THE CHOICE OF THE MODEL IN INVENTORY CONTROL AND THE COST OF SOPHISTICATION

VOLUME 2

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CHAPTER 1

SECTION 1.1 INTRODUCTION

In this volume we derive mathematically the cost equations for the four models (Q,R), (nQ,R,T), (M,T) and (M,R,T) defined in volume 1. In section 1.2 we define the notation. In section 1.3 we give the basic mathematics required for the chapter.

In chapter 2 we consider various approximate versions and the exact model of (Q,R) when the backorder cost is a linear function of the length of time of a backorder. We show that the exact model of (Q,R) gives the least inventory costs, and that the savings are probabily worth the additional computational cost.

In chapter 3 we derive the exact models for (nQ,R,T),(M,T) and (M,R,T) assuming the same backorder cost function as above. In chapters 4 and 5 we derive the cost equations for all four exact models, when the cost of a backorder is in the first case a quadratic function and in the second case an exponential function of the length of time of the backorder, respectively.

In chapter 6 and 7 we derive the models when a backorder is not incurred immediately until after a period of grace has expired.

In chapter 8 we relax the assumption of constant lead times and derive all the models for the different cost functions. In chapter 9 we introduce a probability density function of supply of batch quantity assuming constant lead times.

Finally in chapter 10 we combine the cases of continuous lead times and variable supplies.

Section 1.2 Definitions

Let D be the demand per year.

Let S be the set up or order costs.

Let Q be the batch quantity.

Let M be the maximum order cover.

Let R be the re-order level.

In chapter 2 R is expressed as k standard deviations of stock.

Let T be the review period.

Let L be the lead time.

Let σ be the standard deviation of demand per year.

Let hc be the annual holding cost of a unit of stock.

Let p be the period of grace for which a backorder does not incur a cost.

(p is also used briefly in chapter 2 to indicate a particular ratio).

Let s be the cost dependent only on the number of stockouts.

Let b_1 be the cost of a backorder, independent of the time for which a backorder exists.

Let b_2 be the cost per year relating to the time for which a backorder exists.

Let b_3 be the cost of a backorder per square time for which a backorder exists.

When the backorder cost is an exponential function, the cost of a backorder is $b_1 \exp(b_4 . z)$ where z is the length of time of the backorder.

Let $C_B(t)$ be the backorder cost function where t is the length of time for which a backorder has been backordered, including the cost proportional to number of backorders.

Let $\varrho(x)$ be the steady state probability that the inventory position of the system is x.

Let POR be the probability of placing an order.

Let POUT be the probability of been out of stock.

Let H(L) be the probability density function of Lead time L

Let U(Q) be the probability density function of supply Q.

The probability density function of Normal demand x in a period t is defined as

$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2$$

Let $g\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) = \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2$

Let $f(x,Dt) = \frac{1}{\sqrt{\sigma^2 t}}\, g\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)$

Let $F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) = \int_{\frac{x-Dt}{\sqrt{\sigma^2 t}}}^{\infty} g(v)\, dv$

The notation adopted for the inventory control procedures as defined in the introduction to the thesis, in Vol. 1, are (Q,R), (nQ,R,T),(M,T) and (M,R,T). Some authors use (z,Z) or (s,S) or other notations for (Q,R),or (s,S,T) for (M,R,T), (Z,T) for (M,T). Mnemonic notations have been adopted in this thesis: for example in (Q,R) Q stands for batch quantity and R stands for reorder level. The notation adopted conveys the characteristics of the models better than notations adopted by some authors. In (M,T) and (M,R,T) M stands for maximum order cover, T stands for reorder time and R for reorder level.

SECTION 1.3. BASIC MATHEMATICS

In this section some mathematical results for the Normal distribution would be derived for use in later chapters. All of the results used later are in fact quoted at the appendix of 'Analysis of Inventory Systems' by Hadley and Whitin, Ref 1. Since most of these are stated without proof they are all derived here for the sake of completeness.

Let x be a normal variate with mean D and variance σ^2.

The probability density function of x,

$$= \frac{1}{\sigma} g\left(\frac{x-D}{\sigma}\right)$$

Where $g\left(\frac{x-D}{\sigma}\right) = \dfrac{\exp -\frac{1}{2}\left(\frac{x-D}{\sigma}\right)^2}{\sqrt{2\pi}}$

Hence the moment generating function of x, denoted M.G.F of x

$$= \int_{-\infty}^{\infty} \frac{e^{xt}}{\sigma} g\left(\frac{x-D}{\sigma}\right) dx$$

Substituting for $g\left(\frac{x-D}{\sigma}\right)$

The M.G.F. of x

$$= \int_{-\infty}^{\infty} \frac{e^{xt}}{\sqrt{2\pi\sigma^2}} \exp -\frac{1}{2}\left(\frac{x-D}{\sigma}\right)^2 \, dx$$

Simplifying

$$= \frac{e^{Dt + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-D-\sigma^2 t^2)^2}{2\sigma^2}} \, dx$$

Integrating, the M.G.F. of X

$$= e^{(Dt + \frac{\sigma^2 t^2}{2})}$$

Let $Y = X_1 + X_2 + X_3 + \text{--------} + X_n$ where X_i are identically distributed normal variates with mean D and variance σ^2.

Y is the N^{th} convolution of X.

The moment generating function of Y

$$= (\text{M.G.F. of } X_1) . (\text{M.G.F. of } X_2) \ldots\ldots\ldots (\text{M.G.F. OF } X_n)$$

$$= e^{(nDt + \frac{n\sigma^2 t^2}{2})}$$

which is also the moment generating function of a Normal Variate with mean nD and variance $n\sigma^2$.

Hence the probability density function of Y

$$= \frac{1}{\sqrt{2\pi n\sigma^2}} \exp - \frac{1}{2n\sigma^2} (Y - nD)^2 \qquad 1.3.1.$$

Let $$f(V, Dt) = \frac{\exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2}{\sqrt{2\pi\sigma^2 t}}$$

and let $$g\left(\frac{V-Dt}{\sigma^2 t}\right) = \frac{1}{\sqrt{2\pi}} \exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2$$

Then

$$\int_W^{\infty} (V-W) * f\ (V, Dt)\ dV$$

$$= \int_W^{\infty} \frac{(V-W)\ \exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2}{\sqrt{2\pi\sigma^2 t}}\ dV$$

Re-arranging terms we have

$$= \frac{\sqrt{\sigma^2 t}}{\sqrt{2\pi\sigma^2 t}} \int_W^{\infty} \left(\frac{V-W}{\sqrt{\sigma^2 t}}\right) \times \exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2 \quad dV$$

Simplifying

$$= \frac{1}{\sqrt{2\pi}} \int_W^{\infty} \left(\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) - \left(\frac{W-Dt}{\sqrt{\sigma^2 t}}\right)\right) \exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2 \quad dV$$

But
$$\frac{\partial\left(\exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2\right)}{\partial V}$$

$$= -\frac{1}{\sqrt{\sigma^2 t}}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) \quad \exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2$$

Hence integrating by Parts

$$\int^{\infty} (V-W) * f\ (V, Dt)\ dV$$

$$= \sigma^2 t \left[-\frac{\exp}{\sqrt{2\pi\sigma^2 t}} - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2 \right]_W^{\infty} - (W-Dt) \int_W^{\infty} f(V,Dt)\, dV$$

$$= \sigma^2 t\, f\,(W,Dt) - (W-Dt) \cdot F\left(\frac{W-Dt}{\sqrt{\sigma^2 t}}\right) \qquad 1.3.2.$$

Where $F\left(\frac{W-Dt}{\sqrt{\sigma^2 t}}\right) = \int_V^{\infty} f\,(V,Dt)\, dV$

$$\int_V^{\infty}\int_W^{\infty} (x-W)\, f\,(x,Dt)\, dx\, dW$$

Substituting 1.3.2. into integrals

$$= \int_V^{\infty} \left(\sigma^2 t\, f\,(W,Dt) - (W-Dt) \cdot F\left(\frac{W-Dt}{\sqrt{\sigma^2 t}}\right)\right) dW$$

Integrating

$$= \sigma^2 t \quad F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) - \frac{1}{2}\int_V^{\infty} F\left(\frac{W-x}{\sqrt{\sigma^2 t}}\right) d\,(W-Dt)^2$$

Integrating by Parts

$$= \sigma^2 t\, F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) - \left[\frac{(W-Dt)^2}{2}\quad F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)\right]_V^{\infty}$$

$$- \int_V^{\infty} \frac{1}{2}\left(\frac{V-Dt}{}\right)^2 \cdot f\,(W,Dt)\quad dW$$

Simplifying

$$= \sigma^2 t \left[1 + \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2 \right] F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) + \int_V^\infty \frac{\sigma^2 t}{2} (W-Dt) \, d\, f(W,Dt)$$

$$= \sigma^2 t \left[1 + \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2 \right] F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) + \frac{\sigma^2 t}{2} \Big[(W-Dt) \quad f\,(W,Dt) \Big]_V^\infty$$

$$- \int_V^\infty \frac{\sigma^2 t}{2} \, f\,(W,Dt) \, dW$$

$$= \sigma^2 t \left[1 + \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2 \right] F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) - \frac{\sigma^2 t}{2} (V-Dt) \quad f\left(V, Dt\right)$$

$$- \frac{\sigma^2 t}{2} F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)$$

Hence

$$\int_V^\infty \int_W^\infty (x-w)\, f\,(x,Dt) \quad dx\, dw$$

$$= \frac{\sigma^2 t}{2} \left[\left(1 + \left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2\right) F\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right) - (V-Dt) \quad f\,(V,Dt) \right] \qquad 1.3.3.$$

But

$$\frac{\partial \left(\frac{x - Dt}{\sqrt{\sigma^2 t}}\right)}{\partial t} = -\left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x - Dt)}{2\sigma \; t^{3/2}}\right)$$

Hence

$$\frac{\partial\left(\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2\right)}{\partial t} = \left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)\left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x-Dt)}{2\sigma t^{3/2}}\right)\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2$$

1.3.4.

Differentiating $f(x,Dt)$ with respect to t

$$\frac{\partial f(x,Dt)}{\partial t} = -\frac{1}{2\sqrt{2\pi\sigma^2}\,t^{3/2}}\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 + \frac{\partial\left(\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2\right)}{\sqrt{2\pi\sigma^2 t}\,.\,\partial t}$$

Hence applying 1.3.4.

$$\frac{\partial f(x,Dt)}{\partial t} = \left(\left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x-Dt)}{2\sigma t^{3/2}}\right)\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) - \frac{1}{2t}\right) f\,(x,Dt)$$

1.3.5.

$$F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) = \int_{\frac{x-Dt}{\sqrt{\sigma^2 t}}}^{\infty} \frac{\exp -\frac{1}{2}V^2}{\sqrt{2\pi}}\, dV$$

Hence differentiating $F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)$ with respect to t

$$\frac{\partial F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)}{\partial t} = -\frac{\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2}{\sqrt{2\pi}}\,.\,\frac{\partial\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)}{\partial t}$$

$$= \frac{\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2}{\sqrt{2\pi}}\,,\left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x-Dt)}{2\sigma t^{3/2}}\right)$$

Hence

$$\frac{\partial\left(F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)\right)}{\partial t} = \sqrt{\sigma^2 t}\left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x-Dt)}{2\sigma t^{3/2}}\right) f(x,Dt)$$ 1.3.6

Let $$g(x) = \frac{\exp - \frac{x^2}{2}}{\sqrt{2\pi}}$$

Hence

$$F(x) = \int_x^{\infty} \frac{\exp - \frac{v^2}{2}}{\sqrt{2\pi}}\, dv$$

Hence integrating by Parts

$$\int_k^{\infty} x^n\ F(x)\ dx = \left[\frac{x^{n+1}}{n+1}\ F(x)\right]_k^{\infty} + \int_k^{\infty} \frac{x^{n+1}}{n+1}\ g(x)\ dx$$

If $n = 0$

$$\begin{aligned}\int_k^{\infty} F(x)\ dx &= -k\ F(k) + g(k)\\ &= g(k) - k\,F(k)\end{aligned}$$ 1.3.7.

When $n = 2$

$$\int_k^{\infty} x^2 F(x)\ dx = \left[\frac{x^3}{3}\ F(x)\right]_k^{\infty} + \int_k^{\infty} \frac{x^3}{3}\ g(x)\ dx$$

Simplifying and integrating by Parts

$$= -\frac{k^3}{3} F(k) - \int_k^{\infty} \frac{x^2}{3} \, dg(x)$$

Simplifying

$$= -\frac{k^3}{3} F(k) - \left[\frac{x^2}{3} g(x)\right]_k^{\infty} + \int_k^{\infty} \frac{2x}{3} g(x) \, dx$$

Noting that $dg(x) = -xg(x)\,dx$

$$= -\frac{k^3}{3} F(k) + \frac{k^2}{3} g(k) - \left[\frac{2}{3} g(x)\right]_k^{\infty}$$

and simplifying

$$\int_k^{\infty} x^2 F(x)\,dx = -\frac{k^3}{3} F(k) + \frac{k^2}{3} g(k) + \frac{2}{3} g(k)$$

$$= -\frac{k^3}{3} F(k) + \frac{1}{3}(k^2+2)\, g(k)$$

1.3.8.(a)

If $n = 1$

$$\int_k^{\infty} x\, F(x)\,dx = \frac{1}{2}\left((1-k^3)\, F(k) + kg(k)\right)$$

1.3.8.(b)

Let $Z_o(x,T) = \int_o^T f(x,Dt)\,dt$

Substituting for $f(x,Dt)$

$$= \int_0^T \frac{\exp - \frac{1}{2}\left(\frac{x - Dt}{\sqrt{\sigma^2 t}}\right)^2}{\sqrt{2\pi\sigma^2 t}} \, dt$$

But $\dfrac{\exp - \frac{1}{2}}{\sqrt{\sigma^2 t}} \left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2$

$$=+ \frac{1}{D}\left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x - Dt)}{2\sigma t^{3/2}}\right) e^{-\frac{1}{2}\left(\frac{x - Dt}{\sqrt{\sigma^2 t}}\right)^2}$$

$$- \frac{(x - Dt)}{D \, . \, 2\sigma t^{3/2}} \exp - \frac{1}{2}\left(\frac{x - Dt}{\sqrt{\sigma^2 t}}\right)^2$$

Hence substituting for $\exp - \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2$ in $Zo(X,T)$

$$Zo\,(x,T) = \frac{1}{\sqrt{2\pi}\ D}\int_0^T \left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x-Dt)}{2\sigma t^{3/2}}\right) \exp-\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 dt$$

$$- \frac{1}{\sqrt{2\pi}\,D}\int_0^T \frac{(x - Dt)}{2\,\sigma t^{3/2}} \exp - \frac{1}{2}\left(\frac{x - Dt}{\sqrt{\sigma^2 t}}\right)^2 dt$$

Let $W = \dfrac{x - Dt}{\sqrt{\sigma^2 t}}$

and $m(t) = \dfrac{x - Dt}{\sqrt{\sigma^2 t}}$ in each integral respectively

Hence substituting into $Zo(x,T)$

$$Zo\,(x,T) = \frac{1}{D}\int_{-\infty}^{\frac{x-DT}{\sqrt{\sigma^2 T}}} \frac{e^{-\frac{W^2}{2}}}{\sqrt{2\pi}} \, dW$$

$$- \frac{1}{2D} \int_0^T \frac{m(t)}{t} * \exp - \frac{1}{2} \quad (m(t))^2 \quad dt$$

We now observe that

$$\frac{\partial Z_o(X,T)}{\partial x} = \frac{\partial \left(\int_0^T \dfrac{\exp - \frac{1}{2} \left(\frac{x-DT}{\sqrt{\sigma^2 t}} \right)^2}{\sqrt{2\pi \sigma^2 t}} \, dt \right)}{\partial x}$$

$$= - \int_0^T \frac{\left(\frac{x-DT}{\sqrt{\sigma^2 t}} \right)}{\sqrt{\sigma^2 t}} \; \frac{\exp - \frac{1}{2} \left(\frac{x-DT}{\sqrt{\sigma^2 t}} \right)^2}{\sqrt{2\pi \sigma^2 t}} \quad dt$$

Substituting m(t) into expression

$$\frac{\partial Zo(X,T)}{\partial x} = \frac{-1}{\sigma^2 \sqrt{2\pi}} \int_0^T \frac{m(t)}{t} \exp - \frac{1}{2} \quad (m(t))^2 \; dt \tag{1.3.9.}$$

But it has been noted that

$$Zo\ (X,T) = - \frac{1}{D} \int_{\infty}^{\frac{x-DT}{\sqrt{\sigma^2 t}}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} \quad dw$$

$$- \frac{1}{2D} \int_0^T \frac{m(t)}{t\sqrt{2\pi}} \exp - \frac{1}{2} \quad (m(t))^2 \quad dt$$

Hence applying 1.3.9. into expression,

$$Zo\ (X,T) = - \frac{1}{D} \int^{\frac{x-DT}{\sqrt{\sigma^2 t}}} \frac{e^{\frac{-w^2}{2}}}{\sqrt{2\pi}} \quad dw + \frac{\sigma^2}{2D} \frac{\partial (Zo(X,T))}{\partial x}$$

This is a differential equation. To solve this differential equation we multiply both sides by exp $-(2Dx/\sigma^2)$.

Then we have

$$\exp -\left(\frac{2Dx}{\sigma^2}\right) \cdot Zo(x,T) = \frac{1}{D}\left[F(w)\right]_{\infty}^{\frac{x-DT}{\sqrt{\sigma^2 T}}} \exp - \frac{2DX}{\sigma^2}$$

$$+ \frac{\sigma^2}{2D} \exp - \frac{2Dx}{\sigma^2} \frac{\partial Zo(x,T)}{\partial x} \qquad 1.3.10$$

But we observe that

$$\frac{\partial\left(\exp -\left(\frac{2Dx}{\sigma^2}\right) * Zo(x,T)\right)}{\partial x}$$

$$= -\frac{2D}{\sigma^2} \cdot \exp -\left(\frac{2Dx}{\sigma^2}\right) \cdot Zo\ (x,T) + \exp -\left(\frac{2Dx}{\sigma^2}\right) \frac{\partial Zo\ (x,T)}{\partial x}$$

Substituting into 1.3.10 and re-arranging terms

$$\frac{\partial}{\partial x}\left(\exp - \frac{2Dx}{\sigma^2} \cdot Zo\ (x,T)\right) = -\frac{2}{\sigma^2} \exp - \frac{2DX}{\sigma^2} \quad F\left(\frac{x-Dt}{\sqrt{\sigma^2 T}}\right)$$

(Reference Hadley & Whitin Page 146).

Integrating both sides from x to ∞ and noting that

$$\lim_{x \to \infty} \exp - \frac{2Dx}{\sigma^2} \quad Zo \quad (x,T) = 0$$

We obtain

$$\exp - \left(\frac{2Dx}{\sigma^2}\right) \cdot Zo\ (x,T) = \frac{2}{\sigma^2}\int_x^{\infty} \exp -\left(\frac{2DV}{\sigma^2}\right) F\left(\frac{V-DT}{\sqrt{\sigma^2 T}}\right) \quad dV$$

Integrating by Parts.

$$\exp-\left(\frac{2Dx}{\sigma^2}\right).Zo\ (x,T) = -\frac{1}{D}\left[\exp-\frac{2DV}{\sigma^2}\quad F\left(\frac{V-DT}{\sqrt{\sigma^2 t}}\right)\right]_x^{\infty}$$

$$+\frac{1}{D}\int_x^{\infty} e^{-\frac{2DV}{\sigma^2}}\quad \exp-\left(\frac{V-DT}{\sqrt{\sigma^2 t}}\right)^2 \quad dV$$

Simplifying

$$\exp-\left(\frac{2Dx}{\sigma^2}\right).\ Zo\ (x,T) = +\frac{1}{D}\exp-\left(\frac{2Dx}{\sigma^2}\right).\ F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+\quad\frac{1}{D}\int_x^{\infty}\exp-\frac{\left(\frac{2DV}{\sigma^2}\right)}{\sqrt{\sigma^2 t}}\quad\exp-\left(\frac{V-DT}{\sqrt{\sigma^2 t}}\right)^2\quad dV$$

But $\exp-\left(\frac{2DV}{\sigma^2}\right)\exp-\frac{1}{2}\left(\frac{V-DT}{\sqrt{\sigma^2 T}}\right)^2$

$$= e^{-\frac{1}{2}\left(\frac{V^2-2DTV+D^2T^2+4DTV}{\sigma^2 T}\right)}$$

$$= \exp-\frac{1}{2}\left(\frac{V+DT}{\sqrt{\sigma^2 t}}\right)^2$$

Hence substituting into $\exp-\left(\frac{2Dx}{\sigma^2}\right)Zo\ (x,T)$ and integrating

$$\exp-\left(\frac{2Dx}{\sigma^2}\right)Zo\ (x,T) = \frac{+\exp}{D}-\left(\frac{2Dx}{\sigma^2}\right).\ F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)$$

$$-\frac{F}{D}\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)$$

Hence

$$Zo\ (x, T) = \frac{F}{D}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{1}{D}\exp\left(\frac{2Dx}{\sigma^2}\right) F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)$$

Hence

$$\int_0^T f\ (x, Dt)\ dt = \frac{F}{D}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\exp\left(\frac{2Dx}{\sigma^2}\right)}{D} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right) \qquad 1.3.11$$

The method used to derive equation 1.3.11 is the same method used to derive Jo $(x, T_1\ T_2)$ by Hadley & Whitin in Chapter 3., page 144-148.

Next I would evalue $Z_1(x, T)$ where .

$$Z_1(x, T) = \int_0^\infty t f(x, Dt)\ dt$$

$$Zo(x, T) = \int_0^T \frac{1}{\sqrt{2\pi\sigma^2} t^{\frac{1}{2}}} \exp - \frac{1}{2}\left(\frac{x - Dt}{\sqrt{\sigma^2 t}}\right)^2 dt$$

$$= \int_0^T \frac{t^{-\frac{1}{2}}}{\sqrt{2\pi\sigma^2}} \cdot \exp\ \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 Dt$$

Integrating by Parts and applying 1.3.4.

$$Zo(x, T) = \left[\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2\right]_0^T$$

$$- 2 \int_0^T \frac{t^{\frac{1}{2}}}{\sqrt{2\pi\sigma^2}} \left(\frac{(x-Dt)D}{\sigma^2 t} + \frac{(x-Dt)^2}{2\sigma^2 t^2} \right) \exp - \frac{1}{2} \frac{(x-Dt)^2}{\sigma^2 t} \, dt$$

Simplifying

$$Zo(x,T) = \left. \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2} \left(\frac{x-Dt}{\sqrt{\sigma^2 t}} \right)^2 \right]_0^T + \frac{D^2}{\sigma^2} \int_0^T \frac{t}{\sqrt{2\pi\sigma^2 t}} \exp - \frac{1}{2} \left(\frac{x-Dt}{\sqrt{\sigma^2 t}} \right)^2 dt$$

$$- \frac{x^2}{\sigma^2} \int_0^T \frac{t^{-3/2}}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2} \left(\frac{x-Dt}{\sqrt{\sigma^2 t}} \right)^2 dt$$

Let $Z_1(x,T) = \int_0^T \frac{t}{\sqrt{2\pi\sigma^2 t}} \exp - \frac{1}{2} \left(\frac{x-Dt}{\sqrt{\sigma^2 t}} \right)^2 dt$

Hence substituting $Z_1(x,T)$ into $Z_o(x,T)$

$$Zo(x,T) = \left. \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2} \left(\frac{x-Dt}{\sqrt{\sigma^2 t}} \right)^2 \right]_0^T + \frac{D^2}{\sigma^2} Z_1(x,T)$$

$$- \frac{x^2}{\sigma^2} \int_0^T \frac{t^{-3/2}}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2} \frac{(x-Dt)^2}{\sigma^2 t} \, dt$$

But from 1.3.9

$$\frac{\partial Zo(X,T)}{\partial x} = - \frac{1}{\sqrt{2\pi}} \int_0^T \left(\frac{x-Dt}{\sqrt{\sigma^2 t}} \right) \frac{\exp - \frac{1}{2} \left(\frac{x-DT}{\sqrt{\sigma^2 t}} \right)^2}{\sqrt{\sigma^2 t}} \, dt$$

Hence substituting into Zo (x,T) and simplifying

$$Zo\ (x,T) = + \frac{D^2}{\sigma^2} \ast Z_1\ (x,T)$$

$$+ \frac{x\,\partial Zo\,(x,T)}{\partial x} - \frac{Dx}{\sigma^2} Zo\,(x,T)$$

$$+ \left[\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 \right]_0^T$$

Hence

$$Z_1(x,T) = \frac{\sigma^2}{D^2}\left[- \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 \right]_0^T$$

$$+ \frac{\sigma^2}{D^2}\left(1 + \frac{Dx}{\sigma^2}\right) Zo(x,T) - \frac{\sigma^2 x}{D^2} \frac{\partial (Zo(x,T))}{\partial x}$$

1.3.12

This gives $Z_1(x,T)$ in known quantities.

From 1. 3. 11

$$Zo\,(x,T) = \frac{F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)}{D} - \frac{\exp\left(\frac{2DX}{\sigma^2}\right) F\left(\frac{x+Dt}{\sqrt{\sigma^2 t}}\right)}{D}$$

Hence differentiating with respect to x

$$\frac{\partial Zo(x,T)}{\partial x} = - \frac{\exp - \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2}{D\sqrt{\sigma^2 t 2\pi}} - \frac{2D}{\sigma^2} \exp\left(\frac{2DX}{\sigma^2}\right) F\left(\frac{x+Dt}{\sqrt{\sigma^2 t}}\right)$$

$$+ \frac{1}{D\sqrt{2\pi\sigma^2 t}} \exp - \frac{1}{2}\left(\frac{x-Dt}{\sigma^2 t}\right)^2$$

1.3.13

Hence substituting $Zo(x,T)$ and $\frac{\partial Zo(x,T)}{\partial x}$ into $Z_1(x,T)$ and simplifying we have

$$Z_1(x,T) = \frac{-2\sqrt{\sigma^2 T}}{D^2} \frac{\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 T}}\right)^2}{\sqrt{2\pi}}$$

$$+ \frac{\sigma^2}{D^3}\left(1 + \frac{DX}{\sigma^2}\right) F\left(\frac{x-DT}{\sqrt{\sigma^2 t}}\right) + \frac{1}{D^2}\left(x - \frac{\sigma^2}{D}\right) e^{\frac{2Dx}{\sigma^2}}$$

$$\times F\left(\frac{x+Dt}{\sqrt{\sigma^2 T}}\right) \qquad 1.3.14(a)$$

Next we shall derive the difference equation for higher orders of $Z_n(x,T)$ where $n = o, 1, 2 \ldots\ldots\ldots$

Let $$Z_n(x,T) = \int_o^T \frac{t^n}{\sqrt{2\pi o^2 t}} \exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 dt.$$

Integrating by Parts we have

$$Z_n(x,T) = \frac{t^{n+\frac{1}{2}}}{(n+\frac{1}{2})} \frac{1}{\sqrt{\sigma^2 2\pi}} \cdot \exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 \Bigg]_o^T$$

$$-\int_o^T \frac{t^{n+\frac{1}{2}}}{\sqrt{2\pi\sigma^2}(n+\frac{1}{2})}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)\left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x-Dt)}{2\sigma\, t^{3/2}}\right) *$$

$$\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 dt$$

Simplifying

$$= \frac{2t^{n+\frac{1}{2}}}{\sqrt{2\pi\sigma^2}(2n+1)} \exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 \Bigg]_o^T + \frac{D^2}{(2n+1)\sigma^2}\int_o^T \frac{t^{n+1}}{\sqrt{2\pi\sigma^2 t}} *$$

$$\exp -\frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 dt$$

$$- \frac{x^2}{\sigma^2} \int_0^T \frac{t^{n-1}}{\sqrt{2\pi\sigma^2 t}\,(2n+1)} \exp - \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 \, dt$$

Hence

$$Z_n(x,T)\,(2n+1)\,\frac{\sigma^2}{D^2}$$

$$= \frac{2\sigma^2}{D^2}\,\frac{T^{n+1}}{\sqrt{2\pi\sigma^2 T}} \exp - \frac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2 + Zn+1(x,T) - \frac{x^2}{D^2} Z_{n-1}(x,T)$$

Hence we have

$$Zn+1\,(x,T) = - \frac{2\sigma^2 T^{n+1}}{D^2\sqrt{2\pi\sigma^2 T}} \exp - \frac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$+ \frac{(2n+1)\sigma^2}{D^2} Z_n(x,T) + \frac{x^2}{D^2} Z_{n-1}\,(x,T)$$

$$n = 0,\ 1,\ 2\ \ldots\ldots$$

1.3.14b.

$$\text{Let } F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) = \int_x^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp - \frac{1}{2}\left(\frac{V-Dt}{\sqrt{\sigma^2 t}}\right)^2 \, dV$$

$$\text{Let } R_n(x,T) = \int_0^T t^n F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) dt \quad n = 0,\ 1,\ 2\ \ldots\ldots$$

Integrating by Parts we have

$$R_n(x,T) = \left.\frac{t^{n+1}}{n+1}\, F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)\right]_0^T$$

$$- \frac{1}{\sqrt{2\pi}} \int_0^T \frac{t^{n+1}}{(n+1)\sqrt{\sigma^2 T}} \left(\frac{D}{\sqrt{\sigma^2 t}} + \frac{(x-Dt)}{2\sigma t^{3/2}}\right) \exp - \frac{1}{2}\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)^2 \, dt$$

Hence substituting in $Z_{n+1}\,(x,T)$ and $Z_n(x,T)$

$$R_n(x,T) = \frac{T^{n+1}}{n+1} F\left(\frac{x-Dt}{\sqrt{\sigma^2 T}}\right) - \frac{D}{2(n+1)} Z_{n+1}(x,T)$$

$$-\frac{x}{2(n+1)} Z_n(x,T) \qquad n=0,\ 1,\ 2\ldots\ldots$$

1.3.15

CHAPTER 2

Exact, Inexact, Heuristic And Simple Models Of (Q,R)

Introduction

In this chapter we examine the (Q,R) model at several levels of complexity. The Exact,Inexact, Heuristic and Simple models of (Q,R) are different versions of (Q,R). The Exact and Inexact versions of (Q,R) take into considertion that when there is a stockout, the cost is made up of three items: one is a constant dependent on the number of stockouts only, the second item is proportional to the number of units backordered, and the third is proportional to the time for which each backordered item has been backordered. The Simple and Heuristic models take into consideration only one of the three items, which is the constant cost incurred whenever a stockout occurs.

Hence the additional shortage costs taken into consideration by the Exact and Inexact models over the Heuristic and Simple models are the costs proportional to the number of units backordered, and the costs proportional to the time for which each backordered item has been backordered.

Firstly we compare the Simple model against the Heuristic model. Secondly we compare the Simple ,Heuristic , Inexact models against the Exact model when two of the three cost items are zero in both the Inexact and Exact models. The terms which are zero,are the term proportional to the number of units backordered and the term proportional to the time for which each backordered item has been backordered.

In sections 2.1 and 2.2 we derive the Simple and Heuristic models respectively. In section 2.3 we describe Tate's Exponential approximation to the Heuristic model. In section 2.4 we give the computed results and comparisons between the Simple and the Heuristic models. In section 2.5 we describe the Heuristic model when demand is Uniformly distributed. In sections 2.6 we derive the Exact and Inexact models. In section 2.7 we compare Inexact and Exact models. Finally in section 2.8 we compare the Simple ,Heuristic, Inexact models against the Exact model.

Section 2.1

In this section we consider the simple versions of the (Q,R) inventory control procedure which was described in chapter 1 section 1.
The simple models considered are the batch quantity model which considers batch quantity only, re-order level model which considers both the safety stock only and a heuristic approximation of (Q,R) which considers both the safety stock and batch quantity developed by Tate (ref. 5). Firstly the models are described and the inventory costs are compared later.

(a) Batch quantity only (referred to here as the EBQ model- Economic Batch Quantity)

Inventory cost =Set up cost +stockholding cost

$$C = \frac{DS}{Q} + \frac{Qhc}{2}$$

$$\frac{dC}{dQ} = -\frac{DS}{Q^2} + \frac{hc}{2}$$

A minimum is obtained when $\frac{dC}{dQ} = 0$ Hence $Q = \sqrt{2DS/hc}$ 2.1.1

(b) RE-order level only (referred to here as the ROL model)

This model is the simple one in ref 5. It considers cost of holding safety stock and stockout costs only, in which the cost of each stockout that occurs is taken to be a constant s. k is the safety stock in standard deviations.

σ = standard deviation of demand over lead time.

Cost =Stockholding cost + Shortage cost

$$C = k\sigma hc + \frac{F(k)\ Ds}{Q}$$

$$\frac{dC}{dk} = \sigma hc - \frac{g(k)\ Ds}{Q} \qquad 2.1.2$$

A minimum is obtained when $\frac{dC}{dk} = 0$

Hence $g(k) = \frac{\sigma hcQ}{Ds}$ 2.1.3

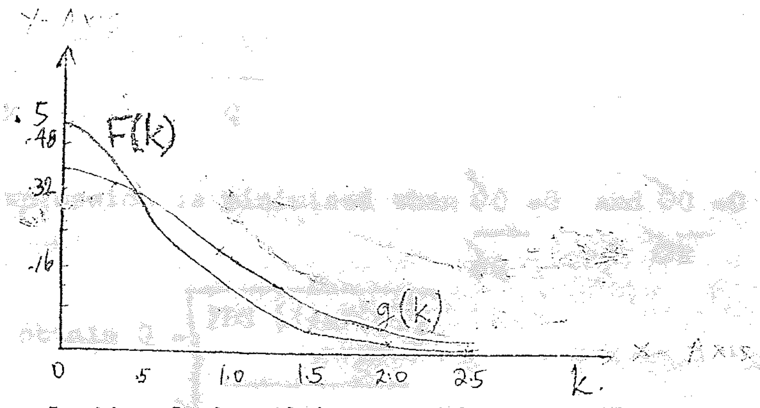
If demand follows the normal distribution then $g(k) = \dfrac{\exp - \frac{k^2}{2}}{\sqrt{2\pi}}$

substituting into 2.1.3

$$k = \sqrt{2 \ln \frac{Ds}{\sqrt{2\pi}\sigma hcQ}} \qquad 2.1.4.$$

Note that the answers contain Q.

When demand in the lead time is normally distributed then k, F(k) and $g(k)$ are related as in the diagram below.



F(k) is defined in this section as the probability that demand in the lead time exceeds k standard deviations of stock.

$$F(k) = \int_k^{\infty} g(x)\, dx$$

where g(x) is the probability that demand equals x in the lead time, expressed in standard deviations.

SECTION 2.2. HEURISTIC MODEL

The heuristic model chosen is a combination of the two simple models described in Section 2 developed by Tate (Ref. 5). The heuristic model considers the average costs of holding batch quantity, cost of ordering goods, cost of holding safety stock and the only shortage factor considered is the probability that demand exceeds the safety stock in the lead time multiplied by the stockout costs.

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + \sigma khc + \frac{DsF(k)}{Q} \qquad 2.2.1$$

Hadley and Whitin (Ref. 1) have two similar models in which the shortage cost is (1) equal to the expected number of backorders multiplied by the cost of a backorder and (2) the shortage cost is equal to the expected number of lost sales multiplied by the cost of a lost sales, rather than as a simple function of the number of backorders as considered in this section. These are better suited in finished goods stock while a constant penalty is better suited to raw materials.

Differentiating with respect to Q and k

$$\frac{\partial C}{\partial Q} = -\frac{DS}{Q^2} + \frac{hc}{2} - \frac{DsF(k)}{Q} \qquad 2.2.2$$

$$\frac{\partial C}{\partial k} = \sigma hc - \frac{Dsg(k)}{Q} \qquad 2.2.3$$

The cost expression is minimised when $\frac{\partial C}{\partial Q} = 0$ and $\frac{\partial C}{\partial k} = 0$

Solving we obtain $Q = \sqrt{\frac{2DS\left(1+\frac{sF(k)}{S}\right)}{hc}}$ 2.2.4

which is a function of k. Let $Q^* = \sqrt{\frac{2DS}{hc}}$ which is the batch quantity obtained from the EBQ model.

Hence $Q = Q^*\sqrt{\frac{1+sF(k)}{S}}$ 2.2.5

and

$$g(k) = \frac{Q\sigma hc}{Ds} \qquad 2.2.6$$

It is difficult to solve these equations for Q and k, for most models of demand distribution. In the special case when the demand is uniformly distributed (section 5), there is no consistent solution for Q and k. When demand is normally distributed

$$g(k) = \frac{\exp -k^2/2}{\sqrt{2\pi}}$$

$$F(k) = \int_k^{\infty} g(x)\,dx$$

substituting into 2.1.2 we have as for the ROL model

$$k = \sqrt{2\log_e \frac{Ds}{\sqrt{2\pi}\,\sigma\,Qhc}} \qquad 2.2.7$$

which is a function of Q. Since it is impossible to solve for Q and k explicitly Tate has suggested an exponential approximation to F(k) (ref. 5) inorder to determine whether even this level of sophistication was worthwhile. With an exponential approximation to F(k) Q and k can be obtained analytically. Tate's exponential approximation is described in the next section.

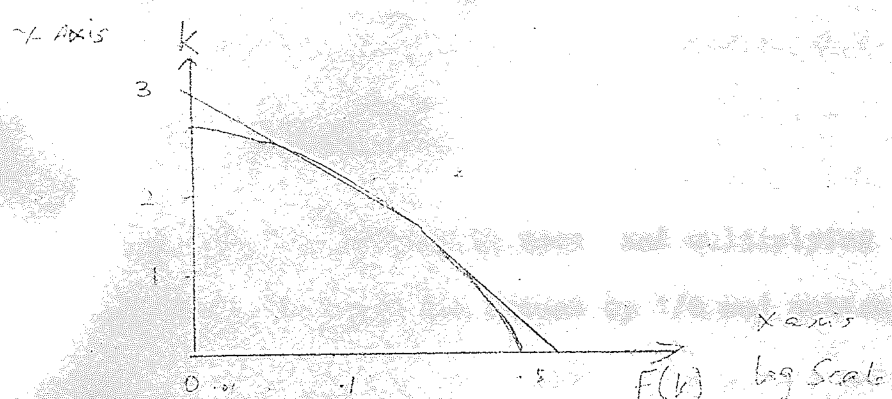
Section.2.3. Tate's Exponential Approximation

Let $F(k) = e^{a-bk}$ $\qquad$ 2.3.1

so that

$$\frac{\partial F(k)}{\partial k} = -be^{a-bk}$$

k is plotted against F(k) with F(k) on a log scale. The graph is divided into sections and each section is approximated by a linear function. the case for two linear functions is given below.



Tate's says that this is not a bad fit since the resulant probability F(k) of a stockout occuring, given a safety stock of k standard deviations of demand during the lead time is described with a maximum error of only 3 percent using two linear logarithmic approximation fitted by eye. The values he gives for two logarithmic approximation are given below.

$a = 5.65$ and $b = 2.49$ for $1.3 \leq k \leq 3.2$

$a = 4.05$ and $b = 1.32$ for $0 \leq k \leq 1.3$

A better fit to F(k) can be obtained by using more than two logarithmic approximation. The result for eight logarithmic approximations is presented here.

$a_1 = -0.68996$	$b_1 = 0.879292$	$0 \le k \le 0.25$
$a_2 = -0.64449$	$b_2 = 1.0581103$	$0.25 \le k \le 0.52$
$a_3 = -0.527307$	$b_3 = 1.279003$	$0.52 \le k \le 0.85$
$a_4 = -0.26906$	$b_4 = 1.57689$	$0.85 \le k \le 1.28$
$a_5 = 0.150199$	$b_5 = 1.906544$	$1.28 \le k \le 1.65$
$a_6 = 0.699972$	$b_6 = 2.238397$	$1.65 \le k \le 2.33$
$a_7 = 1.60653$	$b_7 = 2.700184$	$2.33 \le k \le 4.00$
$a_8 = -8.91797$	$b_8 = 0.03004$	$4.0 \le k \le 8.00$

Applying Tate's exponential approximation

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + k\sigma hc + \frac{Dse^{a-bk}}{Q}$$

Differentiating

$$\frac{\partial C}{\partial Q} = \frac{DS}{Q^2} + \frac{hc}{2} - \frac{Ds\, e^{a-bk}}{Q^2} \qquad 2.3.2$$

$$\frac{\partial C}{\partial k} = \sigma hc - \frac{bDs\, e^{a-bk}}{Q} \qquad 2.3.3$$

Equating both derivatives to zero and multiplying the first derivative by b and the second by 1/Q and subtracting we have

$$-\frac{bDs}{Q^2} + \frac{bhc}{2} - \frac{\sigma hc}{Q} = 0$$

Hence $-\,bDs + \frac{bhcQ^2}{2} - \sigma hcQ = 0$

Hence

$$Q = \frac{\sigma}{b} + \sqrt{\frac{\sigma^2}{b^2} + \frac{2DS}{hc}} \qquad 2.3.4$$

If the standard deviation of demand in the lead time is 0 then $Q = Q^* = \sqrt{\frac{2DS}{hc}}$ which corresponds with economic batch quantity.

From 2.3.3 solving for k

$$k = \frac{1}{b}\left(a - \log_Q \frac{Q\sigma - hc}{bDs}\right) \qquad 2.3.5$$

Tate uses it to show that 30 percent savings can be achieved on some items over the separate calculation of batch quantity and re-order level as if they were independent. The advantage of the exponential approximation is that Q and k can be obtained analytically. As the standard deviation of demand tends to zero Q tends to the batch quantity of the EBQ model. (from 2.3.4).

In the next section we would assign values to the parameters and compute the inventory costs.

Section 2.4 Results and Comparisons

Our criteria for comparing the models as stated in the introduction to the thesis is based on which model gives the least inventory costs bearing in mind the cost of computation of each model on the computer.

The following table gives the values assigned to the parameters.

Lead time in years.	= 0.1, 0.2, 0.4
Holding Costs in £s	= 0.01, 0.1, 1.0
Order cost in £s	= 0.1, 1.0, 10.0
Cost of a stockout in £s	= Leadtime demand * 0.005, 0.05
Demand per year	= 10, 100, 1000, 10000, 100000, 1000000

Savings are defined as the excess of inventory costs obtained by using the EBQ-ROL model or Tate's exponential approximation over the inventory costs obtained by using the Heuristic model of (Q,R).

For all the ranges of parameters considered, the EBQ-ROL model produced the highest inventory costs, while the heuristic model produced the least.

For example when the

Lead time = 0.1 Of a year

Holding cost = 0.1

Order cost= 10.0

Cost of a stockout in £s =0.005* Leadtime demand we have from pages 56-58 the following annual costs.

Demand	Heuristic	Tate's Exponential Approximation	EBQ-ROL
10	4.473	4,473	4.473
100	14.160	14.160	14.160
1000	,45.277	45.291	45.280
10000	158.114	158.422	159.099
100000	836.660	840.225	937.011
1000000	7211.103	7215.283	8366.600

When demand is 1000000, for example, Tate's exponential approximation is only £4.180 more than the inventory costs of the Heuristic model while the EBQ-ROL model is £1150.317 more than the inventory costs of the Heuristic model. Tate's exponential approximation was slightly inferior to the Heuristic model, and much better than the EBQ-ROL model for most of the combinations of parameters.

The results confirm what would be expected because with a good logarithmic approximation of the probability of a stockout, Tate's exponential approximation would reproduce the Heuristic model very accurately, while the EBQ-ROL is inferior to both because batch quantity and reorder level are calculated independently.

As well as producing the highest inventory costs, the EBQ-ROL produces also inconsistent results. We can see from pages 56-58 that the Heuristic and Tate's exponential approximation annual inventory costs increases as Order costs increased for each level of demand while for the EBQ-ROL model the annual costs decreased as order cost increased. The reason is that the optimum values of Q,k can not be obtained for the EBQ-ROL model for some combinations of the parameters.

From equation 2.1.4

$$k = \sqrt{2 \log_e \frac{Ds}{\sqrt{2\pi}\,\sigma\, hcQ}}$$

If for any combination of the parameters $\log_e \frac{Ds}{\sqrt{2\pi}\,\sigma\, hcQ}$ is less than zero then k is set equal to zero, since complex and negative values of k are not allowed.

Also since complex and negative values of k are not permitted, the Heuristic model does not produce the optimum inventory costs in certain cases, also. However we eliminate these inconsistencies in section section 2.8, when we substitute the optimum values of the batch quantity and reorder level obtained for the Simple and Heuristic models into the cost expressions of the Exact model.

The time taken to solve 100 sets of equations for each model is as follows.

Heuristic model	35 seconds
Tate's exponential approximation	25 seconds
EBQ-ROL model	25 seconds

The cost of a minute on the computer is approximately £ 1. As a result the cost of solving one set of equation for each model is approximately the same about $\frac{1}{3}$ of a penny.

Considering the fact that the difference in computational costs between the models is negligible, and also since the Heuristic model produces the least inventory costs, the Heuristic model could always be used in place of the other two models.

SECTION 2.5. UNIFORM DISTRIBUTION

In this section, demand over the lead time is assumed to be a uniform distribution, although it is unlikely that demand in practice, would be uniformly distributed.

In this section we would illustrate that the Heuristic model of Section 2.2. would not give consistent results when demand over the lead time is uniformly distributed.

From Section 2.2.

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + k\sigma hc + \frac{DsF(k)}{Q} \qquad 2.5.1.$$

$$\frac{\partial C}{\partial Q} = -\frac{DS}{Q^2} + \frac{hc}{2} - \frac{DsF(k)}{Q2} \qquad 2.5\ 2.$$

$$\frac{\partial C}{\partial k} = \sigma hc - \frac{Dsg(k)}{Q^2} \qquad 2.5.3.$$

BASIC MATHEMATICS

If V is a uniform variate with mean μ and variance σ^2 and g(v) is the probability density function.

$$g(v) = \frac{1}{s_1 - s_0} \qquad s_0 < v < s_1$$

$$\text{mean of } v = \int_{s_0}^{s_1} v \; dv/(s_1 - s_0)$$

$$= (s_1 + s_0)/2.$$

$$\text{Variance} = \int_{s_0}^{s_1} v^2 \, dv/(s_1 - s_0) - (\text{mean})^2$$

$$= \frac{(s^3_1 - s^3_0)}{3\,(s_1 - s_0)} - \frac{(s_1 + s_0)^2}{4}$$

$$= \frac{(s_1 - s_0)^2}{12.}$$

Probability that

$$v \geqslant m$$

$$= g\,(v \geqslant m) = \int_m^{s_1} \frac{1}{(s_1 - s_0)}\, dv$$

Let v be standardised and let $k = \dfrac{m - \dfrac{(s_1 + s_0)}{2}}{\dfrac{(s_1 - s_0)}{\sqrt{12}}}$; $x = \dfrac{v - \dfrac{(s_1 + s_0)}{2}}{\dfrac{(s_1 - s_0)}{\sqrt{12}}}$

Hence probability that demand is greater than k, F(k)

$$F(k) = \int_k^{\sqrt{3}} dx/\sqrt{12} \quad \text{substituting for v and m}$$

$$F(k) = \frac{(\sqrt{3} - k)}{2\sqrt{3}.} \qquad 2.5.4.$$

Hence

Probability that demand exceeds k standard deviations of stock when demand is uniformly distributed is $(\sqrt{3} - k)/\,2\sqrt{3}$

From 2.5.4.

$$F(k) = \frac{(\sqrt{3} - k)}{2\sqrt{3}} \quad \text{but} \quad \frac{\partial F(k)}{\partial k} = -g(k)$$

Hence $g(k) = \dfrac{1}{2\sqrt{3}}$

Substitute into 2.5.2 and 2.5.3.

$$\frac{\partial C}{\partial Q} = -\frac{DS}{Q^2} + \frac{hc}{2} - \frac{Ds(\sqrt{3}-k)}{2\sqrt{3}\times Q^2}$$

$$\frac{\partial C}{\partial k} = \sigma hc - \frac{Ds}{2\sqrt{3}\cdot Q} \qquad 2.5.5.$$

From 2.5.5.

$$Q = \frac{Ds}{2\,\sigma hc\sqrt{3}} \qquad 2.5.6.$$

From 2.5.5

$$-\frac{DS}{Q2} + \frac{hc}{2} = \frac{Ds(\sqrt{3}-k)}{2\,Q2\sqrt{3}}$$

Hence $$\frac{(\sqrt{3}-k)}{2\sqrt{3}} = -\frac{S}{s} + \frac{Q^2hc}{2Ds}$$

Hence

$$k = \frac{2\sqrt{3}S}{s} - \frac{\sqrt{3}\,hc\,Q^2}{Ds} + \sqrt{3} \qquad 2.5.7.$$

Substitute 2.5.6. into the expression for k in 2.5.7.

$$k = \frac{2\sqrt{3}\,S}{s} - \sqrt{3}\cdot\frac{hc\cdot D^2\cdot s^2}{Ds\cdot 12\,\sigma^2hc^2} + \sqrt{3}$$

$$= \frac{2\sqrt{3}\,S}{s} - \frac{Ds}{4\sqrt{3}\,\sigma^2hc} + \sqrt{3} \qquad 2.5.8.$$

It seems likely that k lies between 0 and $\sqrt{3}$. For example if the probability of stockout $f(k)=0$ then $k=\sqrt{3}$.

Consequently the maximum number of standard deviations of stock it is sensible to keep is $\sqrt{3}$. Now if Q is taken to be the simple EBQ $= \sqrt{\dfrac{2DS}{hc}}$

and substituting into 2.5.7 , $\quad k = \sqrt{3}$.

Hence, when the batch quantity Q is taken as equal to the batch quantity of EBQ model, then $k = \sqrt{3}$ and the probability of a stockout is always zero.

At the other extreme , when 0 standard deviations of stock is held the probability of stockout $F(0) = \frac{1}{2}$. and the batch quantity is

$$Q = \sqrt{\frac{2D(S+s/2)}{hc}}$$

Now the joint minimisation of Q and k gives from 2.5.6

$$Q \doteq \frac{Ds}{2\sigma\ hc\sqrt{3}} \quad \text{and} \quad k = \frac{2\sqrt{3}\ S}{s} - \frac{Ds}{4\sqrt{3}\ \sigma^{2}\ hc} + \sqrt{3}$$

We now compare the three ways of obtaining Q and k. We show that the joint minimisation of Q and k do not necessarily give the least inventory costs.

The following table gives the inventory costs for the three different ways, when the values assigned to the parameters are :

Lead time	= 0.1 of a year.
Order cost in £s	= 10.0
Holding cost in £s	=0.1
Variance Of demand	= 0.1* annual demand

Stockout costs =£ 0.0025 * Lead time demand

Annual Demand	Joint Minimisation	EBQ, $k=\sqrt{3}$	$Q=\sqrt{\frac{2D(S+s/2)}{hc}}$ k=0
10	4.531	4.527	4.473
100	14.518	14.315	14.160
1000	45.278	45.269	45.277
10000	730.3481	143.153	158.114
100000	22829.441	452.691	836.661
1000000	721705.850	1431.534	7211.102

Stockout costs =£ 0.005 * Lead time demand

Annual Demand	Joint Minimisation	EBQ, $k=\sqrt{3}$	$Q=\sqrt{\frac{2D(S+s/2)}{hc}}$ k=0
10	4.5373	4.527	4.478
100	14.318	14.315	14.318
1000	230.957	45.269	50.000
10000	7219.303	143.153	264.575
100000	228223.429	452.691	2280.351
1000000	7216895.755	1431.534	22405.357

Hence judging from the above table of results, the joint minimisation of the Heuristic model need not necessarily give the optimum values of batch quantity and reorder level, when demand is Uniformly distributed.

Section 2.6 Exact and Inexact Models Of (Q,R).

In the last sections we assumed that whenever there was a stockout the cost was constant and independent of the units backordered. In this section it is assumed that when there is a stockout, the cost is made up of three items one is a constant dependent on the number of stockouts only, the second term is proportional to the number of units backordered, and the third is proportional to the time for which each backordered item has been backoredered.

The Exact and Inexact models take into consideration all the three items of shortage costs defined above. We compare the Inexact and Exact models in section 2.7. In section 2.8 we show the differences between all the versions of (Q,R).

The optimization techniques of section 2.2 of Vol. 1. were applied to solve the models where applicable and the computational times are given.

Remembering that (Q,R) was defined in section 1.1. of volume 1 as the model in which review of stock was continuous and an order equal to Q was placed whenever the stock level equalled R, we now proceed to develop the cost expressions for the Exact and Inexact models.

Firstly we develop the steady state probability $p(R+y)$ that the inventory position of the system is R+y. $\left(0 < y < Q. \right)$

A knowledge of this is necessary for the computation of the state probabilities for the net inventory.

The mean rate of demand per year is D. As a result demand in time dt is Ddt if demand occurs. In a time dt the inventory position moves from R+x to R+x−Ddt for all values of x (0 to Q) if demand occurs in dt.

If the inventory position is R+Dt at time t, then it moves to R+Q at time t+dt, since demand triggers the replacement of an order.

Hence

$$\rho(R+Ddt) = \rho(R+2Ddt) = \ . \ . \ . = \rho(R+Q-Ddt) = \rho(R+Q)$$

and since the probability that the system lies in an inventory position is 1.

Hence

$$\int_0^Q \rho(R+y)\,dy = 1$$

It follows therefore that the unique solution of $\rho(r+y) = \frac{1}{Q}$.

The proof is also given in (Ref. 1).

Let f(x,DL) be the probability density function of demand in a lead time L. x follows a normal distribution with mean DL and variance $\sigma^2 L$

$$f(x,DL) = \frac{\exp -\frac{1}{2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2}{\sqrt{2x\sigma^2 L}}$$

Let the re-order level be R and batch quantities be Q.

Let the re-order level R be expressed as k standard deviations of stock. If the inventory position is V at time 0 and the inventory position is x at time t then V−x has been demanded where x can be between o and R+Q.

The method of analysis is exactly similar to that of Hadley and Whitins (Ref.1) chapter 4, except that the analysis here is done in standard deviations of stock.

Let $G_1(x)$ be the probability density function of the quantity on hand x at any time t. x could be less than the re-order level, R or above the re-order level R.

$G_1(x)$ = probability that the system was in state V * probability that V-x was demanded and average over the states of V. If x lies below re-order level R,

$$G_1(x) = \frac{1}{Q}\int_R^{R+Q} \frac{\exp -\frac{1}{2}\left(\frac{V-x-DL}{\sqrt{\sigma^2 L}}\right)^2}{\sqrt{2\pi\sigma^2 L}}\, dV \qquad o < x < R$$

Let $Z = \frac{V-x-DL}{\sqrt{\sigma^2 L}}$ and $k = \frac{R-DL}{\sqrt{\sigma^2 L}}$

then

$$G_1(x) = \frac{1}{Q}\int_{k-\frac{x}{\sqrt{\sigma^2 L}}}^{k+\frac{Q-x}{\sqrt{\sigma^2 L}}} \frac{\exp -\frac{Z^2}{2}\, dZ}{\sqrt{2\pi}}$$

Integrating

$$G_1(x) = \frac{1}{Q}\left(F\left(k - \frac{x}{\sqrt{\sigma^2 L}}\right) - F\left(k + \frac{Q-x}{\sqrt{\sigma^2 L}}\right)\right) \qquad o < x < R \qquad 2.6.1.$$

If x lies above R

$$G_1(x) = \frac{1}{Q}\int_x^{R+Q} \frac{\exp -\frac{1}{2}\left(\frac{V-x-DL}{\sqrt{\sigma^2 L}}\right)^2}{\sqrt{2\pi\sigma^2 L}}\, dV \qquad R < x < R+Q$$

Expressing in standard deviations of stock also

$$G_1(x) = \frac{1}{Q} \int_{\frac{-DL}{\sqrt{\sigma^2 L}}}^{k+\frac{Q-x}{\sqrt{\sigma^2 L}}} \frac{\exp -\frac{z^2}{2}}{\sqrt{2\pi}} dz \qquad R < x < R+Q$$

$$= \frac{1}{Q}\left(F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) - F\left(k + \frac{Q-x}{\sqrt{\sigma^2 L}}\right)\right) \qquad 2.6.2.$$

We shall assume that $F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) = 1$.

Therefore from 2.6.1.

$$G_1(x) = \frac{1}{Q}\left(F\left(k - \frac{x}{\sqrt{\sigma^2 L}}\right) - F\left(k + \frac{(Q-x)}{\sqrt{\sigma^2 L}}\right)\right) \qquad 0 < x < R$$

$$G_1(x) = \frac{1}{Q}\left(1 - F\left(k + \frac{Q-x}{\sigma^2 L}\right)\right) \qquad R < x < R+Q$$

Let y be the number of backorders at time t given that the inventory position was V at time 0. Obviously y+V must have been demanded.

Let $G_2(y)$ be the density function that y is the number of backorders on hand at any time t.

$G_2(y)$ = Probability that demand is y+V averaging over all the states of V.

Hence

$$G_2(y) = \frac{1}{Q} \int_R^{R+Q} \frac{\exp -\frac{1}{2}\left(\frac{V+y-DL}{\sqrt{\sigma^2 L}}\right)^2}{\sqrt{\sigma^2 L\, 2\pi}} dV \qquad y > o$$

Expressing in standard deviations of stock

$$G_2(y) = \frac{1}{Q} \int_{\frac{k+y}{\sqrt{\sigma^2 L}}}^{\frac{k+Q+y}{\sqrt{\sigma^2 L}}} \frac{\exp -\frac{z^2}{2}}{\sqrt{2\pi}} dz$$

Hence

$$G_2(y) = \frac{1}{Q}\left(F\left(\frac{k+y}{\sqrt{\sigma^2 L}}\right) - F\left(k + \left(\frac{y+Q}{\sqrt{\sigma^2 L}}\right)\right)\right) \qquad 2.6.3.$$

Let POUT be the probability that the system is out of stock, which is the probability that y has a value from o to ∞.

$$POUT = \int_0^{\infty} G_2(y)\,dy$$

$$= \frac{1}{Q}\int_0^{\infty}\left(F\left(k+\frac{y}{\sqrt{\sigma^2 L}}\right) - F\left(k + \left(\frac{y+Q}{\sqrt{\sigma^2 L}}\right)\right)\right)dy$$

Simplifying we have

$$= \frac{1}{Q}\int_0^{\infty} F\left(k+\frac{y}{\sqrt{\sigma^2 L}}\right)dy - \frac{1}{Q}\int_0^{\infty} F\left(k+\frac{(y+Q)}{\sqrt{\sigma^2 L}}\right)dy$$

Allowing $Z = k+y/\sqrt{\sigma^2 L}$ and $Z = k + \frac{y+Q}{\sqrt{\sigma^2 L}}$, respectively

$$POUT = \frac{\sqrt{\sigma^2 L}}{Q}\int_k^{\infty} F(Z)\,dZ - \frac{\sqrt{\sigma^2 L}}{Q}\int_{k+Q/\sqrt{\sigma^2 L}}^{\infty} F(Z)\,dZ \qquad 2.6.4.$$

From equation 1.3.7. which states that

$$\int_k^{\infty} F(Z)\,dZ = g(k) - kF(k)$$

and substituting into 2.6.4.

$$\text{POUT} = \frac{\sqrt{\sigma^2 L}}{Q}\left(g(k) - kF(k)\right)$$

$$- \frac{\sqrt{\sigma^2 L}}{Q}\left(g(k+Q/\sqrt{\sigma^2 L}) - (k+Q/\sqrt{\sigma^2 L})F(k+Q/\sqrt{\sigma^2 L})\right) \qquad 4.$$

where g(k) was stated as equal to $\dfrac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}}$ earlier.

Let E(Q,k) be the expected number of backorders incurred per year and B(Q,k) be the expected number of backorders at any time.

E(Q,k) = Demand per year. Probability that system is out of stock

$$= D \times \text{POUT}$$

$$= \frac{D\sqrt{\sigma^2 L}}{Q}\left(g(k) - kF(k)\right) - \frac{D\sqrt{\sigma^2 L}}{Q}\left(g(k+Q/\sqrt{\sigma^2 L}) - (k+\frac{Q}{\sqrt{\sigma^2 L}})F(k+Q/\sqrt{\sigma^2 L})\right) \qquad 2.6.5.$$

Let $\alpha(V) = \sqrt{\sigma^2 L}(g(V) - VF(V))$

Hence $$E(Q,k) = \frac{D}{Q}\left(\alpha(k) - \alpha(k+Q/\sqrt{\sigma^2 L})\right) \qquad 2.6.6.$$

y is the backorders at any time and $G_2(y)$ is the probability that the backorders at any time is y.

Hence
$$B(Q,k) = \int_0^\infty y G_2(y)\,dy$$

Substituting 2.6.3. for $G_2(y)$ in the above expression

$$B(Q,k) = \frac{1}{Q}\int_0^\infty yF(k + \frac{y}{\sqrt{\sigma^2 L}})\,dy - \frac{1}{Q}\int_0^\infty yF(k + \frac{Q}{\sqrt{\sigma^2 L}} + \frac{y}{\sqrt{\sigma^2 L}})\,dy$$

Substituting Z for $k + \frac{y}{\sqrt{\sigma^2 L}}$ and Z for $k + \frac{Q}{\sqrt{\sigma^2 L}} + \frac{y}{\sqrt{\sigma^2 L}}$ respectively

$$B(Q,k) = \frac{\sigma^2 L}{Q} \int_k^\infty (Z-k)F(Z)dZ - \frac{\sigma^2 L}{Q} \int_{k+Q/\sqrt{\sigma^2 L}}^\infty \left(Z-(k + \frac{Q}{\sqrt{\sigma^2 L}})\right)F(Z)dZ \qquad 2.6.7a$$

Expanding the above expression

$$B(Q,k) = \frac{\sigma^2 L}{Q} \int_k^\infty zF(Z)dZ - \frac{k\sigma^2 L}{Q} \int_k^\infty F(Z)dZ - \frac{\sigma^2 L}{Q} \int_{k+Q/\sqrt{\sigma^2 L}}^\infty ZF(Z)dZ$$

$$+ \frac{\sigma^2 L}{Q} \int_{k+Q/\sqrt{\sigma^2 L}}^\infty (k+Q/\sqrt{\sigma^2 L})F(Z)dZ \qquad 2.6.7b$$

From 1.3.8

$$\int_k^\infty ZF(Z)dZ = \tfrac{1}{2}\ ((1 - k^2)F(k) + kg(k))$$

From 1.3.7.

$$\int_k^\infty F(Z)dZ = g(k) - kF(k)$$

Substituting 1.3.7 and 1.3.8 into 2.6.7b

$$B(Q,k) = \frac{\sigma^2 L}{2Q}\left[(1-k^2)F(k) + kg(k)\right]$$

$$- \frac{k\sigma^2 L}{Q}(g(k) - kF(k))$$

$$- \frac{\sigma^2 L}{2Q}\left[\left(1 - (k+Q/\sqrt{\sigma^2 L})^2\right)F(k+Q/\sqrt{\sigma^2 L}) + (k+Q/\sqrt{\sigma^2 L})g(k+Q/\sqrt{\sigma^2 L})\right]$$

$$+ \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) \frac{\sigma^2 L}{Q} \left[g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) - \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) \right]$$

$$\text{Let } \beta(V) = \frac{\sigma^2 L}{2} \left[(1+V^2)F(V) - Vg(V) \right] \qquad 2.6.8.$$

Hence

$$B(Q,k) = \frac{1}{Q}\left(\beta(k) - \beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 2.6.9.$$

The probability that the inventory position was x, when x lied between R and R+Q and when x lies between 0 and R were given in 2.6.1. and 2.6.2. respectively.

Let D(Q,R) be the expected on hand inventory at any time.

$$D(Q,R) = \int_0^R xG_1(x)\,dx + \int_R^{R+Q} xG_1(x)\,dx$$

Substituting for $G_1(x)$ from 2.6.1. & 2.6.2.

$$D(Q,R) = \frac{1}{Q} \int_0^R \left(xF\left(\frac{R-x-DL}{\sqrt{\sigma^2 L}}\right) - xF\left(\frac{R+Q-x-DL}{\sqrt{\sigma^2 L}}\right)\right) dx$$

$$+ \frac{1}{Q} \int_R^{R+Q} \left(x - xF\left(\frac{R+Q-x-DL}{\sqrt{\sigma^2 L}}\right)\right) dx \qquad 2.6.10$$

Simplifying

$$D(Q,R) = \frac{1}{Q} \int_0^R xF\left(\frac{R-x-DL}{\sqrt{\sigma^2 L}}\right) dx - \frac{1}{Q} \int_0^{R+Q} xF\left(\frac{R+Q-x-DL}{\sqrt{\sigma^2 L}}\right) dx$$

$$+ \frac{1}{Q} \int_R^{R+Q} x\,dx$$

Let $V = \dfrac{R-x-DL}{\sqrt{\sigma^2 L}}$ and $\dfrac{R+Q-x-DL}{\sqrt{\sigma^2 L}}$

in each integral

Then $D(Q,R) = \dfrac{\sqrt{\sigma^2 L}}{Q} \displaystyle\int_{\frac{R-DL}{\sigma^2 L}}^{\frac{-DL}{\sigma^2 L}} (\sqrt{\sigma^2} LV-R+DL)F(V)\,dV + \dfrac{\sqrt{\sigma^2 L}}{Q} \int_{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}}$

$* \; (\sigma^2 LV-R-Q+DL)\,dV + \dfrac{(R+Q)^2-R^2}{2Q}$

Remembering that $k = \dfrac{R-DL}{\sqrt{\sigma^2 L}}$

Then expressing R in terms of standard deviations of stock k

$$D(Q,k) = -\frac{\sigma^2 L}{Q}\int_{k}^{\frac{-DL}{\sigma^2 L}} (V-k)F(V)\,dV + \frac{\sigma^2 L}{Q}\int_{k+\frac{Q}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sigma^2 L}} \left(V-\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right)F(V)\,dV \qquad 2.6.11$$

$$+\frac{Q}{2} + k\sqrt{\sigma^2 L}$$

Simplifying

$$D(Q,K) = -\frac{\sigma^2 L}{Q}\int_{k}^{\frac{DL}{\sqrt{\sigma^2 L}}} VF(V)\,dV + \frac{\sigma^2 L}{Q}\int_{k}^{\frac{-DL}{\sqrt{\sigma^2 L}}} kF(V)\,dV + \frac{\sigma^2 L}{Q}\int_{k+\frac{Q}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}} VF(V)\,dV -$$

$$-\frac{\sigma^2 L}{Q}\int_{k+\frac{Q}{\sqrt{\sigma^2 L}}}^{\frac{-DL}{\sqrt{\sigma^2 L}}} \left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)F(V)\,dV \qquad 2.6.12$$

$$+\frac{Q}{2} + k\sqrt{\sigma^2 L}$$

Hence D(Q,k), noting that $F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) = 1$

But from 1.3.7.

$$\int_k^{\infty} F(V)\,dV = -kF(k) + g(k) \qquad 2.6.13$$

and

$$\int_k^{\infty} VF(V)\,dV = \tfrac{1}{2}((1 - k^2)F(k) + kg(k))$$

Hence D(Q,k)

$$= -\frac{\sigma^2 L}{Q}\left(kF(k) - g(k) - \left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) + g\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$- \frac{\sigma^2 L}{2Q}\left((1 - k^2)F(k) + kg(k) - \left(1 - \left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)^2\right)F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) - g\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{\sigma^2 L}{Q}\left(\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) + \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. - \left(1 - \left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)^2\right)F\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right) - g\left(\frac{-DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{Q}{2} + k\sqrt{\sigma^2 L} \qquad 2.6.14$$

Substituting B(Q,k) (2.6.7a) into the above expression and simplifyig

$$D(Q,k) = k\sqrt{\sigma^2 L} + \frac{Q}{2} + \beta(Q,k) \qquad 2.6.15$$

Noting that the expected value of the inventory position, POS(Q,k) is

$$POS(Q,k) = \int_0^{Q} (R+x)\rho(R+x)\,dx$$

Substituting for $\rho(R+x)$ where $\rho(r+x) = \frac{1}{Q}$ as proved earlier

$$POS(Q,k) = \int_0^Q \frac{(R+x)}{Q}\, dx = \left.\frac{Rx}{Q} + \frac{x^2}{2Q}\right]_0^Q$$

Simplifying the expected inventory position

$$POS(Q,k) = R + \frac{Q}{2} \qquad 2.6.16$$

and expected amount on order = DL

the expected number of cycles $= \frac{D}{Q}$ 2.6.16b

Remembering that S is the set up cost, hc is the holding cost, b_1 is the average backorder cost per year, b_2 is the backorder cost per unit per year backordered and s is the stockout cost dependent upon the number of stockouts only, the inventory cost C for the exact model of (Q,R)

$$C = \frac{DS}{Q} + hc\left(\frac{Q}{2} + k\sqrt{\sigma^2 L} + D(Q,k)\right) + b_1 E(Q,k) \qquad 2.6.17$$

$$+ b_2 B(Q,k) + \frac{Ds}{Q} POUT$$

where $POUT = \alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$ from 2.6.4.

where $\alpha(V) = \sqrt{\sigma^2 L}\,(g(V) - VF(V))$

$E(Q,k) = \frac{D}{Q}\left(\alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$ from 2.6.6.

