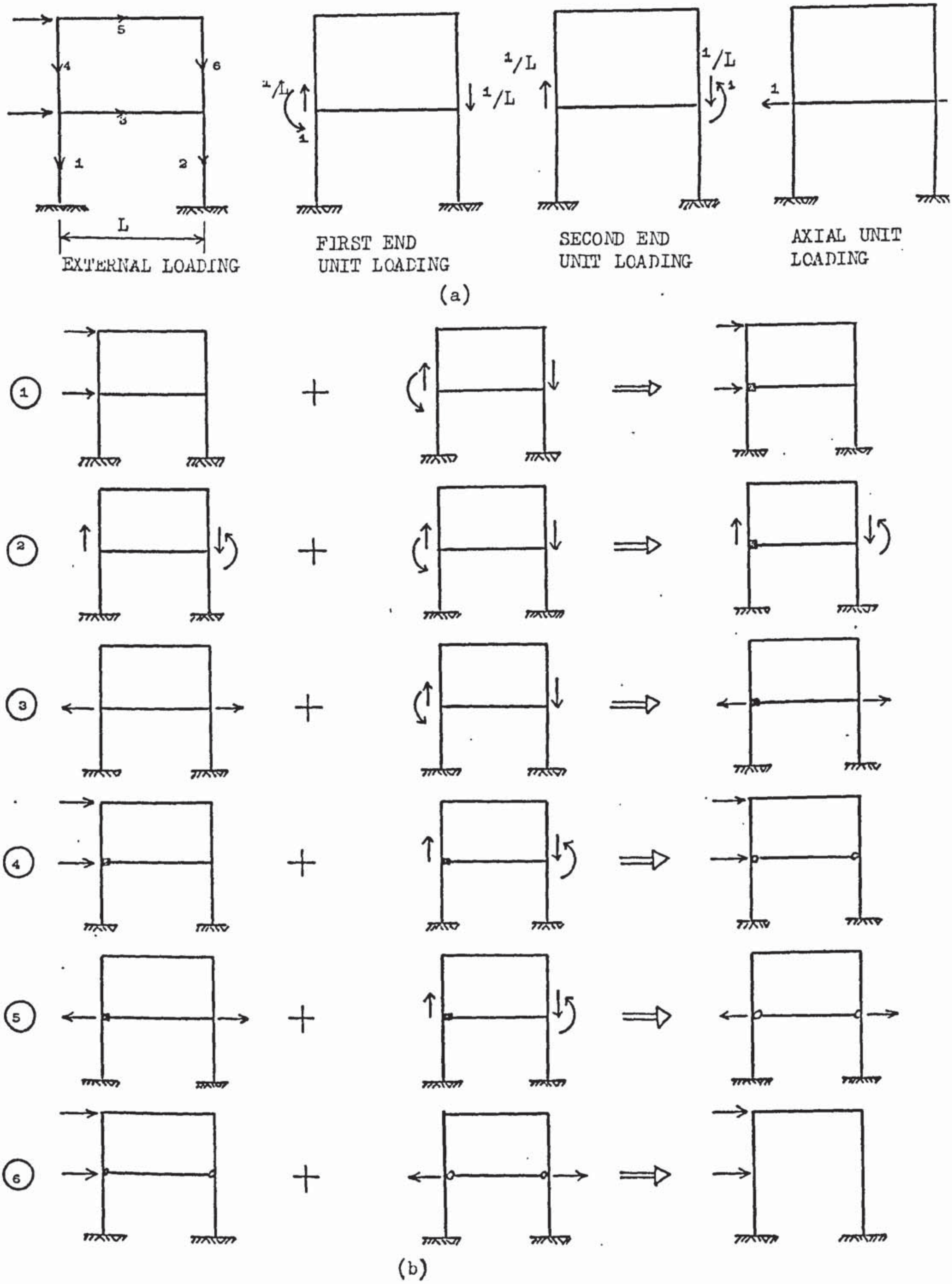
where

$$r_{\alpha_i} = \alpha_i p_i/(1-\alpha_i\, f_{ii}) \qquad 5.20$$

in which $\alpha_i = \delta A_i/A_i$ is the ratio of the change in the area δA_i to its original area A_i and when a member is totally removed $\alpha_i = -1$. f_{ii} and f_{ji} are the forces in members i and j when unit loads are applied at the ends of member i. Hence the total removal of a member in the rigidly jointed structure requires three unit loading cases. This is shown in Figure 5.5(a) for the removal of member 3. It is known that the removal of a member involves firstly the removal of its second moment of area and secondly the removal of its area. Both will be removed in turn. Tho order in which these two basic variables are removed or varied is not important.

As shown in Figure 5.5(b) the process for the variation or removal of the second moment of area and area of the member of rigidly jointed structures consists of 6 stages. No solution of equation is involved in each stage. The only necessary computation is to use the theorems of structural variation and thus substitute the corresponding values of the member forces in the equations 5.12 and 5.16 respectively which is simple.

For instance, in the structure shown in Figure 5.5, member 3 is being removed. In the first stage, the second moment of area of member 3 is removed at the first end of the member while the structure is subject to the external loading. In the second and third stages the same removal is carried out while the structure is subject to the second end unit loading and axial unit loading as



■ REPRESENTS SHEARLESS HINGE

FIGURE 5.5 THE PROCEDURE FOR REMOVAL OF A MEMBER.

shown in Figure 5.5(a). In these three stages the second moment of area at the first end of member 3 is removed. Hence that end of member 3 cannot sustain moment and shear force. That is to say a "shearless hinge" is inserted at the first end of member 3 while the structure is subject to the external and unit loadings respectively. In the fourth stage, the member forces obtained in the second stage are used to remove the second moment of area at the second end of member 3 as shown in Figure 5.5(b). At this stage, the second moment of area of member 3 is totally removed and the member can only sustain axial force i.e. two real hinges are inserted at the ends of member 3. In the fifth stage the second moment of area of member 3 is totally removed and member forces are obtained while the structure is subject to unit axial loading. The member forces obtained at this stage are used to remove the area of member 3 of the structure which is subject to the external loads. The member forces obtained at this stage are exactly equal to the member forces in single storey portal frame shown in Figure 5.5(b) which is subject to the same external loads.

In the case where more than one member of a rigidly jointed frame are to be removed, then it becomes necessary to analyse the parent structure due to external loads and in addition a further 3 × NMR load cases, where NMR is the number of members to be removed. It is obvious that this will apply if section properties of members are varied but not removed. Then each member is removed or varied in turn using its corresponding 3 unit loadings and new member forces are computed while the structure is subject to the external loads and the other unit loads.

5.2.4) VARIATION OF DEFLEXIONS WITH INDIVIDUAL AREAS AND SECOND MOMENT OF AREAS.

The second theorem of structural variation predicts the deflexions throughout a structure when the second moment of area and area of one member is varied independently or totally removed, which results in a structure with a new topology.

The matrix force method can be stated for statically indeterminate structures as follows.

$$\underline{P} = \underline{B}_b\underline{L}_b + \underline{B}_r\underline{L}_r \qquad 5.21$$

and

$$\underline{X}_b = \underline{F}_{bb}\underline{L}_b + \underline{F}_{br}\underline{L}_r \qquad 5.22$$

$$\underline{0} = \underline{F}_{rb}\underline{L}_b + \underline{F}_{rr}\underline{L}_r$$

where $\underline{P}$ is the vector of member forces, $\underline{L}$ is the applied load vector or matrix that corresponds to the nodal displacements $\underline{X}$. The force transformation matrix is $\underline{B}$ while $\underline{F}$ is the overall flexibility matrix. Suffix b refers to theb asic statially determinate structure while suffix r refers to the redundant. The manner in which the matrix force method was formulated indicates that hyperstatic structures form an extension to statically determinate structures.

In the statically indeterminate rigidly jointed frame having a total of t redundants which are taken as internal moment at the ends of the members, the flexibility equation of 5.22 and 5.23 for m loading points becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc} & & & F_{1,k} & \cdots & F_{1,n} \\ & F_{bb} & & \vdots & & \vdots \\ & & & F_{m,k} & \cdots & F_{m,n} \\ \hline F_{k,1} & \cdots & F_{k,m} & F_{k,k} & & F_{k,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ F_{n,1} & \cdots & F_{n,m} & F_{n,k} & \cdots & F_{n,n} \end{array}\right] \left[\begin{array}{c} L_1 \\ L_2 \\ \vdots \\ L_j \\ \vdots \\ L_m \\ \hline M_1 \\ M_2 \\ \vdots \\ M_t \end{array}\right] \qquad 5.24$$

where for convenience, $k = m+1$ and $n = m+t$.

Let us consider one member, say member t, whose end moment was taken as redundant, and take the case when its second moment of area at that end is varied. The moments M_{I_i} in any other redundant moment will be given by equation 5.15 which becomes

$$M_{I_i} = M_i + r_{\beta_t} . m_{f_{it}}$$

Hence equation 5.24 can be written as

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_j \\ \vdots \\ \psi_m \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc|c} & & & F_{1,k} & \cdots & F_{1,n-1} & F_{1,n} \\ & F_{bb} & & \vdots & & \vdots & \vdots \\ & & & F_{m,k} & \cdots & F_{m,n-1} & F_{m,n} \\ \hline F_{k,1} & \cdots & F_{k,m} & F_{k,k} & \cdots & F_{k,n-1} & F_{k,n} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ F_{n-1,1} & & F_{n-1,m} & F_{n-1,k} & \cdots & F_{n-1,n-1} & F_{n-1,n} \\ F_{n,1} & & F_{n,m} & F_{n,k} & \cdots & F_{n,n-1} & F_{n,n} \end{array}\right] \left[\begin{array}{c} L_1 \\ L_2 \\ \vdots \\ L_j \\ L_m \\ \hline M_1 + r_{\beta_t} . m_{f_{1t}} \\ \vdots \\ M_i + r_{\beta_t} . m_{f_{it}} \\ M_{t-1} + r_{\beta_t} . m_{f_{t-1}} \\ M_{I_t} \end{array}\right]$$

5.25

In these equations M_{I_t} is variable because the second moment of area of the member end corresponding to that moment is being varied. Hence both β_t and r_{β_t} are variables. Since the joint deflexions are denoted by them, they are also variables. To differentiate them from the initial deflexions X, they are represented by ψ. On the other hand, although each redundant moment is changing, the moments M_i for $i = 1,2, \ldots t$ are constant and predetermined before varying the second moment of area at the first end of the member t.

Each element of the entire matrix F in equation 5.25 is of the form

$$F = \gamma + \frac{\mu}{I_t'}$$

where γ and μ are constants while the new second moment of area I_t' is a variable. As mentioned above r_{β_t} is dependent on I_t' and has the form $\theta I_t'$ where θ is also a constant. Hence, a displacement ψ_j evaluated from equation 5.24 has the form

$$\psi_j = \gamma + \frac{\mu}{I_t'} + \theta I_t' + (\alpha + \beta/I_t')M_{I_t} \qquad 5.26$$

The quantities $\gamma, \mu, \theta, \alpha$ and β are all constants obtained by multiplying a row of F by the column on the right hand side of equation 5.25 and then collecting the terms. When M_{I_t} is obtained from the last equation of 5.25 and substituted in 5.26 and similar terms collected together, the following equation is obtained

$$\eta_1 + \eta_2 I_t' + \eta_3 I_t'^2 + \eta_4 I_t'^3 + \eta_5 I_t' \psi_j + \eta_6 I_t'^2 \psi_j = 0 \qquad 5.27$$

where $\eta_1, \eta_2 \ldots \eta_6$ are arbitrary constants. This equation 5.27 is obtained for the portal frame in Appendix A and the values of η_i for $i = 1,2,\ldots 6$ are given. When the second moment of area at the end of member t corresponding to M_{I_t} is removed, equation 5.27 remains true for all ψ_j. Hence for $I_t' = 0$, equation 5.27 gives $\eta_1 = 0$, thus

$$\eta_2 I_t' + \eta_3 I_t'^2 + \eta_4 I_t'^3 + \eta_5 I_t'\psi_j + \eta_6 I_t'^2 \psi_j = 0$$

Dividing through by I_t' we obtain

$$\eta_2 + \eta_3 I_t' + \eta_4 I_t'^2 + \eta_5 \psi_j + \eta_6 I_t' \psi_j = 0 \qquad 5.28$$

It can be seen from equation 5.28 that the variation of any displacement ψ_j with changes in the second moment of area at any end of any member where the moment is taken as redundant, is hyperbolic and therefore convex.

However, the hyperbolic relationship between the change of the second moment of area and variation of any displacement ψ_j can also be derived as follows:

If χ_{ji} is any of the displacements of joint j due to the first end unit loading required for the variation of the second moment of area at the first end of the member i, the increase of these unit loads by a factor of r_{β_i} increases the value of the displacement χ_j of joint j to $r_{\beta_i}\ \chi_{ji}$. The new displacement of joint j is obtained by using the principle of superposition

$$\psi_j = x_j + r_{\beta_i}\ \chi_{ji} \qquad 5.29$$

where x_j is the initial displacement due to actual loading and r_{β_i} is the variation factor given by equation 5.12. Substituting this value of r_{β_i} in equation 5.29, it follows that:

$$\psi_j = x_j - \beta_i\ M_{f_i}\ \chi_{ji}/(1+\beta_i\ m_{f_{ii}})$$

which is hyperbolic for the variables ψ_j and β_i.

The second theorem of structural variation is given by equation 5.29. This states that when the second moment of area at any end of member i is changed by an amount δI_i, the new deflexion at any joint j is obtained by adding the deflexion of that joint due to actual loading and the product of the variation factor for member i and the deflexion at j produced by unit loading for that end of the member. As described in the previous section the total change of the second moment of area of that member is carried out in two steps. Further, if the area of member i is also changed, then the variation in the deflexion at any joint j is given in(101) as

$$\theta_j = \Delta_j + r_{\alpha_i}\ \delta_{ji} \qquad 5.30$$

where Δ_j is the deflexion at a joint j due to external loads, θ_j

is this deflexion after changing member i, δ_{ji} is the deflexion at j produced by axial unit loading at the ends of member i after its second moment of area has been changed and r_{α_i} is the variation factor given by equation 5.20.

As explained in the previous section the variation of the second moment of area as well as the area of the member involves 6 stages. Hence at each stage the variation factors which are used in the computation of new member forces are also employed to evaluate the new deflexions of joints. The same procedure which is used for member forces shown in Figure 5.5(b) is also followed for the deflexions.

Hence by the first two theorems of structural variation it becomes possible to predict the forces in the members and the deflexions throughout rigidly jointed plane frames when the second moment of areas and the areas of one or more members are varied or totally removed.

Since statically determinate structures are a special case of statically-indeterminate structures, the same procedure can be employed to derive an expression for the variation of member forces and deflexions with respect to the second moment of areas and area of any member. It is evident that when a member is totally removed care should be taken to avoid the development of a mechanism.

5.2.5) VARIATION OF DEFLEXIONS WITH PROPORTIONAL CHANGES IN AREAS AND SECOND MOMENT OF AREAS.

The third theorem of structural variation predicts the deflexions throughout a structure when the second moment of areas and areas of all the members of the structure are varied proportionally. The stiffness equation has the form

$$\underline{L} = \underline{K}.\underline{X} \qquad 5.31$$

where $\underline{L}$ is the external load vector, $\underline{X}$ is the corresponding displacement vector and $\underline{K}$ is the overall stiffness matrix. In rigidly jointed structures an element K_{ij} of the stiffness matrix $\underline{K}$ consists of accumulative stiffnesses of the form $(\mu A+\nu I)$ contributed by N members.

$$K_{ij} = \sum_{\ell=1}^{N} (\mu_\ell A_\ell + \nu_\ell I_\ell) \qquad 5.32$$

where μ_ℓ and ν_ℓ are the constants which depend on the length, Young's modulus and direction cosines of the member.

In the case where the area and the second moment of area of all the members are varied proportionally, $\alpha = \delta A/A$ and $\beta = \delta I/I$ are kept constant and equal to each other for all members.* The new area A_m^* and the second moment of area I_m^* for member m is given by

$$\begin{aligned} A_m^* &= (1+\alpha)A_m \\ I_m^* &= (1+\alpha)I_m \end{aligned} \qquad 5.33$$

and the new element K_{ij}^* of the stiffness matrix becomes

$$K_{ij}^* = \sum_{\ell=1}^{N} (\mu_\ell A_\ell^* + \nu_\ell I_\ell^*) = (1+\alpha) \sum_{\ell=1}^{N} (\mu_\ell A_\ell + \nu_\ell I_\ell) \qquad 5.34$$

In this manner the overall stiffness matrix becomes

$$\underline{K}^* = (1+\alpha)\underline{K}$$

The new deflexions may be obtained by solving $\underline{L} = \underline{K}^*\underline{X}^*$ which is

$$\underline{X}^* = \underline{K}^{*-1}\,\underline{L}$$

$$\underline{X}^* = \frac{1}{1+\alpha}\,\underline{K}^{-1}.\underline{L}$$

It can be seen that $\underline{K}^{-1}.\underline{L} = \underline{X}$, and it follows that

$$\underline{X}^* = \frac{1}{1+\alpha}\underline{X} \qquad 5.35$$

Equation 5.35 gives the mathematicl expression for the third theorem which states that when the second moment of areas and areas of the members of a structure are varied proportionally, the new deflexion is obtained by dividing the original deflexion by $1+\alpha$. It can be seen from equation 5.35 that the change in the deflexion is hyperbolic.

* See next page

It should be pointed out that proportional changes in the sectional properties of all members of the structure does not change the member forces in these members. This follows from the fact that

$$\underline{k}^* = (1+\alpha)\underline{k}$$

where $\underline{k}^*$ is the new stiffness of the member while $\underline{k}$ is the stiffness before the section properties are varied. The member forces are given by equation 2.6 which is

$$\underline{P} = \underline{k}\underline{A}\underline{X} = \frac{1}{1+\alpha}\ \underline{k}^*.\underline{A}.(1+\alpha)\underline{X}^* = \underline{k}^*\underline{A}\ \underline{X}^*$$

The stresses in these members will change to

$$\sigma_\ell^* = \frac{P_\ell}{A_\ell^*} + \frac{M_\ell}{I_\ell^*}\ y \qquad 5.36$$

where P_ℓ and M_ℓ are the axial force and moment in member ℓ respectively. The stress has changed from σ_ℓ to σ_ℓ^* due to the proportional changes in the sectional properties. Using the expression 5.33

$$\sigma_\ell^* = \frac{1}{1+\alpha}\left(\frac{P_\ell}{A_\ell} + \frac{M_\ell}{I_\ell}.y\right)$$

which is

$$\sigma_\ell^* = \frac{1}{1+\alpha}.\sigma_\ell \qquad 5.37$$

The equations 5.35 and 5.37 give the new deflexions of all the joints and the stresses in all the members of a structure when sectional properties of all members change proportionally. That is to say, if the areas and the second moment of areas of all the members are reduced by half, all the deflexions and stresses are doubled in value. By these formulae, it becomes possible to control the feasibility of the design variables without further analysis in the optimisation procedures which can only operate from feasible points.

*In this section formulae have been derived as if α (areas) and β (second moment of areas) are the same. This has been done for simplicity. Similar expressions for $\alpha \neq \beta$ can however be obtained.

5.3) COMPUTER PROGRAMMING.

By means of these theorems it becomes possible to obtain the analytical results of a derivative structure from the analyses of a large parent structure. To do this, the latter is first analysed for given external loads and the necessary unit loadings, the results being stored on file. These results may then be used to analyse one of the derived structures. For this purpose, the computer programs were written in Fortran IV and run at the ICL 1905E computer at the University of Aston.

As explained in the previous section, variation or removal of each member requires 3 unit load cases. Hence a parent structure is analysed for NLC load cases where NLC = 3 × NMR + 1 in which NMR is the number of members to be removed. The member forces and deflexions due to these load cases are stored in the arrays AF, SF, FEM, SEM and XDD where AF is the axial forces array, SF is the shear forces array, FEM is the first end moments array, SEM is the array which contains the moments at the second ends of the members and XDD is the array which contains the deflexions of the joints. Figure 5.6 shows one of these arrays being that for the member forces where there are 4 members to be removed. As can be seen from the figure each member forces array contains NLC blocks each of which has NOM elements where NOM is the total number of members in the structure. The first block of these arrays contains the member forces due to the external loads, the rest contain the member forces due to the unit loading of the members to be removed. By using the second, third and fourth blocks the first member is removed and the resulting member forces are stored in the first block of the member forces array. It is apparent that the unit loadings for other members to be removed have to be changed. This is carried out by considering each of the unit loadings belonging to the other members to be removed with the unit loadings of the member which is being removed as shown in the figure.

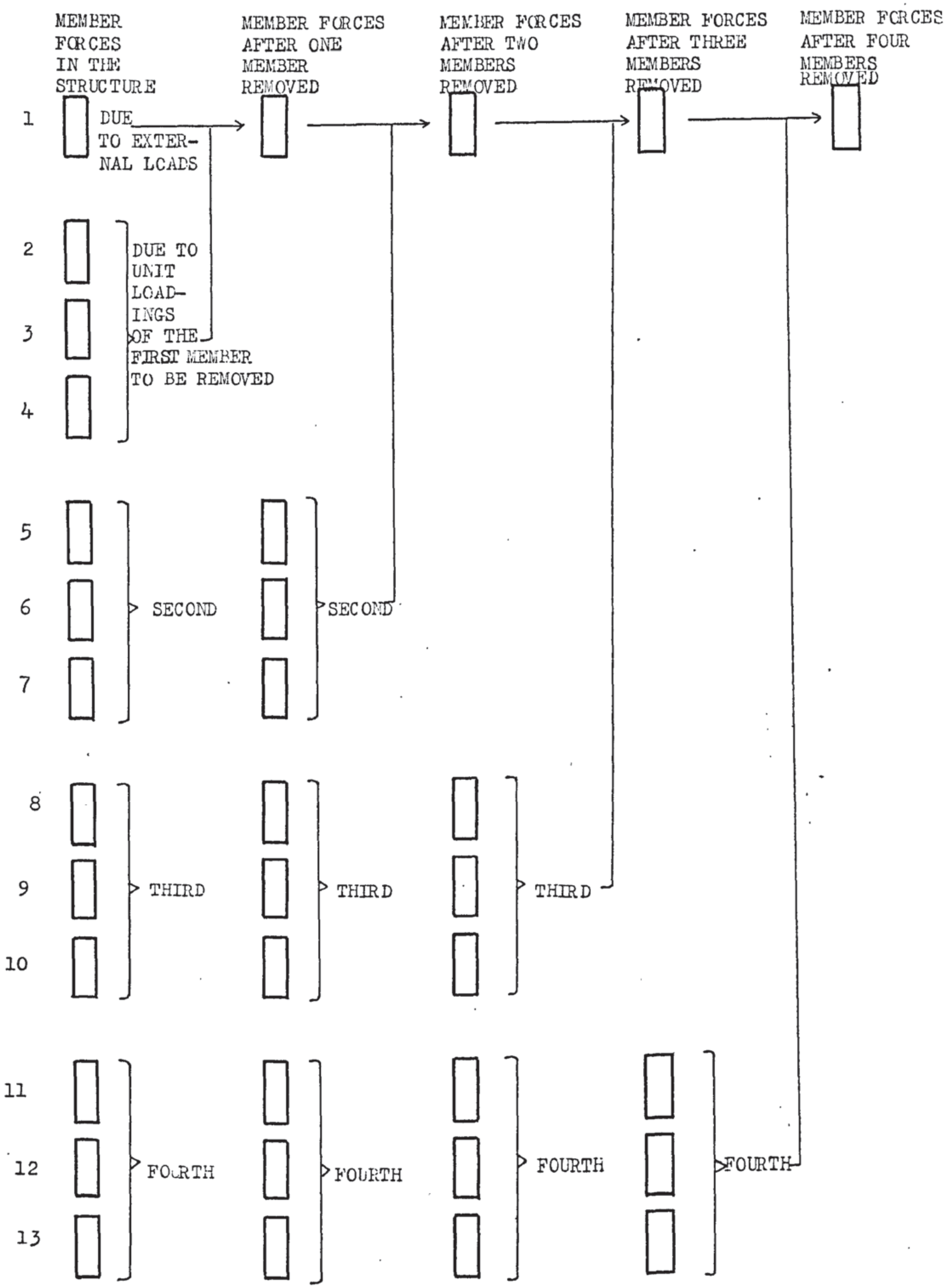


FIGURE 5.6 A DIAGRAM OF THE PROCEDURE FOR REMOVAL OF MORE THAN ONE MEMBER.

A flow diagram of the computer program is given in Figure 5.7. In the flow diagram only the axial force array is shown with the purpose of simplifying the diagram. This also applies to the shear forces, first end moments, second end moments and deflexion arrays. Subroutine MEMREM is called which works as described in Section 5.2.3. There are 4 types of array which are used in the subroutine. The size of each array is equal to NOM. The subscripts represent the loading. For example, the array AF1 contains the axial forces in the members due to the external loads, AF2 contains the axial forces in the members due to the first end unit loading, AF3 contains the axial forces in the members due to the second end unit loading and finally AF4 contains the axial forces in the members due to the axial unit loading. Using these arrays any member of the structure can be varied or removed. Hence as shown in the flow diagram, the member forces and deflexions due to the necessary loadings are inserted in the corresponding arrays, which are used by that subroutine and then by calling it the member is removed and the computed member forces and deflexions are placed back into the first blocks of the corresponding arrays. Then each member force due to the unit loadings of the other members to be removed are inserted in the arrays AF1, SF1, FEM1 and SEM1 and the other arrays remain the same. The subroutine is again called and the member is removed. In this way all the member forces due to the unit loadings of other members to be removed are adjusted. It is apparent that variation of the member will follow the same procedure.

5.4) EXAMPLES ON THE USE OF THE THEOREMS.