$B(Q,k) = \frac{1}{Q}\left(\beta(k) - \beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$ from 2.6.9.

where $\beta(V) = \frac{\sigma^2 L}{2}\left((1 + k^2)F(k) - kg(k)\right)$ from 2.6.8

Substituting the above values into 2.6.17

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\ k\sqrt{\sigma^2 L} + \frac{Db_i}{Q}\left(\alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{(b_2+hc)}{Q}\left(\beta(k) - \beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) + \frac{Ds}{Q}\left(\alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 2.6.18$$

The minimum of the cost expression C is obtained when $\frac{\partial C}{\partial Q}$ and $\frac{\partial C}{\partial k}$ are both equal to zero. In order to be able to obtain the first order derivatives of C, it is necessary to obtain the following derivates

$$\alpha(k) = \sqrt{\sigma^2 L}(g(k) - kF(k)) \text{ from } 2.6.4$$

Differentiating with respect to k

$$\frac{\partial\alpha(k)}{\partial k} = \sqrt{\sigma^2 L}(-\ kg(k) - F(k) + kg(k))$$

Simplifying

$$\frac{\partial\alpha(k)}{\partial k} = -\sqrt{\sigma^2 L}\ F(k) \qquad 2.6.19$$

$$\frac{\partial\alpha(k)}{\partial Q} = 0$$

$$\alpha(k+Q) = \sqrt{\sigma^2 L}\left(g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) - \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \text{ from } 2.6.4$$

Differentiating with respect to k and applying 2.6.19

$$\frac{\partial\alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)}{\partial k} = - \sqrt{\sigma^2 L}\, F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) \qquad 2.6.20$$

Similarly

$$\frac{\partial\alpha\left(k + \frac{Q}{\sigma^2 L}\right)}{\partial Q} = - F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) \qquad 2.6.21$$

From 2.6.8

$$\beta(k) = \frac{\sigma^2 L}{2} \left((1 + k^2)F(k) - kg(k)\right)$$

Differentiating with respect to k

$$\frac{\partial\beta(k)}{\partial k} = \frac{\sigma^2 L}{2} \left(2kF(k) - (1 + k^2)g(k) - g(k) + k^2 g(k)\right)$$

Simplifying

$$= - \sigma^2 L(g(k) - kF(k))$$

Substituting $\alpha(k)$ into the expression

$$\frac{\partial\beta(k)}{\partial k} = - \sqrt{\sigma^2 L}\,\alpha(k) \qquad 2.6.22$$

Differentiating with respect to Q

$$\frac{\partial\beta(k)}{\partial Q} = 0$$

From 2.6.8

$$\beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) = \frac{\sigma^2 L}{2}\left(\left(1 + \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)^2\right)F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. - \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

Differentiating with respect to k

$$\frac{\partial\beta\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)}{\partial k} = \frac{\sigma^2 L}{2}\left(2\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) - \left(1 + \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)^2\right)\right.$$

$$\left. g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) + \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)^2 g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

Simplifying

$$= \sigma^2 L\left(\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) - g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 2.6.24$$

Substituting $\alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$ into expression

$$\frac{\partial\beta\left(k + \frac{Q}{\sigma}\right)}{\partial k} = -\sqrt{\sigma^2 L}\,\alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) \qquad 2.6\ 25$$

Similarly $\frac{\partial\beta\left(k + \frac{Q}{\sigma}\right)}{\partial Q} = -\alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$ 2.6.26

Therefore substituting the above derivatives into the cost expression we obtain

$$\frac{\partial C}{\partial Q} = -\frac{DS}{Q^2} + \frac{hc}{2} + \frac{D}{Q}(b_1 + s)F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{D(b_1+s)}{Q^2}\left(\alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) + \frac{(b_2+hc)}{Q}\,\alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{(b_2+hc)}{Q^2}\left(\beta(k) - \beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 2.6.27$$

$$\frac{\partial C}{\partial K} = \sqrt{\sigma^2 L}\,hc - D\frac{(b_1+s)}{Q}\sqrt{\sigma^2 L}\left(F(k) - F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$- \frac{(b_2+hc)}{Q}\sqrt{\sigma^2 L}\left(\alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 2.6.28$$

C is minimised when $\frac{\partial C}{\partial K}$ and $\frac{\partial C}{\partial Q}$ are both equal to 0. However, the cost expression is not a convex function, hence local minima may exists in the solutions.

By neglecting the $\alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$ and $\beta\left(k + Q/\sqrt{\sigma^2 L}\right)$ in the cost expression, it is possible to produce a modified version of the cost expression which is a convex funtion, which in this context, would be called the Inexact Model of (QR).

Neglecting $\alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$ and $\beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)$ in 2.6.18 we obtain the cost expression for the Inexact model.

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + k\sqrt{\sigma^2 L}\,hc + \frac{Db_1\alpha(k)}{Q} + \frac{DS}{Q}\alpha(k)$$

$$+ \frac{(b_2+hc)}{Q}\beta(k) \qquad 2.6.29$$

The derivatives are

$$\frac{\partial C}{\partial Q} = -\frac{DS}{Q^2} + \frac{hc}{2} - \frac{Db_1\alpha(k)}{Q^2} - \frac{DS}{Q}\alpha(k)$$

$$- \frac{(b_2+hc)}{Q^2}\beta(k) \qquad 2.6.30$$

$$\frac{\partial C}{\partial k} = \sqrt{\sigma^2}Lhc - D\sqrt{\sigma^2 L}\left(\frac{b_1+s}{Q}\right)F(k) - \frac{\sqrt{\sigma^2 L}}{Q}(b_2+hc)\alpha(k) \qquad 2.6.31$$

The heuristic model discussed in section 2.3 of chapter 2 could not be solved explicitly. It is even more difficult to solve the exact and inexact models of this section. We investigated whether the Inexact model which is convex, was more time consuming to solve, and gave better convergence over the exact model which was in general not convex, using Steepest Descent, Powell's iterative method, Simplex method, and the Complex method to solve the equations. These optimization methods have been described in section 2.2 of chapter 1, Vol 1.

The computations were carried out on an ICL 1905 machine and the compilation cost for all the methods were 5p.

The following costs were obtained for solving one set of equations

	Exact	Inexact
Steepest Descent	0.301p	0.300 p
Hadley & Whitin's Iterative technique	-	0.613p
Powell's Iterative technique	0.331p	0.318 p
Simplex method	3.141p	2.081p
Complex method	3.009p	2.075p

Hadley and Whitin's iterative technique is not applicable to the Exact model, because from the derivatives of the Exact model we cannot express Q as a function of k, or k as a function of Q.

This can easily be done with the Inexact model.

From 2.6.30 and 2.6.31 we obtain

$$Q = \sqrt{2D \frac{(S+(s+b_1)\alpha(k)) \quad + 2(b_1 + hc)\beta(k)}{hc}} \qquad 2.6.32$$

$$(D(b_1+s)-k(b_2+hc)\ )F(k) + (b_2+hc)g(k) = Qhc \qquad 2.6.33$$

However equation 2.6.33 is a one dimensional equation in one unknown. As a result Newton's one dimensional iterative technique could be applied on 2.6.33. However when this is tried, convergence is inferior compared to the complex and Powell's methods on the original form of the Inexact model.

Generally the difference in costs between the methods of optimisation for both the Exact and Inexact models were negligible as can be seen from the r-esults presented later. Convergence on the Steepest descent and the Simplex met-hods were very poor. Convergence was very good on both the Complex method and Powell's iterative technique.

Convergence on the Complex method was better than convergence on the Simplex method because Q and k could be constrained as not less than 0 and handled quite easily with the Complex method.

2.7 When to use the Inexact model (Q,k)

From 2.6.18 the cost expression for the exact model is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + k\sqrt{\sigma^2 L}.hc + \frac{Db_1}{Q}\left(\alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{(b_2+hc)}{Q}\left(\beta(k) - \beta\left(k + \frac{Q}{\sqrt{\alpha^2 L}}\right)\right) + \frac{Ds}{Q}\left(\alpha(k) - \alpha\left(k + \frac{Q}{\sqrt{\alpha^2 L}}\right)\right)$$

And from 2.6.29 the cost expression for the inexact model is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + \frac{Db_1}{Q}\alpha(k) + \frac{(b_2+hc)}{Q}\beta(k) + k\sqrt{\sigma^2 L}\,hc$$

$$+ \frac{Ds}{Q}\alpha(k)$$

The Inexact model has been derived from the exact model by neglecting the $\alpha(k + \frac{Q}{\sqrt{\sigma^2 L}})$ and $\beta(k + \frac{Q}{\sqrt{\sigma^2 L}})$ terms.

By varying S set up costs; h_c, holding cost; b_1, backorder costs independent of time of backorder. b_2 backorder cost per year back-ordered and s stockout costs; and $\sqrt{\sigma^2 L}$ standard deviation of demand over the lead time it was found that the most important factor in determining which model of the exact and Inexact models to use was $\sqrt{\sigma^2 L}$ the standard deviation of demand over the lead time.
Also it is possible to prove by analysis that $\sqrt{\sigma^2 L}$ is the important factor

$$\beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) = \frac{\sigma^2 L}{2}\left(\left(\left(1 + \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)^2\right)F\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right) - \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)g\left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$\alpha\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right) = \sqrt{\sigma^2 L}\left(g\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right) - \left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right) F\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

where $g\left(k+Q/\sqrt{\sigma^2 L}\right) = \exp -\frac{1}{2}\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)^2 \Big/ \sqrt{2\pi}$

and $F\left(k+Q/\sqrt{\sigma^2 L}\right) = \int_{k+\frac{Q}{\sqrt{\sigma^2 L}}}^{\infty} g(v)\, dv$

$g\left(k+Q/\sqrt{\sigma^2 L}\right)$ tends to zero as $k+Q/\sqrt{\sigma^2 L}$ tends to ∞ or as $Q/\sqrt{\sigma^2 L}$ tends to infinity which is the same thing as saying that as σ tends to 0.

Similarly $F\left(k+Q/\sqrt{\sigma^2 L}\right)$ tends to zero as σ tends to 0.

This implies that the difference between the Exact and Inexact models diminshes as the standard deviation of demand tends to zero.

The table below shows the inventory costs of the Exact and Inexact models for a change in the values of the parameters by a multiple of 10.
The values assigned to the parameters are;

Demand	=1000
Lead time	=0.1 of a year.
Stockout costs	=0.005 Lead time demand.

From pages 59 and 60

Annual Inventory Costs For Exact Model

	$\sqrt{\sigma^2}$ L =3.162			$\sqrt{\sigma^2}$ L =31.62			$\sqrt{\sigma^2}$L=316.23		
Order cost	hc=0.01	hc=0.1	hc=1.0	hc=0.01	hc=0.1	hc=1.	hc=0.01	hc=0.1	hc=1.0
0.1	1.495	5.151	19.522	2.208	11.185	59.42	9.703	65.492	303.32
1.0	4.539	14.65	47.767	5.152	19.316	75.26	11.80	69.633	310.33
10.	14.193	45.00	142.34	14.66	47.892	153.6	20.15	88.720	353.37

Annual Inventory Costs For Inexact Model

Order cost	hc=0.01	hc=0.1	hc=1.0	hc=0.01	hc=0.1	hc=1.	hc=0.01	hc=0.1	hc=1.0
0.1	1.495	5.151	19.525	2.226	11.198	60.61	9.919	68.059	354.66
1.0	4.539	14.650	47.767	5.153	75.404	70.87	11.94	70.867	357.23
10.	14.19	45.01	142.39	14.66	47.893	153.6	20.30	88.857	381.52

From the above tables, the change in iventory costs of both the Exact and Inexact models and also the savings of the Exact model over the Inexact model was more than the increase or decrease in savings for the same corresponding change in the other parameters, for the same multiple change in standard deviations

Since the standard deviation of demand produces the largest difference between both models, we would investigate further how the difference between the Exact model and Inexact model changes as standard deviation of demand changes.

Let the standard deviation of demand be expressed as a fraction of expected demand over the lead time. Hence $\sqrt{\sigma^2 L} = pDL$ where DL is the expected demand over the lead time. In this context p is diiferent from the p defined as the period of grace used in other chapters. By varying p we can vary the standard deviation of demand. On pages 61 to 66 is presented the annual costs of both models and the differences as p varies.

The difference between Inexact and Exact models diminished as p decreased. When demand was 10 per year the difference obtained when $p < 0.053$ was 0. When demand was 100 the difference between both models was 0 when $p < 0.0018$ was 0. When demand was 1000 the difference between both models was 0 when $p < 0.0006$. When demand was 10000 and above the difference was positive for all the ranges.

This implies that the value of the standard deviation of demand required to produce a difference between the Exact and Inexact models diminishes as demand increases. Also the value of the standard deviation of demand required to produce a difference between both models increases for a constant level of demand.

DEMAND FOLLOWS A NORMAL DISTRIBUTION

EBO RE-ORDER LEVEL MODEL

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEADTIME =0.10*D*SQRT(L)

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED = 0.005*LEADTIME DEMAND

	DEMAND	HC=0.01	HC=0.10	HC=1.0
	10.	0.143	0.453	1.432
	100.	0.496	1.591	5.031
S= 0.1	1000.	2.109	9.370	31.820
	10000.	12.955	83.666	534.519
	100000.	111.992	893.045	7093.807
	1000000.	1142.374	9926.394	85279.562
	10.	0.448	1.416	4.478
	100.	1.432	4.528	14.319
S= 1.0	1000.	4.962	15.910	50.312
	10000.	21.095	93.701	318.198
	100000.	129.554	836.660	5345.191
	1000000.	1119.919	8930.448	70938.074
	10.	1.414	4.473	14.144
	100.	4.478	14.160	44.777
S=10.0	1000.	14.319	45.280	143.189
	10000.	49.619	159.099	503.115
	100000.	210.945	937.011	3181.981
	1000000.	1295.537	8366.600	53451.914

DEMAND FOLLOWS A NORMAL DISTRIBUTION

TATE EXPONENTIAL APPROXIMATION

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEADTIME =0.10*D*SQRT(L)

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED = 0.005*LEADTIME DEMAND

	DEMAND	HC=0.01	HC=0.10	HC=1.0
	10.	0.143	0.453	1.435
	100.	0.496	1.584	5.159
S= 0.1	1000.	2.102	8.402	28.310
	10000.	12.787	72.153	250.791
	100000.	107.911	708.646	2487.719
	1000000.	1045.597	7073.535	24856.959
	10.	0.448	1.416	4.479
	100.	1.432	4.529	14.353
S= 1.0	1000.	4.956	15.842	51.587
	10000.	21.019	84.022	283.102
	100000.	127.867	721.528	2507.911
	1000000.	1079.108	7086.463	24877.194
	10.	1.414	4.473	14.144
	100.	4.478	14.160	44.791
S=10.0	1000.	14.318	45.291	143.533
	10000.	49.561	158.422	515.871
	100000.	210.185	840.225	2831.024
	1000000.	1278.665	7215.283	25079.107

DEMAND FOLLOWS A NORMAL DISTRIBUTION

HEURISTIC MODEL OF (Q,R)

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEADTIME =0.10*D*SQRT(L)

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED = 0.005*LEADTIME DEMAND

	DEMAND	HC=0.01	HC=0.10	HC=1.0
	10.	0.143	0.453	1.432
	100.	0.496	1.581	5.000
S= 0.1	1000.	2.101	8.367	26.458
	10000.	12.787	72.111	228.035
	100000.	107.865	715.382	2240.536
	1000000.	1044.750	7135.292	22365.151
	10.	0.448	1.416	4.448
	100.	1.432	4.528	14.318
S= 1.0	1000.	4.956	15.811	50.000
	10000.	21.015	83.666	264.575
	100000.	127.866	721.110	2280.351
	1000000.	1078.653	7153.822	22405.357
	10.	1.414	4.473	14.144
	100.	4.478	14.160	44.777
S=10.0	1000.	14.318	45.277	143.178
	10000.	49.559	158.114	500.000
	100000.	210.147	836.660	2645.751
	1000000.	1278.664	7211.103	22803.509

DEMAND FOLLOWS A NORMAL DISTRIBUTION

INEXACT MODEL OF (Q,R)

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEADTIME =0.01*D*SQRT(L)

BACKORDER COST PER YEAR = 1.0

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED = .005*LEADTIME DEMAND

	DEMAND	HC=0.01	HC=0.10	HC=1.0
	10.	0.142	0.459	1.421
	100.	0.454	1.465	4.777
S= 0.1	1000.	1.495	5.151	19.525
	10000.	5.388	22.422	117.096
	100000.	23.914	139.609	1091.603
	1000000.	142.899	1114.967	9769.620
	10.	0.447	1.415	4.475
	100.	1.419	4.501	14.238
S= 1.0	1000.	4.539	14.659	47.767
	10000.	14.947	51.507	195.248
	100000.	53.885	224.078	1171.002
	1000000.	238.647	1395.859	10015.113
	10.	1.414	4.472	14.143
	100.	4.475	14.151	44.751
S=10.0	1000.	14.193	45.008	142.377
	10000.	45.390	146.501	478.388
	100000.	149.468	515.748	1952.475
	1000000.	538.829	2259.635	11709.948

EXACT MODEL OF (Q,R)

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEADTIME =0.01*D*SQRT(L)

BACKORDER COST PER YEAR = 1.0

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED = 0.005*LEADTIME DEMAND

	DEMAND	HC=0.01	HC=0.10	HC=1.0
	10.	0.142	0.459	1.424
	100.	0.454	1.465	4.777
S= 0.1	1000.	1.495	5.151	19.522
	10000.	5.388	22.402	117.071
	100000.	23.692	139.365	1021.163
	1000000.	140.294	1111.848	9765.957
	10.	0.447	1.415	4.475
	100.	1.419	4.501	14.238
S= 1.0	1000.	4.539	14.650	47.767
	10000.	14.947	51.507	195.221
	100000.	53.885	223.875	1179.733
	1000000.	236.400	1393.521	10011.796
	10.	1.414	4.472	14.143
	100.	4.475	14.151	44.751
S=10.	1000.	14.193	45.008	142.377
	10000.	45.390	146.501	478.388
	100000.	149.468	515.748	1952.202
	1000000.	538.829	2257.480	11797.198

DEMAND FOLLOWS A NORMAL DISTRIBUTION

EXACT MODEL OF (Q,R)

LEAD TIME IS 0.1 OF A YEAR

BACKORDER COST PER YEAR = 1.0

ORDERING COSTS = 1.0 AND HOLDING COSTS = 0.1

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED =0.00

$0.1 \times \sqrt{\sigma^2 L}/DL$	D=10	D=100	D=1000
0.0000	1.415	4.501	14.645
0.0006	1.421	4.680	17.627
0.0012	1.429	4.880	20.674
0.0018	1.437	5.096	23.800
0.0024	1.447	5.320	26.980
0.0029	1.458	5.558	30.203
0.0035	1.470	5.804	33.449
0.0041	1.484	6.058	36.705
0.0047	1.498	6.319	39.972
0.0053	1.517	6.586	43.246
0.0059	1.530	6.858	46.526
0.0065	1.547	7.135	49.808
0.0071	1.566	7.416	53.090
0.0076	1.586	7.700	56.379
0.0082	1.606	7.988	59.665
0.0088	1.627	8.279	62.952
0.0094	1.649	8.572	66.242
0.0100	1.679	8.868	69.531

DEMAND FOLLOWS A NORMAL DISTRIBUTION

INEXACT MODEL OF (Q,R)

LEAD TIME IS 0.1 OF A YEAR

BACKORDER COST PER YEAR = 1.0

ORDERING COSTS = 1.0 AND HOLDING COSTS = 0.1

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED =0.00

$0.1 \times \sqrt{\sigma^2 L}/\Delta L$	D=10	D=100	D=1000
0.0000	1.415	4.591	14.645
0.0006	1.421	4.680	17.627
0.0012	1.429	4.880	20.690
0.0018	1.437	5.096	23.821
0.0024	1.447	5.323	27.005
0.0029	1.458	5.561	30.235
0.0035	1.470	5.808	33.502
0.0041	1.484	6.063	36.801
0.0047	1.498	6.324	40.127
0.0053	1.517	6.591	43.475
0.0059	1.531	6.863	46.843
0.0065	1.548	7.140	50.229
0.0071	1.567	7.422	53.627
0.0076	1.587	7.707	57.039
0.0082	1.607	7.996	60.462
0.0088	1.628	8.288	63.897
0.0094	1.650	8.583	67.337
0.0100	1.672	8.881	70.786

DEMAND FOLLOWS A NORMAL DISTRIBUTION

DIFFERENCE BETWEEN INEXACT AND EXACT MODEL OF (Q ,R)

LEAD TIME IS 0.1 OF A YEAR

BACKORDER COST PER YEAR = 1.0

ORDERING COSTS = 1.0 AND HOLDING COSTS = 0.1

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED =0.00

$0.1\times\sqrt{\sigma^2 L}/DL$	D=10	D=100	D=1000
0.0000	0.000	0.000	0.000
0.0006	0.000	0.000	0.000
0.0012	0.000	0.000	0.016
0.0018	0.000	0.000	0.021
0.0024	0.000	0.003	0.025
0.0029	0.000	0.003	0.032
0.0035	0.000	0.004	0.053
0.0041	0.000	0.005	0.096
0.0047	0.000	0.005	0.155
0.0053	0.000	0.005	0.229
0.0059	0.001	0.005	0.317
0.0065	0.001	0.005	0.421
0.0071	0.001	0.006	0.537
0.0076	0.001	0.007	0.669
0.0082	0.001	0.008	0.797
0.0088	0.001	0.009	0.945
0.0094	0.001	0.011	1.095
0.0100	0.002	0.013	1.255

DEMAND FOLLOWS A NORMAL DISTRIBUTION

EXACT MODEL OF (Q,P)

LEAD TIME IS 0.1 OF A YEAR

BACKORDER COST PER YEAR = 1.0

ORDERING COSTS = 1.0 AND HOLDING COSTS = 0.1

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED =0.00

$\alpha = \sqrt{\sigma^2 L}/D$	D=10000	D=100000	D=1000000
0.0000	51.377	220.979	1346.606
0.0006	89.957	673.196	6208.189
0.0012	128.338	1094.099	10519.281
0.0018	166.022	1492.994	14565.667
0.0024	202.910	1876.924	18437.644
0.0029	239.181	2250.472	22203.009
0.0035	274.990	2617.390	25878.607
0.0041	310.295	2980.269	29497.571
0.0047	345.273	3333.114	33069.132
0.0053	380.032	3685.383	36608.136
0.0059	414.559	4035.031	40212.198
0.0065	448.938	4382.074	43580.934
0.0071	483.146	4728.465	47051.757
0.0076	517.244	5071.307	50495.318
0.0082	551.254	5420.639	53924.152
0.0088	585.146	5755.174	57349.407
0.0094	618.974	6095.742	60751.238
0.0100	652.794	6434.991	64157.637

DEMAND FOLLOWS A NORMAL DISTRIBUTION

INEXACT MODEL OF (Q,R)

LEAD TIME IS 0.1 OF A YEAR

BACKORDER COST PER YEAR = 1.0

ORDERING COSTS = 1.0 AND HOLDING COSTS = 0.1

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED =0.00

$0.1\times\sqrt{\sigma^2 L}/DL$	D=10000	D=100000	D=1000000
0.0000	51.377	221.172	1348.707
0.0006	90.050	674.394	6298.200
0.0012	128.519	1107.904	10898.922
0.0018	166.686	1524.781	15060.300
0.0024	204.593	1928.678	19147.379
0.0029	242.173	2323.325	23154.066
0.0035	279.518	2710.370	27012.218
0.0041	316.602	3092.058	30844.405
0.0047	353.499	3469.551	34624.625
0.0053	390.230	3843.187	38370.261
0.0059	426.766	4214.415	42091.200
0.0065	463.177	4583.242	45786.870
0.0071	499.458	4950.061	49456.285
0.0076	535.627	5315.541	53125.101
0.0082	571.728	5679.109	56759.414
0.0088	607.893	6041.795	60385.709
0.0094	643.696	6403.634	63955.560
0.0100	679.655	6767.037	67615.896

DEMAND FOLLOWS A NORMAL DISTRIBUTION

DIFFERENCE BETWEEN INEXACT AND EXACT MODEL OF (Q ,R)

LEAD TIME IS 0.1 OF A YEAR

BACKORDER COST PER YEAR = 1.0

ORDERING COSTS = 1.0 AND HOLDING COSTS = 0.1

BACKORDER COST PER UNIT BACKORDERED =0.10

STOCKOUT COST INDEPENDENT OF UNIT BACKORDERED =0.00

$\sqrt{\sigma^2 L}/DL$	D=10000	D=100000	D=1000000
0.0000	0.000	0.193	2.101
0.0006	0.093	1.198	93.011
0.0012	0.181	13.805	289.641
0.0018	0.664	31.787	494.633
0.0024	1.683	51.754	709.735
0.0029	2.992	72.853	951.057
0.0035	4.618	92.980	1133.611
0.0041	6.307	111.789	1346.834
0.0047	8.226	136.437	1555.493
0.0053	10.198	157.804	1762.125
0.0059	12.207	179.384	1879.002
0.0065	14.239	201.168	2196.936
0.0071	16.312	221.596	2404.528
0.0076	18.383	244.234	2629.783
0.0082	20.474	258.470	2826.262
0.0088	22.747	286.621	3036.302
0.0094	24.722	307.892	3204.322
0.0100	26.811	332.046	3458.259

Section 2.8. Comparison Of The Different Versions Of Model (Q,R).

The different versions of model (Q,R) are ;

(1) EBQ-ROL model whose cost equation is

$$C_1 = \frac{DS}{Q} + \frac{Qhc}{2} \qquad 2.8.1$$

$$C_2 = k\sqrt{\sigma^2 L}.hc + \frac{DsF(k)}{Q} \qquad 2.8.2$$

where Q is obtained from C_1 and k is obtained from C_2 .

(2) Heuristic model whose cost equation is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + k\sqrt{\sigma^2 L}.hc + \frac{DsF(k)}{Q} \qquad 2.8.3$$

where Q and k are obtained simultaneously.

(3) Inexact model whose cost equation is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc.k\sqrt{\sigma^2 L} + \frac{Db_1\alpha(k)}{Q} + \frac{(b_2+hc)\beta(k)}{Q} + \frac{Ds\Gamma(k)}{Q} \qquad 2.8.4$$

(4) Exact model whose cost equation is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc.k\sqrt{\sigma^2 L} + Db_1(\alpha(k) - \alpha(k+Q/\sqrt{\sigma^2 L})) + \frac{(b_2+hc)}{Q}\left(\beta(k) - \beta\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$
$$+ \frac{Ds}{Q}(\beta(k) - \beta(k+Q/\sqrt{\sigma^2 L})) \qquad 2.8.5$$

In the EBQ-ROL model Q is independent of the stockout cost, since Q is obtained from 2.8.1. The Heuristic model is a combination of 2.8.1 and 2.8.2 thereby obtaining Q and k simultaneously.

The Exact model differs from the Heuristic model in two aspects; (1) No consideration is given to the 'expected backorders at anytime' in the Heuristic model and (2) the shortage cost of the Heuristic model differs from the shortage costs of the Exact model. The shortage cost of the Heuristic model is equal to the number of periods multiplied by the cost of stockout when the reorder level is broken. The shorage cost of the Exact model is equal to the number of periods multiplied by the cost of a stockout multiplied by the probability of a stockout for any given level of order cover.

Finally the Exact model differs from the Inexact Model in that the Inexact model is derived from the Exact model by dropping the $(k+Q/\sqrt{\sigma^2 L})$

terms in the Exact model.

We compute the annual inventory costs for comparisons by (1) taking the Exact model as the standard and (2) feeding the optimum values of Q and k obtained in each model into the cost expression of the Exact model to obtain the equivalent annual inventory costs.

The following values are assigned to the parameters.

Annual Demand	10,100,1000,10000,100000,1000000.
Lead time	=0.1 of a year.
Stockout cost	= 0.05 Leadtime demand.
Holding cost	= 0.01, 0.1, 1.0.
Order costs	= 0.1, 1.0,10.0.

The annual inventory costs are given on pages 69 to 73

From the results, the Exact model produces the least inventory costs.

The computational costs are;

Versions of (Q,R)	Cost Of Computing One Set Of Parameters.
EBQ-ROL	£0.004
Heuristic	£0.006
Inexact	£0.02
Exact	£0.03

For demands geater than 100 per year the savings in inventory costs using the Exact model over the other models outweigh the extra cost of computation. Hence the Exact model is the right model to use if a (Q,R) model is under consideration.

EBQ-REORDER LEVEL MODEL: ANNUAL INVENTORY COSTS

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEAD TIME = 0.1 * D * SQRT(L)

STOCKOUT COST

=0.05 * LEAD TIME DEMAND

	ANNUAL DEMAND	HC=0.01	HC=0.1	HC=1.0
	10	0.147	0.474	1.492
	100	0.520	1.916	8.082
	1000	2.285	12.019	78.501
S=0.1	10000	14.400	106.074	873.822
	100000	124.583	1039.545	9860.495
	1000000	1248.833	11185.830	99931.360
	10	0.450	1.415	4.481
	100	1.469	4.745	14.922
	1000	5.202	19.163	80.815
S=1.0	10000	22.849	120.193	785.010
	100000	144.004	1060.740	8738.224
	1000000	1245.830	10395.453	98604.950
	10	1.414	4.472	14.144
	100	4.498	14.147	44.809
	1000	14.690	47.446	149.222
S=10.0	10000	52.024	191.628	808.151
	100000	228.491	1201.935	7850.099
	1000000	1440.041	10607.404	87382.236

TATE'S EXPONENTIAL APPROXIMATION TO THE HEURISTIC MODEL

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEAD TIME = 0.1*D *SQRT(L)

STOCKOUT COST = 0.05* LEAD TIME DEMAND

	ANNUAL DEMAND	HC=0.01	HC=0.1	HC=1.0
S=0.1	10	0.147	0.472	1.473
	100	0.520	1.973	6.648
	1000	2.281	11.761	53.277
	10000	14.411	99.286	516.681
	100000	124.430	966.070	5151.179
	1000000	1212.890	9631.636	51496.055
S=1.0	10	0.450	1.415	4.484
	100	1.470	4.719	14.727
	1000	5.197	19.727	66.484
	10000	22.810	117.607	532.773
	100000	144.107	992.863	5166.905
	1000000	1244.300	9660.696	51511.794
S=10.	10	1.414	4.472	14.145
	100	4.497	14.149	44.835
	1000	14.697	47.191	147.291
	10000	51.966	197.276	664.836
	100000	228.096	1176.072	5322.733
	1000000	1441.073	9928.628	51669.051

HEURISTIC MODEL: ANNUAL INVENTORY COSTS

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEAD TIME = 0.1 * D * SQRT(L)

STOCKOUT COST = 0.05 * LEAD TIME DEMAND

	ANNUAL DEMAND	HC=0.01	HC=0.1	HC=1.0
	10	0.147	0.473	1.464
	100	0.520	1.962	6.575
S=0.1	1000	2.286	11.757	52.905
	10000	14.304	99.874	573.179
	100000	121.660	972.805	5765.944
	1000000	1178.703	9700.249	57684.875
	10	0.450	1.415	4.481
	100	1.469	4.730	14.640
S=1.0	1000	5.200	19.621	65.751
	10000	22.856	117.572	529.049
	100000	143.045	998.740	5731.751
	1000000	1216.599	9728.052	57659.492
	10	1.414	4.472	14.144
	100	4.498	14.148	44.811
S=10.0	1000	14.689	47.300	146.397
	10000	52.002	196.206	657.510
	100000	228.557	1175.715	5290.492
	1000000	1430.446	9987.402	57317.446

INEXACT MODEL: ANNUAL INVENTORY COSTS

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEAD TIME = 0.1 * D * SQRT(L)

STOCKOUT COST = 0.05 * LEAD TIME DEMAND

	ANNUAL DEMAND	HC=0.01	HC=0.1	HC=10.
	10	0.142	0.451	1.461
	100	0.476	1.663	6.537
	1000	2.071	10.437	52.818
S=0.1	10000	13.304	92.066	549.979
	100000	111.191	904.471	5128.916
	1000000	1036.974	9033.752	51266.953
	10	0.447	1.415	4.481
	100	1.418	4.509	14.613
	1000	4.757	16.633	65.366
S=1.0	10000	20.708	104.372	527.973
	100000	133.066	920.641	5141.736
	1000000	1116.211	9044.545	51294.349
	10	1.414	4.472	14.144
	100	4.473	14.472	44.809
	1000	14.175	45.087	146.133
S=10.0	10000	47.571	166.350	653.702
	100000	207.083	1043.723	5277.328
	1000000	1332.412	9205.967	51417.420

EXACT MODEL: ANNUAL INVENTORY COSTS

LEAD TIME IS 0.1 OF A YEAR

STANDARD DEVIATION OF DEMAND OVER LEAD TIME = 0.1*D*SQRT(L)

STOCKOUT COST =0.05 * LEADTIME DEMAND

	ANNUAL DEMAND	HC=0.01	HC=0.1	HC=1.0
	10	0.142	0.451	1.461
	100	0.476	1.663	6.528
	1000	2.071	10.436	52.503
S=0.1	10000	13.302	92.055	511.426
	100000	111.142	904.385	5098.044
	1000000	1035.889	9032.652	50959.148
	10	0.447	1.415	4.481
	100	1.418	4.509	14.612
	1000	4.757	16.632	65.282
S=1.0	10000	20.707	104.363	525.035
	100000	133.049	920.529	5111.343
	1000000	1115.777	9043.360	51002.757
	10	1.414	4.472	14.144
	100	4.473	14.147	44.809
	1000	14.175	45.087	146.124
S=10.0	10000	47.571	166.339	653.725
	100000	207.067	1043.629	5252.228
	1000000	1332.232	9205.515	51123.544

CHAPTER 3

MODELS (nQ,R,T),(M,T),(M,R,T). LINEAR BACKORDER COSTS AND CONSTANT LEAD TIMES

Section 3.1 Introduction

In the last chapter we considered the (Q,R) model at various levels of complexity, and it was found that the Exact model gave the least inventory costs. In this chapter we derive the cost equations for the exact models (nQ,R,T), (M,T), and (M,R,T) , since Exact versions give the least inventory costs.

Section 3.2 Definition Of Shortage Costs

As in chapter 2 it is assumed that when there is a stockout the cost is made up of three items : one a constant when a stockout accurs, the second term is proportional to the number of units backordered and the third is proportional to the time for which each backordered item has been backordered.

Section 3.3 Layout Of Chapter

In section 3.4 we derive the exact model of (nQ,R,T) and give the first order derivatives. In section 3.5 we derive the exact equation for (M,T) from first principles but in subsequent chapters we derive (M,T) by taking the limit of (nQ,M,T) as Q tends to 0. The first order derivatives are also given. In section 3.6 we derive the cost equations for model (M,R,T).

Section 3.4 (nQ,R,T)

Model (nQ,R,T) was defined to be the model in which an integer multiple of a fixed quantity Q is ordered only when on hand inventory equals or falls below R at review time T. n is the minimum integer such that the quantity ordered nQ, is such as to bring the inventory position above R.

Let f(x,Dt) be the density function of demand x in a period t.

where x follows a normal distribution. Demands in different periods are independent random variables. The steady state probability that the inventory position of the system is R+y, $0 < y < Q$ is necessary for the calculation of the state probabilities for the net inventory. Let $q(y,z)$ be the probability that if the inventory position is R+y immediately after a given review it will be R+z immediately after the next review.

If $z < y$ then the inventory position can be R+z immediately after a review when it was R+y immediately after the previous review if y-z, y-z+Q, y-z+2Q, ..., units have been demanded in the period between reviews since if d is the demand, then d must satisfy the equations,

$$R+y - d + nQ = R+z \qquad n = 0,1,2, \ldots$$

$$\text{i.e.} \quad d = y - z + nQ \qquad n = 0,1,2, \ldots$$

The probability that the demand is y - z + nQ is f(y - z + nQ, DT). in time T.
Since the probabilities for different n values are mutually exclusive

$$q(y,z) = \sum_{n}^{\infty} f(y - z + nQ,\ DT) \qquad z < y$$

However when $z > y$ the system cannot be in state z unless demand has been at least $y - z + nQ \quad n = 1,2,3, \ldots.$

$$\text{Hence} \quad q(y,z) = \sum_{n=1}^{\infty} f(y - z + nQ,\ DT)$$

The probability ρ(R+z) that the inventory position is R+z is equal to the probabilities that it was R+y multiplied by the transition probabilities from y to z.

$$\rho(R+z) = \int_0^Q \rho(R+y)\,q(y,z)\,dy$$

$$\text{where} \quad q(y,z) = \sum_{n=\varepsilon}^{\infty} f(y - z + nQ,\ DT) \qquad 3.4.1.$$

and $\varepsilon = 0$ if $z < y$

$\varepsilon = 1$ if $z > y$

Also we note that

$$\int_0^z \sum_{n=1}^{\infty} f(y - z + nQ, DT)\,dy + \int_z^Q \sum_{n=0}^{\infty} f(y - z + nQ, DT)\,dy$$

$$= \sum_{n=0}^{\infty} \left(\int_0^z f(nQ + Q - V,\ DT)\,dV + \int_0^{Q-z} f(nQ + V, DT)\,dV \right) \qquad 3.4.2.$$

letting V = z - y and V = y - z in each integral respectively.

3.4.2. is equivalent to $\sum_{n=0}^{\infty} \left(-\int_Q^{Q-z} f(nQ+Z, DT)\,dZ + \int_0^{Q-z} f(nQ + V, DT)\,dV \right)$

Since $\int_0^{\infty} f(\varepsilon, DT)\,d\varepsilon = 1$

Hence the sum of q(y,z) over y is equal to 1 for all z.

Since $\int_0^Q \rho(R+y)\,dy = 1$

Hence $\rho(R+y)$ = constant $0 < y < Q$ is a solution to 3.4.1.

It follows that

$$\rho(R+y) = \frac{1}{Q} \qquad 0 < y < Q \qquad 3.4.3.$$

The method of analysis in this chapter is similar to that of Hadley & Whitin (Ref.1 Ch.5). Let POR be the probability of placing an order at review time. Let R+y be the inventory position at the last review. The probability that at the next review an order is placed is equal to

$$\int_y^{\infty} f(V, DT)\,dV$$

Averaging over the states of y

$$POR = \frac{1}{Q}\int_{o}^{Q}\int_{y}^{\infty} f(V,Dt)\,dV\,dy$$

$$= \frac{1}{Q}\int_{o}^{Q} F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right) dy$$

Substituting V for $\frac{y-DT}{\sqrt{\sigma^2 T}}$

$$= \frac{\sqrt{\sigma^2 T}}{Q} * \int_{-\frac{DT}{\sqrt{\sigma^2 T}}}^{\frac{Q-DT}{\sqrt{\sigma^2 T}}} F(V)\,dV \qquad 3.4.4.$$

Demand can noot be negative so we make probability function 0 when y=0

However we still want the value of the cumulative density to =0 when y =0

and so using 1.3.7

$$POR = \frac{DT}{Q}\left(1 - F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) + F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sqrt{\sigma^2 T}}{Q}\, g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.5.$$

$$\text{where } g(V) = \frac{\exp - \frac{V^2}{2}}{\sqrt{2\pi}}$$

Next we calculate E(Q,R,T) the average number of backorders incurred per year.

If the inventory position of the system is R+y immediately after the review at time t, the expected number of backorders incurred between t and t+L+T

$$= \int_{R+y}^{\infty} (x - R - y) f(x, D(L + T))\,dx$$

Thus if the inventory position is R+y immediately after the review at time t, the expected number of backorders incurred between time t+L and t+L+T is

$$\int_{R+y}^{\infty} (x-R-y)(f(x,D(L+T))-f(x,DL))dx$$

Averaging over the states of y and dividing by T we see that the average number of backorders incurred per year

$$= \frac{1}{T}\int_{0}^{Q} \rho(R+y) \int_{R+y}^{\infty} (x-R-y)(f(x,D(L+T)-f(x,DL))dxdy \qquad 3.4.6.$$

Letting w = R+y in equation 3.4.6.

Hence E(Q,R,T)

$$= \frac{1}{QT}\int_{R}^{R+Q}\int_{w}^{\infty} (x-w)(f(x,D(L+T)-f(x,DL)) \ dxdw$$

where $\rho(R+y) = \frac{1}{Q}$ from 3.4.3.

Simplifying

$$E(Q,R,T) = \frac{1}{QT}\int_{R}^{\infty}\int_{w}^{\infty} (x-w)f(x,D(L+T))dxdw - \frac{1}{QT}\int_{R}^{\infty}\int_{w}^{\infty}(x-w)f(x,DL) \ dxdw$$

$$-\frac{1}{QT}\int_{R+Q}^{\infty}\int_{w}^{\infty} (x-w)f(x,D(L+T))dxdw + \frac{1}{QT}\int_{R+Q}^{\infty} (x-w)f(x,DL)dxdw \qquad 3.4.7$$

Using result 1.3.3 which states that

$$\int_{v}^{\infty}\int_{w}^{\infty} (x-w)f(x,Dt)dxdw$$

$$= \frac{\sigma^2 t}{2} \left((1 + (\frac{V-DT}{\sqrt{\sigma^2 t}})^2) F(\frac{V-DT}{\sqrt{\sigma^2 t}}) - (V-Dt) f(V,Dt) \right)$$

and substituting into 3.4.7

$E(Q,R,T)$

$$= \frac{1}{QT}(G_1(R,T+L) - G_1(R,L) - G_1(R+Q,T+L) + G_1(R+Q,L)) \qquad 3.4.8$$

where $G_1(R,T) = \frac{\sigma^2 T}{2}((1 + (\frac{R-DT}{\sqrt{\sigma^2 T}})^2)F(R,DT) - (R-DT)f(R,DT))$

Let $B(Q,R,T)$ be the expected number of backorders, at any point in time.

At anytime $t + L + \varepsilon$ between $t + L$ and $T + L + T$, the expected number of backorders on the books when the inventory position was $R+y$ immediately after review at time t

is $\int_{R+y}^{\infty} (x - R - y) f(x, D(L + \varepsilon))dx$

Thus the expected number of backorders incurred from $t + L$ to $t + L + T$ is

$$\int_{L}^{L+T} \int_{R+y}^{\infty} (x-R-y) f(x, D\varepsilon) dx d\varepsilon$$

Averaging over the states of y and dividing by T, the expected number of backorders $B(Q,R,T)$

$$= \frac{1}{T} \int_{o}^{Q} \rho(R+y) \int_{L}^{L+T} \int_{R+y}^{\infty} (x-R-y) f(x,Dt) dxdtdy$$

Substituting for $\rho(R+y) = \frac{1}{Q}$ from 3.4.3.

$$B(Q,R,T) = \frac{1}{QT} \int_{o}^{Q} \int_{L}^{L+T} \int_{R+y}^{\infty} (x-R-y) f(x,Dt)\,dx\,dt\,dy$$

Let $w = R+y$

$$B(Q,R,T) = \frac{1}{QT} \int_{Q}^{R+Q} \int_{L}^{L+T} \int_{w}^{\infty} (x-w) f(x,Dt)\,dx\,dt\,dw$$

From equation 1.3.2 which states that

$$\int_{w}^{\infty} (x-w) f(x,Dt)\,dx = \sigma^2 t f(w,Dt) - (w-Dt) F\left(\frac{w-Dt}{\sqrt{\sigma^2 t}}\right)$$

and substituting into B(Q,R,T) we have

$$B(Q,R,T) = \frac{1}{QT} \int_{Q}^{R+Q} \int_{L}^{L+T} \left(\sigma^2 t f(w,Dt) - (w-Dt) F\left(\frac{w-Dt}{\sqrt{\sigma^2 t}}\right)\right) dt\,dw$$

Simplifying

$$B(Q,R,T) = \frac{1}{QT} \int_{Q}^{R+Q} \int_{o}^{L+T} \left(\sigma^2 t f(w,Dt) - (w-Dt) F\left(\frac{w-Dt}{\sqrt{\sigma^2 t}}\right)\right) dt\,dw$$

$$- \frac{1}{QT} \int_{Q}^{R+Q} \int_{o}^{L} \left(\sigma^2 t f(w,Dt) - (w-Dt) F(w,Dt)\right) dt\,dw$$

We shall next evaluate some necessary mathematical results

from equation 1.3.11

$$Z_o(x,T) = \int_{o}^{T} f(x,Dt)\,dt = \frac{F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)}{D} - \frac{e^{\frac{2Dx}{\sigma^2}}}{D} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)$$

Also from equation 1.3.14a

$$Z_1(x,T) = -\frac{2\sqrt{\sigma^2 T}}{D^2}\frac{\exp}{\sqrt{2\pi}} - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$+ \frac{\sigma^2}{D^3}\left(1 + \frac{Dx}{\sigma^2}\right) F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) + \frac{1}{D^2}\left(x - \frac{\sigma^2}{D}\right) e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+Dt}{\sqrt{\sigma^2 T}}\right)$$

From 1.3.15 $\quad R_n(x,T) = \int_o^T t^n F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) dt$

$$= \frac{T^{n+1}}{(n+1)} F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{D}{2(n+1)} Z_{n+1}(x,T) - \frac{x}{2(n+1)} Z_n(x,T) \qquad 3.4.9$$

$n=0,1,2, \ldots$

Where $Z_n(x,T) = \int_o^T t^n f(x,Dt)dt$

When $n = o$

$$R_o(x,T) = \int_o^T F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right)dt$$

and applying 3.4.9.

$$R_o(x,t) = TF\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{D}{2} Z_1(x,T) - \frac{x}{2} Z_o(x,T)$$

Substituting for $Z_1(x,T)$ and $Z_o(x,T)$ from 1.1.14a and 1.4.11 respectively

$$R_o(x,T) = TF\left(\frac{x-Dt}{\sqrt{\sigma^2 T}}\right) - \frac{D}{2}\left\{-\frac{2\sqrt{\sigma^2 T}}{D^2\sqrt{2\pi}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2 + \frac{\sigma^2}{D^3}\left(1 + \frac{Dx}{\sigma^2}\right)\right.$$

$$* F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) + \frac{1}{D^2}\left(x - \frac{\sigma^2}{D}\right) e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)\Bigg)$$

$$- \frac{x}{2D}\left(F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)\right)$$

Simplifying

$$R_o(x.T) = \left(T - \frac{x}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{x-Dt}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{D\sqrt{2\pi}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$+ \frac{\sigma^2}{2D^2} e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.10$$

$$R_1(x,T) = \int_0^T tF\left(\frac{x-DT}{\sqrt{\sigma^2 t}}\right) dt$$

From equation 3.4.9

$$R_1(x,T) = \frac{T^2}{2} F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{D}{4} Z_2(x,T) - \frac{x}{4} Z_1(x,T) \qquad 3.4.11$$

From equation 1.3.14b

$$Z_{n+1}(x,T) = \frac{-2\sigma^2 T^{n+1}}{D^2\sqrt{2\pi\sigma^2 T}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$+ \frac{2n+1}{D^2}\sigma^2 Zn(x,T) + \frac{x^2}{D^2} Z_{n-1}(x,T)$$

when n = 1

$$\text{Hence } Z_2(x,T) = - \frac{2\sigma^2 T^2}{D^2\sqrt{2\sigma^2 T\pi}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$+ \frac{3\sigma^2}{D^2} Z_1(x,T) + \frac{x^2}{D^2} Z_o(x,T) \qquad 3.4.12$$

Substituting $Z_2(x,T)$ in to $R_1(x,T)$

$$R_1(x,T) = \frac{T^2}{2} F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{D}{4}\left(-\frac{2\sigma^2 T^2}{D^2\sqrt{2\pi\sigma^2 T}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2\right.$$

$$\left. + \frac{3\sigma^2}{D^2} Z_1(x,T) + \frac{x^2}{D^2} Z_o(x,T)\right) - \frac{x}{4} Z_1(x,T)$$

Simplifying

$$R_1(x,T) = \frac{T^2}{2} F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{3\sigma^2}{4D} + \frac{x}{4}\right) Z_1(x,T) - \frac{x^2}{4D} Z_o(x,T)$$

$$+ \frac{\sqrt{\sigma^2 T}}{2D\sqrt{2\pi}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

Substituting for $Z_1(x,T)$ and $Z_o(x,T)$ from 1.1.14a and 1.3.11 respectively

$$R_1(x,T) = \frac{T^2}{2} F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{3\sigma^2}{4D} + \frac{x}{4}\right)\left(-\frac{2\sqrt{\sigma^2 T}}{D^2\sqrt{2\pi}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2\right.$$

$$\left. + \frac{\sigma^2}{D^3}\left(1 + \frac{Dx}{\sigma^2}\right) F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) + \frac{1}{D^2}\left(x - \frac{\sigma^2}{D}\right) e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)\right)$$

$$- \frac{x^2}{4D^2}\left(F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)\right) + \frac{\sqrt{\sigma^2 T}}{2D\sqrt{2\pi}} \exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

Simpifying

$$R_1(x,T) = \tfrac{1}{2}\left(T^2 - \frac{x^2}{D^2} - \frac{2\sigma^2 x}{D^3} - \frac{3\sigma^4}{2D^4}\right) F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{\sqrt{\sigma^2 T}}{2D^2}\left(DT + \frac{3\sigma^2}{D} + x\right)\frac{\exp}{\sqrt{2\pi}} - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$- \frac{\sigma^2}{2D^3}\left(x - \frac{3\sigma^2}{2D}\right) e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.13$$

From 3.4.8

$$B(Q,R,T) = \frac{1}{QT}\int_R^{R+Q}\int_o^{L+T}\left(\sigma^2 t f(w,Dt) - (w-Dt)F\left(\frac{w-Dt}{\sqrt{\sigma^2 T}}\right)\right) dt\,dw$$

$$- \frac{1}{QT}\int_R^{R+Q}\int_o^{L}\left(\sigma^2 t f(w,Dt) - (w-Dt)F\left(\frac{w-Dt}{\sqrt{\sigma^2 t}}\right)\right) dt\,dw \qquad 3.4.14.$$

Substituting Z_1 , R_o and R_1 into expression

$$B(Q,R,T) = \frac{1}{QT}\int_R^{R+Q} (\sigma^2 Z_1(w,L+T) - wR_o(w,L+T) + DR_1(W,L+T))\,dw$$

$$- \frac{1}{QT}\int_R^{R+Q} (\sigma^2 Z_1(w,L) - wR_o(w,L) + DR_1(w,L))\,dw \qquad 3.4.15$$

Let $G_2(w,T) = \int_o^T\int_w^\infty (x-w) f(x,Dt)\,dx\,dt$

$$= \sigma^2 Z_1(w,T) - wR_o(w,T) + DR_1(w,T) \qquad 3.4.16.$$

From 1.3.14a, 3.4.10 and 3.4.13, substituting for $Z_1(w,T)$, $R_o(w,T)$ and $R_1(w,T)$ into 3.4.16

$$G_2(w,T) = \frac{\sigma^4}{D^3}\left(1 + \frac{Dw}{\sigma^2}\right) F\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right) - \frac{2\sigma^2\sqrt{\sigma^2 T}}{D^2}\, g\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{\sigma^2}{D^2}\, e^{\frac{2Dw}{\sigma^2}} \left(w - \frac{\sigma^2}{D}\right) F\left(\frac{w+DT}{\sqrt{\sigma^2 T}}\right)$$

$$- w\left(T - \frac{w}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right) - w\,\frac{\sqrt{\sigma^2 T}}{D}\, g\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{w\sigma^2}{2D^2}\, e^{\frac{2Dw}{\sigma^2}}\, F\left(\frac{w+DT}{\sqrt{\sigma^2 T}}\right) + \frac{D}{2}\left(T^2 - \frac{w^2}{D^2} - \frac{2\sigma^2 w}{D^3} - \frac{3\sigma^4}{2D^4}\right) F\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{\sqrt{\sigma^2 T}}{2D}\left(DT + \frac{3\sigma^2}{D} + w\right) g\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^2}{2D^2}\left(w - \frac{3\sigma^2}{2D}\right) e^{\frac{2Dw}{\sigma^2}}\, F\left(\frac{w+DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.17a$$

Simplifying

$$G_2(w,T) = \left(\frac{\sigma^4 + 2D^4T^2}{4D^3} + \frac{w(\sigma^2 - 2D^2T)}{2D^2} + \frac{w^2}{2D}\right) F\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \tfrac{1}{2}\left(\sigma T^{3/2} - \frac{\sigma^3 T^{\frac{1}{2}}}{D^2} - \frac{T^{\frac{1}{2}} w}{D}\right) g\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.17b$$

$$- \frac{\sigma^4}{4D^3}\, e^{\frac{2Dw}{\sigma^2}}\, F\left(\frac{w+DT}{\sqrt{\sigma^2 T}}\right)$$

Substituting $G_2(w,L)$ and $G_2(W,L+T)$ into 3.4.15

$$B(Q,R,T) = \frac{1}{QT}\int_R^{R+Q} \left(G_2(W,L+T) - G_2(W,L)\right) dw \qquad 3.4.18$$

But $\int_R^{\infty} e^{\frac{2DW}{\sigma^2}} \; F\left(\frac{W+DT}{\sqrt{\sigma^2 T}}\right) dW$

integrating by parts

$$= F\left(\frac{W+DT}{\sqrt{\sigma^2 T}}\right) \frac{\sigma^2}{2D} e^{\frac{2DW}{\sigma^2}} \Bigg]_R^{\infty} + \frac{\sigma^2}{2D} \int_R^{\infty} e^{\frac{2DW}{\sigma^2}} \; g\left(\frac{W+DT}{\sqrt{\sigma^2 T}}\right) dW$$

Noting that $e^{\frac{2DW}{\sigma^2}} \; g\left(\frac{W+DT}{\sqrt{\sigma^2 T}}\right) = g\left(\frac{W-DT}{\sqrt{\sigma^2 T}}\right)$

and simplifying and integrating

$$\int_R^{\infty} e^{\frac{2DW}{\sigma^2}} F\left(\frac{W+DT}{\sqrt{\sigma^2 T}}\right) dW = -\frac{\sigma^2}{2D} e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^2}{2D} F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.19$$

Integrating 3.4.17b from R to ∞ and substituting 1.3.7, 1.3.8b, 3.4.19 into integration we have

$$G_3(R,T) = \int_R^{\infty} G_2(W,T)\, dW \qquad 3.4.20$$

$$= \left[\left(\frac{\sigma^4 + 2D^4T^2}{4D^3}\right)\right] \times \left[g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right]$$

$$+ \frac{1}{2}\left(\frac{\sigma^2 - 2D^2T}{2D^2}\right)\left((\sigma^2 T + D^2T^2 - R^2) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \sqrt{\sigma^2 T}\,(R+DT)\, g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right)$$

$$+ \frac{1}{6D}\left[(D^3T^3 + 3DT^2\sigma^2 - R^3) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \sqrt{\sigma^2 T}\,(R^2 + RDT + D^2T^2 + 2\sigma^2 T)\, g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right]$$

$$+ \frac{1}{2}\left(\sigma T^{3/2} - \frac{\sigma^3 T^{\frac{1}{2}}}{D^2}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \frac{T^{\frac{1}{2}}}{D}\left(\sigma^2 T\, g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + DT\sqrt{\sigma^2 T}\, F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right)$$

$$- \frac{\sigma^4}{8D^4} F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^4}{8D^4} e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)$$

Simplifying we have

$$G_3(R,T) = \left(\frac{D^2T^3}{6} - \frac{\sigma^4 R}{4D^3} - \frac{DT^2R}{2} - \frac{\sigma^2R^2}{4D^2} + \frac{\sigma^2T^2}{4} + \frac{TR^2}{2} - \frac{R^3}{6D} - \frac{\sigma^6}{8D^4} \right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \left(\frac{DT^{5/2}\sigma}{6} - \frac{\sigma T^{3/2}R}{3} + \frac{\sigma T^{\frac{1}{2}}R^2}{6D} + \frac{\sigma^3 T^{3/2}}{12D} + \frac{\sigma^3 T^{\frac{1}{2}}R}{4D^2} + \frac{\sigma^5 T^{\frac{1}{2}}}{4D^3} \right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^6}{8D^4} \cdot e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.21$$

Substituting $G_3(R,T)$ into 3.4.18

$$B(Q,R,T) = \frac{1}{QT}\left(G_3(R+L+T) - G_3(R,L) - G_3(R+Q,L+T) + G_3(R+Q,L)\right) \qquad 3.4.22$$

Let D(Q,R,T) be the expected on hand inventory

If the inventory position of the system was R+y immediately after the review at time t , the expected value of the net inventory at any time ε between t+L and t+L+T.

$$= \int_0^\infty (R+y-x) f(x, D\varepsilon)\,dx$$

where x has been demanded in ε.

The expected value of the on hand inventory at any time is the expected net inventory plus the expected number of backorders.

Hence the expected on hand inventory at time ε

$$= R+y-D\varepsilon + \int_{R+y}^\infty (x-R-y) f(x, D\varepsilon)\,dx$$

Integrating from L to T+L and averaging over the states of y and dividing T we obtain the expected on hand inventory.

$$D(Q,R,T) = \frac{1}{QT}\int_0^Q \int_L^{L+T} (R+y-D\varepsilon)\,d\varepsilon\,dy + \frac{1}{QT}\int_0^Q \int_L^{L+T} \int_{R+y}^{\infty} (x-R-y) f(x,D\varepsilon)\,dx\,d\varepsilon\,dy$$

3.4.23

Integrating and simplifying

$$D(Q,R,T) = \frac{Q}{2} + R - DL - \frac{DT}{2} + B(Q,R,T) \qquad 3.4.24$$

where $B(Q,R,T) = \frac{1}{QT}\int_0^Q \int_L^{L+T} \int_{R+y}^{\infty} (x-R-y) f(x,D\varepsilon)\,dx\,d\varepsilon\,dy$

Thus the inventory cost for model (nQ,R,T)

$$C = \frac{Rc}{T} + \frac{S.POR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + b_1 E(Q,R,T)$$

$$+ (hc + b_2) B(Q,R,T).$$

where Rc is the review cost.

Next we develop the probability of a stockout which is not however given in Hadley & Whitin (Ref. 1). If we decide to include the stockout cost s which depends upon the number of stockouts only, then we need to develop the probability of a stockout POUT. If the inventory position was R+y at time t, immediately after a review,

the probability that there is a stockout at anytime between t+L and t+L+T is the probability that demand exceeds R+y at t+L+ε.

$$= \int_{R+y}^{\infty} f(x, D(L+\varepsilon))\,dx$$

Thus the probability of a stockout

$$= \int_{L}^{L+T} \int_{R+y}^{\infty} f(x, D\varepsilon)\,dx\,d\varepsilon$$

Averaging the states of y, probability of stockout

$$= \frac{1}{Q} \int_{o}^{Q} \int_{L}^{L+T} \int_{R+y}^{\infty} f(x, D\varepsilon)\,dx\,d\varepsilon\,dy \qquad 3.4.25$$

Integrating with respect to x

$$\text{PouT} = \frac{1}{Q} \int_{o}^{Q} \int_{L}^{L+T} F(R+y, D\varepsilon)\,d\varepsilon\,dy \qquad 3.4.26$$

By differentiating 3.4.26 with respect to T we immediately see that

$$\lim_{T \to 0} \text{POUT OF } (Q,R,T) = \text{POUT OF } (Q,R). \text{ (Equation 3.1.17)}$$

Substitute equation $R_o(x,T)$ of 3.4.10 into 3.4.26 we have

$W = R+y$

$$\text{PouT} = \frac{1}{Q} \int_{R}^{R+Q} (Ro(W,L+T) - Ro(W,L))\,dW \qquad 3.4.27$$

Noting that substituting for Ro(W,T) from 3.4.10

$$\text{Let } G_4(R,T) = \int_{R}^{\infty} Ro(W,T)\,dw$$

$$G_4\ (R,T) = \int_{R}^{\infty}\left(\left(T - \frac{W}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{W-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{D}\, g\left(\frac{W-DT}{\sqrt{\sigma^2 T}}\right)\right.$$

$$+ \frac{\sigma^2}{2D^2} e^{\frac{2DW}{\sigma^2}} F\left(\frac{w+DT}{\sqrt{\sigma^2 T}}\right)\Bigg) dW \qquad 3.4.28$$

Integrating and applying 1.3.7, 1.3.8b and 3. 4.9 we have

$$G_4(R,T) = \left(T - \frac{\sigma^2}{2D^2}\right)\sqrt{\sigma^2 T}\left(g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right)$$

$$- \frac{1}{2D}\left(\left(\sigma^2 T + D^2T^2 - R^2\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \sqrt{\sigma^2 T}\;(R+DT)\, g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right)$$

$$+ \frac{\sigma^2 T}{D} f\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^4}{4D^3}\left(F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) e^{\frac{2DR}{\sigma^2}}\right)$$

Simplifying

$$G_4(R,T) = \left(\frac{(R-DT)^2}{2D} + \frac{\sigma^2 R}{2D^2} + \frac{\sigma^4}{4D^3}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{\sqrt{\sigma^2 T}}{2}\left(T - \frac{\sigma^2}{D^2} - \frac{R}{D}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^4}{4D^3} F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) e^{\frac{2DR}{\sigma^2}} \qquad 3.4.29$$

Hence PouT from 3.4.27

$$\text{Pour} = \frac{1}{Q}\left(G_4(R,L+T) - G_4(R,L) - G_4(R+Q,L+T) + G_4(R+Q,L)\right)$$

If the stockout cost s, is a constant cost incurred whenever a stockout occurs is included in C, then C becomes:

$$C = \frac{R_c}{T} + \frac{S.PoR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + b_r E(Q,R,T)$$

$$+ \ (hc + b_2)B(Q,R,T) + \frac{s}{T}\ \mathrm{PouT} \qquad 3.4.30$$

Substituting 3.4.8 for $E(Q,R,T)$, 3.4.22 for $B(Q,R,T)$ and 3.4.20 for we have

$$C = \frac{R_c}{T} + \frac{S.POR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + \frac{b_1}{QT}\Big(G_1(R,T+L) - G_1(R,L)$$

$$- G_1(R+Q,T+L) + G_1(R+Q,L)\Big)$$

$$+ \frac{(hc+b_2)}{QT}\Big(G_3(R,T+L) - G_3(R,L) - G_3(R+Q,T+L) + G_3(R+Q,L)\Big)$$

$$+ \frac{s}{QT}\ (G_4(R,T+L) - G_4(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)) \qquad 3.4.31$$

The steepest descent and Powell's iterative technique requires the first order derivatives for optimisation.

In order to be able to obtain the first order derivates of C it necessary first to obtain the first order derivatives of POR, G_1, G_3 and G_4.

From 3.4.5.

$$POR = \frac{DT}{Q}\left(1 - F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) + F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{\sqrt{\sigma^2 T}}{Q}\ g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

Differentiating with respect to Q

$$\frac{\partial POR}{\partial Q} = -\frac{DT}{Q^2}\left(1 - F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) + \frac{DT}{Q}\cdot\frac{1}{\sqrt{\sigma^2 T}}\ g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{1}{\sqrt{\sigma^2 T}} g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{Q^2} g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{Q\sqrt{\sigma^2 T}}\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

Simplifying

$$\frac{\partial POR}{\partial Q} = -\frac{DT}{Q^2}\left(1 - F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) + \frac{\sqrt{\sigma^2 T}}{Q^2} g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.32$$

Differentiating POR with respect to T and using 1.3.4

$$\frac{\partial POR}{\partial T} = \frac{D}{Q}\left(1 - F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) - \frac{DT}{Q}\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{(Q-DT)}{2\sigma T^{3/2}}\right) g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{(Q-DT)}{2\sigma T^{3/2}}\right) g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma}{2QT^{\frac{1}{2}}} g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sqrt{\sigma^2 T}}{Q}\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{(Q-DT)}{2\sigma T^{3/2}}\right) g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

Simplifying

$$\frac{\partial POR}{\partial T} = \frac{D}{Q}\left(1 - F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) - \frac{\sigma}{2QT^{\frac{1}{2}}} g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.33$$

$$\frac{\partial POR}{\partial R} = 0 \qquad 3.4.34$$

From 3.4.8

$$G_1(R,T) = \frac{\sigma^2 T}{2}\left(\left(1 + \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)^2\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right)$$

Differentiating with respect to R and from 1.3.2 and 1.3.3

$$\frac{\partial G_1(R,T)}{\partial R} = - \left(\sqrt{\sigma^2}T g\left(\frac{R-DT}{\sqrt{\sigma^2 t}}\right) - (R-DT)F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right)$$

Differentiating with respect to T and using 1.3.4., 1.3.5., 1.3.6.

$$\frac{\partial G_1(R,T)}{\partial T} = \frac{\sigma^2}{2}\left[\left(1 + \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)^2\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right.$$

$$+ \frac{\sigma^2 T}{2}\left[- 2\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{R-DT}{2\sigma T^{3/2}}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right.$$

$$+ \left(1 + \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)^2\right)\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{R-DT}{2\sigma T^{3/2}}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) +$$

$$\left.\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{R-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)^2\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{R-DT}{2\sigma T^{3/2}}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right]$$

Simplifying

$$\frac{\partial G_1(R,T)}{\partial T} = \frac{\sigma^2}{2}\left[\left(1 + \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)^2\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right]$$

$$+ \sqrt{\sigma^2 T}\left(\frac{D}{2} + \frac{R}{2T}\right)\left(g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right) \qquad 3.4.35$$

From 3.4.21

$$G_3(R,T) = \left(\frac{D^2T^3}{6} - \frac{\sigma^4 R}{4D^3} - \frac{DT^2R}{2} - \frac{\sigma^2R^2}{4D^2} + \frac{\sigma^2T^2}{4} + \frac{TR^3}{2} - \frac{\sigma^6}{8D^4} - \frac{R^3}{6D}\right) *$$

$$F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+\left(\frac{DT^{5/2}\sigma}{6}-\frac{\sigma T^{3/2}R}{3}+\frac{\sigma T^{\frac{1}{2}}R^2}{6D}+\frac{\sigma^3 T^{3/2}}{12D}+\frac{\sigma^3 T^{\frac{1}{2}}R}{4D^2}+\frac{\sigma^5 T^{\frac{1}{2}}}{4D^3}\right)\times$$

$$g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)+\frac{\sigma^6}{8D^4}\,e^{\frac{2DR}{\sigma^2}}\,F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)$$

Differentiating with respect to Q and applying 3.4.17 and the fact that

$$G_3(R,T) = \int_R^\infty G_2(W,T)\,dW$$

then

$$\frac{\partial G_3(R,T)}{\partial R} = -\,G_2(R,T)$$ and substituting in 3.4.17b

$$-\frac{\partial G_3(R,T)}{\partial R} = \left(\frac{\sigma^4+2D^4T^2}{4D^3}+\frac{\sigma^2-2D^2T}{2D^2}+\frac{R^2}{2D}\right)F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+\ \frac{1}{2}\left(\sigma T^{3/2}-\frac{\sigma^3 T^{\frac{1}{2}}}{D^2}-\frac{\sigma T^{\frac{1}{2}}R}{D}\right)g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$-\ \frac{\sigma^4}{4D^3}\,e^{\frac{2DR}{\sigma^2}}\,F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.4.36$$

Differentiating $G_3(R,T)$ with respect to T and applying the fact that

$$G_3(R,T) = \int_R^\infty G_2(W,T)\,dW$$

thus $$\frac{\partial G_3(R,T)}{\partial T} = \int_R^\infty \frac{\partial G_2(W,T)}{\partial T}\,dW \qquad 3.4.37a$$

From 3.4.16

$$G_2(W,T) = \int_0^T \left(\sqrt{\sigma^2 t}\, g\left(\frac{W-DT}{\sqrt{\sigma^2 T}}\right) - (W-Dt)\, F\left(\frac{W-Dt}{\sqrt{\sigma^2 t}}\right)\right) dt \qquad 3.4.37b$$

Thus

$$\frac{\partial G_2(w,T)}{\partial T} = \sqrt{\sigma^2 T}\, g\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right) - (w-DT)\, F\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right)$$

and substituting into 3.4.27 and applying 1.3.3.

$$\frac{\partial G_3(R,T)}{\partial T} = \frac{\sigma^2 T}{2}\left(\left(1 + \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)^2\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right) \qquad 3.4.38$$

From 3.4.28

$$G_4(R,T) = \int_R^\infty Ro(W,T)\, dW$$

Thus $\dfrac{\partial G_4(R,T)}{\partial R} = -\, Ro(R,T)$ 3.4.39a

where Ro(R,T) from 3.4.10

$$Ro(R,T) = \left(T - \frac{R}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{D}\, g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{\sigma^2}{2D^2}\, e^{\frac{2DR}{\sigma^2}}\, F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)$$

Differentiating $G_4(R,T)$ with respect to T and applying 3.4.28

$$\frac{\partial G_4(R,T)}{\partial T} = \int_R^{\infty} \frac{\partial Ro(w,T)\,dw}{\partial T} \qquad 3.4.39\ b.$$

and from 1.3.15

$$R_n(w,T) = \int_o^T t^n F\left(\frac{w-Dt}{\sqrt{\sigma^2 t}}\right) dt \qquad n=o,1,2, \ldots.$$

thus $\dfrac{\partial Rn(w,T)}{\partial T} = T^n F\left(\dfrac{w-DT}{\sqrt{\sigma^2 T}}\right)$ $\quad n = 0,1,2, \ldots$

and applying this to 3.4.39

$$\frac{\partial G_4(R,T)}{\partial T} = \int_R^{\infty} F\left(\frac{w-DT}{\sqrt{\sigma^2 T}}\right) dw$$

and applying 1.3.7

$$\frac{\partial G_4(R,T)}{\partial T} = \sqrt{\sigma^2 T}\left(g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right) \qquad 3.4.40$$

Hence differentiating the cost expression 3.4.31 with respect to Q,R,T respectively

we obtain

$$\frac{\partial C}{\partial Q} = \frac{S}{T}\cdot\frac{\partial POR}{\partial Q} + \frac{hc}{2} - \frac{b_1}{TQ^2}\Big(G_1(R\ T+L) - G_1(R,L) - G_1(R+Q,T+L)$$

$$- G_1(R+Q,L)\Big) + \frac{b_1}{QT}\left(-\frac{\partial G_1}{\partial Q}(R+Q,T+L) + \frac{\partial G_1}{\partial Q}(R+Q,L)\right)$$

$$+\frac{(hc+b_2)}{QT}\left(-\frac{\partial G_3}{\partial Q}(R+Q,T+L)+\frac{\partial G_3}{\partial Q}(R+Q,L)\right)$$

$$-\frac{(hc+b_2)}{Q^2T}(G_3(R,T+L)-G_3(R,L)-G_3(R+Q,T+L)+G_3(R+Q,L))+\frac{s}{QT}\left(-\frac{\partial G_4}{\partial Q}(R+Q,T+L)\right.$$
$$\left.+\frac{\partial G_4}{\partial Q}(R+Q,L)\right)$$

$$-\frac{s}{Q^2T}(G_4(R,T+L)-G_4(R,L)-G_4(R+Q,T+L)+G_4(R+Q,L))$$

$$\frac{\partial C}{\partial R}=hc+\frac{b_1}{QT}\left(\frac{\partial G_1}{\partial R}(R,T+L)-\frac{\partial G_1}{\partial R}(R,L)-\frac{\partial G_1}{\partial R}(R+Q,T+L)+\frac{\partial G_1}{\partial R}(R+Q,L)\right)$$

$$+\frac{(b_2+h_c)}{QT}\left(\frac{\partial G_3}{\partial R}(R,T+L)-\frac{G_3}{R}(R,L)-\frac{G_3}{R}(R+Q,T+L)+\frac{G_3}{R}(R+Q,L)\right)$$

$$+\frac{s}{QT}\left(\frac{\partial G_4}{\partial R}(R,T+L)-\frac{\partial G_4}{\partial R}(R,L)-\frac{\partial G_4}{\partial R}(R+Q,T+L)+\frac{\partial G_4}{\partial R}(R+Q,L)\right)$$

$$\frac{\partial C}{\partial T}=-\frac{Rc}{T^2}-\frac{s\ POR}{T^2}+\frac{s}{T}\frac{\partial PoR}{\partial T}-\frac{Dhc}{2}-\frac{b_1}{QT^2}(G_1(R,T+L)-G_1(R,L)$$

$$-G_1(R+Q,T+L)+G_1(R+Q,L))+\frac{b}{QT}\left(\frac{\partial G_1}{\partial T}(R,T+L)-\frac{\partial G_1}{\partial T}(R+Q,T+L)\right)$$

$$-\frac{(hc+b_2)}{QT^2}(G_3(R,T+L)-G_3(R,L)-G_3(R+Q,T+L)+G_3(R+Q,L))$$

$$+\frac{(hc+b_2)}{QT}\left(\frac{\partial G_3}{\partial T}(R,T+L)-\frac{\partial G_3}{\partial T}(R+Q,T+L)\right)$$

$$-\frac{s}{QT^2}(G_4(R,T+L)-G_4(R,L)-G_4(R+Q,T+L)+G_4(R+Q,L))$$

$$+\frac{s}{QT}\left(\frac{\partial G_4}{\partial T}(R,T+L)-\frac{\partial G_4}{\partial T}(R+Q,T+L)\right) \qquad 3.4.41$$

3.5. Model (M,T)

In section 1.2 model (M,T) was defined as the model in which an order is placed at each review time T and the order is such as to bring the inventory level to M. We could derive model (M,T) in either of two ways. One directly from basic principles and the other from model (nQ,R,T).

In this section it is be derived from basic principles but in later chapters it would be derived from model (nQ,R,T).

Let E(M,T) be the average number of backorders per year.

Since the inventory position is M immediately after reviewing and placing an order at time t

the expected number of backorders at time t+L+T

$$= \int_M^\infty (x-M)f(x,D(L+T))dx$$

and time t+L, expected number of backorders

$$= \int_M^\infty (x-M)f(x,DL)dx$$

Hence the average number of backorders incurred from t+L to t+L+T

$$= \int_M^\infty (x-M)(f(x,D(L+T))-f(x,DL))dx \qquad 3.5.1$$

Applying equation 1.3.2.

$$E(M,T) = \sqrt{\sigma^2(T+L)}\; g\left(\frac{M-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right) - (M-D(T+L))F\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$-\sqrt{\sigma^2 L}\; g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - (M-DL)F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

Let $R_5(M,T) = \sqrt{\sigma^2 T}\, g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - (M-DT)F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$ 3.5.2.

Thus $E(M,T) = \frac{1}{T}(G_5(M,T+L) - G_5(M,L))$ 3.5.3

Let $B(M,T)$ be the expected number of backorders at any point in time. At any time $t+L+\varepsilon$ between $t+L$ and $t+L+T$, the expected number of backorders on the books

$$= \int_M^\infty (x-M)f(x,D(L+\varepsilon))dx$$

Thus the expected number of backorders incurred from T+L to t+L+T is

$$\int_L^{L+T} \int_M^\infty (x-M)f(x,D\varepsilon)dxd\varepsilon$$

Hence the expected number of backorders, per year, at any point in time

$$= \frac{1}{T}\int_L^{L+T} \int_M^\infty (x-M)f(x,D\varepsilon)dxd\varepsilon \quad 3.5.4$$

Applying 3.4.17a

$$B(M,T) = \frac{1}{T}(G_2(M,T+L) - G_2(M,L)) \quad 3.5.5$$

where from 3.4.17

$$G_2(M,T) = \left(\frac{(\sigma^4+2D^4T^2)}{4D^3} + \frac{M(\sigma^2-2D^2T)}{2D^2} + \frac{M^2}{2D}\right) *$$

$$F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{1}{2}\left(\sigma T^{3/2} - \frac{\sigma^3 T^{\frac{1}{2}}}{D^2} - \frac{\sigma T^{\frac{1}{2}}M}{D}\right) g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{\sigma^4}{4D^3} e^{\frac{2DM}{\sigma^2}} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right) \quad 3.5.6$$

Let $D(M,T)$ be the expected on hand inventory. The expected on hand inventory $D(M,T)$ is equal to the net inventory plus the expected number of backorders.

If M is the inventory position immediately after reviewing and placing an order, and since everything on order will have arrived in a procurement lead time and nothing which is not on order can arrive in this time, the expected net inventory immediately after the arrival of a procurement must be M-DL, where DL is the expected lead time demand. Also since D is the expected rate of demand per year, the expected demand in an interval dt = D dt.

Hence the expected on hand inventory at time ε

$$t+L < t+L+\varepsilon < t+L+T$$

$$= \int_{M}^{\infty} (x-M) f(x, D\varepsilon) + M - DL - D\varepsilon$$

Integrating from L to T+L and dividing by T to get the on hand inventory per year

$$D(M,T) = \frac{1}{T} \int_{L}^{T+L} (x-R) f(x, D\varepsilon) dx d\varepsilon + \frac{1}{T} \int_{L}^{T+L} (R-DL-D\varepsilon) d\varepsilon$$

From 3.5.4 and integrating

$$D(M,T) = \frac{1}{T} (M-DL-\frac{DT}{2}) + B(M,T) \qquad 3.5.7$$

$$\text{Thus Inventory costs} = \frac{R_c}{T} - \frac{S}{T} + hcD(M,T) + b_1 E(M,T) + b_2 B(M,T)$$

Hence

$$C = \frac{Rc}{T} + \frac{S}{T} + hc\,(M-DL-\frac{DT}{2}) + b_1 E(M,T) + (b_2+hc) B(M,T) \qquad 3.5.8$$

If we decided to include the stockout costs dependent upon the number of stockouts only then we need to develop POUT the probability of a stockout.

The probability that there is a stockout at any time $t+L+\varepsilon$, between $t+L$ and $t+L+T$ is the probability that demand exceeds M at $t+L+\varepsilon$

$$= \int_{M}^{\infty} f(x, D(L+\varepsilon))\,dx$$

Integrating over the ranges of ε, the probability of a stockout is

$$\text{POUT} = \int_{L}^{T+L} \int_{M}^{\infty} f(x, D\varepsilon)\,dx\,d\varepsilon \qquad 3.5.9$$

Integrating

$$\text{POUT} = \int_{L}^{T+L} F\left(\frac{M-D\varepsilon}{\sqrt{\sigma^2 \varepsilon}}\right) d\varepsilon$$

Applying 3.4.10

$$\text{POUT} = \text{Ro}(M, T+L) - \text{Ro}(M, L) \qquad 3.5.10$$

$$\text{where } \text{Ro}(R,T) = \left(T - \frac{M}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{D}\, g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{\sigma^2}{2D^2}\, e^{\frac{2DR}{\sigma^2}}\, F\left(\frac{M\ DT}{\sqrt{\sigma^2 T}}\right)$$

The inventory cost for model (M,T) becomes

$$C = \frac{R_c + S}{T} + hc\left(M - \cancel{DL} - \frac{DT}{2}\right) + b_1 E(M,T) + (b_2 + hc)B(M,T) + \frac{s}{T}\,\text{PouT} \qquad 3.5.11$$

Substituting 3.5.3 for E(M,T) and 3.5.5 for B(M,T) and 3.5.10 for POUT

We obtain

$$C = \frac{R_C + S}{T} + h_C\left(M - L - \frac{DT}{2}\right) + \frac{b_1}{T}\left(G_5(M,T+L) - G_5(M,L)\right)$$

$$+ \frac{(b_2 + h_C)}{T}\left(G_2(M,T+L) - G_2(M,L)\right) + \frac{s}{T}\left(Ro(M,T+L) - Ro(M,L)\right) \qquad 3.5.12$$

The Simplex technique and Powell's iterative tchnique require the first order derivatives for optimisation.

In order to obtain the first order derivatives of C we need to obtain the first order derivatives of $G_5(M,T)$, $G_2(M,T)$, $Ro(M,T)$

From 3.5.2.

$$G_5(M,T) = \sqrt{\sigma^2 T}\; g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - (M-DT)\,F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

Differentiating with respect to M

$$\frac{\partial G_5(M,T)}{\partial M} = -\frac{\sqrt{\sigma^2 T}}{\sqrt{\sigma^2 T}}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.5.13$$

Simplifying

$$\frac{\partial G_5(M,T)}{\partial M} = -F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.5.14$$

Differentiating with respect to T and applying 1.3.4, and 1.3.6

$$\frac{\partial G_5(M\ T)}{\partial T} = \frac{\sigma}{2T^{\frac{1}{2}}}\, g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \sqrt{\sigma^2 T}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{(M\ DT)}{2\sigma T^{3/2}}\right)$$

$$g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + DF\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - (M-DT)\left(\frac{D}{\sqrt{\sigma^2 T}} + \frac{(M-DT)}{2\sigma T^{3/2}}\right) \times$$

$$g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

Simplifying

$$\frac{\partial G_5(M,T)}{\partial T} = \frac{\sigma}{2T^{\frac{1}{2}}}\, g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + DF\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.5.15$$

From 3.5.6

$$G_2(M,T) = \left(\left(\frac{\sigma^4+2D^4T^2}{4D^3}\right) + \frac{(\sigma^2-2D^2T)}{2D^2}M + \frac{M^2}{2D}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{1}{2}\left(\sigma T^{3/2} - \frac{\sigma^3 T^{\frac{1}{2}}}{D^2} - \frac{\sigma T^{\frac{1}{2}} M}{D}\right) g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^4}{4D^3}\, e^{\frac{2DM}{\sigma^2}}\, F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

Differentiating with respect to M

$$\frac{\partial G_2(M,T)}{\partial M} = \left(\frac{(\sigma^2-2D^2T)}{2D^2} + \frac{M}{D}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{1}{\sqrt{\sigma^2 T}}\left(\left(\frac{\sigma^4+2D^4T^2}{4D^3}\right) + \left(\frac{\sigma^2-D^2T}{D^2}\right) M + \frac{M^2}{2D}\right) g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{\sigma T^{\frac{1}{2}}}{2D} \; g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - \frac{1}{\sqrt{\sigma^2 T}}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)\left(\sigma T^{3/2} - \frac{\sigma^3 T^{\frac{1}{2}}}{D^2} - \frac{\sigma T^{\frac{1}{2}} M}{D}\right) *$$

$$g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^2}{2D^2} \, e^{\frac{2DM}{\sigma^2}} \; F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^4}{4D^3\sqrt{\sigma^2 T}} \, e^{\frac{2DM}{\sigma^2}} \; g\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

Simplifying

$$\frac{\partial G_2(M,T)}{\partial M} = -\left[\left(T - \frac{M}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{D} \times\right.$$

$$\left. g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^2}{2D^2} \, e^{\frac{2DM}{\sigma^2}} \; F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)\right] \qquad 3.5.16$$

Applying 3.4.27b

$$-\frac{\partial G_2(M,T)}{\partial T} = \sqrt{\sigma^2 T} \; g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

From 3.4.10

$$Ro(M,T) = \left(T - \frac{M}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{D} \; g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{\sigma^2}{2D^2} \; e^{\frac{2DM}{\sigma^2}} \; F \; \frac{M+DT}{\sigma^2 T} \qquad 3.5.17$$

Remembering that

$$Ro(M,T) = \int_0^T F\left(\frac{M-DT}{\sqrt{\sigma^2 t}}\right) dt \qquad 3.5.18$$

Differentiating Ro(M,T) with respect to T

$$\frac{\partial Ro(M,T)}{\partial T} = F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \qquad 3.5.19$$

Also differentiating 3.5.18 with respect to M

$$\frac{\partial Ro(M,T)}{\partial M} = -\int_0^T \frac{1}{\sqrt{\sigma^2 T}} \; g\left(\frac{M-Dt}{\sqrt{\sigma^2 t}}\right) dt$$

and applying 1.3.11

$$\frac{\partial Ro(M,T)}{\partial M} = -\left(\frac{F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)}{D} - \frac{e^{\frac{2DM}{\sigma^2}} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)}{D}\right) \qquad 3.5.20$$

Now differentiating the cost expression C with respect to M and T

$$\frac{\partial C}{\partial M} = hc + \frac{b_1}{T}\left(\frac{\partial G_5}{\partial M}(M,T+L) - \frac{\partial G_5}{\partial M}(M,L)\right)$$

$$+ \frac{(b_2+hc)}{T}\left(\frac{\partial G_2}{\partial M}(M,T+L) - \frac{\partial G_3}{\partial M}(M,L)\right) + \frac{s}{T}\left(\frac{\partial Ro}{\partial M}(M,T+L) - \frac{\partial Ro}{\partial M}(M,L)\right)$$

$$\frac{\partial C}{\partial T} = -\frac{(Rc+S)}{T^2} - \frac{Dhc}{2} - \frac{b_1}{T^2}\left(G_5(M,T+L) - G_5(M,L)\right)$$

$$+ \frac{b_1}{T}\frac{\partial G_5}{\partial T}(M,T+L) - \frac{(b_2+hc)}{T^2}\left(G_5(M,T+L) - G_5(M,L)\right)$$

$$- \frac{s}{T^2}\left(Ro(M,T+L) - Ro(M,L)\right) + \frac{s}{T}\frac{\partial Ro}{\partial T} R_0(M,T+L) \qquad 3.5.21$$

3.6. Model (M,R,T)

In section 1.1 model (M,R,T) was defined as the model in which at a review time when the inventory position is less than or equal to R, some quantity is ordered which is sufficient to bring the inventory position up to M.

Let R+x be the inventory position immediately after a review at time t and let $G_6(R+x,T)$ be the expected number of backorders incurred from t+L to t+L+T if the inventory position is R+x at time t. Hence

$$G_6(M+x,T) = \int_{R+x}^{\infty} (v - (R+x))(f(V,D(T+L) - f(V,DL))dV \qquad 3.6.1$$

From 3.5.1 and 3.5.2

$$G_6(R+x,T) = G_5(R+x,T+L) - G_5(R+x,L) \qquad 3.6.2$$

Let $G_7(R+x,T)$ be the expected number of unit years of shortage incurred from t+L to t+L+T. At time $t+L+\varepsilon$, $t+L < t+L+\varepsilon < t+L+T$ the expected number of unit years of shortage

$$= \int_{R+x}^{\infty} (V-R+x) f(V,D(L+\varepsilon)) dV$$

Integrating over the states of ε we obtain $G_7(R+x,T)$

$$= \int_{L}^{T+L} \int_{R+x}^{\infty} (V-R-x) f(V,D(L+\varepsilon)) dV d\varepsilon \qquad 3.6.3$$

From 3.5.4

$$G_7(R+x,T) = G_2(R+x,T+L) - G_2(R+x,L) \qquad 3.6.4$$

Let $D(R+x,T)$ be the expected unit years of storage incurred from $t+L$ to $t+L+T$. Following the same analysis as for $D(M,T)$ of section 3.5 which led to equation 3.5.7, we obtain equivalently,

$$D(R+x,T) = \int_L^{L+T} (R+x-D\varepsilon)\,d\varepsilon + G_7(R+x,T)$$

Integrating

$$D(R+x,T) = T(R+x-DL-\frac{DT}{2}) + G_7(R+x,T) \qquad 3.6.5$$

Let $G_8(R+x,T)$ be the expected cost of carrying inventory and backorders for a length of T time units i.e. from $t+L$ to $t+L+T$

$$\text{Hence } G_8(R,x,T) = hcT(R+x-DL-\frac{DT}{2}) + b_1G_6(R+x,T) + (b_2+hc)G_7(R+x,T) \qquad 3.6.6$$

If the stockout cost s is included in $G_8(R+x,T)$ following the same analysis as for 3.5.10, the cost of stockout

$$= s\left(Ro(R+x,T+L) - Ro(R+x,L)\right) \text{ from } 3.5.10$$

Let $G_9(R+x,T) = Ro(+x,T+L) - Ro(R+x,L)$

Thus

$$G_8(R+x,T) = hcT(R+x-DL - \frac{DT}{2}) + b_1G_6(R+x,T)$$
$$+ (b_2+hc)G_7(R+x,T) + s.G_9(R+x,T) \qquad 3.6.7$$

If an order is placed at time t and the next order is placed at time $t+nT$ $n \geq 1$, then given that no order has been placed since time t then the probability that the inventory position lies between x and x+dx is the probability that M-R-x units have been demanded in n periods.

The probability that M-R-x units is demanded in n periods is $f^n(M-R-x, DT)$ when $f^n(y,T)$ is the n-fold convolution of $f(y, DT)$ From equation 1.3.1.

$$f^n(M-R-x, DT) = \frac{\exp -\frac{1}{2}\left(\frac{M-R-x-nDT}{\sqrt{n\sigma^2 T}}\right)^2}{\sqrt{2\pi n\sigma^2 T}} \qquad 3.6.8$$

The expected cost of carrying inventory for the first period is $G_8(M,T)$ since the inventory position is M at time t immediately after a review. At other review times nT $n \geqslant 1$, inventory position is between M and R say i.e. R+y, $o < y < M-R$. and this occurs with probability

$f^n(M-R-y, DT)$

Thus the expected cost per cycle

$$\sum_{n=1}^{\infty} \int_{o}^{M-R} G_8(R+y,T)\ f^n(M-R-y, DT)\,dy + G_8(M,T) \qquad 3.6.9.$$

The cycle will be precisely one period if the demand is greater than M-R before the 1st review. The probability of this occurring is $F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right)$.

The probability that a cycle will be $n(n \geqslant 2)$ periods is that M-R-y units have been demanded in n-1 periods and more than y units are demanded in the nth period.

Thus the probability that a cycle contains precisely n periods

$$= \int_{o}^{M-R} f^{(n-1)}(M-R-y, DT)\,F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right) dy \qquad n \geqslant 2 \qquad 3.6.10$$

Thus the expected length of a cycle is T times the expected number of periods per cycle.

$$= T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty}\int_0^{M-R} n\, f^{(n-1)}(M-R-y, DT)\cdot F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right)dy\right] \qquad 3.6.11$$

The review cost Rc is incurred in every period but the ordering cost S is incurred only when an order occurs. Thus the inventory cost is from 3.6.9 and 3.6.11.

$$C = \frac{Rc}{T} + \frac{S + \sum_{n=1}^{\infty}\int_0^{M-R} G_8(R+y,T)\cdot f^n(M-R-y,DT)dy + G_8(M,T)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty}\int_0^{M-R} n\, f^{(n-1)}(M-R-y,DT)\cdot F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right)dy\right]} \qquad 3.6.12$$

As before solving for Q and k by Steepest Descent requires 1st derivatives. We need to develop the first order derivatives of $G_8(R,T)$ before we can obtain the first order derivative of C.

From 3.6.7

$$G_8(R,T) = T\left(R-DL-\frac{DT}{2}\right) + b_1 G_6(R,T)$$
$$+ (b_2 + hc)\, G_7(R,T) + s\,G_9(R,T)$$

where $G_6(R,T) = G_5(R,T+L) - G_5(R,L)$ from 3.6.2

where $G_7(R,T) = G_2(R,T+L) - G_2(R,L)$ from 3.6.4.

and $G_9(R,T) = Ro(R,T+L) - Ro(R,L)$ from 3.6.7

Thus differentiating $G_8(R,T)$ with respect to R

we obtain
$$\frac{\partial G_8(R,T)}{\partial R} = T + b_1\left(\frac{\partial G_5(R,T+L)}{\partial R} - \frac{\partial G_5(R,L)}{\partial R}\right)$$
$$+ (b_2 + hc)\left(\frac{\partial G_2(R,T+L)}{\partial R} - \frac{\partial G_2(R,L)}{\partial R}\right) + s\left(\frac{\partial Ro(R,T+L)}{\partial R} - \frac{\partial Ro(R,L)}{\partial R}\right) \qquad 3.6.13$$

where the derivatives of G_5 , G_2 , Ro with respect to R are given in 3.5.4, 3.5.6, 3.5.20 respectively.

Differentiating $G_8(R,T)$ with respect to T we obtain

$$\frac{\partial G_8(R,T)}{\partial T} = \left(R-DL-\frac{DT}{2}\right)-DT + b_1\frac{\partial G_5(R,T+L)}{\partial T}$$

$$+ \quad (b_2 + hc)\frac{\partial G_2(R,T+L)}{\partial T} + s\frac{\partial R_o(R,T+L)}{\partial T}$$

3.6.14

where the derivatives of G_5 , G_2 and Ro with respect to T are given in 3.5.15 and 3.4.27b and 3.5.19 respectively.

$$\text{Let NUM} = S + \sum_{n=1}^{\infty}\int_0^{M-R} G_8(R+Y,T) * f^n(M-R-Y,DT)\,dY + G_8(M,T)$$

3.6.15

$$\text{Hence } \frac{\partial NUM}{\partial M} = \sum_{n=1}^{\infty} G_8(M,T)\, f^n(0,DT) + \frac{\partial G_8(M,T)}{\partial M}$$

3.6.16

and remembering that the value of the normal density at zero demand is 0

then $\dfrac{\partial NUM}{\partial M} = \dfrac{\partial G_8(M,T)}{\partial M}$ 3.6.17

Differentiating NUM with respect to T

then

$$\frac{\partial NUM}{\partial T} = \sum_{n=1}^{\infty}\int_0^{M-R}\left(\frac{\partial G_8(R+Y,T)}{\partial T}\quad f^n\ (M-R-Y,DT)\right.$$

$$\left. + G_8(R+Y,T)\frac{\partial f^n}{\partial T}(M-R-Y,DT)\right)dY + \frac{\partial G_8(M,T)}{\partial T}$$

3.6.18

and from 1.3.1 and 1.3.5

$$\frac{\partial f^n}{\partial T}\ (M-R-Y,DT) = \left(\frac{1}{2}\left(\frac{nD}{\sqrt{n\sigma^2 T}} + \frac{(M-R-Y-nDT)}{\sqrt{n\sigma^2}\,T^{3/2}}\right) * \left(\frac{M-R-Y-nDT}{\sqrt{n\sigma^2 T}}\right)\right.$$

$$- \frac{1}{2T} \Bigg) \quad \frac{\exp - \frac{1}{2n\sigma^2 T}(M-R-Y-DnT)^2}{\sqrt{2\pi n\sigma^2 T}}$$ 3.6.19

Differentiating NUM with respect to R

$$\frac{\partial NUM}{\partial R} = -\sum_{n=1}^{\infty} G_8(R,T)\, f^n(0,DT) = 0$$ 3.6.20

$$f(0,DT) -$$

$$\text{Let } DEM = T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty}\int_0^{M-R} n f^{n-1}(M-R-Y,DT)\, F\left(\frac{Y-DT}{\sqrt{\sigma^2 T}}\right) dy\right]$$ 3.6.21

Differentiating with respect to M

$$\frac{\partial DEM}{\partial M} = T\left[\frac{1}{\sqrt{\sigma^2 T}}\, g\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty} n f^{n-1}(0,DT)\, F\left(\frac{Y-DT}{\sqrt{\sigma^2 T}}\right)\right]$$ 3.6.22

Simplifying and noting that $f(0,DT) = 0$

$$\frac{\partial DEM}{\partial M} = \frac{T}{\sqrt{\sigma^2 T}}\, g\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right)$$ 3.6.23

Differentiating DEM with respect to R

$$\frac{\partial DEM}{\partial R} = -\frac{T}{\sqrt{\sigma^2 T}} \times g\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right)$$ 3.6.24

Differentiating DEM with respect to T and applying 1.3.6

$$\frac{\partial DEM}{\partial T} = \frac{DEM}{T} + T\left[\frac{D}{\sqrt{\sigma^2 T}} + \frac{(M-R-DT)}{2\sigma\, T^{3/2}}\right) g\left(\frac{(M-R-DT)}{\sqrt{\sigma^2 T}}\right)$$

$$+\sum_{n=2}^{\infty}\int_{0}^{M-R}\left(n f^{n-1}(M-R-Y,DT)\, x\left(\frac{D}{\sqrt{\sigma^2 T}}+\frac{Y}{2\sigma T^{3/2}}\right) g\left(\frac{Y-DT}{\sqrt{\sigma^2 T}}\right)\right.$$

$$\left.+\, n\frac{\partial f^{n-1}}{\partial T}(M-R-Y,DT)\quad F\left(\frac{Y-DT}{\sqrt{\sigma^2 T}}\right)\right)dy\Bigg] \qquad 3.6.25$$

substituting NUM 3.6.15 and DEM into the cost expression 3.6.12

$$C = \frac{Rc}{T} + \frac{NUM}{DEM} \qquad 3.6.26$$

Thus the first order derivatives of C are

$$\frac{\partial C}{\partial R} = \frac{DEM.\frac{\partial NUM}{\partial R} - NUM\frac{\partial DEM}{\partial R}}{(DEM)^2}$$

$$\frac{\partial C}{\partial M} = \frac{DEM\frac{\partial NUM}{\partial M} - NUM\frac{\partial DEM}{\partial M}}{(DEM)^2}$$

$$\frac{\partial C}{\partial T} = \frac{-Rc}{T^2} + \frac{DEM\frac{\partial NUM}{\partial T} - NUM\frac{\partial DEM}{\partial T}}{(DEM)^2} \qquad 3.6.27$$

CHAPTER 4

CONSTANT LEAD TIMES AND QUADRATIC BACKORDER COST TERMS

In this chapter, we compare the performance of the models when the backorder cost expression is a quadratic function: $b_1 + b_2t + b_3t^2$ where t is the length of time for which the backorder exists. The approach of the previous chapter did not take into account the length of time for which a backorder exists, in deriving the cost equations of (Q,R).

Instead of considering the durations of individual backorders and their distribution, the required results are arrived at by calculating B(Q,R,T) the expected number of backorders at any time. However the approach of the previous chapter can not be used to calculate the expected stockout costs when the backorder cost function is more complicated than the linear form assumed in the previous chapter.

Hence in this chapter and the following chapters we calculate the expected backorder cost using the distribution function of the length of time for which a backorder exists.

Firstly in 4.1 we give the basic mathematics rquired for this chapter, develop the cost expression for the (Q,R) model assuming linear backorder cost term in 4.2 and reconcile the result obtained with that of Section 2.6 of Chapter 2. We then proceed to calculate the inventory costs when the backorder cost is a quadratic function of the length of time of a backorder for the (Q,R), (nQ,R,T), (M,T), (M,R,T) models in 4.3,4.4, 4.5, and 4.6 respectively.

SECTION 4.1 BASIC MATHEMATICS

Remembering that from 1.3.14b

that

$$Z_n(x,T) = -\frac{2\sigma^2 T^{n+1}}{D^2\sqrt{2\pi\sigma^2 T}} \exp -\tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$+ \frac{\sigma^2(2n+1)}{D^2} Zn(x,T)$$

$$+ \frac{X^2}{D^2} Zn\text{-}1(x,T) \qquad n = 0,1,2 \ldots\ldots \tag{4.1.1}$$

Where $Zn(X,T) = \int_0^T t^n f(X,Dt)\,dt$

From 1.3.11

$$Zo(X,T) = \frac{F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)}{D} - \frac{\exp\left(\frac{2Dx}{\sigma^2}\right) F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)}{D} \tag{4.1.2}$$

From 1.3.14a

$$Z_1(X,T) = \frac{\sigma^2}{D^3}\left(1 + \frac{Dx}{\sigma^2}\right) * F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{2\sqrt{\sigma^2 T}}{D^2} g\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{1}{D^2}\left(X - \frac{\sigma^2}{D}\right) e^{\frac{2DX}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right) \tag{4.1.3}$$

From 4.1.1, letting n = 1

$$Z_2(X,T) = -\frac{2\sigma^2 T^2}{D^2\sqrt{2\pi\sigma^2 T}} \exp -\tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2 + \frac{3\sigma^2}{D^2} Z_1(X,T)$$

$$+ \frac{X^2}{D^2} Z_0(X,T) \tag{4.1.4.}$$

Similarly

$$Z_3(X,T) = \frac{-2\ \sigma^2 T^3}{D^2\sqrt{2\pi\sigma^2 T}} \exp -\tfrac{1}{2}\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right)^2 + \frac{5\sigma^2}{D^2} Z_2(X,T)$$

$$+ \frac{x^2}{D^2} \quad Z_1(x,T) \qquad 4.1.5$$

Substituting for $Z_2(X,T)$ in $Z_3(X,T)$

we have

$$Z_3(X,T) = -\frac{2\sigma^2 T^3}{D^2\sqrt{2\pi\sigma^2 T}} \exp -\tfrac{1}{2}\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right)^2 - \frac{5\sigma^2}{D^2}\left(\frac{-2\sigma^2 T^2}{D^2\sqrt{2\pi\sigma^2 T}} \times \right.$$

$$\exp - \tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2$$

$$\left. + \frac{3\sigma^2}{D^2} Z_1(X,T) + \frac{X^2}{D^2} Zo(X,T)\right) + \frac{X^2}{D^2} Z_1(X,T)$$

Simplifying and remembering that

$$g\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right) = \frac{1}{\sqrt{2\pi}} \exp -\tfrac{1}{2}\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right)^2$$

we have

$$Z_3(X,T) = -\frac{2\sqrt{\sigma^2 T}}{D^2}\left(T^2 + \frac{5\sigma^2 T}{D^2}\right) g\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ Z_1(X,T)\left(\frac{X^2}{D^2} + \frac{15\sigma^4}{D^4}\right) + \frac{5\sigma^2 X^2}{D^4} \quad Zo\ (X,T) \qquad 4.1.6.$$

From 4.1.1, when n = 3

$$Z_4(X,T) = -\frac{2\sigma^2 T^4}{D^2\sqrt{2\pi\sigma^2 T}} \exp -\tfrac{1}{2}\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right)^2 + \frac{7\sigma^2}{D^2} Z_3(X,T)$$

$$+ \frac{X^2}{D^2} \quad Z_2\ (X,T) \qquad 4.1.7.$$

Substituting for $Z_3(X,T)$ and $Z_2(X,T)$ then

$$Z_4(X,T) = -\frac{2\sqrt{\sigma^2 T}}{D^2} \; T^3 g\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right) + \frac{7\sigma^2}{D^2}\left(-\frac{2\sqrt{\sigma^2 T}}{D^2}\left(T^2+5\sigma^2 T\right) g\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right)\right.$$

$$\left. + Z_1(X,T)\left(\frac{X^2}{D^2}+\frac{15\sigma^4}{D^4}\right) + \frac{5\sigma^2 X^2}{D^4} Zo(X,T)\right)$$

$$+ \frac{X^2}{D^2}\left(-\frac{2\sqrt{\sigma^2 T}}{D^2}\, T g\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right) + \frac{3\sigma^2}{D^2} Z_1(X,T) + \frac{X^2}{D^2} Zo(X,T)\right)$$

Simplifying

$$Z_4(X,T) = -\frac{2\sqrt{\sigma^2 T}}{D^2}\left(T^3 + \frac{7\sigma^2 T^2}{D^2} + \frac{35\sigma^4 T}{D^4} + \frac{X^2 T}{D^2}\right) g\left(\frac{X-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ Z_1(X,T)\left(\frac{7\sigma^2 X^2}{D^4} + \frac{105\sigma^6}{D^6} + \frac{3\sigma^2 X^2}{D^4}\right)$$

$$+ \left(\frac{35\sigma^4 X^2}{D^6} + \frac{X^4}{D^4}\right) Zo(X,T) \qquad 4.1.8.$$

From 1.3.15

$$Rn(M,L) = \int_0^L t^n \, F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) dt$$

$$= \frac{L^{n+1}}{n+1} F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{2(n+1)} Z_{n+1}(M,L) \qquad 4.1.9.$$

$$- \frac{M}{2(n+1)} Z_n(M,L) \qquad n = 0,1,2,\ldots$$

Thus $$Ro(M,L) = LF\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{2} Z_1(M,L)$$

$$- \frac{M}{2} Zo(M,L) \qquad 4.1.10.$$

also $$R_1(M,L) = \frac{L^2}{2} F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{4} Z_2(M,L)$$

$$- \frac{M}{4} Z_1(M,L) \qquad 4.1.11$$

Substitute 4.1.4 for $Z_2(M,L)$ in $R_1(M,L)$

Thus $$R_1(M,L) = \frac{L^2}{2} F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{4}\left(-\frac{2}{D^2}\sqrt{\sigma^2 L}\,.\,L\,.\,g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. + \frac{3}{D^2}\sigma^2 Z_1(M,L) + \frac{M^2}{D^2} Z_o(M,L)\right) - \frac{M}{4} Z_1(M,L) \qquad 4.1.12$$

Similarly letting $n = 2$ in 4.1.9

$$R_2(M,L) = \frac{L^3}{3} F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{6} Z_3(M,L) - \frac{M}{6} Z_2(M,L) \qquad 4.1.13.$$

Substitute 4.16 for $Z_3(M,L)$ and 4.14 for $Z_2(M,L)$

We have

$$R_2(M,L) = \frac{L}{3}^3 F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{6}\left(-\frac{2\sigma^2 L}{D^2}\left(L^2 + \frac{5\sigma^2 L}{D^2}\right) *\right.$$

$$\left. g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)\right) + Z_1(M,L)\left(\frac{M^2}{D^2} + \frac{15\sigma^4}{D^4}\right) + \frac{5\sigma^2 M^2}{D^4} Zo(M,L)\Bigg)$$

$$- \frac{M}{6}\left(-\frac{2\sqrt{\sigma^2 L}}{D^2}\,.\,L\,.\,g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) + \frac{3\sigma^2}{D^2} Z_1(M,L) + \frac{M^2}{D^2} Zo(M,L)\right) \qquad 4.1.14$$

Simplifying

$$R_2(M,L) = \frac{L}{3}^3 F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{2\sqrt{\sigma^2 L}}{D^2} g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)\left[-\frac{D}{6}\left(L^2 + \frac{5\sigma^2 L}{D^2}\right)\right.$$

$$\left. - \frac{ML}{6}\right] - \left[\frac{D}{6}\left(\frac{M^2}{D^2} + \frac{15\sigma^4}{D^4}\right) + \frac{3\sigma^2 M}{6D^2}\right] Z_1(M,L)$$

$$-\left(\frac{5\sigma^2 M^2}{6D^3} + \frac{M^3}{6D^2}\right) Zo(M,L) \qquad 4.1.15.$$

From 4.1.9 when $n = 3$

$$R_3(M,L) = \frac{L^4}{4} F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{8} Z_4(M,L) - \frac{M}{8} Z_3(M,L) \qquad 4.1.16.$$

Substitute 4.1.8 for $Z_4(M,L)$ and 4.1.6 for $Z_3(M,L)$ then we have

$$R_3(M,L) = \frac{L^4}{4} F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{D}{8}\left(-\frac{2\sqrt{\sigma^2 L}}{D^2}\left(L^3 + \frac{7\sigma^2 L^2}{D^2} + \frac{35\sigma^4 L}{D^4} + \frac{M^2 L}{D^2}\right) g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) + Z_1(ML)\left(\frac{7\sigma^2 M^2}{D^4} + \frac{105\sigma^6}{D^6} + \frac{3\sigma^2 M^2}{D^4}\right) + \left(\frac{35\sigma^4 M^2}{D^6} + \frac{M^4}{D^4}\right) Z_0(M,L)\right)$$

$$- \frac{M}{8}\left(-\frac{2\sqrt{\sigma^2 T}}{D^2}\left(L^2 + \frac{5\sigma^2 L}{D^2}\right) g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) + Z_1(M,L) \times \left(\frac{M^2}{D^2} + \frac{15\sigma^4}{D^4}\right) + \frac{5\sigma^2 M^2}{D^4} Z_0(M,L)\right)$$

Simplifying we have

$$R_3(M,L) = \frac{L^4}{4} F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{2\sqrt{\sigma^2 L}}{D^2} g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)\left[-\frac{D}{8}\left(L^3 + \frac{7\sigma^2 L^2}{D^2} + \frac{35\sigma^4 L}{D^4} + \frac{M^2 L}{D^2}\right) - \frac{M}{8}\left(L^2 + \frac{5\sigma^2 L}{D^2}\right)\right]$$
$$+ Z_1(M,L)\left(-\frac{D}{8}\left(\frac{7\sigma^2 M^2}{D^4} + \frac{105\sigma^6}{D^6} + \frac{3\sigma^2 M^2}{D^4}\right) - \frac{M}{8}\left(\frac{M^2}{D^2} + \frac{15\sigma^4}{D^4}\right)\right]$$
$$+ Z_0(M,L)\left[-\frac{D}{8}\left(\frac{35\sigma^4 M^2}{D^6} + \frac{M^4}{D^4}\right) - \frac{5\sigma^2 M^3}{8D^4}\right] \qquad 4.1.17$$

SECTION 4.2 MODEL (Q.R) LINEAR BACKORDER COSTS

Let $C_B(t)$ be the cost of a backorder which has been outstanding for time t.

If a backorder is incurred at time Z, $Z < L$ then L-Z is the time for which the backorder lasts.

Let R + y $0<y<Q$ be the inventory level at time o, then if the system is out of stock in the time interval Z to Z + dZ after the reorder point R is reached then R + y was demanded in time Z and a demand occurred in time dZ.

This probability is $D.dZ \dfrac{1}{\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\dfrac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right)^2$

4.2.1.

Hence the probability that t = L-Z, length of time of a backorder $= \dfrac{D}{\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\dfrac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right)^2 dZ \quad 0<z<L$

giving an inventory level R + Y at time O.

Expected cost of a backorder = $C_B(L-Z)$ x probability of there been a backorder lasting L-Z

$$= \frac{D.C_B(L-Z)}{\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right)^2 dZ$$

Hence the expected backorder cost per cycle

$$= D\int_0^Q \int_0^L \frac{C_B(L-Z)}{\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right)^2 dZ\,dY \qquad 4.2.2.$$

For the linear backorder cost function, $C_\beta(t) = b_1 + b_2 t$

When the cost function is linear $C_\beta(L-Z) = b_1 + b_2(L-Z)$ and substituting into 4.2.2, expected backorder costs per cycle

$$= \frac{D}{\sqrt{\sigma^2 L}} \int_0^Q \int_0^L \left(b_1 + b_2(L-Z)\right) g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right) dZ\,dY \qquad 4.2.3.$$

Simplifying

Expected backorder costs per cycle

$$= \frac{D}{\sqrt{\sigma^2 L}} \int_0^Q \int_0^L (b_1 + b_2 L) g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right) dZ\,dY$$

$$- \frac{Db_2}{\sqrt{\sigma^2 L}} \int_0^Q \int_0^L Z\, g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right) dZ\,dY \qquad 4.2.4.$$

Letting $v = \dfrac{R+Y-DZ}{\sqrt{\sigma^2 L}}$

Expected backorder costs

$$= - \int_0^Q \int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{\frac{R+Y-DL}{\sqrt{\sigma^2 L}}} (b_1 + b_2 L)\, g(v)\, dv\,dY + b_2 \int_0^Q *$$

$$\int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{\frac{R+Y-DL}{\sqrt{\sigma^2 L}}} \left(\frac{R+Y - v\sqrt{\sigma^2 L}}{D}\right) g(v)\, dv\,dY$$

Simplifying we have

$$- \int_0^Q \int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{\frac{R+Y-DL}{\sqrt{\sigma^2 L}}} \left(b_1 + b_2 L - b_2 \left(\frac{R+Y}{D}\right)\right) g(v)\ dv\,dY$$

$$- b_2 \int_0^Q \int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{\frac{R+Y-DL}{\sqrt{\sigma^2 L}}} v\,g(v)\ dv\ dY \qquad 4.2.5.$$

Integrating and noting that $\int_R^Q Vg(V)\ dV = g(R)$

We have the expected backorder costs equal to

$$= \int_0^Q \left(b_1 + b_2 L - b_2 \left(\frac{R+Y}{D}\right)\right)\left(F\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) - F\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right)\right) dY$$

$$- b_2 \int_0^Q \left(g\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) - g\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right)\right) dY \qquad 4.2.6.$$

Assume that $F\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) = 0$

Then expected backorder costs equals

$$\int_0^Q \left(b_1 + b_2 L - b_2 \left(\frac{R+Y}{D}\right)\right) F\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) dY + b_2 \int_0^Q g\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) dY$$

4.2.7.

noting that $b_2L - b_2\left(\frac{R+Y}{D}\right) = -b_2\left(\frac{R+Y-DL}{D}\right)$

then expected backorder costs equals

$$\int_0^Q \left(b_1 - \frac{b_2}{D}\sqrt{\sigma^2 L}\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right)\right) F\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) dy + b_2 \int_0^Q g\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) dY$$

Let $V = (R+Y-DL) / \sqrt{\sigma^2 L}$

then expected backorder costs equals

$$\sqrt{\sigma^2 L} \int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} \left(b_1 - \frac{b_2\sqrt{\sigma^2 L}.V}{D}\right) F(V)\, dV + b_2\sqrt{\sigma^2 L} \int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} g(V)\, dV \qquad 4.2.8.$$

Applying 1.3.7 and 1.3.8 and integrating, expected

backorder costs equals

$$- \sigma^2 L\, b_1 \Big(g(V) - V\, F(V)\Big)\Big|_{R-DL/\sqrt{\sigma^2 L}}^{R+Q-DL/\sqrt{\sigma^2 L}}$$

$$- b_2\, \sigma^2 L\; F(V)\Big|_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}}$$

$$+ \frac{b_2\, \sigma^2 L}{2D} \Big[(1-V^2)F(V) + Vg(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} \qquad 4.2.9.$$

Simplifying, expected backorder costs per cycle

$$= -\sqrt{\sigma^2 L}\, b_1 \Big[g(V) - V\, F(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} - \frac{b_2 \sigma^2 L}{2D} \Big[(1+V^2)F(V) - Vg(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}}$$

Simplifying we have

$$= b_1\sqrt{\sigma^2 L}\left[\left(g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.\right.$$

$$\left.\left. - \left(g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)\right]$$

$$+ \frac{b_2\sigma^2 L}{2D}\left[\left(\left(1+\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)\right.$$

$$\left. - \left(1+\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right]$$

4.2.10.

From 2.6.6 and 2.6.8 and letting

$$k = \frac{R-DL}{\sqrt{\sigma^2 L}}$$

it can be seen that the number of cycles multiplied by the expected backorder costs per cycle equals the expected backorder costs obtained in 2.6.18.

Thus this is a simple check on both pieces of mathematics.

SECTION 4.3 (Q,R) QUADRATIC BACKORDER COST

We shall extend this method of analysis to derive the backorder costs when the backorder cost is a quadratic function of the length of time of backorder.

$$C_\beta(t) = b_1+b_2t+b_3t^2 \qquad 4.3.1.$$

Expected backorder costs per cycle and applying equation 4.23

$$= \frac{D}{\sqrt{\sigma^2 L}} \int_0^Q\int_0^L \left(b_1+b_2(L-Z)+b_3(L-Z)^2\right) g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right) dZ\, dY$$

Simplifying, expected backorder costs per cycle

$$= \frac{D}{\sqrt{\sigma^2 L}} \int_0^Q \int_0^L (b_1+b_2L+b_3L^2-Z(b_2+2b_3L)+b_3Z^2)g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 L}}\right)dZ\, dY \qquad 4.3.2.$$

Let $V = \dfrac{R+Y-DZ}{\sqrt{\sigma^2 L}}$

then we have

$$-\int_0^Q \int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{R+Y-DL} \left[(b_1+b_2L+b_3L^3) - \left(\frac{R+Y-\sqrt{\sigma^2 L}V}{D}\right) * (b_2+2b_3L) + b_3\left(\frac{R+Y-\sqrt{\sigma^2 L}V}{D}\right)^2\right] g(V)\, dV\, dY$$

Simplifying we have

$$-\int_0^Q \int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{\frac{R+Y-DL}{\sqrt{\sigma^2 L}}} \left(b_1+b_2L+b_3L^2 - \frac{(R+Y)(b_2+2b_3L)}{D} + \frac{b_3(R+Y)^2}{D^2}\right) * \quad g(V)\, dV\, dY$$

$$-\int_0^Q \int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{\frac{R+Y-DL}{\sqrt{\sigma^2 L}}} \sqrt{\sigma^2 L}\left(\frac{V}{D}(b_2+2b_3L) - \frac{2b_3(R+Y)V}{D^2}\right) g(V)\, dV\, dY$$

$$-\frac{b_3\sigma^2 L}{D^2}\int_0^Q \int_{\frac{R+Y}{\sqrt{\sigma^2 L}}}^{\frac{R+Y-DL}{\sqrt{\sigma^2 L}}} V^2 g(V) dV.dY$$

Integrating with respect to V and applying 1.3.7 and 1.3.8 we have

$$-\int_0^Q \left(b_1+b_2L+b_3L^2 - \frac{(R+Y)(b_2+2b_3L)}{D} + \frac{b_3(R+Y)^2}{D^2}\right) * \left(F\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) - F\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right)\right) dY$$

$$-\sqrt{\frac{\sigma^2}{2}L}\int_0^Q \left(\frac{(b_2+2b_3L)}{D} - \frac{2b_3(R+Y)}{D^2}\right)\left(g\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) - g\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right)\right) dY$$

$$-\frac{b_3\sigma^2 L}{D^2}\int_0^Q \left(F\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) + \left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) - \left(F\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) + \left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right)\right)\right) dY \qquad 4.3.3$$

Assuming that $F\left(\frac{R+Y}{\sqrt{\sigma^2 L}}\right) = 0$ as in Section 2

then we have

$$+\int_0^Q \left(b_1+b_2L+b_3L^2 - \frac{(R+Y)(b_2+2b_3L)}{D} + \frac{b_3(R+Y)^2}{D^2}\right) F\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) dY$$

$$+\sqrt{\sigma^2 L}\int_0^Q \left(\frac{(b_2+2b_3L)}{D} - \frac{2b_3(R+Y)}{D^2}\right) g\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}}\right) dY$$

$$+\frac{b_3\sigma^2L}{D^2}\int_0^Q\left(F\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)+\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)g\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)\right)dy$$

4.3.4

Simplifying we have

$$+\int_0^Q\left(b_1+b_2L+b_3L^2-\frac{(R+Y)(b_2+2b_3L)}{D}+\frac{b_3(R+Y)^2}{D^2}+\frac{b_3\sigma^2L}{D^2}\right)\times F\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)dY$$

$$+\int_0^Q\sqrt{\sigma^2L}\left(\left(\frac{b_2+2b_3L)}{D}-\frac{2b_3(R+Y)}{D^2}\right)+\frac{\sigma^2Lb_3}{D^2}\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)\right)g\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)dY$$

4.3.5.

Noting that $b_1+b_2L+b_3L^2-(R+Y)(b_2+2b_3L)+\frac{b_3(R+Y)^2}{D^2}+\frac{b_3\sigma^2L}{D^2}$

$$=b_1+\frac{b_3\sigma^2L}{D^2}+\frac{b_3\sigma^2L}{D^2}\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)^2-\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)\frac{b_2}{D}\sqrt{\sigma^2L}$$

and noting that

$$\sqrt{\sigma^2L}\left(\frac{b_2}{D}+\frac{2b_3L}{D}\right)-\frac{2b_3(R+Y)}{D^2}*\sqrt{\sigma^2L}+\frac{\sigma^2Lb_3}{D^2}\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)$$

$$=\frac{\sqrt{\sigma^2L}b_2}{D}-\frac{b_3\sigma^2L}{D^2}\left(\frac{R+Y-DL}{\sigma^2L}\right)$$

then substituting into 4.3.15

we have

$$\int_0^Q\left(b_1+\frac{b_3}{D^2}\sigma^2L+\frac{b_3}{D^2}\sigma^2L\left(\frac{R+Y-DL}{\sigma^2L}\right)^2-\frac{b_2}{D}\sqrt{\sigma^2L}\left(\frac{R+Y-DL}{\sigma^2L}\right)\right)F\left(\frac{R+Y-DL}{\sqrt{\sigma^2L}}\right)dY$$

$$+ \int_0^Q \left(\frac{\sqrt{\sigma^2 L}\, b_2}{D} - \frac{b_3 \sigma^2 L}{D^2} \left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}} \right) \right) g\left(\frac{R+Y-DL}{\sqrt{\sigma^2 L}} \right) \, dY \qquad 4.3.6.$$

Let $V = \dfrac{R+Y-DL}{\sqrt{\sigma^2 L}}$.

then we have re-arranging

$$\sqrt{\sigma^2 L} \int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} \left(b_1 + \frac{b_3 \sigma^2 L}{D^2} \right) F(V) \, dV - \frac{b_2 \sigma^2 L}{Q} \int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} VF(V) \, dV$$

$$+ \frac{b_3 \sigma^3 L^{3/2}}{D^2} \int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} V^2 F(V) \, dV + \frac{b_2 \sigma^2 L}{QD} \int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} g(V) \, dV$$

$$- \frac{\sigma^3 L^{3/2} b_3}{D^2} \int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} Vg(V) \, dV \qquad 4.3.7.$$

Integrating and applying 1.3.7 and 1.3.8 we have

$$- \sqrt{\sigma^2 L} \left(b_1 + \frac{b_3 \sigma^2 L}{D^2} \right) \Big[g(V) - VF(V) \Big]_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}}$$

$$+ \frac{b_2 \sigma^2 L}{2D} \Big[(1-V^2) F(V) + Vg(V) \Big]_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}}$$

$$-\frac{b_3\sigma^3L^{3/2}}{3D^2}\Big[(V^2+2)g(V)-V^3F(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}$$

$$-\frac{b_2\sigma^2L}{D}\Big[F(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}+\frac{\sigma^3L^{3/2}b_3}{D^2}\Big[g(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}$$

4.3.8.

Simplifying

$$=-\sqrt{\sigma^2L}\,b_1\Big[(g(v)-VF(V))\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}+b_2\sigma^2L\Big[(1-V^2-2)F(V)+vg(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}$$

Simplifying

$$=-\sqrt{\sigma^2L}\,b_1\Big[2(V)-VF(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}-\frac{b_2\sigma^2L}{2D}\Big[(1+V^2)F(V)-vg(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}$$

$$-\frac{b_3\sigma^3L^{3/2}}{3D^2}\Big[(V^2+2)g(V)-V(3+V^2)F(V)\Big]_{\frac{R-DL}{\sqrt{\sigma^2L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2L}}}$$

4.3.9.

Substituting $\alpha(V)$ and $\beta(V)$ from 2.6.4 and 2.6.8 respectively into 4.2.19 and letting

$$\theta(V) = \frac{\sigma^3}{3} L^{3/2}((V^2+2)g(V) - V(3+V^2)F(V)) \qquad 4.3.10.$$

Then we have expected backorders per cycle to be

$$b_1\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{b_2}{D}\left(\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{b_3}{D^2}\left(\theta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \theta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 4.3.11.$$

Number of cycles = D/Q

Hence expected backorder costs per year

$$= \frac{b_1 D}{Q}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{b_2}{Q}\left(\left(\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)\right)$$

$$+ \frac{b_3}{D.Q}\left(\theta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \theta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 4.3.12.$$

Hence from equation 2.6.18 the inventory costs for model (Q,R) with quadratic backorder cost terms and substituting for k in 4.2.22.

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc.k\sqrt{\sigma^2 L} + \frac{D}{Q}b_1\left(\alpha(k) - \alpha\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{(b_2+hc)}{Q}\left(\beta(k) - \beta\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right) + \frac{b_3}{DQ}\left(\theta(k) - \theta\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{Ds}{Q}\left(\alpha(k) - \alpha\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 4.3.13.$$

With the only additional term to 2.6.18 being the b_3 factor.

SECTION 4.4 MODEL (nQ,R,T) QUADRATIC COST TERMS

We now derive the backorder costs using the length of time of a backorder.

From equation 4.3.1

$C_\beta(t) = b_1+b_2t+b_3t^2$ where t is the length of time of a backorder.

For the periodic review model, the variance for a time interval of length $t = \sigma^2 t$.

If the inventory position of the system is R+Y immediately after the review at time t, then the expected backorder costs at time t+L.

(Following the same analysis of Section 2 Chapter 4.)

$$= \frac{1}{Q}\int_0^Q D\int_0^L D\int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dZ\, dt\, dY \qquad 4.4.1$$

Similarly the expected backorder costs at time t+L+T

$$= \frac{1}{Q}\int_0^Q D\int_0^{L+T} D\int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dZ\, dt\, dY \qquad 4.4.2$$

Noting that $C_B(t) = b_1+b_2t+b_3t^2$

and substituting into 4.4.1 and 4.4.2.

expected backorder costs at time t+L

$$= \frac{1}{Q}\int_0^Q D\int_0^L D\int_0^t \frac{\left(b_1+b_2(t-z)+b_3(t-z)^2\right)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dZ\, dt\, dY \qquad 4.4.3$$

and at time t+L+T

$$= \frac{1}{Q} \int_0^Q D \int_0^{L+T} D \int_0^t \frac{\left(b_1 + b_2(t-z) + b_3(t-z)^2\right)}{\sqrt{\sigma^2 t}} g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz\ dt\ dy$$

4.4.4.

Dealing with 4.4.3 first and applying 4.3.2 and 4.3.6, 4.4.3 integral is equal to

$$\frac{D}{Q} \int_0^Q \int_0^L \left(b_1 + \frac{b_3\sigma^2 t}{D^2} + \frac{b_3\sigma^2 t}{D^2}\left(\frac{R+Y-Dt}{\sqrt{\sigma^2 t}}\right)^2 - \frac{b_2\sqrt{\sigma^2 t}}{D}\left(\frac{R+Y-Dt}{\sqrt{\sigma^2 t}}\right)\right) F\left(\frac{R+Y-Dt}{\sqrt{\sigma^2 t}}\right)$$

$$dt\ dY$$

$$+\frac{D}{Q} \int_0^Q \int_0^L \left(\frac{\sqrt{\sigma^2 t}\, b_2}{D} - \frac{b_3\sigma^2 t}{D^2}\left(\frac{R+Y-Dt}{\sqrt{\sigma^2 t}}\right)\right) g\left(\frac{R+Y-Dt}{\sqrt{\sigma^2 t}}\right) dt\ dY$$

4.4.5.

Integrating with respect to Y and applying 4.3.6 and

4.3.9 we have

$$\frac{D\sqrt{\sigma^2 L}}{Q} b_1 \int_0^L \left(g\left(\frac{R-Dt}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R-Dt}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R-Dt}{\sqrt{\sigma^2 L}}\right)\right) dt$$

$$- \frac{D\sqrt{\sigma^2 L}\, b_1}{Q} \int_0^L \left(g\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-Dt}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 L}}\right)\right) dt$$

$$+\frac{Db_2\sigma^2}{2DQ} \int_0^L t \left(\left(1 + \left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)^2\right) F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) - \left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\right) dt$$

$$-\frac{b_2\sigma^2 D}{2DQ}\int_0^L t\left(\left(1+\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)^2\right)F\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)-\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)g\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)\right)dt$$

$$+\frac{b_3\sigma^3 D}{3D^2 Q}\int_0^L t^{3/2}\left(\left(\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)^2+2\right)g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)-\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\left(3+\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)^2\right) * F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\right)dt$$

$$-\frac{b_3\sigma^3 D}{3D^2 Q}\int_0^L t^{3/2}\left(\left(\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)^2+2\right)g\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)-\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right) * \left(3+\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)^2\right)F\left(\frac{R+Q-Dt}{\sqrt{\sigma^2 t}}\right)\right)dt \qquad 4.4.6.$$

We shall integrate the b_1 factor first

$$\frac{\sigma^2 L b_1 D}{Q}\int_0^L \left(g\left(\frac{R-Dt}{\sqrt{\sigma^2 L}}\right)-\left(\frac{R-Dt}{\sqrt{\sigma^2 L}}\right)F\left(\frac{R-Dt}{\sqrt{\sigma^2 L}}\right)\right)dt$$

Let $V = \dfrac{R-Dt}{\sqrt{\sigma^2 L}}$

then we have

$$-\frac{\sqrt{\sigma^2 L}\, b_1 \sqrt{\sigma^2 L}\, D}{DQ}\int_{\frac{R}{\sqrt{\sigma^2 L}}}^{\frac{R-DL}{\sqrt{\sigma^2 L}}}\left(g(V)-VF(V)\right)dV \qquad 4.4.7.$$

Integrating and applying 1.3.8 we have

$$-\frac{b_1}{D}\frac{\sigma^2 LD}{Q}\Big[F(V)-\tfrac{1}{2}(1-V^2)F(V)+Vg(V)\Big]_{\frac{R}{\sqrt{\sigma^2 L}}}^{\frac{R-DL}{\sqrt{\sigma^2 L}}}$$

simplifying, then we have

$$\frac{Db_1\sigma^2 L}{2DQ}\left(\left(1+\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 4.4.8.$$

Then the b_1 factor gives from 4.4.6 and 4.4.8

$$b_1\frac{\sigma^2 LD}{2DQ}\left(\left(1+\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)^2\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$-b_1\frac{\sigma^2 L.D}{2DQ}\left(\left(1+\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2\right)F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)-\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

4.4.9.

which agrees with $E(Q,R,T)$ of Chapter 3 equation 3.4.8

$(=\frac{1}{D}E(Q,R,T))$.

Taking the b_2 factor and the expression for $\frac{R-Dt}{\sqrt{\sigma^2 L}}$ only

we have

$$\frac{b_2}{2Q}\int_0^L \sigma^2 t\left(\left(1+\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)^2\right)F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)-\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\right)dt \qquad 4.4.10.$$

Simplifying we have

$$\frac{b_2.D}{2DQ}\int_0^L \sigma^2 t\left(\left(1+\frac{R^2}{\sigma^2 t}-\frac{2DRT}{\sigma^2 t}+\frac{D^2t^2}{\sigma^2 t}\right)F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)-\left(\frac{R}{\sqrt{\sigma^2 t}}-\frac{Dt}{\sqrt{\sigma^2 t}}\right)g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\right)dt$$

4.4.11.

From 1.3.14

$$Zn(R,L)=\int_0^L \frac{t^n}{\sqrt{\sigma^2 t}}\,g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)dt$$

and $Rn\,(R,L) = \int_0^L t^n\, F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) dt$

Then integrating we have

$$\frac{Db_2}{2QD}\left(R^2 R_0(R,L)+(\sigma^2-2DR)R_1(R,L)+D^2R_2(R,L)\right.$$
$$\left.-\sigma^2 RZ_1(R,L)+D\sigma^2 Z_2(R,L)\right)$$

4.4.12.

Substitute for $R_0(R,L)$, $R_1(R,L)$, $R_2(R,L)$ and $Z_2(R,L)$ from 4.1.10, 4.1.12, 4.1.15, 4.1.4 respectively we have

$$\frac{Db_2}{2QD}\Bigg[R^2\left(LF\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\frac{D}{2}\,Z_1(R,L)-\frac{R}{2}\,Z_0(R,L)\right)$$
$$+(\sigma^2-2DR)\left(\frac{L^2}{2}F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)+\frac{2\sqrt{\sigma^2 L}.L}{4D}\,g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\frac{3\sigma^2}{4D}\,Z_1(R,L)\right.$$
$$\left.-\frac{R^2}{4D}Z_0(R,L)-\frac{R}{4}Z_1(R,L)\right)$$
$$+D^2\left\{\frac{L^3}{3}F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)+\frac{2\sqrt{\sigma^2 L}}{D^2}\,g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{DL^2}{6}+\frac{5\sigma^2 L}{D}+\frac{RL}{6}\right)\right\}$$
$$-D^2\left(\left(\frac{R^2}{6D}+\frac{15\sigma^4}{6D^3}+3\,\frac{\sigma^2 R}{6D^2}\right)Z_1(R,L)+\left(\frac{5\sigma^2R^2}{6D^3}+\frac{R^3}{6D^2}\right)Z_0(R,L)\right)$$
$$-\sigma^2 RZ_1(R,L)+D\sigma^2\left(-\frac{2\sqrt{\sigma^2 L}.L}{D^2}\,g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)+\frac{3\sigma^2}{D^2}\,Z_1(R,L)\right.$$
$$\left.\left.+\frac{R^2}{D^2}\,Z_0(R,L)\right)\Bigg]$$

4.4.13.

Simplifying we have

$$\frac{b_2}{2Q}\left(R^2L+(\sigma^2-2DR)\frac{L^2}{2}+\frac{D^2L^3}{3}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$-\frac{2b_2\sqrt{\sigma^2L}}{2QD^2}\left(-(\sigma^2-2R)\frac{DL}{4}-D^2\left(\frac{DL^2}{6}+\frac{5\sigma^2L}{D}+\frac{RL}{6}\right)+D\sigma^2L\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{b_2}{2Q}\left(-\frac{DR^2}{2}-\frac{3\sigma^2}{4D}(\sigma^2-2DR)-D^2\left(\frac{R^2}{6D}+\frac{15\sigma^4}{6D^3}+\frac{3\sigma^2R}{6D^2}\right)-\frac{R}{4}(\sigma^2-2DR)\right.$$

$$\left.-\sigma^2R+\frac{3\sigma^4}{D}\right)Z_1(R,L)\quad+$$

$$\frac{b_2}{2Q}\left(-\frac{R^3}{2}-\frac{R^2}{4D}(\sigma^2-2DR)-D^2\left(\frac{5\sigma^2R^2}{6D^3}+\frac{R^3}{6D^2}\right)+\frac{\sigma^2R^2}{D}\right)Z_0(R,L)$$

4.4.14.