The computer program described in the previous section which uses the first two theorems of structural variation, was utilised to show the application of these theorems. A number of rigidly jointed frames are considered, derivatives of these frames are shown and the

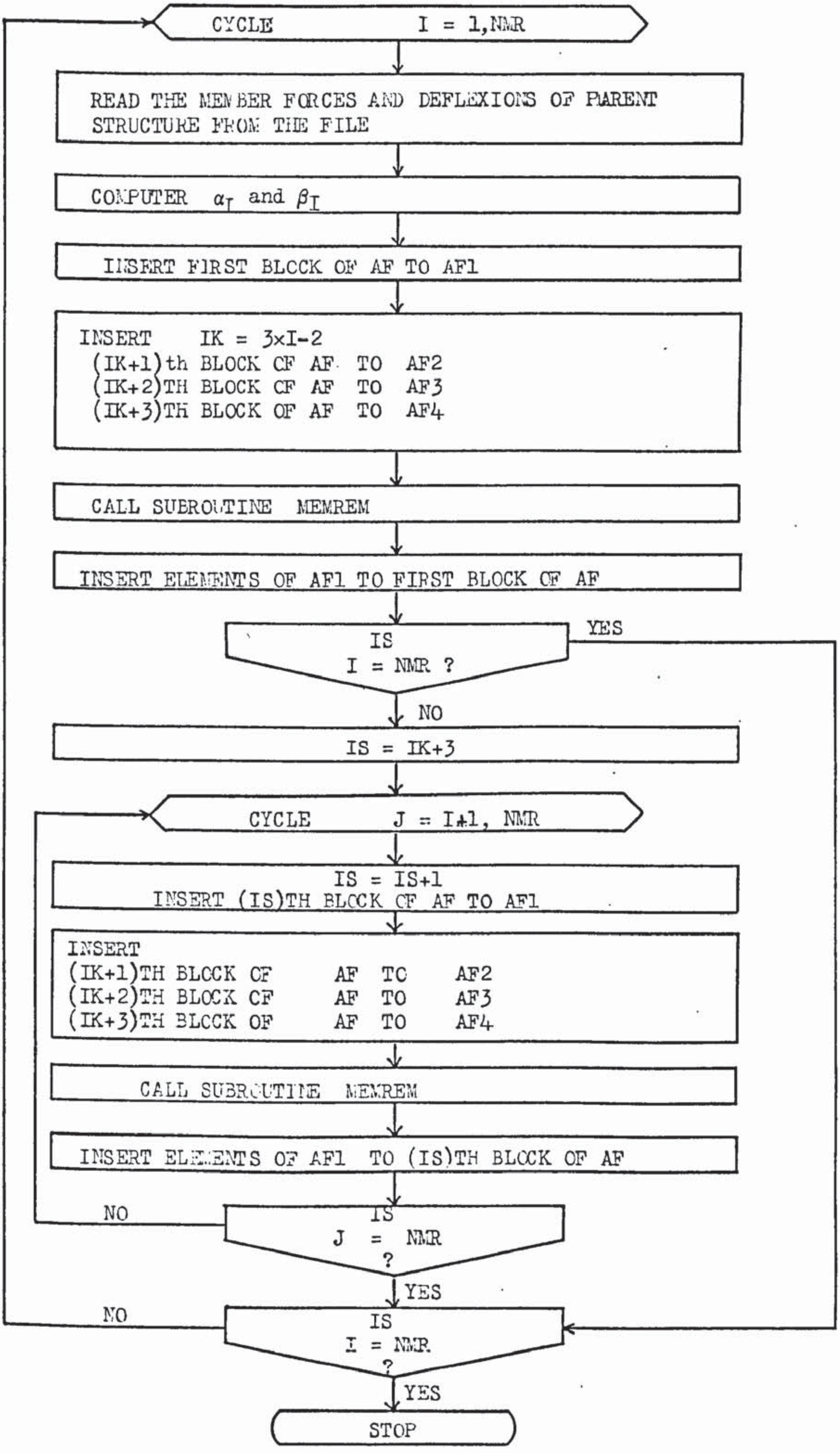


FIGURE 5.7 A FLOW-DIAGRAM FOR STRUCTURAL VARIATION.

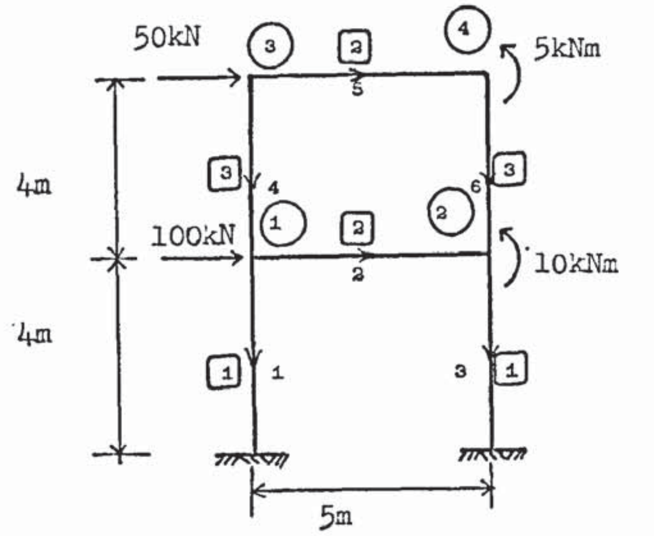
results are tabulated. This shows that all the structures are inter-related and it is possible to obtain the analysis of one frame from the analysis of another.

5.4.1) EXAMPLE 1.

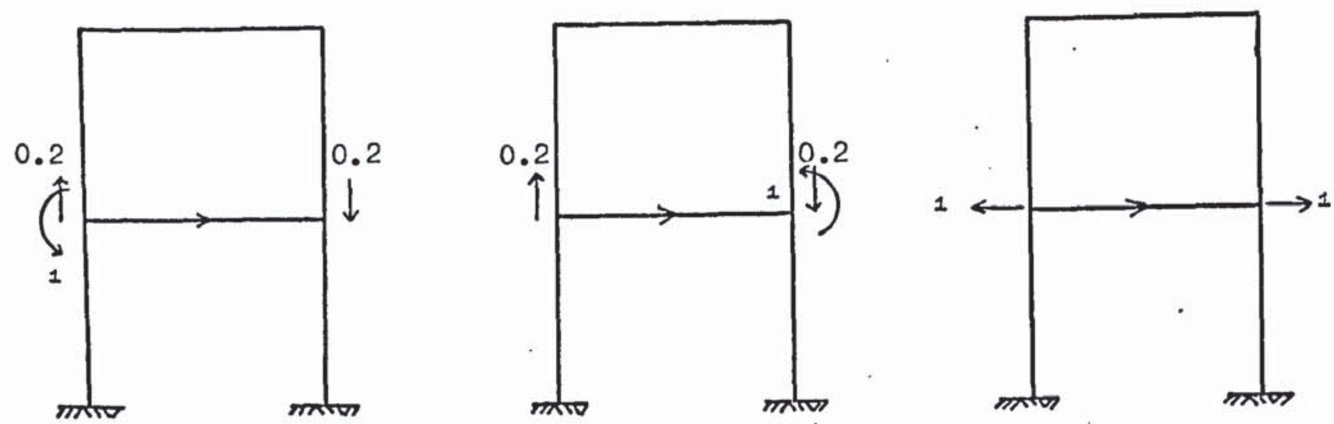
The two storey rigidly jointed frame of Figure 5.8(a) is subject to horizontal loads of 100 kN and 50 kN acting at joints 1 and 3 respectively and also two moments of 10 kNm and 5 kNm acting at joints 2 and 4 respectively. Dimensions and member numbering are shown in Figure 5.8(a). The frame consists of two groups. Members 1 and 3 are in group 1 with areas of 4×10^4 mm^2 and second moment of areas of 0.75×10^8mm^4, members 2 and 5 are in group 2 with areas of 2×10^4 mm^2 and second moment of areas of 0.3×10^8 mm^4 and members 4 and 6 are in group 3 with areas of 3×10^4 mm^2 and second moment of areas of 0.6×10^8 mm^4. The modulus of elasticity of the material is 207 kN/mm^2.

This frame was analysed by the matrix displacement method while subjecting it to the external loads together with the unit loadings as shown in Figure 5.8(b). The stiffness equations are constructed and solved only once, even though 3 loading cases are considered. The first two theorems of structural variations are then used to predict the member forces and deflexions of the derivative portal frame when member 2 of the two storey frame is removed. The deflexions and the member forces obtained for the portal frame shown in Figure 5.8(c) which is the derivative of the two storey frame, are given in the tables 5.1 and 5.2. The portal frame shown in Figure 5.8(c) was also analysed separately and the deflexions and the member forces obtained are shown in the Tables 5.3 and 5.4 respectively.

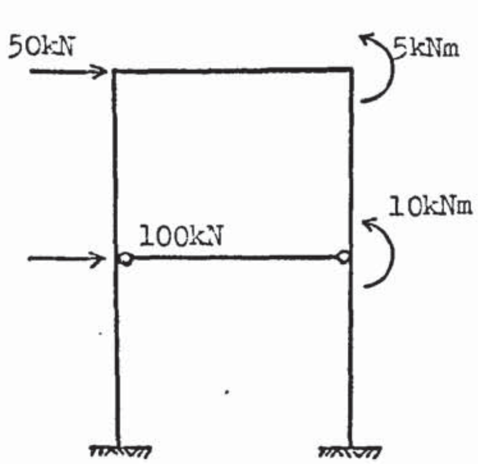
It can be seen by comparing Tables 5.1, 5.2 and 5.3, 5.4, that the deflexions and member forces obtained by structural variation is exact. The third theorem of structural variation can be



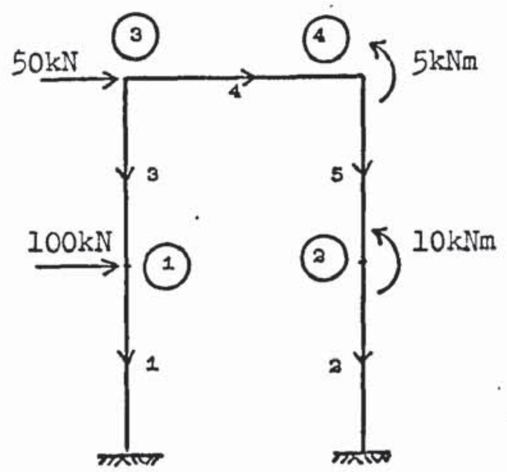
(a) STRUCTURE AND LOADING



(b) UNIT LOADINGS



(d) DERIVATIVE STRUCTURE OBTAINED REMOVING THE SECOND MOMENT OF AREA OF MEMBER 2



(c) PORTAL FRAME OBTAINED BY REMOVING MEMBER 2

FIGURE 5.8: RIGIDLY JOINTED GROUND STRUCTURE.

JOINT NO	HORIZONTAL DEF.(mm)	VERTICAL DEF. (mm)	ROTATION (rad.)
1	109.209	0.02090	-0.03635
2	79.988	-0.02091	-0.03246
3	203.465	0.04903	-0.00945
4	203.412	-0.04903	-0.01984

TABLE 5.1 DEFLEXIONS OF THE PORTAL FRAME OBTAINED BY STRUCTURAL VARIATION.

MEMBER NO	AXIAL FORCE (kN)	SHEAR FORCE(kN)	FIRST END MOMENTS kN mm	SECOND END MOMENTS kN mm
1	43.56793	-106.1898	71239.026	353520.737
2	-43.56793	- 43.8101	-38399.324	213639.562
3	43.56794	- 6.1900	95999.268	-71239.026
4	-43.80989	43.56797	-95999.268	-121840.438
5	-43.56794	- 43.80898	126840.438	48399.324

TABLE 5.2 MEMBER FORCES OF THE PORTAL FRAME OBTAINED BY STRUCTURAL VARIATION

JOINT NO	HORIZONTAL DEF.(mm)	VERTICAL DEFL.(mm)	ROTATION (rad.)
1	109.208	0.02104	-0.03636
2	79.9876	-0.02104	-0.03246
3	203.4655	0.04911	-0.009434
4	203.4126	-0.04911	-0.019837

TABLE 5.3 DEFLEXIONS OF THE PORTAL FRAME OBTAINED BY THE MATRIX DISPLACEMENT METHOD.

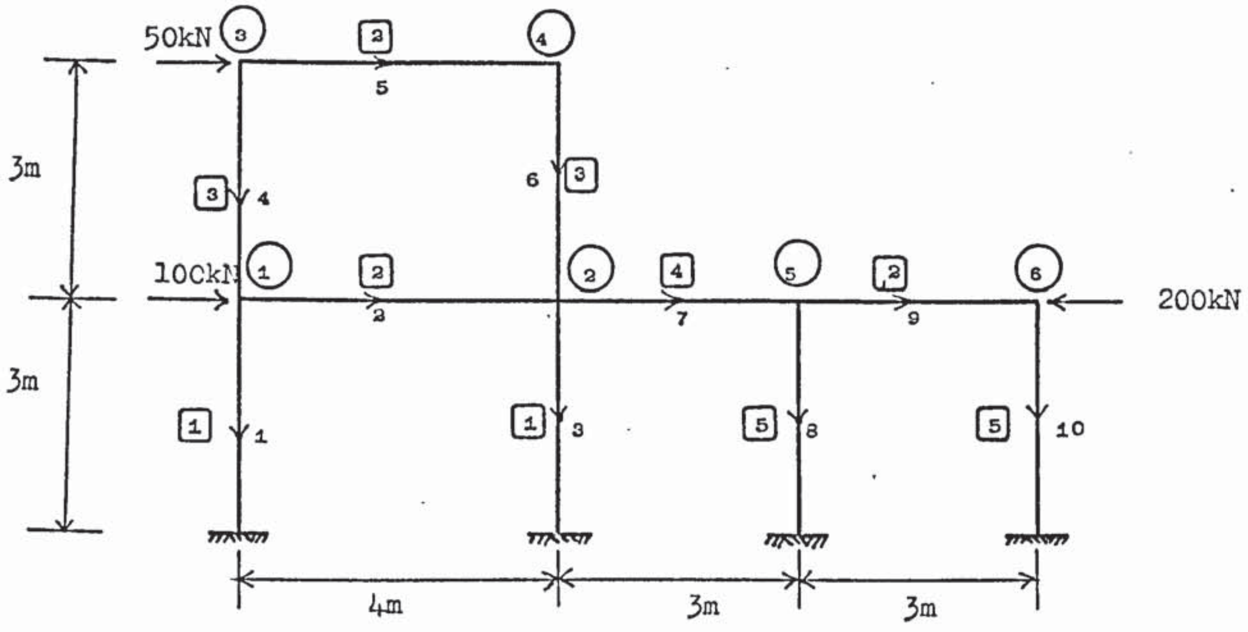
MEMBER NO	AXIAL FORCE (kN)	SHEAR FORCE (kN)	FIRST END MOMENTS (kN mm)	SECOND END MOMENTS (kN mm)
1	43.5679	-106.1900	71239.1653	353520.903
2	-43.5679	-43.8099	-38399.461	213639.389
3	43.5679	- 6.1900	95999.234	-71239.165
4	-43.8099	43.5679	-95999.234	-121840.467
5	-43.5679	-43.8099	126840.467	48399.461

TABLE 5.4 MEMBER FORCES OF THE PORTAL FRAME OBTAINED BY THE MATRIX DISPLACEMENT METHOD.

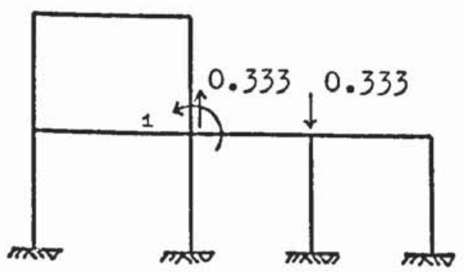
used to change the areas and the second moment of areas of the members that will eventually lead to a structure which is totally different from the original structure, in shape and in cross sectional properties of the members. As described in the previous section, the strcture shown in Figure 5.8(d) was also obtained at the fourth stage of the process of removal of member 2 which corresponds to the removal of the second moment of area of member 2. The same example was also solved by structural variation by firstly removing the area and secondly the second moment of area of member 2 whence exactly the same results as shown in Tables 5.1 and 5.4 were obtained. Hence, it may be concluded that in rigidly jointed structures the order in which the cross-sectional properties are removed or varied does not affect the results. However, it is apparent that the case where the area is removed first does not give the structure shown in 5.8(d) as an intermediate step.

5.4.2) EXAMPLE 2.

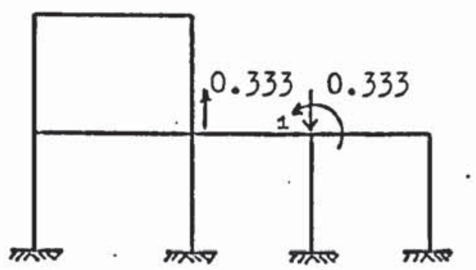
The structure of Figure 5.9(a) consists of a two storey frame and a portal frame connected by one member. The dimensions and member numbering are shown in the figure. The members 1 and 3 are in group 1 with areas of 2×10^4 mm^2 and second moment of areas of 4×10^8 mm^4, members 2, 5 and 9 are in group 2 with areas of 1×10^4 mm^2 and second moment of areas of 2×10^8 mm^4, members 4 and 6 are in group 3 with areas of 1.5×10^4 mm^2 and second moment of areas of 3×10^8 mm^4, member 7 is in group 4 with an area of 1.8×10^4 mm^2 and second moment of area of 3.4×10^8 mm^4 and members 8 and 10 are in group 5 with areas of 3×10^4 mm^2 and the second moment of areas of 6×10^8 mm^4. The structure is subject to a horizontal loading of 100, 50 and 200 kN acting joints 1, 3 and 6 respectively. The elastic modulus of the material is 207 kN/mm^2.



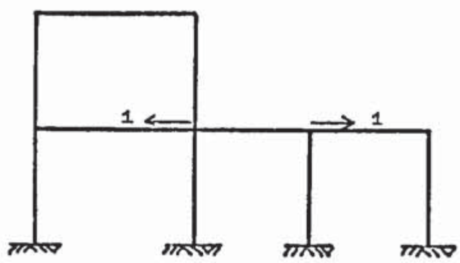
(a) STRUCTURE AND DIMENSIONS



FIRST END UNIT LOADING

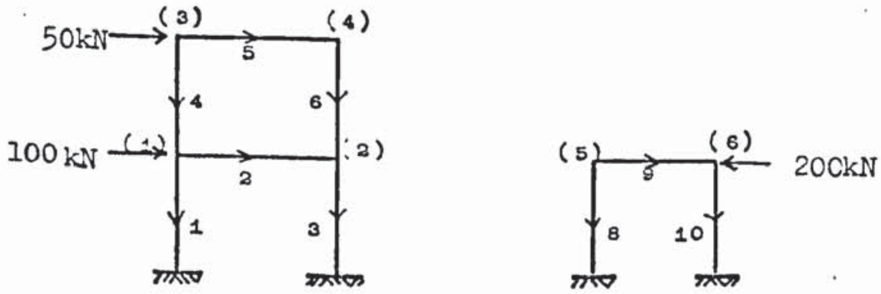


SECOND END UNIT LOADING



AXIAL UNIT LOADING

(b) UNIT LOADINGS



(c) DERIVATIVE STRUCTURES

FIGURE 5.9 A GROUND STRUCTURE.

JOINT NO	HORIZONTAL DEFL.(mm)	VERTICAL DEFL.(mm)	ROTATION (rad)
1	5.048	0.07511	-0.001994
2	5.000	-0.07516	-0.001977
3	12.041	0.1168	-0.001454
4	11.9449	-0.1168	-0.001448
5	- 3.573	-0.0316	0.001194
6	- 3.7159	0.0321	0.001253

TABLE 5.5 DEFLEXIONS OF THE FRAMES

MEMBER NO	AXIAL FORCE kN	SHEAR FORCE kN	FIRST END MOMENTS kN mm.	SECOND END MOMENTS kN mm.
1	103.7746	-75.3379	57789.990	168223.865
2	-24.6896	60.6490	-121499.960	-121094.941
3	-103.7340	-74.6584	57308.667	166666.568
4	43.1255	-50.027	86372.621	63710.062
5	-49.9724	43.1255	36844.382	-86129.533
6	-43.1255	-49.9724	86129.529	63787.841
8	-66.3133	98.5891	-98563.893	-197205.320
9	-98.5933	-66.2725	98563.794	100254.014
10	66.2725	101.4066	-100254.003	-203966.015

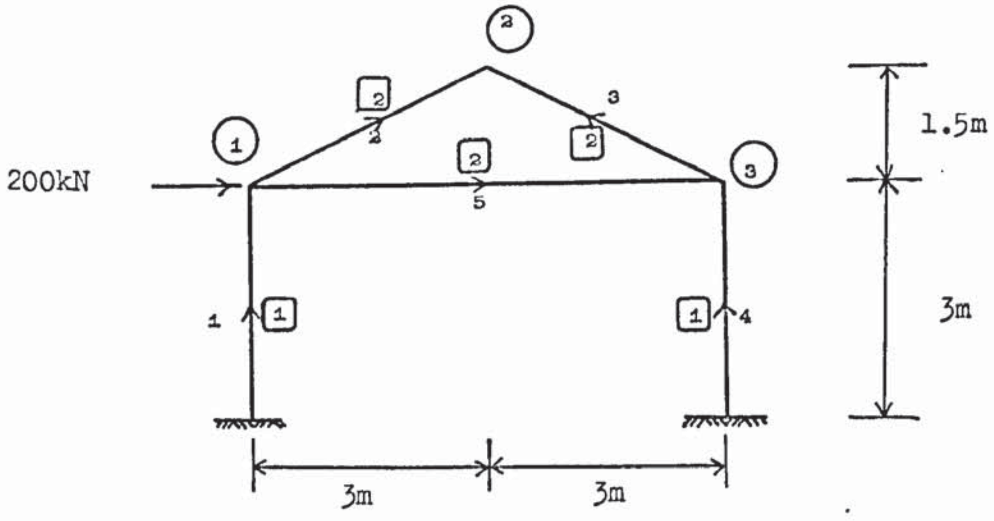
TABLE 5.6 MEMBER FORCES OF THE FRAMES.

This structure was also analysed by the matrix displacement method considering the four load cases, first of which is the external loading and the following three of which are the unit loadings. Then using the theorems of structural variations member 7 was removed, which yielded two different structures as shown in Figure 5.9(c). These were a two storey frame and a portal frame. The deflexions and member forces obtained by the formulae of structural variations for these two frames are given together in Tables 5.5 and 5.6. The numbering of the joints and members are shown in Figure 5.9(c). These two frames were also analysed separately and identical deflexions and member forces were computed. It was noticed that separate analysis of these two frames took 131 seconds of computer time while the results obtained by structural variation took 13 seconds. This verifies that in structures where some of the members are varied or totally removed, computation of the new deflexions and member forces by structural variations saves computer time. Hence, in such problems the use of structural variation is preferable.

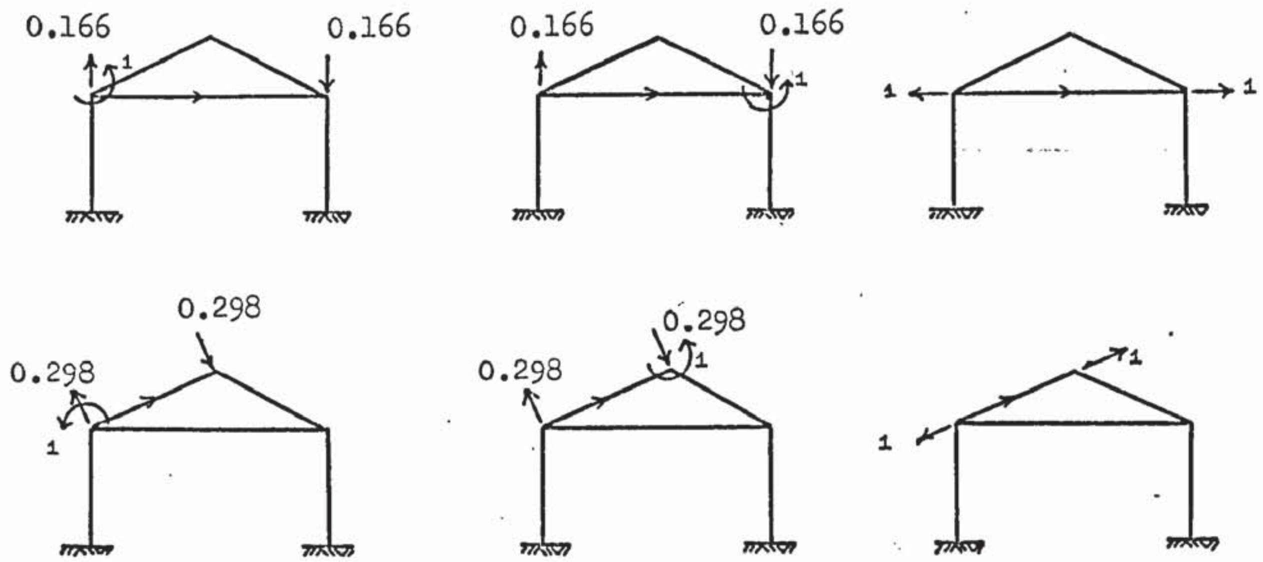
5.4.3) <u>EXAMPLE 3.</u>

The rigidly jointed structure shown in Figure 5.10(a) is subject to a horizontal load of 200 kN acting at joint 1. Dimensions and member numbering are shown in the figure. The elastic modulus of the material is 207 kN/mm^2. The members are divided into two groups, members 1 and 4 are of the same section and in group 1 with areas of 1×10^4 mm^2 and second moment of areas of 1×10^8 mm^4 and members 2,3 and 5 are of the same section belonging to group 2 with areas of 0.5×10^4 mm^2 and second moment of areas of 0.6×10^8 mm^4.

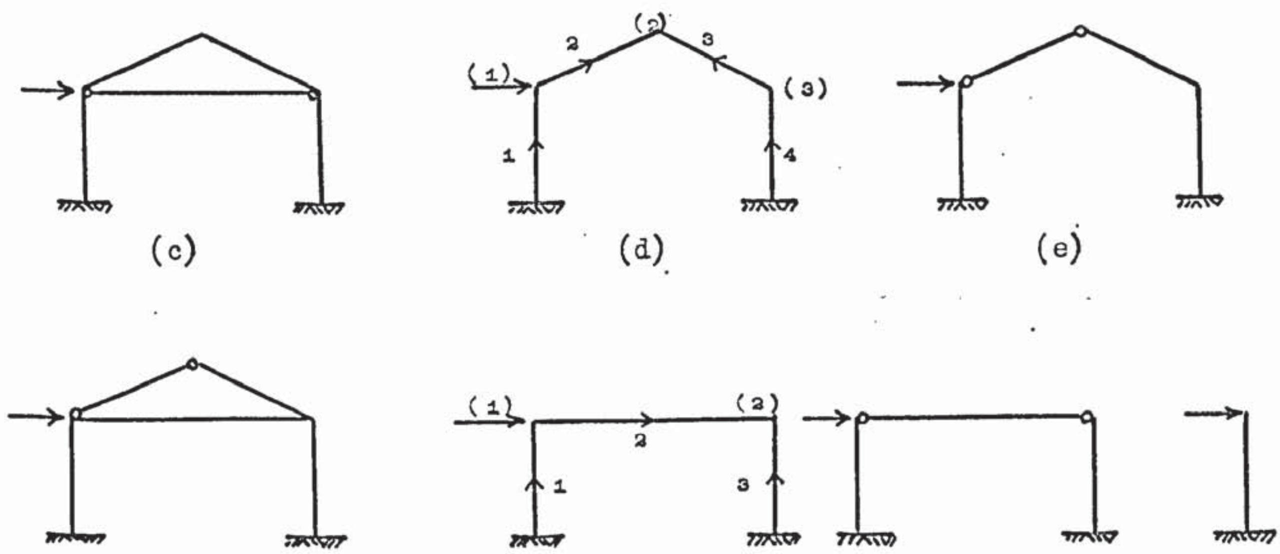
The frame was analysed to obtain the member forces and deflexions under the external force and the further 6 load cases



(a) STRUCTURE AND DIMENSIONS



(b) UNIT LOADINGS



(c) (d) (e)

(f) (g) (h) (i)

FIGURE 5.10 GROUND STRUCTURE AND DERIVATIVES.

JOINT NO	HORIZONTAL DEFL.(mm)	VERTICAL DEFL.(mm)	ROTATION (rad.)
1	26.956	0.0446	-0.00832
2	23.455	6.713	0.00413
3	19.81	-0.0446	-0.00837

TABLE 5.7 DEFLEXIONS OF THE PITCHED ROOF FRAME

MEMBER NO	AXIAL FORCE kN	SHEAR FORCE kN	FIRST END MOMENTS kN mm	SECOND END MOMENTS kN mm
1	30.743	-133.2166	257217.442	142432.327
2	-45.9532	57.4256	-142432.326	-50179.296
3	-73.5125	-2.3072	-42440.681	50179.296
4	-30.7432	-66.7834	157909.555	42440.682

TABLE 5.8 MEMBER FORCES IN THE PITCHED ROOF FRAME

JOINT NO	HORIZONTAL DEFL.(mm)	VERTICAL DEFL.(mm)	ROTATION (rad.)
1	22.792	0.0467	-0.007885
2	22.217	-0.0467	-0.007625

TABLE 5.9 DEFLEXIONS OF THE PORTAL FRAME

MEMBER NO	AXIAL FORCE kN	SHEAR FORCE kN	FIRST END MOMENT kN mm	SECOND END MOMENT kN mm
1	32.2236	-100.778	205624.86	96708.82
2	-99.3129	32.0492	-96708.82	-95586.55
3	-32.2236	- 99.222	201420.927	95586.55

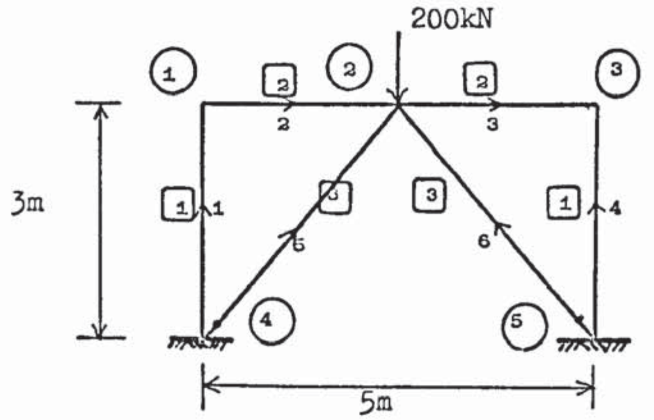
TABLE 5.10 MEMBER FORCES IN THE PORTAL FRAME.

shown in Figure 5.10(b). It then becomes possible, by employing the theorems of structural variations to predict the behaviour of at least 7 structures which are shown in Figure 5.10(c,d,e,f,...i). The 7 structures themselves may have an infinite combination of sectional properties. The removal of the second moment of area of member 5 yields the structure shown in Figure 5.10(c). Further, if the area of this member is also removed, the behaviour of the pitched roof frame is obtained. The results are given in Tables 5.7 and 5.8. The structure of Figure 5.10(e) is obtained by reducing the second moment of area of member 2 to zero in the pitched roof frame of Figure 5.10(d). In the case where member 2 is removed completely in the structure of Figure 5.10(a) the portal frame of Figure 5.10(f) is obtained. The results are shown in Tables 5.9 and 5.10 respectively. Further, removal of member 5 yields the cantilever shown in Figure 5.10(i).

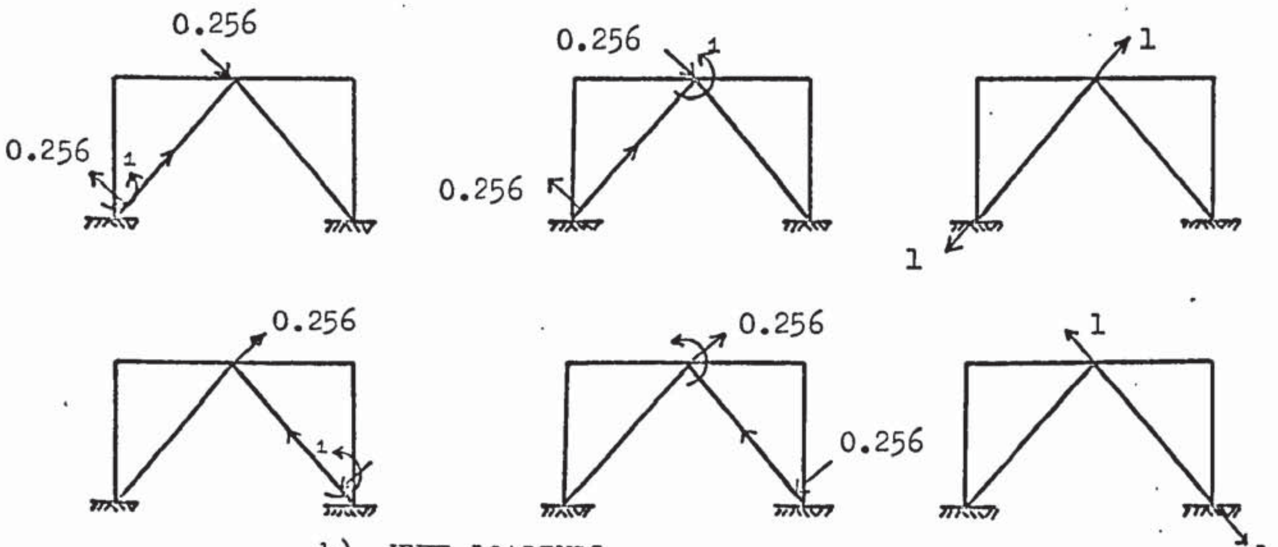
5.4.4) EXAMPLE 4.

The structure whose dimensions and member numbering are shown in Figure 5.11(a) is subject to a vertical load of 200 kN acting at joint 2. This structure consists of 3 groups. Members 1 and 4 in group 1 with areas of 1.6×10^4 mm^2 and second moment of areas of 3×10^8 mm^4, members 2 and 3 are in group 2 with areas of 0.8×10^4 mm^2 and second moment of areas of 1.5×10^8 mm^4 and inclined members 5 and 6 are in group 3 with areas of 0.5×10^4 mm^2 and the second moment of areas of 0.5×10^8 mm^4. The elastic modulus of the material is 207 kN/mm^2.

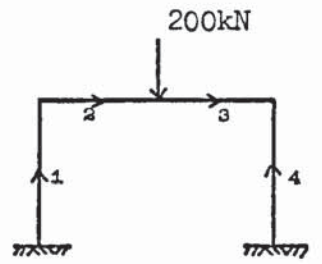
As seen from the figure the vertical displacement of joint 2 will be small due to the fact that joint 2 is supported by the inclined members. Removal of these members leads to the portal frame shown in Figure 5.11(c) where the vertical displacement of joint 2 is relatively large. The structure of Figure 5.11(a) was



a) STRUCTURE AND DIMENSIONS



b) UNIT LOADINGS



c) DERIVATIVE STRUCTURE

FIGURE 5.11 GROUND STRUCTURE AND DERIVATIVE.

analysed by the matrix displacement method under the external loads and the load cases shown in Figure 5.11(b). It can be seen from Figure 5.11(a) that the inclined members are connected to the supports. In such cases it becomes necessary to consider an additional joint in the member which is very near to the support in order to apply the first or second end unit loadings. In this way the generality of the theorems is preserved. Insertion of these additional joints generates a new small member with one end supported. The length of these small members can be taken as $0.01L \sim 0.001L$ where L is the length of the main member which is connected to a support. Now, it becomes possible by the first two theorems of structural variation to remove the inclined members 5 and 6 and obtain the vertical deflexion of joint 2 of the portal frame shown in Figure 5.11(c). This was carried out and it was found that the vertical displacement of joint 2 was 0.5718 mm before the removal of the inclined members and it increased to 5.967 mm after their removal. In the case where this value of vertical displacement is not allowed and permissible displacement of the joint 2 is given as 1 mm, it becomes necessary to apply the third theorem of structural variation. Equation 5.35 can be used for this purpose:

$$x_2^* = \frac{1}{1+\alpha} x_2$$

it follows that

$$\alpha = \frac{x_2 - x_2^*}{x_2^*}$$

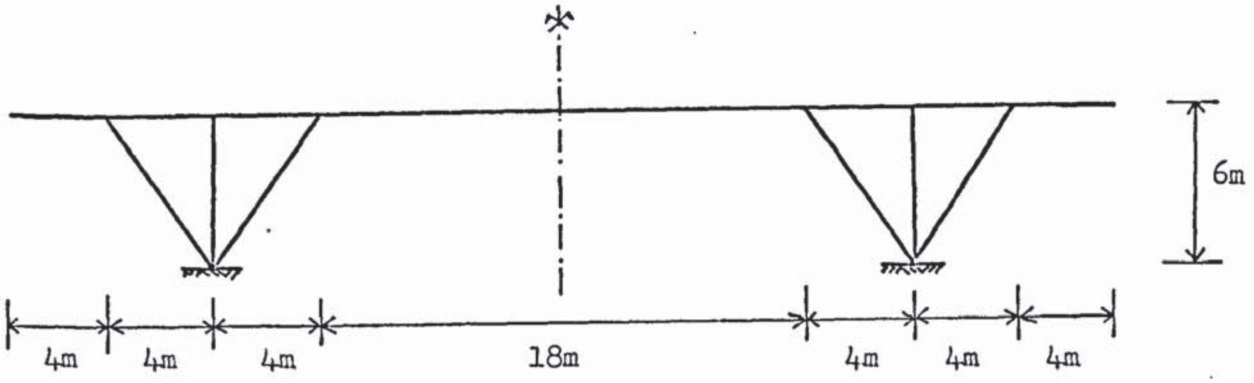
substituting $x_2^* = 1$ mm and $x_2 = 5.967$ mm, α is obtained to be 4.967. Hence, if the vertical deflexion of joint 2 in the structure shown in Figure 5.11(a) should not exceed 1 mm after removing the inclined members 5 and 6, it is necessary to increase proportionally the areas and the second moment of areas of the members to 5.967 times their initial value. It is obvious that this will increase the weight of the frame considerably. This example shows that the theorems of the structural variation can be used to examine the influence of each member in the structure on the weight of the structure which is the case in the topological design of structures.

In all the examples considered here member removal has been carried out. As explained in the previous sections in the case where a member is varied, α or β has a value which is different from -1 and the process remains the same. However, in the following example member variations are considered as well as member removals.

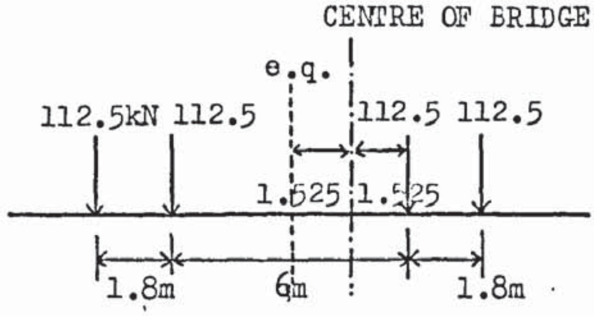
5.5) APPLICATION IN STRUCTURAL DESIGN.

In structural engineering, the most important feature of a structure is considered to be its shape. Hitherto this has been decided upon by a designer where experience has been the main factor. The theorems of structural variations may be employed to achieve this aim. They may be used in conjunction with the topological design of structures where a ground structure is developed by joining the nodes to each other which are arranged to cover the feasible space. Then, insignificant members and joints are removed to obtain the actual shape of the structure. On the other hand, even without considering this method, it may be possible to find the optimum shape of a structure only by means of these theorems. This is shown in the following example.

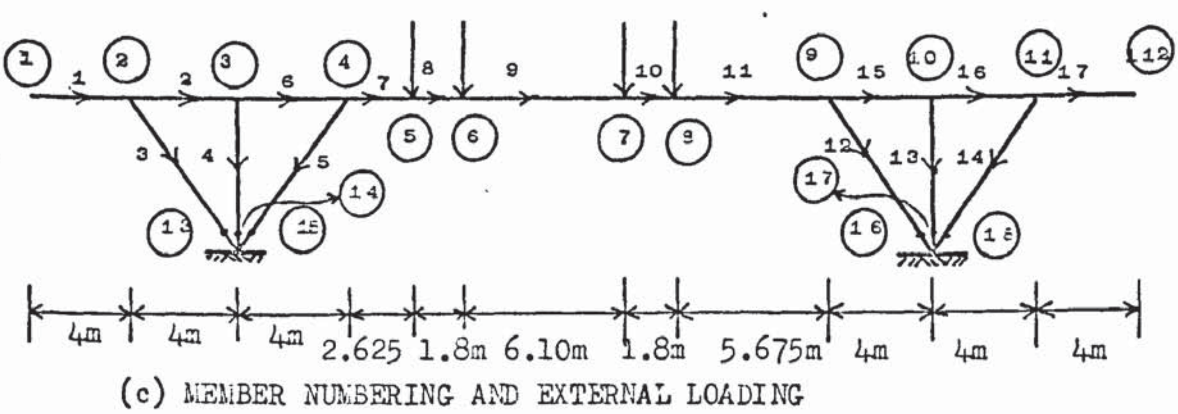
The structure of Figure 5.12(a) is a practical frame commonly used in structural engineering as a highway bridge. This bridge is subject to the vehicle loads shown in Figure 5.12(b) which is specified by B.S.153 as HB abnormal loading. Dimensions and member numbering are given in Figure 5.12(c). The Young's modulus of the material is 207 kN/mm^2. The bridge consists of 7 groups. The beam is in group 1, while the 6 columns belong to different groups. Derivatives of this bridge are shown in Figure 5.13(a,b,c,d,e,h) each of which has a different shape but may be used for the same purpose. 5 universal sections shown in Table 5.11 are considered to be available which can be adopted for each of the groups in these bridges.



(a) A HIGHWAY BRIDGE AND DIMENSIONS

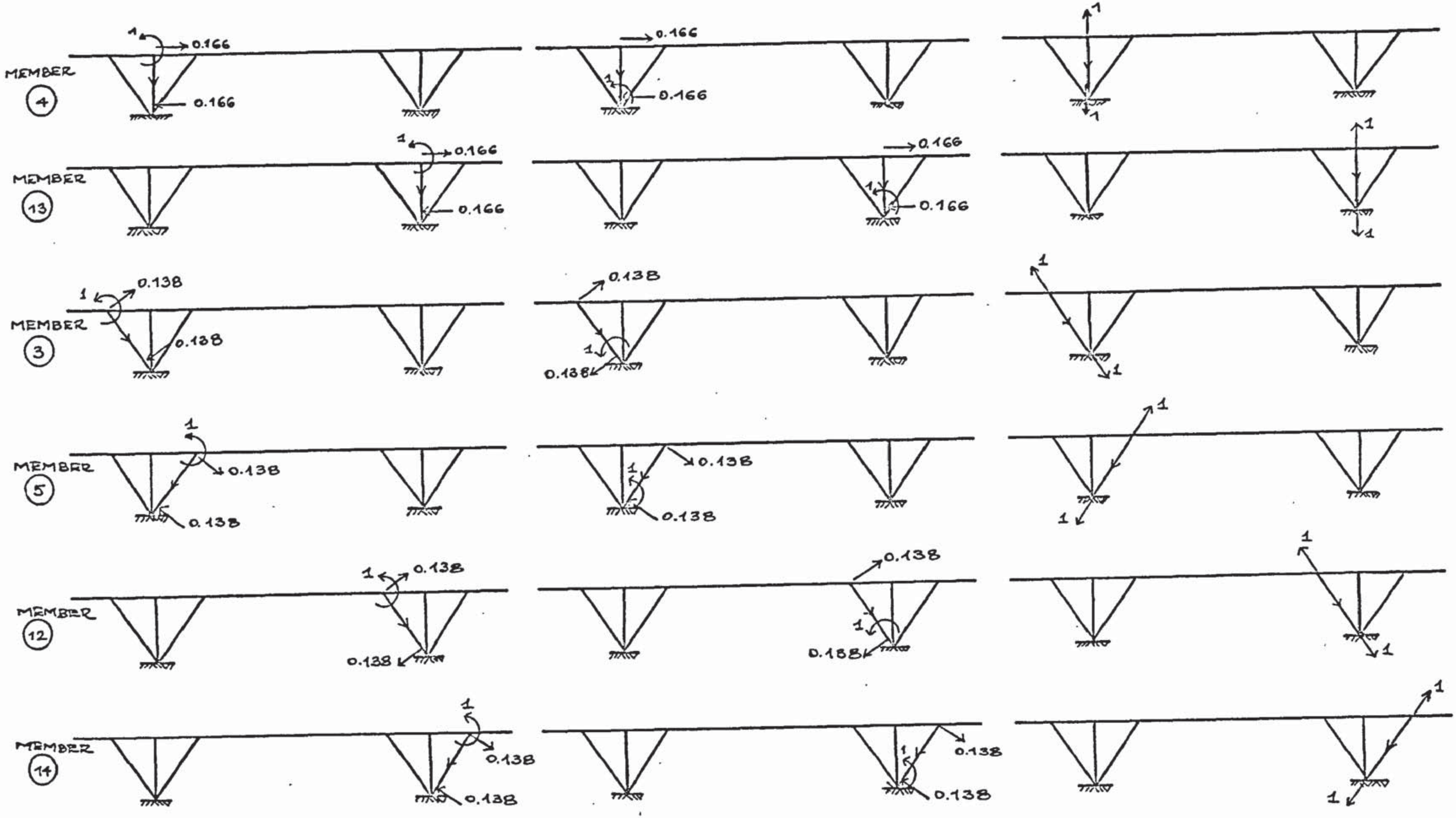


(b) HB ABNORMAL LOADING FROM B.S.153



(c) MEMBER NUMBERING AND EXTERNAL LOADING

FIGURE 5.12: A HIGHWAY BRIDGE.



(d) UNIT LOADINGS

FIGURE 5.12. A HIGHWAY BRIDGE AND LOADINGS

In the case where minimum material is chosen to be the objective while the limitations on deflexions are imposed by B.S.449, and the combined stresses in the members should not exceed 0.165 kN/mm^2, the question of selecting the optimum design out of these 7 bridges becomes important and cannot be carried out by experience. It is apparent that there are 5 sections which can be adopted for the beam and each of the columns. Analysis of all the bridges considering all possible combinations of these 5 sections would require a tremendous amount of computing time. Instead, to achieve this objective the theorems of structural variation may be employed. In order to do this, the bridge of Figure 5.12(a) was analysed once with the beam and the columns having the lowest available sections of Table 5.11. The load matrix consisted of the external loads and the unit loads shown in Figure 5.12(d). The theorems of structural variation were then used for the beam on each of the sections of Table 5.11. For convenience of programming the column sections were kept intact. The member forces and deflexions obtained for the bridge with 5 different beam sections were stored on file in the computer. The program described in the previous section was then utilised to obtain the member forces and deflexions in the derivative bridges of Figure 5.13. Further, each column of these bridges was varied considering 5 available sections in turn while keeping the sections of other columns the same. For example, the first bridge of Figure 5.13 can be obtained by removing columns 3 and 13 of the parent bridge. Then each column remaining can have 5 different sections. It can be seen that there are 625 member variations involving the columns of this particular bridge. For each of these variations, the deflexions and the stresses were checked against their imposed limits. In this way it became possible to find out the feasible set of sections which gave the bridge minimum weight. The same procedure was applied to all the other bridges. The results obtained are illustrated in

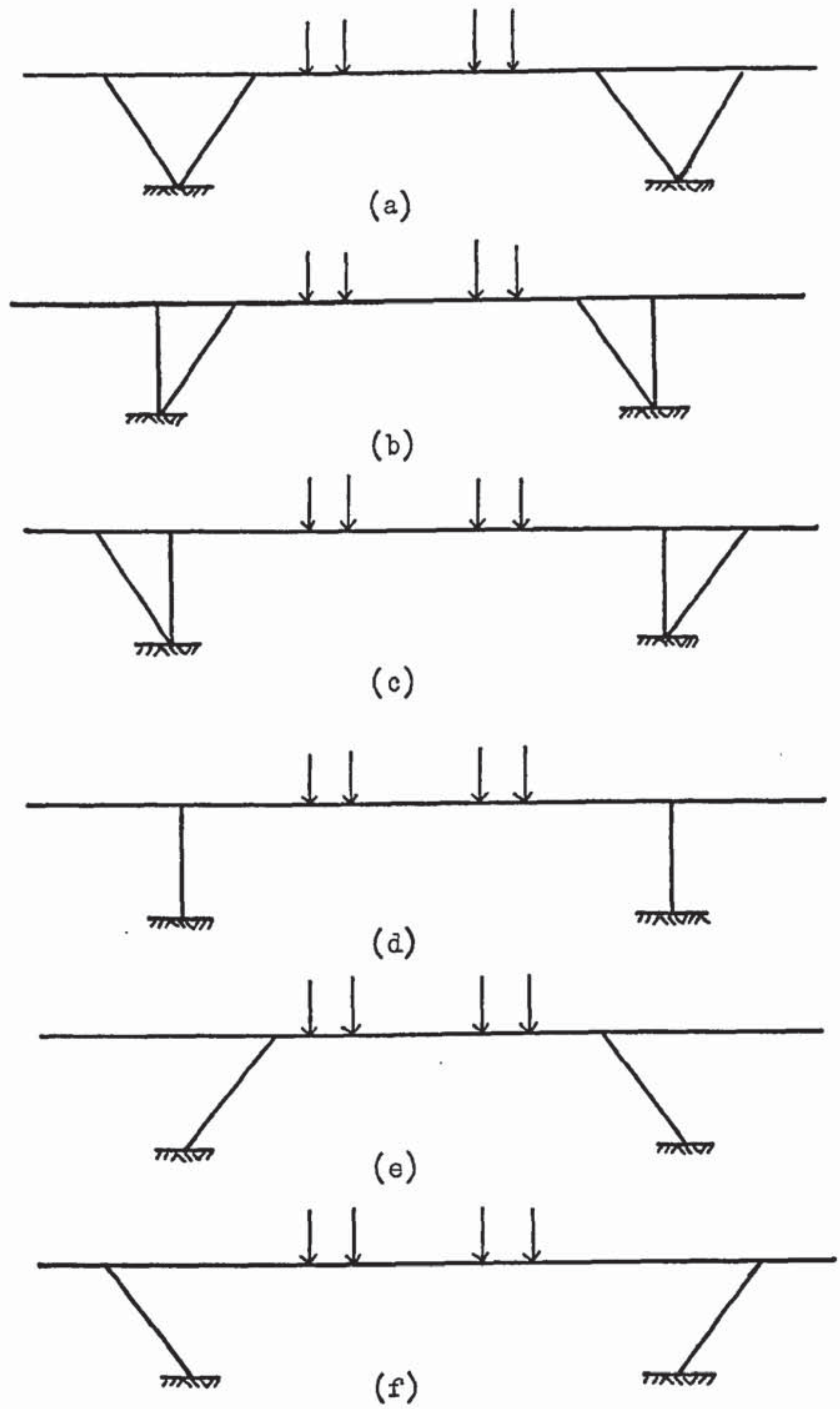


FIGURE 5.13 DERIVATIVE BRIDGES.

SECTION	AREA (cm^2)	SECOND MOMENT OF AREA (cm^4)	SECTION MODULUS (cm^3)	WEIGHT (kg)
305x127	47.4	6142.0	470.3	37
457x152	34.9	28731.0	1404.0	74
533x210	155.6	68719.0	2794.0	122
762x267	220.2	189341.0	5374.0	173
914x305	322.5	406504.0	9490.0	253

TABLE 5.11 UNIVERSAL BEAMS.