Substitute for $Z_1(R,L)$ and $Z_0(R,L)$ from 4.1.3. and 4.1.2. respectively then we have

$$\frac{b_2}{2Q}\left(R^2L+(\sigma^2-2DR)\frac{L^2}{2}+\frac{D^2L^3}{3}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$-\frac{b_2 2\sqrt{\sigma^2L}}{2QD^2}\left(-\frac{\sigma^2DL}{4}+\frac{D^2RL}{2}-\frac{D^3L^2}{6}-5D\sigma^2L-\frac{D^2RL}{6}+D\sigma^2L\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{b_2}{2Q}\left(-\frac{DR^2}{2}-\frac{3\sigma^4}{4D}+\frac{6\sigma^2 R}{4}-\frac{DR^2}{6}-\frac{15\sigma^4}{6D}-\frac{\sigma^2 R}{2}-\sigma^2 R-\frac{3\sigma^4}{D}-\frac{\sigma^2 R}{4}+\frac{2DR^2}{4}\right)$$

$$*\left(\frac{\sigma^2}{D^3}\left(1+\frac{DR}{\sigma^2}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-2\frac{\sqrt{\sigma^2 L}}{D^2}\,g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\frac{+1}{D^2}\left(R-\frac{\sigma^2}{D}\right)e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+\frac{b_2}{2QD}\left(-\frac{R^3}{2}-\frac{R^2\sigma^2}{4D}+\frac{R^3}{2}-\frac{5\sigma^2 R^2}{6D}-\frac{R^3}{6}+\frac{\sigma^2 R^2}{D}\right)*$$

$$\left(F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 4.4.15.$$

Simplifying we have

$$\frac{b_2}{Q}\left(\frac{D^2L^3}{6}-\frac{\sigma^4 R}{6D^3}-\frac{DL^2R}{2}-\frac{\sigma^2 R^2}{4D^2}+\frac{\sigma^2 L^2}{4}+\frac{LR^2}{2}-\frac{R^3}{6D}\right.$$

$$\left.-\frac{\sigma^6}{8D^4}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\frac{b_2}{Q}\left(\sqrt{\sigma^2 L}\left(\frac{DL^2}{6}-\frac{LR}{3}+\frac{R^2}{6D}+\frac{\sigma^2 L}{12D}+\frac{\sigma^2 R}{4D^2}+\frac{\sigma^4}{4D^3}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+\frac{\sigma^6}{8D^4 Q}e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

$$=\frac{b_3}{Q}\times G_3(R,L) \qquad 4.4.16.$$

(which is equal to, from 3.4.21, $G_3(R,L)$).

From equation 4.4.6. considering the b_3 factor and expression for $\frac{R-Dt}{\sqrt{\sigma^2 L}}$ only we have

$$\frac{b_3}{3DQ}\int_0^L \sigma^3 t^{3/2}\left(\left(\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)^2 + 2\right) g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) - \left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\left(3+\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)^2\right) * F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\right) dt \qquad 4.4.17.$$

Expanding

$$= \frac{b_3}{3DQ}\int_0^L \left(\sqrt{\sigma^2 t}\,(R^2-2RDt+D^2t^2)+2\sigma^2 t\right) g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)$$

$$-\left(\frac{\sigma^2 tR}{D^2} - \frac{D\sigma^2 t^2}{D^2} + \frac{R^3}{3D^2} - \frac{3R^2Dt}{3D^2} + \frac{3RD^2t^2}{3D^2} - \frac{D^3t^3}{3D^2}\right) F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) dt$$

4.4.18.

Re-arranging in Powers of t we have

$$\frac{b_3}{3DQ}\int_0^L \frac{1}{\sqrt{\sigma^2 t}}\left(R^2\sigma^2 t - t^2(2D\sigma^2 R - 2\sigma^4) + D^2\sigma^2 t^3\right) g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) dt$$

$$-\frac{b_3}{3D^2Q}\int_0^L \left(\left(\sigma^2 R - DR^2\right)t + t^2\left(RD^2 - D\sigma^2\right) + \frac{R^3}{3} - \frac{t^3D^3}{3}\right) F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) dt \qquad 4.4.19$$

Integrating, we have, and noting that $\int_0^L t^n\, g\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right) dt$

$= Z_n(R,T)$ and $R_n(X,T) = \int_0^L t^n\, F\left(\frac{X-Dt}{\sqrt{\sigma^2 t}}\right) dt$

We have

$$\frac{b_3}{3DQ}\left(R^2\sigma^2 Z_1(R,L) - \left(2D\sigma^2 R - 2\sigma^4\right)Z_2(R,L) + D^2\sigma^2 Z_3(R,L)\right)$$

$$-\frac{b_3}{DQ}\left(\left(\sigma^2 R - DR^2\right)R_1(R,L) + (RD^2 - D\sigma^2)R_2(R,L) + \frac{R^3}{3}R_0(R,L)\right.$$

$$\left. -\frac{D^2}{3}R_3(R,L)\right) \qquad 4.4.20.$$

Substitute for $Z_2(R,L)$, $Z_3(R,L)$, $R_1(R,L)$, $R_2(R,L)$, $R_3(R,L)$ and $R_0(R,L)$., from 4.1.4, 4.1.6, 4.1.12, 4.1.15, 4.1.17, 4.1.10 respectively.

then we have, noting that $g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) = \frac{1}{\sqrt{2\pi}} \exp - \frac{1}{2}\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)^2$)

$$\frac{b_3}{3DQ}\left[\frac{R^2\sigma^4}{D^3}\left(1+\frac{DR}{\sigma^2}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \frac{2R^2\sigma^2\sqrt{\sigma^2 L}}{D^2}\, g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$+\frac{R^2\sigma^2}{D^2}\left(R-\frac{\sigma^2}{D}\right)e^{\frac{2DR}{\sigma^2}}\, F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

$$+(2D\sigma^2 R - 2\sigma^4)\frac{2\sqrt{\sigma^2}L.L}{D^2}\, g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \frac{6\sigma^2}{D^2}(D\sigma^2 R - \sigma^4)Z_1(R,L)$$

$$-\frac{2R^2}{D^2}(D\sigma^2 R - \sigma^4)Z_0(R,L)$$

$$-2D^2\sigma^2\frac{\sqrt{\sigma^2 L}}{D^2}\left(L^2 + \frac{5\sigma^2 L}{D^2}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + D^2\sigma^2\left(\frac{R^2}{D^2} + \frac{15\sigma^4}{D^4}\right)Z_1(R,L)$$

$$\left. + \frac{5D^2\sigma^4 R^2}{D^4}Z_0(R,L)\right]$$

$$-\frac{b_3}{DQ}\left[\left(\sigma^2R-DR^2\right)\frac{L^2}{2}F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)+\frac{D}{2}\left(\frac{\sigma^2R-DR^2}{D^2}\right)\sqrt{\sigma^2L}.L\ \ g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)\right.$$

$$\left(\frac{R}{4}+\frac{3\sigma^2}{4D}\right)\left(\sigma^2R-DR^2\right)Z_1(R,L)-\frac{R^2}{4D}(\sigma^2R-DR^2)Z_o(R,L)$$

$$+(RD^2-D\sigma^2)\left(\frac{L^3}{3}F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)+\frac{2\sqrt{\sigma^2L}}{D^2}\left(\frac{DL^2}{6}+\frac{5\sigma^2L}{6D}+\frac{RL}{6}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)\right.$$

$$-(RD^2-D\sigma^2)\left(\frac{R^2}{6D}+\frac{15\sigma^4}{6D^3}+\frac{\sigma^2R}{6D}\right)Z_1(R,L)$$

$$-\left(RD^2-D\sigma^2\right)\left(\frac{5}{6}\sigma^2R^2+\frac{R^3}{6D^2}\right)Z_o(R,L)$$

$$+\frac{R^3}{3}LF\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\frac{DR^3}{6}Z_1(R,L)-\frac{R^4}{6}Z_o(R,L)$$

$$-\frac{D^3}{3.}\frac{L^4}{4.}\ \ F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\frac{2\sqrt{\sigma^2L}D^3}{3D^2}\left(+\frac{DL^3}{8}+\frac{7\sigma^2L^2}{8D}+\frac{35\sigma^4L}{8D^3}\right.$$

$$\left.+\frac{R^2L^2}{8D}+\frac{RL^2}{8}+\frac{5\sigma^2RL^2}{8D^2}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)+\frac{D^4}{24}\left(\frac{R\sigma^2L^2}{D^4}+\frac{105\sigma^6}{D^6}\right.$$

$$\left.+\frac{3\sigma^2R^2}{D^4}+\frac{D^3R}{24}\left(\frac{R^2}{D^2}+\frac{15\sigma^4}{D^4}\right)\right)Z_1(R,L)$$

$$\left.+\frac{D^3}{8}\left(\frac{35\sigma^4R^2}{8D^5}+\frac{R^4}{8D^3}-\frac{5\sigma^2R^3}{8D^3}\right)Z_o(R,L)\right] \qquad 4.4.21.$$

Simplifying we have

$$\frac{Db_3}{Q}Z_o(R,L)\left[-\frac{7\sigma^2R^3}{12D^3}+\frac{2\sigma^4R^2}{3D^4}-\frac{R^3}{12D^2}+\frac{5\sigma^4R^2}{6D^4}+\frac{5\sigma^2R^3}{8D^3}+\frac{R^4}{6D^2}-\frac{R^4}{24D^2}-\frac{35\sigma^4R^2}{24D^4}\right]$$

$$+\frac{Db_3}{Q}Z_1(R,L)\left[\frac{7\sigma^2R^2}{12D^2}-\frac{R^3}{12D}-\frac{21\sigma^3R}{12D^3}+\frac{2\sigma^6}{D^4}-\frac{\sigma^2R^2}{4D^2}\right.$$

$$+\frac{\sigma^2 R^2}{6D^2}+\frac{15\sigma^6}{6D^4}+\frac{R^3}{8D}+\frac{15\sigma^4 R}{8D^3}-\frac{7\sigma^2 R^2}{24D^2}-\frac{105\sigma^6}{24D^4}-\frac{3\sigma^2 R^2}{24D^2}\Bigg]$$

$$-\frac{2\sqrt{\sigma^2 T}}{D^2}\; g\left(\frac{R-DL}{\sqrt{\sigma^2 t}}\right)\Bigg\{-\frac{7\sigma^2 RL}{12D^2}+\frac{2}{3}\frac{\sigma^4 L}{D^2}-\frac{R^2 T}{12}+\frac{\sigma^6 L^2}{6}$$

$$+5\frac{\sigma^4 L}{6D^2}+\frac{L^2 RD}{8}+\frac{5\sigma^2 RL}{8D}-\frac{D^2 L^3}{24}-\frac{7\sigma^2 L^2}{24}-\frac{35\sigma^4 L}{24D^2}-\frac{R^2 L}{24}\Bigg\}+$$

$$F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left[-\frac{L^2\sigma^2 R}{2D^2}+\frac{L^2 R^2}{2D}-\frac{L^3 R}{3}+\frac{L^3\sigma^2}{3D}-\frac{R^3 L}{3D^2}+\frac{E^3 D}{12}\right]$$

4.4.22.

Simplifying we have

$$\frac{Db_3}{Q}\; Z_0(R,L)\left\{\frac{R^4}{24D}{}^2+\frac{\sigma^2 R^3}{24D^3}+\frac{\sigma^4 R^2}{24D^4}\right\}$$

$$+\frac{Db_3}{Q}\; Z_1(R,L)\left\{\frac{R^3}{24D}+\frac{\sigma^2 R^2}{12D^2}+\frac{\sigma^4 R}{8D^3}+\frac{\sigma^6}{8D^4}\right\}$$

$$-2D\frac{\sqrt{\sigma^2 T}}{D^2}\; g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left\{\frac{\sigma^2 LR}{24D}+\frac{\sigma^4 L}{24D^2}-\frac{R^2 L}{8}-\frac{\sigma^2 L^2}{8}+\frac{L^2 RD}{8}-\frac{D^2 L^3}{24}\right\}\cdot\frac{b_3}{Q}$$

$$+\frac{Db_3}{Q}\; F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left\{-\frac{L^2\sigma^2 R}{2D^2}+\frac{L^2 R^2}{2D}-\frac{L^3 R}{3}+\frac{E^3\sigma^2}{3D}-\frac{R^3 L}{3D^2}+\frac{L^4 D}{12}\right\}$$

4.4.23.

Substitute 4.1.2. for $Z_0(R,L)$ and 4.1.3. for $Z_1(R,L)$ we have

$$\frac{b_3}{Q}\left(F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\right)\times\left(\frac{R^4}{24D^2}+\frac{\sigma^2 R^3}{24D^3}+\frac{\sigma^4 R^2}{24D^4}\right)$$

$$+\frac{b_3 D}{Q}\left(\frac{\sigma^2}{D^3}\left(1+\frac{DR}{\sigma^2}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\frac{2\sqrt{\sigma^2 L}}{D^2}\,g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)+\frac{1}{D^2}\left(R-\frac{\sigma^2}{D}\right)\right.$$

$$\left.\times\, e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+DL}{\sigma^2 L}\right)\right)\cdot\left(\frac{R^3}{24D}+\frac{\sigma^2 R^2}{12D^2}+\frac{\sigma^4 R}{8D^3}+\frac{\sigma^6}{8D^4}\right)$$

$$-2D\frac{\sqrt{\sigma^2 T}}{D^2}\,g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{\sigma^2 LR}{24D}+\frac{\sigma^4 L}{24D^2}-\frac{R^2 L}{8}-\frac{\sigma^2 L^2}{8}+\frac{L^2 RD}{7}-\frac{D^2 L^3}{24}\right)$$

$$+DF\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left\{-\frac{\sigma^2 L^2 R}{2D^2}+\frac{L^2 R^2}{2D}-\frac{L^3 R}{3}+\frac{L^3\sigma^2}{3D}-\frac{R^3 L}{3D^2}+\frac{L^4 D}{12}\right\}$$

4.4.24.

Simplifying we have

$$\frac{b_3 D}{Q}\left(\frac{R^4}{12D^3}+\frac{\sigma^2 R^3}{6D^4}+\frac{\sigma^4 R^2}{4\,D^5}+\frac{\sigma^6 R}{4D^6}+\frac{\sigma^8}{8D^7}-\frac{L^2\sigma^2 R}{2D^2}+\frac{L^2 R^2}{2D}\right.$$

$$\left.-\frac{RL^3}{3}+\frac{L^3\sigma^2}{3D}-\frac{R^3 L}{3D^2}+\frac{L^4 D}{12}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\frac{b_3}{Q}\cdot 2\sqrt{\sigma^2 L}\,D.\ g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{\sigma^2 RL}{24D^3}+\frac{\sigma^4}{24}\frac{R}{D^4}-\frac{R^2 L}{8D^2}-\frac{\sigma^2 L^2}{8D^2}+\frac{L^2 R}{8D}\right.$$

$$\left.-\frac{L^3}{24}+\frac{R^3}{24D^3}+\frac{\sigma^2 R^2}{12D^4}+\frac{\sigma^4 R}{8D^5}+\frac{\sigma^6}{8D^6}\right)$$

$$- \frac{1}{8} \frac{\sigma^8}{D^6} \frac{b_3}{Q} e^{\frac{2DR}{\sigma^2}} \quad F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

4.4.25.

If we define $G_{11}(R,L)$ in such a way that $\frac{b_3}{Q} \times G_{11}(R,L)$

equals equation 4.4.25. then from 4.4.6, expected backorder costs at time t+L for the b_3 factor

equals $\frac{b_3}{Q} G_{11}(R,L) - \frac{b_3}{Q} G_{11}(R+Q,L)$. 4.4.26.

Similarly from 4.4.4, expected backorder costs at time t+L+T for the b_3 factor

$$= \frac{b_3}{Q}(G_{11}(R_7T+L) - G_{11}(R+Q, \ t+T)) \qquad 4.4.27.$$

No of cycles $= \frac{1}{T}$

Hence expected backorder costs per year from 3.4.31. excluding the cost based on number of stockouts is

$$= \frac{b_1}{QT}\left(G_1(R_1T+L) - G_1(R,L) - G_1(R+Q,T+L) + G_1(R+Q,L)\right)$$

$$+\frac{b_2}{QT}\left(G_3(R,T+L)-G_3(R,L)-G_3(R+Q,T+L)+G_3(R+Q,L)\right)$$

$$+\frac{b_3}{QT}\left(G_{11}(R,T+L)-G_{11}(R,L)-G_{11}(R+Q,T+L)+G_{11}(R+Q,L)\right)$$

4.4.28.

From equation 3.4.31, the inventory cost for the case of quadratic backorder costs is equal to the inventory costs for the linear backorder costs plus the b_3 factor.

$$C=\frac{Rc}{T}+\frac{S.POR}{T}+hc\left(Q+R-DL-\frac{DT}{2}\right)+\frac{b_1}{QT}\left(G_1(R,T+L)-G_1(R,L)\right.$$
$$\left.-G_1(R+Q,T+L)+G_1(R,L)\right)$$
$$+\frac{(hc+b_2)}{QT}\left(G_3(R,T+L)-G_3(R,L)-G_3(R+Q,T+L)+G_3(R+Q,L)\right)$$
$$+\frac{b_3}{QT}\left(G_{11}(R+T+L)-G_{11}(R,L)-G_{11}(R+Q,T+L)?G_{11}(R+Q,L)\right)$$
$$+\frac{S}{QT}\left(G_4(R,T+L)-G_4(R,L)-G_4(R+Q,T+L)+G_4(R+Q,L)\right) \qquad 4.4.29.$$

The additional first order derivatives required are the derivatives of $G_{11}(R,T)$ with respect to R,T in order to be able to give the first order derivatives of C.

$$G_{11}(R,T)=D\left(\frac{R^4}{12D^3}+\frac{\sigma^2R^3}{6D^4}+\frac{\sigma^3R^2}{4D^5}+\frac{\sigma^6R}{4D^6}+\frac{\sigma^8}{8D^7}-\frac{\sigma^2RL^2}{2D^2}\right.$$
$$\left.+\frac{L^2R^2}{2D}-\frac{RL^3}{3}+\frac{L^3\sigma^2}{3D}-\frac{R^3L}{3D^2}+\frac{L^4D}{12}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$
$$-2\sqrt{\sigma^2}LD\left(\frac{\sigma^2RL}{24D^3}+\frac{\sigma^4L}{24D^4}-\frac{R^2L}{8D^2}-\frac{\sigma^2L^2}{8D^2}+\frac{L^2R}{8D}-\frac{L^3}{24}+\frac{R^3}{24D^3}+\frac{\sigma^2R^2}{12D^4}+\frac{\sigma^2R}{8D^5}+\frac{\sigma^6}{8D^6}\right)$$
$$\times\, g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$- \frac{1}{8} \frac{\sigma^8}{D^6} \cdot e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) \qquad 4.4.30.$$

Differentiating with respect to R we have

$$\frac{\partial G_{11}(R,T)}{\partial R} = D\left(\frac{R^3}{3D^3} + \frac{\sigma^2 R^2}{2D^4} + \frac{\sigma^4 R}{2D^5} + \frac{\sigma^6}{4D^6} - \frac{\sigma^2 L^2}{2D^2} + \frac{L^2 R}{D} - \frac{L^3}{3} - \frac{R^2 L}{D^2}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-D\left(\frac{R^4}{12D^3} + \frac{\sigma^2 R^3}{6D^4} + \frac{\sigma^4 R^2}{4D^5} + \frac{\sigma^6 R}{4D^6} + \frac{\sigma^8}{8D^7} - \frac{\sigma^2 RL^2}{2D^2} + \frac{L^2 R^2}{2D} - \frac{RL^3}{3} + \frac{L^3\sigma^2}{3D} - \frac{R^3 L}{3D^2}\right.$$

$$\left. + \frac{L^4 D}{12}\right) \frac{1}{\sqrt{\sigma^2 L}} \; g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-2\sqrt{\sigma^2 L}\, D\left(\frac{\sigma^2 L}{24D^3} - \frac{RL}{4D^2} + \frac{L^2}{8D} + \frac{R^2}{12D^3} + \frac{\sigma^2 R}{6D^4} + \frac{\sigma^2}{8D^5}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+2D\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{\sigma^2 RL}{24D^3} + \frac{\sigma^4 L}{24D^4} - \frac{R^2 L}{8D^2} - \frac{\sigma^2 L^2}{8D^2} + \frac{L^2 R}{8D} - \frac{L^3}{24} + \frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4}\right.$$

$$\left. + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right).$$

$$- \frac{1}{4} \cdot \frac{\sigma^6}{D^5} e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) + \frac{1}{8} \cdot \frac{1}{\sqrt{\sigma^2 L}} \cdot \frac{\sigma^8}{D^6} \, g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right). \qquad 4.4.31.$$

Simplifying we have

$$\frac{\partial G_{11}(R,L)}{\partial R}=D\left(\frac{R^3}{3D^3}+\frac{\sigma^2R^2}{2D^4}+\frac{\sigma^4R}{2D^5}+\frac{\sigma^6}{4D^6}-\frac{\sigma^2L^2}{2D^2}+\frac{L^2R}{D}-\frac{L^3}{3}-\frac{R^2L}{D^2}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$-\frac{D}{\sqrt{\sigma^2L}}\;g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)\left(-\frac{2}{3}\;\frac{\sigma^2RL^2}{D^2}+\frac{\sigma^2L^3}{3D}+\frac{\sigma^2R^2L}{3D^3}+\frac{\sigma^4}{2}\frac{RL}{D^3}+\frac{\sigma^4L^2}{6D^3}+\frac{8\sigma^6L}{D^5}\right)$$

$$-\frac{1}{4}\cdot\frac{\sigma^6}{D^5}e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+DL}{\sqrt{\sigma^2L}}\right)\qquad 4.4.32a$$

Let $\frac{\partial G_{11}(R,L)}{\partial R} = -G_{12}(R,L)$. 4.4.32b

Thus $-G_{12}(R,L)$ equals 4.4.32.

From 4.4.17

$$\frac{\partial G_{11}(R,L)}{\partial L}=\frac{\sigma^3}{D^2}L^{3/2}\left(\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)^2+2\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)\left(3+\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

4.4.33.

SECTION 4.5 MODEL(M,T)

For the linear backorder costs we derived model (M,T) from basic principles. However in this section we shall derive model (M,T) for the quadratic backorder costs, from Model (nQ,R,T) by taking its limit as $Q \to 0$ and setting R to M.

$$\text{Model } (M,T) = \lim_{Q \to 0} (nQ,M,T) \qquad 4.5.1.$$

From 4.4.29 Inventory cost for model (nQ,R,T).

$$C = \frac{Rc}{T} + S.\frac{POR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + \frac{b_1}{QT}\Big(G_1(R,T+L)$$

$$-G_1(R,L) - G_1(R+Q,T+L) + G_1(R,L)\Big)$$

$$+ \frac{(hc+b_2)}{QT}\Big(G_3(R,T+L) - G_3(R,L) - G_3(R+Q,T+L) + G_3(R+Q,L)\Big)$$

$$+\frac{b_3}{QT}\Big(G_{11}(R,T+L) - G_{11}(R,L) - G_{11}(R+Q,T+L) + G_{11}(R+Q,L)\Big)$$

$$+\frac{s}{QT}\Big(G_4(R,T+L) - G_4(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)\Big)$$

From 3.4.4.

$$\lim_{Q \to 0} POR = 1 \qquad 4.5.2.$$

From 3.4.35

$$\lim_{Q \to 0} \frac{G_1(R+Q, T+L)}{Q}$$

$$= -\left(\sqrt{\sigma^2(T+L)}\; g\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right) - \left(R-D(T+L)\right) F\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)\right)$$

$$= -G_5(R, T+L) \qquad 4.5.3.$$

From 3.4.36

$$\lim_{Q \to 0} \frac{G_3(R+Q, L)}{Q} = -G_2(R, L) \qquad 4.5.4.$$

From 4.4.32.

$$\lim_{Q \to 0} \frac{G_{11}(R+Q, L)}{Q} = -G_{12}(R, L) \qquad 4.5.5.$$

From 3.4.29

$$\lim_{Q \to 0} \frac{G_4(R+Q, L)}{Q} = -R_0(R, L) \qquad 4.5.6.$$

Hence from 4.5.1, the inventory cost for model (M, T).

$$C = \frac{Rc+S}{T} + hc\left(M - DL - \frac{DT}{2}\right) + \frac{b_1}{T}\left(G_5(M, T+L) - G_5(M, L)\right)$$

$$+ \frac{(b_2 + hc)}{T}\left(G_2(M, T+L) - G_2(M, L)\right) + \frac{b_3}{T}\Big(G_{12}(M, T+L)$$

$$- G_{12}(M, L)\Big) + \frac{s}{T}\left(R_0(M, T+L) - R_0(M, L)\right) \qquad 4.5.7.$$

The additional first order derivatives required are the derivatives of $G_{12}(M,T+L)$ and $G_{12}(M,L)$ with respect to M AND T, the first order derivatives for (M,T) of Section 3.5.

From 4.4.32

$$G_{12}(M,L)=-D\left(\frac{M^3}{3D^3}+\frac{\sigma^2M^2}{2D^4}+\frac{\sigma^4M}{2D^5}+\frac{\sigma^6}{4D^6}-\frac{\sigma^2L^2}{2D^2}+\frac{L^2M}{D}-\frac{L^3}{3}-\frac{M^2L}{D^2}\right)F\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{D}{\sqrt{\sigma^2L}}\,g\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)\left(-\frac{2}{3}\;\frac{\sigma^2ML^2}{D^2}+\frac{\sigma^2L^3}{3D}+\frac{\sigma^2M^2L}{3D^3}-\frac{\sigma^4ML}{2D^4}+\frac{\sigma^4L^2}{6D^3}+\frac{8\sigma^6L}{D^5}\right)$$

$$+e^{\frac{2DM}{\sigma^2}}F\left(\frac{M+DL}{\sqrt{\sigma^2L}}\right)\times\frac{\sigma^6}{4D^5}$$

Differentiating with respect to M

$$\frac{\partial G_{12}(M,L)}{\partial M}=-D\left(\frac{M^2}{D^3}+\frac{\sigma^2M}{D^4}+\frac{\sigma^4}{2D^5}+\frac{L^2}{D}-\frac{2ML}{D^2}\right)F\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{D}{\sqrt{\sigma^2L}}\left(\frac{M^3}{3D^3}+\frac{\sigma^2M^2}{2D^4}+\frac{\sigma^4M}{2D^5}+\frac{\sigma^6}{4D^6}-\frac{\sigma^2L^2}{2D^2}+\frac{L^2M}{D}-\frac{L^3}{3}-\frac{M^2L}{D^2}\right)g\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{D}{\sqrt{\sigma^2L}}\,g\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)\left(-\frac{2}{3}\;\frac{\sigma^2L^2}{D^2}+\frac{2\sigma^2ML}{3D^3}+\frac{\sigma^4L}{2D^4}\right)$$

$$-\frac{D}{\sigma^2L}\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)g\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)\left(-\frac{2}{3}\;\frac{\sigma^2ML^2}{D^2}+\frac{\sigma^2L^3}{3D}+\frac{\sigma^2M^2L}{3D^3}+\frac{\sigma^4ML}{2D^4}+\frac{\sigma^4L^2}{6D^3}+\frac{8\sigma^6L}{D^5}\right)$$

$$-\frac{1}{2}\,\frac{\sigma^4}{D^4}\,e^{\frac{2DM}{\sigma^2}}F\left(\frac{M+DL}{\sqrt{\sigma^2L}}\right)-\frac{1}{4}\,\frac{\sigma^6}{D^5}\,\frac{1}{\sqrt{\sigma^2L}}\,g\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)$$

4.5.8.

Simplifying we have

$$-D\left(\frac{m^2}{D^3}+\frac{\sigma^2 m}{D^4}+\frac{\sigma^4}{2D^5}+\frac{L^2}{D}-\frac{2mL}{D^2}\right)F\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\frac{D}{\sqrt{\sigma^2 L}}\,g\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)\left(-\frac{\sigma^2 L^2}{D^2}-\frac{2m^2 L}{3D^2}+\frac{\sigma^2 mL}{D^3}-\frac{15}{2}\cdot\frac{\sigma^4 m}{D^5}+\frac{16}{2}\cdot\frac{\sigma^4 L}{D^4}\right)$$

$$-\frac{\sigma^4}{2D^4}\,e^{\frac{2Dm}{\sigma^2}}\,F\left(\frac{m-DL}{\sigma^2 L}\right) \qquad 4.5.9.$$

Differentiating $G_{12}(m,L)$ with respect to L

$$\frac{\partial G_{12}(m,L)}{\partial L}=-D\left(\frac{-\sigma^2 L}{D^2}+\frac{2Lm}{D}-\frac{L^2}{1}-\frac{m^2}{D^2}\right)F\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\frac{D}{2}\left(\frac{m^3}{3D^3}+\frac{\sigma^2 m^2}{2D^4}+\frac{\sigma^4 m}{2D^5}+\frac{\sigma^6}{4D^6}-\frac{\sigma^2 L^2}{2D^2}+\frac{L^2 m}{D}-\frac{L^3}{3}-\frac{m^2 L}{D^2}\right)$$

$$\times\frac{D}{\sqrt{\sigma^2 L}}\left(D+\frac{m}{L}\right)g\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\frac{D}{\sqrt{\sigma^2 L}}\,g\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)\left(-\frac{4\sigma^2 mL}{3D^2}+\frac{\sigma^2 L^2}{D}+\frac{\sigma^2 m^2}{3D^3}+\frac{\sigma^4 L}{2D^4}+\frac{\sigma^4 L}{3D^3}+\frac{8\sigma^6}{D^5}\right)$$

$$+\frac{D}{\sqrt{\sigma^2 L}}\left(-\frac{1}{2\sigma L^{3/2}}+\frac{1}{2\sigma^2 L}\left(\frac{m^2}{L}-D^2 L\right)\right)g\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\frac{D}{\sqrt{\sigma^2 L}}\left(\frac{m}{2L}-\frac{D}{2}\right)g\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)\times\frac{\sigma^6}{D^6} \qquad 4.5.10.$$

Simplifying we have

$$\frac{\partial G_{12}(M,L)}{\partial L} = \left(\frac{(M-DL)^2}{D} + \frac{\sigma^2 L}{D}\right) F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\left(\frac{M-DL}{D}\right) g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) \qquad 4.5.11.$$

SECTION 4.6. MODEL (M,R,T) QUADRATIC COST TERMS

In Section 3.6 we developed $G_8(R+X,T)$ where $G_8(R+X,T)$ was the expected cost of carrying inventory and backorders for a length of T from t+L to t+T+L where R+X is the inventory position at time t. Then the cost of a backorder was a linear function of the length of time of backorder.

From 3.6.6.

$$G_8(R+X,T) = T\left(R+X-DL-\frac{DT}{2}\right)+b_1\ G_6(R+X,T)$$

$$+(b_2+hc)G_7(R+X,T)$$

The cost per backorder

$$C_B(t) = b_1+b_2t+b_3t^2.$$

If the inventory position of the system is R+Y immediately after review at time t, then the expected backorder costs at time t+L as in 4.4.1.

$$= D\int_0^L \mathcal{D} \int_0^t \frac{1}{\sqrt{\sigma^2 t}} C_B(t-Z)\ g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dZ\ dt. \tag{4.6.1.}$$

Similarly the expected backorder costs at time t+L+T

$$= D\int_0^{L+T} \mathcal{D} \int_0^t \frac{1}{\sqrt{\sigma^2 t}} C_B(t-Z)\ g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dZ\ dt \tag{4.6.2.}$$

Defining $G_{13}(R+Y,T)$ as expected costs of a backorder from t+L to t+L+T for the quadratic cost case

$$G_{13}(R+Y,T) = D\int_0^{T+L} D\int_0^t \frac{C_\beta(t-Z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dZ\, dt$$

$$-D\int_0^L D\int_0^t \frac{1}{\sqrt{\sigma^2 t}}\, C_\beta(t-Z)\, g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dZ\, dt \qquad 4.6.3.$$

From equations 4.4.3. and 4.4.4, $G_{13}(R+Y,T)$ is the $\lim_{Q \to 0}$ of (4.4.4-4.4.3) Hence using 4.5.1 and 4.5.7.

$$G_{13}(R+Y,T)=b_1\big(G_5(R+Y,T+L)-G_5(R+Y,L)\big)+b_2\big(G_2(R+Y,T+L)$$

$$-G_2(R,L)\big)+b_3\big(G_2(R+Y,T+L)-G_{12}(R+Y,L)\big) \qquad 4.6.4.$$

Let $G_{14}(R+Y,T)$ be the expected cost of carrying inventory and backorders including the cost of a stockout dependent on the number of stockouts only. Noting that the only difference between $G_{13}(R+Y,T)$ and $G_8(R+Y,T)$ is the b_3 factor and using

$$G_{14}(R+Y,T)=hcT\left(R+Y-DL-\frac{DT}{2}\right)+b_1\big(G_5(R+Y,T+L)-G_5(R+Y,L)\big)$$

$$+\big(b_2+hc\big)\big(G_2(R+Y,T+L)-G_2(R+Y,L)\big)$$

$$+b_3\left(G_{12}(R+Y,T+L)-G_{12}(R+Y,L)\right)$$

$$+s\left(R_0(R+Y,T+L)-R_0(R+Y,L)\right) \qquad 4.6.5.$$

Replacing $G_8(R+Y,T)$ in 3.6.12 by $G_{14}(R+Y,T)$,

we obtain the inventory cost for quadratic backorder cost terms to be

$$C=\frac{\frac{Rc+S}{T}+\sum_{n=1}^{\infty}\int_0^{M-R} G_{14}(R+Y,T)\times f^n(M-R-Y,DT)dY+G_{14}(M,T)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2T}}\right)+\sum_{n=2}^{\infty}\int_0^{M-R} n\, f^{(n-1)}(M-R-Y,DT)F\left(\frac{Y-DT}{\sqrt{\sigma^2T}}\right)dY\right]}$$

4.6.6.

As before the Steepest descent requires the first order derivatives.

The additional derivatives required in order to calculate the derivatives of C apart from those given in Section 3.6. are the derivatives of $G_{14}(Z,T)$ with respect to Z and T.

From 4.6.4.

$$G_{14}(Z,T)=hcT\left(Z-DL-\frac{DT}{2}\right)+b_1(G_5(Z,T+L)$$

$$-G_5(Z,L))+(b_2+hc)\left(G_2(Z,T+L)-G_2(Z,L)\right)$$

$$+b_3\left(G_{12}(Z,T+L)-G_{12}(Z,L)\right)$$

$$+s\left(R_0(Z,T+L)-R_0(Z,L)\right)$$

Differentiating with respect to Z we have

$$\frac{\partial G_{14}(Z,T)}{\partial Z} = hcT + b_1\left(\frac{\partial G_5(Z,T+L)}{\partial Z}-\frac{\partial G_5(Z,L)}{\partial Z}\right)$$

$$+(b_2+hc)\left(\frac{\partial G_5(Z,T+L)}{\partial Z} -\frac{\partial G_5(Z,L)}{\partial Z}\right)$$

$$+b_3\left(\frac{\partial G_{12}(Z,T+L)}{\partial Z} -\frac{\partial G_{12}(Z,L)}{\partial Z}\right)$$

$$+ s\left(\frac{R_0(Z,T+L)}{\partial Z} - \frac{R_0(Z,L)}{\partial Z}\right) \qquad 4.6.7.$$

Differentiating with respect to T

$$\frac{\partial G_{14}(Z,T)}{\partial T}=hc\left(Z-DL-\frac{DT}{2}\right)-\frac{DhcT}{2}+b_1\frac{\partial G(Z,T+L)}{\partial Z}$$

$$+\frac{(b_2+hc)\partial G_2(Z,T+L)}{\partial T}+\frac{b_3\partial G_{12}(Z,T+L)}{\partial T}$$

$$+ \frac{s\partial R_0(Z,T+L)}{\partial T}. \qquad 4.6.8.$$

Chapter 5

Constant Lead Times And Exponential Cost Terms

Section 5 Model (Q,R)

In this chapter we now compare the performance of the models when the backorder cost is an exponential function $b_1 \exp(b_4 t)$ where $b_4 > 0$ and t is the time for which the backorder exists.

The method of chapter 4 which took into consideration the length of time for which the backorder exists directly, would be used to derive the expected backorder costs.

Let $C_B(t)$ be the cost of a backorder which lasts for time t

$$C_B(t) = b_1 e^{b_4 t} \qquad b_4 > 0 \tag{5.1.1.}$$

From 4.2.2. expected backorder cost per cycle

$$D \int_0^Q \int_0^L \frac{C_B(L-z)}{\sqrt{2\pi\sigma^2 L}} \exp - \tfrac{1}{2}\left(\frac{R+y - Dz}{\sqrt{\sigma^2 L}}\right)^2 dz\, dy \tag{5.1.2.}$$

where $R+y$, $0<y<Q$, is the inventory position at time 0. and $\sqrt{\sigma^2 L}$ is the standard deviation of demand over the lead time.

From 5.1.1.

$$C_B(L-z) = b_1 e^{b_4(L-z)} \tag{5.1.3.}$$

Substituting into 5.1.2.
we have:

$$C_{15}(Q,R) = D \int_0^Q \int_0^L \frac{b_1 e^{b_4(L-Z)}}{\sqrt{2\pi\sigma^2 L}} \exp - \tfrac{1}{2}\left(\frac{R+y - Dz}{\sqrt{\sigma^2 L}}\right)^2 dz\; dy$$

Simplifying and noting that

$$\left(b_4 (L-z) - (R+y - Dz)^2 / 2\sigma^2 L\right)$$

$$= b_4 L - \frac{1}{2\sigma^2 L}\left((R+y)^2 - 2Dz \cdot (R+y) \; D^2 z^2 + 2\sigma^2 b_4 z L\right)$$

then

$$G_{15}(Q,R) = \frac{D b_4 e^{b_4 L}}{\sqrt{\sigma^2 L}} \int_0^Q \int_0^L \exp - \frac{1}{2\sigma^2 L}\Big((R+y)^2 - 2Dz(R+y)$$

$$+ D^2 z^2 + 2\sigma^2 L b_4 z\Big) \, dz \, dy$$

Simplifying

$$G_{15}(Q,R) = \frac{D b_4 e^{b_4 L}}{\sqrt{\sigma^2 L}} * \int_0^Q \int_0^L \exp \frac{-1}{2\sigma^2 L} \left[\left((R+y) - \frac{\sigma^2 b_4 L}{D} \right)^2 \right.$$

$$\left. - 2Dz\left(R+y - \frac{\sigma^2 b_4 L}{D}\right) + D^2 z^2 - \left(\frac{\sigma^4 b_4^2 L^2}{D^2} - 2\sigma^2 \frac{L b_4 * (R+y)}{D} \right) \right] dz \, dy$$

Simplifying

$$G_{15}(Q,R) = \frac{D b_4 e^{b_4 L}}{\sqrt{\sigma^2 L}} * \int_0^Q \exp\left(\frac{\sigma^2 b_4^2 L}{2D^2} - \frac{b_4 * (R+y)}{D} \right) \int_0^L *$$

$$\exp - \frac{1}{2} \left(\frac{R+y - \frac{\sigma^2 b_4 L}{D} - Dz}{\sqrt{\sigma^2 L}} \right)^2 dz \, dy \qquad 5.1.4.$$

Let $$v = \frac{R+y - \frac{\sigma^2 b_4 L}{D} - Dz}{\sqrt{\sigma^2 L}}$$

Thus $$dv = \frac{-D}{\sqrt{\sigma^2 L}} dz$$

Substitute v into 5.1.4

$$G_{15}(Q,R) = -b_4 * e^{b_4 L} \int_0^Q \exp\left(\frac{\sigma^2 b_4^2 L}{2D^2} - \frac{b_4 * (R+y)}{D} \right) * \int_{\frac{R+y-\sigma^2 L b_4}{\sqrt{\sigma^2 L}}}^{\frac{R+y-\frac{\sigma^2 L b_4}{D} - DL}{\sqrt{\sigma^2 L}}} g(v)\, dv$$

Integrating with respect to v

$$G_{15}(Q,R) = -b_1 e^{b_4 L} \cdot \int_0^{Q} \exp\left(\frac{\sigma^2 b_4 L}{D} - b_4' \frac{(R+Y)}{D}\right) *$$

$$\left[F\left(\frac{R+y-\frac{\sigma^2 L b_4}{D}}{\sqrt{\sigma^2 L}}\right) \quad F\left(\frac{R+Y-\sigma^2\frac{b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}\right)\right] dy$$

Let

$$F\left(\frac{R+Y-\sigma^2\frac{b_4 L}{D}}{\sqrt{\sigma^2 L}}\right) = 0$$

Thus

$$G_{15}(Q,R) = b_1 e^{b_4 L} \cdot \int_0^{Q} \exp\left(\frac{\sigma^2 b_4 L}{2D^2} - b_4 \frac{(R+Y)}{D}\right) *$$

$$F\left(\frac{R+y-\frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}\right) dy \qquad 5.1.5$$

Define G_{16} $(R+y, L)$ such that

$$G_{16}(Q,R) = \int_0^{Q} G_{15}(R+y, L)\, dy \qquad 5.1.6$$

Thus

$$G_{16}(R+y,L) = \frac{D}{\sqrt{\sigma^2 L}}\int_0^{L} \frac{C_8(L-z)}{\sqrt{2\pi}} * \exp -\frac{1}{2}\left(\frac{R+y-Dz}{\sqrt{\sigma^2 L}}\right)^2 dz \qquad 5.1.7a$$

Applying

5.1.5

$$G_{16}(R+y,\ L) = b_1 e^{b_4 L} * \exp\left(\frac{\sigma^2 b_4^2 L}{2D^2} - \frac{b_4(R+y)}{D}\right) *$$

$$F\left(\frac{R+y - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}\right) \qquad 5.1.7b$$

Let $V = \dfrac{R+y-\frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}$

Substitute into 5.1.5, then :

$$G_{15}(Q,R) = \sqrt{\sigma^2 L}\, b_1 e^{b_4 L} \int_{\left(R - \frac{\sigma^2 b_4 L}{D}\right)/\sqrt{\sigma^2 L}}^{\left(R+Q - \frac{\sigma^2 b_4 L}{D} - DL\right)/\sqrt{\sigma^2 L}} \exp \frac{\sigma^2 b_4^2}{2D^2} * \exp \frac{-b_4}{D} *$$

$$\left(\sqrt{\sigma^2 L}\ V + \frac{\sigma^2 L b_4}{D} + DL\right) * F(V)\,dV$$

Simplifying we have

$$G_{15}(Q,R) = b_1 \sqrt{\sigma^2 L} * e^{\frac{-\sigma^2 b_4 L}{2D^2}} * \int_{\left(R - \frac{\sigma^2 b_4 L}{D} - DL\right)/\sqrt{\sigma^2 L}}^{\left(R+Q - \frac{\sigma^2 b_4 L}{D} - DL\right)/\sqrt{\sigma^2 L}} e^{-\frac{b_4\sqrt{\sigma^2 L}}{D}} F(V)\,d$$

Integrating by parts we have

$$G_{15}(Q,R) = b_1\sqrt{\sigma^2 L}\; e^{-\frac{\sigma^2 b_4 L}{2D^2}} \left[-\frac{D}{b_4\sqrt{\sigma^2 L}} \times e^{-\frac{b_4\sqrt{\sigma^2 L}\times V}{D}} F(V) \right]_{\frac{R-\frac{\sigma^2 b_4 L}{D}-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-\frac{\sigma^2 b_4 L}{D}-DL}{\sqrt{\sigma^2 L}}}$$

$$-\frac{Db_1}{b_4}\frac{1}{\sqrt{2\pi}} \exp -\frac{\sigma^2 b_4 L}{2D^2} \int_{\frac{R-\frac{\sigma^2 b_4 L}{D}-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-\frac{\sigma^2 b_4}{D}L-DL}{\sqrt{\sigma^2 L}}} \exp\left[-\frac{b_4 V\sqrt{\sigma^2 L}}{D} - \frac{V^2}{2}\right] dV \qquad 5.1.8$$

Noting that $-\frac{\sigma^2 b_4^2}{2D^2} - \frac{b_4 V\sqrt{\sigma^2 L}}{D} - \frac{V^2}{2}$

$$= -\frac{1}{2}\left(V^2 + \frac{2 b_4 V\sqrt{\sigma^2 L}}{D} + \frac{\sigma^2 L b_4^2}{D^2}\right)$$

$$= \left(V + b_4\frac{\sqrt{\sigma^2 L}}{D}\right)^2$$

then $G_{15}(Q,R)$

$$= -\frac{Db_1}{b_4}\left[\exp\left[-\frac{\sigma^2 b^2{}_4}{2D^2} + b_4\frac{\sqrt{\sigma^2 L}.V}{D}\right] F(V)\right]_{\frac{R-\frac{b_4\sigma^2 L}{D}-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-\frac{b_4\sigma^2 L}{D}-DL}{\sqrt{\sigma^2 L}}},$$

$$-\frac{Db_1}{b_4\sqrt{2\pi}} \int_{\frac{R-\frac{\sigma^2 b_4 L}{D}-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-\frac{\sigma^2 b_4 L}{D}-DL}{\sqrt{\sigma^2 L}}} \exp -\tfrac{1}{2}\left(V + b_4\frac{\sqrt{\sigma^2 L}}{D}\right)^2 dV$$

let $x = V + \frac{b_4\sqrt{\sigma^2 L}}{D}$ and substitute into integral

then

$$G_{15}(Q,R) = -\frac{Db_1}{b_4}\left[\exp -\left(\frac{\sigma^2 b^2{}_4 L}{2D^2} + \frac{b_4\sqrt{\sigma^2 L}}{D}\right) F(V)\right]_{\left(R - \frac{\sigma^2 b_4 L}{D} - DL\right)/\sqrt{\sigma^2 L}}^{R+Q-\frac{\sigma^2 b_4 L}{D} - DL/\sqrt{\sigma^2 L}}$$

$$-\frac{Db_1}{b_4}\int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} \exp\left(\frac{x^2}{2}\right) dx$$

Integrating we have

$$G_{15}(Q,R) = -\frac{Db_1}{b_4}\left[\exp - \frac{\sigma^2 b^2{}_4 L}{2D^2} \cdot \exp \frac{b_4 V\sqrt{\sigma^2 L}}{D} F(V)\right]_{\frac{R - \frac{\sigma^2 L b_4}{D} - DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}}$$

5.1.9

$$+ \frac{Db_1}{b_4}\left[F(V)\right]_{(R-DL)/\sqrt{\sigma^2 L}}^{(R+Q-DL)/\sqrt{\sigma^2 L}}$$

Expanding

$$G_{15}(Q,R) = \frac{Db_1}{b_4}\left[\exp -\left(\frac{\sigma^2 L b^2{}_4}{2D^2}\right) + \frac{b_4}{D}\left(R - \frac{\sigma^2 L b_4}{D} - DL\right) F\left(\frac{R - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$- \exp - \left(\frac{\sigma^2 b_4{}^2 L}{2D^2} + \frac{b_4}{D} \left(\frac{R+Q - \sigma^2 b_4 L - DL}{D} \right) \right) F \left(\frac{R+Q - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}} \right) \Bigg]$$

$$- \frac{Db_1}{b_4} \left(F\left(\frac{R-DL}{\sqrt{\sigma^2 L}} \right) - F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}} \right) \right)$$

5.1.10

The limit of $G_{15}(Q,R)$ as $b_4 \to 0$ should equal the expected number of backorder $E(Q,k)$ incurred per year for model (Q,R) of equation 2.6.7. We now show that this is so.

$$b_1 e^{b_4 t} \longrightarrow b_1 \text{ as } b_4 \longrightarrow 0$$

$$\lim_{b_4 \to 0} G_{15}(Q,R) = \Bigg[Db_1 \Bigg[- \left(\frac{\sigma^2 b_4 L}{D^2} + \frac{R}{D} - 2 \frac{\sigma^2 b_4 L}{D} - L \right) *$$

$$\exp \left(- \frac{\sigma^2 b_4 L}{2D^2} - \frac{b_4}{D} \left(R - \frac{\sigma^2 b_4 L}{D} - DL \right) \right) *$$

$$F \left(\frac{R - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}} \right) + \frac{\sqrt{\sigma^2 L}}{D} * \exp \left(- \frac{\sigma^2 b^2{}_4 L}{2D^2} \right.$$

$$\left. - \frac{b_4}{D} * \left(R - \frac{\sigma^2 b_4 L}{D} - DL \right) \right) g \left(\frac{R - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}} \right) \Bigg]_{R+Q}^{R}$$

5.1.11

and b_4 is set equal to 0

Simplifying

$$\lim_{b_4 \to 0} G_{15}(Q,R) = Db_1 \Bigg[\left(-\frac{R}{D} \times F\left(\frac{R - DL}{\sqrt{\sigma^2 L}} \right) \right.$$

$$+ \frac{\sqrt{\sigma^2 L}}{D} \times g\left(\frac{R - DL}{\sqrt{\sigma^2 L}} \right) + \left(\frac{R+Q}{D} \right) F\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}} \right)$$

$$- \frac{\sqrt{\sigma^2 L}}{D} \; g\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}} \right) \Bigg] \qquad 5.1.12$$

Simplifying

$$\lim_{b_4 \to 0} G_{15}(Q,R) = \sqrt{\sigma^2 L} \quad b_1 \left(g\left(\frac{R-DL}{\sqrt{\sigma^2 L}} \right) - \left(\frac{R - DL}{D} \right) \times F\left(\frac{R - DL}{\sqrt{\sigma^2 L}} \right) \right) \qquad 5.1.13$$

$$-\sqrt{\sigma^2 L}\; b_1 \left(g\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}} \right) - \left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}} \right) F\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}} \right) \right) \qquad 5.1.14$$

from 2.6.6 $\quad \alpha(v) = \sqrt{\sigma^2 L}\,(g(v) - vF(v))$

nus

$$\lim_{b_4 \to 0} G_{15}(Q,R) = b_1 \left(\alpha\left(\frac{R - DL}{\sqrt{\sigma^2 L}} \right) - \alpha\left(\frac{R + Q - DL}{\sqrt{\sigma^2 L}} \right) \right) \qquad 5.1.15$$

which reconciles with equation 2.6.7 $\qquad$ 5.1.16

Number of cycles $= \dfrac{D}{Q}$

Hence expected cost of backorders per year $= \frac{D}{Q} * G_{15}(Q,R)$ 5.1.17

Replacing all the cost terms dependent upon the length of time of a backorder in equation 2.6.18 by 5.1.13

Then the inventory costs for model (Q,R) with exponential cost terms and noting that

$$k = (R-DL)/\sqrt{\sigma^2 L}$$

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\left(R - DL\right) + hc\,B\,(Q,R) + \frac{D}{Q}\quad G_{15}(Q,R)$$

$$\frac{Ds}{Q}\left(\alpha\left(\frac{R - DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 5.1.18$$

In order to be able to obtain the first order derivatives of C, the first order derivatives of $G_{15}(Q,R)$ with respect to Q and R would be required.

Differentiating $G_{15}(Q,R)$ with respect to Q and applying 5.1.5

$$\frac{\partial G_{15}(Q,R)}{\partial Q} = b_1 e^{b_4 L} * \exp\left(\frac{\sigma^2 b_4^2 L}{2D^2} - \frac{b_4 (R+Q)}{D}\right) * F\left(\frac{R+Q - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}\right) \qquad 5.1.19$$

Differentiating $G_{15}(Q,R)$ with respect to R from 5.1.9

$$\frac{\partial G_{15}(Q,R)}{\partial R} = \frac{Db_1}{b_4}\left[\frac{b_4}{D} * \exp-\left(\frac{\sigma^2 b_4^2 L}{2D^2} + \frac{b_4}{D}\left(R - \frac{\sigma^2 b_4 L - DL}{D}\right)\right) * F\left(\frac{R - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}\right) - \frac{1}{\sqrt{2\pi\sigma^2 L}} * \exp-\left(\frac{\sigma^2 b_4^2 L}{2D^2} + \frac{b_4}{D}\left(R - \frac{\sigma^2 b_4 L - DL}{D}\right)\right) \cdot \exp - \frac{1}{2\sigma^2 L}\left(R - \frac{\sigma^2 b_4 L}{D} - DL\right)^2\right]$$

$$- \frac{Db_1}{b_4}\left[\frac{b_4}{D} \exp - \left(\frac{\sigma^2 b^2{}_4 L}{2D^2} + \frac{b_4}{D}\left(R+Q - \frac{\sigma^2 b_4 L}{D} - DL \right)\right) F \left(\frac{R+Q - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}} \right) \right.$$

$$\left. \frac{1}{\sqrt{2\pi\sigma^2 L}} \exp - \left(\frac{\sigma^2 b^2{}_4 L}{2D^2} + \frac{b_4}{D}\left(R+Q - \sigma^2 b_4 L - DL \right)\right) . \exp - \frac{1}{2\sigma^2 L}\left(R+Q - \frac{\sigma^2 b_4 L}{D} - DL \right)^2 \right]$$

$$+ \frac{Db_1}{b_4}\frac{1}{\sqrt{\sigma^2 L}} \left(g\left(\frac{R-DL}{\sqrt{\sigma^2 L}} \right) - g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}} \right) \right) \qquad 5.1.20$$

Simplying

$$\frac{\partial G_{15}(Q,R)}{\partial R} = \left[b_1 \exp + \left(\frac{\sigma^2 b^2{}_4 L}{2D^2} - \frac{b_4}{D} \times (R-DL) \right) F \left(\frac{R - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}} \right) \right.$$

$$\left. - \frac{Db_1}{b_4\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R-DL}{\sqrt{\sigma^2 L}} \right)^2 \right]$$

$$- \left[b_1 \exp \left(\frac{\sigma^2 b^2{}_4 L}{2D^2} - \left(\frac{R+Q - DL}{D} \right) \times b_4 \right) \times F \left(\frac{R+Q - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}} \right) \right.$$

$$\left. - \frac{Db_1}{b_4\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}} \right)^2 \right]$$

$$+ \left(\frac{Db_1}{b_4\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R-DL}{\sqrt{\sigma^2 L}} \right)^2 - \frac{Db_1}{b_4\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}} \right)^2 \right)$$

Simplying

$$\frac{\partial G_{15}(Q,R)}{\partial R} = b_1 \exp \frac{\sigma^2 b^2{}_4 L}{2D^2} \left[\exp - \frac{b_4}{D} \times (R-DL) \; F \left(\frac{R - \frac{\sigma^2 b_4 L}{D} DL}{\sqrt{\sigma^2 L}} \right) \right.$$

$$\left. - \exp - \frac{b_4}{D}(R+Q - DL) \times F \left(\frac{R \; Q - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}} \right) \right] \qquad 5.1.21$$

Noting that from equation 2.6.21

$$\frac{\partial \alpha\left(\frac{R - DL}{\sqrt{\sigma^2 L}}\right)}{\partial R} = - F\left(\frac{R - DL}{\sqrt{\sigma^2 L}}\right)$$

And
$$\frac{\partial \alpha\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}}\right)}{\partial R} = -F\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}}\right)$$

Then the first order derivatives of C are

$$\frac{\partial C}{\partial Q} = \frac{-DS}{Q^2} + \frac{hc}{2} - \frac{D G_{1,5}(Q,R)}{Q^2} + \frac{D\,\partial G_{15}(Q,R)}{Q\,\partial Q}$$

$$-\frac{Ds}{Q}\left(\alpha\left(\frac{R - DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{Ds}{Q} F\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}}\right)$$

$$\frac{\partial C}{\partial R} = hc + \frac{D}{Q}\frac{\partial G_{1,5}(Q,R)}{\partial R} - \frac{Ds}{Q}\left(F\left(\frac{R - DL}{\sqrt{\sigma^2 L}}\right) - F\left(\frac{R+Q - DL}{\sqrt{\sigma^2 L}}\right)\right)$$

5.1.22

In this section we derive the expected backorder costs when the cost of a backorder is $C_B(t) = b_1 e^{b_4 t}$ $b_4 > 0$ when t is the length of time of a backorder.

$\sigma^2 z$ is the Variance of demand over a period z.

As in 4.4.1 If the inventory position of the system immediately after the review at time t, is $R+Y$, the expected backorder cost at time t, is from 4.4.1

$$= \frac{1}{Q}\int_0^Q D \int_0^L D \int^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}} * g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 L}}\right) dz\,dt\,dY \qquad 5.2.1$$

Similarly the expected backorder costs at time $t+L+T$

$$= \frac{1}{Q}\int_0^Q D \int_0^{L+T} D \int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}} * g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz\,dt\,dY \qquad 5.2.2$$

$$C_B(L-z) = b_1 e^{b_4(L-z)} \qquad b_4 > 0 \qquad 5.2.3$$

Let

$$G_{17}(Q, R, L) = \frac{1}{Q}\int_0^Q D \int_0^L D \int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}} * g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz\,dt\,dY \qquad 5.2.4$$

Substitute for $C_B(t-z)$ in $G_{17}(Q, R, L)$

we have

$$G_{17}(Q,R,L) = \frac{1}{Q}\int_0^Q D \int_0^L D \int_0^t \frac{b_1 e^{b_4(L-z)}}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz\,dt\,dY \qquad 5.2.5$$

Integrating and applying 5.1.7

we have

$$G_{17}(Q,R,L) = \frac{D}{Q}\int_0^Q \int_0^L G_{16}(R+Y, t)\,dt\,dY \qquad 5.2.6$$

Noting that $G_{15}(Q,R) = \int_0^Q G_{16}(R+Y, t)\,dY$ from 5.1.6

Integrating $G_{17}(Q,R,L)$ with respect to Q and substituting for $G_{15}(Q,R)$

$$G_{17}(Q,R,L) = \frac{D}{Q}\int_0^L \frac{b_1}{b_4}\left(\exp -\left(\frac{b_4 R}{D} - b_4 t - \frac{\sigma^2 t\, b_4^2}{2D^2}\right) F\left(\frac{R - \frac{\sigma^2 b_4 t - Dt}{D}}{\sqrt{\sigma^2 t}}\right)\right.$$

$$\left. - F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)\right) dt$$

$$- D\int_0^L \frac{b_1}{b_4}\left(\exp -\left(\frac{b_4(R+Q)}{D} - b_4 t - \frac{\sigma^2 t b_4^2}{2D^2}\right) F\left(\frac{R+Q - \frac{\sigma^2 b_4 t - Dt}{D}}{\sqrt{\sigma^2 t}}\right)\right.$$

$$\left. - F\left(\frac{R+Q - Dt}{\sqrt{\sigma^2 t}}\right)\right) dt \qquad 5.2.7$$

Let

$$G_{18}(R,L) = \int_0^L \frac{b_1}{b_4}\left(\exp -\left(\frac{b_4}{D}R - b_4 t - \frac{\sigma^2 t b_4^2}{2D^2}\right) \times \right.$$

$$\left. F\left(\frac{R - \frac{\sigma^2 b_4 t - Dt}{D}}{\sqrt{\sigma^2 t}}\right) - F\left(\frac{R - Dt}{\sqrt{\sigma^2 t}}\right)\right) dt \qquad 5.2.8$$

Thus $$G_{17}(Q,R,L) = \left(G_{18}(R,L) - G_{18}(R+Q,L)\right)/Q \qquad 5.2.9$$

Re - arranging the exponential terms of 5.2.8

$$G_{18}(R,L) = D\int_0^L \frac{b_1}{b_4} \cdot \exp\left[t\left(\frac{\sigma^2 b_4^2}{2D^2} + b_4\right) - \frac{b_4 R}{D}\right] F\left(\frac{R - \frac{\sigma^2 b_4 t - Dt}{D}}{\sqrt{\sigma^2 t}}\right)$$

Integrating by parts

$$\frac{G_{18}(R,L)}{D} = \frac{b_1}{b_4}\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\left[e^{t\left(\frac{\sigma^2 b_4^2}{2D^2} + b_4\right) - \frac{b_4 R}{D}}\right. \quad *$$

$$F\left(\frac{\frac{R-\sigma^2 t b_4 - Dt}{D}}{\sqrt{\sigma^2 t}}\right)\Bigg]_0^L$$

$$-\frac{b_1}{2b_4}\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\int_0^L \exp\left(t\left(\frac{\sigma^2 b_4^2}{2D^2}+b_4\right)-\frac{b_4 R}{D}\right)$$

$$\exp -\frac{1}{2}\left(\frac{R-\frac{\sigma^2 b_4 z - Dz}{D}}{\sqrt{\sigma^2 t}}\right)^2 \times \left(\frac{D^2+\sigma^2 b_4}{D\sqrt{\sigma^2 t}}+\frac{R}{\sigma t^{3/2}}\right)dz$$

$$-\frac{b}{b_4}\int_0^L F\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)dt$$

Noting that

$$t\left(\frac{\sigma^2 b_4^2}{2D^2}+b_4\right)-\frac{b_4 R}{D}\quad -\frac{1}{2\sigma^2 t}\left(R-\frac{\sigma^2 b_4 t - Dt}{D}\right)^2$$

$$= -\frac{1}{2\sigma^2 t}\left(-2\sigma^2 t^2\left(\frac{\sigma^2 b_4^2}{2D^2}+b_4\right)+\frac{2\sigma^2 t b_4 R}{D}+R^2-\frac{2tR\sigma^2 b_4}{D}\right.$$

$$\left. -2DtR+\frac{t^2\sigma^4 b_4^2}{D}+\frac{2t^2 D\sigma^2 b_4}{D}+D^2t^2\right)$$

And simplifying

$$= -\frac{1}{2\sigma^2 t}\left(R^2-2DRt+D^2t^2\right) = -\frac{1}{2}\left(\frac{R-Dt}{\sqrt{\sigma^2 t}}\right)^2$$

Hence simplifying

$$\frac{1}{D} G_{18}(R,L) = \frac{b_1}{b_4}\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\left[e^{t\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right) - \frac{b_4 R}{D}} F\left(\frac{R - \sigma^2 b_4 t - Dt}{\frac{D}{\sqrt{\sigma^2 t}}}\right)\right]_0^L$$

$$- \frac{b_1}{b_4}\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\int_0^L \left(\frac{D^2 + \sigma^2 b_4}{2D\sqrt{\sigma^2 t}} + \frac{R}{2\sigma t^{3/2}}\right) *$$

$$\frac{1}{\sqrt{2\pi}} \exp - \frac{1}{2}\left(\frac{R - Dt}{\sqrt{\sigma^2 t}}\right)^2 dt$$

$$- \frac{b}{b_4 2} \int_0^L F\left(\frac{R - Dt}{\sqrt{\sigma^2 t}}\right) dt \qquad 5.2.10$$

From 4.1.1.

$$Z_n(x,T) = \int_0^T \frac{t^n \exp - \frac{1}{2}\left(\frac{x - Dt}{\sqrt{\sigma^2 t}}\right)^2}{\sqrt{\sigma^2 t\, 2\pi}} dt$$

and

$$Z_{n+1}(x,T) = - \frac{2\sigma^2 T^{n+1}}{D^2\sqrt{2\pi\sigma^2 T}} \exp - \frac{1}{2}\left(\frac{x - DT}{\sqrt{\sigma^2 T}}\right)^2 - \frac{\sigma^2 (2n+1)}{2D^2} Z_n(x,T) + \frac{x^2}{D^2} Z_{n-1}(x,T)$$

Hence

$$Z_1(x,T) = - \frac{2\sigma^2 T}{D^2\sqrt{2\pi\sigma^2 T}} \exp - \frac{1}{2}\left(\frac{x - DT}{\sqrt{\sigma^2 T}}\right)^2 + \frac{\sigma^2}{D^2} Z_0(x,T) + \frac{x^2}{D^2} Z_{-1}(x,T) \qquad 5.2.11$$

Hence $$\frac{x^2}{D^2} * Z_{-1}(x,T) = Z_1(x,T) + \frac{2 * \sigma^2 T}{D^2\sqrt{2\pi\sigma^2 T}} \exp - \frac{1}{2}\left(\frac{x - DT}{\sqrt{\sigma^2 T}}\right)^2 - \frac{\sigma^2}{D^2} Z_0(x,T)$$

Substitute $Z_1(x,T)$ and $Z_0(x,T)$, from 4.1.3 and 4.1.2 respectively then $$\frac{x^2}{D^2} Z_{-1}(x,T) = \frac{\sigma^2}{D^3}\left(1+\frac{Dx}{\sigma^2}\right) F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - \frac{2\sigma^2 T}{D^2\sqrt{2\pi\sigma^2 T}} *$$

$$\exp -\tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2 + \frac{1}{D^2}\left(x - \frac{\sigma^2}{D}\right) * e^{\frac{2Dx}{\sigma^2}} \cdot F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{2\sigma^2 T}{D^2\sqrt{2\pi\sigma^2 T}} \exp -\tfrac{1}{2}\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right)^2 - \frac{\sigma^2}{D^3}\left(F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) - e^{\frac{2Dx}{\sigma^2}} \cdot F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)\right)$$

5.2.12

Simplifying

$$\frac{x^2}{D^2} Z_{-1}(x,T) = \frac{x}{D^2} * F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) + \frac{x}{D^2} e^{\frac{2Dx}{\sigma^2}} F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)$$

5.2.13

Thus $$Z_{-1}(x,T) = \frac{1}{x}\left(F\left(\frac{x-DT}{\sqrt{\sigma^2 T}}\right) + e^{\frac{2Dx}{\sigma^2}} * F\left(\frac{x+DT}{\sqrt{\sigma^2 T}}\right)\right)$$

5.2.14

Also noting that $$R_0(x,T) = \int_0^T F\left(\frac{x-Dt}{\sqrt{\sigma^2 t}}\right) dt$$

from 5.2.10

$$\frac{G_{118}(R,L)}{D} = \left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\frac{b_1}{b_4}\left[e^{t\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right) - \frac{b_4 R}{D}} * F\left(\frac{R - \sigma^2 b_4 t - Dt}{\frac{D}{\sqrt{\sigma^2 t}}}\right)\right]_0^L$$

$$-\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\frac{b_1}{b_4}\left[\frac{D^2+\sigma^2 b_4}{2D} Z_0(R,L) + \frac{R}{2} Z_{-1}(R,L)\right] - \frac{b_1}{b_4} R_0(R,L)$$

5.2.15.a

Substitute for $Z_0(R,L)$ from 4.1.2 and $Z_{-1}(R,L)$ from 5.2.14
and $R_0(R,L)$ from 4.1.10

we have:

$$G_{\frac{18}{D}}(R,L) = \frac{b_1}{b_4}\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\left[e^{\left(\frac{\sigma^2 b^2{}_4 + 2D^2 b_4}{2D^2}\right)} - \frac{b_4 R}{D} \quad F\left(\frac{R - t\,(\sigma^2 b_4 + D^2)/D}{\sqrt{\sigma^2 t}}\right)\right]_0^L$$

$$- \frac{b_1}{b_4}\left(\frac{2D^2}{\sigma^2 b_4 + 2D^2 b_4}\right)\left(\frac{D^2 + \sigma^2 b_4}{2D^2}\right)\left[F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - e^{\frac{2DR}{\sigma^2}} \quad F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\right] - \frac{b_1}{2 b_4} *$$

$$\left(\frac{2D^2}{\sigma^2 b_4 + 2D^2 b_4}\right)\left(F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + e^{\frac{2DR}{\sigma^2}} * F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\right) - \frac{b_1}{b_4}\left(\left(L - \frac{R}{D} - \frac{\sigma^2}{2D^2}\right) * F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left.\frac{2\sigma^2 T}{D\sqrt{2\pi\sigma^2 T}} \exp -\tfrac{1}{2}\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + \frac{\sigma^2}{2D^2} * e^{\frac{2DR}{\sigma^2}} * F\left(\frac{R+DL}{\sigma^2 L}\right)\right) \qquad 5.2.15.b$$

Re-arranging terms:

$$G_{\frac{18}{D}}(R,L) = \frac{2b_1 D^2}{\sigma^2 b_4^3 + 2D^2 b_4^2} \quad e^{L\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right)} \cdot e^{\frac{-b R}{D}} * F\left(\frac{R - \left(\frac{\sigma^2}{D} b_4 + D\right)}{\sqrt{\sigma^2 L}}\right)$$

$$+ \frac{b_1}{b_4} \; F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left[-\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right) * \left(\frac{D^2 + \sigma^2 b_4}{2D^2}\right) - \frac{1}{2}\left(\frac{2D^2}{\sigma^2 b_4{}^2 + 2D^2 b_4}\right)\right.$$

$$\left. - \left(L - \frac{R}{D} - \frac{\sigma^2}{2D^2}\right)\right] + e^{\frac{2DR}{\sigma^2}} \; F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\left[\frac{b_1}{b_4}\left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right)\left(\frac{D^2 + \sigma^2 b_4}{2D^2}\right)\right.$$

$$\left. - \left(\frac{2D^2}{\sigma^2 b_4^2 + 2D^2 b_4}\right) * \frac{b_1}{2 b_4} \quad - \frac{\sigma^2}{2D^2} * \frac{b_1}{b_4}\right] - 2\sqrt{\frac{\sigma^2 L}{b_4 D}} * g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

Simplifying

$$G_{18}(R,L) = \frac{2D^3 b_1}{(\sigma^2 b_4^3 + 2D^2 b_4^2)} \quad e^{L\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right)} \cdot e^{-\frac{b_4}{D}} * F\left(\frac{R - DL\left(1 + \frac{\sigma^2 b_4}{D^2}\right)}{\sqrt{\sigma^2 L}}\right),$$

$$- \frac{b_1}{b_4} \; F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left[DL - R - \frac{\sigma^2}{2D} + \frac{D}{b_4}\right] \qquad \frac{-\sigma^4 b_1 b_4^2}{2D(\sigma^2 b_4^2 + 2D^2 b_4) b_4} *$$

$$e^{\frac{2RD}{\sigma^2}} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) - \frac{2\sqrt{\sigma^2 L}\, b_1}{b_4} \cdot g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) \qquad 5.2.16$$

From 5.2.9.