Table 5.12. For comparison, each of the 7 bridges were also designed by the method described in Chapter 2 where a continuous set of sections was considered to be available. The optimum areas for each group of members in the bridges are shown in Table 5.13. In this case the weight of the frames were obtained by multiplying the lengths and the section areas of the members and adding them together for all the members in the bridge. For comparison the weights shown in Table 5.12 were also computed in this way. The optimum results of Table 5.13 were rounded off considering the discrete set of sections of Table 5.11. It was found that the bridge type which had minimum weight had changed. The optimum shapes obtained in three ways are shown in Table 5.14. It can be seen from the table that for this particular example the optimum shape obtained by the optimisation procedure, which considers a continuous set of sections, then rounding off to available sections comes out to be the same with the optimum shape obtained by means of structural variations. However it can also be seen that the optimum shape obtained by structural variations is lighter than the optimum shape obtained by rounding off to available sections. This is due to the fact that to round off to available sections does not necessarily yield the optimum discrete sets. This can be verified by comparing the optimum discrete sections for the bridges B and C shown in Table 5.14. Finally, it should be pointed out that it is the beam which dominates the minimum weight design of these bridges because of their long spans.

The facility for member removal can be used to advantage in carrying out an actual costing of a structure. This is considered now for the case of the bridge example. The present day prices in the steel bridge construction are approximately as follows:

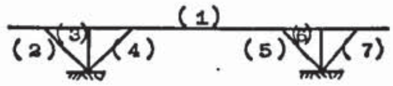
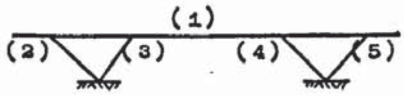
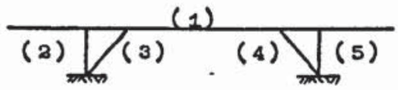
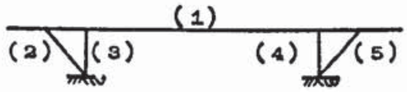
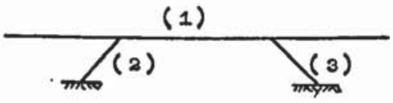
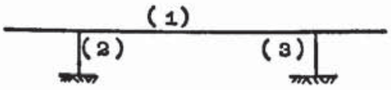
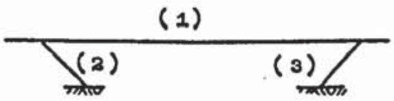
BRIDGE TYPES	GROUP NO	SECTIONS	WEIGHT $\times 10^8 mm^3$
	1 2 3 4 5 6 7	762x267 305x127 305x127 305x127 305x127 305x127 305x127	11.18442
	1 2 3 4 5	762x267 305x127 305x127 305x127 305x127	10.61562
	1 2 3 4 5	762x267 305x127 305x127 305x127 305x127	10.50081
	1 2 3 4 5	914x305 305x127 305x127 305x127 305x127	14.79741
	1 2 3	762x267 762x267 762x267	12.42417
	1 2 3	914x305 914x305 914x305	17.4150
	1 2 3	NONE OF THE AVAILABLE SECTIONS IS FEASIBLE	

TABLE 5.12 MINIMUM WEIGHTS OF THE BRIDGES OBTAINED BY STRUCTURAL VARIATION CONSIDERING DISCRETE SET OF SECTIONS SHOWN IN TABLE 5.11.

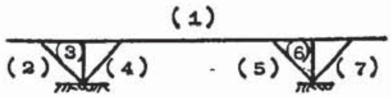
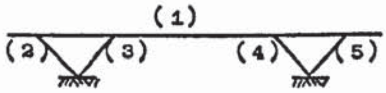
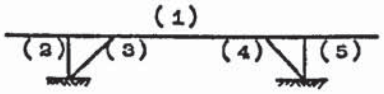
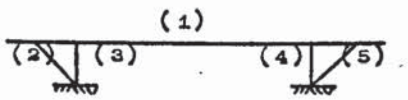
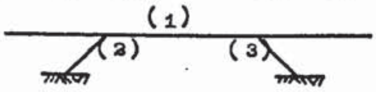
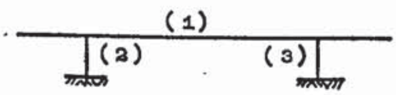
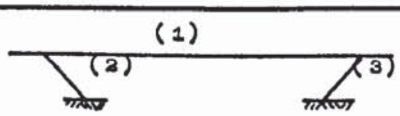
BRIDGE TYPES	GROUP NO	OPTIMUM AREAS $\times 10^2$ mm^2	WEIGHT$\times 10^8$mm^3
	1 2 3 4 5 6 7	215.640 5.240 2.724 38.203 60.609 7.259 3.437	9.89197
	1 2 3 4 5	213.268 5.924 42.896 89.024 11.309	10.03284
	1 2 3 4 5	216.95 7.916 41.695 62.396 10.462	9.97304
	1 2 3 4 5	312.369 37.610 61.567 63.504 35.164	14.39474
	1 2 3	212.956 228.481 213.84	12.13378
	1 2 3	305.999 304.939 282.054	16.37395
	1 2 3	409.699 400.274 349.133	22.61145

TABLE 5.13 MINIMUM WEIGHTS OF THE BRIDGES OBTAINED BY OPTIMIZATION PROCEDURE OF CHAPTER 2 WHICH CONSIDERS CONTINUOUS SET OF SECTION AVAILABLE TO CHOOSE FROM.

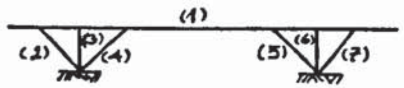
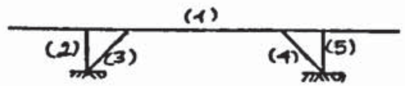
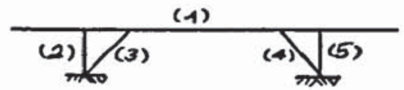
		OPTIMUM SHAPE	GROUP No.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^8 mm^3$
A	BY OPTIMISATION PROCEDURE OF CHAPTER 2 WHICH CONSIDERS CONTINUOUS SET OF SECTION TO CHOOSE FROM.		1	215.64	
			2	5.24	
			3	2.72	
			4	38.21	
			5	60.61	
			6	7.26	
			7	3.44	9.89197
B	BY ROUNDING OFF THE RESULTS OBTAINED IN (A) CONSIDERING DISCRETE SET OF SECTIONS OF TABLE 5.11.		1	220.20	
			2	47.40	
			3	47.40	
			4	94.50	
			5	47.40	10.84333
C	BY STRUCTURAL VARIATION CONSIDERING DISCRETE SET OF SECTIONS OF TABLE 5.11.		1	220.20	
			2	47.40	
			3	47.40	
			4	47.40	
			5	47.40	10.50081

TABLE 5.14. OPTIMUM SHAPES OBTAINED BY DIFFERENT METHODS

COST OF MATERIAL	:	£180 PER TON
COST OF TRANSPORTATION	:	£ 20 PER TON
COST OF ERECTION	:	£ 50 PER TON
COST OF WELDING + CUTTING + PAINTING AT SHOP	:	£150 PER TON

These figures are employed to generate arbitrary formulae for each and every concomitant of steel bridge construction. It can be assumed that the cost of transportation of a universal beam is a function of its width and depth. There is also a relationship between this cost and the length of a beam. Because of the discrete nature of transportation facilities, such a relationship is also discrete. This is due to the fact that the length of lorries available varies discretely. Therefore, the cost of transportation immediately increases, if the length of the beam exceeds a certain limit, even though its weight may be small. The same argument applies to the cost of erection. When the length of the beam exceeds a certain limit, a crane with a larger capacity may be required even though that crane cannot be used to full capacity because of weight reasons. In welding too it is considered that when the height of a beam is more than a certain limit, then the cost of its welding and cutting increases in a discrete manner. The cost of welding and cutting is taken as a function of the thickness, the width and the depth of a beam.

Since, the nature of these cost formulae is a subject of a separate research, the present work limits itself only to two different sets of arbitrary formulae. These are given in Tables 5.15 and 5.16. In the first set of formulae the first order terms of variables were considered while the second set of formulae contained variables of higher order. These formulae are the employed to express the total cost of the structure. Thus:

FIRST SET OF FORMULII		
TRANSPORTATION	$\ell \leq 8m$	$CT = 2.306a.\ell(B+D)$
	$\ell > 8m$	$CT = 4.6131.\ell(B+D)$
ERECTION	$\ell \leq 8m$	$CE = 4.8756.\ell(2B+D)$
	$\ell > 8m$	$CE = 7.1978.\ell(2B+D)$
WELDING + CUTTING	$D \leq 0.6$	$CW = 532.506.t^{\frac{1}{2}}(4B+2D)$
	$D > 0.6$	$CW = 336.718.t^{\frac{1}{2}}(4B+2D)$

TABLE 5.15

SECOND SET OF FORMULII		
TRANSPORTATION	$\ell \leq 6m$	$CT = 1.1124.\ell.D^2.B^{-1}$
	$\ell > 6m$	$CT = 2.2248.\ell.D^2.B^{-1}$
ERECTION	$\ell < 7m$	$CE = 3.5739.\ell.(2D^{\frac{1}{2}}-B^3)$
	$\ell \geq 7m$	$CE = 7.1478.\ell.(2D^{\frac{1}{2}}-B^3)$
WELDING + CUTTING	$D \leq 0.5m$	$CW = 5652.8.t^{\frac{1}{4}}.B^2.D^{-\frac{1}{2}}$
	$D > 0.5m$	$CW = 3914.09.t^{\frac{1}{4}}.B^2.D^{-\frac{1}{2}}$

TABLE 5.16

WHERE

ℓ is the LENGTH OF BEAM IN METRES
B IS THE WIDTH OF BEAM IN METRES
D IS THE DEPTH OF BEAM IN METRES
t IS THE THICKNESS OF BEAM IN METRES.

$$TC = \sum_{i=1}^{NOM} (CM+CT+CE+CW)$$

where

TC is the total cost of structure

CM is the cost of material of a member

CT is the cost of transportation of a member

CE is the cost of erection of a member

CW is the cost of welding and cutting of a member

NOM is the total number of members in the structure.

This expression is used as an objective function in the bridge examle considered. When a member is removed or its section is changed, the cost of the bridge is adjusted while satisfying the stress and deflexion requirements. The costs obtained, by using these two sets of formulii, for each bridge are given in Table 5.17. The various shapes are also shown in Table 5.18 with the different objectives chosen. This table clearly shows that the optimum shape obtained for minimum weight is not necessarily optimum from the cost point of view. When the cost is taken as an objective, the cost formula becomes important due to the fact that they influence the optimum shape. It is possible, however, to make use of the theorems of structural variation to extend the design problem to that of cost optimisation. This indeed can be taken as a topic for future research.

5.6) CONCLUSIONS.

It has been shown that the theorems of structural variations, which are proved in this chapter can be used to predict the behaviour of a variety of derived rigidly jointed and pin jointed structures from the analysis of a general rigidly jointed structure. Hence, by means of these theorems it was confirmed that structures are closely related to each other. The significance of these theorems

BRIDGE TYPE	MIN.COST BY 1ST SET OF FOR.	MIN.COST BY 2ND SET OF FOR.
	£3574.34	£3927.23
	£3391.38	£3591.08
	£3487.84	£3730.93
	£4256.25	£4683.91
	£3332.70	£3651.28
	£4092.32	£4569.63

TABLE 5.17 MINIMUM COSTS OF BRIDGES

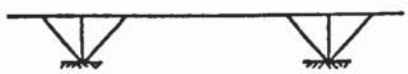
OBJECTIVES CHOSEN	OPTIMUM SHAPES
MINIMUM WEIGHT CONSIDERING CONTINUOUS SET OF SECTIONS	
MINIMUM WEIGHT CONSIDERING DISCRETE SET OF SECTIONS	
MINIMUM COST CONSIDERING FIRST SET OF FORMULII	
MINIMUM COST CONSIDERING SECOND SET OF FORMULII	

TABLE 5.18 OPTIMUM SHAPES FOR THE OBJECTIVES CHOSEN.

becomes apparent in the case where it is necessary to carry out a fresh analysis each time there is an alteration in the sectional properties of any member or in the shape of a structure.

It has been demonstrated that it is possible by means of these theorems to insert real hinges at the ends of any member in a rigidly jointed structure. This corresponds to removal of the second moment of area of that member. Further, it is possible to insert a hinge at any point of any member in the structure. This can be achieved by considering the point where a hinge is to be inserted as a small member. Then the second moment of area of that small member is removed. As a result it may be concluded that these theorems can be applied to elastic-plastic analysis of structures, which makes it possible to compute the moments throughout a structure when a plastic hinge occurs at any point without carrying out the solution of stiffness equations after adding a column and row corresponding to that hinge.

From the definition of the stiffness of a member it is evident that these theorems are equally true when the moduli of elasticity E of members vary. Hence, it becomes possible to compute the member forces and deflexions of a structure from the analysis of another structure made out of a different material.

Finally, it has been shown that these theorems can be used to find the order of significance of the member in a structure when minimum material or minimum cost is taken as an objective. It is this matter which is of prime importancein the design of structures in which the geometry and the topology are fundamental design parameters. This will be discussed in more detail in the next chapter.

C H A P T E R 6

SHAPE OPTIMISATION.

6.1) INTRODUCTION.

Some of the recent structural optimisation work has been devoted to obtaining the optimum design configuration. This has been partly due to the progress made in algorithms for optimum design of structures having fixed geometry. As concluded in Section 1.4 of Chapter 1, there are two methods which were followed and found practical in topological design of structures. The first is to select the initial geometry by considering the coordinates of the joints in a structure as design variables. During the topological design cycles, those members having zero sectional properties, and joints having zero coordinates are then deleted. The joints which are retained are moved until an optimum geometry is found. The second is to form a ground structure and commence the design from this structure. This is carried out by first arranging a network of nodes to cover the design space as suggested by Dorn et al.[97]. From these, a ground structure is produced by joining every node to every other node. The optimum shape of the structure is then obtained by removing those members and joints from the ground structure which do not have a significant function. This approach has proved to be powerful especially since the theorems of structural variations have been proposed by Majid and Elliott[101]. By means of these new structural principles it became possible to forecast the manner in which the numerous members of a ground structure or the subsequent derivative structures should be removed. Later, they described an approach[102] for the topological design of pin jointed structures. The method aims at selecting the shape of the lightest structure, while imposing permissible deflexions and stresses as design criteria.

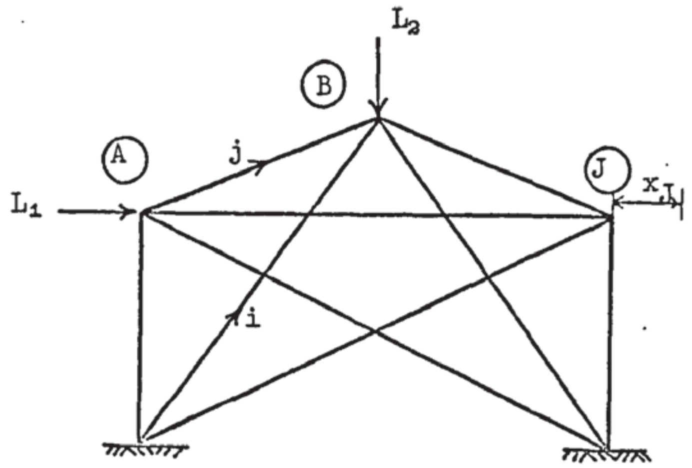
The method proposed in this thesis also starts with a ground structure and utilizes the design algorithm of Chapter 2 to obtain the optimum sectional properties of the members in this fixed shape structure. It then makes use of the theorems of structural variation given in detail in Chapter 5 of this thesis to decide upon a policy of removing members and/or joints, thus changing the shape of the structure. This topological design procedure aims at selecting the shape of the lightest rigidly jointed structure, while satisfying stiffness, stress and deflexion constraints.

6.2) THE DESIGN POLICY.

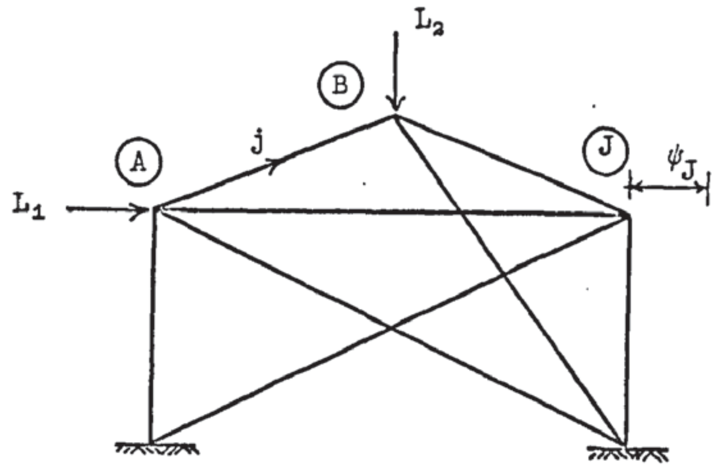
As shown in Chapter 5, it is possible to forecast the forces and deflexions throughout a rigidly jointed structure by the theorems of structural variaton, when one or more of its members are varied or totally removed. This fact can be employed to calculate, in advance, the material saving to be avhieved by altering the topology of a structure. This is carried out by first predicting the weight of the new feasible structure with each member removed in turn. It is obvious that the member which reduces the weight most significantly has to be removed first. In this way, members are arranged in a benefit vector. Hence, the order of members in the benefit vector gives the corresponding order of the savings in weight of the structure when they are removed. This vector is constructed by considering both stress and deflexion constraints. The cases where each is dominant are given in the following sections.

6.2.1) THE DOMINATING STRESS CONSTRAINTS.

A simple ground structure shown in Figure 6.1(a) is subject to a horizontal force L_1 and a vertical load L_2. Under this loading, the displacement in a general joint A is x_A and the forces in member j are $\underline{P} = \{p_j \; S_j \; M_{AB_j} \; M_{BA_j}\}$. If a member is removed from this structure, as shown in Figure 6.1(b), the member forces in member j change to become $\underline{P}_I = \{p_{j_I} \; S_{j_I} \; M_{AB_{jI}} \; M_{BA_{jI}}\}$ and the stress in j is given by



a) STRUCTURE AND LOADING



b) STRUCTURE AFTER REMOVING MEMBER i

FIGURE 6.1 A SIMPLE GROUND STRUCTURE.

$$\sigma_j = \frac{p_{jI}}{A_j} + \frac{M_{jI}}{I_j}.y \qquad 6.1$$

where p_{jI} is the axial force, M_{jI} is the largest of either end moment, σ_j is the combined stress in member j and A_j, I_j are the area and second moment of area of member j respectively. The allowable combined direct and bending stress in this member is σ_j^* and for fully stressed conditions its area and second moment of area may be changed by δA_j and δI_j respectively. Defining

$$\frac{\delta A_j}{A_j} = \frac{\delta I_j}{I_j} = \nu_j \qquad 6.2$$

the new area and second moment of area become:

$$A_{n_j} = (1+\nu_j)A_j$$
$$I_{n_j} = (1+\nu_j)I_j \qquad 6.3$$

and

$$\sigma_j^* = \frac{p_{jI}}{A_{n_j}} + \frac{M_{jI}}{I_{n_j}}.y \qquad 6.4$$

Equations 6.1, 6.3 and 6.4 give

$$\sigma_j^* = \frac{1}{1+\nu_j}\sigma_j$$

It follows that

$$\nu_j = \frac{\sigma_j}{\sigma_j^*} - 1 \qquad 6.5$$

Equation 6.5 represents the fully stressed condition in member j. When $\sigma_j = 0$ then $\nu_j = -1$ and $\delta A_j = -A_j$ and $\delta I_j = -I_j$ indicating that this member can be removed altogether. For $\sigma_j \leqslant \sigma_j^*$, ν_j is negative and so $\delta A_j, \delta I_j$ and the area and second moment of area of the member can be reduced by the same proportion.

When all the N members are altered, the total change δv_i in the volume of the structure due to the removal of member i is

Note: It is recognised that if the value of ν is different for each member, then further changes in the bending moment and force distributions will occur. To overcome this problem the value of ν is taken to be constant for all members in the frame and equal to the maximum numerical values.

$$\delta v_i = \sum_{j=1}^{N} \ell_j \, \delta A_j = \sum_{j=1}^{N} \ell_j \, A_j \, \nu_j$$

Using equation 6.5, it follows that

$$\delta v_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left\{ \ell_j \, A_j \left[\frac{\sigma_j}{\sigma_j *} - 1 \right] \right\} \qquad 6.5$$

where ℓ_j is the length of member j. The removal of i reduces the volume by $\ell_\ell \, A_\ell$ and the net saving becomes $\ell_i A_i - \delta v_i$. Hence, the new volume v_i of the structure is given by

$$v_i = v_o + \delta v_i - \ell_i A_i \qquad 6.7$$

where v_o is the original volume before removing member i and is given by

$$v_o = \sum_{\substack{j=1 \\ j \neq i}}^{N} \ell_j A_j + \ell_i A_i \qquad 6.8$$

Substituting equations 6.6 and 6.8 in equation 6.7, the new volume is obtained as

$$v_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \ell_j A_j \; \frac{\sigma_j}{\sigma_j *} \qquad 6.9$$

In the case where several members in the structure are grouped together, so that they can have the same sectional properties, equation 6.9 becomes

$$v_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \ell_j A_k \cdot \; \frac{\sigma_k}{\sigma_k *} \qquad 6.10$$

where σ_k is the most critical combined stress associated with area group k, A_k is the adopted area for group k and ℓ_j is the length of member j which belongs to group k.

When any member is removed in the structure, the member forces are computed as described in Section 5.2.3 of Chapter 5. In this process, it is possible that r_β may become infinite in which case both member forces and v_i become infinite also. This indicates that the removal of that member converts the structure or part of it into a mechanism. In such cases, care should be taken, so that such members may not be removed. This can be achieved by placing them at the end of the benefit vector.

Equation 6.9 makes it possible to forecast, in advance, the weight of the new structure when any member of the original structure is removed. Carrying this out for each member of the original structure, the possible savings in its weight due to removal of each member can be obtained. As a result, it becomes possible to forecast the order in which members should be removed. This is accomplished by scanning the members so that they may be placed in the benefit vector in the order of their decreasing benefit.

6.2.2) <u>THE DOMINATING DEFLECTION CONSTRAINTS.</u>

As shown in Section 5.2.4 of Chapter 5, when any member is removed from the structure the deflexions are obtained throughout together with the member forces using the same variation factors. Now, when member i is removed from the structure of Figure 6.1(a), the deflexion x_J at joint J becomes ψ_J. It is apparent that this deflexion should not exceed the allowable deflexion Δ_J at joint J. Hence, member i can only be removed provided that all the members are proportionally increased by a constant factor of γ to prevent ψ_J from exceeding Δ_J. The third theorem of structural variation states

$$\Delta_J = \frac{1}{1+\gamma_{Ji}} \psi_J \qquad 6.11$$

it follows that

$$\gamma_{Ji} = (\psi_J - \Delta_J)/\Delta_J \qquad 6.12$$

where the suffixes J and i indicate that the all round factor γ is calculated for the condition in which member i is being removed and the deflexion at J is becoming critical. A similar expression can be obtained for each deflexion constraint. The largest γ_{Ji} is denoted by γ_i^*. This gives the value by which all the areas and second moment of areas of the resulting structure should be increased so that member i can be removed without violating any of the deflexion constraints. This process is carried out for each member and the value of each γ^* is determined.

Now, it becomes possible to forecast the volume of the resulting structure. In the case where member m is removed, the predicted volume v_m of the new structure is given by

$$v_k = \sum_{i=1}^{N} \ell_i A_i (1+\gamma_m^*) - \ell_m A_m (1+\gamma_m^*) \qquad 6.13$$

This is obtained by adding the original volume $\Sigma\, \ell A$ and the increase in this volume $\Sigma\, \gamma \ell A$ and substracting the volume saved by removing member m. The equation 6.13 can be used to forecast the new volume of the structure due to the removal of each member.

6.2.3) THE PREPARATION OF THE BENEFIT VECTOR.

The two methods described in the previous two sections are combined using the "Mini-Max" principle for the preparation of a single overall benefit vector. This is carried out firstly, by computing the new volumes of the resulting structure utilizing equations 6.9 and 6.13 when its members are removed in turn. Secondly, the larger value of two volumes corresponding to removal of each member is chosen to ensure that weight is not saved by violating the feasibility. An inspection of these volumes enables the members in the benefit vector to be arranged in order. This is carried out in such a way that the first member in the list may be removed with the largest reduction in the volume of the structure. In this way all the members are graded in the benefit vector which only gives the order in which members may be removed from the structure so that it results in a feasible and lighter structure. However, in the case where the removal of each member does not result in a lighter structure, the one which generates the least increase in the volume of the structure takes the first row in the benefit vector. This does not introduce any error into the design procedure due to the fact that the benefit bector only forecasts the manner in which the members should be removed. The process of member removal is terminated as soon as this entails increases in the optimum overall weight of the structure. The preparation of the benefit vector is explained in detail by K.I.Majid[1], where an example is also given.

6.3) DESIGN PROCEDURE.

As shown in Figure 6.2, the design procedure is initiated by developing the ground structure. There are two ways of doing this. The first is to arrange a network of nodes, then join every node to every other node. The second is to combine a number of candidate structures where engineering experience can be utilized to advantage. Unless the grouping of members is an architectural necessity, each member is given a different group member. In this way, the existence of each member is also decided by the optimisation procedure, depending on its significant from the minimum weight point of view.