$G_{17}(Q,R,L) = G_{18}(R,L) - G_{18}(R+Q,L)/Q$

From 5.2.3.

expected backorder cost at t+L, averaging over the states of y

$$= \frac{1}{Q}\left(G_{18}(R,L) - G_{18}(R+Q,L)\right) \qquad 5.2.17.$$

and the expected backorder cost at t+L+T averaging over the states of y

$$= \frac{1}{Q}\left(G_{18}(R,L+T) - G_{18}(R+Q,L+T)\right) \qquad 5.2.18.$$

Hence the expected backorder cost per year $G_{19}(Q,R,T)$

$$= \frac{1}{Q}\left(G_{18}(R.L+T) - G_{18}(R,L) - G_{18}(R+Q,L+T) + G_{18}(R+Q,L)\right) \qquad 5.2.19.$$

Replacing the expected backorder costs dependent on the length of time for which the backorder exits by $G_{19}(Q,R,T)$ in equation 3.4.31 we obtain the inventory cost $\mathcal{C}$ for exponential backorder cost

$$\mathcal{C} = \frac{Rc}{T} + \frac{S.POR}{T} + hc\,\frac{Q}{2} + R\text{-}DL\text{-}\frac{DT}{2} + \frac{hc}{QT}\,G_3(R,T+L) - G_3(R,L) - G_3(R+Q,L))$$
$$+ \frac{1}{QT}\left(G_{18}(R,L+T) - G_{18}(R,L) - G_{18}(R+Q,L+T) + G_{18}(R+Q,L)\right) + \frac{s}{QT}\,(G_4(R,T+L)$$
$$- G_4(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)) \qquad 5.2.20.$$

The additional first order derivatives of $\mathcal{C}$ are the derivatives of $G_{18}(R,Z)$ with respect to R and Z.

From 5.2.9.

Differentiating $G_{18}(R,Z)$ with respect to Z we have

$$\frac{\partial G_{18}(R,Z)}{\partial Z} = \frac{D b_1}{b_4}\, \exp -\left(\frac{b_4 R}{D} - b_4 Z - \sigma^2 \frac{Z b_4^2}{2 D^2}\right) F\left(\frac{R - \sigma^2 \frac{b_4}{D} Z - D Z}{\sqrt{\sigma^2 Z}}\right) \qquad 5.2.21.$$

Differentiating $G_{18}(R,Z)$ with respect to R

$$\frac{\partial G_{18}(R,Z)}{\partial R} = -\frac{2 D^2 b_1}{b_4\left(\sigma^2 b_4^3 + 2 D^2 b_4^2\right)}\, \exp\left(\frac{\sigma^2 b_4^2 + 2 D^2 b_4}{2 D^2} - \frac{b_4 R}{D}\right) F\left(\frac{R - Z\left(D + \sigma^2 b_4/D\right)}{\sqrt{\sigma^2 Z}}\right) - \frac{2 D^3 b_1}{\left(\sigma^2 b_4^3 + 2 D^2 b_4^2\right)} * \exp -\tfrac{1}{2}\left(\frac{R - Z\left(D + \frac{\sigma^2}{D} b_4\right)}{\sqrt{\sigma^2 Z}}\right)^2$$

$$\exp\left(\frac{\sigma^2 b_4^2 Z + 2 D^2 b_4 Z}{2 D^2} - \frac{b_4 R}{D}\right) - D b_1 \exp - \tfrac{1}{2}\left(\frac{R-DZ}{\sqrt{\sigma^2 Z}}\right)^2 \left[\left(\frac{R-D}{D}\right)\right.$$

$$\left. - \frac{1}{b_4} + \frac{\sigma^2}{2 D^2}\right] + \frac{b_1}{b_4} F\left(\frac{R-DZ}{\sqrt{\sigma^2 Z}}\right) - \frac{2 D^2}{\sigma^2} * \frac{\sigma^4 b_4^2 b_1}{2 D^2 (\sigma^2 b_4^2 + 2 D^2 b_4) b_4}$$

$$* e^{\frac{2 DR}{\sigma^2}} * F\left(\frac{R+DZ}{\sqrt{\sigma^2 Z}}\right) + 2 \frac{b_1}{b_4}\left(\frac{R-DZ}{\sigma^2 Z}\right) \frac{\exp -\frac{1}{2}\left(\frac{R-DZ}{\sqrt{\sigma^2 Z}}\right)^2}{\sqrt{2\pi}} + \frac{\sigma^4 b_4 b_1}{2 D(\sigma^2 b_4^2 + 2 D^2 b_4) b_4} *$$

$$\frac{\exp-\frac{1}{2}\left(\frac{R-DZ}{\sqrt{\sigma^2 Z}}\right)^2}{\sqrt{2\pi \sigma^2 Z}} \qquad 5.2.22.$$

Simplifying

$$\frac{\partial G_{18}(R,Z)}{\partial R} = \frac{-2D^2 b_1}{b_4(\sigma^2 b_4^2 + 2 D^2 b_4)} \exp\left[Z\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2 D^2}\right) - \frac{b_4 R}{D}\right]$$

$$F\left(\frac{R- Z\left(D+ \frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 Z}}\right) + \frac{b_1}{b_4}(R-DZ) \frac{\exp-\frac{1}{2}\left(\frac{R-DZ}{\sqrt{\sigma^2 Z}}\right)^2}{\sqrt{2\pi \sigma^2 Z}} + \frac{b_1}{b_4} F\left(\frac{R-DZ}{\sqrt{\sigma^2 Z}}\right)$$

$$- \frac{\sigma^2 b_4^2 b_1}{b_4 (\sigma^2 b_4^2 + 2 D^2 b_4)} \exp \frac{2DR}{\sigma^2} * F\left(\frac{R+DZ}{\sqrt{\sigma^2 Z}}\right)$$

5.2.23.

Let $G_{19}(R,Z) = - \frac{\partial G_{18}(R,Z)}{\partial R}$ 5.2.24.

Section 5.3. Model (M,T) Exponential cost terms.

Model (M,T) is now derived from the cost equations for model (Q,R,T) when the cost of a backorder was an exponential function of the length of time of a backorder given in section 5.2.

Model (M,T) = $\lim_{Q \to 0}$ of model (Q,M,T) 5.3.1.

From equation 5.2.20 the cost equation for model (Q,M,T)

$$C = \frac{Rc}{T} + \frac{S.POR}{T} + hc\left(\frac{Q}{2} + M - DL - \frac{DT}{2}\right) + \frac{hc}{QT}\left(G_3(M,T+L) - G_3(M,L) - G_3(M+Q,T+L)\right.$$

$$\left. + G_3(M+Q,L)\right) + \frac{1}{QT}\left(G_{18}(M,L+T) - G_{18}(M,L) - G_{18}(M+Q,L+T) + G_{18}(M+Q,L)\right) + \frac{s}{QT}$$

$$\left(G_4(M,T+L) - G_4(M,L) - G_4(M+Q,T+L) + G_4(M+Q,L)\right) \qquad 5.3.2.$$

From 3.4.4.

$$\lim_{Q\to 0} POR = 1 \qquad 5.3.3.$$

From 3.4.36

$$\lim_{Q\to 0} \frac{G_3(M+Q,L)}{Q} = -G_2(M,L) \qquad 5.3.4.$$

From 5.2.24.

$$\lim_{Q\to 0} \frac{G_{18}(M+Q,L)}{Q} = -G_{19}(M,L) \qquad 5.3\ .6.$$

From 3.4.29.

$$\lim_{Q\to 0} \frac{G_4(M+Q,L)}{Q} = -R_0(M,L) \qquad 5.3.7.$$

Hence applying the limit of (Q,M,T) as $Q \to 0$

Inventory cost for model (M,T)

$$C = \frac{(Rc + S)}{T} + hc\left(M - DL - \frac{DT}{2}\right) + \frac{hc}{T}\left(G_2(M,T+L) - G_2(M,L)\right) + \frac{1}{T}\left(G_{19}(M,L+T)\right.$$

$$\left. - G_{19}(M,L)\right) + \frac{s}{T}\left(R_0(M,L+T) - R_0(M,L)\right) \qquad 5.3.8.$$

The additional derivatives required in order to obtain the derivatives of C are the derivatives of $G_{19}(M,L+T)$ and $G_{19}(M,L)$ with respect to M & T

From 5.2.23. & 5.2.24.

$$-G_{19}(M,T) = \frac{-2D^2 b_1}{b_4(\sigma^2 b_4^2 + 2D^2 b_4)} \exp\left[T\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right) - \frac{b_4 M}{D}\right] \quad *$$

$$F\left(\frac{M-T\left(D+\sigma^2\frac{b_4}{D}\right)}{\sqrt{\sigma^2 T}}\right)+\frac{b_1}{b_4}(M-DT)\frac{\exp-\frac{1}{2}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)^2}{\sqrt{2\pi\sigma^2 T}}-\frac{b_1}{b_4}F\left(\frac{M-DT}{\sigma^2 T}\right)$$

$$-\frac{\sigma^2 b_4 b_1}{b_4(\sigma^2 b_4^2+2D^2 b_4)}\exp\left(\frac{2\cdot DM}{\sigma^2}\right)F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right) \qquad 5.3.9.$$

Differentiating with respect to T applying 5.2.6.

$$\frac{\partial^2 G_{17}(Q,R,T)}{\partial Q\partial T}=G_{16}(R+Q,T)$$

$$\text{Applying 5.2.9} \quad = -\frac{\partial^2 G_{18}(R+Q,T)}{\partial Q\,\partial T} \qquad 5.3.10.$$

$$\text{Applying 5.2.24} \quad = \frac{\partial G_{19}(R+Q,T)}{\partial T}$$

Hence differentiating G_{19} (M,T) with respect to T

From 5.1.6.

$$\frac{\partial G_{19}(M,T)}{\partial T}=G_{16}(M,T)$$

$$= D\, b_1\, .\, e^{b_4 T}\cdot\exp\left(\frac{\sigma^2\cdot b_4^2}{2D^2}-\frac{b_4 M}{D}\right)\cdot F\left(\frac{M-\frac{\sigma^2 b_4}{D}-DT}{\sqrt{\sigma^2 T}}\right) \qquad 5.3.11.$$

Differentiating G_{19} (M,T) with respect to M

$$-\frac{\partial G_{19}(M,T)}{\partial M}=\frac{2\,b_1\,D}{\left(\sigma^2 b_4^2+2D^2 b_4\right)}\exp\left[T\left(\frac{\sigma^2 b_4^2+2D^2 b_4}{2D^2}\right)-\frac{b_4 M}{D}\right]*$$

$$F\left(\frac{M-T\left(D+\sigma^2\cdot\frac{b_4}{D}\right)}{\sqrt{\sigma^2 T}}\right)+\frac{+2D^2 b_1}{b_4(\sigma^2 b_4^2+2D^2 b_4)}*\exp\left(T\left(\frac{\sigma^2 b_4^2+2D^2 b_4}{2D^2}\right)-\frac{b_4 M}{D}\right)$$

$$*\frac{1}{\sqrt{2\pi\sigma^2 T}}\exp-\frac{1}{2}\left(\frac{M-T\left(D+\frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 T}}\right)^2+\frac{b_1}{b_4}\exp-\frac{1}{2}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)^2\cdot\frac{1}{\sqrt{2\pi\sigma^2 T}}$$

$$- \frac{b_1}{b_4} \frac{(M-DT)^2}{\sigma^2 T} * \frac{\exp - \frac{1}{2}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)^2}{\sqrt{2\pi\sigma^2 T}} - \frac{b_1}{b_4} \frac{\exp - \frac{1}{2}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)^2}{\sqrt{2\pi\sigma^2 T}} - \frac{2 b_4^2 b_1 D \cdot e^{\frac{2DM}{\sigma^2}} F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)}{b_4(\sigma^2 b_4 + 2 D^2 b_4)}$$

$$+ \frac{\sigma^2 \cdot b_4^2 \cdot b_1}{b_4(\sigma^2 b_4^2 + 2 D^2 b_4)} \frac{\exp - \frac{1}{2}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)^2}{\sqrt{2\pi\sigma^2 T}} \quad 5.3.12.$$

Simplifying

$$\frac{\partial G_{19}(M,T)}{\partial M} = \left(\frac{2 b_1 D}{(\sigma^2 b_4^2 + 2 D^2 b_4)}\right) * \exp\left[T\left(\frac{\sigma^2 b_4 + 2 D^2 b_4}{2 D^2}\right) - \frac{b_4 M}{D}\right]$$

$$F\left(\frac{M-T\left(D+\frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 T}}\right) - \frac{b_1}{b_4} * \left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)^2 \exp -\frac{1}{2}\left(\frac{-DT}{\sqrt{\sigma^2 T}}\right)^2 \Big/ \sqrt{2\pi\sigma^2 T}$$

$$- \frac{2 b_4 b_1 D}{(\sigma^2 b_4^2 + 2 D^2 b_4)} * e^{\frac{2DM}{\sigma^2}} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

Section 5.4. Model (M,R,T) Exponential cost terms.

In section 3.7. we developed G_8 $(R + x, T)$ where G_8 $(R+x, T)$ was the expected cost of carrying inventory and backorders for a length of T from t+L to t+L+T where $R+x$ is the inventory position at time t, for linear backorder cost. In this section we develop the corresponding cost G_{20} $(R+x, T)$ when the cost per backorder is an exponential function of the length of time of back -order.

The Cost per backorder

$C_B(t) = b_1 e^{b_4 t}$ $b_4 > 0$ where t is the length of time of backorders.

5.4.1.

If the inventory position of the system is R+y immediately after review at time t, then the expected backorder costs at time t+L as in 4.4.1

$$= D \int_0^{L/D} \int_0^t \frac{1}{\sqrt{\sigma^2 t}} C_B(t-z)\, g\left(\frac{R+y - Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt$$

5.4.2.

Similarly the expected backorder costs at time t+L+T

$$= D\int_0^{L+T} D\int_0^t \frac{1}{\sqrt{\sigma^2 t}} \quad C_\beta(t-z)\ g\left(\frac{R+y-Dz}{\sqrt{\sigma^2\ t}}\right) \quad dz\ dt \qquad 5.4.3.$$

Hence

$$G_{20}(R+y,T) = D\int_L^{L+T} D\int_0^t \frac{C_\beta(t-z)}{\sqrt{\sigma^2 t}}\ g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right)\ dz\ dt \qquad 5.4.4.$$

From 5.2.4.

$$\frac{\partial G_{17}(Q,R,L)}{\partial Q} = D\int_0^L D\int_0^t \frac{C_\beta(t-z)}{\sqrt{\sigma^2 t}} * g\left(\frac{R+Q-Dz}{\sqrt{\sigma^2\ t}}\right)\ dz\ dt \qquad 5.4.5.$$

From 5.2.9.

$$G_{17}(Q,R,L) = G_{18}(R,L) - G_{18}(R+Q,L) \qquad 5.4.6.$$

Hence

$$\frac{\partial G_{17}(Q,R,L)}{\partial Q} = -\frac{\partial G_{18}(R+Q,L)}{\partial Q} \qquad 5.4.7.$$

From 5.2.24.

$$\frac{\partial G_{17}(Q,R,L)}{\partial Q} = -\frac{\partial G_{18}(R+Q,L)}{\partial Q} = G_{19}\ (R+Q,L) \qquad 5.4.8.$$

Hence from 5.4.4.

$$G_{20}\ (R+y,T) = G_{19}\ (R+y,T+L) - G_{19}\ (R+y,L) \qquad 5.4.9.$$

Hence when the cost of a stockout dependent only on the number of stockouts is included and the cost of carrying inventories is included (then from 4.6.5)

$$G_{21}(R+y,T) = hc(R+y-DL-\frac{DT}{2}) + hc\ (G_2(R+y,T+L) - G_2\ (R+y,L)) + (G_{19}(R+y,T+L) - G_{19}(R+y,L)) + s(R_0\ (R+y,T+L) - R_0(R+y,L)) \qquad 5.4.10.$$

Hence the inventory cost for model (M,R,T) is obtained by replacing G_{14} (R+y, T) by G_{21} (R+y,T) in equation 4.6.6.

$$C = \frac{Rc}{T} + \frac{S + \sum_{n=1}^{\infty} \int_{0}^{M-R} G21\ (R+y,T) f^{n}\ (M-R-y,DT)\ dy + G_{21}\ (M,T)}{\left[T\ F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty} \int_{0}^{M-R} n\ f^{n-1}\left(M-R-y,DT\right) \propto F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right)\right]}$$

5.4.11.

The additional derivatives required in order to calculate the derivatives of C apart from those given in section 3.6. are the derivatives of $G_{21}\ (Z,T)$ with respect to Z and T.

From 5.4.10.

$$G_{21}\ (Z,T) = hc.T(Z-DL-\frac{DT}{2}) + hc(G_2(Z,T+L) - G_2(Z,L)) + \left(G_{19}(Z,T+L) - G_{19}(Z,L)\right)$$
$$+s(R_0(Z,T+L) - R_0(Z,L)) \qquad 5.4.12.$$

Differentiating with respect to Z

$$\frac{\partial G_{21}(Z,T)}{\partial Z} = hcT + hc\left(\frac{\partial G_2}{\partial Z}(Z,T+L) - \frac{\partial G_2(Z,L)}{\partial Z}\right) + \left(\frac{\partial G_{19}\ (Z,T+L)}{\partial Z} - \frac{\partial G_{19}(Z,L)}{\partial Z}\right)$$
$$+ s\left(\frac{\partial R_0(Z,T+L)}{\partial Z} - \frac{\partial R_0\ (Z,L)}{\partial Z}\right) \qquad 5.4.13.$$

Differentiating with respect to T

$$\frac{\partial G_{21}(Z,T)}{\partial T}\ hc(Z-DL-\frac{DT}{2}) - \frac{DhcT}{2} + hc\frac{\partial G_2(Z,T+L)}{\partial T} + \frac{\partial G_{19}\ (Z,T+L)}{\partial T}$$
$$= s\frac{\partial R_0\ (Z,T+L)}{\partial T}$$

Chapter 6

Period of Grace

Section 6.1. Introduction.

In the previous chapters we have assu ed that as soon as a backorder is incurred the cost of the backorder begins from that moment. In this chapter and the next chapter we shall assume that there is a period of grace after which the cost of the backorder is incurred.
For example if the lead time is L, and the period of grace is p and a back-order occurs at time z then the cost of the backorder begins at time p+z and the length of time that counts in computing the cost of a backorder is L - (z+p).
Also backorders that are incurred after time L-p do not incurr any backorder cost. In this chapter we shall derive the expected backorder costs for the four different models when the cost of a backorder is a quadratic function of the length of time for which a backorder is computable, i.e. $C_B(t) = b_1 + b_2 t + b_3 t^2$. The corresponding linear cost would not derived directly but would be obtained by making $b_3 = 0$ in the corresponding quandratic cost expressions.

Section 6.2. Model (Q,R).

If a backorder is incurred at time z, z<L then L-z is the time for which the backorder lasts but if the period of grace is p then L-z-p is the length of time which bears a backorder cost.
Let R+y, 0<y<Q be the inventory level at time 0, then if the system is out of stock in the time interval z to z+dz after the reorder R is reached then R+y was demanded in time z and a demand occurred in time dz.
This probability is

$$\frac{D\,dz}{\sqrt{2\pi\sigma^2 L}}\ \exp -\frac{1}{2}\left(\frac{R+y-Dz}{\sqrt{\sigma^2 L}}\right)^2 \qquad 6.2.1.$$

Length of time of a backorder is L-z, but the length of time of a backorder which bears a backorder cost = L-z-p. $\quad L > z+p \qquad$ 6.2.2.

Hence expected cost of a backorder occuring at z to z+dz

$$= \frac{C_B(L-z-p)}{\sqrt{2\pi\sigma^2 L}}.\ \exp -\frac{1}{2}\left(\frac{R+y-Dz}{\sqrt{\sigma^2 L}}\right)^2 dz \qquad 6.2.3.$$

Hence the expected backorder cost per cycle $G_{22}(Q,R)$

$$G_{22}(Q,R) = D\int_0^Q\int_0^{L-p} \frac{C_B(L-z-p)}{\sqrt{2\pi\sigma^2 L}}.g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 L}}\right) dz\ dy \qquad 6.2.4.$$

Let t = z+p and substitute into $G_{22}(Q,R)$ then

$$G_{22}(Q,R) = D\int_0^Q\int_p^L \frac{C_B(L-t)}{\sqrt{2\pi\sigma^2 L}} \cdot g\left(\frac{R+y-D(t-p)}{\sqrt{\sigma^2 L}}\right) dt\ dy \qquad 6.2.5$$

When the cost of a backorder is a quadratic function of the length of time of a backorder then $C_B(L-t) = b_1 + b_2(L-t) + b_3(L-t)^2$ 6.2.6

Substitute into $G_{22}(Q,R)$

$$G_{22}(Q,R) = D\int_0^Q\int_p^L \frac{\left(b_1+b_2(L-t)+b_3(L-t)^2\right)}{\sqrt{2\pi\sigma^2 L}} g\left(\frac{R+y-D(t-p)}{\sqrt{\sigma^2 L}}\right) dt\ dy \qquad 6.2.7.$$

But

$$D\int_0^Q\int_0^L \frac{\left(b_1 + b_2(L-t) + b_3(L-t)^2\right)}{\sqrt{2\pi\sigma^2 L}}\, g\left(\frac{R+y-Dt}{\sqrt{\sigma^2 L}}\right) dt\, dy$$

has already been derived in section 4.3
What we would need to derive now in order to obtain $G_{22}(Q,R)$ is

$$D\int_0^Q\int_0^p \frac{(b_1+b_2(L-t) + b_3(L-t)^2)}{\sqrt{2\pi\sigma^2 L}}\, g\left(\frac{R+y+Dp-Dt}{\sqrt{\sigma^2 L}}\right) dt\, dy \qquad 6.2.8.$$

Let $$G_{23}(R+Dp+y,p) = \int_0^p \frac{D.\,C_B(L-t)}{\sqrt{2\pi\sigma^2 L}}\, g\left(\frac{R+y+Dp-Dt}{\sqrt{\sigma^2 L}}\right) dt \qquad 6.2.9.$$

Let $Z = \dfrac{R+y+Dp-Dt}{\sqrt{\sigma^2 L}}$

Thus $dZ = \dfrac{-D\,dt}{\sqrt{\sigma^2 L}}$ and substituting into 6.2.9.

$$G_{23}(R+y+Dp,p) = \frac{-1}{\sqrt{2\pi}}\int_{\frac{R+y+Dp}{\sqrt{\sigma^2 L}}}^{\frac{R+y}{\sqrt{\sigma^2 L}}} \left(b_1 + b_2\left(L - \frac{R+y+Dp-Z\sqrt{\sigma^2 L}}{D}\right) + b_3\left(L - \frac{R+y+Dp-Z\sqrt{\sigma^2 L}}{D}\right)^2\right) g\left(\frac{R+y+Dp-DZ}{\sqrt{\sigma^2 L}}\right) dZ \qquad 6.2.10$$

Making the same assumption as was made in section 4.2, that $F\left(\frac{R+y}{\sqrt{\sigma^2 L}}\right) = 0$

then intergrating G_{23} (R+y+Dp,p) 6.2.11

Hence the expected cost of backorders per cycle = $G_{22}(Q,R)$. From 6.2.7.

$$G_{22}(Q,R) = \frac{D}{\sqrt{2\pi\sigma^2 L}} \int_0^Q\int_0^L \left(b_1 + b_2(L-t) + b_3(L-t)^2\right) * g\left(\frac{R+y+Dp-Dt}{\sqrt{\sigma^2 L}}\right) dt\, dy \qquad 6.2.12$$

which is the same as the expected cost per cycle derived in section 4.3 except that safety stock R is increased in this case to R^1 = R+Dp.
From 4.3.11, expected backorders cost per cycle $G_{22}(Q,R)$

$$= b_1\left(\alpha\left(\frac{R+Dp-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q+Dp-DL}{\sqrt{\sigma^2 L}}\right)\right)$$
$$+ b_2\left(\beta\left(\frac{R+Dp-DL}{\sqrt{\sigma^2 L}}\right) - \beta\left(\frac{R+Q+Dp-DL}{\sqrt{\sigma^2 L}}\right)\right)$$
$$+ b_3\left(\phi\left(\frac{R+Dp-DL}{\sqrt{\sigma^2 L}}\right) - \phi\left(\frac{R+Dp+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 6.2.13$$

Number of cycles = D/Q
Hence expected backorder costs per year = $\frac{D}{Q}\; G_{22}(Q,R)$

From 4.3.13 the inventory cost for model (Q,R) when p is the period of grace noting that $k = \frac{R-DL}{\sqrt{\sigma^2 L}}$, is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hck\sqrt{\sigma^2 L} + \frac{hc}{Q}\left(\beta(k) - \beta\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+\frac{Db_1}{Q}\left(\alpha\left(k+\frac{Dp}{L}\right) - \alpha\left(k+\frac{Dp+Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{b_2}{Q}\left(\beta\left(k+\frac{Dp}{\sqrt{\sigma^2 L}}\right) - \beta\left(k+\frac{Dp+Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+\frac{b_3}{DQ}\left(\theta\left(k+\frac{Dp}{\sqrt{\sigma^2 L}}\right) - \theta\left(k+\frac{Dp+Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{Ds}{Q}\left(\alpha(k) - \alpha\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 6.2.14$$

The derivatives of all the factors involved in the cost expression have already been given in section 4.3

Section 6.3. Model (Q,R,T).

In the section we would derive the expected backorder costs when the cost of a backorder is a quadratic function of the length of time of back -order less the period of grace.
Let p be the period of grace. Hence if the inventory position is R+y immediately after the review at time t, then the expected backorder costs at time t+L from 4.4.1

$$= \frac{1}{Q}\int_0^{\frac{Q}{D}}\int_0^{\frac{L}{D}}\int_0^{t-p} \frac{C_\beta(t-z-p)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt\, dy \qquad 6.3.1.$$

Similarly the expected backorder costs at time t+L+T

$$= \frac{1}{Q}\int_0^{\frac{Q}{D}}\int_0^{\frac{L+T.}{D}}\int_0^{t-p} \frac{C_\beta(t-z-p)}{\sqrt{\sigma^2 t}} * g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt\, dy \qquad 6.3.2.$$

Applying 6.2.4. and 6.2.12
the expected backorder costs at time t+L

$$= \frac{1}{Q}\int_0^{\frac{Q}{D}}\int_0^{\frac{L}{D}}\int_0^{t} \frac{C_\beta(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y+Dp-Dz}{\sigma^2 t}\right) dz\, dt\, dy \qquad 6.3.3.$$

Similarily the expected backorder costs at time t+L+T

$$= \frac{1}{Q}\int_0^{\frac{Q}{D}}\int_0^{\frac{L+T}{D}}\int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}} g\left(\frac{R+y+Dp-Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt\, dy \qquad 6.3.4a$$

which are exactly equal to equations 4.4.1 and 4.4.2 except that safety stock R is increased to R+Dp.

Hence from 3.4.31 the expected cost of backorders per year excluding the cost based on number of stockouts

$$= \frac{b_1}{QT}\left(G_1(R+Dp,\ T+L) - G_1(R+Dp,L) - G_1(R+Q+Dp,T+L) + G_1(R+Q+Dp,L)\right)$$

$$+\frac{b_2}{QT}\left(G_3(R+Dp,T+L) - G_3(R+Dp,L) - G_3(R+Q+Dp,T+L) + G_3(R+Q+Dp,L)\right)$$

$$+\frac{b_3}{QT}\left(G_{11}(R+Dp,T+L) - G_{11}(R+Dp,L) - G_{11}(R+Q+Dp,T+L)\right.$$

$$\left.+ G_{11}(R+Q+Dp,L)\right) \qquad 6.3.4b$$

Hence the inventory cost for model (nQ,R,T) from 4.4.2

$$C = \frac{R_c}{T} + \frac{SPOR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + \frac{hc}{QT}\left(G_3(R,T+L)\right.$$

$$\left. - G_3(R,L) - G_3(R+Q,T+L) + G_3(R+Q,L)\right) + \frac{b_1}{QT}\left(G_1(R+Dp,T+L) - G_1(R+Dp,L)\right.$$

$$\left. -G_1(R+Q+Dp,T+L) + G_1(R+Q+Dp,L)\right) + \frac{b_2}{QT}\left(G_3(R+Dp,T+L) - G_3(R+Dp,L) - G_3(R+Q+Dp,\right.$$

$$\left. T+L) + G_3(R+Q+Dp,L)\right) + \frac{b_3}{QT}\left(G_{11}(R+Dp,\ T+L) - G_{11}(R+Dp,L) - G_{11}(R+Q+Dp,T+L)\right.$$

$$\left. + G_{11}(R+Q+Dp,L)\right) + \frac{s}{QT}\left(G_4(R,T+L) - G_4(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)\right)$$

6.3.5.

The first order deviations of the factors involved in the cost expression have already been given in section 4.4.

Section 6.4. Model(M,T).

The cost expression for model (M,T) would be derived from model (nQ,R,T)

$$\text{Model } (M,T) = \lim_{Q \to 0} (nQ,M,T) \qquad 6.4.1.$$

We noted in section 6.3 that the expected backorder costs when the period of grace was p, was the same as the expected backorders cost for the corresponding model (Q,R,T) except that safety stock was increased from R to R+Dp

From 6.4.5. inventory cost for model (Q,R,T) with period of grace

$$C = \frac{R_c}{T} + \frac{S.POR}{T} + \frac{hc}{2}\left(Q+R-DL-\frac{DT}{2}\right) + \frac{hc}{QT}\left(G_3(R,T+L) - G_3(R,L) - G_3(R+Q,T+L) + G_3(R+Q,L)\right) + \frac{b_1}{QT}\left(G_1(R+Dp,T+L) - G_1(R+Dp,L) - G_1(R+Q+Dp,T+L) + G_1(R+Q+Dp,L)\right)$$

$$+ b_2\left(G_3(R+Dp,T+L) - G_3(R+Dp,L) - G_3(R+Q+Dp,T+L) + G_3(R+Q+Dp,L)\right) + \frac{b_3}{QT}$$

$$\left(G_{11}(R+Dp,T+L) - G_{11}(R+Dp,L) - G_{11}(R+Q+Dp,T+L) + G_{11}(R+Q+Dp,L)\right) + \frac{s}{QT}$$

$$\left(G_4(R,T+L) - G_4(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)\right) \qquad 6.4.2.$$

From 3.4.4

$$\lim_{Q \to 0} POR = 1 \qquad 6.4.3.$$

From 4.5.3

$$\lim_{Q \to 0} \frac{G_3(R+Q+Dp,L)}{Q} = -G_2(R+Dp,L) \qquad 6.4.4.$$

From 4.5.4.

$$\lim_{Q \to 0} \frac{G_1(R+Q+Dp,L)}{Q} = -G_5(R+Dp,L) \qquad 6.4.5.$$

From 4.4.32

$$\lim_{Q \to 0} \frac{G_{11}(R+Q+Dp,L)}{Q} = -G_{12}(R+Dp,L) \qquad 6.4.6.$$

3.4.29

$$\lim_{Q \to 0} \frac{G_4(R+Q+Dp,L)}{Q} = -R_0(R+Dp,L) \qquad 6.4.7$$

Hence taking the limit of C as $Q \to 0$ and setting R to M we obtain the inventory cost for model (M,T)

$$C = \left(\frac{R_c+S}{T}\right) + hc\left(M-DL-\frac{DT}{2}\right) + hc\left(G_2(M,T+L) - G_2(M,L)\right) + \frac{b_1}{T}\left(G_5(M+Dp,T+L) - G_5(M+Dp,L)\right)$$

$$+ \frac{b_2}{T}\left(G_2(M+Dp,T+L) - G_2(M+Dp,L)\right) + \frac{b_3}{T}\left(G_{12}(M,+Dp,T+L) - G_{12}(M+Dp,L)\right) + \frac{s}{T}\left(R_0(M+Dp,T+L)\right.$$

$$\left. - R_0(M+Dp,L)\right) \qquad 6.4.8.$$

The first order derivatives of the factors involved in C have already been given in section 4.4.

Section 6.5. Model (M,R,T)

Let G_{23} (R+y,T) be the cost of carrying backorders from t+L to t+L+T. In this section every backorder experiences a period of grace of time p units.

Hence from equation 4.6.3. expected costs of backorders from t+L to t+L+T

$$G_{23}(R+y,T) = D\int_{L}^{L+T}_{D}\int_{0}^{t-p} C_{\beta}\frac{(t-z-p)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt \qquad 6.5.1.$$

Applying 6.2.4. and 6.2.12

$$G_{23}(R+y,T) = \int_{L}^{L+T}_{D}\int_{0}^{t}D.\, \frac{C_{\beta}(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y+Dp-Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt \qquad 6.5.2.$$

Applying 5.4.4 which states that $G_{20}(R+y,T) = \int_{L}^{L+T}_{D}\int_{0}^{t} D\, \frac{C_{\beta}(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right) dz\,dt$

6.5.3.

then $G_{23}(R+y,T) = G_{20}(R+y+Dp,T)$. 6.5.4.

Hence the cost of carrying inventory and backorders and cost of stock-out dependent only on the number of stockouts, defined as G_{24} (R+y,T) here

Applying 5.4.10

$$G_{24}(R+y,T) = hcT\left(R+y-DL-\frac{DT}{2}\right) + G_{20}(R+y+Dp,T) + s\left(R_o(R+y,T+L) - R_o(R+y,L)\right) \qquad 6.5.5.$$

Hence the inventory cost for model (M,R,T) is obtained by replacing G_{14} (R+y,T) by G_{24}(R+y,T) in equation 4.6.6.

$$C = \frac{R_c}{T} + S + \frac{\sum_{n=1}^{\infty}\int_{0}^{M-R} G_{24}(R+y,T) f^{n}(M-R-y,DT)\, dy + G_{24}(M,T)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty} n f^{n-1}(M-R-y,DT) * F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right)\right]}$$

6.5.6.

Section 6.6. Corresponding cost equations for linear cost terms.

Now we shall derive the cost equations when the cost of a backorder is a linear function of the length of time of backorder less the period of grace.
By setting $b_3 = 0$ in the corresponding cost equations we would obtain the cost equations for the linear cost.

For model (Q,R) the cost equation is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hck\sqrt{\sigma^2 L} + \frac{hc}{Q}\left(\beta(k) - \beta\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right) + \frac{D}{Q} \times b_1\left(\alpha\left(k+\frac{Dp}{\sqrt{\sigma^2 L}}\right) - \alpha\left(k+\frac{Dp+Q}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+\frac{b_2}{Q}\left(\beta\left(k+\frac{Dp}{\sqrt{\sigma^2 L}}\right) - \beta\left(k+\frac{Dp+Q}{\sqrt{\sigma^2 L}}\right)\right) + \frac{Ds}{Q}\left(\alpha(k) - \alpha\left(k+\frac{Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 6.6.1.$$

From equation 6.4.5. the inventory cost for model (Q,R,T)

$$C = \frac{Rc}{T} + \frac{S,POR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + \frac{hc}{QT}\left(G_3(R,T+L) - G_3(R,L) - G_3(R+Q,T+L)\right.$$

$$\left. + G_3(R+Q,L)\right) + \frac{b_1}{QT}\left(G_1(R+Dp,T+L) - G_1(R+Dp,L) - G_1(R+Q+Dp,T+L) + G_1(R+Q+Dp,L)\right)$$

$$+\frac{b_2}{QT}\left(G_3(R+Dp,T+L) - G_3(R+Dp,L) - G_3(R+Q+Dp,T+L) + G_3(R+Q+Dp,L)\right) + \frac{s}{QT}\left(G_4(R,\right.$$

$$\left. T+L) - G_4(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)\right) \qquad 6.6.2.$$

The inventory cost for model (M,T) from 6.4.8.

$$C = \left(\frac{Rc+s}{T}\right) + hc\left(M-DL-\frac{DT}{2}\right) + hc\left(G_2(M,T+L) - G_2(M,L)\right)$$

$$+\frac{b_1}{T}\left(G_5(M+Dp,T+L) - G_5(M+Dp,L)\right) + \frac{b_2}{T}\left(G_2(M+Dp,T+L) - G_2(M+Dp,L)\right) + \frac{s}{T}$$

$$\left(R_0(M,T+L) - R_0(M,L)\right) \qquad 6.6.3.$$

For model (M,R,T) let G_{25} (R+y,T) be the expected cost of carrying inventory and backorders including the cost dependent on stockouts only. Note that R becomes R+Dp in b_1 & b_2 factors

$$G_{25}(R+y,T) = hcT\left(R+y-DL-\frac{DT}{2}\right) + hc\left(G_2(R+y,T+L) - G_2(R+y,L)\right) + b_1\left(G_5(R+y+Dp,T+L)\right.$$

$$\left. - G_5(R+y+Dp,L)\right) + b_2\left(G_2(R+y+Dp,T+L - G_2(R+y+Dp,L)\right) + \frac{s}{T}\left(R_0(R+y+Dp,T+L)\right.$$

$$\left. - R_0(R+y+Dp,L)\right) \qquad 6.6.4.$$

From 4.6.6.

$$C = \frac{\frac{R_c}{T} + S + \sum_{n=1}^{\infty} \int_0^{M-R} G_{25}(R+y,T) * f^n(M-R-y,DT)\,dy + G_{25}(M,T)}{T\left[\sum_{n=2}^{\infty} \int_0^{M-R} nf^{n-1}(M-R-y,DT)\, F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right) + F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right)\right]} \qquad 6.6.5.$$

Chapter 7.

In this chapter we would derive the cost equations of the models when the cost of a backorder is an exponential function of the length of time of a backorder after the period of grace has expired.
Let p be the period of grace after a backorder occurs.
For model (Q,R) making use of 5.1.2. the expected backorder cost per cycle

$$= D\int_0^{Q}\int_0^{L-p} \frac{C_B(L-z-p)}{\sqrt{2\pi\sigma^2 L}} \exp -\frac{1}{2}\left(\frac{R+y-Dz}{\sigma^2 L}\right)^2 dz\,dy \qquad 7.1.$$

where $C_B(L-z) = b_1 e^{b_4(L-z)}$ $\quad b_4 > 0$

Let $v = Z+p$ then expected backorder cost per cycle

$$= D\int_0^{Q}\int_p^{L} \frac{C_B(L-v)}{\sqrt{2\pi\sigma^2 L}} \exp -\frac{1}{2}\left(\frac{R+y+Dp-Dv}{\sqrt{\sigma^2 L}}\right)^2 dv\,dy \qquad 7.2.$$

From 6.2.9. and 6.2.11.

The expected backorder cost per cycle $= D\int_0^{Q}\int_0^{L} \frac{C_B(L-v)}{\sqrt{2\pi\sigma^2 L}} \exp -\frac{1}{2}\left(\frac{R+y+Dp-Dv}{\sqrt{\sigma^2 L}}\right)^2 dv\,dy$ 7.3.

which is equal to the expected backorder cost obtained when there was no grace but safety stock was increased from R to R+Dp

Hence expected backorder cost per cycle from 5.1.2.

$$= G_{15}(Q, R+Dp) \qquad 7.4.$$

Hence inventory cost for model (Q,R) with period of grace operating, from 5.1.18

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc(R-DL) + \frac{D}{Q}G_{15}(Q\ R+Dp) + \frac{Ds}{Q}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 7.5.$$

Differentiating with respect to Q & R

$$\frac{\partial C}{\partial Q} = \frac{DS}{Q^2} + \frac{hc}{2} - \frac{D\,G_5(Q,R+Dp)}{Q^2} - \frac{Ds}{Q^2}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+\frac{D}{Q}\frac{\partial G_{15}(Q,R+Dp)}{\partial Q} + \frac{Ds}{Q}F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$$

$$\frac{\partial C}{\partial R} = hc + \frac{D}{Q}\frac{\partial G_{15}(Q,R+Dp)}{\partial R}$$

$$- \frac{Ds}{Q^2}\left(F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 7.6.$$

For model (ΛQ,R,T), from 5.2.1., the expected backorder costs at time t+L would be

$$\frac{1}{Q}\int_0^{\frac{Q}{D}}\int_0^{\frac{L}{D}}\int_0^{t-p} C_\beta\frac{(t-z-p)}{\sqrt{\sigma^2 t}} * g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right) dz\ dt\ dy \qquad 7.7.$$

and the expected backorder costs a time t+L+T would be

$$\frac{1}{Q}\int_0^{\frac{Q}{D}}\int_0^{\frac{L+T}{D}}\int_0^{t-p} C_\beta\frac{(t-z-p)}{\sqrt{\sigma^2 t}}\ g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right) dz\ dt\ dy \qquad 7.8.$$

Hence let v= z+p, then the expected backorder costs from time t+L to t+L+T

$$= \frac{1}{Q}\int_0^{\frac{Q}{D}}\int_L^{\frac{L+T}{D}}\int_p^{t} \frac{C_\beta(t-v)}{\sqrt{\sigma^2 t}}\ g\left(\frac{R+t+Dp-Dv}{\sqrt{\sigma^2 t}}\right) dz\ dt\ dy. \qquad 7.9.$$

making use of 6.2.9. & 6.2.11

Expected backorder costs from time t+L to t+L+T

$$= \int_0^{\frac{Q}{D}}\frac{}{Q}\int_L^{\frac{L+T}{D}}\int_0^{t} \frac{C_\beta(t-v)}{\sqrt{\sigma^2 t}}\ g\left(\frac{R+y+Dp-Dv}{\sqrt{\sigma^2 t}}\right) dv\ dt\ dy \qquad 7.10$$

which is equal to the expected backorder costs for model (ΛQ,R,T) except that R is increased from R to R+Dp.

Hence from 5.2.19 and 5.2.20. the inventory cost for model (ΛQ,R,T)

$$C = \frac{Rc}{T} + \frac{SPQR}{T} + hc\left(Q + R - DL - \frac{DT}{2}\right) + \frac{hc}{QT}\big(G_3(R,T+L) - G_3(R,L) - G_3(R,L) - G_3(R+Q,T+L) + G_3(R+Q,L)\big) + \frac{1}{QT}\big(G_{18}(R+Dp,L+T) - G_{18}(R+Dp,L) - G_{18}(R+Q+Dp,L+T) + G_{18}(R+Q+Dp,L)\big) + \frac{s}{QT}\big(G_4(R,T+L) - G_4(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)\big) \qquad 7.11.$$

Since model (M,T) $= \lim\limits_{Q\to 0}$ (ΛQ,R,T) $\Big/$ R=M 7.12.

and from 5.5.6.

$$\lim_{Q\to 0} \frac{G_{18}(R+Q+Dp,L)}{Q}\Bigg|_{R=M,} = G_{19}(M+Dp,L) \qquad 7.13.$$

Hence from 5.3.8., inventory cost for model (M,T)

$$C = \left(\frac{Rc+S}{T}\right) + hc\left(M - DL - \frac{DT}{2}\right) + \frac{hc}{T}\big(G_2(M,T+L) - G_2(M,L)\big) + \frac{1}{T}\big(G_{19}(M+Dp,T+L) - G_{19}(M+Dp,L)\big) + \frac{s}{T}\big(R_0(M,L+T) - R_0(M,L)\big) \qquad 7.14.$$

For model (M,R,T)

Let G_{26} (R+y,T) be cost of carrying inventory and backorders and the cost of stockout dependent only on there been a stockout from t+L to t+L+T when R+y is the inventory position at time t.
However the cost of carrying backorders from t+L to t+L+T making use of 5.4.4.

$$= D\int_L^{L+T} \int_0^{t-p} C_\beta \frac{(t-z-p)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y-Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt \qquad 7.15.$$

let v = z+p

Then cost

$$= D\int_L^{L+T} \int_p^{t} C_\beta \frac{(t-v)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y+Dp-Dv}{\sqrt{\sigma^2 t}}\right) dv dt \qquad 7.16.$$

Integrating and making use of 6.2.9. & 6.2.11

$$= D\int_L^{L+T} \int_0^{t} C_\beta \frac{(t-v)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+y+Dp-Dv}{\sqrt{\sigma^2 t}}\right) dv dt \qquad 7.17$$

Making use of 5.4.4.

$$= G_{20}(R+y+Dp,T) \qquad 7.18$$

Making use of 5.4.10

$$G_{26}(R+y,T) = hcT(R+y-DL-DT) + hc(G_2(R+y,T+L) - G_2(R+y,L)) + (G_{19}(R+y+Dp, T+L) - G_{19}(R+y+Dp,L)) + s(R_0(R+y,T+L) - R_0(R+y,L)) \qquad 7.19.$$

Hence the inventory cost for model (M,R,T) is obtained by replacing $G_{14}(R+y,T)$ by $G_{26}(R+y,T)$ in equation 4.6.6.

$$C = \frac{\frac{Rc}{T} + S + \sum_{n=1}^{\infty} \int_0^{M-R} G_{26}(R+y,T)\, f^n(M-R-y,DT)\, dy + G_{26}(M,T)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty} \int_0^{M-R} {}_n f^{n-1}(M-R-y,DT) * F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right) dy\right]}$$

7.20.

CHAPTER 8

CONSTANT LEAD TIMES

Section 8.1 Introduction

In this chapter we would relax the assumption of constant lead time and assume that lead time is a random variable. This would allow orders to overdue one another. However this is not likely to happen in practice.

We assume in the analysis that lead times are gamma distributed. Hence the range of values of lead times may not always be less than T for the periodic models. However we shall assume that lead times are independent random variables and that orders are received in the same sequence in which they were placed.

One difference between the transaction reporting model (Q,R) and the periodic review models (nQ,R,T),(M,T) and (M,R,T) is that in the periodic review models, the time between review models is T while in model (Q,R) there is a positive probability that two orders will be placed in an arbitrary small time interval.

We shall compute the inventory ~~costs~~ equations for all models assuming that the cost of a backorder is a quadratic function of the length of time of a backorder and also when the cost is an exponential function. The corresponding inventory cost can be obtained from the quadratic cost directly for the case when the cost of a backorder is a linear function, by putting $b_3 = 0$.

The cost equations could be derived by either calculating the inventory costs for a lead time demand for a fixed lead time and averaging over the probabilities of lead time or by computing directly the various expected values by making use of the marginal distribution of lead time demand. The approach in this chapter is to average the inventory costs for a fixed lead time over the probabilities of lead time. This approach is chosen because it is mathematically easier.

In the derivation of the equations of this chapter extensive use was made of the mathematical results in 'Inventory Control With Normal Demand and Gamma Lead Times' by Burgin, Operational Research Quarterly Vol. 23 No. 1. In this paper Burgin derives an exact expression for protection and potential lost sales for a reorder level system of inventory control in which demand is Normally distributed and lead time is Gamma distributed. However he does not touch on any of the inventory control procedures we are considering in this chapter.

Section 8.2 Basic Mathematics

Some of the mathematical results here can be seen in Burgin (Ref. 8). However for the sake of completeness the equations are derived here from first principles. The notation is the same and mathematical results derived by Burgin are asterisked.

Now we start deriving the mathematical results.

$$\text{Let } \frac{1}{\sqrt{\sigma^2 L}} g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) = \frac{1}{\sqrt{2\pi\sigma^2 L}} \exp -\tfrac{1}{2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2 \qquad 8.2.1$$

$$\text{Let } H(L) = \frac{\alpha^k L^{k-1} \exp-(\alpha L)}{\Gamma(k)} \qquad L>0 \qquad 8.2.2$$

Hence

$$\int_0^\infty \frac{H(L)}{\sqrt{\sigma^2 L}}\, g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) dL \qquad 8.2.3$$

Substituting for H(L) and $g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)$ we have

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 L}} \exp-\tfrac{1}{2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2 \frac{\alpha^k L^{k-1} \exp-(\alpha L)}{\Gamma(k)}\, dL \qquad 8.2.4$$

Simplifying

*

$$= \frac{\alpha^k}{\Gamma(k)} \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\tfrac{1}{2}\left(\frac{x^2}{\sigma^2 L} - \frac{2Dx}{\sigma^2} + \frac{D^2 L}{\sigma^2}\right)\right\} \exp-(\alpha L) L^{k-3/2}\, dL \qquad 8.2.5$$

Re-arranging the exponential term

$$= \frac{\alpha^k}{\Gamma(k)} \int_0^\infty \frac{L^{k-3/2}}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{Dx}{\sigma^2}\right) \exp\left\{\frac{-x^2}{2\sigma^2 L} - L\left(\frac{2\alpha\sigma^2 + D^2}{2\sigma^2}\right)\right\} dL \qquad 8.2.6$$

But

$$* \int_0^{\infty} L^{v-1} \exp\left(-\frac{B}{L} - YL\right) dL = 2\left(\frac{B}{Y}\right)^{v/2} K_v(2\sqrt{(BY)})$$

$B, Y > 0$

8.2.7

where $K_v\left(2\sqrt{(BY)}\right)$ denotes the Bessel function of imaginary argument.

$$* \text{ Hence } \int_0^{\infty} H(L) g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) \frac{1}{\sqrt{\sigma^2 L}} \, dL = \int_0^{\infty} \frac{e^{-\alpha L} L^{k-1} \alpha^k}{\sqrt{\sigma^2 L}\,\Gamma(k)} g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) dL$$

$$= \frac{\alpha^k}{\sigma\sqrt{2\pi}\,\Gamma(k)} \int_0^{\infty} L^{k-3/2} \exp\left(\frac{Dx}{\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2 L} - L\left(\frac{2\alpha\sigma^2 + D^2}{2\sigma^2}\right)\right) dL$$

$$* = \frac{\alpha^k \exp\left(\frac{Dx}{\sigma^2}\right)}{\sigma\sqrt{2\pi}\;\Gamma(k)} \left[2\left(\frac{x^2}{2\alpha\sigma^2 + D^2}\right)^{\frac{1}{2}(k-\frac{1}{2})} K_{k-\frac{1}{2}}\left(\frac{x}{\sigma^2}(2\alpha\sigma^2 + D^2)^{\frac{1}{2}}\right)\right]$$

8.2.8.

* If k is an integer then

$$K_{k-\frac{1}{2}}(z) = K_{\frac{1}{2}}(z) \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!\,(k-j-1)!} (2z)^{-j}$$

8.2.9

Where $$K_{\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{\sqrt{2}} (z)^{-\frac{1}{2}} \exp(-z)$$ 8.2.10

Hence $$K_{k-\frac{1}{2}}(z) = \sqrt{\pi} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!\,(k-j-1)!} (2z)^{-j-\frac{1}{2}} \exp-(z)$$

8.2.11.

Next we derive

$$\int_0^{\infty} H(L)\, F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) dL$$ 8.2.12

$$= \int_0^\infty \frac{\alpha^k \, L^{k-1} \exp\,(\alpha L)}{\Gamma(k)} * F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) dL$$

8.2.13

$$\text{But} \int L^k \exp^{-\alpha L} \, dL = \sum_{z=1}^{k+1} \frac{k! \, L^{k-z+1} e^{-\alpha L}}{\alpha^z \, (k+1-z)!}$$

8.2.14

Hence from 8.2.14

$$\int_0^\infty \frac{\alpha^k L^{k-1} \exp^{-\alpha L} \, F\left(\frac{x-DL}{(\sigma^2 L)^{\frac{1}{2}}}\right)}{\Gamma(k)} dL$$

$$= \frac{\alpha^k}{\Gamma(k)} \int_0^\infty F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) d\left(-\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} L^{k-z} e^{-\alpha L}\right)$$

8.2.15

Integrating by parts we have

$$= -\frac{\alpha^k}{\Gamma(k)} \left[F\left(\frac{x-DL}{(\sigma^2 L)^{\frac{1}{2}}}\right) \sum_{z=1}^{k} \frac{(k-1)! \, L^{k-z} \, e^{-\alpha L}}{\alpha^z \, (k-z)!} \right]_0^\infty$$

$$+ \frac{\alpha^k}{2\,\Gamma(k)} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \int_0^\infty L^{k-z} \, e^{-\alpha L} \, g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{D}{\sqrt{\sigma^2 L}} + \frac{x}{\sigma \, L^{3/2}}\right) dL$$

8.2.16

Simplifying

We have

$$\frac{\alpha^k}{2\Gamma(k)\sigma} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \int_0^\infty \left(D \, L^{k-z-\frac{1}{2}} + xL^{k-z-1\frac{1}{2}}\right) e^{-\alpha L} *$$

$$g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) \quad dL$$

Integrating and applying 8.2.8 we have

$$\int_0^\infty H(L) \ast F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) dL$$

$$= \frac{\alpha^k e^{\left(\frac{Dx}{\sigma^2}\right)}}{\sqrt{2\pi}\, 2\sigma \Gamma(k)} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left[2D\left(\frac{x^2}{2\alpha\sigma^2+D^2}\right)^{\frac{1}{2}(k-z+\frac{1}{2})} K_{k-z+\frac{1}{2}}\left(\frac{x}{\sigma^2}\left(2\alpha\sigma^2+D^2\right)^{\frac{1}{2}}\right)\right.$$

$$\left. + \; 2x\left(\frac{x^2}{2\alpha\sigma^2+D^2}\right)^{\frac{1}{2}(k-z-\frac{1}{2})} K_{k-z-\frac{1}{2}}\left(\frac{x}{\sigma^2}\left(2\alpha\sigma^2+D^2\right)^{\frac{1}{2}}\right)\right]$$

8.2.17

Let $\theta^2 = 2\alpha\sigma^2 + D^2$

then
$$\ast \quad \int_0^\infty H(L)\, F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) dL = \frac{\alpha^k}{\Gamma(k)} \int_0^\infty L^{k-1} e^{-\alpha L} F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) dL$$

$$= \frac{\alpha^k e^{\frac{Dx}{\sigma^2}}}{\sqrt{2\pi}\, 2\sigma\, \Gamma(k)} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left[2D\left(\frac{x}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right)\right.$$

$$\left. + \; 2x\left(\frac{x}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right)\right]$$

8.2.18

Now dealing with

$$\ast \quad \int_Q^\infty e^{\left(\frac{Dx}{\sigma^2}\right)} \cdot \left(\frac{x}{\theta}\right)^{Y+\frac{1}{2}} K_{k-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) dx$$

and

Substituting for $K_{k-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right)$ from 8.2.11

$$= \int_Q^\infty e^{\left(\frac{Dx}{\sigma^2}\right)}\left(\frac{x}{\theta}\right)^{Y+\frac{1}{2}} \left(\sqrt{\pi} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!(k-j-1)!}\left(\frac{2x\theta}{\sigma^2}\right)^{-j-\frac{1}{2}} \ast \right.$$

$$\left. \exp -\left(\frac{x\theta}{\sigma^2}\right)\right) dx$$

Simplifying

$$=\sqrt{\pi}\left(\frac{\sigma^2}{2\,\theta^2}\right)^{Y+\frac{1}{2}} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!(k-j-1)!} \int_Q^{\infty} \left(\frac{2x\theta}{\sigma^2}\right)^{Y-j} \exp\left(\frac{-x}{\sigma^2}\ (\theta-D)\right) dx$$

8.2.19

This integral can be evaluated explicitly as follows:

$$\int_Q^{\infty} \left(\frac{2x\theta}{\sigma^2}\right)^{Y-j} \exp\left(-x\left(\frac{(\theta-D)}{\sigma^2}\right)\right) dx$$

integrating by parts

$$=-\left(\frac{2\theta}{\sigma^2}\right)^{Y-j} \sum_{i=0}^{Y-j} \frac{(Y-j)!}{(Y-j-i)!} \frac{x^{Y-j-i}}{((\theta-D)/\sigma^2)^{i+1}} \exp\left(-x\left(\frac{\theta-D}{\sigma^2}\right)\right)\Bigg]_Q^{\infty}$$

8.2.20

Simplifying

$$=\left(\frac{2\theta}{\sigma^2}\right)^{Y-j} \sum_{i=0}^{Y-j} \frac{(Y-j)!}{(Y-j-i)!} \frac{Q^{Y-j-i}}{((\theta-D)/\sigma^2)^{i+1}} \exp\left(-Q\left(\frac{\theta-D}{\sigma^2}\right)\right)$$

8.2.21

Hence 8.2.19 gives

$$= \sqrt{\pi} \left(\frac{\sigma^2}{2\theta^2}\right)^{Y+\frac{1}{2}} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!(k-j-1)!} \left(\frac{2\theta}{\sigma^2}\right)^{Y-j} *$$

$$\left[\sum_{i=0}^{Y-j} \frac{(Y-j)!}{(Y-j-i)!} \frac{Q^{Y-j-i}}{((Q-D)/\sigma^2)^{i+1}} \exp\left(-Q\left(\frac{\theta-D}{\sigma^2}\right)\right)\right]$$

8.2.22

Simplifying we have

$$\sqrt{\pi} \sum_{j=o}^{k-1} \sum_{i=o}^{Y-j} \frac{(k+j-1)!}{j!(k-j-1)!} \frac{(Y-j)!}{(Y-j-i)!} \left(\frac{\theta-D}{\sigma^2}\right)^{-i-1} \left(\frac{\sigma^2}{2\theta^2}\right)^{Y+\frac{1}{2}} \left(\frac{2\theta}{\sigma^2}\right)^{Y-j} *$$

$$Q^{Y-j-i} \quad \exp\left(-Q\left(\frac{\theta-D}{\sigma^2}\right)\right)$$

8.2.23

Simplifying we have

$$\frac{\sqrt{\pi}}{\theta^Y} \left(\frac{\sigma^2}{2\theta^2}\right)^{\frac{1}{2}} \sum_{j=o}^{k-1} \sum_{i=o}^{Y-j} \frac{(k+j-1)!}{j!(k-j-1)!} \frac{(Y-j)!}{(Y-j-i)!} \left(\frac{\theta-D}{\sigma^2}\right)^{j-Y-1} \left(\frac{2\theta}{\sigma^2}\right)^{-j}$$

$$* \left(Q\left(\frac{\theta-D}{\sigma^2}\right)\right)^{Y-j-i} \quad \exp\left(-Q\left(\frac{\theta-D}{\sigma^2}\right)\right)$$

8.2.24

Let $\lambda = \left(\frac{\theta-D}{\sigma^2}\right)$

8.2.25

Then we have

$$* \quad \frac{\sqrt{\pi}}{\theta^Y} \left(\frac{\sigma^2}{2\theta^2}\right)^{\frac{1}{2}} \sum_{j=o}^{k-1} \sum_{i=o}^{y-j} \frac{(k+j-1)!}{j!(k-j-1)!} \frac{(Y-j)!}{(Y-j-i)!} \lambda^{j-Y-1} \left(\frac{2\theta}{\sigma^2}\right)^{-j}$$

$$(\lambda Q)^{Y-j-i} \quad \exp\left(-Q\left(\frac{\theta-D}{\sigma^2}\right)\right)$$

8.2.26

$$* \text{Let} \sum_{i=0}^{Y-j} \frac{(\lambda Q)^{Y-j-i}}{(Y-j-i)!} \exp(-\lambda Q) = \bar{Q}\left(2Q\lambda/2(Y-j+1)\right)$$

8.2.27

where $\bar{Q}(X^2/v)$ is the chi-squared distribution function

Hence 8.2.26

$$= \frac{\sqrt{\pi}\left(\frac{\sigma^2}{2}\right)^{\frac{1}{2}}}{(\lambda\theta)^{Y+1}} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!(k-j-1)!} (Y-j)!\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \bar{Q}\left(2\lambda Q/2(Y-j+1)\right)$$

8.2.28

Hence

$$* \int_Q^{\infty} e^{\frac{Dx}{\sigma^2}} \left(\frac{x}{\theta}\right)^{Y+\frac{1}{2}} K_{k-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) dx$$

$$= \sqrt{\frac{\pi\sigma^2}{2}} \left(\frac{1}{\lambda\theta}\right)^{Y+1} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!(k-j-1)!} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \bar{Q}\left(\frac{2\lambda Q}{2(Y-j+1)}\right) * (Y-j)!$$

8.2.29

Also we need

$$\int_0^{\infty} H(L) \cdot F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) e^{\frac{2DR}{\sigma^2}} dL$$

Applying 8.2.15 we have

$$= \frac{\alpha^k}{\Gamma(k)} \int_0^{\infty} e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) d\left(-\sum_{z=1}^{k} \frac{(k-1)!\, L^{k-z}}{\alpha^z (k-z)!} e^{-\alpha L}\right)$$

$$= \frac{\alpha^k}{\Gamma(k)} \left[F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) - \sum_{z=1}^{k} \frac{(k-1)!\ \ L^{k-z}\ e^{-\alpha L}}{\alpha^z\ (k-z)!}\ e^{\frac{2DR}{\sigma^2}} \right]_0^{\infty}$$

$$+ \frac{\alpha^k}{\Gamma(k)} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \int_0^{\infty} L^{k-z}\ e^{-\alpha L} g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{x}{2\sigma L^{3/2}} - \frac{D}{2\sigma L^{\frac{1}{2}}}\right)\ dL$$

8.2.30a

Integrating and applying 8.2.8 we have

$$\int_0^{\infty} H(L) \cdot F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) * e^{\frac{DR}{\sigma^2}}\ dL = e^{\frac{2DR}{\sigma^2}} \int_0^{\infty} \frac{e^{-\alpha L}\alpha^k\ L^{k-1}}{\Gamma(k)} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) dL$$

$$= \frac{\alpha^k}{2\sigma\Gamma(k)} * \cdot \frac{1}{\sqrt{2\pi}} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left[-2D\left(\frac{x}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) \right.$$

$$\left. + \quad 2x \left(\frac{x}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) \right]$$

Now we need 8.2.30b

$$\int_0^{\infty} H(L) * F\left(\frac{x - D(L+T)}{\sqrt{\sigma^2 (L+T)}}\right) \quad dL$$

Substituting for H(L) we have

$$= \frac{\alpha^k}{\Gamma(k)} \int_0^{\infty} L^{k-1}\ e^{-\alpha L}\ F\left(\frac{x-D(L+T)}{\sqrt{\sigma^2 (L+T)}}\right) \quad dL$$

Let $z = L+T$

Substituting and simplifying the expression we obtain

$$\frac{\alpha^k}{\Gamma(k)} e^{\alpha T} \sum_{j=0}^{k-1} (-T)^j \int_0^{\infty} \binom{k-1}{j} z^{k-1-j} e^{-\alpha z} F\left(\frac{x-Dz}{\sqrt{\sigma^2 z}}\right) \quad dz$$

Applying 8.2.18

then

$$\int_0^{\infty} H(L) * F\left(\frac{x-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) dL$$

$$= \frac{e^{\left(\frac{Dx}{\sigma^2}+\alpha T\right)}}{2\sigma\sqrt{2\pi}\,\Gamma(k)}\alpha^k \sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} \sum_{z=1}^{k-j} \frac{(k-1-j)!}{\alpha^z (k-j-z)!}$$

$$\left(2D\left(\frac{x}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) + 2x\left(\frac{x}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{x}{\theta}\right)\right)$$

Now we want 8.2.31

$$\int_0^{\infty} \frac{H(L)}{\sqrt{\sigma^2(L+T)}} * g\left(\frac{x-D(T+L)}{\sqrt{\sigma^2(L+T)}}\right) dL$$

Substituting for H(L) and $g\left(\frac{x-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$ we obtain

$$\frac{\alpha^k}{\Gamma(k)} \int_0^{\infty} \frac{L^{k-1}}{\sqrt{2\pi\sigma^2(L+T)}} \exp-(\alpha L)\, \exp-\tfrac{1}{2}\left(\frac{x-D(L+T)}{\sqrt{\sigma^2(T+L)}}\right)^2 dL$$

8.2.32

Let $z = L+T$

then the expression becomes

$$\frac{\alpha^k}{\Gamma(k)} \int_0^{\infty} \frac{(z-T)^{k-1}}{\sqrt{2\pi\sigma^2 Z}} \exp\left(-\alpha(z-T)\right) \exp -\tfrac{1}{2}\left(\frac{x-Dz}{\sqrt{\sigma^2 z}}\right)^2 dz$$

Noting that $(z-T)^{k-1} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} Z^{k-1-j}\, T^j$

8.2.33

Then We obtain

$$\frac{\alpha^k}{\Gamma(k)} \frac{e^{\alpha T}}{\sqrt{2\pi\sigma^2}} \sum_{j=0}^{k-1} (-T)^j \int_0^{\infty} \binom{k-1}{j} z^{k-3/2-j} e^{-\alpha z} \exp -\tfrac{1}{2}\left(\frac{x-Dz}{\sigma^2 z}\right)^2 dz$$

Applying 8.2.8 to $\int_0^{\infty} \frac{H(L)}{\sqrt{\sigma^2(L+T)}}\, g\left(\frac{x-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) dL$ we obtain

$$= \frac{e^{\left(\alpha T+\frac{Dx}{\sigma^2}\right)}}{\sigma\sqrt{2\pi}} \frac{\alpha^k}{\Gamma(k)} * \sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} \left(2\left(\frac{x}{\theta}\right)^{(k-j-\frac{1}{2})} K_{k-j-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right)\right)$$

8.2.34

And finally we want

$$\int_0^{\infty} H(L) * F\left(\frac{x+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) * e^{\frac{2Dx}{\sigma^2}} dL$$

Substituting for H(L)

$$\int_0^{\infty} \frac{e^{-\alpha L} L^{k-1} \alpha^k}{\Gamma(k)} F\left(\frac{x+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) e^{\frac{2Dx}{\sigma^2}} dL$$

Let $z = L+T$

Then we have

$$\int_0^{\infty} \frac{e^{-\alpha(z-T)}(z-T)^{k-1}}{\Gamma(k)} * \alpha^k * F\left(\frac{x+Dz}{\sqrt{\sigma^2 z}}\right) e^{\frac{2Dx}{\sigma^2}} dz$$

Simplifying we have

$$\frac{\alpha^k \exp(\alpha T)}{\Gamma(k)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-T)^j \int_0^\infty z^{k-1-j} \, e^{-\alpha z} \, F\left(\frac{x+Dz}{\sqrt{\sigma^2 z}}\right) e^{\frac{2Dx}{\sigma^2}} \, dz$$

Applying 8.2.30 b

we have

$$\frac{\alpha^k}{2\sigma \, \Gamma(k)} \sum_{j=0}^{k-1} \sum_{z=1}^{k-j} (-T)^j \frac{(k-j-1)!}{\alpha^z (k-j-z)!} \binom{k-1}{j}$$

$$\left(-\; 2D\left(\frac{x}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) + 2x\left(\frac{x}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right)\right) \qquad 8.2.35$$

SECTION 8.3 MODEL (Q,R) QUADRATIC COST

Let the period of grace be p, where the period of grace is the period for which a backorder bears no cost. Let H(L) be the probability density function of the lead time L

$$H(L) = \frac{\alpha^k L^{k-1} \exp(-\alpha L)}{\Gamma(k)} \qquad 0 \leq L \leq \infty$$

8.3.1.