As shown in Figure 6.2, the procedure for the topological design of rigidly jointed structures employs two approaches which are described in this thesis. The first is the structural optimisation procedure of Chapter 2 which is used to obtain the optimum sectional properties for the structures with fixed shape. The second is the theorems of structural variations described in

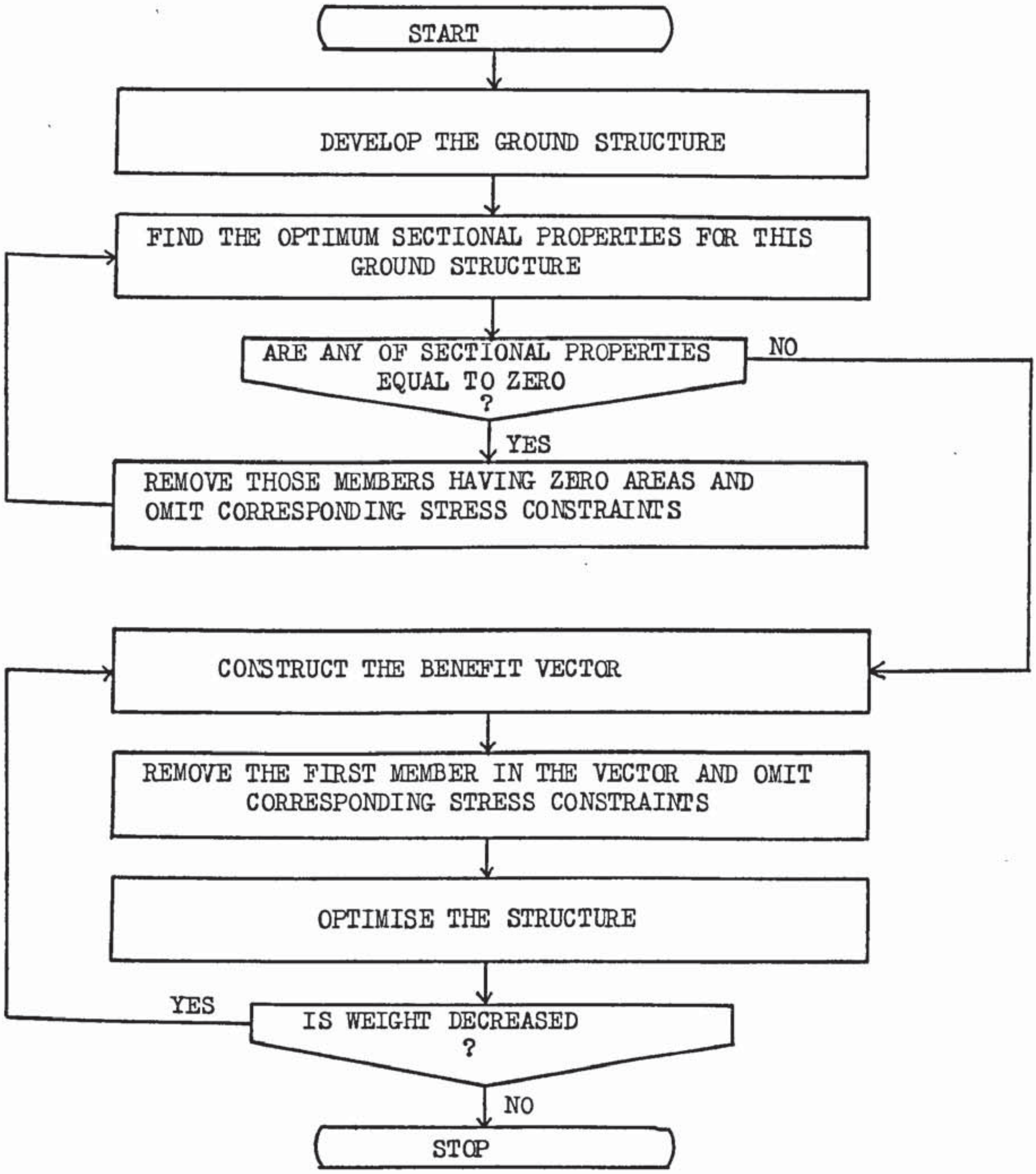


FIGURE 6.2 THE FLOW-DIAGRAM OF THE PROCEDURE FOR TOPOLOGICAL DESIGN OF STRUCTURES.

Chapter 5 which are utilized to construct the benefit vector.

After developing the ground structure, initial values are chosen for the sectional areas of the members. Since the structural optimisation procedure of Chapter 2 can operate from feasible or infeasible initial design points, the sectional properties of members can be set at their lower bounds which are given in safe load tables. Using this initial design point, the weight of the ground structure is minimised and the optimum areas for the sections of the members are obtained. It is possible that some of these areas may turn out to be very small and such members are deleted. Consequently, the corresponding design variables and stress constraints are altered. It may also be possible to remove some joints as a result of the removal of these members which are connected to them. Consequently the corresponding deflexion constraints are also reduced. It is apparent that after carrying out these removals, the optimum point is no longer feasible for the resulting structure which has a new topology. However, this point can be employed as an initial design point for the next topological cycle. This procedure is repeated until all the design variables have values different from zero. It can be seen that this stage does not correspond to the final design. It may be possible that the removal of some members in this structure generates a reduction in its overall weight. Hence, when this stage is reached, the members in the structure are arranged in the required order in the benefit vector. The less advantageous members are then removed, provided that this does not entail an increase in the weight of the structure. This process is continued until it becomes impossible to decrease the overall weight of the structure. In the final design several derived structures become available. Amongst these there may be a structure which is not the lightest but which is more economical from the overall construction point of view. Obviously, this latter structure will interest engineers the most.

6.4) DESIGN EXAMPLES WITHOUT CONSIDERING ARCHITECTURAL CONSTRAINTS.

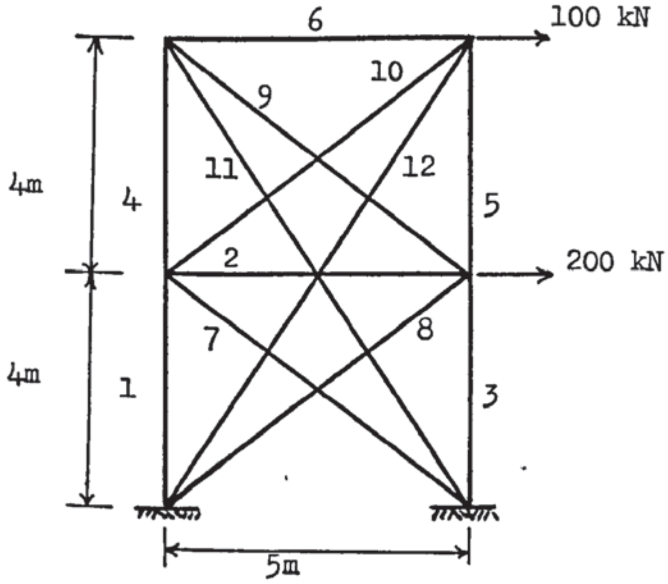
The proposed design procedure was applied to a number of examples to demonstrate its generality and efficiency. In all these examples the stiffness, stress and the deflexion constraints were considered. However, the configuration of a structure is not only affected by these constraints but also by functional and architectural constraints. This is due to the fact that they affect significantly the existence or the removal of members or joints, consequently the appearance of a structure. Architectural constraints are of secondary importance in the design of pin-jointed structures. This is due to the fact that such structures usually appear as a truss skeleton in bridges or roof structures where the main objective is to carry the external loads safely. In contrary to pin jointed structures these constraints are particularly important in the topological design of rigidly jointed structures. One of the reasons for this is that rigidly jointed structures are not only built to sustain the external loads safely but also to satisfy the service requirements adequately. In some cases these two purposes can contradict each other. For example, in a two storey frame the existence of the beam which separates the two stories is required from architectural point of view while this might not be necessary to carry the external loads safely.

Hence, the design examples considered were divided into two groups. In the first group architectural constraints were not considered and optimum shapes were obtained by only considering the stiffness, stress and deflexion constraints. As a result, the shape of the final designs were unusual from the architectural point of view. In the second group in addition to stiffness, stress and deflexion constraints, architectural requirements were also considered.

6.4.1) EXAMPLE 1.

The ground structure shown in Figure 6.3(a) was developed by linking the joints of a two storey frame to each other. This structure consisted of 12 members. These were not grouped together so that the existence of each one could also be decided by the optimisation procedure. The modulus of elasticity of the material was 207 kN/mm^2. The structure was subjected to two horizontal loads of 100 kN and 200 kN acting at the second and first storey respectively. The design requirements included deflexion and stress constraints. The limitations on deflexions were imposed by B.S.449 which restricted the sway of the frame to 12.3 mm at the first floor and 24.6 mm at the second floor. The vertical deflexions were limited to 13.8 mm. The combined axial and bending stresses in the members were restricted to 0.165 kN/mm^2.

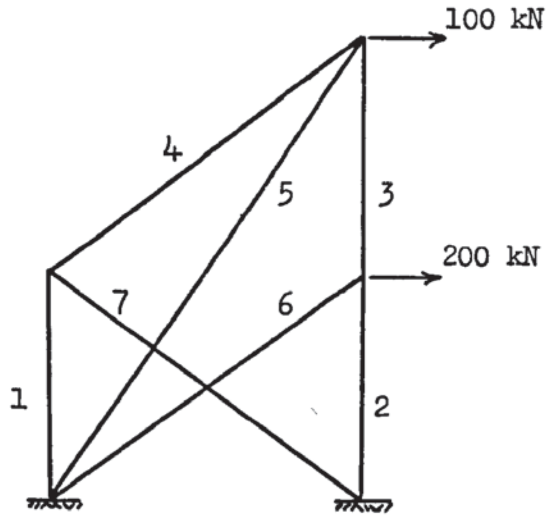
The design was initiated from the ground structure shown in Figure 6.3(a). The weight of this structure was minimised and the optimum sectional areas are given in Table 6.1(a). As can be seen in the table, members 2,4,6,9,11 have areas which are very near to zero. It can be seen that the values of the areas are of the same order as the convergence rate which was taken as 0.001. Since, such areas can be assumed to be zero, these members need not be included in the ground structure. Indeed, this is the approach adopted by Dobbs and Felton [99]. The structure obtained after removing such members is shown in Figure 6.3(b). The optimum sectional areas are also given in Table 6.1(b). As seen from the table none of the members has an area close to zero. Hence, it becomes necessary to forecast the member to be removed which will generate reduction in the weight of the structure. To achieve this, the benefit vector shown in Table 6.1(c) was constructed. As can be seen from this benefit vector removal of a member 5 is the most beneficial. This was carried out and the structure shown in Figure 6.3(c) was obtained. The optimum member



a) GROUND STRUCTURE

MEMBER NO.	AREAS mm^2	WEIGHT $\times 10^8 mm^3$
1	348.12	
2	0.003	
3	2243.03	
4	0.0076	
5	833.37	
6	0.0062	
7	277.12	
8	1669.98	
9	0.0018	
10	283.50	
11	0.0078	
12	816.85	0.35985

a) OPTIMUM MEMBER AREAS FOR THE GROUND STRUCTURE



b) THE STRUCTURE OBTAINED AFTER ONE TOPOLOGICAL ITERATION

MEMBER NO	AREAS mm^2	WEIGHT $\times 10^8 mm^3$
1	840.40	
2	1993.17	
3	644.61	
4	599.90	
5	339.05	
6	1662.62	
7	593.94	0.35402

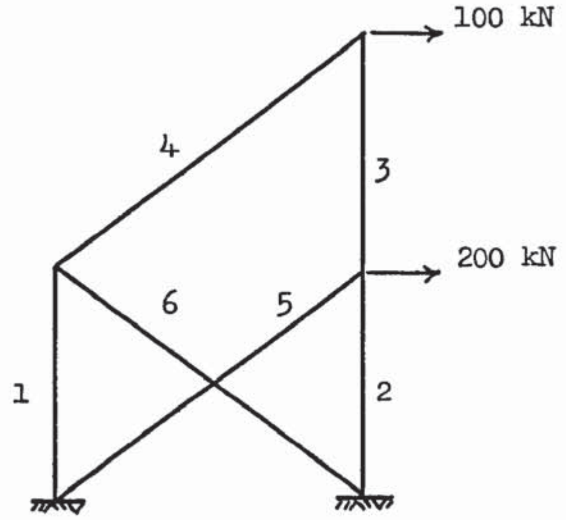
b) OPTIMUM MEMBER AREAS FOR THE STRUCTURE SHOWN IN FIG. 6.3(b)

MEMBER ORDER	5	4	7	3	1	6	2

c) BENEFIT VECTOR

FIGURE 6.3

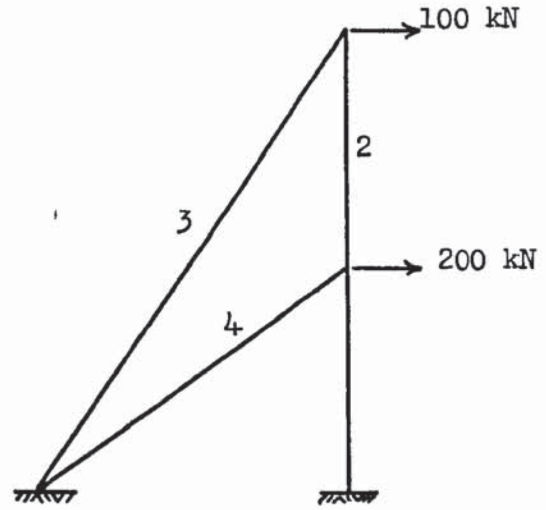
TABLE 6.1.



c) FINAL DESIGN

MEMBER NO.	AREAS mm^2	WEIGHT $\times 10^8 mm^3$
1	1164.80	
2	1823.30	
3	515.89	
4	800.00	
5	165.72	
6	809.60	0.34934

d) OPTIMUM MEMBER AREAS FOR THE STRUCTURE SHOWN IN FIGURE 6.3(c).



d) DERIVED STRUCTURE

MEMBER NO.	AREAS mm^2	WEIGHT $\times 10^8 mm^3$
1	2479.07	
2	1025.79	
3	1220.15	
4	1676.41	0.36260

e) OPTIMUM MEMBER AREAS FOR THE STRUCTURE SHOWN IN FIGURE 6.3(d).

FIGURE 6.3 DESIGN EXAMPLE 1

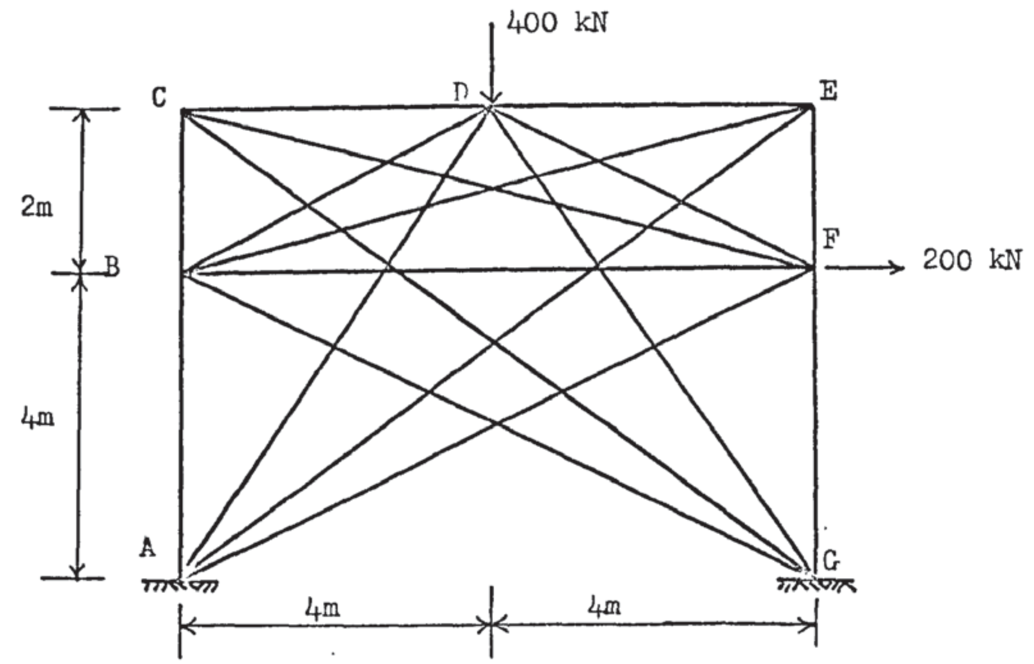
TABLE 6.1 OPTIMUM MEMBER AREAS FOR THE STRUCTURES SHOWN IN FIGURE 6.3.

areas for this structure are given in Table 6.1(d). It can be seen that by removing member 5, reduction was achieved in the weight of the structure, while satisfying the design limitations.

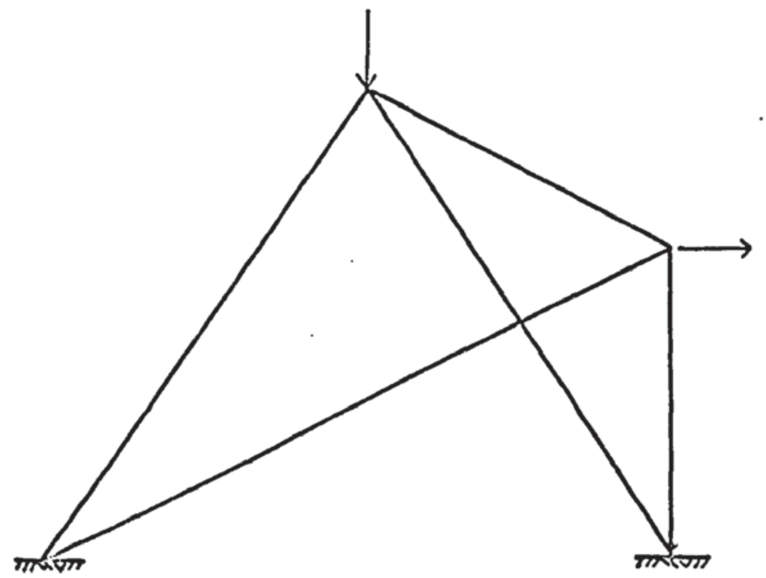
Although, this member removal procedure was also applied to the structure of Figure 6.3(c), no improvement was obtained in the weight of the structure. It was noticed that the stress constraints were dominant in the design problem and the final design was fully stressed. The minimum volume of this structure obtained was 0.34934×10^8 mm^3. As a further stage, the structure shown in Figure 6.3(d) was obtained by removing member 4 from the structure of Figure 6.3(b) where member 4 was the second member in the benefit vector belonging to this structure. It can be seen that although this structure is heavier than the final design shown in Figure 6.3(c), it is more economical from the overall construction point of view. This shows that the lightest structure is not necessarily the cheapest one. Furthermore, it can easily be concluded that the final shape of the structure depends on the oojective chosen. In the case where economy is taken as an objective, the benefit vector can be prepared in such a way that the first member in it gives the greatest reduction in the cost of the structure.

6.4.2) EXAMPLE 2.

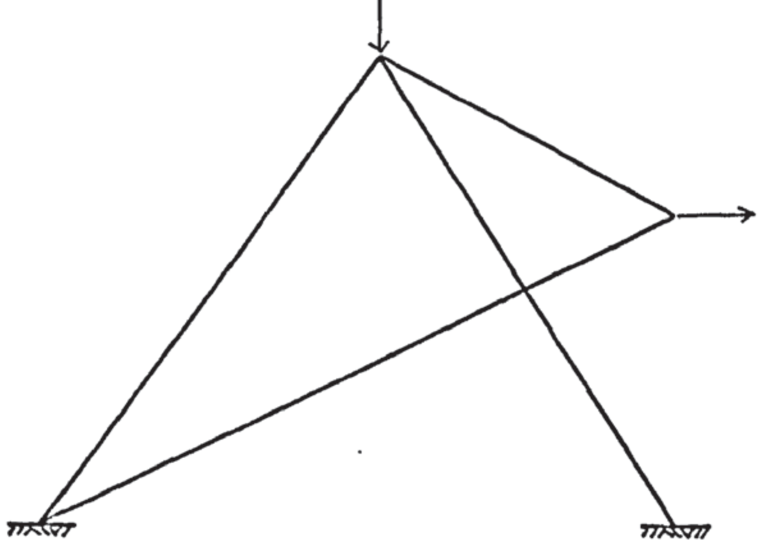
As a further example, the ground structure of Figure 6.4(a) was selected to obtain the optimum structure that sustains the external loads which are also shown in the figure. This rigidly jointed ground structure consists of 7 joints each of which is connected to every other joint by a member so that there is a total of 17 members. The vertical load is 400 kN acting at joint D and the horizontal load is 200 kN acting at joint F. The deflexions due to these loads were restricted by B.S.449 which imposed 12.3 mm on the horizontal deflexion of joints B and F and 18.46 mm on joints C,D



a) GROUND STRUCTURE



b) THE STRUCTURE OBTAINED AT FIRST ITERATION OF TOPOLOGICAL DESIGN $W = 0.40852 \times 10^{8}$ mm^3



c) FINAL DESIGN $W = 0.34336 \times 10^{8}$ mm^3

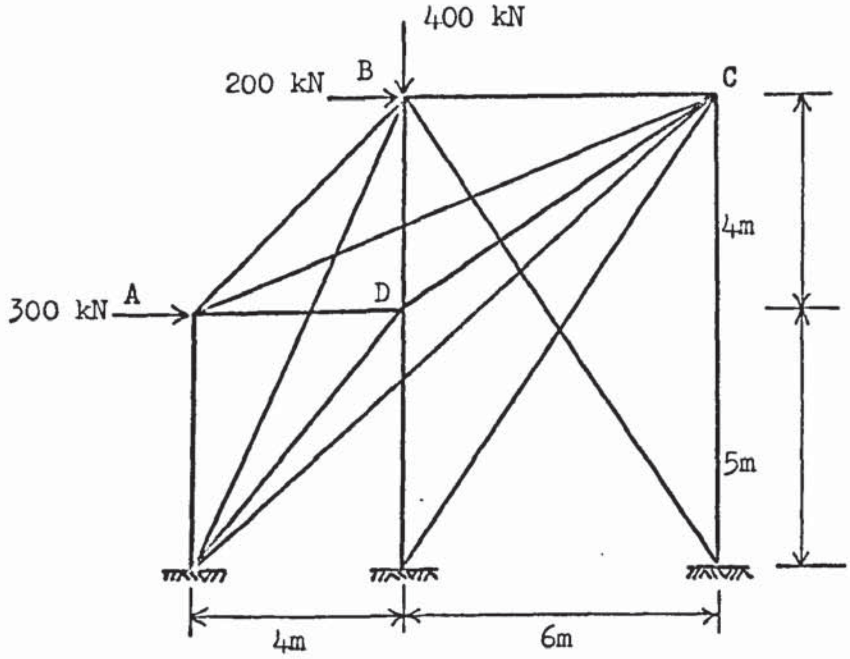
and E. The vertical deflexions throughout were limited to 22.22 mm. The modulus of elasticity of the material was 207 kN/mm^2. The combined axial and bending stresses in the members were restricted to 0.165 kN/mm^2.

The design was initiated from the ground structure of Figure 6.4(a). The final design was obtained after two topological iterations and is shown in Figure 6.4(c). This structure consists of 4 members and has the minimum volume of 0.34336×10^8 mm^3. The stress constraints were dominant and the final design was fully stressed. It can be seen that the design procedure is quite effective and it requires few iterations to reach the final design. The same example was also designed applying external moments at the same joints as the direct loads. It was noticed that the final shape of the structure did not change. However, the weight of the structure increased.

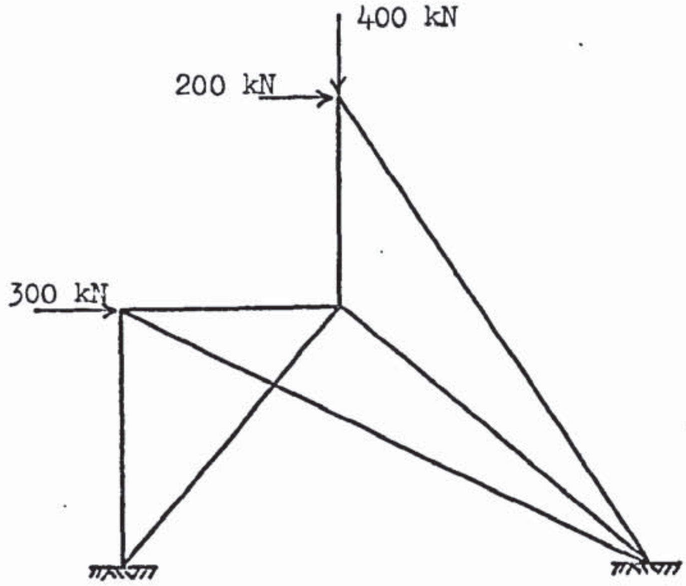
It can be seen that the final shape obtained is not usual. Furthermore, perhaps it is not accepted from the architectural point of view. However, as far as the minimum weight is concerned, this is the best shape which can be obtained from the ground structure to carry the external loads shown in Figure 6.4(a). It can be seen that the vertical load is carried by two inclined members which are in compression and the horizontal load is carried by the other 2 members which are in tension. It is known that the final shape depends upon the original shape of the ground structure[102].

6.4.3) EXAMPLE 3.