In the analysis k would take on integer values only.

The conditional distribution of demand , over the lead time L

$$\frac{1}{\sqrt{\sigma^2 L}}\, g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) = \frac{\exp\left[-\frac{1}{2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2\right]}{\sqrt{2\pi\sigma^2 L}}$$

8.3.2

Let R+Y, $0<Y<Q$ be the inventory position at time 0, then if the system is out of stock in the time interval z to z+dz after the re-order point is reached then R+Y was demanded in time z and a demand occured in time dz.

Length of Time of backorder = L-z

Length of time of backorder which bears a cost = L-z-p

Cost of a backorder = C_β(L-z-p)

$$= b_1 + b_2(L-z-p) + b_3(L-z-p)^2 \qquad 8.3.3$$

Applying 4.2.2 expected backorder cost per cycle $G_{27}(Q,R)$

$$= D\int_0^\infty \int_0^Q \int_0^{L-p} \frac{C_\beta(L-z-p)}{\sqrt{\sigma^2 L}} * H(L)\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 L}}\right) dz\, dY\, dL$$

8.3.4

Let $\gamma = x + p$

Hence

$$G_{27}(Q,R) = D\int_0^{\infty}\int_0^{Q}\int_p^{L} \frac{C_B(L-\gamma)}{\sqrt{\sigma^2 L}} H(L)\, g\left(\frac{R+Y+Dp-D\gamma}{\sqrt{\sigma^2 L}}\right) d.\,d\gamma\, dL$$

8.3.5

Making use of 6.2.9 and6.2.11

$$G_{27}(Q,R) = D\int_0^{\infty}\int_0^{Q}\int_0^{L} \frac{C_B(L-\gamma)}{\sqrt{2\pi}\sqrt{\sigma^2 L}} H(L)\exp-\tfrac{1}{2}\left(\frac{R+Y+Dp-D\gamma}{\sqrt{\sigma^2 L}}\right)^2 dv\,d\gamma\, dL$$

8.3.6

Substituting for $C_B(L-\gamma)$ from 8.3.3

$$G_{27}(Q,R) = D\int_0^{\infty}\int_0^{Q}\int_0^{L} (b_1+b_2(L-\gamma)+b_3(L-\gamma)^2)\frac{H(L)}{\sqrt{\sigma^2 L.2\pi}}\exp-\tfrac{1}{2}\left(\frac{R+Y+Dp}{\sqrt{\sigma^2 L}} - \frac{D\gamma}{\sqrt{\sigma^2 L}}\right)^2 dv\,d\gamma\, dL$$

8.3.7

Integrating with respect to γ and making use of the results of section 4.3 especially equation 4.3.6 which states

$$D\int_0^{L} \frac{(b_1+b_2(L-\gamma) + b_3(L-\gamma)^2)}{\sqrt{\sigma^2 L}} \exp-\tfrac{1}{2}\left(\frac{R+Y+Dp-Dv}{\sqrt{\sigma^2 L}}\right)^2 dv$$

8.3.8

$$= \left(b_1 + \frac{b_3\sigma^2 L}{D^2} + \frac{b_3\sigma^2 L}{D^2}\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2 L}}\right)^2 - \frac{b_2\sqrt{\sigma^2 L}}{D}\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$F\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{\sqrt{\sigma^2 L}b_2}{D} - \frac{b_3\sigma^2 L}{D^2}\left(\frac{R+Y+Dp-DL}{\sigma^2 L}\right)\right) g\left(\frac{R+Y+Dp-DL}{\sqrt{\sigma^2 L}}\right)$$

8.3.9

Substituting in-to $G_{27}(Q,R)$ of 8.3.7 and changing the range of integration of Y

$$= \int_0^{\infty} H(L) \int_{R+Dp}^{R+Q+Dp} \left(b_1 + \frac{b_3^3\sigma^2 L}{D^2} + \frac{b_3\sigma^2 L}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2 - \frac{b_2\sqrt{\sigma^2 L}}{D}\left(\frac{x-DL}{\sigma^2 L}\right)\right) *$$

$$F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) \quad dx\,dL$$

$$- \int_0^{\infty} H(L) \int_{R+Dp}^{R+Q+Dp} \left(\frac{\sqrt{\sigma^2 L}\, b_2}{D} - \frac{b_3\sqrt{\sigma^2 L}}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)\right) g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) \quad dx\,dL$$

8.3.10

$$\text{Let} \quad G_{27}(Q,R) = \int_{R+Dp}^{R+Q+Dp} G_{28}(x) \quad dx$$

8.3.11

Hence from 8.3.10

$$G_{28}(x) = \int_0^{\infty} H(L) \left(b_1 + \frac{b_3\sqrt{\sigma^2 L}}{D^2} + \frac{b_3\sigma^2 L}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2 - \frac{b_2(x-DL)}{D}\right) *$$

$$F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) \quad dL$$

$$- \int_0^{\infty} H(L) \left(\frac{b_2\sigma L^{\frac{1}{2}}}{D} - \frac{b_3\sigma^2 L}{D^2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)\right) g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right) \quad dL$$

8.3.12

Substituting for H(L) from 8.3.1 and simplifying

$$G_{28}(x) = \frac{\alpha^k}{\Gamma(k)} \int_0^\infty \left(\frac{b_1 L^{k-1}}{} + \frac{b_3 \sigma^2 L^k}{D^2} + \frac{b_3 x^2 L^{k-1}}{D^2} \right.$$

$$\left. - 2 \frac{b_3 x L^k}{D} + \frac{b_3 L^{k+1}}{D^2} - \frac{b_2 x L^{k-1}}{D} + b_2 L^k \right) e^{-\alpha L} \; F\left(\frac{x - DL}{\sqrt{\sigma^2 L}}\right) \quad dL$$

$$- \frac{\alpha^k}{\Gamma(k)} \int_0^\infty \left(\frac{\sigma L^{k-\frac{1}{2}} b_2}{D} - \frac{b_3 \sigma L^{k-\frac{1}{2}} x}{D^2} + \frac{b_3 \sigma L^{k+\frac{1}{2}}}{D} \right) e^{-\alpha L} \, g\left(\frac{x - DL}{\sqrt{\sigma^2 L}}\right) \quad dL$$

8.3.13

Re-arranging terms

$$G_{28}(x) = \frac{\alpha^k}{\Gamma(k)} \left[L^{k-1} \left(b_1 + \frac{b_3 x^2}{D^2} - \frac{b_2 x}{D} \right) \right.$$

$$\left. + L^k \left(\frac{b_3 \sigma^2}{D^2} - \frac{2 b_3 x}{D} + b_2 \right) + \frac{b_3 L^{k+1}}{D^2} \right] e^{-\alpha L} \quad F\left(\frac{x - DL}{\sqrt{\sigma^2 L}}\right) \quad dL$$

$$- \frac{\alpha^k \sigma}{\Gamma(k)} \int_0^\infty \left[L^{k-\frac{1}{2}} \left(\frac{b_2}{D} - \frac{b_3 x}{D^2} \right) + \frac{b_3 L^{k+\frac{1}{2}}}{D} \right]$$

$$e^{-\alpha L} \quad g\left(\frac{x - DL}{\sqrt{\sigma^2 L}}\right) \, dL$$

8.3.14

Integrating and applying 8.2.18 and 8.2.8 and let

$$Q^2 = 2\alpha^2 \sigma^2 + D^2$$

$$G_{28}'(x) = \frac{e^{\frac{Dx}{\sigma^2}}}{2\sigma} \frac{\alpha^k}{\Gamma(k)\sqrt{2\pi}} \left[\left(\frac{b_1 + b_3 x^2}{D^2} - \frac{b_2 x}{D} \right) \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \right.$$

$$\left(2D \left(\frac{x}{\theta} \right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) + 2x \left(\frac{x}{\theta} \right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right)$$

$$+ \left(\frac{b_3 \sigma^2}{D^2} - \frac{2b_3 x}{D} + b_2 \right) \sum_{z=1}^{k+1} \frac{k!}{\alpha^z (k-z+1)!} \left(2D \left(\frac{x}{\theta} \right)^{k-z+1\frac{1}{2}} \right.$$

$$\left. K_{k-z+1\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) + 2D \left(\frac{x}{\theta} \right)^{k-z+\frac{1}{2}} K_{k-z+1\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right)$$

$$+ \frac{b_3}{D^2} \sum_{z=1}^{k+2} \frac{(k+1)!}{\alpha^z (k-z+2)!} \left(2D \left(\frac{x}{\theta} \right)^{k-z+2\frac{1}{2}} K_{k-z+2\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right.$$

$$\left. \left. + 2x \left(\frac{x}{\theta} \right)^{k-z+1\frac{1}{2}} \cdot K_{k-z+1\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right) \right]$$

$$- \frac{\alpha^k e^{\frac{Dx}{\sigma^2}}}{\Gamma(k)\sqrt{2\pi}} \left[2 \left(\frac{b_2}{D} - \frac{b_3 x}{D^2} \right) \left(\frac{x}{\theta} \right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right.$$

$$\left. + \frac{2b_3}{D} \left(\frac{x}{\theta} \right)^{k+1\frac{1}{2}} K_{k+1\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right] \qquad 8.3.15$$

re-arranging terms

$$G_{28}(x) = \frac{\alpha^k e^{\frac{Dx}{\sigma^2}} b_1}{2\sigma \, \Gamma(k)} \left[\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left(2D \left(\frac{x}{\theta} \right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right. \right.$$

$$\left. \left. + 2x \left(\frac{x}{\theta} \right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{x\theta}{\sigma^2} \right) \right) \right]$$

$$+ \frac{e^{\frac{Dx}{\sigma^2}}}{2\sigma 2\pi \Gamma(k)} \frac{b_2}{D} \left[-\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left(2D\theta \left(\frac{x}{\theta}\right)^{k-z+1\frac{1}{2}} K_{k-z+\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right. \right.$$

$$\left. + 2\theta \left(\frac{x}{\theta}\right)^{k-z+1\frac{1}{2}} K_{k-z-\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right) + \sum_{z=1}^{k+1} \frac{k!}{\alpha^z (k-z+1)!} \left(2D^2 \left(\frac{x}{\theta}\right)^{k-z+1\frac{1}{2}} \right.$$

$$\left. K_{k-z+1\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) + 2\theta D \left(\frac{x}{\theta}\right)^{k-z+1\frac{1}{2}} K_{k-z+\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right)$$

$$\left. - \frac{2\sigma}{\sqrt{2\pi}} \left(2 \left(\frac{x}{\theta}\right)^{k-1\frac{1}{2}} K_{k-1\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right) \right]$$

$$+ \frac{\alpha^k e^{\frac{Dx}{\sigma^2}} b_3}{2\sigma\sqrt{2\pi}\,\Gamma(k)} \left[\sum_{z=0}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left(\frac{2\theta^2}{D} \left(\frac{x}{\theta}\right)^{k-z+2\frac{1}{2}} K_{k-z+\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right. \right.$$

$$\left. + \frac{2}{D^2} \left(\frac{x}{\theta}\right)^{k-z+2\frac{1}{2}} K_{k-z-\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right) + \sum_{z=1}^{k+1} \frac{k!}{\alpha^z (k-z+1)!}$$

$$\left(\frac{2\sigma^2}{D} \left(\frac{x}{\theta}\right)^{k-z+1\frac{1}{2}} K_{k-z+1\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) + \frac{2\theta\sigma^2}{D^2} \left(\frac{x}{\theta}\right)^{k-z+1\frac{1}{2}} K_{k-z+1\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right)$$

$$- \sum_{z=1}^{k+1} \frac{k!}{\alpha^z (k-z+1)!} \left(\frac{4}{D^2} \left(\frac{x}{\theta}\right)^{k-z+2\frac{1}{2}} K_{k-z+2\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right.$$

$$\left. + \frac{4\theta^2}{D} \left(\frac{x}{\theta}\right)^{k-z+2\frac{1}{2}} K_{k+z+1\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right)$$

$$+ \sum_{z=1}^{k+2} \frac{(k+1)!}{\alpha^z (k-z+2)!} \left(\frac{2}{D} \left(\frac{x}{\theta}\right)^{k-z+2\frac{1}{2}} K_{k-z+2\frac{1}{2}} \left(\frac{x\theta}{\sigma^2}\right) \right.$$

$$+ \frac{2\theta}{D^2} \left(\frac{x}{\theta}\right)^{k-z+2\frac{1}{2}} K \left(\frac{x\theta}{\sigma^2}\right) \Bigg)$$

$$+ \frac{4\sigma}{\sqrt{2\pi}\,D} \left(\frac{x}{\theta}\right)^{k+1\frac{1}{2}} K_{k+1\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) + \frac{4\sigma}{\sqrt{2\pi}\,D} \left(\frac{x}{\theta}\right)^{k+1\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{x\theta}{\sigma^2}\right) \Bigg]$$

8.3.16

from 8.3.11

$$G_{27}(Q,R) = \int_{R+Dp}^{R+Q+Dp} G_{28}(x)\,dx$$

where $G_{27}(Q,R)$ is the expected backorder cost per cycle

Expanding

$$G_{27}(Q,R) = \int_{R+Dp}^{\infty} G_{28}(x)\,dx - \int_{R+Q+Dp}^{\infty} G_{28}(x)\,dx$$

8.3.17

Integrating $G_{28}(x)$ with respect to x from z to ∞ and applying 8.2.4

$$\int_{z}^{\infty} G_{28}(x)\,dx$$

8.3.18

$$= \frac{2\alpha^k b_1}{4\,\Gamma(k)} \left[\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left[2D \sum_{j=0}^{k-z} \frac{(k-z+j)!}{j!(k-z-j)!}\right.\right.$$

$$\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \times Q\left(\frac{2\lambda Q}{2(k-z-j+1)}\right) \frac{\gamma}{(\theta\lambda)^{k-z+1}},$$

$$+ \theta \sum_{j=0}^{k-z-1} \frac{(k-z-1-j)!}{j!(k-z-1-j)!} \frac{1}{(\theta\lambda)^{k-z+1}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} Q\left(\frac{2\lambda Q}{2(k-z-j+1)}\right)$$

$$+ \frac{2\alpha^k b_2}{4D\,\Gamma(k)} \left[-\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left[D\theta \sum_{j=0}^{k-z} \frac{(k-z+j)!}{j!(k-z-j)!}.\right.\right.$$

$$\frac{1}{(\theta\lambda)^{k-z+2}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \tilde{Q}\left(\frac{2\lambda Q}{2(k-z-j+1)}\right)$$

$$+\theta \sum_{j=0}^{k-z-1} \frac{(k-z-1+j)!}{j!\,(k-z-1-j)!} \frac{(k-z-j)!}{(\theta\lambda)^{k-z+2}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \tilde{Q}\left(\frac{2\lambda Q}{2(k-z-j+1)}\right)\Bigg]$$

$$+ \sum_{z=1}^{k+1} \frac{k!}{\alpha^{z}(k-z+1)!} \Bigg[D^2 \sum_{j=0}^{k-z+1} \frac{(k-z+1+j)!}{j!\,(k-z+1-j)!} \frac{(k-z+1-j)!}{(\theta\lambda)^{k-z+2}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}$$

$$\tilde{Q}\left(\frac{2\lambda Q}{2(k-z+2-j)}\right) + 2\theta D \sum_{j=0}^{k-z} \frac{(k-z+j)!}{j!\,(k-z-j)!} \frac{(k-z+1-j)!}{(\theta\lambda)^{k-z+2}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \tilde{Q}\left(\frac{2\lambda Q}{2(k-z+2-j)}\right)\Bigg]$$

$$-2\sigma^{-2} \sum_{j=0}^{k} \frac{(k+j)!}{j!\,(k-j)!} \frac{(k-j)!}{(\theta\lambda)^{k+1}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \tilde{Q}\left(\frac{2\lambda Q}{2(k+1-j)}\right)\Bigg]$$

$$+\frac{\alpha^k b_3}{2\,\Gamma(k)} \Bigg[\sum_{z=0}^{k} \frac{(k-1)!}{\alpha^{z}(k-z)!} \Bigg[\frac{2\theta^2}{D} \sum_{j=0}^{k-z} \frac{(k-z+j)!}{j!\,(k-z-j)!} \frac{(k-z+2-j)!}{(\theta\lambda)^{k-z+3}}$$

$$\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \tilde{Q}\left(\frac{2\lambda Q}{2(k-z+3-j)}\right)$$

$$+\frac{2}{D^2} \sum_{j=0}^{k-z-1} \frac{(k-z++j)(k-z+2-j)!}{j!\,(k-z-1-j)!\,(\theta\lambda)^{k-z+3}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \tilde{Q}\left(\frac{2\lambda Q}{2(k-z+3-j)}\right)\Bigg]$$

$$+\sum_{z=1}^{k+1}\frac{k!}{{}^{z}(k-z+1)!}\Bigg[\frac{2\sigma^2}{D}\sum_{j=o}^{k-z+1}\frac{(k-z+1+j)!}{j!\,(k+1-z-j)!}(k-z+1-j)!$$

$$\frac{1}{(\theta\lambda)^{k-z+2}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\,\Phi_Q\left(\frac{2\lambda_Q}{2(k-z+2-j)}\right)$$

$$+\frac{2\theta\sigma^2}{D^2}\sum_{j=o}^{k-z}\frac{(k-z+j)!}{j!(k-z-j)!}(k-z+1-j)!\;\frac{1}{(\theta\lambda)^{k-z+2}}\,\Phi_Q\left(\frac{2\lambda_Q}{2(k-z+2-j)}\right)$$

$$-\sum_{z=1}^{k+1}\frac{k!}{\alpha^{z}(k-z+1)}\Bigg[\frac{4}{D^2}\sum_{j=o}^{k-z+2}\frac{(k-z+2+j)!}{j!(k-z+2-j)!}(k-z+2-j)!$$

$$\frac{1}{(\theta\lambda)^{k-z+3}}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}\,\Phi_Q\left(\frac{2\lambda_Q}{2(k-z-j+3)}\right)$$

$$+\frac{4\theta^2}{D}\sum_{j=o}^{k-z+1}\frac{(k-z+1+j)!}{j!(k-z+1-j)!}(k-z+2-j)!\left(\frac{1}{\theta\lambda}\right)^{k-z+3}\left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j}$$

$$\Phi_Q\left(\frac{2\lambda_Q}{2(k-z-j+3)}\right)\Bigg]$$

$$+\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^{z}(k-z+2)!}\Bigg[\frac{2}{D}\sum_{j=o}^{k-z+2}\frac{(k-z+2+j)!}{j!(k-z+2-j)!}(k-z+2-j)!$$

$$\frac{1}{(\theta\lambda)^{k-z+3}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \mathcal{Q}\left(\frac{2Q}{2(k-z+3-j)}\right) + \frac{2\theta}{D^2} \sum_{j=0}^{k-z+1} \frac{(k-z+2+j)}{j!\,(k-z+1-j)!}$$

$$\left(\frac{1}{\theta\lambda}\right)^{k-z+3} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \mathcal{Q}\left(\frac{2Q}{2(k-z+3-j)}\right)$$

$$+ \frac{2\sigma^2}{D} \sum_{j=0}^{k+1} \frac{(k+1+j)!}{j!\,(k+1-j)!} \frac{(k+1-j)!}{(\theta\lambda)^{k+2}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \mathcal{Q}\left(\frac{2\lambda Q}{2(k+2-j)}\right)$$

$$+ \frac{2\sigma^2\theta}{D} \sum_{j=0}^{k} \frac{(k+j)!}{j!\,(k-j)!} \frac{(k+1-j)!}{(\theta\lambda)^{k+2}} \left(\frac{2\theta}{\lambda\sigma^2}\right)^{-j} \mathcal{Q}\left(\frac{2\lambda Q}{2(k+2-j)}\right)\Bigg]$$

8.3.19

Let $G_{29}(Z)$ equal the b_1 factor of 8.3.19 8.3.20

Let $G_{30}(Z)$ equal the b_2 factor of " 8.3.21

Let $G_{31}(Z)$ equal the b_3 factor of " 8.3.22

Hence from 8.3.17

The expected backorder cost per cycle

$$= b_1 \left(G_{29}(R+Dp) - G_{29}(R+Q+Dp)\right) + b_2 \left(G_{30}(R+Dp) - G_{30}(R+Q+Dp)\right) + b_3 \left(G_{31}(R+Dp) - G_{31}(R+Q+Dp)\right) \quad 8.3.23$$

Hence the expected cost per year

$$= \frac{b_1}{Q}\left(G_{29}(R+Dp) - G_{29}(R+Q+Dp)\right) + \frac{b_2}{Q}\left(G_{30}(R+Dp) - G_{31}(R+Q+Dp)\right) + \frac{b_3}{Q}\left(G_{31}(R+Dp) - G_{31}(R+Q+Dp)\right)$$

8.3.24

From 2.6.14 the expected on hand inventory for a given lead time L

$$D(Q,k) = k\sqrt{\sigma^2 L} + \frac{Q}{2} + B(Q,k) \qquad 8.3.25$$

from 2.6.9

$$B(Q,k) = \frac{1}{Q}\left(\beta(k) - \beta\left(\frac{k+Q}{\sqrt{\sigma^2 L}}\right)\right) \qquad 8.3.26$$

from 2.6.8

where $$\beta(k) = \frac{\sigma^2}{2}\left((1+k^2)\cdot F(k) - k\,g(k)\right) \qquad 8.3.27$$

Hence $$D(Q,k) = k\sqrt{\sigma^2 L} + \frac{Q}{2} + \frac{\sigma^2 L}{2}\left((1+k^2)\cdot F(k) - k\,g(k)\right)$$

$$- \frac{\sigma^2 L}{2}\left(\left(1+\frac{(k+Q)^2}{\sigma^2 L}\right) F\left(\frac{k+Q}{\sqrt{\sigma^2 L}}\right) - \left(k + \frac{Q}{\sqrt{\sigma^2 L}}\right) g\left(\frac{k+Q}{\sqrt{\sigma^2 L}}\right)\right)$$

8.3.28

Remembering that $k = \frac{R-DL}{\sqrt{\sigma^2 L}}$ then substituting for k

then D(Q,R,L)

$$= \frac{Q}{2} + R - DL + \frac{\sigma^2 L}{2}\left(\left(1 + \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. - \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right) - \frac{\sigma^2 L}{2}\left(\left(1+\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. + \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

8.3.29

But $$\frac{\sigma^2 L}{2}\left(\left(1 + \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$= \int_R^\infty \left(\sqrt{\sigma^2 L}\; g\left(\frac{v-DL}{\sqrt{\sigma^2 L}}\right) - (v-DL) * F\left(\frac{v-DL}{\sqrt{\sigma^2 L}}\right)\right) dv$$

8.3.30

$$\text{Hence } D(Q,R) = \frac{Q}{2} + R-DL + \int_R^{R+Q} \left(\sqrt{\sigma^2 L}\; g\left(\frac{V-DL}{\sigma^2 L}\right) - (V-DL) F\left(\frac{V-DL}{\sqrt{\sigma^2 L}}\right)\right) dV$$

where $D(Q,R)$ is the inventory on hand for a given lead time L. Hence expected on hand inventory

$$= \int_0^{\infty} H(L)\, D(Q,R,L)\; dL \qquad 8.3.31$$

$$= \int_0^{\infty} \left(\frac{Q}{2} + R\right) H(L)\, dL - D\int_0^{\infty} LH(L)\, dL + \frac{1}{Q}\int_0^{\infty}\int_R^{R+Q} H(L)\sqrt{\sigma^2 L}$$

$$\left(g\left(\frac{V-DL}{\sigma^2 L}\right) - (V-DL) * F\left(\frac{V-DL}{\sqrt{\sigma^2 L}}\right)\right) dV\, dL \qquad 8.3.32$$

$$\text{Noting that } \int_0^{\infty} L\, e^{-\alpha L} \frac{L^{k-1}\alpha^k}{\Gamma(k)}\, dL = \frac{k}{\alpha} \qquad 8.3.33$$

And also noting that

$$\int_0^{\infty}\int_R^{R+Q} H(L)\sqrt{\sigma^2 L}\left(g\left(\frac{V-DL}{\sqrt{\sigma^2 L}}\right) - (V-DL)\; F\left(\frac{V-DL}{\sqrt{\sigma^2 L}}\right)\right) dV\, dL$$

is the co-efficient of b_2 in 8.3.10

then integrating 8.3.32

We have

$$D(Q,R) = \frac{Q}{2} + R - \frac{Dk}{\alpha} + \frac{1}{Q}\left(G_{30}(R) - G_{30}(R+Q)\right)$$

$$\text{Cost of ordering} = \frac{DS}{Q}, \qquad 8.3.34$$

Inventory costs for model (Q,R)

$$C = \frac{DS}{Q} + hc\cdot D(Q,R) + \frac{b_1}{Q}\left(G_{29}(R+Dp) - G_{29}(R+Q+Dp)\right)$$

$$+ \frac{b_2}{Q}\left(G_{30}(R+Dp) - G_{30}(R+Q+Dp)\right) + \frac{b_3}{Q}\left(G_{31}(R+Dp) - G_{31}(R+Q+Dp)\right)$$

8.3.35

Substituting for D(Q,R) from 8.3.34

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\left(R - \frac{Dk}{\alpha}\right) + \frac{b_1}{Q}\left(G_{29}(R+Dp) - G_{29}(R+Q+Dp)\right)$$

$$+ \frac{b_2}{Q}\left(G_{30}(R+Dp) - G_{30}(R+Q+Dp)\right) + \frac{b_3}{Q}\left(G_{31}(R+Dp) - G_{31}(R+Q+Dp)\right)$$

$$+ \frac{hc}{Q}\left(G_{30}(R) - G_{30}(R+Q)\right) \qquad 8.3.36$$

The corresponding cost when no period of grace p is operating

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\left(R - \frac{Dk}{\alpha}\right) + \frac{b_1}{Q}\left(G_{29}(R) - G_{29}(R+Q)\right)$$

$$+ \frac{(hc + b_2)}{Q}\left(G_{30}(R) - G_{30}(R+Q)\right) + \frac{b_3}{Q}\left(G_{31}(R) - G_{31}(R+Q)\right)$$

$$8.3.37$$

SECTION 8.4 MODEL (Q,R,T).

In this section we derive the inventory costs for Model (Q,R,T) assuming that the cost of a backorder is a quadratic function of the length of time of a backorder, and the lead time is a random variable and follows a gamma distribution.

Probability density function of L, lead time

$$H(L) = \frac{\alpha^k L^{k-1} e^{-\alpha L}}{\Gamma(k)} \qquad L > 0 \qquad 8.4.1.$$

cost of a backorder $C_B(t) = b_1 + b_2 t + b_3 t^2$ 8.4.2.

If the inventory position of the system is R+Y immediately after the review at time t, then the expected backorder costs at time t+L, where L, is the lead time for order placed at time t, from 4.4.1

$$= \frac{1}{Q}\int_0^Q D\int_0^{L_1} D\int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz\, dt\, dY$$

And averaging over the states of L we have

$$\frac{1}{Q}\int_0^\infty D\int_0^Q D\int_0^{L_1} H(L_1)\int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-DZ}{\sqrt{\sigma^2 t}}\right) dz\, dt dY\, dL$$

8.4.3

For the order place at time t+T, the lead time is L_2 with probability $H(L_2)$.

Hence expected backorder cost averaging over states of L_2.

$$= \frac{1}{Q}\int_0^\infty D\int_0^Q D\int_0^{L_2+T}\int_0^t H(L_2)\, \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz dt dY dL$$

8.4.4

Substitute for $C_\beta(t-z)$ from 8.4.2 we have

$$\frac{1}{Q}\int_0^{\infty}\int_0^{Q} D\int_0^{L_1} D\int_0^{t} \frac{H(L_1)\left(b_1+b_2(L-z)+b_3(L-z)^2\right)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz\,dt\,dy\,dL_1 \qquad 8.4.5$$

and

$$\frac{1}{Q}\int_0^{\infty}\int_0^{Q} D\int_0^{L_2+T} D\int_0^{t} \frac{H(L_2)\left(b_1+b_2(L-z)+b_3(L-z)^2\right)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dz\,dt\,dy\,dL_2 \qquad 8.4.6$$

Hence the expected backorder cost per cycle

$$= \frac{1}{Q}\int_0^{\infty}\int_0^{\infty}\int_0^{Q} D\int_{L_1}^{L_2+T} D\int_0^{t} \frac{H(L_1)H(L_2)}{\sqrt{\sigma^2 t}}\left(b_1+b_2(t-z)+b_3(t-z)^2\right) g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right)$$
$$dz\,dt\,dy\,dL_1\,dL_2 \qquad 8.4.7$$

Integrating with respect to dz, dt, dy in that order we obtain the expected backorder cost for a given L_1 and L_2 and this has been evaluated in section 4.4 of chapter 4.

Hence substituting equation 4.4.28 (which is the expected backorder cost for a given L_1 and L_2) into 8.4.7 we obtain

$$\frac{b_1}{Q}\int_0^{\infty}\int_0^{\infty} H(L_1)\,H(L_2)\left(G_1(R-T+L_2)-G_1(R,L_1)-G_1(R+Q,T+L_2)-G_1(R+Q,L_1)\right) dL_1\,dL_2$$

$$+\ \frac{b_2}{Q}\int_0^{\infty}\int_0^{\infty} H(L1)(H(L_2)\left(G_3(R,T+L_2)-G_3(R,L)-G_3(R+Q,T+L_2)+G_3(R+Q,L_1)\right) dL_1\,dL_2$$

$$-\ \frac{b_3}{Q}\int_0^{\infty}\int_0^{\infty} H(L_1)H(L_2)\left(G_{11}(R,T+L_2)-G_{11}(R,L)-G_{11}(R+Q,T+L_2)+G_{11}(R+Q,L_1)\right) dL_1\,dL_2$$

$$+ \frac{b_2}{Q}\int_0^\infty\int_0^\infty H(L_1)H(L_2)\Big(G_3(R,T+L_2)-G_3(R,L_1)-G_3(R+Q,T+L_2)+G_3(R+Q,L_1)\Big)\,dL_1dL_2$$

$$- \frac{b_3}{Q}\int_0^\infty\int_0^\infty H(L_1)H(L_2)\Big(G_{11}(R,T+L_2)-G_{11}(R,L_1)-G_{11}(R+Q,T+L_2)+G_{11}(R+Q,L_1)\Big)\,dL_1dL_2$$

8.4.8

Simplifying we have, the expected backorder cost per cycle to be

$$\frac{b_1}{Q}\int_0^\infty H(L_2)\Big(G_1(R,T+L_2) - G_1(R+Q,T+L_2)\Big)\,dL_2$$

$$- \frac{b_1}{Q}\int_0^\infty H(L_1)\Big(G_1(R,L_1) - G_1(R+Q,L_1)\Big)\,dL_1$$

$$+ \frac{b_2}{Q}\int_0^\infty H(L_2)\Big(G_3(R,T+L_2) - G_3(R+Q,T+L_2)\Big)\,dL_2$$

$$- \frac{b_2}{Q}\int_0^\infty H(L_1)\Big(G_3(R,L_1) - G_3(R+Q,L_1)\Big)\,dL_1$$

$$+ \frac{b_3}{Q}\int_0^\infty H(L_2)\Big(G_{11}(R,T+L_2) - G_{11}(R+Q,T+L_2)\Big)\,dL_2$$

$$- \frac{b_3}{Q}\int_0^\infty H(L_1)\Big(G_{11}(R,t) - G_{11}(R+Q,L_1)\Big)\,dL_1$$

8.4.9

In order to be able to evaluate the expressions involving L_1 we need to evaluate the following integrals

$$G_{32}(R) = \int_0^{\infty} H(L) \quad G_1(R,L) \quad dL \tag{8.4.10}$$

$$G_{32}(R) = \int_0^{\infty} H(L) \quad G_3(R,L) \quad dL \tag{8.4.11}$$

$$G_{34}(R) = \int_0^{\infty} H(L) \quad G_{11}(R,L) \quad dL \tag{8.4.12}$$

From 3.4.8

$$G_1 \ (R,L) = \frac{\sigma^2 L}{2}\left(\left(1 + \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) * g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

Simplifying

$$G_1 \ (R,L) = \left(\frac{\sigma^2 L}{2} + \left(\frac{R^2 - 2DLR + D^2L^2}{2}\right)\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \frac{(R-DL)\sqrt{\sigma^2 L}}{2} g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) \tag{8.4.13}$$

Multiplying by H(L)

Hence $H(L) \ G_1 \ (R,L) =$

$$\frac{\alpha^k e^{-\alpha L}}{2\,\Gamma(k)} \left(\sigma^2 L^k + R^2 L^{k-1} - 2RDL^k + D^2 L^{k+1}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \left(RL^{k-\frac{1}{2}} - DL^{k+\frac{1}{2}}\right) . \sigma . g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) \quad dL \tag{8.4.14}$$

Substituting into 8.4.10 we have

$$G_{32}(R) = \frac{\alpha^k}{2}\frac{1}{\Gamma(k)}\int_0^{\infty}\left(R^2L^{k-1} + (\sigma^2-2RD)L^k + D^2L^{k+1}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) dL$$

$$- \frac{\alpha^k\sigma}{\Gamma(k)}\int_0^{\infty}\left(RL^{k-\frac{1}{2}} - DL^{k+\frac{1}{2}}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) dL$$

8.4.15

Integrating and applying 8.2.18 and 8.2.8 and let $\theta^2 = 2\alpha\sigma^2 + D^2$

$$G_{32}(R) = \frac{\alpha^k}{2\,\Gamma(k)}\Bigg[\frac{e^{2DR/\sigma^2}}{2\sigma}\Bigg(R^2\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^z(k-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)\Bigg)$$

$$+ \left(\left(\sigma^2 - 2DR\right)\sum_{z=0}^{k+1}\frac{k!}{\alpha^z(k+1-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+1\frac{1}{2}}K_{k-z+1\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)\right)$$

$$+\left(D^2\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^z(k+2-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k+5/2-z}K_{k+5/2-z}\left(\frac{R\theta}{\sigma^2}\right)\right.\right.$$

$$\left.\left.+ 2R\left(\frac{R}{\theta}\right)^{k-z+3/2}K_{k-z+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right)\right)\Bigg]$$

$$- \frac{2\alpha^k\sigma}{\sqrt{2\pi}\,\Gamma(k)}\left[e^{\frac{DR}{\sigma^2}}\left(R\left(\frac{R}{\theta}\right)^{k+\frac{1}{2}}K_{k+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) - D\left(\frac{R}{\theta}\right)^{k+3/2}K_{k+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right)\right]$$

8.4.16

From 3.4.176.

$$G_3(R,L) = \left(\frac{D^2L^3}{6} - \frac{\sigma^4 R}{2D^3} - \frac{DL^2R}{2} - \frac{\sigma^2R^2}{4D^2} + \frac{\sigma^2L^2}{4} + \frac{R^2L}{2}\right.$$

$$-\frac{R^3}{6D} - \frac{\sigma^6}{8D^4}\Big) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+ \frac{DL^{5/2}\sigma}{6} - \frac{\sigma L^{3/2} R}{3} + \frac{\sigma L^{\frac{1}{2}} R^2}{6D} + \frac{\sigma^3 L^{3/2}}{2D}$$

$$+ \frac{\sigma^3 L^{\frac{1}{2}} R}{4D^2} + \frac{\sigma^5 L^{\frac{1}{2}}}{4D^3}\Big) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + \frac{\sigma^6}{8D^4} e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

8.4.17

Simplifying

$$G_3(R,L) = \left[-\left(\frac{\sigma^4 R}{2D^3} + \frac{\sigma^2 R^2}{4D^2} + \frac{R^3}{6D} + \frac{\sigma^6}{8D^4}\right) + \frac{R^2 L}{2} + \left(-\frac{DR}{2} + \frac{\sigma^2}{4}\right) L^2 + \frac{D^2 L^3}{6}\right] F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+ \left[\left(\frac{\sigma^5}{4D^3} + \frac{\sigma^3 R}{4D^2} + \frac{\sigma R^2}{6D}\right) L^{\frac{1}{2}} + \left(\frac{\sigma^3}{12D} - \frac{\sigma R}{3}\right) L^{3/2} + \frac{D\sigma L^{5/2}}{6}\right] g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + \frac{\sigma^6}{8D^4} e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

8.4.18

$$H(L)\, G_3(R,L) = \frac{\alpha^k}{\Gamma(k)} \left[-\left(\frac{\sigma^4 R}{2D^3} + \frac{\sigma^2 R^2}{4D^2} + \frac{R^3}{6D} + \frac{\sigma^6}{8D^4}\right) L^{k-1} + \frac{R^2 L^k}{2} + \left(\frac{\sigma^2}{4} - \frac{DR}{2}\right)^{k+1} + \frac{D^2 L^{k+2}}{6}\right] e^{-\alpha L} F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+ \left[\left(\frac{\sigma^5}{4D^3} + \frac{\sigma^3 R}{4D^2} + \frac{\sigma R^2}{6D}\right) L^{k-\frac{1}{2}} + \left(\frac{\sigma^3}{12D} - \frac{\sigma R}{3}\right) L^{k+\frac{1}{2}} + \frac{D\sigma L^{k+3/2}}{6}\right]$$

$$g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + \frac{\sigma^6 L^{k-1} \alpha^k}{8D^4\, \Gamma(k)} e^{\frac{2DR}{\sigma^2}} F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

8.4.19

substituting for $H(L)\, G_3(R,L)$ in 8.4.11 and integrating and applying 8.2.18, 8.2.8, 8.2.6 we have

$$G_{33}(R) = \frac{\alpha^{k} e^{\frac{DR}{\sigma^2}}}{2\sigma \, \Gamma(k)\sqrt{2\pi}} \left[-\left(\frac{\sigma^4 R}{2D^3} + \frac{\sigma^2 R^2}{4D^2} + \frac{R^3}{6D} + \frac{\sigma^6}{8D^4}\right) \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^{z}(k-z)!} \right.$$

$$\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+ \frac{R^2}{2} \sum_{z=1}^{k+1} \frac{k!}{\alpha^{z}(k+1-z)!} \left(2D\left(\frac{R}{\theta}\right)^{k-z+3/2} K_{k-z+3/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+ \left(\frac{\sigma^2}{4} - \frac{DR}{2}\right) \sum_{z=1}^{k+2} \frac{(k+1)!}{\alpha^{z}(k+2-z)!} \left(2D\left(\frac{R}{\theta}\right)^{k-z+5/2} K_{k-z+5/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z+3/2} K_{k-z+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$\left. + \frac{D^2}{6} \sum_{z=1}^{k+3} \frac{(k+2)!}{\alpha^{z}(k+3-z)!} \left(2D\left(\frac{R}{\theta}\right)^{k-z+7/2} K_{k-z+7/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z+5/2} K_{k-z+5/2}\left(\frac{R\theta}{\sigma^2}\right)\right) \right]$$

$$+ \frac{2\, e^{DR/\sigma^2} \alpha^{k}}{\sqrt{2\pi}\, \Gamma(k)} \left[\left(\frac{\sigma^5}{4D^3} + \frac{\sigma^3 R}{4D^2} + \frac{\sigma R^2}{6D}\right)\left(\frac{R}{\theta}\right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \right.$$

$$\left. + \left(\frac{\sigma^3}{12D} - \frac{\sigma R}{3}\right)\left(\frac{R}{\theta}\right)^{k+3/2} K_{k+3/2}\left(\frac{R\theta}{\sigma^2}\right) + \frac{D\sigma}{6}\left(\frac{R}{\theta}\right)^{k+5/2} K_{k+5/2}\left(\frac{R\theta}{\sigma^2}\right) \right]$$

$$+ \frac{\sigma^6 \alpha^{k} e^{DR/\sigma^2}}{8D^4 \, \Gamma(k)\, 2\sqrt{2\pi\sigma^2}} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^{z}(k-z)!} \left(2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}} - 2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} \right) K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \Bigg]$$

8.4.20

From 4.4.30

$$G_{11}(R,L) = \left(\frac{R^4}{12D^3} + \frac{\sigma^2 R^3}{6D^4} + \frac{\sigma^4 R^2}{4D^5} + \frac{\sigma^6 R}{4D^6} + \frac{\sigma^8}{8D^7} - \frac{\sigma^2 R L^2}{2D^2} + \frac{L^2 R^2}{2D} - \frac{RL^3}{3} \right.$$

$$\left. + \frac{L^3\sigma^2}{3D} - \frac{R^3 L}{3D^2} + \frac{L^4 D}{12} \right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- 2\sqrt{\sigma^2 L}\left(\frac{\sigma^2 RL}{24D^3} + \frac{\sigma^4 L}{24D^4} - \frac{R^2 L}{8D^2} - \frac{\sigma^2 L^2}{8D^2} + \frac{L^2 R}{8D} - \frac{L^3}{24} + \frac{R^3}{24D^3} \right.$$

$$\left. + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6} \right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{\sigma^8}{8D^7} e^{\frac{2DR}{\sigma^2}} f\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

8.4.21

Simplifying

$$G_{11}(R,L) = \left[\left(\frac{R^4}{12D^3} + \frac{\sigma^2 R^3}{6D^4} + \frac{\sigma^4 R^2}{4D^5} + \frac{\sigma^6 R}{4D^6} + \frac{\sigma^8}{8D^7} \right) - \frac{R^3}{3D^2} \times L \right.$$

$$\left. + \left(-\frac{\sigma^2 R}{2D^2} + \frac{R^2}{2D} \right) L^2 + \left(\frac{\sigma^2}{3D} - \frac{R}{3} \right) L^3 + \frac{DL^4}{12} \right] F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- 2\sqrt{\sigma^2 L}\left[\left(\frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6} \right) + \left(\frac{\sigma^2 R}{24D^3} + \frac{\sigma^4}{24D^4} - \frac{R^2}{8D^2} \right) L \right.$$

$$\left. + \left(-\frac{\sigma^2}{8D^2} + \frac{R}{8D} \right) L^2 - \frac{L^3}{24} \right] g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{\sigma^8}{8D^7} e^{\frac{2DR}{\sigma^2}} \times F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

8.4.22

Multiplying by H(L)

$$H(L)\,G_{11}\,(R,L) = \frac{\alpha^k\,e^{-\alpha L}}{\Gamma(k)}\left[\left(\frac{R^4}{12D^3}+\frac{\sigma^2R^3}{6D^4}+\frac{\sigma^4R^2}{4D^5}+\frac{\sigma^6R}{4D^6}+\frac{\sigma^8}{8D^7}\right)L^{k-1} - \frac{R^3}{3D^2}\,L^k\right.$$

$$\left.+\left(\frac{R^2}{2D}-\frac{\sigma^2R}{2D^2}\right)L^{k+1}+\left(\frac{\sigma^2}{3D}-\frac{R}{3}\right)L^{k+2}+\frac{DL^{k+3}}{12}\right]\;F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$-\frac{2\,\alpha^k\,e^{-\alpha L}}{\Gamma(k)}\left[\left(\frac{R^3}{24D^3}+\frac{\sigma^2R^2}{12D^4}+\frac{\sigma^2R}{8D^5}+\frac{\sigma^6}{8D^6}\right)L^{k-\frac{1}{2}}+\left(\frac{\sigma^2R}{24D^3}+\frac{\sigma^4}{24D^4}-\frac{R^2}{8D^2}\right)L^{k+\frac{1}{2}}\right.$$

$$\left.+\left(\frac{R}{8D}-\frac{\sigma^2}{8D^2}\right)L^{k+3/2}-\frac{L^{k+5/2}}{24}\right]\cdot g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$-\frac{\sigma^8}{8D^7}\,e^{\frac{2DR}{\sigma^2}}\,\frac{\alpha^kL^{k-1}}{\Gamma(k)}\;F\left(\frac{R+DL}{\sqrt{\sigma^2L}}\right)$$

8.4.23

substituting into 8.4.12 and integrating and applying 8.2.18, 8,2,8, 8.2.21 we have

$$G_{34}(R) = \frac{\alpha^k\,e^{\frac{DR}{\sigma^2}}}{2\sigma\,\Gamma(k)\sqrt{2\pi}}\left[\left(\frac{R^4}{12D^2}+\frac{\sigma^2R^3}{6D^4}+\frac{\sigma^4R^2}{4D^5}+\frac{\sigma^6R}{4D^6}+\frac{\sigma^8}{8D^7}\right)\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^z(k-z)!}\right.$$

$$\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)+2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$-\frac{R^3}{3D^2}\sum_{z=1}^{k+1}\frac{k!}{\alpha^z(k+1-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+3/2}+2R\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+\left(\frac{R^2}{2D}-\frac{\sigma^2R}{2D^2}\right)\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^z(k+2-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+5/2}K_{k-z+5/2}\left(\frac{R\theta}{\sigma^2}\right)+2R\left(\frac{R}{\theta}\right)^{k-z+3/2}K_{k-z+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+\left(\frac{\sigma^2}{3D}-\frac{R}{3}\right)\sum_{z=1}^{k+3}\frac{(k+2)!}{\alpha^{z}(k+3-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+7/2}K_{k-z+7/2}\left(\frac{R\theta}{\sigma^2}\right)+2R\left(\frac{R}{\theta}\right)^{k-z-5/2}K_{k-z+5/2}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$\left.+\frac{D}{12}\sum_{z=1}^{k+4}\frac{(k+3)!}{\alpha^{z}(k+4-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+9/2}K_{k-z+9/2}\left(\frac{R\theta}{\sigma^2}\right)+2R\left(\frac{R}{\theta}\right)^{k-z-7/2}K_{k-z+7/2}\left(\frac{R\theta}{\sigma^2}\right)\right)\right]$$

$$-\frac{2\alpha^k e^{\frac{DR}{\sigma^2}}}{\Gamma(k)\sqrt{2\pi}}\left[\left(2\left(\frac{R}{\theta}\right)^{k+\frac{1}{2}}K_{k+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)\times\left(\frac{R^3}{24D^3}+\frac{\sigma^2R^2}{12D^4}+\frac{\sigma^2R}{8D^5}+\frac{\sigma^6}{8D^6}\right)\right.$$

$$+\left(\frac{\sigma^2R}{24D^3}+\frac{\sigma^4}{24D^4}-\frac{R^2}{8D^2}\right)\times\left(2\left(\frac{R}{\theta}\right)^{k+3/2}K_{3/2+k}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

8.4.24

$$\left.+2\left(\frac{R}{8D}-\frac{\sigma^2}{8D^2}\right)\left(\frac{R}{\theta}\right)^{k+5/2}K_{k+5/2}\left(\frac{R\theta}{\sigma^2}\right)-\frac{2}{24}\left(\frac{R}{\theta}\right)^{k+7/2}K_{k+7/2}\left(\frac{R\theta}{\sigma^2}\right)\right]$$

$$-\frac{\alpha^k}{\Gamma(k)}\cdot\frac{\sigma^8}{8D^7}\frac{1}{2\sigma\sqrt{2\pi}}\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^{z}(k-z)!}\left(-2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)+2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^3}\right)\right)$$

8.4.25

Also to be able to evaluate the expressions involving L_2 we need to evaluate the following integrals.

$$G_{35}(R,T)=\int_0^\infty H(L)\,G_1(R,L+T)\,dL \qquad 8.4.26$$

$$G_{36}(R,T)=\int_0^\infty H(L)\,G_3(R,L+T)\,dL \qquad 8.4.27$$

$$G_{37}(R,T)=\int_0^\infty H(L)\,G_{11}(R,L+T)\,dL \qquad 8.4.28$$

from 8.4.13

$$G_1(R,L+T) = \left(\frac{\sigma^2}{2}(L+T) + \frac{(R^2 - 2DR(L+T) + D^2(L+T)^2}{2}\right)$$

$$F\left(\frac{R - D(L+T)}{\sqrt{\sigma^2(T+L)}}\right) - \frac{\sqrt{\sigma^2(L+T)}}{2} \quad (R-D(L+T))\,g\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.4.29

Simplifying

$$= \frac{1}{2}\left(\sigma^2 L + \sigma^2 T + R^2 - 2DRT - 2DRL + D^2L^2 + 2D^2LT\right.$$

$$\left.+ D^2T^2\right) F\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(T+L)}}\right) - \frac{\sqrt{\sigma^2(L+T)}}{2}\left(R-D(L+T)\right) g\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.4.30

$$H(L)G_1(R,L+T) = \frac{\alpha^k}{2\,\Gamma(k)}\left(\left(\sigma^2T+R^2-2DRT+D^2T^2\right)L^{k-1} + \left(\sigma^2-2DR+2D^2T\right)L^k\right.$$

$$\left.+ D^2L^{k+1}\right) F\left(\frac{R-D(L+T)}{\sqrt{\sigma^2\,(T+L)}}\right)$$

$$- \frac{\alpha^k e^{-\alpha L}}{2\,\Gamma(k)}\sqrt{\sigma^2(L+T)}\left(RL^{k-1} - DL^k - DTL^{k-1}\right) g\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(T+L)}}\right)$$

8.4.31

Substituting into 8.4.23 and integrating and applying 8.2.34, 8.2.31 we have

$$G_{35}(R,T) = \frac{e^{\left(\frac{\alpha T+DR}{\sigma^2}\right)}\alpha^k}{2\sigma\,\Gamma(k)\sqrt{2\pi}}\left[\left(\sigma^2T + R^2 - 2DRT + D^2T^2\right)\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}\right.$$

$$\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\alpha^z(k-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+ (\sigma^2 - 2DR + 2D^2T) \sum_{j=o}^{k} (-T)^j \binom{k}{j} \sum_{z=1}^{k+1-j} \frac{(k-j)!}{\alpha^z (k+1-j-z)!} \left(2D\left(\frac{R}{\phi}\right)^{k-j-z+3/2} K_{k-j-z+3/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\phi}\right)^{k-j-z+\frac{1}{2}} K_{k-z-j+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+ D^2 \sum_{j=o}^{k+1} (-T)^j \binom{k-1}{j} \sum_{z=1}^{k+2-j} \frac{(k+1-j)!}{\alpha^z (k+2-j-z)!} \left(2D\left(\frac{R}{\phi}\right)^{k-j-z+5/2} K_{k-j-z+5/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-j-z+3/2} K_{k-z-j+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right)\Bigg]$$

$$- \frac{2\sigma \alpha^k e^{\left(\alpha T + \frac{DR}{\sigma^2}\right)}}{\Gamma(k) \sqrt{2\pi}} \left[(R-DT) \sum_{j=o}^{k-1} (-T)^j \binom{k-1}{j} \left(\frac{R}{\theta}\right)^{k+\frac{1}{2}-j} K_{k+\frac{1}{2}-j}\left(\frac{R\theta}{\sigma^2}\right)\right.$$

$$\left. - D \sum_{j=o}^{k} (-T)^j \binom{k}{j} \left(\frac{R}{\theta}\right)^{k+3/2-j} K_{k+3/2-j}\left(\frac{R\theta}{\sigma^2}\right)\right]$$

8.4.32

rom 8.4.18, substituting L+T for L, we have

$$G_3(R, L+T) = \left[-\left(\frac{\sigma^4 R}{2D^3} + \frac{\sigma^2 R^2}{4D^2} + \frac{R^3}{6D} + \frac{\sigma^6}{8D^4}\right) + \frac{R^2 T}{2} + \frac{R^2 L}{2} + \left(\frac{\sigma^2}{4} - \frac{DR}{2}\right)\right.$$

$$\left. \sum_{i=o}^{2} \binom{2}{j} T^i L^{2-i} + \frac{D^2}{6} \sum_{i=o}^{3} T^i L^{3-i} \right] F\left(\frac{R-D(L+T)}{\sqrt{\sigma^2 (L+T)}}\right)$$

$$+ \sqrt{(T+L)} \left[\left(\frac{\sigma^5}{4D^3} + \frac{\sigma^3 R}{4D^2} + \frac{\sigma R^2}{6D}\right) + \left(\frac{\sigma^3}{12D} - \frac{\sigma R}{3}\right) T + \left(\frac{\sigma^3}{12D} - \frac{\sigma R}{3}\right) L\right.$$

$$+\frac{D\sigma}{6}\sum_{i=o}^{2}\binom{2}{i}T^{i}L^{2-i}\Bigg]g\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)+\frac{\sigma^6}{8D^4}e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.4.33.

Multiplying by H(L) from 8.4.1

$H(L)G_3(R,L)$

$$=\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\Bigg[-\left(\frac{\sigma^4 R}{2D^3}+\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}+\frac{\sigma^6}{8D^4}-\frac{RT}{2}\right)L^{k-1}$$

$$+\frac{R^2L^k}{2}+\left(\frac{\sigma^2}{4}-\frac{DR}{2}\right)\sum_{i=o}^{2}\binom{2}{i}T^iL^{1-i+k}+\frac{D^2}{6}\sum_{i=o}^{3}\binom{3}{i}T^iL^{2-i+k}\Bigg]*$$

$$F\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

$$+\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\sqrt{(T+L)}\Bigg[\left(\frac{\sigma^5}{4D^3}+\frac{\sigma^3R}{4D^2}+\frac{\sigma R^2}{6D}+\frac{\sigma^3T}{12D}-\frac{\sigma RT}{3}\right)L^{k-1}+\left(\frac{\sigma^3}{12D}-\frac{\sigma R}{3}\right)L^k$$

$$+\frac{D\sigma}{6}\sum_{i=o}^{2}\binom{2}{i}T^iL^{1-i+k}\Bigg]g\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)+\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}e^{\frac{2DR}{\sigma^2}}F\left(\frac{R+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.4.34

Substituting into $G_{36}(R,T)$ equation 8.4.24, integrating and applying 8.2.31, 8.2.34, 8.2.35

$$G_{36}(R,T)=\frac{\alpha^k e^{\left(\alpha T+\frac{DR}{\sigma^2}\right)}}{\Gamma(k)\;2\sigma\sqrt{2\pi}}\Bigg[-\left(\frac{\sigma^4R}{2D^3}+\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}+\frac{\sigma^6}{8D^4}-\frac{RT}{2}\right)*$$

$$\sum_{j=o}^{k-1}(-T)^j\binom{k-1}{j}\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\alpha^z(k-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)+2R\left(\frac{R}{\theta}\right)^{k-j-z-\frac{1}{2}}*\right.$$

$$\left.K_{k-j-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)+\frac{R^2}{2}\sum_{j=o}^{k}(-T)^j\binom{k}{j}\sum_{z=1}^{k+1-j}\frac{(k-j)!}{\alpha^z(k+1-j-z)!}$$

$$\left(2D\left(\frac{R}{\phi}\right)^{k-j-z+3/2} K_{k-j-z+3/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+\left(\frac{\sigma^2}{4}-\frac{DR}{2}\right)\sum_{i=o}^{2}\binom{2}{i}T^i\sum_{j=o}^{k+1-i}(-T)^j\binom{k+1-i}{j}\sum_{z=1}^{k+2-i-j}\frac{(k+1-i-j)!}{\alpha^z(k+2-j-z)!}$$

$$\left(2D\left(\frac{R}{\phi}\right)^{k-i-z+5/2-j} K_{k-i-z+5/2-j}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-i-z+3/2-j} K_{k-i-z+3/2-j}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$+\frac{D^2}{6}\sum_{i=o}^{3}\binom{3}{i}T^i\sum_{j=o}^{k-i+2}(-T)^j\binom{k-i+2}{j}\sum_{z=1}^{k-i+3-j}\alpha^z\frac{(k+2-i-j)}{(k-i+3-z-j)!}$$

$$\left.\left(2D\left(\frac{R}{\theta}\right)^{k-i-z+7/2-j} K_{k-i-z+7/2-j}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-i-z+5/2-j} K_{k-i-z+5/2-j}\left(\frac{R\theta}{\sigma^2}\right)\right)\right]$$

$$+\frac{2\alpha^k e^{\left(\alpha T+\frac{DR}{\sigma^2}\right)}}{2\pi}\left[\left(\frac{\sigma^5}{4D^3}+\frac{\sigma^3 R}{4D^2}+\frac{\sigma R^2}{6D}+\frac{\sigma^3 T}{12D}-\frac{\sigma RT}{3}\right)\sum_{j=o}^{k-1}(-T)^j\binom{k-1}{j}\right.$$

$$\left(\frac{R}{\phi}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + \left(\frac{\sigma^3}{12D}-\frac{\sigma R}{3}\right)\sum_{j=o}^{k}(-T)^j\binom{k}{j}\left(\frac{R}{\theta}\right)^{k-j+3/2} *$$

$$K_{k-j+3/2}\left(\frac{R\theta}{\sigma^2}\right)$$

$$\left.+\frac{2D\sigma}{6}\sum_{i=o}^{2}\binom{2}{i}T^i\left(\frac{R}{\theta}\right)^{k-i+3/2} K_{k-i+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right]$$

$$+\frac{\alpha^k}{\Gamma(k)}\frac{e^{\left(\alpha T+\frac{DR}{\sigma^2}\right)}}{\sigma\sqrt{2\pi}}\sum_{j=o}^{k-1}(-T)^j\binom{k-1}{j}\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\alpha^z(k-j-z)!}\left(D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right.$$

$$+ \quad R\left(\frac{R}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\Bigg)\Bigg) \qquad 8.4.35$$

From 8.4.22 substituting L+T for L and re-arranging terms.

$$G_{11}(R,L+T) = \left[\left(\frac{R^4}{12D^3} + \frac{\sigma^2R^3}{6D^4} + \frac{\sigma^4R^2}{4D^5} + \frac{\sigma^6R}{4D^6} + \frac{\sigma^8}{8D^7} - \frac{R^3T}{3D^2}\right) - \frac{R^3L}{3D^2}\right.$$

$$+\left(\frac{R^2}{2D^2} - \frac{\sigma^2R}{2D^2}\right)\sum_{i=o}^{2}\binom{2}{i}T^i L^{2-i} + \left(\frac{\sigma^2}{3D} - \frac{R}{3}\right)\sum_{i=o}^{3}\binom{3}{i} T^i L^{3-i}$$

$$\left. + \frac{D}{12}\sum_{i=o}^{4}\binom{4}{i} T^i L^{4-i}\right] F\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

$$- 2\sigma(T+L)^{\frac{1}{2}}\left[\left(\frac{R^3}{24D^3} + \frac{\sigma^2R^2}{12D^4} + \frac{\sigma^2R}{8D^5} + \frac{\sigma^6}{8D^6} + \frac{\sigma^2RT}{24D^3} + \frac{\sigma^4T}{24D^4} - \frac{R^2T}{8D^2}\right)\right.$$

$$+\left(\frac{\sigma^2R}{24D^3} + \frac{\sigma^4}{24D^4} - \frac{R^2}{8D^2}\right) L + \left(\frac{R}{8D} - \frac{\sigma^2}{8D^2}\right)\sum_{i=o}^{2}\binom{2}{i} T^iL^{2-i}$$

$$\left. - \frac{1}{24}\sum_{i=o}^{3}\binom{3}{i}T^iL^{3-i}\right] g\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$- \frac{\sigma^8}{8D^7} * e^{2DR/\sigma^2} * F\left(\frac{R+D(L+T)}{\sqrt{\sigma^2(T+L)}}\right) \qquad 8.4.36$$

$$H(L)\, G_{11}(R,T+L) = \frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\left[\left(\frac{R^4}{12D^3} + \frac{\sigma^2R^3}{6D^4} + \frac{\sigma^4R^2}{4D^5} + \frac{\sigma^6R}{4D^6} + \frac{\sigma^8}{8D^7} - \frac{R^3T}{3D^2}\right)L^k\right.$$

$$- \frac{R^3L^k}{3D^2} + \left(\frac{R^2}{2D^2} - \frac{\sigma^2R}{2D^2}\right)\sum_{i=o}^{2}\binom{2}{i} T^iL^{k-i+1}\left(\frac{\sigma^2}{3D} - \frac{R}{3}\right)\sum_{i=o}^{3}\binom{3}{i}T^iL^{2-i+k}$$

$$\left. + \frac{D}{12}\sum_{i=o}^{4}\binom{4}{i} T^iL^{3-i+k}\right] F\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$- \frac{2\sigma \; \alpha^k e^{-\alpha L}}{\Gamma(k)} (T+L)^{\frac{1}{2}} \Bigg[\left(\frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6} + \frac{\sigma^2 RT}{24D^3} + \frac{\sigma^4 T}{24D^4} - \frac{R^2 T}{8D^2} \right) L^{k-1}$$

$$+ \left(\frac{\sigma^2 R}{24D^3} + \frac{\sigma^4}{24D^4} - \frac{R^2}{8D^2} \right) L^k + \left(\frac{R}{8D} - \frac{\sigma^2}{8D^2} \right) \sum_{i=0}^{2} \binom{2}{i} T^i L^{1-i+k}$$

$$- \frac{1}{24} \sum_{i=0}^{3} \binom{3}{i} T^i L^{2-i+k} \Bigg] * g\left(\frac{R-D(T+L)}{\sqrt{\sigma^2 (T+L)}} \right)$$

$$- \frac{\sigma^8}{8D^7} L^{k-1} e^{\frac{2DR}{\sigma^2}} \frac{\alpha^k e^{-\alpha L}}{\Gamma(k)} * F\left(\frac{R+D(T+L)}{\sqrt{\sigma^2 (T+L)}} \right)$$

8.4.37

Substituting into 8.4.28, and integrating, and applying 8.2.34, 8.2.31 and 8.2.35 we have

$$G_{37}(R,T) = \frac{\alpha^k e^{\left(\alpha T + \frac{DR}{\sigma^2}\right)}}{2\sigma \; \Gamma(k) \sqrt{2\pi}} \Bigg[\left(\frac{R^4}{12D^3} + \frac{\sigma^2 R^3}{6D^4} + \frac{\sigma^4 R^2}{4D^5} + \frac{\sigma^6 R}{4D^6} + \frac{\sigma^8}{8D^7} - \frac{R^3 T}{3D^2} \right)$$

$$\sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} \sum_{z=1}^{k-j} \frac{(k-1-j)!}{\alpha^z (k-j-z)!} \left(2D \left(\frac{R}{\theta} \right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}} \left(\frac{R\theta}{\sigma^2} \right) \right.$$

$$\left. + \; 2R \left(\frac{R}{\theta} \right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}} \left(\frac{R\theta}{\sigma^2} \right) \right)$$

$$- \frac{R^3}{3D^2} \sum_{j=0}^{k} (-T)^j \binom{k}{j} \sum_{z=1}^{k-j} \frac{(k-j)!}{\alpha^z (k+1-j-z)!} \left(2D \left(\frac{R}{\theta} \right)^{k-j-z+3/2} K_{K-j-z+3/2} \left(\frac{R\theta}{\sigma^2} \right) \right.$$

$$\left. + \; 2R \left(\frac{R}{\theta} \right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}} \left(\frac{R\theta}{\sigma^2} \right) \right) + \left(\frac{R^2}{2D^2} - \frac{\sigma^2 R}{2D^2} \right) \sum_{i=0}^{2} \binom{2}{i} T^i \sum_{j=0}^{k-i+1} (-T)^j \binom{k-i+1}{j}$$

$$\sum_{z=o}^{k-i+2} \frac{(k-i+1-j)!}{\alpha^z (k-i+2-j-z)!} \left(2D\left(\frac{R}{\theta}\right)^{k-i-j-z+3/2} K_{k-i-j-z+3/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-i-j-z+\frac{1}{2}} K_{k-i-j-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right) + \left(\frac{\sigma^2}{3D} - \frac{R}{3}\right) \sum_{i=o}^{3} \binom{3}{i} T^i \sum_{j=o}^{k-i+2} (-T)^j \Big(k$$

$$\sum_{z=o}^{k-i+3} \frac{(k-i+2-j)!}{\alpha^z (k-i+3-j-z)!} \left(2D\left(\frac{R}{\theta}\right)^{k-i-j-z+3/2} K_{k-i-j-z+3/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-i-j-}\right.$$

$$\left. K_{k-i-j-z+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right) + \frac{D}{12} \sum_{i=o}^{4} \binom{4}{i} T^i \sum_{j=o}^{k-i+3} (-T)^j \binom{k-i+3}{j} \sum_{z=o}^{k-i+3}$$

$$\frac{(k-i+3-j)!}{\alpha^z (k-i+4-j-z)!} \left(2D\left(\frac{R}{\theta}\right)^{k-i-j-z+7/2} K_{k-i-j-z+7/2}\left(R\theta/\sigma^2\right) + 2R\left(\frac{R}{\theta}\right)^{k-i-j-z+5/2} K_{k-i-j-z+5/2}\left(R\theta/\sigma^2\right)\right)\Bigg]$$

$$- \frac{\sqrt{2}\sigma \alpha^k e^{\left(\alpha T + \frac{DR}{\sigma^2}\right)}}{\sqrt{\pi}\ \Gamma(k)} \Bigg[\left(\frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6} + \frac{\sigma^2 RT}{24D^3} + \frac{\sigma^4 T}{24D^4} - \frac{R^2 T}{8D^2}\right)$$

$$\sum_{j=o}^{k-1} (-T)^j \binom{k-1}{j} \left(2\left(\frac{R}{\theta}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + \left(\frac{\sigma^2 R}{24D^3} + \frac{\sigma^4}{24D^4} - \frac{R^2}{8D^2}\right)\right.$$

$$\sum_{j=o}^{k} (-T)^j \binom{k}{j} \left(2\left(\frac{R}{\theta}\right)^{k-j+3/2} K_{k-j+3/2}\left(\frac{R\theta}{\sigma^2}\right) + 2\left(\frac{R}{8D} - \frac{\sigma^2}{8D^2}\right) \sum_{i=o}^{2} \binom{2}{i} T^i\right.$$

$$\sum_{j=o}^{k-i+1} \binom{k-i+1}{j} (-T)^j \left(\frac{R}{\theta}\right)^{k-j+5/2-i} K_{k-j+5/2-i}\left(\frac{R\theta}{\sigma^2}\right)$$

$$-\frac{2}{24}\sum_{i=0}^{3}\binom{3}{i}T^{i}\sum_{j=0}^{k-i+2}\binom{k-i+2}{j}(-T)^{j}\left(\frac{R}{\theta}\right)^{k-j+7/2}K_{k-j+7/2}\left(\frac{R\theta}{\sigma^{2}}\right)\Bigg]$$

$$-\frac{\sigma^{8}}{8D^{7}}\,\frac{\alpha^{k}\left(\alpha T+\frac{DR}{\sigma^{2}}\right)}{2\sigma\,\Gamma(k)\sqrt{2\pi}}\left[\sum_{j=0}^{k-1}(-T)^{j}\binom{k-1}{j}\sum_{z=1}^{k-i}\frac{(k-1-j)!}{\alpha^{z}(k-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^{2}}\right)+2R\left(\frac{R}{\theta}\right)^{k-j-z-\frac{1}{2}}K_{k-j-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^{2}}\right)\right)\right]$$

8.4.38

Hence the expected backorder cost per cycle from 8.4.8

$$=\frac{b_1}{Q}\Big(G_{35}(R,T)-G_{32}(R)-G_{35}(R+Q,T)+G_{32}(R+Q)\Big)$$

$$+\frac{b_2}{Q}\Big(G_{36}(R,T)-G_{33}(R)-G_{36}(R+Q,T)+G_{33}(R+Q)\Big)$$

$$+\frac{b_3}{Q}\Big(G_{37}(R,T)-G_{34}(R)-G_{37}(R+Q,T)+G_{34}(R+Q)\Big)$$

8.4.39

Probability of a stockout, POR from 3.4.5

$$POR=\frac{DT}{Q}\left(1-F\left(\frac{Q-DT}{\sqrt{\sigma^{2}T}}\right)\right)+F\left(\frac{Q-DT}{\sqrt{\sigma^{2}T}}\right)-\frac{\sqrt{\sigma^{2}T}}{Q}\cdot g\left(\frac{Q-DT}{\sqrt{\sigma^{2}T}}\right)$$

8.4.40

which is independent of lead time.