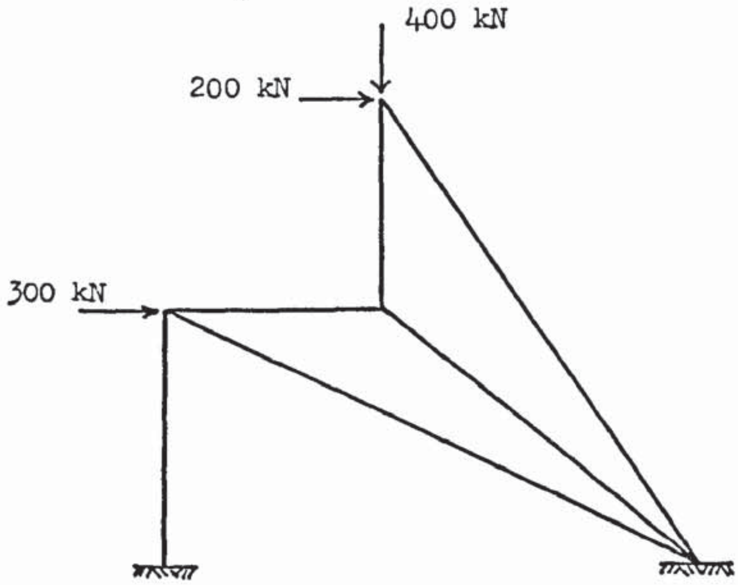
The ground structure shown in Figure 6.5(a) was developed in order to obtain a structure that has the optimum shape under the external loads which are also shown in the figure. The horizontal loads are 300 kN and 200 kN acting at joints A and B respectively while the vertical loads is 400 kN acting at joint B. The horizontal



a) GROUND STRUCTURE



b) STRUCTURE OBTAINED AT FIRST ITERATION OF TOPOLOGICAL DESIGN $W = 0.54676 \times 10^8\ mm^3$



c) FINAL DESIGN $W = 0.54641 \times 10^8\ mm^3$

FIGURE 6.5 DESIGN EXAMPLE 3

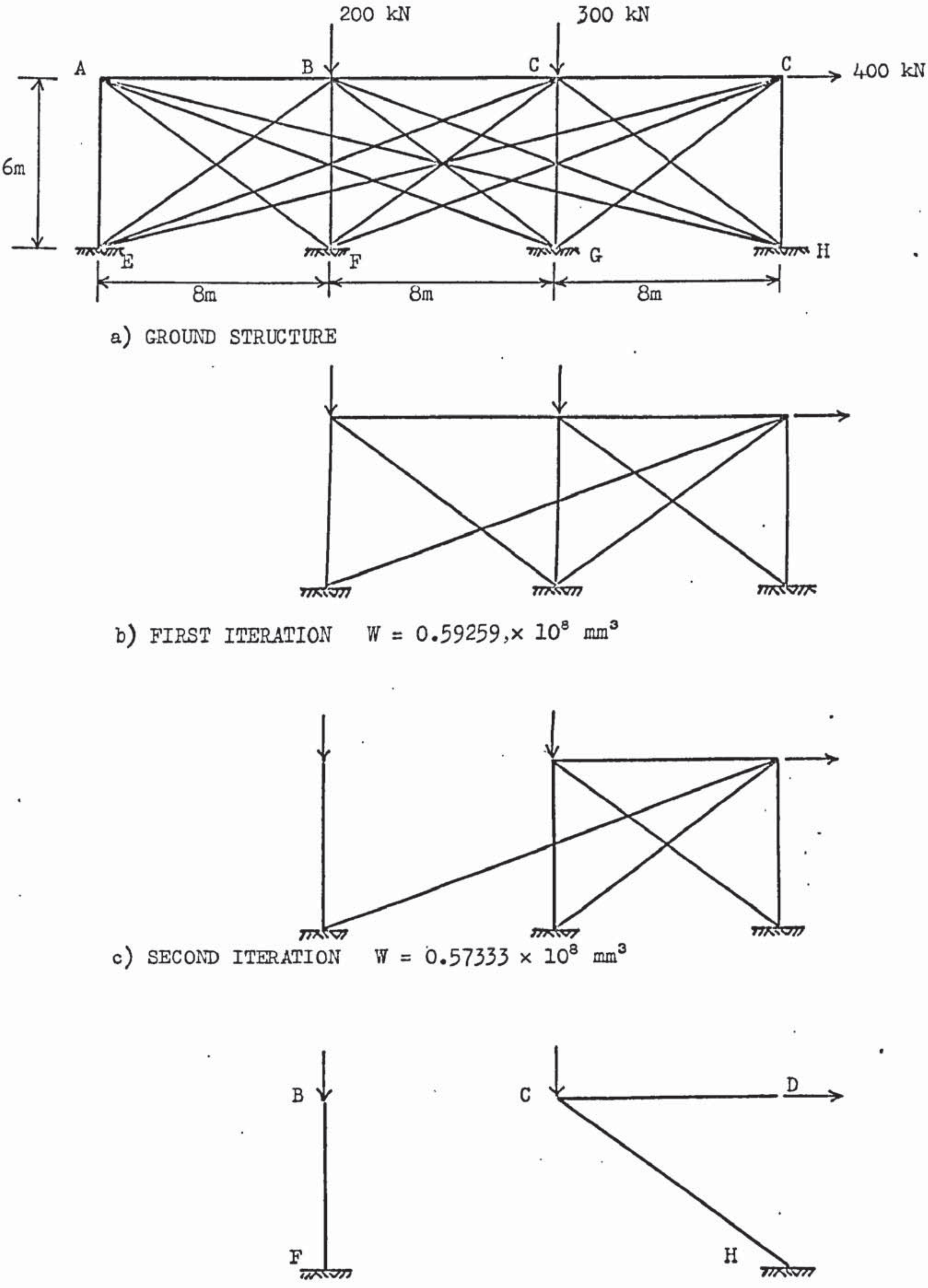
deflexions due to these loads were limited to 15.33 mm at joints A and D and 27.69 mm at joints B and C. The vertical deflexions at joints A and D were limited to 11.11 mm and at joints B and C to 16.66 mm. The modulus of elasticity of the material was 207 kN/mm^2. The combined axial and bending stresses in the members were restricted to 0.165 kN/mm^2.

The ground structure which consisted of 16 members was designed by the proposed procedure which reduced the total number of members to 6 in the final design. This final design is shown in Figure 6.5(c). It had a minimum volume of 0.54641×10^8 mm^3. Although the design procedure was continued, none of the structures derived from this particular structure were lighter. It was noticed that the structure of Figure 6.5(c) was also fully stressed. Once again, the final design obtained is unusual from the architectural point of view.

6.4.4) EXAMPLE 4.

The ground structure of Figure 6.6(a) was developed by linking each joint of the 3 bay one storey frame to every other joint. This ground structure was employed to obtain the optimum structure for the external loads shown in Figure 6.6(a). The dimensions of the frame are also given in the figure. The modulus of elasticity of the material was 207 kN/mm^2. The combined stresses in the members were limited to 0.165 kN/mm^2. The sway of the frame was limited to 18.46 mm while vertical deflexions throughout the structure were limited to 22.22 mm.

The design was initiated from the ground structure of Figure 6.6(a) which consisted of 19 members. After 3 iterations the final design was obtained. As seen from the Figure 6.6(d) it consisted of only 3 members. During the iterations those members which had area values more than the selected convergence ratio were



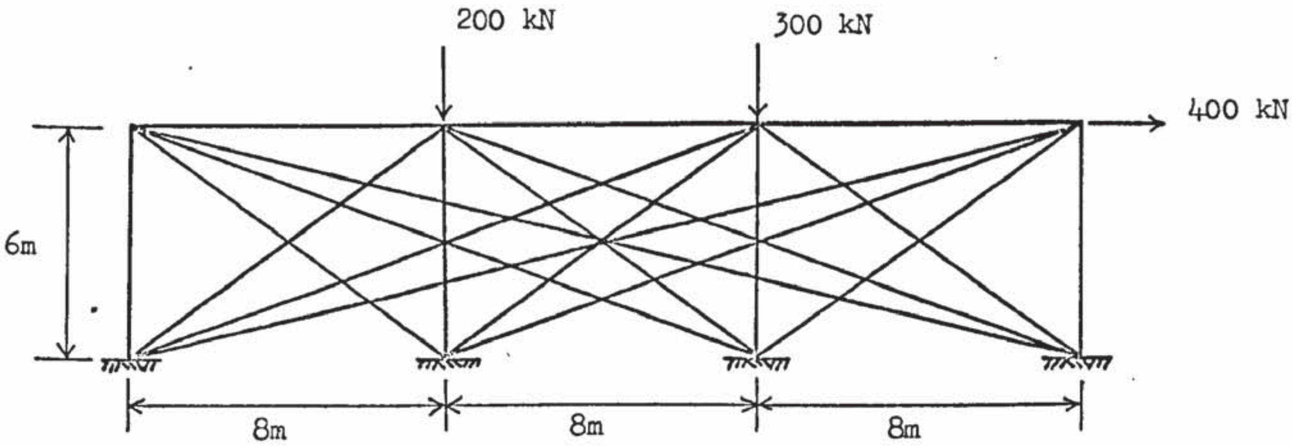
a) GROUND STRUCTURE

b) FIRST ITERATION $W = 0.59259 \times 10^8$ mm^3

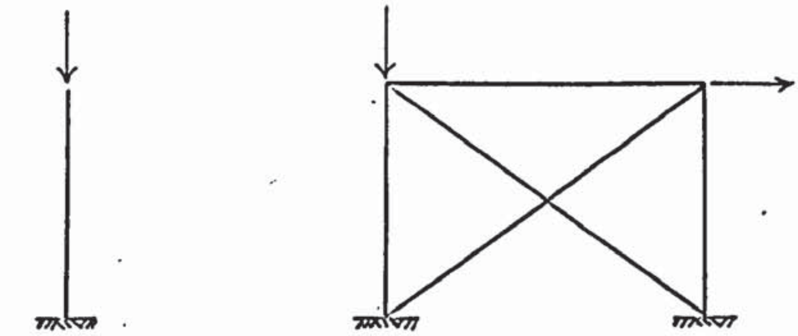
c) SECOND ITERATION $W = 0.57333 \times 10^8$ mm^3

d) FINAL DESIGN $W = 0.57218 \times 10^8$ mm^3 .

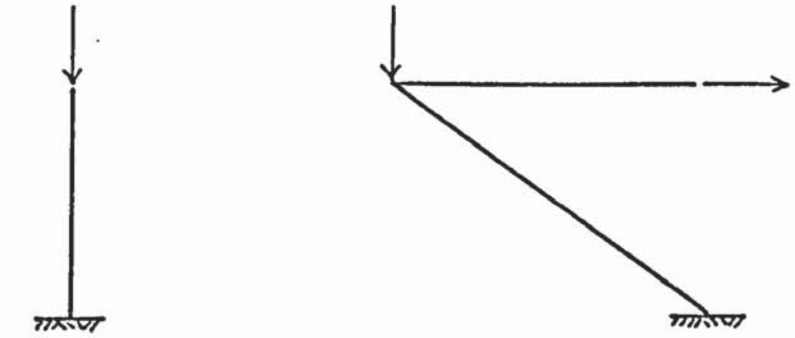
FIGURE 6.6 DESIGN EXAMPLE 4.



a) GROUND STRUCTURE



b) FIRST ITERATION $\quad W = 0.60191 \times 10^8 \ \mathrm{mm}^3$



c) FINAL DESIGN $\quad W = 0.57218 \times 10^8 \ \mathrm{mm}^3$

FIGURE 6.7 DESIGN EXAMPLE 4.

not removed. It was noticed that those members had areas such as 0.4 mm^2 while the member with largest areas was around 2500 mm^2. Hence, the design process was repeated by starting from the same ground structure but this time members with relatively small areas were also removed. The structures obtained at each iteration are shown in Figure 6.7(b),(c). It can be seen that removal of members which have relatively small areas, reduces the number of iterations involved in the topological design procedure whereas the final design remains unchanged. Although the final design is unusual, it is predictable. It is known that the best structure to carry a vertical load is a column and this obtained for the vertical load of 200 kN. As a result, the final design which is shown in the Figure 6.6(d) consists of two parts. The first part is the column which carries the vertical load. The second part is a cantilever which carries the vertical load of 300 kN acting at joint C and the horizontal load of 400 kN acting at joint D. It can be seen that the members of this structure are subjected to no moments. The member CD is in tension and the member CH is in compression. Hence, it is obvious that this structure has the best shape for that particular loading.

It can easily be seen by inspecting the final designs obtained that they have the global shape under the load considered as far as the weight is concerned. It was only in the example considered here that the final design was statically determinate. This was due to the fact that no moment was developed in the members of this final structure under the external loading considered. Since, the members are subject to direct forces only, it may be seen that a pin jointed structure could be used to carry the external loads. However, it can also be seen that such a pin jointed structure needs a member DH to make the structure stable. However, such a structure will be heavier than the rigidly jointed structure of Figure 6.6(d).

As mentioned previously, architectural constraints were not included in all the design examples considered. As a result the final designs obtained were unusual from the architectural point of view. In the next section this aspect is given priority.

6.5) DESIGN EXAMPLES CONSIDERING ARCHITECTURAL CONSTRAINTS.

It has been shown in the previous section that exclusion of architectural constraints in the topological design of rigidly jointed structures led to the final designs which had unusual shapes. However, it has also been shown that when the weight of a structure was taken as an objective and the design criteria only consisted of the stiffness, stress and the deflexion limitations, these final designs had the best shape that can be obtained from the selected ground structure. In the examples considered in this section in addition to these constraints architectural requirements are also included in the design criteria. The inclusion of such requirements can be carried out in two ways. The first is to impose severe limitations on the deflexions of those joints which are not wanted to be removed and group the members which are to be retained with the vital members of the ground structure. The second way is to arrange the ground structure in such a way that it does not include those members which contravene the functional and architectural constraints. This is carried out by first developing the ground structure in the usual way and then removing those members whose existence violates the architectural constraints. It is apparent that in both ways members which have to be retained due to the architectural constraints are not considered in the preparation of the benefit vector.

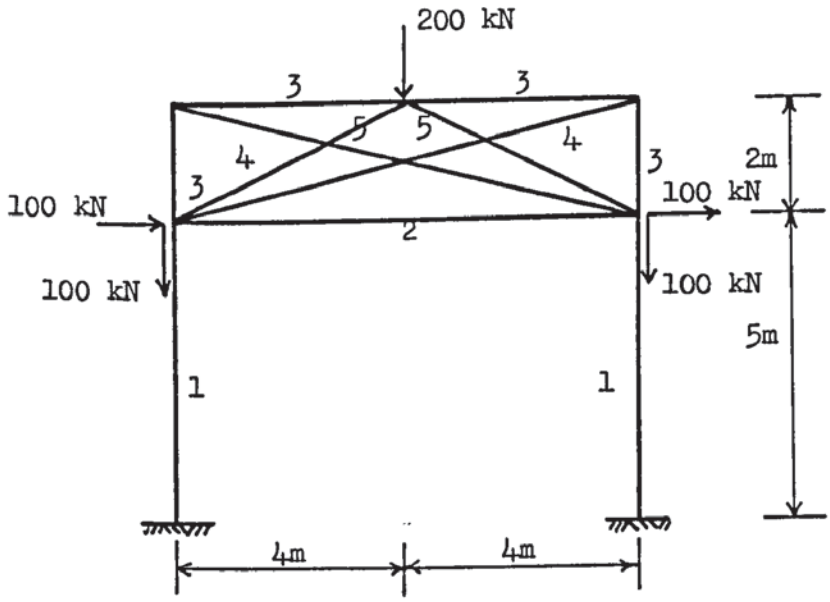
In the case where no inclined member is allowed in the ground structure due to the severity of architectural constraints, then the ground structure is reduced to only beams and columns. It has been shown that the proposed design method can be employed in the design of such structures. It is possible to eliminate some of the columns so that a better design may be obtained.

6.5.1) DESIGN OF ONE BAY FRAME.

The structure of Figure 6.8(a) was utilized as a ground structure for the design of a one bay frame which is subject to the external loads shown in Figure 6.8(a). This reduced ground structure was obtained from the general ground structure which was developed in the manner described in the last section. This was carried out by first linking each joint to every other joint, then removing those members which were not allowed because of the functional requirements. It was assumed that inclined members were not allowed in the part of the structure up to a height of 5 metres. The final ground structure consists of 11 members. The dimensions of the frame are given in Figure 6.8(a). The modulus of elasticity of the material was 207 kN/mm^2. The combined stresses in members were limited to 0.165 kN/mm^2. The sway of the frame was restricted to 50.0 mm while the vertical displacements throughout the structure were limited to 20.0 mm.

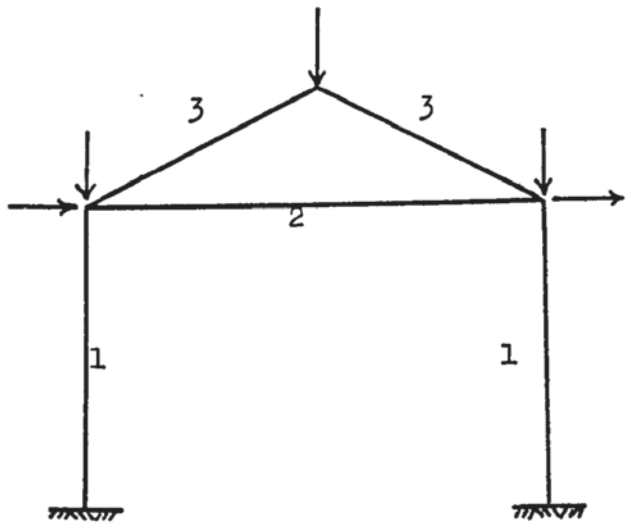
The design was initiated from the ground structure of Figure 6.8(a) and optimum areas for each group are also given in the figure. It can be seen that the areas required for the members in groups 3 and 4 were reduced to zero. As a result removal of those members resulted in the tied pitched roof frame shown in Figure 6.8(b). The minimum volume of this frame was obtained to be 0.236094×10^9 mm^3. The design was then continued by removing the member in group 2 and the pitched roof frame shown in the Figure 6.8(c) was obtained. It was found that this frame had the minimum volume of 0.273016×10^9 mm^3 which was heavier than the frame of Figure 6.8(b).

In the case where practical reasons do not allow the members of group 5 in the ground structure of Figure 6.8(a), then the frame shown in Figure 6.8(a) can be selected as a ground structure to carry the external loads considered. There are two structures which



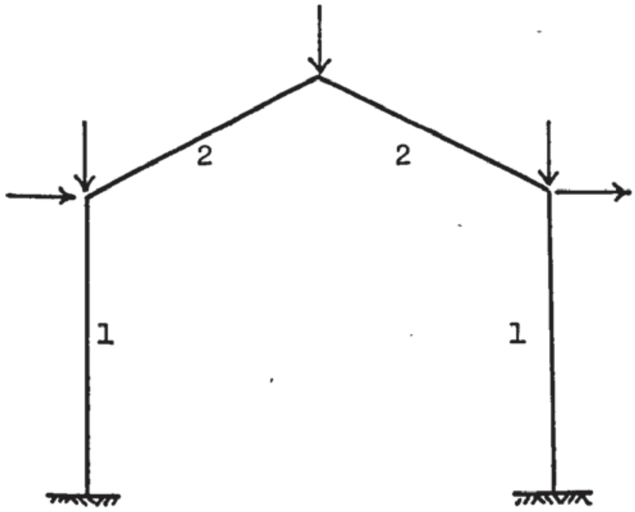
GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$mm^3
1	142.90	
2	79.76	
3	0.02	
4	0.02	
5	33.42	0.236673

a) GROUND STRUCTURE



GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$ mm^3
1	143.52	
2	78.42	
3	33.36	0.236094

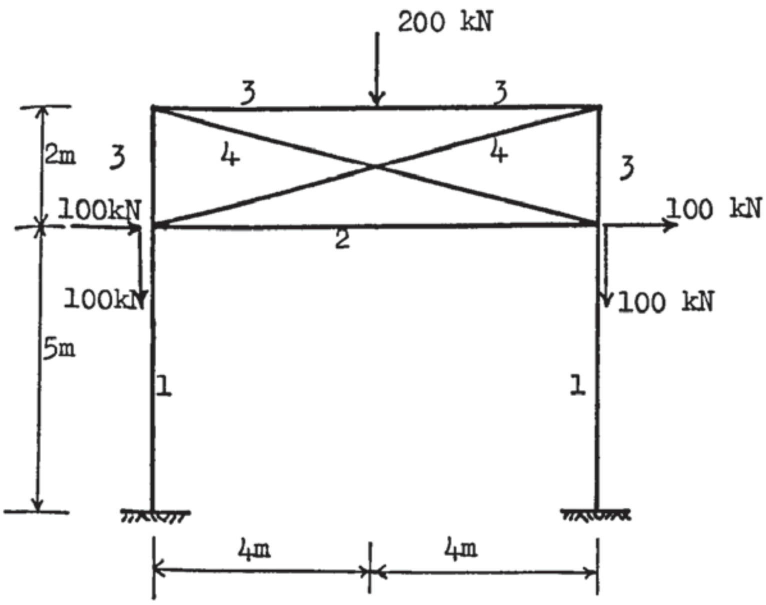
b) FINAL DESIGN



GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$mm^3
1	164.54	
2	121.27	0.273016

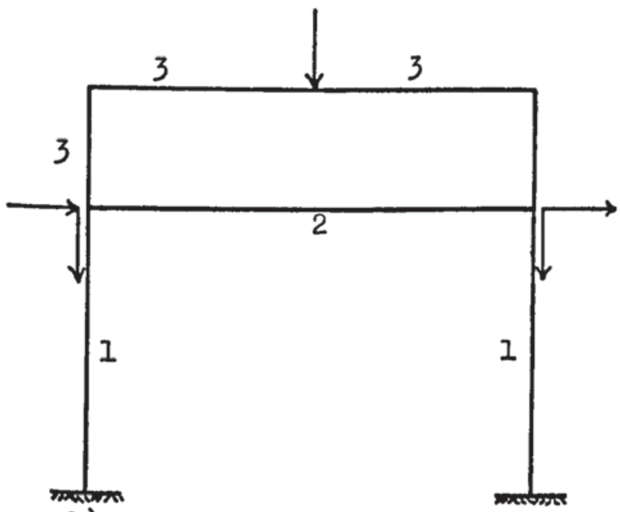
c)

FIGURE 6.8 DESIGN OF ONE BAY FRAME.



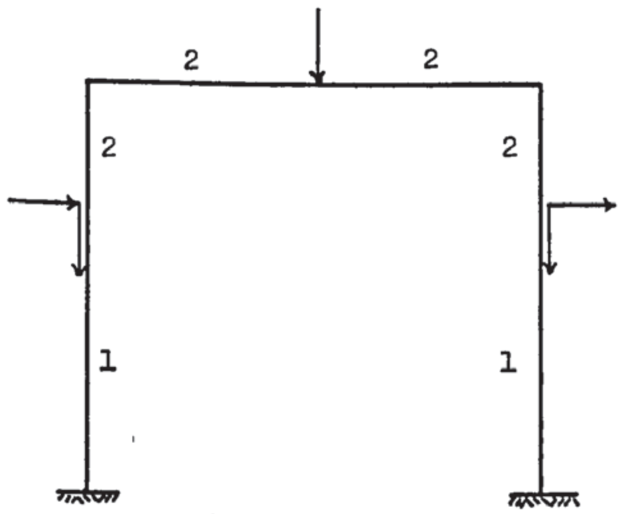
a) GROUND STRUCTURE

GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9 mm^3$
1	445.65	
2	9.17	
3	103.35	
4	4.84	0.284979



b)

GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9 mm^3$
1	168.88	
2	1.55	
3	105.99	0.297309



c)

GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9 mm^3$
1	173.62	
2	105.02	0.299742

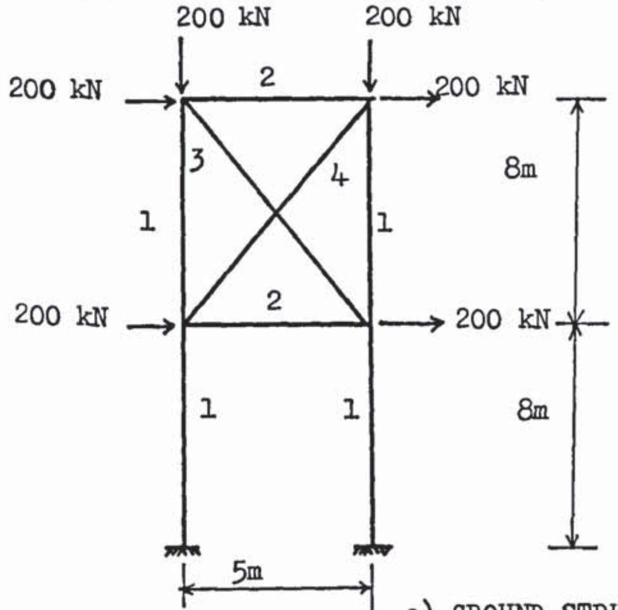
FIGURE 6.9 DESIGN OF ONE BAY FRAME.

can be derived from this ground structure without violating the architectural constraints. These are shown in Figures 6.9(b) and 6.9(c) respectively. It was found that both had the volumes more than the ground structure. Hence, the ground structure itself is the final design. It can be seen by comparing the volumes of the frames shown in Figures 6.8(b) and 6.9(a) that to carry the external loads considered, the pitched roof frame shown in Figure 6.8(b) is a better design than the frame shown in Figure 6.9(a).

6.5.2) DESIGN OF A TWO STOREY FRAME.

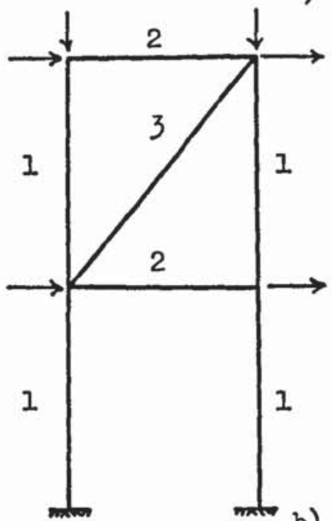
The ground structure of Figure 6.10(a) was developed for the design of a two storey frame. It is assumed that inclined members are not allowed in the first storey due to architectural reasons. Furthermore, due to the functional reasons columns and beams are not to be removed. Hence, the ground structure consists of 8 members, 2 of which are inclined members obtained by linking the joints of the second storey. This ground structure was employed to obtain the optimum structure for the external loads shown in Figure 6.10(a). The dimensions of the frame are also given in the figure. The combined stresses in the members were limited to 0.165 kN/mm^2. The sway of the frame was restricted to 25.0 mm at the first storey and to 100.0 mm at the second storey. The modulus of elasticity of the material was 207 kN/mm^2.