The expected on hand inventory for fixed lead times L_1, L_2 from 3.4.24

$$D(Q,R,T)=\frac{Q}{2}+\left(R-DL_1-\frac{DT}{2}+B(Q,R,T)\right)$$

8.4.41

where $B(Q,R,T) = \frac{1}{QT}\Big(G_3(R,L_2+T) - G_3(R,L_1)$

$$- G_3(R+Q,L_2+T) + G_3(R+Q,L_1)\Big) \qquad 8.4.42$$

Substituting L_1 for L and L_2+T for $L+T$ in $B(Q,R,T)$, and averaging $D(Q,R,T)$ over the states of L_1 and L_2 we have

$$\int_0^\infty\int_0^\infty D(Q,R,T)\,H(L_1)\,H(L_2)\,dL_1\,dL_2 \qquad 8.4.43$$

Substituting for $D(Q,R,T)$, integral

$$= \int_0^\infty\int_0^\infty\Big[\Big(\frac{Q}{2} + R - DL_1 - \frac{DT}{2}\Big) + \frac{1}{QT}\Big(G_3(R,L_2+T) - G_3(R,L_1) - G_3(R+Q,L_2+T)$$

$$+ G_3(R+Q,L_1)\Big)\,H(L_1)\,H(L_2)\Big]\,dL_1\,dL_2 \qquad 8.4.44$$

Noting that $G_{36}(R,T) = \int_0^\infty G_3(R,L_2+T)\,H(L_2)\,dL_2$

and $G_{33}(R) = \int_0^\infty G_3(R,L_1)\,H(L_1)\,dL_1$

then on hand inventory, $D(Q,R,T)$,

$$= \frac{Q}{2} + R - \frac{Dk}{\alpha} + \Big(G_{36}(R,T) - G_{33}(R) - G_{36}(R+Q,T) + G_{33}(R+Q)\Big)$$

Number of cycles $= \frac{1}{T}$

Hence inventory cost for Model (Q,R.T) when lead time is continuous and the cost of a backorder is a quadratic function of the length of a backorder is

$$C = \frac{R_c}{T} + \frac{S.POR}{T} + h_c\Big(\frac{Q}{2} + R - \frac{Dk}{\alpha} - \frac{DT}{2}\Big) +$$

$$\frac{b_1}{QT}\Big(G_{35}(R,T) - G_{32}(R) - G_{35}(R+Q,T) + G_{32}(R+Q)\Big)$$

$$+\frac{(h_c + b_2)}{QT}\Big(G_{36}(R,T) - G_{33}(R) - G_{36}(R+QT) + G_{33}(R+Q)\Big)$$

$$\frac{b_3}{QT}\Big(G_{37}(R,T) - G_{34}(R) - G_{37}(R+Q,T) + G_{34}(R+Q)\Big) \qquad 8.4.46$$

SECTION 8.5 Model (M,T)

We shall derive the cost equation for model (M,T) by averaging the cost equations for fixed lead times over the states of the lead times.

The cost equations when the cost of a backorder is a quadratic function of the length of time of a backorder is given by 4.5.7, for fixed lead times. From 4.5.7 the cost equations for model (M,T) excluding the cost of stockout dependent only on the number of stockouts.

$$C = \frac{Rc + S}{T} + c\left(M - DL - \frac{DT}{2}\right) + \frac{b_1}{T}\left(G_5(M, T+L) - G_5(M, L)\right)$$

$$+ \frac{(hc + b_2)}{T}\left(G_2(M, T+L_2) - G_2(M, L)\right) + \frac{b_3}{T}\left(G_{12}(M, T+L_2)\right.$$

$$\left. - G_{12}(M, L_1)\right) \qquad 8.5.1.$$

Let

$$G_{38}(R) = \int_0^\infty H(L)\, G_5(R, L)\, dL \qquad 8.5.2$$

$$G_{39}(R) = \int_0^\infty H(L)\, G_2(R, L)\, dL \qquad 8.5.3$$

$$G_{40}(R) = \int_0^\infty H(L)\, G_{12}(R, L)\, dL \qquad 8.5.4$$

$$G_{41}(R, T) = \int_0^\infty H(L)\, G_5(R, T+L)\, dL \qquad 8.5.5$$

$$G_{42}(R, T) = \int_0^\infty H(L)\, G_2(R, T+L)\, dL \qquad 8.5.6$$

$$G_{43}(R, T) = \int_0^\infty H(L)\, G_{12}(R, T+L)\, dL \qquad 8.5.7$$

From 3.5.2

$$G_5(M, L) = \sqrt{\sigma^2 L}\; g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - (M - DL) \times F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) \qquad 8.5.8$$

Multiplying by H(L) where $H(L) = \dfrac{\alpha^k e^{-\alpha L} L^{k-1}}{\Gamma(k)}$

Hence

$$H(L)\, G_5\, (M,L) = \frac{\alpha^k e^{-\alpha L}}{\Gamma(k)} \left[\sigma L^{k-\frac{1}{2}}\, g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \left(ML^{k-1} - DL^k\right) * F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) \right] \qquad 8.5.9$$

Hence $\int_0^\infty H(L)\, G_5(M,L)\, dL$, applying 8.2.8 and 8.2.18

$$G_{38}(k) = \frac{\alpha^k \sigma\, e^{\frac{DM}{\sigma^2}}}{\sqrt{2\pi}} \left[2\left(\frac{M}{\theta}\right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \right]$$

$$- \frac{\alpha^k e^{\frac{DM}{\sigma^2}}}{2\sigma\sqrt{2\pi}\,\Gamma(k)} \left[M \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left(2D\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \right) + D \sum_{z=1}^{k+1} \frac{k!}{\alpha^z (k+1-z)!} \left(2D\left(\frac{M}{\theta}\right)^{k-z+3/2} K_{k-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \right) \right]$$

8.5.10

From 3.5.6

$$G_2(M,L) = \left(\frac{\sigma^4}{4D^3} + \frac{DL^2}{2} + \frac{\sigma^2}{2D^2} - L + \frac{M^2}{2D}\right) * F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+ \frac{1}{2}\left(\sigma L^{3/2} - \frac{\sigma^3 L^{\frac{1}{2}}}{D^2} - \frac{\sigma L^{\frac{1}{2}} M}{D} \right) g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{\sigma^4}{4D^3} e^{\frac{2DM}{\sigma^2}} * F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right) \qquad 8.5.11$$

Simplifying

$$G_2(M,L) = \left[\left(\frac{\sigma^4}{4D^3} + \frac{\sigma^2}{2D^2} + \frac{M^2}{2D}\right) - L + \frac{DL^2}{2}\right] F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+ \frac{1}{2}\left(- L^{\frac{1}{2}} \left(\frac{\sigma^3}{D^2} + \frac{\sigma M}{D}\right) + \sigma L^{3/2}\right) g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{\sigma^4}{4D^3} e^{\frac{2DM}{\sigma^2}} \times F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

8.5.12

Multiplying by $H(L) = \dfrac{\alpha^k e^{-\alpha L}}{\Gamma(k)}$

$$H(L)\, G_2(M,L) = \frac{e^{-\alpha L}\alpha^k}{\Gamma(k)}\left[\left(\frac{\sigma^4}{4D^3} + \frac{\sigma^2}{2D^2} + \frac{M^2}{2D}\right) L^{k-1} - L^k + \frac{DL^{k+1}}{2}\right] F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{1}{2}\,\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\left[\left(\frac{\sigma^3}{D^2} + \frac{\sigma M}{D}\right) L^{k-\frac{1}{2}} - \sigma L^{k+\frac{1}{2}}\right] g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{\sigma^4}{4D^3} \times e^{\frac{DM}{\sigma^2}} \times F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right) \times \frac{e^{-\alpha L} L^k \alpha^k}{\Gamma(k)}$$

8.5.13

Hence

$\int_0^\infty H(L)\, G_2(M,L)\, dL$ applying 8.2.8, 8.2.16, 8.2.30

we have

$$G_{39}(R) = \frac{\alpha^k e^{\frac{DM}{\sigma^2}}}{2\sigma\sqrt{2\pi}\,\Gamma(k)}\left[\left(\frac{\sigma^4}{4D^3} + \frac{\sigma^2}{2D^2} + \frac{M^2}{2D}\right)\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right.\right.$$

$$\left. 2M\left(\frac{M}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right) - \sum_{z=1}^{k+1} \frac{k!}{\alpha^z (k+1-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+3/2} K_{k-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)\right.$$

$$\left. 2M\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+ \frac{D}{2} \sum_{z=1}^{k+2} \frac{(k+1)!}{\alpha^{z}(k+2-z)!} \left(2D\left(\frac{M}{\theta}\right)^{k-z+5/2} K_{k-z+5/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z+3/2} K_{k-z+3/2}\left(\frac{M\theta}{\sigma^2}\right) \right)$$

$$- \frac{\alpha^k e^{\frac{DM}{\sigma^2}}}{2\sqrt{2\pi}\,\Gamma(k)} \left[2\left(\frac{\sigma^3}{D^2} + \frac{\sigma M}{D}\right) \left(\frac{M}{\theta}\right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) - \sigma \left(\frac{M}{\theta}\right)^{k+3/2} K_{k+3/2}\left(\frac{M\theta}{\sigma^2}\right) \right]$$

$$- \frac{\sigma^4 \alpha^k e^{\frac{DM}{\sigma^2}}}{\sqrt{2\pi}\, 4D^3 \quad 2\sigma \ \Gamma(k)} \left[\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^{z}(k-z)!} \left(2M\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) - 2D\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}} \right.\right.$$

$$\left.\left. K_{k-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \right) \right]$$

8.5.14

From 4.4.32

$$G_{12}(M,L) = -\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4 M}{2D^5} - \frac{\sigma^6}{4D^6} - \frac{\sigma^2 L^2}{2D^2} + \frac{L^2 M}{D} - \frac{L^3}{3} - \frac{M^2 L}{D^2}\right)$$

$$F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) + \frac{1}{\sqrt{\sigma^2 L}}\, g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)\left(- \frac{2}{3}\frac{\sigma^2 ML^2}{D^2} + \frac{\sigma^2 L^3}{3D} + \frac{\sigma^2 M^2 L}{3D^3} + \frac{\sigma^4 ML}{2D^4}\right.$$

$$\left. + \frac{\sigma^4 L^2}{6D^3} + \frac{8\sigma^6 L}{D^5}\right) + \frac{\sigma^6}{4D^6} e^{\frac{2DM}{\sigma^2}} \quad F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

8.5.15

Simplifying

$$G_{12}(M,L) = -\left[\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4 M}{2D^5} + \frac{\sigma^6}{4D^6}\right) - \frac{M^2 L}{D^2} + L^2\left(\frac{M}{D} - \frac{\sigma^2}{2D^2}\right)\right.$$

$$\left. - \frac{L^3}{3}\right] F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) + \frac{1}{\sqrt{\sigma^2}}\, g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) \left[\left(\frac{\sigma^2 M^2}{3D^3} + \frac{\sigma^4 M}{2D^4}\right) * L^{\frac{1}{2}}\right.$$

$$+\ \frac{8\sigma^6}{D^5}\ L^{\frac{1}{2}} + L^{3/2}\left[\left(-\frac{2\sigma^2 M}{3\ D^2} + \frac{\sigma^4}{6D^3}\right) + \frac{\sigma^2}{3D} L^{5/2}\right]$$

$$+\ \frac{\sigma^6}{4D^6}\ e^{\frac{2DM}{\sigma^2}}\qquad F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

8.5.16.

$$H(L)\ G_{12}(M,L) = \frac{\alpha^k\ e^{-\alpha L}}{\Gamma(k)}\left[\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4 M}{2D^5} + \frac{\sigma^6}{4D^6}\right) L^{k-1} - \frac{M^2 L^k}{D^2}\right.$$

$$\left. +\ L^{k+1}\left(\frac{M}{D} - \frac{\sigma^2}{2D^2}\right) - \frac{L^{k+2}}{3}\right]\ F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\ \frac{1}{\sigma}\, g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)\left[\left(\frac{\sigma^2 M^2}{3D^3} + \frac{\sigma^4 M}{2D^4}\right) L^{k-\frac{1}{2}} + \frac{8\sigma^6}{D^5}\ L^{k-\frac{1}{2}}\right.$$

$$\left. +\ L^{k+\frac{1}{2}}\left(\frac{\sigma^4}{6D^3} - 2/3\ \frac{\sigma^2 M}{D^2}\right) + \frac{\sigma^2}{3D}\ L^{k+3/2}\right]\ \frac{e^{-\alpha L}\,\alpha^k}{\Gamma(k)}$$

$$+\ \frac{\sigma^6}{4D^6}\ e^{\frac{2DM}{\sigma^2}}\ \frac{\alpha^k\ e^{-\alpha L}}{\Gamma(k)}\ L^{k-1}\quad F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

8.5.17

Hence $\int_0^\infty H(L)\ G_{12}(M,L)\ dL$ applying 8.2.8, 8.2.16, 8.2. 30

$$= G_{40}(M) = \frac{\alpha^k\ e^{\frac{DM}{\sigma^2}}}{2\sigma\ \Gamma(k)\sqrt{2\pi}}\left[\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^3}{2D^4} + \frac{\sigma^4 M}{4D} + \frac{\sigma^6}{4D^5}\right)\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^z (k-z)!}\right.$$

$$\left(2D\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$-\ \frac{M^2}{D^2}\sum_{z=1}^{k+1}\frac{k!}{\alpha^z (k+1-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+3/2} K_{k-z+3/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+\ \left(\frac{M}{D} - \frac{\sigma^2}{2D^2}\right)\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^z (k+2-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+5/2} K_{k-z+5/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z+3/2}\right.$$

$$\left. K_{k-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$- \frac{1}{3} \sum_{z=1}^{k+3} \frac{(k+1)!}{\alpha^{z}(k+3-z)!} \left(2D\left(\frac{M}{\theta}\right)^{k-z+7/2} K_{k-z+7/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z+5/2} K_{k-z+5/2}\left(\frac{M\theta}{\sigma^2}\right) \right) \Bigg]$$

$$+ \frac{e^{\frac{DM}{\sigma^2}}}{\sqrt{2\pi\sigma^2}} \frac{\alpha^k}{\Gamma(k)} \Bigg[2\left(\frac{\sigma^2 M^2}{3D^3} + \frac{\sigma^4 M}{2D4} + \frac{8\sigma^6}{D^5}\right) \left(\frac{M}{\theta}\right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)$$

$$+ 2\left(\frac{\sigma^4}{6D^3} - \frac{2\sigma^2 M}{3D^2}\right) \left(\frac{M}{\theta}\right)^{k+3/2} K_{k+3/2}\left(\frac{M\theta}{\sigma^2}\right)$$

$$+ \frac{2\sigma^2}{3D} \left(\frac{M}{\theta}\right)^{k+5/2} K_{k+5/2}\left(\frac{M\theta}{\sigma^2}\right) \Bigg]$$

$$+ \frac{\sigma^6 \, e^{\frac{DM}{\sigma^2}}}{2\sigma \cdot 4D^6 \cdot} \frac{\alpha^k}{\Gamma(k)} \frac{1}{\sqrt{2\pi}} \Bigg[\sum_{z=1}^{k} \frac{(k+1)!}{\alpha^{z}(k-z)!} \left(-2D\left(\frac{M}{\theta}\right)^{k-z+\frac{1}{2}} + 2M\left(\frac{M}{\theta}\right)^{k-z-\frac{1}{2}} \right) K_{k-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \Bigg]$$

8.5.18

From 8.5.8

$$G_5(M, L+T) = \sigma\ (L+T)^{\frac{1}{2}} g\left(\frac{M - D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) - \left(M - D(L+T)\right) \times F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.5.19

Simplifying

$$G_5(M, L+T) = \sigma (L+T)^{\frac{1}{2}} g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) - \left[(M-DT) - DL\right] F \frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}$$

8.5.20

Multiplying by H(L)

$$H(L)\ G_5(M, L+T) = \frac{\sigma \alpha^k e^{-\alpha L}}{\sqrt{2\pi}\,\Gamma(k)} L^{k-1} (L+T)^{\frac{1}{2}} g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

$$- \frac{\alpha^k e^{-\alpha L}}{\Gamma(k)} \left[(M-DT)\ L^{k-1} - DL^k \right] F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2 (L+T)}} \right)$$

8.5.21

Hence

$\int_0^\infty H(L)\ G_5(M, L+T)\ dL$ applying 8.2.34, 8.2.31

$$G_{41}(M,T) = \frac{\sigma}{\sqrt{2\pi}} \frac{e^{\left(\alpha T + \frac{DM}{\sigma^2}\right)} \alpha^k}{\Gamma(k)} \sum_{j=o}^{k-1} (-T)^j \binom{k-1}{j} \left(2\left(\frac{M}{\theta}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \right)$$

$$\frac{1}{2\sigma\sqrt{2\pi}} \frac{\alpha^k e^{\left(\alpha T + \frac{DM}{\sigma^2}\right)}}{\Gamma(k)} \left[\sum_{j=o}^{k-1} (-T)^j \binom{k-1}{j} \sum_{z=1}^{k-j} \frac{(k-1-j)!}{\alpha^z (k-j-z)!} \left(2D\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} \right.\right.$$

$$\left. K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) - 2M\left(\frac{M}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \right) * (M-DT)$$

$$- D \sum_{j=o}^{k} (-T)^j \binom{k}{j} \sum_{z=1}^{k+1-j} \frac{(k-j)!}{\alpha^z (k+1-j-z)!} \left(2D\left(\frac{M}{\theta}\right)^{k-j-z+3/2} K_{k-j-z+3/2}\left(\frac{M\theta}{\sigma^2}\right) \right.$$

$$\left.\left. + 2M\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \right) \right]$$

8.5.22

From 8.5.16 substituting L+T for L and simplifying

$$G_{12}(M, L+T) = \left[\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4 M}{2D^5} + \frac{\sigma^6}{4D^6} - \frac{M^2 T}{D^2} \right) - \frac{M^2 L}{D^2} \right.$$

$$\left. + \sum_{i=o}^{2} \binom{2}{i} T^i L^{2-i} \left(\frac{M}{D} - \frac{\sigma^2}{2D^2} \right) - \frac{1}{3} \sum_{i=o}^{3} \binom{3}{i} T^i L^{3-i} \right] F\left(\frac{M-DL}{\sqrt{\sigma^2 L}} \right)$$

$$+ \frac{1}{\sigma}\, g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2 (L+T)}} \right) \left[\left(\frac{\sigma^2 M^2}{3D^3} + \frac{\sigma^4 M}{2D^4} + \frac{8\sigma^6}{D^5} \right) (L+T)^{\frac{1}{2}} \right.$$

$$+\left(\frac{\sigma^4}{6D^3}-\frac{2\sigma^2 M}{3D^2}\right)(T+L)^{3/2}+\sigma^2(L+T)^{\frac{1}{2}}\sum_{i=o}^{2}\binom{2}{i}T^iL^{2-i}\Bigg]$$

$$+\frac{\sigma^6}{4D^6}e^{\frac{2DM}{\sigma^2}}F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

8.5.23

Multiplying by H(L) we have

$$H(L)\,G_{12}(M,L+T)=-\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\left[\left(\frac{M^3}{3D^3}+\frac{\sigma^2M^2}{2D^4}+\frac{\sigma^4 M}{2D^5}+\frac{\sigma^6}{4D^6}-\frac{M^2T}{D^2}\right)L^{k-1}\right.$$

$$\left.+\frac{M^2L^k}{D^2}+\sum_{i=o}^{2}\binom{2}{i}T^iL^{k+1-i}\left(\frac{M}{D}-\frac{\sigma^2}{2D^2}\right)-\frac{1}{3}\sum_{i=o}^{3}\binom{3}{i}T^iL^{k-i+2}\right]\times$$

$$F\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{\alpha^k}{\sigma}\frac{e^{-\alpha L}}{\Gamma(k)}(L+T)^{\frac{1}{2}}g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)\left[\left(\left(\frac{\sigma^2M^2}{3D^3}+\frac{\sigma^4M}{2D^4}+\frac{8\sigma^6}{D^5}\right)+\frac{\sigma^4T}{6D^3}-\frac{2\sigma^2MT}{3D^2}\right)L^{k-1}\right.$$

$$\left.+\left(\frac{\sigma^4}{6D^3}-\frac{2\sigma^2M}{3D^2}\right)L^k+\sigma^2\sum_{i=o}^{2}\binom{2}{i}T^iL^{k-i+1}\right]$$

$$+\frac{\sigma^6}{4D^6}e^{\frac{2DM}{\sigma^2}}\frac{\alpha^k e^{-\alpha L}L^{k-1}}{\Gamma(k)}F\left(\frac{M+DL}{\sqrt{\sigma^2L}}\right),$$

8.5.24

$\int_o^{\infty}$ H(L) $G_{12}(M,L+T)$ dL applying 8.2.31, 8.2.34 and 8.2.35

$$G_{43}(M,T)=\frac{\alpha^k e^{\left(\alpha T+\frac{DM}{\sigma^2}\right)}}{\sqrt{2\pi}\,\Gamma(k)2\sigma}\left[\left(\frac{M^3}{3D^3}+\frac{\sigma^2M^2}{2D^4}+\frac{\sigma^4M}{2D^5}\right.\right.$$

$$+\left(\frac{\sigma^6}{4D^6} - \frac{M^2 T}{D^2}\right) \sum_{j=o}^{k-1} \binom{k-1}{j} (-T)^j \sum_{z=1}^{k-j} \frac{(k-1-k)!}{\alpha^z (k-j-z)!}$$

$$\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+\frac{M^2}{D^2} \sum_{j=o}^{k} \binom{k}{j} (-T)^j \sum_{z=1}^{k+1-j} \frac{(k-j)!}{\alpha^z (k+1-j-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+3/2}\right.$$

$$\left.K_{k-j-z-3/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+\left(\frac{M}{D} - \frac{\sigma^2}{2D^2}\right) \sum_{i=o}^{2} \binom{2}{i} T^i \sum_{j=o}^{k+1-i} (-T)^j \binom{k+1-i}{j}$$

$$\sum_{z=1}^{k+2-i-j} \frac{(k+1-i-j)!}{\alpha^z (k+2-i-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+5/2} K_{k-j-z+5/2}\left(\frac{M\theta}{\sigma^2}\right)\right.$$

$$\left.+ 2M\left(\frac{M}{\theta}\right)^{k-j-z+3/2} K_{k-j-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$-\frac{1}{3} \sum_{i=o}^{3} \binom{3}{i} T^i \sum_{j=o}^{k+2-i} (-T)^j \binom{k-i+2}{j} \sum_{z=1}^{k+3-i-j} \frac{(k+2-i-j)!}{\alpha^z (k+3-i-z)!}$$

$$\left.\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+7/2} K_{k-j-z+7/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z+5/2} K_{k-j-z+5/2}\left(\frac{M\theta}{\sigma^2}\right)\right)\right]$$

$$+ \frac{e^{\left(\alpha T + \frac{DM}{\sigma^2}\right)}}{\sqrt{2\pi}\ \sigma^2} \frac{\alpha^k}{\Gamma(k)} \left[\sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} \cdot \left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4 M}{2D^5} + \frac{\sigma^6}{4D^6} - \frac{M^2 T}{D^2} \right) \right.$$

$$2\left(\frac{M}{\theta}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2\left(\frac{\sigma^4}{6D^3} - \frac{2\sigma^2 M}{3D^2}\right) \sum_{j=0}^{k} (-T)^j \binom{k}{j} \left(\frac{M}{\theta}\right)^{k-j+3/2}$$

$$K_{k-j+3/2}\left(\frac{M\theta}{\sigma^2}\right) + 2\sigma^2 \sum_{i=0}^{2} \binom{2}{i} T^i \sum_{j=0}^{k-i+1} (-T)^j \binom{k-i+1}{j} \left(\frac{M}{\theta}\right)^{k-i+5/2-j}$$

$$\left. K_{k-i+5/2-j}\left(\frac{R\theta}{\sigma^2}\right) \right]$$

$$+ \frac{\sigma^6}{4D^6} \frac{\alpha^k e^{\left(\frac{DM + \alpha T}{\sigma^2}\right)}}{\Gamma(k)\, 2\sigma \sqrt{2\pi}} \left[\sum_{j=0}^{k-1} \binom{k-1}{j} (-T)^j \sum_{z=1}^{k-j} \frac{(k-1-j)!}{\alpha^z (k-j-z)!} \right.$$

$$\left. \left(- 2D\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) \right) \right] \qquad 8.5.25$$

From 8.5.12, substituting L+T for L

$$G_2(M, L+T) = \left[\left(\frac{\sigma^4}{4D^3} + \frac{\sigma^2}{2D^2} + \frac{M^2}{2D} - T \right) - L + \frac{D}{2} \sum_{i=0}^{2} \binom{2}{i} T^i L^{2-i} \right] *$$

$$F\left(\frac{M - D(L+T)}{\sqrt{\sigma^2 (L+T)}}\right) + \frac{\sqrt{(L+T)}}{2} \left[-\left(\frac{\sigma^3}{D^2} + \frac{\sigma M}{D} - \sigma T \right) + \sigma L \right] g\left(\frac{M - D(L+T)}{\sqrt{\sigma^2 (L+T)}}\right)$$

$$- \frac{\sigma^4}{4D^3} \cdot e^{\frac{2DM}{\sigma^2}} \cdot F\left(\frac{M + D(L+T)}{\sqrt{\sigma^2 (L+T)}}\right) \qquad 8.5.26$$

Multiplying by $H(L)$, we have

$$H(L)\,G_2(M,L+T) = \left[\left(\frac{\sigma^4}{4D^3} + \frac{\sigma^2}{2D^2} + \frac{M^2}{2D} - T\right) L^{k-1} - L^k\right.$$

$$\left. + \frac{D}{2}\sum_{i=0}^{2}\binom{2}{i} T^i L^{k+1-i}\right] F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$+ \frac{1}{2}\cdot(L+T)^{\frac{1}{2}}\left[-\left(\frac{\sigma^3}{D^2} + \frac{\sigma M}{D} - \sigma T\right) L^{k-1} + \sigma L^k\right] g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$- \frac{\sigma^4}{4D^3}\,\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\, e^{2DM/\sigma^2}\, F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

$\int_0^\infty H(L)\,G_2(M,L+T)\,dL$, applying 8.2.31, 8.2.34 and 8.2.35

we have

$$G_{42}(M,T) = \frac{\alpha^k e^{\left(\alpha T + \frac{DM}{\sigma^2}\right)}}{2\sigma\,\Gamma(k)\sqrt{2\pi}}\left[\left(\frac{\sigma^4}{4D^3} + \frac{\sigma^2}{2D^2} + \frac{M^2}{D} - T\right)\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j} *\right.$$

$$\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\alpha^z (k-j-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$-\sum_{j=0}^{k}(-T)^j\binom{k}{j}\sum_{z=1}^{k+1-j}\frac{(k-j)!}{\alpha^z (k+1-j-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+3/2} K_{k-j-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)\right.$$

$$\left. + 2M\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+ \frac{D}{2} \sum_{i=o}^{2} \binom{2}{i} T^{i} \sum_{j=o}^{k+1-i} (-T)^{j} \binom{k+1-i}{j} \sum_{z=i}^{k+2-i-j} \frac{(k+1-i-j)!}{\alpha^{z}(k+2-i-j)!} \; *$$

$$\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+5/2-i} K_{k-j-z-i+5/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-i+3/2} K_{k-j-z-i+3/2}\left(\frac{M\theta}{\sigma^2}\right)\right)\Bigg]$$

$$+ \frac{\alpha^{k} e^{\left(\alpha T+\frac{DM}{\sigma^2}\right)}}{2\sqrt{2\pi}\,\Gamma(k)} \Bigg[- 2\left(\frac{\sigma^3}{D^2} + \frac{\sigma M}{D} - \sigma T\right) \sum_{j=o}^{k-1} T^{j} \binom{k-1}{j}\left(\frac{M}{\theta}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)$$

$$+ \sigma \sum_{j=o}^{k} (-T)^{j} \binom{k}{j} \left(\frac{M}{\theta}\right)^{k-j+3/2} K_{k-j+3/2}\left(\frac{M\theta}{\sigma^2}\right)\Bigg]$$

$$- \frac{\sigma^4}{4D^3} \frac{\alpha^{k} e^{(\alpha T + DM/\sigma^2)}}{(k) \quad 2\sigma \quad 2} \sum_{j=o}^{k-1} (-T)^{j} \binom{k-1}{j} \sum_{z=1}^{k-j} \frac{(k-1-j)!}{\alpha^{z}(k-j-z)!}$$

$$\left(- 2D\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right) \qquad 8.5.27$$

Hence averaging the inventory cost of (M,T) over the states of L_1 and L_2 then we have from equation 8.5.1

$$C = \frac{Rc + S}{T} + hc\left(M - \frac{Dk}{\alpha} - \frac{DT}{2}\right) + \frac{b_1}{T}\left(G_{41}(R,T) - G_{38}(R)\right)$$

$$+ \frac{(hc + b_2)}{T}\left(G_{42}(R,T) - G_{39}(R)\right) + \frac{b_3}{T}\left(G_{43}(R,T) - G_{40}(R)\right) \qquad 8.5.28$$

SECTION 8.6. MODEL (M,R,T)

In this section we derive the cost equations for continuous lead times when the cost of a backorder is a quadratic function of the length of time of a backorder by averaging over the states of the lead times, the corresponding cost for fixed lead times.

In section 4.6. equation 4.6.5, G_{14} (R+Y,T) is the expected cost of carrying inventory and backorders including the cost of a stockout dependent on the number of stockouts only, for fixed lead times, L_1 and L_2. From equation 4.6.5 substituting L_1 for L and L_2+T for L+T we have

$$
\begin{aligned}
G_{14}(R+Y,T) &= hc\, T\left(R+Y-DL_1-\frac{DT}{2}\right) + b_1\,\big(G_5(R+Y,T+L_2) \\
&\quad - G_5(R+Y,L_1)\big) + (b_2+hc)\,\big(G_2(R+Y,T+L_2) - G_2(R+Y,L_1)\big) \\
&\quad + b_3\,\big(G_{12}(R+Y,T+L_2) - G_{12}(R+Y,L_1)\big) \\
&\quad + s\,\big(R_o(R+Y,T+L_2) - R_o(R+Y,L_1)\big)
\end{aligned}
\qquad 8.6.1.
$$

We shall exclude the cost dependent on the number of stockouts in determining the inventory costs for (M,R,T)

$$
\begin{aligned}
\text{Let } G_{44}(R+Y,T) &= \int_0^\infty\!\!\int_0^\infty H(L_1)H(L_2)\,\big(G_{14}(R+Y,T) - s(R_o(R+Y,T+L_2) \\
&= R_o(R+Y,L_1)\big)\big)DL_1\, dL_2
\end{aligned}
\qquad 8.6.2.
$$

Substituting for G_{14}(R+Y,T)

$$
\begin{aligned}
G_{44}(R+Y,T) &= \int_0^\infty\!\!\int_0^\infty H(L_1)H(L_2) \times \Big(hcT\left(R+Y-DL_1-\frac{DT}{2}\right) \\
&\quad + b_1\,\big(G_5(R+Y,T+L_2) - G_5(R+Y,L_1)\big) + b_2\,\big(G_2(R+Y,T+L_2) \\
&\quad - G_2(R+Y,L_1)\big) + b_3\,\big(G_{12}(R+Y,T+L_2) - G_{12}(R+Y,L_1)\big)\,dL_1\, dL_2
\end{aligned}
\qquad 8.6.3
$$

From 8.5.2 to 8.5.7

$$G_{44}(R+Y,T) = hcT\left(R+Y - \frac{Dk}{\alpha} - \frac{DT}{2}\right) + b_1\left(G_{41}(R+Y,T) - G_{38}(R+Y)\right) + b_2\left(G_{42}(R+Y,T) - G_{39}(R+Y)\right) + b_3\left(G_{43}(R+Y,T) - G_{40}(R+Y)\right) \qquad 8.6.4$$

The inventory cost for model (M,R,T) is obtained by replacing $G_{14}(R+Y,T)$ by $G_{44}(R+Y,T)$ in equation 4.6.6.

Hence

$$C = \frac{Rc}{T} + \frac{\left(S + \sum_{m=1}^{\infty} \int_0^{M-R} G_{44}(R+Y,T)\, f^{m}(M-R-Y,DT)\, dY + G_{44}(M,T)\right)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{m=1}^{\infty}\int_0^{M-R} m\, f^{m-1}(M-R-Y,DT)\, F\left(\frac{Y-DT}{\sqrt{\sigma^2 T}}\right) dY\right]} \qquad 8.6.5$$

SECTION 8.7. MODEL (Q,R). EXPONENTIAL COST TERMS

In section 5.1 we derived the cost equations when the cost of a backorder was a function of the length of time of a backorder, and the lead times were fixed.

In this section we derive the cost equations for the continuous lead times by averaging over the states of the lead times the cost equations obtained in section 5.1.

From 5.1.18 the cost equations excluding the cost dependent on the number of stockouts only is

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\left(R-DL+\frac{B(Q,R)}{Q}\right) + \frac{D}{Q}G_{15}(Q,R) \qquad 8.7.1$$

From 2.6.9 substituting $\frac{R-DL}{\sqrt{\sigma^2}}$ for

$$B(Q,R) = \frac{1}{Q}\left(\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 8.7.2$$

From 2.6.8

$$\beta(V) = \frac{\sigma^2 L}{2} * \left((1+V^2) * F(v) - Vg(V)\right) \qquad 8.7.3.$$

From 5.1.10

$$G_{15}(Q,R) = \frac{Db_1}{b_4}\left[\exp -\left(\frac{\sigma^2 L b_4^2}{2D^2} + \frac{b_4}{D}\left(R - \frac{\sigma^2 L\, b_4}{D} - DL\right)\right)\right.$$

$$F\left(\frac{R - \frac{\sigma^2 b_4 L}{D} - DL}{\sigma^2 L}\right)$$

$$\left. - \exp -\left[\frac{\sigma^2 b_4^2 L}{2D^2} + \frac{b_4}{D}\left(R+Q - \frac{\sigma^4 b_4 L}{D} - DL\right)\right] F\left(\frac{R+Q - \frac{\sigma^2 L b_4}{D} - DL}{\sqrt{\sigma^2 L}}\right)\right]$$

$$- \frac{Db_1}{b_4}\left(F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad 8.7.4.$$

Let $G_{15}(Q,R) = \left(G_{45}(R,L) - G_{45}(R+Q,L)\right)$ 8.7.5.

where $G_{45}(R,L) = \dfrac{Db_1}{b_4} \exp -\left(\dfrac{b_4}{D} \cdot \left(R - DL\right) - \dfrac{\sigma^2 b_4 L}{2D^2}\right) *$

$$F\left(\frac{R - \frac{\sigma^2 b_4 L}{D} - DL}{\sqrt{\sigma^2 L}}\right) - \frac{Db_1}{b_4} * F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) \qquad 8.7.6$$

The integral of $\frac{Q}{2} + \left(R - DL + B(Q,R)\right)$ has been given in 8.3.34

$$= \frac{Q}{2} + R - \frac{Dk}{\alpha} + \frac{1}{Q}\left(G_{30}(R) - G_{30}(R+Q)\right) \qquad 8.7.7$$

We need next the integral of $\int_0^\infty G_{45}(R,L)\, H(L)\, dL$ in order to obtain the cost equation (Q,R) when lead time is a variable.

Simplifying $G_{45}(R,L)$

$$G_{45}(R,L) = \frac{Db_1}{b_4} \exp \frac{-b_4 R}{D} \exp\left[L\left(D + \frac{\sigma^2 b_4}{2D^2}\right)\right] F\left(\frac{R - L\left(\frac{\sigma^2 b_4}{D} + D\right)}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{Db_1}{b_4} * F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) \qquad 8.7.8$$

Multiplying by H(L) where $H(L) = e^{-\alpha L} \alpha^k L^{k-1} / \Gamma(k)$

We have

$$H(L)\, G_{45}(R,L) = \frac{Db_1}{b_4} \exp - \frac{b_4 R}{D} \frac{\alpha^k}{\Gamma(k)} \exp L\left(\left(D + \frac{\sigma^2 b_4}{D^2}\right) - \alpha\right) F\left(\frac{R-L\left(\frac{\sigma^2 b_4}{D}+D\right)}{\sqrt{\sigma^2 L}}\right)$$

$$- D \frac{\alpha^k L^{k-1} e^{-\alpha L} b_1}{\Gamma(k)\ b_4} F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) \qquad 8.7.9$$

Let $G_{46}(R) = \int_0^\infty H(L)\, G_{45}(R,L)\, dL \qquad 8.7.10$

Let $\gamma^2 = 2\left(\left(D + \frac{\sigma^2 b_4}{D^2}\right) - \alpha\right) \times \sigma^2 + \left(\frac{\sigma^2 b_4}{D} + D\right)^2$ 8.7.11

Applying 8.2.18

$$G_{46}(R) = \frac{Db_1}{b_4} \frac{\exp\left(\frac{-b_4}{D}R + \left(\frac{\sigma^2 b_4}{D} + D\right) \cdot R\right)}{\sqrt{2\pi}\ \Gamma(k)} \frac{\alpha^k}{2\sigma} \sum_{z=1}^{k} \frac{(k-1)!}{\left(D + \frac{\sigma^2 b_4}{D^2} - \alpha\right)^z (k-z)!}$$

$$\left(2\left(\frac{\sigma^2 b_4}{D} + D\right)\left(\frac{R}{\gamma}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{M\gamma}{\sigma^2}\right) + 2R\left(\frac{R}{\gamma}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{M\gamma}{\sigma^2}\right)\right)$$

$$- \frac{D\alpha^k b_1 e^{\frac{DR}{\sigma^2}}}{2\sigma\ b_4\ \Gamma(k)\sqrt{2\pi}} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!} \left[2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}\right.$$

$$\left. K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right]$$ 8.7.12

Hence $\frac{D}{Q} \int_0^\infty G_{15}(Q,R)\ H(L)\ dL$

$$= \frac{D}{Q} \int_0^\infty \left(G_{45}(R,L) - G_{45}(R+Q,L)\right) H(L)\ dL$$

$$= \frac{D}{Q} \left(G_{46}(R) - G_{46}(R+Q)\right)$$ 8.7.13

Hence the inventory costs for continuous lead time

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\left(R - \frac{Dk}{\alpha}\right) + \frac{hc}{Q}\left(G_{30}(R) - G_{30}(R+Q)\right)$$

$$+ \frac{D}{Q}\left(G_{46}(R) - G_{46}(R+Q)\right)$$ 8.7.14

SECTION 8.8 MODEL (Q,R,T) EXPONENTIAL COST TERMS

The inventory cost for model (Q,R,T) when the cost of a backorder is an exponential function of the length of time of a backorder for fixed lead times is given in 5.2.20.

From 5.2.20 substituting L_1 for L and L_2+T for L+T and excluding the cost dependent on the number of stockouts.

We have

$$C = \frac{Rc}{T} + \frac{S.POR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + \frac{hc}{QT}\Big(G_3(R,T+L_2)$$

$$- G_3(R,L_1) - G_3(R+Q,\ T+L_2) + G_3(R+Q,L_1)\Big)$$

$$+ \frac{1}{QT}\left(G_{18}(R,L_2+T) - G_{18}(R,L_1) - G_{18}(R+Q,L_2+T) + G_{18}(R+Q,L_1)\right)$$

8.8.1

Noting that

$$G_{33}(R) = \int_0^{\infty} H(L)\ G_3(R,L)\ dL \text{ from } 8.4.11$$

and

$$G_{36}(R,T) = \int_0^{\infty} H(L)\ G_3(R,T+L)\ dL \text{ from } 8.4.27$$

and

Let

$$G_{47}(R) = \int_0^{\infty} H(L)\ G_{18}(R,L)\ dL \qquad 8.8.2$$

$$G_{48}(R,T) = \int_0^{\infty} H(L)\ G_{18}(R,L+T)\ dL \qquad 8.8.3$$

From 5.2.16

$$G_{18}(R,L) = \frac{2D^2 b_1}{\left(\sigma^2 b^3{}_4 + 2D^2 b_4{}^2\right)} e^{L\left(\frac{\sigma^2 b_4{}^2 + 2D^2 b_4}{2D^2}\right)} \cdot e^{-\frac{b_4 R}{D}} \quad F\left(\frac{R - L\left(D + \frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{b_1}{b_4} F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) * \left(L - \left(\frac{R}{D} + \frac{\sigma^2}{2D^2} - \frac{1}{b_4}\right)\right)$$

$$- \frac{\sigma^4 b_1 b_4}{2D^2(\sigma^2 b_4{}^2 + 2D^2 b_4) b_4} * e^{\frac{2DR}{\sigma^2}} * F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) - \frac{2\sqrt{\sigma^2 L}}{D b_4} g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

8.8.4

Multiplying by H(L)

$$H(L)G_{18}(R,L) = \frac{2D^2 b_1 \; e^{\frac{-b_4 R}{D}} \alpha^k L^{k-1}}{\left(\sigma^2 b_4{}^3 + 2D^2 b_4{}^2\right)} e^{L\left(\frac{\sigma^2 b_4{}^2 + 2D^2 b_4 - 2D^2\alpha}{2D^2}\right)}$$

$$F\left(\frac{R - L\left(D + \frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{b_1}{b_4} \frac{\alpha^k e^{-\alpha L}}{\Gamma(k)} \left[L^k - \left(\frac{R}{D} + \frac{\sigma^2}{2D^2} - \frac{1}{b_4}\right) L^{k-1}\right] F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{\sigma^4 b_1 b_2 \; e^{\frac{2DR}{\sigma^2}} \; e^{-\alpha L} L^{k-1}}{2D^2(\sigma^2 b_4{}^2 + 2D^2 b_4) \; \Gamma(k)} * F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) - \frac{2\sqrt{\sigma^2 L} \, b_1 \alpha^k \; e^{-\alpha L} L^{k-1}}{D b_4 \; * \; \Gamma(k)} g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

Hence $\int_0^\infty H(L)\, G_{18}(R,L)\, dL$ applying 8.2.8, 8.2.16 and 8.2.31

Let $\varepsilon^2 = 2\left(\frac{\sigma^2 b_4{}^2 + 2D^2 b_4 - 2D^2\alpha}{2D^2}\right)$

8.8.5.

$$G_{47}(R) \quad \frac{2D^2 b_1 \alpha^k}{2\sigma(\sigma^2 b_4^3+2D^2 b_4^2)} \frac{e^{R\left(D+\frac{\sigma^2 b_4}{D}\right)} e^{-\frac{b_4 R}{D}}}{\Gamma(k)\sqrt{2\pi}} \sum_{z=1}^{k} \frac{(k-1)!}{\left(\frac{\sigma^2 b_4^2+2D^2 b_4-2D\frac{2}{\alpha}}{2D^2}\right)!\,(k-z)!}$$

$$\left(2\left(D+\frac{\sigma^2 b_4}{D}\right)\left(\frac{R}{\varepsilon}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\varepsilon}{\sigma^2}\right) + 2R\left(\frac{R}{\varepsilon}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{R\varepsilon}{\sigma^2}\right)\right)$$

$$-\frac{b_1\alpha^k e^{\frac{DR}{\sigma^2}}}{2\sigma b_4 * \Gamma(k)}\left[\sum_{z=1}^{k+1} \frac{(k)!}{\alpha^z (k+1-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+3/2} K_{k-z+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right.\right.$$

$$\left.\left.+ 2R\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right) - \left(\frac{R}{D}+\frac{\sigma^2}{2D^2}-\frac{1}{b_4}\right)\sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!}\right.$$

$$\left.\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)\right]$$

$$-\frac{\sigma^4 b_1 b_4^2 \alpha^k e^{\frac{DR}{\sigma^2}}}{\sqrt{2\pi}\; 2D^2 (\sigma^2 b_4^2+2D^2 b_4) b_4 \Gamma(k)} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right.$$

$$\left.+ 2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$-\frac{2 b_1 \alpha^k \sigma e^{\frac{DR}{\sigma^2}}}{\sqrt{2\pi}\, D b_4 \Gamma(k)} \left(\frac{R}{\theta}\right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \qquad 8.8.6.$$

From 5.2.16

$$G_3(R,T+L) = \frac{2D^2 b_1}{(\sigma^2 b_4^3+2D^2 b_4^2)} e^{T\left(\frac{\sigma^2 b_4^2+2D^2 b_4}{2D^2}\right)} e^{\frac{-b_4 R}{D}} e^{L\left(\frac{\sigma^2 b_4^2+2D^2 b_4}{2D^2}\right)}$$

$$\Phi\left(\frac{R-(L+T)\left(D+\frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 L+T}}\right)$$

$$- \frac{b_1}{b_4} \; F\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(T+L)}}\right)\left(L - \left(\frac{R}{D} + \frac{\sigma^2}{2D^2} - \frac{1}{b_4} - T\right)\right)$$

$$- \frac{\sigma^4 b_1 b_4^2 \; e^{\frac{2DR}{\sigma^2}}}{2D^2(\sigma^2 b_4^2 + 2D^2 b_4) b_4} \; F\left(\frac{R+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) - \frac{2\sqrt{\sigma^2(L+T)}\, b_1}{D b_4} \; g\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.8.7.

Multiplying by H(L) where $H(L) = L^{k-1} \alpha^k e^{-\alpha L} / \Gamma(k)$

$$H(L)\; G_3(R,T+L) = \frac{2D^2 b_1 \alpha^k L^{k-1}}{(\sigma^2 b_4^3 + 2D^2 b_4^2)} e^{T\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right)} e^{-\frac{b_4 R}{D}} \cdot e^{L\frac{(\sigma^2 b_4^2 + 2D^2 b_4)}{2D^2}}$$

$$F\left(\frac{R - (L+T)(D + \frac{\sigma^2 b_4}{D})}{\sqrt{\sigma^2(L+T)}}\right) \cdot e^{-\alpha L}$$

$$- \frac{b_1}{b_4} \; \frac{\alpha^k e^{-\alpha L}}{\Gamma(k)} \left[L^k - \left(\frac{R}{D} + \frac{\sigma^2}{2D^2} - \frac{1}{b_4} - T\right) L^{k-1}\right] F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{\sigma^4 b_1 b_4^2 \; e^{\frac{2DR}{\sigma^2}} L^{k-1} \; e^{-\alpha L}}{2D^2 (\sigma^2 b_4^2 + 2D^2 b_4) b_4} \quad F \; \frac{R+D(L+T)}{\sigma^2(L+T)}$$

$$- \frac{2\alpha^k e^{-\alpha L} L^{k-1} \sqrt{\sigma^2(L+T)}}{D b_4 \; \Gamma(k)} \quad g\left(\frac{R-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.8.8.

$\int_0^\infty H(L)\; G_3(R,T+L)\; dL$ applying 8.2.18, 8.2.8 and 8.2.30b

8.2.34 and 8.2.35

$$G_{48}(R,T) = \frac{2D^2 b_1 \alpha^k e^{T\left(\frac{\sigma^2 b_4 + 2D^2 b_4}{2D^2}\right)} e^{\left(-\frac{b_4 R}{D} + \frac{DR}{\sigma^2} + \alpha T\right)}}{2\sigma\; (\sigma^2 b_4^3 + 2D^2 b_4)\; \Gamma(k) \sqrt{2\pi}} \sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j}$$

$$\sum_{z=1}^{k-1-j} \frac{(k-1-j)!}{\left(\frac{\sigma^2 b_4^2+2D^2 b_4}{2D^2}-\alpha\right)^z (k-z)!}\left(2\left(D+\frac{\sigma^2 b_4}{D}\right)\left(\frac{R}{\Xi}\right)^{k-z+\frac12-j} K_{k-z+\frac12-j}\left(\frac{R\Xi}{\sigma^2}\right)\right.$$

$$\left. + \; 2R\left(\frac{R}{\Xi}\right)^{k-z-\frac12-j} K_{k-z-\frac12-j}\left(\frac{R\Xi}{\sigma^2}\right)\right)$$

$$-\frac{b_1}{b_4}\,\frac{\alpha^k e^{\frac{DR}{\sigma^2}}}{\sqrt{2\pi}\,\Gamma(k)\,2\sigma}\left[\sum_{z=1}^{k}\frac{k!}{\alpha^z(k-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+3/2}K_{k-z+3/2}\left(\frac{R\theta}{\sigma^2}\right)\right.\right.$$

$$\left. + \; 2R\left(\frac{R}{\theta}\right)^{k-z+\frac12}K_{k-z+\frac12}\left(\frac{R\theta}{\sigma^2}\right)\right) - \left(\frac{R}{D}+\frac{\sigma^2}{2D^2}-\frac{1}{b_4}-T\right)\sum_{z=1}^{k-1}\frac{(k-1)!}{\alpha^z(k-z)!}$$

$$\left.\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac12}K_{k-z+\frac12}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z-\frac12}K_{k-z-\frac12}\left(\frac{R\theta}{\sigma^2}\right)\right)\right]$$

$$-\frac{\sigma^4 b_1 b_4^2\, e^{\frac{DR}{\sigma^2}}\, e^{+\alpha T}}{4\sigma\sqrt{2\pi}\,D^2\left(\sigma^2 b_4^2+2D^2 b_4\right)b_4\,\Gamma(k)}\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}\sum_{z=1}^{k-1-j}\frac{(k-1-j)!}{\alpha^z(k-z)!}$$

$$\left(-2D\left(\frac{R}{\theta}\right)^{k-z+\frac12-j}K_{k-z+\frac12-j}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-z-j-\frac12}K_{k-z-j-\frac12}\left(\frac{R\theta}{\sigma^2}\right)\right)$$

$$\frac{-4\alpha^k\sigma}{Db_4}\,\frac{e^{\left(\alpha T+\frac{DR}{\sigma^2}\right)}}{\Gamma(k)\sqrt{2\pi}}\sum_{j=0}^{k-1}T^j\binom{k-1}{j}\; k-j+3/2\;\left(\frac{R}{\theta}\right)^{k-j+3/2}K_{k-j+3/2}\left(\frac{R\theta}{\sigma^2}\right)$$

8.8.9.

Hence the inventory costs for model (nQ,R,T) with continuous lead times is obtained by substituting 8.4.11, 8.4.27 and 8.8.2 and 8.8.3 into the integral of 8.8.1

$$C = \frac{R_c}{T} + \frac{S\ POR}{T} + hc\left(\frac{Q}{2} + R - \frac{Dk}{\alpha} - \frac{DT}{2}\right)+$$

$$+ \frac{hc}{QT} \left(G_{36}(R,T) - G_{33}(R) - G_{36}(R+Q,T) + G_{33}(R+Q)\right)$$

$$+ \frac{1}{QT} \left(G_{48}(R,T) - G_{47}(R) - G_{48}(R+Q,T) + G_{47}(R+Q)\right) \tag{8.8.10}$$

SECTION 8.9 MODEL (M,T) EXPONENTIAL COST

The inventory cost for model (M,T) when the cost is an exponential function of the length of a backorder for fixed lead times is given in 5.3.8. Substituting L_1 for L, and $L_2 + T$, inventory cost for fixed lead times excluding the cost dependent on numbers of stockouts is

$$C = \frac{(Rc + S)}{T} + hc\left(M - DL_1 - \frac{DT}{2}\right) + \frac{hc}{T}\Big(G_2(M,T+L_2) - G_2(M,L_1)\Big)$$
$$+ \frac{1}{T}\Big(G_{19}(M,L_2+T) - G_{19}(M,L_1)\Big) \qquad 8.9.1$$

The inventory costs when the lead times are continuous random variables are obtained by averaging the cost for the fixed lead times over the states of the lead times.

From 8.5.3

$$G_{39}(R) = \int_0^{\infty} H(L)\, G_2(R,L)\, dL \qquad 8.9.2$$

$$G_{42}(R,T) = \int_0^{\infty} H(L)\, G_2(R,L+T)\, dL \qquad 8.9.3$$

Let $$G_{49}(R) = \int_0^{\infty} G_{19}(M,L)\, H(L)\, dL \qquad 8.9.4$$

$$G_{50}(R,T) = \int_0^{\infty} H(L)\, G_{19}(M,L+T)\, dL \qquad 8.9.5$$

From 5.3.9

$$G_{19}(M,L) = \frac{2Db_1}{b_4(\sigma^2 b_4^2 + 2D^2 b_4)} \exp\left[L\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right) - \frac{b_4 M}{D}\right] F\left(\frac{M - L\left(D + \frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 L}}\right)$$
$$+ \frac{b_1}{b_4}\,\frac{(M-DL)}{D\sqrt{\sigma^2 L}}\; g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{b_1}{b_4 D}\, F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$
$$- \frac{\sigma^2 b_4^2 b_1\, e^{\frac{2DM}{\sigma^2}}}{Db_4(\sigma^2 b_4^2 + 2D^2 b_4)} * F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right) \qquad 8.9.6$$

Multiplying H(L) where $H(L) = \alpha^k L^{k-1} e^{-\alpha L} / \Gamma(k)$

$$H(L)\, G_{19}(M,L) = \frac{2Db_1 \alpha^k}{b_4(\sigma^2 b_4^{\,2}+2D^2 b_4)\,\Gamma(k)} \exp\left[L\left(\frac{\sigma^2 b_4^{\,2}+2D^2 b_4}{2D^2} - \alpha\right) - \frac{b_4 M}{D}\right] *$$

$$L^{k-1} \; F\left(\frac{M - L\left(D + \frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 L}}\right) + \frac{b_1}{Db_4}\frac{e^{-\alpha L}}{\sqrt{\sigma^2 L}}\left(ML^{k-1} - DL^k\right) g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{b_1}{b_{4D}} * e^{-\alpha L} L^{k-1} * F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) - \frac{\sigma^2 b_4^{\,2} b_1 e^{-\alpha L} L^{k-1} \alpha^k}{Db_4(\sigma^2 b_4^{\,2}+2D^2 b_4)\,\Gamma(k)} e^{\frac{2DM}{\sigma^2}} \; F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

8.9.7

$\int_0^\infty H(L)\, G_{19}(M,L)\, dL$ applying 8.2.8, 8.2.16 and 8.2.31

and letting $\varepsilon^2 = 2\left(\frac{\sigma^2 b_4^{\,2} + 2D^2 b_4 - 2D^2\alpha}{2D^2}\right)$

$$G_{49}(R) = \frac{2Db_1\alpha^k e^{\left(M\left(D+\frac{\sigma^2 b_4}{D}\right) - \frac{b_4 M}{D}\right)}}{2\sigma b_4(\sigma^2+2D^2 b_4)\,\Gamma(k)\sqrt{2\pi}} \sum_{z=1}^{k} \frac{(k-1)!}{\left(\frac{\sigma^2 b_4^{\,2}+2D^2 b_4 - 2D^2\alpha}{2D^2}\right)!\;(k-z)!}$$

$$\left(2\left(D + \frac{\sigma^2 b_4}{D}\right) * \left(\frac{R}{\varepsilon}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\varepsilon}{\sigma^2}\right) + 2R\left(\frac{R}{\varepsilon}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{R\varepsilon}{\sigma^2}\right)\right)$$

$$- \frac{b_1 e^{\frac{DR}{\sigma^2}}}{\sigma\sqrt{2\pi}\, b_4\,\Gamma(k).\,D}\left[2R\left(\frac{R}{\theta}\right)^{k-\frac{1}{2}} K_{k-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) - 2D\left(\frac{R}{\theta}\right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right]$$

$$\frac{1}{\sqrt{2\pi}\,2\sigma}\,\frac{b_1}{b_4 D}\,\frac{e^{\frac{DR}{\sigma^2}}}{\Gamma(k)} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z (k-z)!}\left[2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right.$$

$$\left. + 2R\left(\frac{R}{\theta}\right) K_{k-z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right)\right] - \frac{\sigma^2 b_4^{\,2} b_1 e^{\frac{DR}{\sigma^2}} \alpha^k}{Db_4(\sigma^2 b_4^{\,2}+2D^2 b_4)}\,\frac{1}{\Gamma(k)} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^z\left(\frac{\sigma^2 b_4^{\,2}+2D^2 b_4-2D^2\alpha}{2D^2}\right)(k-z)!}$$

$$\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}} K_{k-z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\phi}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}\left(\frac{R\phi}{\sigma^2}\right)\right)$$

8.9.8.

From 8.9.6, substituting L+T for L we have

$$G_{19}(M,L+T) = \frac{2Db_1}{b_4(\sigma^2 b_4{}^2+2D^2 b_4)} e^{\left[L\left(\frac{\sigma^2 b_4{}^2+2D^2 b_4}{2D^2}\right) - \frac{b_4 M}{D}\right]}$$

$$F\left(\frac{M-(L+T)\left(D+\frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2(L+T)}}\right) + \frac{b_1}{b_4} * \left(\frac{(M-DT)-DL}{D\sqrt{\sigma^2(L+T)}}\right) \quad \frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}$$

$$- \frac{b_1}{Db_4} * F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) - \frac{\sigma^2 b_4{}^2 b_1 e^{\frac{2DM}{\sigma^2}}}{Db_4(\sigma^2 b_4{}^2+2D^2 b_4)} * F\left(\frac{M+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.9.9

Multiplying by H(L)

$$H(L)\,G_{19}(M,L+T) = \frac{2Db_1\alpha^k L^{k-1}}{b_4(\sigma^2 b_4{}^2+2D^2 b_4)\,\Gamma(k)} e^{\left[T\left(\frac{\sigma^2 b_4{}^2+2D^2 b_4}{2D^2}\right) - \frac{b_4 M}{D}\right]}$$

$$e^{L\left(\frac{\sigma^2 b_4{}^2+2D^2 b_4}{2D^2}-\alpha\right)} F\left(\frac{M-(L+T)\left(D+\frac{\sigma^2 b_4}{D}\right)}{\left(\sigma^2(L+T)\right)^{\frac{1}{2}}}\right) + \frac{b_1}{b_4} \frac{e^{-\alpha L}\alpha^k}{\Gamma(k)} \left(\frac{(M-DT)L^{k-1}-DL^k}{D\sqrt{\sigma^2(T+L)}}\right)$$

$$g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) - \frac{b_1}{Db_4} \frac{e^{-\alpha L}L^{k-1}\alpha^k}{\Gamma(k)} * F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

$$- \frac{\sigma^2 b_4{}^2 b_1 e^{-\alpha L}L^{k-1}\alpha^k}{Db_4(\sigma^2 b_4{}^2+2D^2 b_4)\,\Gamma(k)} F\left(\frac{M+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

8.9.10

$\int_0^\infty H(L)\,G_{19}(M,L+T)\,dL$ applying 8.2.18, 8.2.8, 8.2.30 and 8.2.34 and 8.2.35

$$G_{50}(M,T) = \frac{2Db_1\alpha^k e^{\left(T\left(\frac{\sigma^2 b_4^{\,2}+2D^2b_4}{2D^2} - b_4 M + \alpha T\right) + \frac{DM}{\sigma^2}\right)}}{2\sigma\,\Gamma(k)\; b_4(\sigma^2 b_4^{\,2}+2D^2 b_4)\;2} \sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} *$$

$$\sum_{z=1}^{k-j} \frac{(k-1-j)!}{\left(\frac{\sigma^2 b_4^{\,2}+2D^2 b_4}{2D^2}-\alpha\right)^z} \frac{1}{(k-z)!} \left(2\left(D+\frac{\sigma^2 b_4}{D}\right)\left(\frac{M}{\varepsilon}\right)^{k-j-z+\frac{1}{2}} * K_{k-j-z+\frac{1}{2}}\left(\frac{M\varepsilon}{\sigma^2}\right)\right.$$

$$\left. + \; 2M\left(\frac{M}{\varepsilon}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\varepsilon}{\sigma^2}\right)\right)$$

$$+ \frac{2b_1\alpha^k e^{\frac{DM}{\sigma^2}} e^{\alpha T}}{Db_4\,\Gamma(k)\;\sqrt{2\pi\sigma^2}} \sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} \left(\frac{M}{\varepsilon}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}\left(\frac{M\varepsilon}{\sigma^2}\right)$$

$$\frac{-\,b_1\alpha^k e^{\frac{DM}{\sigma^2}+\alpha T}}{\sqrt{2\pi}\,Db_2\;\;2\sigma\,\Gamma(k)} * \sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} \sum_{z=1}^{k-j} \frac{(k-1-j)!}{\alpha^z (k-j-z)!} \left(2D\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}}\right.$$

$$\left. K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$\frac{-\,\sigma^2 b_4^{\,2} b_1 \alpha^k e^{\frac{DM}{\sigma^2}+\alpha T}}{2\sigma D b_4(\sigma^2 b_4^{\,2}+2D^2 b_4)\,\Gamma(k)\sqrt{2\pi}} \sum_{j=0}^{k-1} (-T)^j \binom{k-1}{j} \sum_{z=1}^{k-j} \frac{(k-1-j)!}{\alpha^z (k-j-z)!} *$$

$$\left(-\,2D\left(\frac{M}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-\frac{1}{2}} K_{k-j-z-\frac{1}{2}}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

8.9.11

Hence integrating 8.9.1 over the states of L_1 and L_2, the inventory cost for model (M,T) when the lead times are continuous random variables is

$$C = \frac{(\beta c + S)}{T} + hc\left(M - \frac{Dk}{\alpha} - \frac{DT}{2}\right) + \frac{hc}{T}\left(G_{42}(R,T) - G_{39}(R)\right)$$

$$+ \frac{1}{T}\left(G_{50}(R,T) - G_{49}(R)\right)$$

8.9.12

SECTION 8.10 MODEL (M,R,T) EXPONENTIAL COST

In this section we derive the cost equations for the continuous lead times when the cost of a backorder is an exponential function of the length of time of a backorder by averaging over the states of the lead times, the corresponding cost for the fixed lead times.

In section 5.4., equation 5.4.10 $G_{21}(R+Y,T)$ is the expected cost of carrying inventory and backorders including the cost of a stockout dependent on the number of stockouts only, for fixed lead times L_1 and L_2. From equation 5.4.10, substituting L_1 for L and L_2+T for $T+L_2$.

$$G_{21}(R+Y,T) = hcT\left(R+Y-DL_1-\frac{DT}{2}\right) + hc\,(G_2(R+Y,T+L_2)$$

$$-\,G_2(R+Y,L_1)) + (G_{19}(R+Y,T+L_2) - G_{19}(R+Y,L_1))$$

$$+\quad s(Ro(R+Y,T+L_2) - Ro(R+Y,L_1)) \qquad 8.10.1$$

We shall exclude the cost dependent on the number of stockouts in determining the inventory costs for (M,R,T)

$$\text{Let } G_{51}(R+Y,T) = \int_0^{\infty}\int_0^{\infty} H(L_1)H(L_2)\Big(G_{21}(R+Y,T)-s(Ro(R+Y,L_2+T)$$

$$-\,Ro(R+Y,L_1))\Big)dL_1\,dL_2 \qquad 8.10.2.$$

Substituting for G_{21} (R+Y,T), we have

$$G_{51}(R+Y,T)=\int_0^{\infty}\int_0^{\infty} H(L_1)(H(L_2)\Big[hcT\left(R+Y-DL_1-\frac{DT}{2}\right)$$

$$+\,hc\big(G_2(R+Y,L_2T) - G_2(R+Y,L_1)\big) + \big(G_{19}(R+Y,T+L_2)-G_{19}(R+Y,L_1)\big)\Big]dy \qquad 8.10.3$$

Applying 8.9.2 to 8.9.5

$$G_{51}(R+Y,T) = hcT\left(R+Y - \frac{DK}{\alpha} - \frac{DT}{2}\right) + hc.\left(G_{42}(R+Y,T) - G_{39}(R)\right) + \left(G_{50}(R+Y,T) - G_{49}(R+Y)\right) \qquad 8.10.4$$

Hence the inventory cost for model (M,R,T) is obtained by replacing $G_{21}(R+Y,T)$ by $G_{51}(R+Y,T)$ in equation 5.4.11

$$C = \frac{R_c}{T} + S + \frac{\sum_{n=1}^{\infty}\int_0^{M-R} G_{51}(R+Y,T)\, f^n(M-R-Y,DT)\, dY + G_{51}(M,T)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=1}^{\infty}\int_0^{M-R} n\, f^{n-1}(M-R-Y,DT) * F\left(\frac{Y-DT}{\sqrt{\sigma^2 T}}\right) dY\right]} \qquad 8.10.5$$

SECTION 8.11 LINEAR COST TERM

The expected backorder ~~cost~~ when the cost of a backorder is a linear function of the length of a backorder can be obtained from the corresponding expected backorder cost when the cost is a quadratic function.

When the cost of a backorder is a quadratic function

$$C_\beta(t) = b_1 + b_2 t + b_3 t^2$$

The corresponding linear cost per length of time of a backorder is obtained by setting $b_3 = 0$.

Hence by setting $b_3 = 0$ in the corresponding quadratic inventory costs we have for Model (Q,R) from 8.3.37

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\left(R - \frac{Dk}{\alpha}\right) + \frac{b_1}{Q}\left(G_{29}(R) - G_{29}(R+Q)\right)$$

$$+ \frac{(hc + b_2)}{Q}\left(G_{30}(R) - G_{30}(R+Q)\right)$$

8.11.1

For Model (Q,R,T) from 8.4.46

$$C = \frac{Rc}{T} + \frac{S.POR}{T} + hc\left(\frac{Q}{2} + R - \frac{Dk}{\alpha} - \frac{DT}{2}\right) +$$

$$\frac{b_1}{QT}\left(G_{35}(R,T) - G_{32}(R) - G_{35}(R+Q,T) + G_{32}(R+Q)\right)$$

$$+ \frac{(hc+b_2)}{QT}\left(G_{36}(R,T) - G_{33}(R) - G_{36}(R+Q,T) + G_{33}(R+Q)\right)$$

8.11.2

for model (M,T) from 8.5.28

$$C = \frac{Rc+S}{T} + hc\left(M - \frac{Dk}{\alpha} - DT\right) + \frac{b_1}{T}\left(G_{41}(R,T) - G_{38}(R)\right)$$

$$+ \frac{(hc+b_2)}{T}\left(G_{42}(R,T) - G_{39}(R)\right)$$

8.11.3

Hence from 8.6.5

$$C = \frac{\frac{R_c}{T} + S + \sum_{n=1}^{\infty} \int_0^{M-R} G_{52}(R+Y,T) \, f^{n}(M-R-Y,DT) \, dY + G_{52}(M,T)}{T \left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=1}^{\infty} \int_0^{M-R} n f^{n-1}(M-R-Y,DT) \, F\left(\frac{Y-DT}{\sqrt{\sigma^2 T}}\right) dY \right]} \qquad 8.11.4$$

CHAPTER 9

Random Supply And Constant Lead Times.

It has been assumed in the previous chapters that the batch quantity ordered was always equal to the quantity received.
For example in (Q,R), Q could be thought of as a random variable when scrap exists. In this case R the reorder level becomes the controllabe variable in the inventory control procedure while in (nQ,R,T) R and T become the controlable variables.