The design was initiated from the ground structure of Figure 6.10(a). Due to the severity of architectural constraints, there are only two structures which can be extracted from this ground structure. These are shown in Figures 6.10(b) and 6.10(c). Both frames were optimised and minimum volumes were obtained to be 1.743685×10^9 mm^3 and 1.870216×10^9 mm^3 respectively. It can be seen that these two frames are heavier than the ground structure. As a result, the ground structure has the optimum shape. This shows



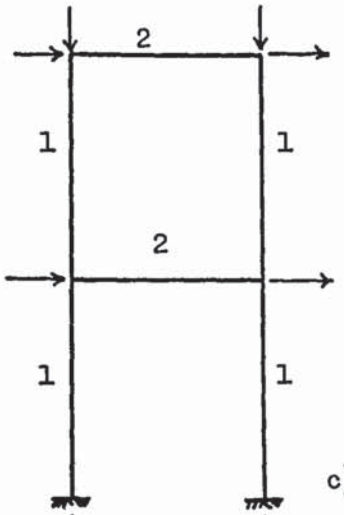
a) GROUND STRUCTURE

GROUP NO.	AREAS $\times 10^2\ mm^2$	WEIGHT $\times 10^9 mm^3$
1	488.36	
2	16.25	
3	28.20	
4	71.97	1.673496



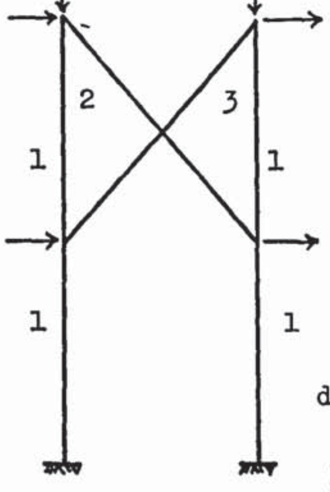
b)

GROUP NO.	AREAS $\times 10^2\ mm^2$	WEIGHT $\times 10^9 mm^3$
1	486.99	
2	46.44	
3	137.66	1.743685



c)

GROUP NO.	AREAS $\times 10^2\ mm^2$	WEIGHT $\times 10^9 mm^3$
1	457.72	
2	405.52	1.870216



d)

GROUP NO.	AREAS $\times 10^2\ mm^2$	WEIGHT $\times 10^9 mm^3$
1	488.03	
2	39.95	
3	78.36	1.673297

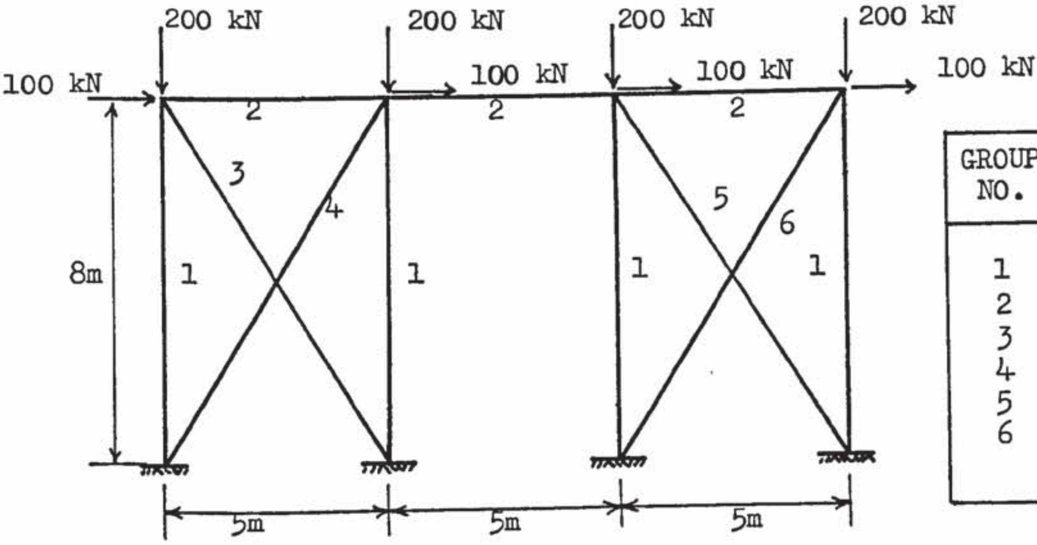
FIGURE 6.10 DESIGN OF TWO STOREY FRAME

that in the case where the architectural constraints dominates the design problem, it may be possible that no improvement can be achieved in the shape of the ground structures. However, if these constraints are partly relaxed, a better design may be obtained. This was shown by removing the beams in the ground structure shown in Figure 6.10(a). This led to the structure of Figure 6.10(d). This minimum volume of this structure was obtained to be 1.673297×10^9 mm^3 which was less than the volume of the ground structure.

6.5.3) DESIGN OF A ONE STOREY 3 BAY FRAME.

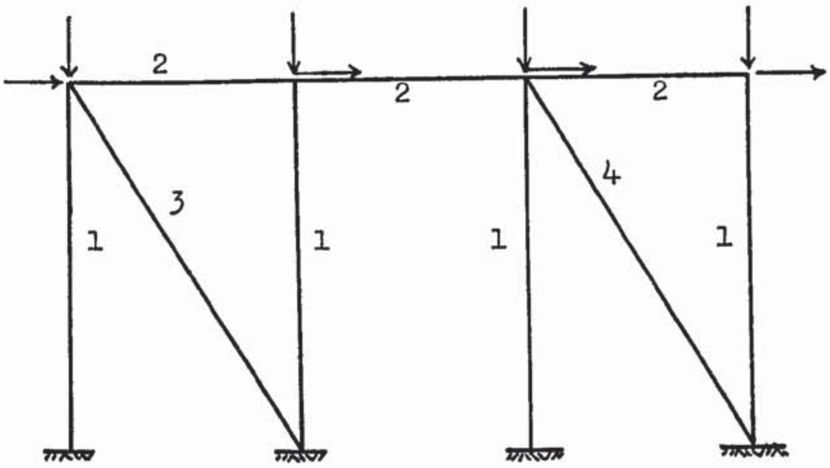
The ground structure shown in Figure 6.11(a) was utilized to obtain the optimum one storey 3 bay frame for the external loads shown in Figure 6.11(a). This ground structure was developed by assuming that inclined members were not allowed in the second bay due to the architectural reasons. The dimensions of the frame are also given in the figure. The modulus of elasticity of the material was 207 kN/mm^2. The combined stresses in the members were limited to 0.165 kN/mm^2. The sway of the frame was restricted to 100.0 mm while vertical deflexions throughout the structure were limited to 20.0 mm.

The design was initiated from the ground structure of Figure 6.11(a). The minimum volume of this frame was obtained to be 0.102977×10^9 mm^3. It was found that removal of the members belonging to the groups 4 and 6 was the most beneficial. This was carried out and at the end of the first topological iteration the structure shown in Figure 6.11(b) was obtained. This structure has the minimum volume of 0.102864×10^9 mm^3 which is slightly lighter than the ground structure. There are only two structures which can be derived from this structure without violating the architectural constraints. These are shown in Figures 6.11(c) and 6.11(d). Both



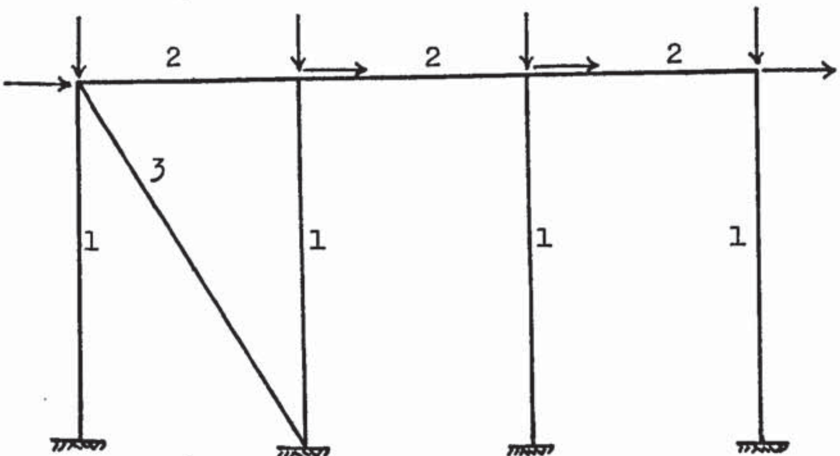
GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^9 mm^3$
1	14.51	
2	6.41	
3	21.45	
4	0.37	
5	27.25	
6	0.67	0.102977

a) GROUND STRUCTURE



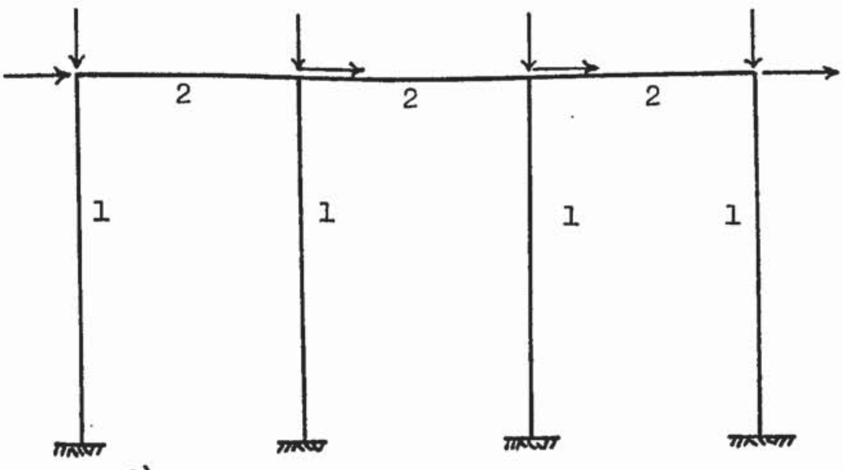
GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^9 mm^3$
1	14.30	
2	6.80	
3	21.95	
4	27.96	0.102864

b) FINAL DESIGN



GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^9 mm^3$
1	31.30	
2	21.26	
3	51.93	0.181054

c)



GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^9 mm^3$
1	231.550	
2	0.798	0.742165

d)

FIGURE 6.11 DESIGN OF A ONE STOREY 3 BAY FRAME frames were optimised and minimum volumes were obtained to be 0.181054×10^9 mm^3 and 0.842165×10^9 mm^3 respectively. It can be seen that these two frames are heavier than the structure of Figure 6.11(b). Hence, this structure is the final design. The examples considered so far verify that the final shape of the structure depends on the ground structure chosen as well as the severity of the architectural constraints. It has also been shown that in the case of severe architectural requirements, it may be possible that no improvement can be achieved in the shape of the ground structure. Hence, it becomes obvious that architectural constraints play very important part in the toplogical design of rigidly jointed structuresdue to the fact that they affect the shape of the ground structure.

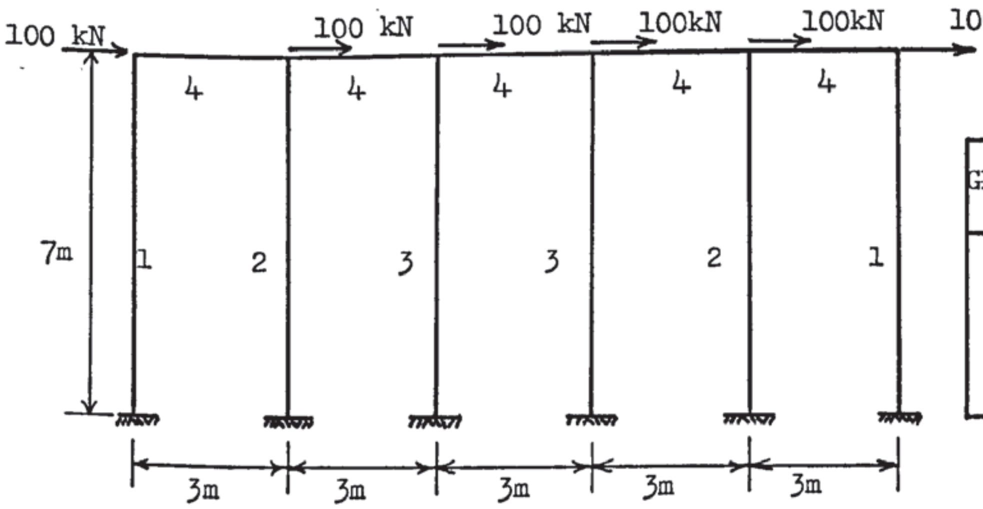
6.5.4) DESIGN OF ONE STOREY MULTIBAY FRAME.

It may be possible that due to the service requirements inclined members are not allowed in the ground structure. In such cases the proposed design procedure can also be employed to obtain a better design. In order to demonstrate this, the structure of Figure 6.12(a) was selected as a ground structure. Since, it is possible to increase the number of columns to cover the design space, then the proposed design procedure can be used to eliminate those columns which are not necessary to carry the external loads, in order to obtain the optimum column distribution.

The ground structure shown in Figure 6.12(a) has 6 joints and 11 members 6 of which are columns. A ground structure with this number of joints would require 66 members if every joint were to be connected to every other joint. The use of such a ground structure therefore reduces the size of the initial problem. This ground structure was employed to obtain the optimum structure for the wind loading which is distributed to each column as shown in Figure 6.12(a). The dimensions of the frame are also given in the figure. The sway of the frame was limited to 100.0 mm while vertical deflexions throughout the structure were limited to 20.0 mm. The combined stresses in the members were restricted to 0.165 kN/mm^2. The modulus of elasticity of the material was 207 kN/mm^2.

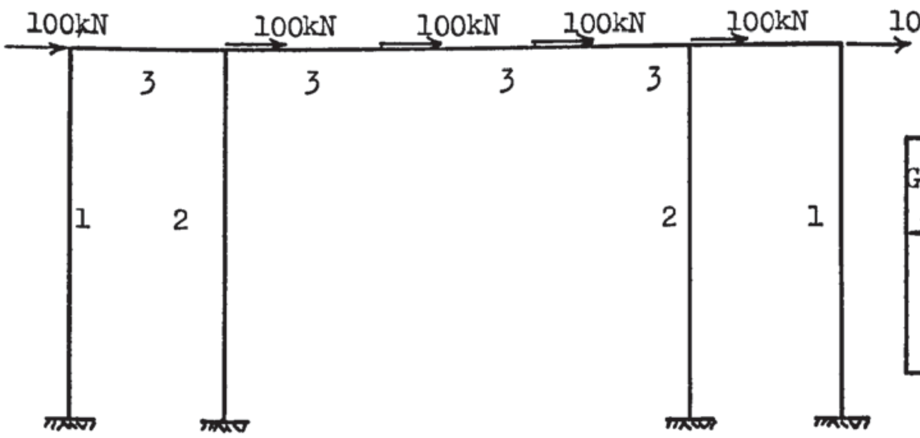
The design was initiated from the ground structure shown in Figure 6.12(a) and the minimum volume of this structure was obtained to be 7.20586×10^8 mm^3. The benefit vector was then prepared and it was found that removal of columns belonging to group 3 was the most beneficial. This was carried out and at the end of first topological iteration the structure shown in Figure 6.12(b) was obtained. This structure has the minimum volume of 6.69219×10^8 mm^3 which is lighter than the ground structure. Further improvement in the shape can only be obtained without violating the architectural constraint by removing the columns beloning to group 2. This was carried out and the ground structure was reduced to the portal frame shown in Figure 6.12(c). This frame has the minimum volume of 6.19494×10^8 mm^3. It is apparent that no further improvement can be achieved in the shape. Hence, this portal frame is the final design and has the global shape under the horizontal loading considered.

It can be seen that in the design example considered when a column was removed, the horizontal forces acting on it were not moved to the columns adjacent. This was in order to keep the external loading pattern unchanged during the design process. However, the proposed design procedure was reapplied to the same example but this time when a column was removed, the wind loading was redistributed to the remaining columns. This process is shown in Figures 6.13(a), (b) and (c). It was found that in this particular example this redistribution of forces had no effect on the final design obtained but the optimum member areas were changed slightly.



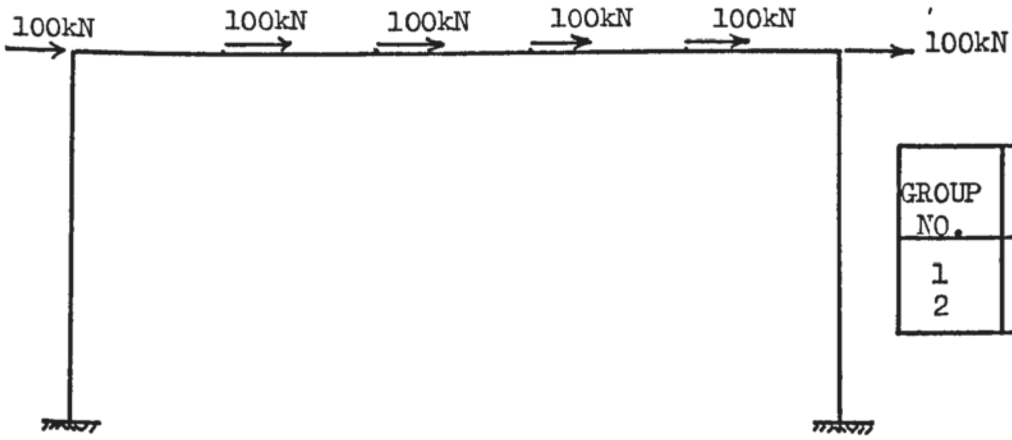
GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^8 mm^3$
1	111.36	
2	154.64	
3	156.12	
4	86.41	7.20586

a) GROUND STRUCTURE



GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^8 mm^3$
1	6.23	
2	361.74	
3	102.71	6.69219

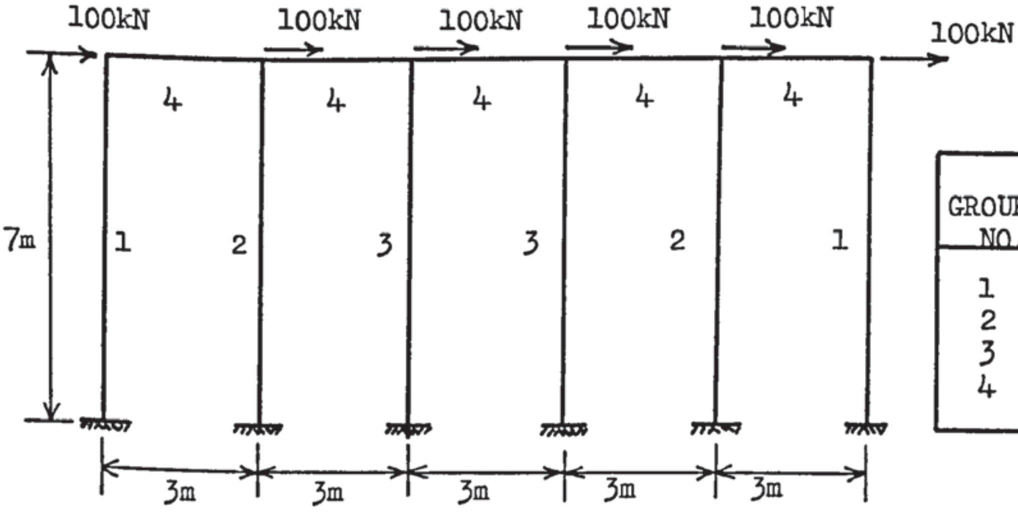
b)



GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^8 mm^3$
1	424.52	
2	16.77	6.19494

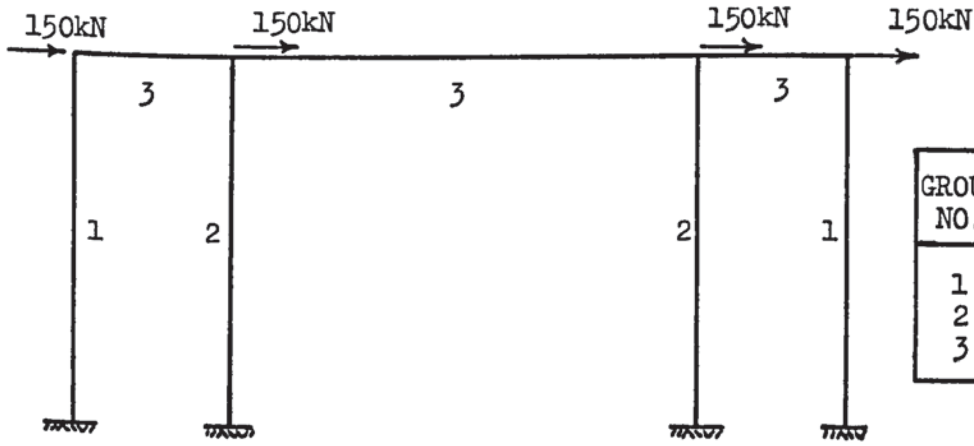
c) FINAL DESIGN

FIGURE 6.12 DESIGN OF A ONE STOREY MULTIBAY FRAME



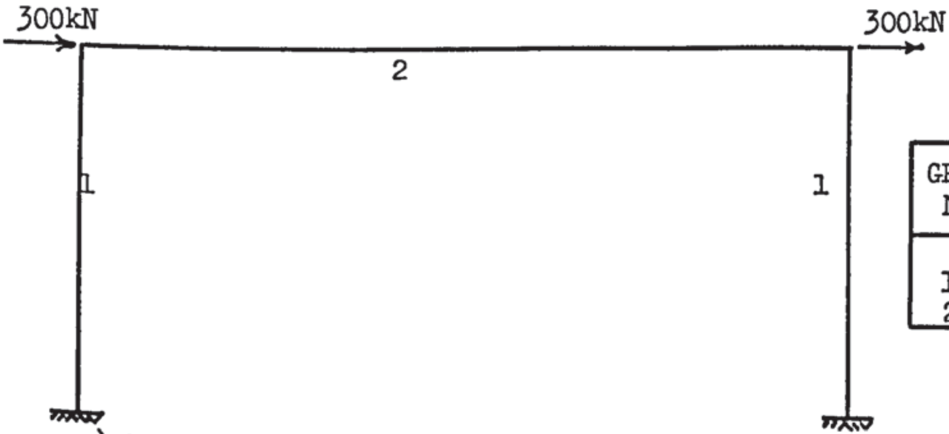
GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^8$ mm^3
1	111.36	
2	154.64	
3	156.12	
4	86.41	7.20586

a) GROUND STRUCTURE



GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^8 mm^3$
1	6.83	
2	355.10	
3	110.59	6.72585

b)



GROUP NO.	AREAS $\times 10^2 mm^2$	WEIGHT $\times 10^8 mm^3$
1	424.94	
2	9.06	6.08511

c) FINAL DESIGN

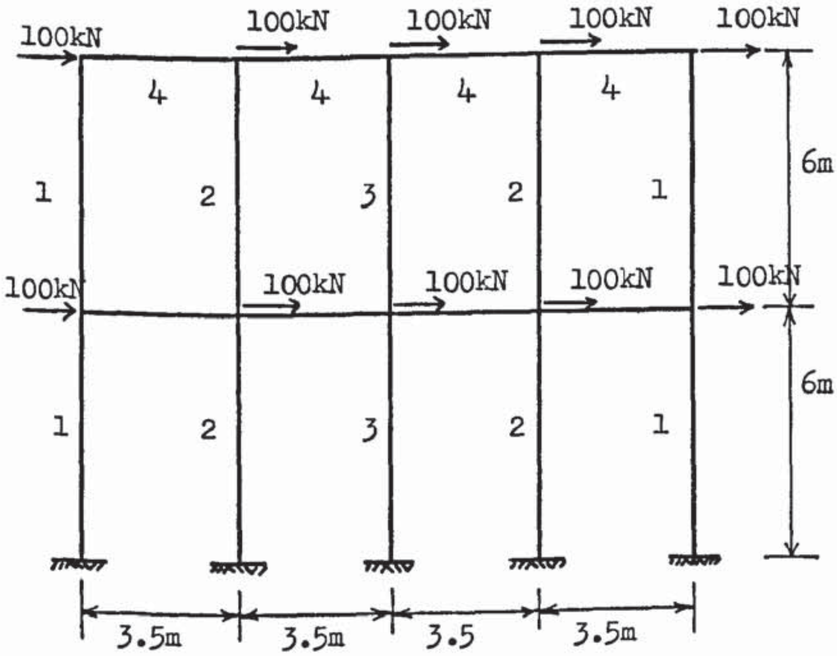
FIGURE 6.13 DESIGN OF A ONE STOREY MULTIBAY FRAME.

As a result it may be concluded that the proposed design procedure can be employed in the design of structures where the shape is variable to obtain the optimum column distribution.

6.5.5) DESIGN OF TWO STOREY MULTIBAY FRAME.

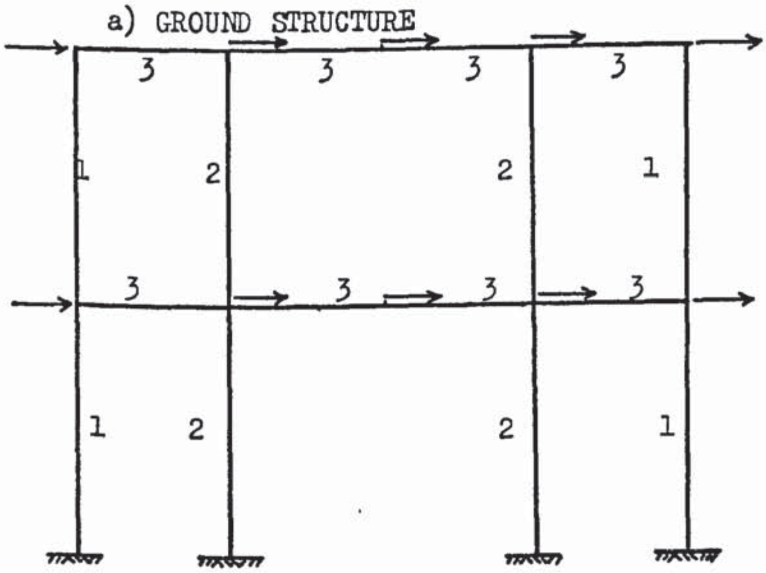
As a further example, the two storey multibay frame of Figure 6.14(a) is considered as a ground structure, to demonstrate the application of the design procedure in finding the optimum column distribution. External loading is taken as wind loading which is distributed to each column as shown in Figure 6.14(a). The dimensions of the frame are also given in the figure. The combined stresses in members were restricted to 0.165 kN/mm^2. The modulus of elasticity of the material was 207 kN/mm^2. The sway of the frame was limited to 100.0 mm at the second storey and to 50.0 mm at the first storey while vertical deflexions throughout the structure were limited to 20.0 mm.

The design was initiated from the ground structure shown in Figure 6.14(a) and minimum volume for this structure was obtained to be 1.711015 $\times$ 10^9 mm^3. The benefit vector was then prepared and it was found that removal of columns belonging to group 3 was the most beneficial. It should be pointed out that in such design problems, if the benefit vector gives one of the columns in the first storey to be removed, then the columns in the upper storeys which are supported by this particular column, have to be removed as well. The removal of columns beloning to group 3 leads the structure shown in Figure 6.14(b) which has the minimum volume of 1.709489 $\times$ 10^9 mm^3. It can be seen that this frame is slightly ligher than the ground structure. However, although the decrease in weight is small, the saving in the cost of the frame is considerable. This is due to the fact that this frame contains fewer members. The two storey one bay frame shown in Figure 6.14(c) was



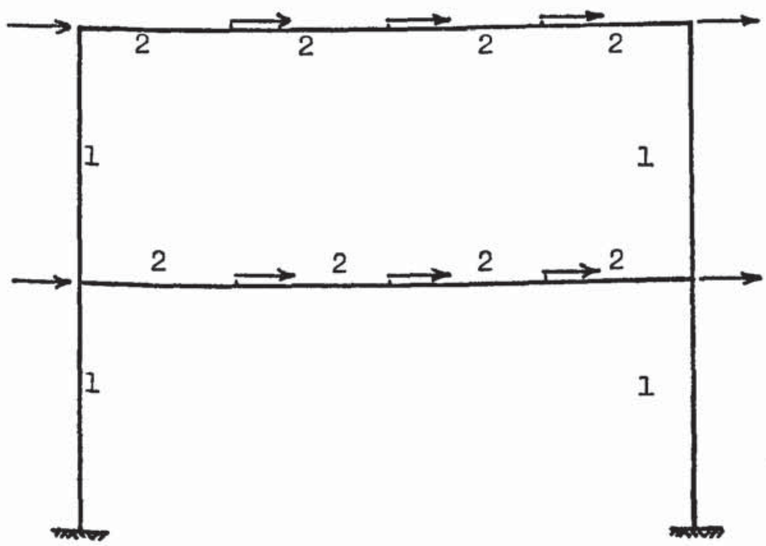
a) GROUND STRUCTURE

GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$ mm^3
1	175.05	
2	230.42	
3	230.68	
4	164.66	1.711015



b) FINAL DESIGN

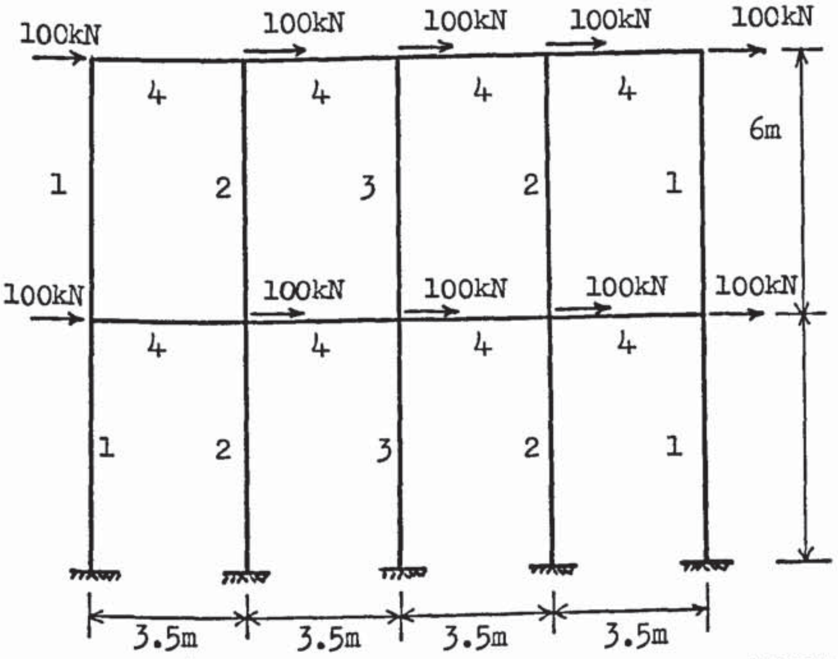
GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$ mm^3
1	71.26	
2	403.43	
3	203.59	1.709489



GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$ mm^3
1	702.48	
2	34.86	1.783604

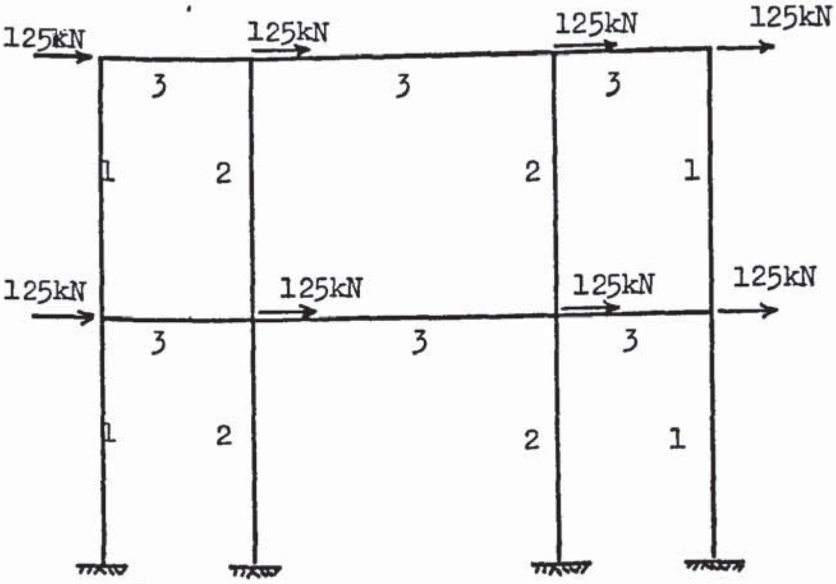
c)

FIGURE 6.14 DESIGN OF TWO STOREY MULTIBAY FRAME.



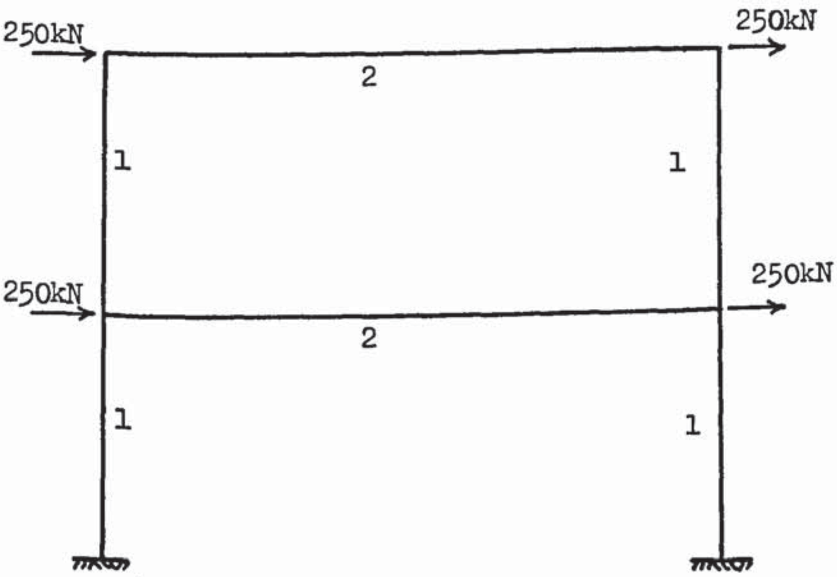
GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$mm^3
1	175.05	
2	230.42	
3	230.68	
4	164.66	1.711015

a) GROUND STRUCTURE



b)

GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$mm^3
1	65.73	
2	407.01	
3	203.63	1.704744



GROUP NO.	AREAS $\times 10^2$ mm^2	WEIGHT $\times 10^9$ mm^3
1	706.62	
2	0.20	1.696423

c) FINAL DESIGN

FIGURE 6.15 DESIGN OF TWO STOREY MULTIBAY FRAME obtained by removing the columns belonong to group 2 of the frame shown in Figure 6.14(b). This frame has the minimum volume of 1.783604×10^9 mm^3. This shows that no further improvement can be obtained in the shape of the frame. Consequently, the frame of Figure 6.14(b) is the final design.

The proposed design procedure was reapplied to the same example but this time when a column was removed, the wind loading was redistributed to the remaining columns. The process is shown in Figures 6.14(a), (b) and (c) and it can be seen that the final design has changed. Hence it may be concluded that in such design problems redistributions of forces to the remaining members could improve the final design.

It may therefore be concluded that the proposed design procedure can be utilized in the design of structures which consists of beams and columns to eliminate those columns which are redundant from the weight point of view, in order to obtain a better design.

6.6) CONCLUSIONS.

It has been shown that the topology of rigidly jointed structures can efficiently be treated as a design parameter and as a result of topological changes considerable weight reduction can often be achieved. The design examples considered have shown that the number of topological iterations required to obtain the final design is small. This proves the effectiveness of the way in which the design problem is formulated as described in Chapter 2. Further the final designs obtained were global. Although, the design procedure was continued after reaching the final design, no relative minima were identified. It was found that when the stress constraints dominate the design problem, the final design is fully stressed.

Although in the examples considered in Section 6.4 the members are not grouped together, it is possible to do so. However, it should be pointed out that the manner in which they are grouped together affects the shape of the final design. Furthermore, the final shape of a structure depends upon the shape of the initial ground structure.

It was found that the shape of a rigidly jointed structure at its final stage could be considerably different from that of an initial ground structure. It was also found that these shapes may be difficult to select intuitively and also unusual from the architectural point of view. However, it was later shown that the proposed design procedure can also be employed in the topological design of rigidly jointed structures where the design criteria includes these architectural constraints as well as stress and deflexion constraints. It was found that depending on the severity of architectural constraints it may not be possible to obtain a better design than the ground structure. However, it was shown that in such cases when architectural constraints are partly relaxed, a better design can be obtained. The design procedure was also successfully employed in the design of structures to find the optimum column distribution.

It has been shown that the structure with the least weight is not always the cheapest to construct. Indeed, in one of the examples considered, a structure obtained was slightly heavier than the final design but it contained less members. It can be seen that this structure is better than the final design from the constructional stand point. However, this is the case when minimum cost is taken as an objective instead of minimum weight. In such cases the benefit vector can be prepared in such a way that the first member in it gives the greatest reduction in the cost of the structure.

Although the design procedure described in this chapter was only applied to rigidly jointed plane frames, it is general and may be extended to space structures.

CHAPTER 7

SUGGESTIONS FOR FURTHER WORK.

It has been shown in this thesis that the formulation of the design problem by the matrix displacement method, considering the displacements of joints in a structure as design variables is effective. This formulation makes it possible to produce a general computer program for automatic optimum design of realistic, rigidly jointed multi-storey frames. Furthermore, in rigidly jointed structures, expressing the second moment of area and the section modulus in terms of the sectional area by approximate relationship; this formulation generates a convex design problem. Therefore, the optimum solution obtained is global.

Instead of using approximate relationship between the sectional properties, they may be treated as independent design variables. This approach, decreases the order of nonlinearity of the design problem, but it trebles the number of variables for the sections adopted for groups. As a result the computer time and storage required are increased.

The approximating programming was proved to be very effective for obtaining the solution of design problem. For the solution of each linearised problem, two phase-revised simplex method was employed and found to be very powerful. However, the dual simplex method can be employed to obtain the solution. This does not involve the concept of artificial variables. As a result, obtaining no feasible solution is avoided. However, in the use of dual simplex method, each equality constraint has to be replaced with two inequality constraints. On the other hand, the number of constraints can be reduced by carrying out the redundancy rules and removing redundant constraints from the problem.

In the design procedure described in Chapter 2, buckling constraints were not considered. It is known that short structural members in compression fail by crushing or yielding of the material. On the other hand, slender structural members in compression yield by buckling before the crushing stress has been reached. Hence, the design procedure described in Chapter 2 may be extended to include the buckling criteria.

The finite element method may be used to formulate the design problem where the displacements of nodes of elements can be considered as design variables. In this way it is possible to design complex structures such as plates and shells. Eventually this design procedure may be extended to cover complete structures consisting of frames and slabs, where the optimum slab thickness can be obtained, as well as, the optimum sectional properties of frames.

The theorems of structural variation have been proved capable for calculating in advance the effect of variation or removal of members upon the behaviour of rigidly jointed structures. It is, therefore possible to calculate the volume of a derived structure by studying its parent structure. These theorems may also be used to vary the properties of structural members for the purpose of design economy.

By using these theorems, it is possible to obtain the behaviour of a rigidly jointed structure, when a hinge is inserted at any point along its members. This fact may be applied to elastic-plastic analysis of structures, in which it is necessary to analyse the structure for each time a plastic hinge develops in the structure. By the theorems of structural variations, one analysis is enough to carry out an entire elastic-plastic analysis of a structure. Hence, it is possible to avoid a considerable amount of computation.

These theorems may be extended to cover space structures, plates and shells. Using this approach, it may be possible to study behaviour of the complete structure consisting of frames and slabs, when some or all of its members or slabs are removed.

The work presented on minimum weight design, with shape as a design variable for rigidly jointed structures is encouraging. The extension of the theorems of structural variation to include rigidly jointed space frames and complete structure does not involve any fundamental difficulties. It is, therefore possible to produce an automatic design procedure, based on these theorems, to design rigidly jointed space structures and complete structures with fixed or variable shape for minimum weight.

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APPENDIX A.

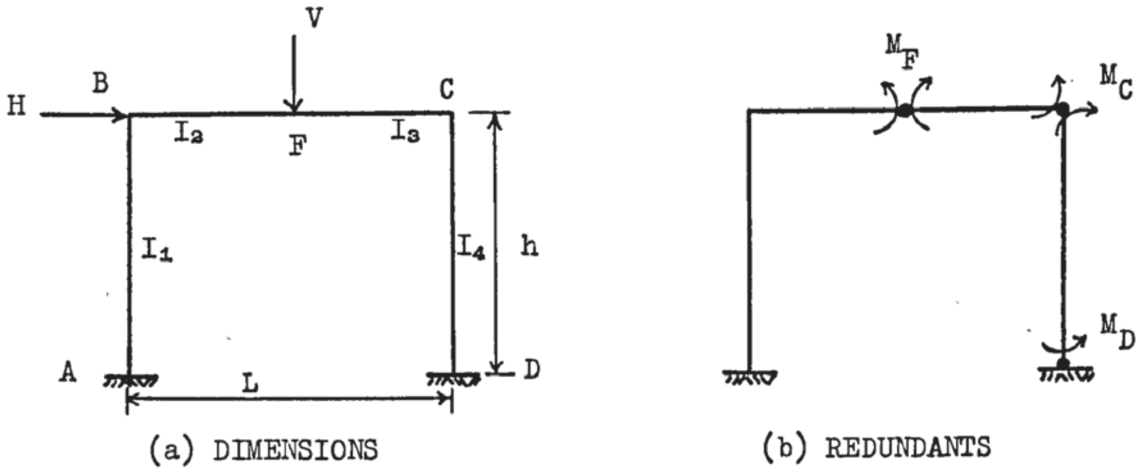

FIGURE A.1: A PORTAL FRAME

The hyperbolic relationship between the variation of any displacement and the second moment of area is derived by considering the portal frame of Figure A.1(a). As shown in Figure A.1(b) the internal moments M_F, M_C and M_D are taken as the redundants. The overall flexibility equations of a structure are given as follows:

$$\begin{bmatrix} X_b \\ X_r \end{bmatrix} = \begin{bmatrix} B_b' f B_b & B_b' f B_r \\ B_r' f B_b & B_r' f B_r \end{bmatrix} \begin{bmatrix} L_b \\ L_f \end{bmatrix} \qquad \text{A.1}$$

where X_b is the deflexion vector under the external loads L_b and X_r is the deflexion vector under the redundant forces L_r. f is member flexibility matrix for the structure. B_b and B_r are the force transformation matrices in the basic statically determinate structure due to the external and redundant forces respectively. The overall flexibility equations of A.1 for the portal frame is of the form:

$$\begin{bmatrix} y_F \\ x_B \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{E} \begin{array}{c} \begin{array}{cc} \overbrace{\hspace{5cm}}^{F_{bb}} & \overbrace{\hspace{9cm}}^{F_{br}} \end{array} \\ \left[\begin{array}{cc:ccc} \frac{L^2h}{4I_1} + \frac{L^3}{24I_1} & \frac{Lh^2}{4I_1} & \frac{Lh}{I_1} + \frac{5L^2}{24I_2} & -\frac{3Lh}{4I_1} - \frac{L^2}{12I_2} & -\frac{Lh}{4I_1} \\ \frac{Lh^2}{4I_1} & \frac{h^3}{3I_1} & \frac{h^2}{I_1} & -\frac{5h^2}{6I_1} & -\frac{h^2}{3I_1} \\ \hdashline \frac{Lh}{I_1} + \frac{5L^2}{24I_2} & \frac{h^2}{I_1} & \frac{4h}{I_1} + \frac{14L}{12I_1} + \frac{L}{6I_3} & -\frac{3h}{I_1} - \frac{5L}{12I_2} + \frac{L}{12I_3} & -\frac{h}{I_1} \\ -\frac{3Lh}{4I_1} - \frac{L^2}{12I_2} & -\frac{5h^2}{6I_1} & -\frac{3h}{I_1} - \frac{5L}{12I_2} & \frac{7h}{3I_1} + \frac{L}{6I_2} + \frac{L}{6I_3} & \frac{5h}{6I_1} - \frac{h}{6I_1} \\ -\frac{Lh}{4I_1} & -\frac{h^2}{3I_1} & -\frac{h}{I_1} & \frac{5h}{6I_1} - \frac{h}{6I_4} & \frac{h}{3I_1} + \frac{h}{3I_4} \end{array}\right] \\ \begin{array}{cc} \underbrace{\hspace{5cm}}_{F_{rb}} & \underbrace{\hspace{9cm}}_{F_{rr}} \end{array} \end{array} \begin{bmatrix} V \\ H \\ \hline M_F \\ M_C \\ M_D \end{bmatrix} \tag{A.2}$$

In the case where the second moment of area I_2 of member 2 is varied to I_2' at the point F, the moment M_F becomes M_{F_I} and the other redundant moments is given by equation 5 as shown in Chapter 5

$$M_{I_i} = M_i + r_{\beta_i} \, m_{f_i}$$

and consequently the deflexions of the portal frame considered will also vary. The next deflexions are denoted ψ to differentiate them from the initial deflexions X. The equation A.2 becomes

$$\begin{bmatrix} \psi_F \\ \psi_B \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{E} \left[\begin{array}{c:c} F_{bb} & F_{br} \\ \hdashline F_{rb} & F_{rr} \end{array}\right] \begin{bmatrix} V \\ H \\ M_{F_I} \\ M_C + r_{\beta_F} \, m_{f_c} \\ M_D + r_{\beta_F} \, m_{f_d} \end{bmatrix} \tag{A.3}$$

The varied displacement ψ_F is obtained by multiplying the first row of F by the column on the right hand side of equation A.3.

$$E\psi_F = \left(\frac{L^2h}{4I_1} + \frac{L^3}{24I_1}\right)V + \frac{Lh^2}{4I_1}H + \left(\frac{Lh}{I_1} + \frac{5L^2}{24I_2'}\right)M_{F_I} + \left(-\frac{3Lh}{4I_1} - \frac{L^2}{4I_1}\right)\left(M_C + r_{\beta_F} m_{f_c}\right)$$

$$+ \left(-\frac{Lh}{4I_1}\right)\left(M_D + r_{\beta_F} m_{f_D}\right)$$

This equation is simplified to

$$E\psi_F = \phi + \frac{\theta}{I_2'} + \rho\, r_{\beta_F} + d.\ \frac{r_{\beta_F}}{I_2'} + \left(a + \frac{b}{I_2'}\right)M_{F_I} \qquad (A.4)$$

where the constants $\phi, \theta, \rho, d, a, b$ are given as

$$\phi = \frac{Lh^2}{4I_1}V + \frac{Lh^2}{4I_1}H - \frac{3Lh}{4I_1}M_C - \frac{Lh}{4I_1}M_D$$

$$\theta = \frac{L^3}{24}V - \frac{L^2}{12}M_C$$

$$\rho = -\frac{3Lh}{4I_1}m_{f_c} - \frac{Lh}{4I_1}m_{f_D}$$

$$d = -\frac{L^2 m_{f_c}}{12}$$

$$a = \frac{Lh}{I_1}$$

$$b = \frac{5L^2}{24}$$

Furthermore, the third equation, in equation A.3 is of the form:

$$0 = \left(\frac{Lh}{I_1} + \frac{5L^2}{24I_2'}\right)V + \frac{h^2}{I_1}H + \left(\frac{4h}{I_1} + \frac{14L}{12I_2'} + \frac{L}{6I_3}\right)M_{F_I} +$$

$$+ \left(-\frac{3h}{I_1} - \frac{5L}{12I_2'} + \frac{L}{12I_3}\right)\left(M_C + r_{\beta_F} m_{f_c}\right) - \frac{h}{I_1}\left(M_D + r_{\beta_F} m_{f_D}\right)$$

which is simplified to

$$0 = m + \frac{n}{I_2'} + s.r_{\beta_F} + t.\ \frac{r_{\beta_F}}{I_2'}\left(\alpha + \frac{\beta}{I_2'}\right)M_{F_I} \qquad A.5$$

where the constants m, n, s, t, α and β are given as

$$m = \frac{Lh}{I_1} V + \frac{h^2}{I_1} H + \left(-\frac{3h}{I_2} + \frac{L}{12I_3} \right) M_C - \frac{h}{I_1} M_D$$

$$n = \frac{5L^2}{24} V - \frac{5L}{12} M_C$$

$$s = \left(-\frac{3h}{I_1} + \frac{L}{12I_3} \right) m_{f_c} - \frac{h}{I_1} m_{f_D}$$

$$t = \frac{5L}{12} m_{f_c}$$

$$\alpha = \frac{4h}{I_1} + \frac{L}{6I_3}$$

$$\beta = \frac{14L}{12}$$

The equation A.5 gives M_{F_I} in the form:

$$M_{F_I} = \frac{-\left(m + \frac{n}{I_2'} + s.r_{\beta_F} + t.\frac{r_{\beta_F}}{I_2'} \right)}{\alpha + \frac{\beta}{I_2'}} \qquad \text{A.6}$$

Substituting M_{F_I} into equation A.4, it follows that:

$$E\left(\alpha + \frac{\beta}{I_2'}\right)\psi_F = \left(\alpha + \frac{\beta}{I_2'}\right)\left(\phi + \frac{\theta}{I_2'} + \rho.r_{\beta_F} + d\frac{r_{\beta_F}}{I_2'}\right)$$
$$-\left(a + \frac{b}{I_2'}\right).\left(m + \frac{n}{I_2'} + s.r_{\beta_F} + t.\frac{r_{\beta_F}}{I_2'}\right)$$

which is simplified to

$$E\alpha\psi_F + \frac{E\beta}{I_2'}\psi_F = A + \frac{B}{I_2'} + C.\frac{r_{\beta_F}}{I_2'} + D.r_{\beta_F} + \frac{G}{I_2'^2} + F.\frac{r_{\beta_F}}{I_2'^2} \qquad \text{A.7}$$

where the constants A,B,C,D,G and F are given as

$A = \alpha\phi - am$

$B = \alpha\theta - \beta\phi - an - bm$

$C = \alpha d + \beta\rho + at - bs$

$D = \alpha\rho - as$

$G = \beta\theta - bn$

$F = \beta d - bt.$

Multiplying both sides of equation A.7 by $I_2'^2$, it follows that

$$E\alpha\, I_2'^{2}\, \psi_F + E\beta\, I_2'\, \psi_F = AI_2'^{2} + BI_2' + C.r_{\beta_F}.I_2' + Dr_{\beta_F} I_2'^{2} + G + F.r_{\beta_F} \qquad \text{A.8}$$

r_{β_F} is given in Chapter 5 by equation 5.12 which is of the form $\nu\, I_2'$ where ν is a constant. Substituting this in A.8 and collecting the similar terms, it is found that

$$\eta_1 + \eta_2\, I_2' + \eta_3\, I_2'^{2} + \eta_4\, I_2'^{3} + \eta_5\, I_2'\, \psi_F + \eta_6\, I_2'^{2}\, \psi_F = 0 \qquad \text{A.9}$$

where $\eta_1, \eta_2 \ldots \eta_6$ are constants and given as

$$\eta_1 = G, \quad \eta_3 = C\nu + A, \quad \eta_5 = -\beta E$$

$$\eta_2 = F\nu + B, \quad \eta_4 = D\nu, \quad \eta_6 = -E\alpha.$$