In (M,T) and (M,R,T) the quantity ordered varies with each review. However, since the same quantity ordered is not necessarily received, M the maximum order cover varies as a result. The probability density function of Supply is assumed to be gamma distributed.

The use of the probability density function makes R the only controllable variable in (Q,R). However the approach of this chapter is such that the expected value of Q denoted $\bar{Q}$ as well as R could be used as the controllabe variables. By assigning the desired values of μ and v to the probability density function, the desired value of the expected value of Q can be obtained. The same analysis applies to the other models.

In chapters 3 and 4 we developed the cost equations for fixed batch quantities and fixed maximum order cover M. The cost equations when the supply is random are obtained by averaging the cost equations for fixed supply over the states of the batch quantity. In section 9.1 we develop the basic mathematics required for the chapter. We develop the cost equations for models (Q,R), (nQ,R,T) (M,T) and (M,R, T) when the backorder cost is a quadratic function, in sections 9.2, 9.3, 9.4, 9.5 respectively. In sections 9.6, 9.7, 9.8, 9.9 we develop the cost equations when the backorder cost is an exponential function for the models.

Section 9.1 Basic mathematics

Let U(Q) be the probability density function of supply.

Let $U(Q) = \dfrac{\exp(-\mu Q)\, Q^{v-1}}{\mu^{v}\, \Gamma(v)}$ $\quad Q>0 \qquad$ 9.1.1

Let $g\left(\dfrac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) = \dfrac{\exp-\frac{1}{2}\left(\dfrac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2}{\sqrt{2\pi}}$ $\qquad$ 9.1.2

Now we derive

$$\int_0^\infty U(Q)\, g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) dQ \qquad 9.1.3$$

Substituting for U(Q) and $g\left(\dfrac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$ we have

$$\int_0^\infty \frac{\exp-\mu Q \quad Q^{v-1}}{\Gamma(v)} \exp-\frac{1}{2}\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2 dQ \qquad 9.1.4$$

Simplifying we have

$$\frac{1}{\sqrt{2\pi}} \exp\left(\frac{2R\mu\sigma^2 L+\mu^2\sigma^4 L^2 -2D\mu\sigma^2 L^2}{2\sigma^2 L}\right) \int_0^\infty \frac{Q^{v-1}\mu^{v}}{\Gamma(v)} \cdot \exp-\frac{1}{2}\left(\frac{Q-(DL-R-\mu\sigma^2 L)}{\sqrt{\sigma^2 L}}\right)^2 dQ \qquad 9.1.5$$

But $\displaystyle\int_0^\infty \exp Qt.\exp-\frac{1}{2}\left(\frac{Q-(DL-R-\mu\sigma^2 L)}{\sqrt{\sigma^2 L}}\right)^2 dQ$

$$= \exp\left((DL-R-\mu\sigma^2 L)t + \frac{\sigma^2 t^2}{2} \right) \qquad 9.1.6$$

Expanding

$$= \sum_{j=0}^{\infty} \frac{\left[(DL-R-\mu\sigma^2 L)t + \frac{\sigma^2 t^2 L}{2}\right]^j}{j!}$$

9.1.7.

The coefficient of t^{v-1} $\displaystyle\sum_{j=0}^{(v-1)/2} \binom{v-1-j}{j} A^{v-1-2j} B^j$

9.1.8.

Where $A=(DL-R-\mu\sigma^2 L)$ and $B = \sigma^2 L/2.0$

Hence

$$\frac{\int_0^{\infty} Q^{v-1} \exp -\frac{1}{2}\left(\frac{Q-(DL-R-\mu\sigma^2 L)}{\sqrt{\sigma^2 L}}\right)^2 dQ}{\mu^v \Gamma(v).} \cdot e^{\left(R\mu + \mu^2 \frac{\sigma^2 L}{2} - D\mu L\right)}$$

$$= \frac{\sigma^2 L}{\Gamma(v)} \mu^v e^{\left(R\mu + \frac{\mu^2 \sigma^2 L}{2} - D\mu L\right)} \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} (DL-R-\mu\sigma^2 L)^{v-1-2i} \left(\frac{\sigma^2 L}{2}\right)^i$$

9.1.9.

Hence

$$\mu^v \frac{\int_0^{\infty} e^{-\mu Q} Q^{v-1} \; e^{-\frac{1}{2}\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2} dQ}{\Gamma(v) \sqrt{2\pi}}$$

$$= \frac{\sqrt{\sigma^2 L}}{\Gamma(v)} \mu^v \; e^{\left(R\mu + \frac{\mu^2 \sigma^2 L}{2} - D L\right)} \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} (DL-R-\mu\sigma^2 L)^{v-1-2i} \left(\frac{\sigma^2 L}{2}\right)^i$$

9.1.10

$$\int_0^{\infty} U(Q) \, F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) dQ$$

9.1.11

Substituting for U(Q) we have

$$\int_0^{\infty} \frac{e^{-\mu Q} Q^{v-1}}{\Gamma(v)} \; F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) dQ$$

9.1.12

Noting that

$$d\left(-\sum_{z=1}^{v} \frac{(v-1)! \, Q^{v-z} e^{-\mu Q}}{\mu^z (v-z)!}\right) = e^{-\mu Q} Q^{v-1}$$

9.1.13.

Then integrating by parts we have

$$\frac{\mu^v}{\Gamma(v)}\left[-\sum_{z=1}^{v} \frac{(v-1)! \, Q^{v-z} e^{-\mu Q}}{\mu^z (v-z)} F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right]_0^{\infty} + \frac{\mu^v}{\Gamma(v)} \int_0^{\infty} \sum_{z=1}^{v} \frac{(v-1)!}{\mu^z} \frac{Q^{v-z} e^{-\mu Q}}{\Gamma(v)(v-z)!} \cdot$$

$$\frac{g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)}{\sqrt{\sigma^2 L}} dQ$$

9.1.14

Simplifying we have

$$\frac{\mu^{v}}{\Gamma(v)} \sum_{z=1}^{v} \frac{(v-1)!}{\mu^{z}(v-z)!} \frac{Q^{v-z} e^{-\mu Q}}{\sqrt{\sigma^{2} L}} g\left(\frac{R+Q-DL}{\sqrt{\sigma^{2} L}}\right) dQ \qquad 9.1.15$$

Applying 9.1.10 and integrating we have 9.1.11 equals

$$\frac{\mu^{v}}{\Gamma(v)} \sum_{z=1}^{v} \frac{(v-1)!}{\mu^{z}(v-z)!} \sum_{i=0}^{v-z/2} \binom{v-z-i}{i} \left(DL-R-\mu\sigma^{2} L\right)^{v-z-2i} \left(\frac{\sigma^{2} L}{2}\right)^{i} *$$

$$e^{\left(R\mu+\mu^{2}\cdot\frac{\sigma^{2} L}{2} - D\mu L\right)} .$$

9.1.16

$$\text{Now } \frac{\mu^{v}}{\Gamma(v)} \int_{0}^{\infty} Q^{v-1} e^{-\mu Q} e^{\frac{2D(R+Q)}{\sigma^{2}}} F\left(\frac{R+Q+DL}{\sqrt{\sigma^{2} L}}\right) dQ$$

9.1.18

Simplifying

$$= \frac{\mu^{v}}{\Gamma(v)} e^{\frac{2DR}{\sigma^{2}}} \int_{0}^{\infty} Q^{v-1} e^{-\left(\mu-\frac{2D}{\sigma^{2}}\right)Q} F\left(\frac{R+Q+DL}{\sqrt{\sigma^{2} L}}\right) dQ$$

9.1.19

Noting that

$$d\left(-\sum_{z=1}^{v} \frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^{2}}\right)^{z}} Q^{v-z} e^{-\left(\mu-\frac{2D}{\sigma^{2}}\right)Q}\right) = e^{-\left(\mu-\frac{2D}{\sigma^{2}}\right)Q} * Q^{v-1}$$

then integrating by parts we have

$$e^{\frac{2DR}{\sigma^{2}}} \frac{\mu^{v}}{\Gamma(v)} \left(- \sum_{z=1}^{v} \frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^{2}}\right)^{z}} Q^{v-z} e^{-\left(\mu-\frac{2D}{\sigma^{2}}\right)Q} * F\left(\frac{R+Q+DL}{\sqrt{\sigma^{2} L}}\right)\right)_{0}^{\infty}$$

$$+ \frac{\mu^{v}}{\Gamma(v)} \frac{e^{\frac{2DR}{\sigma^{2}}}}{\sqrt{\sigma^{2} L}} \int_{0}^{\infty} \sum_{z=1}^{v} \frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^{2}}\right)^{z}} Q^{v-z} * \exp-\left(\mu-\frac{2D}{\sigma^{2}}\right) \exp-\frac{1}{2}\left(\frac{R+Q+DL}{\sqrt{\sigma^{2} L}}\right)^{2} dQ$$

9.1.20.

Simplifying we have

$$\frac{\mu^{v} \exp\frac{2DR}{\sigma^{2}}}{\Gamma(v)\ \sigma^{2} L} \int_{0}^{\infty} \sum_{z=1}^{v} \frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^{2}}\right)^{z}} Q^{v-z} e^{-\left(\mu-\frac{2D}{\sigma^{2}}\right)*Q} * \exp-\frac{1}{2}\left(\frac{R+Q+DL}{\sqrt{\sigma^{2} L}}\right)^{2} dQ$$

Simplifying we have

$$\frac{\mu^{v}}{\Gamma(v)\sqrt{\sigma^{2} L}} \sum_{z=1}^{v} \frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^{2}}\right)^{z}} \int_{0}^{\infty} Q^{v-z} * \exp-\frac{1}{2}\left(\frac{Q-(DL-R-\mu\sigma^{2} L)}{\sqrt{\sigma^{2} L}}\right)^{2} dQ \ *$$

$$\exp\left(R\mu+\mu^{2}\frac{\sigma^{2} L}{2}-D\mu L\right) \qquad 9.1.21$$

Applying 9.1.9. we have

$$\frac{\mu^v}{\Gamma(v)} \exp\left(R\mu+\frac{\mu^2\sigma^2 L}{2}-D\mu L\right) \sum_{z=1}^{v} \frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^2}\right)^z (v-z)!} \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} \left(DL-R-\mu\sigma^2 L\right)^{v-1-2i} \left(\frac{\sigma^2 L}{2}\right)^i$$

9.1.22.

Hence

$$\frac{\mu^v}{\Gamma(v)} \int_0^Q Q^{v-1}\, e^{-\mu Q}\, e^{\frac{2D(R+Q)}{\sigma^2}}\, F\left(\frac{R+Q+DL}{\sqrt{\sigma^2 L}}\right) dQ$$

$$= \frac{\mu^v}{\Gamma(v)} e^{\left(R\mu+\mu^2\frac{\sigma^2}{2}L-D\mu L\right)} \sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^2}\right)^z (v-z)!} \binom{v-z-i}{i} \left(DL-R-\mu\sigma^2 L\right)^{v-z-2i} \left(\frac{\sigma^2 L}{2}\right)^i$$

9.1.23

Section 9.2. Model (Q,R).

From 4.3.13 the inventory cost for model (Q,R) with quadratic backorder cost substituting $(R-DL)/\sqrt{\sigma^2 L}$ for k

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + (R-DL)hc + \frac{Db_1}{Q}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{(b_2 + hc)}{Q} \times \left(\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. - \beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{b_3}{DQ}\left(\theta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \theta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) + \frac{Ds}{Q}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

9.2.1.

where from 3.1.4:

$$\alpha(k) = \sqrt{\sigma^2 L}\,(g(k) - kF(k))$$

9.2.3.

from 3.1.8.

$$\beta(k) = \frac{\sigma^2 L}{2}\left((1+k^2)\,F(k) - k\,g(k)\right)$$

9.2.3.

from 4.3.10.

$$\theta(k) = \frac{\sigma^3 L^{3/2}}{3}\left((k^2 + 2)\, g(k) - k(3 + k^2)\, F(k)\right)$$

9.2.4.

Let $$G_{53}(R,L) = \int_0^\infty \frac{U(Q)}{Q}\,\alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) dQ$$

9.2.5.

substituting for U(Q) and $\frac{R+Q-DL}{\sqrt{\sigma^2 L}}$ from 9.1.1. and 9.2.2. respectively.

$$G_{53}(R,L) = \sqrt{\sigma^2 L} \int_0^\infty \frac{e^{-\mu Q}\, Q^{v-2}}{\mu^v\,\Gamma(v)} \left(g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) dQ$$

9.2.6.

Simplifying

$$G_{53}(R,L) = \sqrt{\sigma^2 L} \int_0^\infty \mu^v \frac{e^{-\mu Q}}{\Gamma(v)} Q^{v-2} \left(g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) dQ$$

9.2.6.

Integrating and applying 9.1.10 and 9.1.16.

$$G_{53}(R,L) = \frac{\sigma^2 L \mu^v}{\Gamma(v)} e^{\left(R\mu + \frac{\mu^2\sigma^2 L}{2} - D\mu L\right)} \sum_{i=0}^{\frac{v-2}{2}} \binom{v-2-i}{i} \left(DL-R-\mu\sigma^2 L\right)^{v-2-2i} \left(\frac{\sigma^2 L}{2}\right)^i$$

$$-\frac{\mu^v}{\Gamma(v)} \cdot e^{\left(R\mu + \frac{\mu^2\sigma^2 L}{2} - D\mu L\right)} * (R-DL) \sum_{z=1}^{v-1} \sum_{i=0}^{\frac{v-1-z}{2}} \frac{(v-2)!}{\mu^z (v-1-z)!} \binom{v-z-i-1}{i} \left(DL-R-\mu\sigma^2 L\right) \left(\frac{\sigma^2 L}{2}\right)^i$$

$$-\frac{\mu^v}{\Gamma(v)} e^{\left(R\mu + \frac{\mu^2\sigma^2 L}{2} - D\mu L\right)} \sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\mu^z (v-z)!} \binom{v-z-i}{i} \left(DL-R-\mu\sigma^2 L\right)^{v-z-2i} \left(\frac{\sigma^2 L}{2}\right)^i$$

9.2.8

From 9.2.3. substituting $(R+Q-DL)/\sqrt{\sigma^2 L}$ for k

$$\beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) = \frac{\sigma^2 L}{2} * \left(\left(1+\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2 F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) * g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)\right)$$

9.2.9.

Simplifying we have

$$\beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) = \left(\frac{\sigma^2 L}{2} + Q^2 + Q(2R-2DL) + \left(R^2 - DLR + D^2 * L^2\right)\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\sqrt{\frac{\sigma^2 L}{2}} \left((R-DL) + Q\right) g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$$

Hence

9.2.10

$$\frac{U(Q)}{Q} \beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) = \mu^v \frac{e^{-\mu Q}}{\Gamma(v)} \left[\left(\sigma^2 L + R^2 - DLR + D^2 * L^2\right) Q^{v-2} + 2(R-DL) * Q^{v-1} + Q^v\right] F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \frac{\sqrt{\sigma^2 L}}{2} \mu^v e^{-\mu Q}\left[(R-DL)Q^{v-2} + Q^{v-1}\right] g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$$

9.2.11.

Let $G_{54}(R,L) = \int_0^{\infty} U(Q)\beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) dQ$ 9.2.12

Substituting for $\frac{U(Q)}{Q}\beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$ in 9.2.12. Integrating and Applying 9.1.10 and 9.1.16 We have

$$G_{54}(R,L) = \frac{\mu^v}{\Gamma(v)} e^{\left(R\mu+\mu^2\frac{\sigma^2}{2}L-D\mu L\right)}\Bigg[(\sigma^2 L+R^2-DLR+D^2 L^2) * \sum_{Z=1}^{v-1}\frac{(v-2)!}{\mu^Z(v-1-Z)!}$$

$$\sum_{i=0}^{(v-Z-1)/2}\binom{v-Z-i-1}{i}(DL-R-\mu\sigma^2 L)^{v-1-Z-2i}\left(\frac{\sigma^2 L}{2}\right)^i + 2(R-DL)\sum_{Z=1}^{v}\frac{(v-1)!}{\mu^Z(v+1-Z)!}\sum_{i=0}^{\frac{v-Z}{2}}\binom{v-Z-i}{i} *$$

$$(DL-R-\mu\sigma^2 L)^{v-Z-2i} * \left(\frac{\sigma^2 L}{2}\right)^i$$

$$+\sum_{Z=1}^{v+1}\frac{(v)!}{\mu^Z(v+2-Z)!}\sum_{i=0}^{\frac{v-Z+1}{2}}\binom{v+1-Z-i}{i} * (DL-R-\sigma^2\mu L)^{v+1-Z-2i}(\sigma^2 L/2)^i\Bigg]$$

$$-\frac{\sigma^2 L}{2\Gamma(v)} e^{\left(R\mu+\frac{\sigma^2\mu^2}{2}L-D\mu L\right)}\Bigg[(R-DL)\sum_{i=0}^{(v-2)/2}\binom{v-2-i}{i} * (DL-R-\sigma^2\mu L)^{v-2-2i}\left(\frac{\sigma^2 L}{2}\right)^i$$

$$+\sum_{i=0}^{\frac{v-1}{2}}\binom{v-i-1}{i}(DL-R-\sigma^2\mu L)^{v-1-2i}\left(\frac{\sigma^2 L}{2}\right)^i\Bigg] \quad 9.2.13.$$

From 9.2.4. substituting $\frac{R+Q-DL}{\sqrt{\sigma^2 L}}$ for k

$$\theta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) = \frac{\sigma^3 L^{3/2}}{3}\left(\left(\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2+2\right) g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) *\right.$$

$$\left.\left(3+\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2\right)\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) \quad 9.2.14.$$

Simplifying

$$\theta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) = \frac{\sigma^3 L^{3/2}}{3}\left(\left(\frac{Q^2+2Q(R-DL)+(R^2-2DL+D^2 L^2+2\sigma^2 L)}{\sigma^2 L}\right)\right.$$

$$g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \frac{1}{\sigma^3 L^{3/2}}\left(Q^3+3Q^2(R-DL)+3Q\left((R-DL)^2+\sigma^2 L\right)+\left((R-DL)^2+3\sigma^2 L\right)\right.$$

$$\cdot(R-DL)\Big) \quad \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) \qquad 9.2.15.$$

Multiplying by $U(Q)\phi\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$

$$\frac{U(Q)}{Q}\phi\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) = \frac{e^{-\mu Q}}{3\Gamma(v)}\sigma L^{\frac{1}{2}}\Big(Q^v + 2Q^{v-1}\cdot(R-DL) + \big((R-DL)^2 + 2\sigma^2\cdot L\big)Q^{v-2} \quad g\left(\frac{R+Q-DL}{\sigma^2 L}\right)$$

$$-\frac{e^{-\mu Q}\mu^v}{3\Gamma(v)}\Big(Q^{v+1} + 3(R-DL)\,Q^v + 3\big((R-DL)^2 + \sigma^2\cdot L\big)\,Q^{v-1} + \big((R-DL)^2 + 3\sigma^2 L\big)(R-DL)\cdot$$

$$Q^{v-2}\Big)F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) \qquad 9.2.16.$$

$$\text{Let } G_{55}(R,L) = \int_0^\infty U(Q)\phi\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)dQ \qquad 9.2.17$$

Substituting for $U(Q)\phi\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$ integrating and applying 9.1.10 and 9.1.16. we have

$$G_{55}(R,L) = \frac{\mu^v\sigma^2 L}{3\Gamma(v)}\cdot e^{\left(R\mu+\frac{\mu^2\sigma^2}{2}L-D\mu L\right)}\Bigg[\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}\left(DL-R-\mu\sigma^2 L\right)^{v-2i}\left(\frac{\sigma^2 L}{2}\right)^i$$

$$+2(R-DL)\sum_{i=0}^{(v-1)/2}\binom{v-i-1}{i}\left(DL-R-\mu\sigma^2 L\right)^{v-2i-1}\left(\frac{\sigma^2 L}{2}\right)^i + \left((R-DL)^2+2\sigma^2 L\right)\cdot$$

$$\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i}\left(DL-R-\mu\sigma^2 L\right)\left(\frac{\sigma^2 L}{2}\right)^i\Bigg] - \mu^v\frac{e^{\left(R\mu+\frac{\mu^2\sigma^2 L}{2}-D\mu L\right)}}{3\Gamma(v)}\Bigg[$$

$$\sum_{z=1}^{v-2}\sum_{i=0}^{\frac{v+1}{2}}\binom{v+1-i}{i}\frac{(v+1)!}{}\left(DL-R-\sigma^2\mu L\right)^{v+2-z-2i}\left(\frac{\sigma^2 L}{2}\right)^i + 3(R-DL)\sum_{z=1}^{v+1}$$

$$\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}\frac{v!}{\mu^z(v+2-z)!}\left(DL-R-\mu\sigma^2 L\right)^{v-z-2i}\left(\frac{\sigma^2 L}{2}\right)^i + 3\left((R-DL)^2+\sigma^2 L\right)\sum_{z=1}^{v}\sum_{i=0}^{\frac{v-1}{2}}\binom{v-i-1}{i}\cdot$$

$$\frac{(v-1)!}{\mu^z(v-z-1)!}\left(DL-R-\mu\sigma^2 L\right)^{v-2-z-2i}\left(\frac{\sigma^2}{2}L\right)^i + \left((R-DL)^2+3\sigma^2 L\right)\cdot$$

$$(R-DL) \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-2}{2}} \binom{v-2-i}{i} \frac{(v-1)!}{\mu^{z}(v-z-i)!} (DL-R-\mu\sigma^2 L)^{v-2-z-2i} \left(\frac{\sigma^2 L}{2}\right)^{i} \Big] \quad 9.2.18$$

Hence averaging the inventory costs for fixed Q, (equation 9.2.1) over the states of Q we have the inventory costs for (Q,R), when Q is a random variable and backorder cost is a quadratic function of the length of time of a backorder.

$$C = \frac{DS\mu}{(v-1)} + \frac{hcv}{2\mu} + (R-DL)hc + Db_1\left(\frac{\mu\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)}{(v-1)} - G_{53}(R,L)\right) + (b_2 + hc) *$$

$$\left(\frac{\mu\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)}{(v-1)} - G_{54}(R,L)\right) + \frac{b_3}{D}\left(\frac{\theta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\mu}{(v-1)} - G_{55}(R,L)\right) + Ds\left(\frac{\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)}{(v-1)} - G_{53}(R,L)\right)$$

9.2.19.

Section 9.3. (Q,R,T) Quadratic backorder costs.

The inventory costs for fixed Q and constant lead time is given in section 4.4. equation 4.4.29. The inventory costs when the supply is a random variable are obtained by averaging the inventory costs when Q is fixed over the states of Q. The inventory costs when Q is fixed is

$$C = \frac{Rc}{T} + \frac{S.POR}{T} + hc\left(\frac{Q}{2} + R-DL-\frac{DT}{2}\right) + \frac{b_1}{QT}\Big(G_1(R,T+L) - G_1(R,L) - G_1(R+Q,T+L)$$

$$+ G_1(R+Q,L)\Big) + \frac{(hc + b_2)}{QT}\left(G_3(R,T+L) - G_3(R,L) - G_3(R+Q,T+L) + G_3(R+Q,L)\right)$$

$$+\frac{b_3}{QT}\Big(G_{11}(R,T+L) - G_{11}(R,L) - G_{11}(R+Q,T+L) + G_{11}(R+Q,L)\Big) + \frac{s}{QT}\Big(G_4(R,T+L) - G_4$$

$$(R,L) - G_4(R+Q,T+L) + G_4(R+Q,L)\Big) \quad 9.3.1.$$

To be able to obtain the inventory costs when supply is a random variable we need the following

The p.d.f of Q, $U(Q) = \mu^{v} e^{-\mu Q} Q^{v-1} / \Gamma(k)$

$$G_{60}(R,T) = \int_0^\infty \frac{U(Q)}{Q} G_1(R+Q,T)\, dQ \quad 9.3.2.$$

$$G_{61}(R,T) = \int_0^\infty \frac{U(Q)}{Q} G_3(R+Q,T)\, dQ \quad 9.3.3.$$

$$G_{62}(R,T) = \int_0^\infty \frac{U(Q)}{Q} G_{11}(R+Q,T)\, dQ \quad 9.3.4.$$

$$G_{63}(R,T) = \int_0^\infty \frac{U(Q)}{Q} G_4(R+Q,T)\, dQ \quad 9.3.5.$$

From 3.4.8. $G_1(R+Q,T) = \frac{\sigma^2 T}{2}\left(\left(1 + \left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)^2\right)F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)\right)$ 9.3.6.

Simplifying

$$G_1(R+Q,T) = \frac{1}{2}\left[\sigma^2 T + Q^2 - 2Q*(DT-R) + (R-DT)^2\right] F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{\sqrt{\sigma^2 T}}{2}(R+Q-DT)\, g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)$$

9.3.7.

Multiplying by U(Q) we have

$$\frac{U(Q)\, G_1(R+Q,T)}{Q} = \frac{\mu^v e^{-\mu Q}}{(v)}\left(\left(\sigma^2 T + (R-DT)^2 * Q^{v-2} - 2Q^{v-1}(DT-R) + Q^v\right)F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)\right.$$

9.3.8.

$$-\frac{\sqrt{\sigma^2 T}}{2}\mu^v \frac{e^{-\mu Q}}{\Gamma(v)}\left((R-DT)\,Q^{v-2} + Q^{v-1}\right) g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$\int_0^\infty U(Q)\, G_1(R+Q,T)\, dQ$, applying 9.1.10. and 9.1.16. we obtain

$$G_{60}(R,T) = \frac{\mu^v}{\Gamma(v)} e^{\left(R\mu + \frac{\mu^2\sigma^2}{2}L - D\mu L\right)}\left[\left(\sigma^2 T + (R-DT)^2\right)\sum_{z=1}^{v-1}\sum_{i=0}^{(v-z-1)/2}\frac{(v-2)!}{\mu^z (v-z)!} *\right.$$

$$\binom{v-z-i-1}{i} * (DT-R\ \sigma^2\mu T)^{v-z-2i-1}\left(\frac{\sigma^2 * T}{2}\right)^i - 2(DT-R)\sum_{z=1}^{v}\sum_{i=0}^{\frac{v-z}{2}}\frac{(v-1)!}{\mu^z (v-z)!}$$

$$\binom{v-z-i}{i} * (DT-R-\sigma^2\mu T)^{v-z-2i}\left(\frac{\sigma^2 * T}{2}\right)^i + \sum_{z=1}^{v+1}\sum_{i=0}^{v+1-z}\frac{v!}{\mu^z (v+2-z)!}\binom{v+1-z-i}{i} *$$

$$\left.(DT-R-\sigma^2\mu T)^{v+1-z-2i}\left(\frac{\sigma^2 * T}{2}\right)^i\right] - \frac{\sigma^2 * T}{2}\frac{\mu^v}{\Gamma(v)} e^{\left(R\mu + \frac{\mu^2\sigma^2}{2}L - D\mu L\right)}\left[(R-DT) *\right.$$

$$\left.\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i}(DT-R-\sigma^2\mu T)^{v-2-2i}\left(\frac{\sigma^2 T}{2}\right)^i + \sum_{i=0}^{\frac{v-1}{2}}\binom{v-i-1}{i}(DT-R-\mu\sigma^2 T)^{v-2i-1} * \left(\frac{\sigma^2 T}{2}\right)^i\right]$$

9.3.9.

From 3.4.21.

$$G_3(R+Q,T) = \left(\frac{D^2 * T^3}{6} - \frac{\sigma^4 (R+Q)}{4D^3} - \frac{DT^2(R+Q)}{2} - \sigma^2\frac{(R+Q)^2}{4D^2} + \frac{\sigma^2 * T^2}{4} + T\frac{(R+Q)^2}{2} - \frac{(R+Q)^3}{6D}\right.$$

$$\left.- \frac{\sigma^6}{8D^4}\right) F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) + \sigma T^{\frac{1}{2}}\left(\frac{D * T^2}{6}\, T\frac{(R+Q)}{3} + \frac{(R+Q)^2}{6D} + \frac{\sigma^2 * T}{12D} + \sigma^2\frac{(R+Q)}{4D^2} + \frac{\sigma^4}{4D^3}\right) *$$

$$g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^6}{8D^4} e^{\frac{2D(R+Q)}{\sigma^2}} F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.3.10.$$

Simplifying

$$G_3(R+Q,T) = \left(\left(\frac{D^2T^3}{6} - \frac{\sigma^4 R}{4D^3} - \frac{DT^2 R}{2} - \frac{\sigma^2 R^2}{4D^2} + \frac{\sigma^2 T^2}{4} + \frac{TR^2}{2} + \frac{R^3}{6D} - \frac{\sigma^6}{8D^4}\right)\right.$$

$$+ Q\left(-\frac{\sigma^4}{4D^3} - \frac{DT^2}{2} - \frac{2R\sigma^2}{4D^2} + \frac{2TR}{2} + \frac{3R^2}{6D}\right) + Q^2\left(-\frac{\sigma^2}{4D^2} + \frac{T}{2} + \frac{3R}{6D} + \frac{Q^3}{6D}\right)\Bigg) F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \sigma T^{1/2}\left(\left(\frac{DT^2}{6} - \frac{TR}{3} + \frac{R^2}{6D} + \frac{\sigma^2 T}{12D} + \frac{\sigma^2 R}{4D^2} + \frac{\sigma^4}{4D^3}\right) + Q\left(-\frac{T}{3} + \frac{2R}{6D} + \frac{\sigma^2}{4D^2}\right) + \frac{Q^2}{6D}\right) *$$

$$g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^6}{8D^4} e^{\frac{2D(R+Q)}{\sigma^2}} * F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.3.11.$$

Multiplying by $\frac{U(Q)}{Q}$ we have

$$\frac{U(Q)\, G_3\,(R+Q,T)}{Q} = \frac{\mu^v e^{-\mu Q}}{\Gamma(v)}\left(\left(\left(\frac{D^2T^3}{6} - \frac{\sigma^4 R}{4D^3} - \frac{DT^2 R}{2} - \frac{\sigma^2 R^2}{4D^2} + \frac{\sigma^2 T^2}{4} + \frac{TR^2}{2} + \frac{R^3}{6D}\right.\right.\right.$$

$$\left.-\frac{\sigma^6}{8D^4}\right) Q^{v-2} + \frac{Q^{v+1}}{6D}\left(-\frac{\sigma^4}{4D^2} - \frac{DT^2}{2} - \frac{R\sigma^2}{2D^2} + TR + \frac{R^2}{2D}\right) + Q^v\left(-\frac{\sigma^2}{4D^2} + \frac{T}{2} + \frac{R}{2D} + \frac{Q^{v+1}}{6D}\right)\Bigg) *$$

$$F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) + \sqrt{\sigma^2 T}\left(\left(\frac{DT^2}{6} - \frac{TR}{3} + \frac{R^2}{6D} + \frac{\sigma^2 T}{12D} + \frac{\sigma^2 R}{4D^2} + \frac{\sigma^4}{4D^3}\right) Q^{v-2} + Q^{v-1} *\right.$$

$$\left.\left(-\frac{T}{3} + \frac{2R}{6D} + \frac{\sigma^2}{4D^2}\right) + \frac{Q^{v+1}}{6D}\right) g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)\Bigg) + \frac{\sigma^6}{8D^4}\frac{Q^{v-2}}{\Gamma(v)} e^{\left(-\mu Q + \frac{2D(R+Q)}{\sigma^2}\right)} F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.3.12.$$

Hence

$\int_0^\infty \frac{U(Q)}{Q} G_3\ (R+Q,T)\ dQ$, applying 9.1.10., 9.1.16. and 9.1.23. we have

$$G_{61}(R,T) = \frac{\mu^v}{\Gamma(v)} e^{\left(R\mu + \mu^2\frac{\sigma^2 T}{2} - D\mu T\right)}\Bigg[\left(\frac{D^2T^3}{6} - \frac{\sigma^4 R}{4D^2} - \frac{DT^2 R}{2} - \frac{\sigma^2 R^2}{4D^2} + \frac{\sigma^2 T^2}{4} + \frac{TR^2}{2}\right.$$

$$\left.+\frac{R^3}{6D} - \frac{\sigma^6}{8D^4}\right)\sum_{z=1}^{v-1}\sum_{i=0}^{\frac{v-z-1}{2}} \frac{(v-2)!}{\mu^z (v-1-z)!}\binom{v-z-i-1}{i} \times (DT-R-\mu\sigma^2 L)^{v-1-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i$$

$$+\left(\frac{-\sigma^4}{4D^2} - \frac{DT^2}{2} - \frac{R\sigma^2}{2D^2} + TR + \frac{R^2}{2D}\right)\sum_{z=1}^{v}\sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\mu^z (v-z)!}\binom{v-z-i}{i} * (DT-R-\mu\sigma^2 T)^{v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i$$

$$+\left(\frac{-\sigma^2}{4D^2} + \frac{T}{2} + \frac{R}{2D}\right)\sum_{z=1}^{v+1}\sum_{i=0}^{(v+1-z)/2} \frac{v!}{\mu^z (v-z+1)!}\binom{v+1-z-i}{i} (DT-R-\mu\sigma^2 T)^{v+1-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i$$

$$+\frac{1}{6D}\sum_{z=1}^{v+1}\sum_{i=0}^{(v+1-z)/2} \frac{(v+1)!}{\mu^z (v-z+2)!}\binom{v+1-z-i}{i}\left(DT-R-\mu\sigma^2 T\right)^{v+1-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i\Bigg]$$

$$+ \frac{\sigma^2 T \mu^v e^{(R\mu+\mu^2\frac{\sigma^2}{2}T - D\mu T)}}{\Gamma(v)} \left[\left(\frac{D^2T^2}{6} - \frac{TR}{3} + \frac{R^2}{6D} + \frac{\sigma^2 T}{12D} + \frac{\sigma^2 R}{4D^2} + \frac{\sigma^4}{4D^3} \right) \sum_{i=0}^{\frac{v-2}{2}} \binom{v-2-i}{i} * (DT-R-\mu\sigma^2T)^{v-2-2i} \left(\frac{\sigma^2 T}{2}\right)^i + \left(-\frac{T}{3} + \frac{R}{3D} + \frac{\sigma^2}{4D^2}\right) \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i}\binom{v-1-i}{i} (DT-R-\mu\sigma^2 T)^{v-1-2i} \left(\frac{\sigma^2 T}{2}\right)^i + \frac{1}{6D}\sum_{i=0}^{v} \binom{v-i}{i} (DT-R-\mu\sigma^2T)^{v-2i} \left(\frac{\sigma^2T}{2}\right)^i \right] + \frac{\sigma^6}{8D^4}\frac{\mu^v}{\Gamma(v)} e^{(R\mu + \frac{\mu^2\sigma^2 T}{2} - D\mu T)} \sum_{z=1}^{v-1} \sum_{i=0}^{v-2/2} \frac{(v-2)!}{\left(\mu - \frac{2D}{\sigma^2}\right)^z (v-1-z)!}$$

$$\binom{v-2-i}{i} * (DT-R-\mu\sigma^2T)^{v-2-2i} \left(\frac{\sigma^2 T}{2}\right)^i \qquad 9.3.13.$$

From 4.4.30.

$$G_{11}(R+Q,T) = \left[\frac{(R+Q)^4}{12D^3} + \frac{\sigma^2 * (R+Q)^3}{6D^4} + \frac{\sigma^4 (R+Q)^2}{4D^5} + \frac{\sigma^6 (R+Q)}{4D^6} + \frac{\sigma^8}{8D^7} - \frac{\sigma^2T^2(R+Q)}{2D^2} + \frac{T^2(R+Q)^2}{2D} - \frac{T^3(R+Q)}{3} + \frac{T^3\sigma^2}{3D} - \frac{(R+Q)^3}{3D^2} + \frac{T^4 D}{12} \right] F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) - 2\sqrt{\sigma^2 T} \left[\frac{\sigma^2 T(R+Q)}{2\ 4D^3} + \frac{\sigma^4 T}{24D^4} - \frac{(R+Q)^2 T}{8D^2} - \frac{\sigma^2 T^2}{8D^2} + \frac{T^2(R+Q)}{8D} - \frac{T^3}{24} + \frac{(R+Q)^3}{24D^3} + \frac{\sigma^2(R+Q)^2}{12D^4} + \frac{\sigma^2(R+Q)}{8D^5} + \frac{\sigma^6}{8D^6} \right]$$

$$g\left(\frac{R+Q-DT}{\sqrt{\sigma^2T}}\right) - \frac{\sigma^8}{8D^7} e^{\frac{2D(R+Q)}{\sigma^2}} F\left(\frac{R+Q+DT}{\sqrt{\sigma^2T}}\right) \qquad 9.3.14.$$

Simplifying we have

$$G_{11}(R+Q,T) = \left[\frac{1}{12D^3} \sum_{w=0}^{4} \binom{4}{w} R^{4-w} Q^w + \left(\frac{\sigma^2}{6D^4} - \frac{1}{3D^2}\right) \sum_{w=0}^{3} \binom{3}{w} R^{3-w} Q^w + \left(\frac{\sigma^4}{4D^5} + \frac{T^2}{2D}\right) \sum_{w=0}^{2} \binom{2}{w} Q^w R^{2-w} + \left(\frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3}\right) Q + \frac{\sigma^8}{8D^7} + \frac{\sigma^6}{4D^6} - \frac{\sigma^2T^2}{2D^2} - \frac{T^3}{3} + \frac{T^3\sigma^2}{3D} + \frac{T^4 D}{12} \right)\Bigg] F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- 2\sqrt{\sigma^2 T} \left[\left(\frac{\sigma^2 TR}{24D^3} + \frac{\sigma^4 T}{24D^4} - \frac{R^2 T}{8D^2} - \frac{\sigma^2 T^2}{8D^2} + \frac{T^2 R}{8D} - \frac{T^3}{24} + \frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6} \right) + Q \left(\frac{\sigma^2 T}{24D^3} - \frac{2RT}{8D^2} + \frac{T^2}{8D} + \frac{3R^2}{24D^3} + \frac{2\sigma^2 R}{12D^4} + \frac{\sigma^2}{8D^5} \right) + Q^2\left(-\frac{T}{8D^2} + \frac{3R}{24D^3} + \frac{\sigma^2}{12D^4} \right) + \frac{Q^3}{24D^3} \right] g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^8 e^{\frac{2D(R+Q)}{\sigma^2}}}{8D^7} F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right)$$

Multiplying by $\frac{U(Q)}{Q}$ we have 9.3.15.

$$\frac{U(Q)\ G_{11}\ (R+Q,T)}{Q} = \frac{\mu^v e^{-\mu Q}}{\Gamma(v)} \left[\frac{1}{12D^3} \sum_{w=0}^{4} \binom{4}{w} R^{4-w} Q^{w+v-2} + \left(\frac{\sigma^2}{4D^4} - \frac{1}{3D^2}\right) \sum_{w=0}^{3} \binom{3}{w} R^{3-w} * Q^{w+v-2} + \left(\frac{\sigma^4}{4D^5} + \frac{T^2}{2D}\right) \sum_{w=0}^{2} R^{2-w} Q^{w+v-2} + \left(\frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3}\right) Q^{v-1} + \left(\frac{\sigma^8}{8D^7} + \frac{\sigma^6}{4D^6}\right.\right.$$

$$- \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3} + \frac{T^3\sigma^2}{3D} + \frac{T^4 D}{12}\Big) Q^{v-2}\Big] F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\mu^v e^{-\mu Q}}{\Gamma(v)}\, 2\sqrt{\sigma^2 T}\Big[\Big(\frac{\sigma^2 TR}{24D^3}$$

$$+ \frac{\sigma^4 T}{24D^4} - \frac{R^2 T}{8D^2} - \frac{\sigma^2 T^2}{8D^2} + \frac{T^2 R}{8D} - \frac{T^3}{24} + \frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6}\Big) Q^{v-2} + Q^{v-1}\Big(\frac{\sigma^2 T}{24D^3} -$$

$$\frac{RT}{4D^2} + \frac{T^2}{8D} + \frac{R^2}{8D^3} + \frac{\sigma^2 R}{6D^4} + \frac{\sigma^2}{8D^5}\Big) + Q^v\Big(-\frac{T}{8D^2} + \frac{R}{8D^3} + \frac{\sigma^2}{12D^4}\Big) + \frac{Q^{v+1}}{24D^3}\Big] g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^8}{8D^7} e^{\frac{2D(R+Q)}{\sigma^2}} *$$

$$F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.3.16.$$

Hence

$\int_0^\infty \frac{U(Q)}{Q} G_3(R+Q,T)\,dQ$, applying 9.1.10., 9.1.16. and 9.1.23. we have

$$G_{62}(R,T) = \frac{\mu^v}{\Gamma(v)} e^{\left(R\mu + \frac{\mu^2\sigma^2}{2}T - D\mu T\right)}\Big[\frac{1}{12D^3}\sum_{w=0}^{4}\sum_{z=1}^{w+v-1}\sum_{i=0}^{(w+v-z)/2}\binom{4}{w}R^{4-w}\binom{w+v-z-i}{i} *$$

$$\frac{(w+v-2)!}{(w+v-1-z)!} * (DT-R-\mu\sigma^2 T)^{w+v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i + \left(\frac{\sigma^2}{6D^4} - \frac{1}{3D^2}\right)\sum_{w=0}^{3}\sum_{z=1}^{w+v-1}\sum_{i=0}^{\frac{w-v-z}{2}}\binom{3}{w}R^{3-w}$$

$$\binom{w+v-z-i}{i}\frac{(w+v-2)!}{\mu^z(w+v-1-z)!}(DT-R-\mu\sigma^2 T)^{w+v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i$$

$$+ \left(\frac{\sigma^4}{4D^5} + \frac{T^2}{2D}\right)\sum_{w=0}^{2}\binom{2}{w}R^{2-w}\binom{w+v-z-i}{i}\frac{(w+v-2)!}{\mu^z(w+v-1-z)!}(DT-R-\mu\sigma^2 T)^{w+v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i$$

$$+ \left(\frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3}\right)\sum_{z=1}^{v}\sum_{i=0}^{\frac{v-z}{2}}\frac{(v-1)!}{\mu^z(v-z)!}\binom{v-z-i}{i}(DT-R-\mu\sigma^2 T)^{v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i$$

$$+ \left(\frac{\sigma^8}{8D^7} + \frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3} + \frac{T^3\sigma^2}{3D} + \frac{T^4 D}{12}\right)\sum_{z=1}^{v-1}\sum_{i=0}^{\frac{v-z-i}{2}}\frac{(v-2)!}{\mu^z(v-1-z)!}\binom{v-z-i-1}{i}(DT-R-\mu\sigma^2 T)^{v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i\Big]$$

$$- \frac{2\mu^v\sigma^2 T}{\Gamma(v)} e^{\left(R\mu + \frac{\mu^2\sigma^2}{2}T - D\mu T\right)}\Big[\Big(\frac{\sigma^2 TR}{24D^3} + \frac{\sigma^4 T}{24D^4} - \frac{R^2 T}{8D^2} - \frac{\sigma^2 T^2}{8D^2} + \frac{T^2 R}{8D} - \frac{T^3}{24} + \frac{R^3}{24D^2}$$

$$+ \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6}\Big)\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i}(DT-R-\mu\sigma^2 T)^{v-2-2i} * \left(\frac{\sigma^2 T}{2}\right)^i + \Big(\frac{\sigma^2 T}{24D^3} - \frac{RT}{4D^2} + \frac{T^2}{8D} + \frac{R^2}{8D^3}$$

$$\frac{\sigma^2 R}{6D^4} + \frac{\sigma^2}{8D^5}\Big)\sum_{i=0}^{\frac{v-1}{2}}\binom{v-1-i}{i}(DT-R-\mu\sigma^2 T)^{v-1-2i}\left(\frac{\sigma^2 T}{2}\right)^i + \left(-\frac{T}{8D^2} + \frac{R}{8D^3} + \frac{\sigma^2}{12D^4}\right)\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}$$

$$(DT-R-\mu\sigma^2 T)^{v-2i}(\sigma^2 T)^i + \frac{1}{24D^3}\sum_{i=0}^{\frac{v+1}{2}}\binom{v+1-i}{i} * (DT-R-\mu\sigma^2 T)^{v+1-2i}\left(\frac{\sigma^2 T}{2}\right)^i\Big]$$

$$- \frac{\sigma^8 \mu^v}{8D^7} \frac{e^{\left(R\mu + \frac{\mu^2 \sigma^2 T}{2} - D\mu T\right)}}{\Gamma(v)} \sum_{z=1}^{v} \sum_{i=0}^{\frac{v-1}{2}} \frac{(v-1)!}{\left(\mu - \frac{2D}{\sigma^2}\right)^z (v-z)!} \binom{v-1-i}{i} \left(DT - R - \mu\sigma^2 T\right)^{v-1-2i} \left(\frac{\sigma^2 T}{2}\right)^i \qquad 9.3.17$$

From 3.4.29.

$$G_4(R+Q.T) = \left[\frac{(R+Q-DT)^2}{2D} + \frac{\sigma^2(R+Q)}{2D^2} + \frac{\sigma^4}{2D^3}\right] F\left(\frac{R+Q-DT}{\sigma^2 T}\right) + \sqrt{\frac{\sigma^2 T}{2}} *$$

$$\left(\left(T - \frac{\sigma^2}{D^2}\right) - \frac{(R+Q)}{D}\right) g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^4}{4D^3} F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) e^{\frac{D(R+Q)}{\sigma^2}}$$

Simplifying we have

$$G_4(R+Q,T) = \left[\left(\frac{\sigma^4}{2D^3} + \frac{\sigma^2}{2D^2} + \frac{(R-DT)^2}{2D}\right) + Q\left(\frac{\sigma^2}{2D^2} + \frac{2\,(R-DT)}{2D}\right) + \frac{Q^2}{2D}\right] F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) + \sqrt{\frac{\sigma^2 T}{2}}\left(\left(T - \frac{\sigma^2}{D^2}\right.\right.$$

$$\left.\left. - \frac{R}{D}\right) - \frac{Q}{D}\right) g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^4}{4D^3} F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.3.18a$$

Multiplying by $\frac{U(Q)}{Q}$ we have $U(Q)\; G_4\; (R+Q,T)/Q$

$$\frac{U(Q)\; G_4(R+Q,T)}{Q} = \mu^v \frac{e^{-\mu Q}}{\Gamma(v)} \left[\left(\frac{\sigma^4}{2D^3} + \frac{\sigma^2}{2D^2} + \frac{(R-DT)^2}{2D}\right) Q^{v-2} + Q^{v-1}\left(\frac{\sigma^2}{2D^2} + \frac{2(R-DT)}{2D}\right)\right.$$

$$\left. + \frac{Q^v}{2D}\right] F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) + \sqrt{\frac{\sigma^2 T}{2}}\left(\left(T - \frac{\sigma^2}{D^2} - \frac{R}{D}\right) Q^{v-2} - \frac{Q^{v-1}}{D}\right) \frac{e^{-\mu Q}\mu^v}{\Gamma(v)}\; g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{\sigma^4}{4D^3} \frac{\mu^v e^{-\mu Q}}{\Gamma(v)} F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) * Q^{v-2} \qquad 9.3.18b$$

Hence we obtain

$$\int_0^\infty \frac{U(Q)\quad G_4(R+Q,T)}{Q}\, dQ$$ applying 9.1.10., 9.1.16 and 9.1.23 :

$$G_{63}(R,T) = \mu^v \frac{e^{\left(R\mu + \frac{\mu^2 \sigma^2 T}{2} - D\mu T\right)}}{\Gamma(v)} \left[\left(\frac{\sigma^4}{2D^3} + \frac{\sigma^2}{2D^2} + \frac{(R-DT)^2}{2D}\right) \sum_{z=1}^{v-1} \sum_{i=0}^{\frac{v-z-1}{2}} \binom{v-1-z-i}{i}\right.$$

$$\frac{(v-2)!}{\mu^z (v-1-z)!} \left(DT - R - \mu\sigma^2 T\right)^{v-1-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i + \left(\frac{\sigma^2}{2D^2} + \frac{(R-DT)}{D}\right) \sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\mu^z (v-z)!}$$

$$\binom{v-z-i}{i} \left(DT - R - \mu\sigma^2 T\right)^{v-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i + \frac{1}{2D} \sum_{z=1}^{\frac{v+1}{2}} \sum_{i=0}^{\frac{v+1-z}{2}} \binom{v+1-z-i}{i} \frac{v!}{\mu^z (v+1-z)!} \left(DT - R - \mu\sigma^2 T\right)^{v+1-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i$$

$$+ \frac{\sigma^2 T}{2} \mu^v \frac{e^{\left(R\mu + \frac{\mu^2 \sigma^2 T}{2} - D\mu T\right)}}{\Gamma(v)} \left[\left(T - \frac{\sigma^2}{D^2} - \frac{R}{D}\right) \sum_{i=0}^{v-2/2} \binom{v-2-i}{i} * \left(DT - R - \mu\sigma^2 T\right)^{v-2-2i} \left(\frac{\sigma^2 T}{2}\right)^i\right.$$

$$- \frac{1}{D} \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} * \left(DT - R - \sigma^2 T\right)^{v-1-2i} \left(\frac{\sigma^2 T}{2}\right)^i - \frac{\sigma^4}{4D^3} \frac{\mu^v e^{\left(R\mu + \mu^2 \frac{\sigma^2}{2} T - D\mu T\right)}}{\Gamma(v)} \sum_{z=1}^{v-1} \sum_{i=0}^{\frac{v-z-1}{2}}$$

$$\frac{(v-2)!}{\left(\mu-\frac{2D}{\sigma^2}\right)^z (v-z)!}\binom{v-z-i}{i}(DT-R-\mu\sigma^2T)^{v-z-2i+1} * \left(\frac{\sigma^2 T}{2}\right)^i$$

9.3.19.

The probability of ordering, POR at every review, from 3.4.5.

$$POR = \frac{DT}{Q} * \left(1-F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) + F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sqrt{\sigma^2 T}}{Q}\, g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

Simplifying

$$POR = \frac{DT}{Q} + \left(1 - \frac{DT}{Q}\right) F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sqrt{\sigma^2 T}}{Q}\, g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

Multiplying by U(Q)

$$U(Q)\ POR = \mu^v \frac{e^{-\mu Q}}{\Gamma(v)} DTQ^{v-2} + (Q^{v-1} - DTQ^{v-2})\ e^{-\mu Q}\ F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) - \mu^v \frac{e^{-\mu Q}\sqrt{\sigma^2 T}\ Q^{v-2}}{\Gamma(v)}$$

Let $POR_T = \int_0^\infty U(Q) POR\, dQ$, applying 9.1.10 and 9.1.16.

$$POR_T = \frac{DT\mu^2}{(v-1)(v-2)} + \mu^v e^{\left(R\mu+\mu^2\frac{\sigma^2}{2}T - D\mu T\right)}\left[\sum_{z=1}^{v}\sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\mu^z (v-z)!}\binom{v-z-i}{i}(DT-R-\mu\sigma^2 T)^{v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^i\right.$$

$$- DT \sum_{z=1}^{v-1}\sum_{i=0}^{(v-z-1)/2} \frac{(v-2)!}{\mu^z (v-1-z)!}\binom{v-1-z-i}{i} * (DT-R-\mu\sigma^2 T)^{v-1-z-2i} * \left(\frac{\sigma^2 T}{2}\right)^i - \frac{\mu^v}{\Gamma(v)}\ \sigma^2 T\ e^{\left(R\mu+\mu^2\frac{\sigma^2 T}{2} - D\mu T\right)}$$

$$\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i} * (DT-R-\mu\sigma^2 T)^{v-2-2i}\left(\frac{\sigma^2 T}{2}\right)^i$$

9.3.20.

Hence applying all the above integrals the inventory costs for random supply is

$$C = \frac{R_c}{T} + \frac{SPOR_T}{T} + hc\left(\frac{v}{2\mu} + R - DL - \frac{DT}{2}\right) + \frac{b_1}{T}\left(G_1(R,T+L)\frac{\mu}{(v-1)} - \frac{G_1(R,L)\mu}{(v-1)} - G_{60}\right.$$

$$\left.(R,T+L) + G_{60}(R,L)\right) + \frac{(hc+b_2)}{T}\left(\frac{\mu}{(v-1)} G_3(R,T+L) - \frac{\mu}{(v-1)} G_3(R,L) - G_{61}(R,T+L)\right.$$

$$\left. + G_{61}(R,L)\right) + \frac{b_3}{T}\left(\frac{\mu}{(v-1)} G_{11}(R,T+L) - \frac{\mu}{(v-1)} G_{11}(R,L) - G_{62}(R,T+L) + G_{62}(R,L)\right)$$

$$+ \frac{S}{T}\left(\frac{\mu}{(v-1)} G_4(R,T+L) - \frac{\mu}{(v-1)} G_4(R,L) - G_{63}(R,T+L) + G_{63}(R,L)\right)$$

9.3.21.

Section 9.4. Model (M,T)

Since the supply is a random variable, the maximum re-order cover M would vary. Similarily M follows a gamma distribution. T becomes the only control parameter. The probability density function of M, U(M)

$$U(M) = \frac{e^{-\mu M} M^{v-1} \mu^{v}}{\Gamma(v)} \qquad M>0 \qquad 9.4.1.$$

The inventory costs for fixed M (maximum re-order cover) was given in section 4.5 equation 4.5.7., when the cost of a backorder was a quadratic function of the length of time of a backorder, for fixed M.

$$C = \frac{R_c}{T} + S + hc\left(M - DL - \frac{DT}{2}\right) + \frac{b_1}{T}\left(G_5(M,T+L) - G_5(M,L)\right) + \frac{(b_2 + hc)}{T}\left(G_2(M,T+L) - G_2(M,L)\right) + \frac{b_3}{T}\left(G_{12}(M,T+L) - G_{12}(M,L)\right) + \frac{s}{T}\left(R_0(M,T+L) - R_0(M,L)\right) \qquad 9.4.2.$$

From 3.5.2.

$$G_5(M,T) = \sqrt{\sigma^2 T}\, g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - (M-DT)\, F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.4.3.$$

Multiplying by U(M)

$$U(M)\, G_5(M,T) = \frac{\sigma^2 T\, e^{-\mu M} M^{v-1} \mu^{v}}{\Gamma(v)}\, g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - \frac{e^{-\mu M}\mu^{v}}{\Gamma(v)}\left[M^{v} - DTM^{v-1}\right] F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.4.4$$

$$\text{Let } G_{56}(T) = \int_0^{\infty} U(M)\, G_5(M,T)\, dM \qquad 9.4.5.$$

Substituting for $U(M)\, G_5(M,T)$ integrating and applying 9.1.10. and 9.1.16.

$$G_{56}(T) = \frac{\sigma^2 T}{\Gamma(v)} \mu^{v} e^{\left(\frac{\mu^2 \sigma^2 T}{2} - D\mu T\right)} \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} \left(DT - \mu\sigma^2 T\right)^{v-1-2i} \left(\frac{\sigma^2 T}{2}\right)^{i}$$

$$+ \frac{\mu^{v}}{\Gamma(v)} e^{\left(\frac{\mu^2\sigma^2 T}{2} - D\mu T\right)} \left[DT \sum_{z=1}^{v} \frac{(v-1)!}{\mu^{z}(v-z)!} \sum_{i=0}^{\frac{v-z}{2}} \binom{v-z-i}{i} \left(DT - \mu\sigma^2 T\right)^{v-z-2i} \left(\frac{\sigma^2 T}{2}\right)^{i} \right.$$

$$\left. - \sum_{z=1}^{v+1} \sum_{i=0}^{\frac{v+1-z}{2}} \frac{v!}{\mu^{z}(v+1-z)!} \binom{v+1-z-i}{i} \left(DT - \mu\sigma^2 T\right)^{v+1-z-2i} \left(\frac{\sigma^2 T}{2}\right)^{i} \right] \qquad 9.4.6.$$

From 3.5.6.

$$G_2(M,T) = \left(\frac{\sigma^4}{4D^3} + \frac{DT^2}{2} + \frac{\sigma^2 M}{2D^2} - TM + \frac{M^2}{2D}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma}\, T^2}{2}\left(\left(T - \frac{\sigma^2}{D^2}\right) - \frac{M}{D}\right) g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{\sigma^4}{4D^3} e^{\frac{2DM}{\sigma^2}} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

Multiplying by U(M)

$$U(M)\, G_2(M,T) = \left[\left(\frac{\sigma^4}{4D^3} + \frac{DT^2}{2}\right) + \left(\frac{\sigma^2}{2D^2} - T\right) * M^{v} + \frac{M^{v+1}}{2D}\right] \times \frac{e^{-\mu M}\mu^{v}}{\Gamma(v)} F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma\, T^{1/2}}{2} *$$

$$\left(\left(T - \frac{\sigma^2}{D^2}\right) M^{v-1} - \frac{M^v}{D}\right) \frac{e^{-\mu M} \mu^v}{\Gamma(v)} g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - \frac{e^{-\mu M} \mu^v}{4D^3 \Gamma(v)} M^{v-1} * e^{\frac{2DM}{\sigma^2}} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

9.4.7.

Let $G_{57}(T) = \int_0^\infty U(M)\, G_2(M,T)\, dM$ 9.4.8.

Substituting for $U(M)\, G_2(M,T)$, integrating and applying 9.1.10., 9.1.16 and 9.1.23. we have

$$G_{57}(T) = \mu^v \frac{e^{\left(\frac{\mu^2\sigma^2 T}{2} - D\mu T\right)}}{\Gamma(v)} \left[\left(\frac{\sigma^4}{4D^3} + \frac{DT^2}{2}\right) \sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\mu^z (v-z)!} \binom{v-z-i}{i} (DT-\mu\sigma^2 T)^{v-z-2i} * \left(\frac{\sigma^2 T}{2}\right)^i + \frac{1}{2D} \sum_{z=1}^{v+2} \frac{(v+1)!}{\mu^z (v+2-z)!} \sum_{i=0}^{\frac{v+2-z}{2}} \binom{v+2-z-i}{i} (DT-\mu\sigma^2 T)^{v+2-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i\right]$$

$$+ \left(\frac{\sigma^2}{2D^2} - T\right) \sum_{z=1}^{v+1} \sum_{i=0}^{(v+1-z)/2} \frac{v!}{\mu^z (v+1-z)!} \binom{v+1-z-i}{i} (DT-\mu\sigma^2 T)^{v+1-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i$$

$$\frac{\sigma^2 T}{2} \mu^v \frac{e^{\left(\frac{\mu^2\sigma^2 T}{2} - D\mu T\right)}}{\Gamma(v)} \left[\left(T - \frac{\sigma^2}{D^2}\right) \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} (DT-\mu\sigma^2 T)^{v-1-2i} \left(\frac{\sigma^2 T}{2}\right)^i\right.$$

$$\left. - \frac{1}{D} \sum_{i=0}^{\frac{v}{2}} \binom{v-i}{i} (DT-\mu\sigma^2 T)^{v-2i} \left(\frac{\sigma^2 T}{2}\right)^i\right] - \mu^v \frac{e^{\left(\frac{\mu^2\sigma^2 T}{2} - D\mu T\right)}}{\Gamma(v)\, 4D^3} \sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\left(\mu - \frac{2D}{\sigma^2}\right)^z (v-z)!} *$$

$$(DT-\mu\sigma^2 T)^{v-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i$$

9.4.9.

From 4.4.32. substituting T for L

$$G_{12}(M,T) = -\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4 M^2}{2D^5} + \frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} + \frac{T^2 M}{D} \quad \frac{T^3}{3} - \frac{TM^2}{D^2}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{1}{\sqrt{\sigma^2 T}} g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \left(\frac{-2}{3} \frac{\sigma^2 MT^2}{D^2} + \frac{\sigma^2 T^3}{3D} + \frac{\sigma^2 MT}{3D^3} + \frac{\sigma^4 MT}{2D^4} + \frac{\sigma^4 T^2}{6D^3} + \frac{8\sigma^6 T}{D^5}\right) + \frac{\sigma^6}{4D^6}$$

$$e^{\frac{2DM}{\sigma^2}} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

9.4.10.

Multiplying by U(M) and simplfying

$$U(M)\, G_{12}(M,T) = -\frac{e^{-\mu M} \mu^v}{\Gamma(v)} \left[\frac{M^{2+v}}{3D^3} + M^{1+v}\left(\frac{\sigma^2}{2D^4} - \frac{T}{D^2}\right) + M^v \left(\frac{\sigma^4}{2D^5} + \frac{T^2}{D}\right) + M^{v+1} * \left(\frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3}\right)\right] F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{e^{-\mu M} \mu^v}{\sqrt{\sigma^2 T}\, \Gamma(v)} * g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) \left(\left(-\frac{2}{3} \frac{\sigma^2 T^2}{D^2} + \frac{\sigma^2 T}{3D^3} + \frac{\sigma^4 T}{2D^4}\right) M^v\right.$$

$$\left. + \left(\frac{\sigma^2}{3D} T^3 + \frac{\sigma^4}{6D^3} + \frac{8\sigma^6 T}{D^5}\right) R^{v-1}\right) + \frac{\sigma^6}{4D^6} e^{\frac{2DM}{\sigma^2}} \frac{e^{-\mu M} R^{v-1} \mu^v}{\Gamma(v)} * F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

9.4.11.

Let $G_{58}(T) = U(M)\, G_{12}(M,T)\, dM$ 9.4.12a

then substituting for $U(M)\, G_{12}(M, L+T)$ integrating and applying 9.1.10, 9.1.16. and 9.1.23, we have

$$G_{58}(T) = -\frac{\mu^v}{\Gamma(v)} e^{\left(\frac{\mu^2\sigma^2 T}{2} - D\mu T\right)} \left[\frac{1}{3D^3} \sum_{z=1}^{3+v} \sum_{i=0}^{(v-z+3)/2} \frac{(v+2)!}{\mu^z (v+3-z)!} \binom{v+3-z-i}{i} * \right.$$

$$(DT-\mu\sigma^2 T)^{v+3-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i + \left(\frac{\sigma^2}{2D^4} - \frac{T}{D^2}\right) \sum_{z=1}^{2+v} \sum_{i=0}^{(v+2-z)/2} \frac{(v+1)!}{\mu^z (v+2-z)!} \binom{v+2-z-i}{i} (DT-\mu\sigma^2 T)^{v+2-z-2i} *$$

$$\left(\frac{\sigma^2 T}{2}\right)^i + \left(\frac{\sigma^4}{2D^5} + \frac{T^2}{D}\right) \sum_{z=1}^{v+1} \sum_{i=0}^{(v+1-z)/2} \frac{v!}{\mu^z (v+1-z)!} \binom{v+1-z-i}{i} (DT-\mu\sigma^2 T)^{v+1-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i$$

$$\left. + \left(\frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3}\right) \sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\mu^z (v-z)!} \binom{v-z-i}{i} (DT-\mu\sigma^2 T)^{v-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i \right]$$

$$+ \frac{\mu^v}{(v)} e^{(\mu^2\sigma^2 T - D\mu T)} \left[\left(-\frac{2}{3}\frac{\sigma^2 T^2}{D^2} + \frac{\sigma^2 T}{3D^3} + \frac{\sigma^4 T}{2D^4}\right) \sum_{i=0}^{\frac{v}{2}} \binom{v-i}{i} (DT-\mu\sigma^2 T)^{v-2i} * \left(\frac{\sigma^2 T}{2}\right)^i \right.$$

$$\left. + \left(\frac{\sigma^2 T^3}{3D} + \frac{\sigma^4 T^2}{6D^3} + \frac{8\sigma^6 T}{D^5}\right) \sum_{i=0}^{\frac{v-1}{2}} \binom{v-i-1}{i} * (DT-\sigma^2 T)^{v-2i} \left(\frac{\sigma^2 T}{2}\right)^i \right] + \frac{\sigma^6}{4D^6} \frac{e^{\left(\frac{\mu^2\sigma^2 T}{2} - D\mu T\right)}}{\Gamma(v)} *$$

$$\sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\left(\mu - \frac{2D}{\sigma^2}\right)^z (v-z)!} \binom{v-z-i}{i} \binom{v-z-i}{i} (DT-\mu\sigma^2 T)^{v-z-2i} \left(\frac{\sigma^2 T}{2}\right)^i$$ 9.4.12b

From 3.4.10.

$$R_0(M,T) = \left(T - \frac{M}{D} - \frac{\sigma^2}{2D^2}\right) F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sqrt{\sigma^2 T}}{D}\, g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^2}{2D^2} e^{\frac{2DM}{\sigma^2}} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$ 9.4.13.

Multiplying by $U(M)$, we have

$$U(M) R_0(M,T) = \frac{\mu^v e^{-\mu M}}{\Gamma(v)} \left[\left(T - \frac{\sigma^2}{2D^2}\right) M^{v-1} - \frac{M^v}{D}\right] F\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$ 9.4.14.

$$+ \sqrt{\sigma^2 T} \frac{\mu^v e^{-\mu M} M^{v-1}}{D\,\Gamma(v)} g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\mu^v e^{-\mu M} \sigma^2 e^{\left(\frac{2DM}{\sigma^2} - \mu M\right)}}{2D^2} F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right)$$

Let $G_{59}(T) = \int_0^\infty U(M)\, R_0(M,T)\, dM$ 9.4.15.

Substituting for $U(M)\, R_0(M,T)$, integrating and applying 9.1.10, 9.1.16 and 9.1.23.

$$G_{59}(T) = \frac{\mu^v e^{(\mu^2\sigma^2 T - D\mu T)}}{\Gamma(v)} \left[\left(T - \frac{\sigma^2}{2D^2}\right) \sum_{z=1}^{v} \frac{(v-1)!}{\mu^z (v-z)!} \sum_{i=0}^{\frac{v-z}{2}} \binom{v-z-i}{i} * \right.$$

$$(DT-\mu\sigma^2T)^{v-Z-2i}\left(\frac{\sigma^2T}{2}\right)^i - \frac{1}{D}\sum_{Z=1}^{v+1}\frac{v!}{\mu^Z(v+1-Z)!}\sum_{i=0}^{(v+1-Z)/2}\binom{v+1-Z-i}{i}(DT-\mu\sigma^2T)^{v+1-Z-2i}\left(\frac{\sigma^2T}{2}\right)^i$$

$$+\mu^v\frac{e^{(\mu^2\sigma^2T-D\mu T)}}{\Gamma(v)} * \frac{\sigma^2T}{D}\sum_{i=0}^{v-1}\binom{v-1-i}{i} * (DT-\mu\sigma^2T)^{v-1-2i}\left(\frac{\sigma^2T}{2}\right)^i + \mu^v\frac{\sigma^2}{2D^2}$$

$$e^{\left(\mu^2\frac{\sigma^2}{2}T-D\mu T\right)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\frac{(v-1)!}{\left(\mu-\frac{2D}{\sigma^2}\right)^Z(v-Z)!}\binom{v-Z-i}{i} * (DT-\mu\sigma^2T)^{v-Z-2i}\left(\frac{\sigma^2T}{2}\right)^i$$

9.4.16

Hence averaging the inventory costs for fixed M (equation 9.3.2.) over the states of M we obtain the inventory costs for variable M

$$C = \frac{R_c}{T} + S + \frac{hc*v}{\mu} - hc\left(DL+\frac{DT}{2}\right) + \frac{b_1}{T}\left(G_{56}(T+L) - G_{56}(L)\right) + \frac{\left(b_2+hc\right)}{T}\left(G_{57}(T+L) - G_{57}(L)\right) + \frac{b_3}{T}\left(G_{58}(T+L) - G_{58}(L)\right) + \frac{s}{T}\left(G_{59}(T+L) - G_{59}(L)\right)$$

9.4.17.

Section 9.5. (M,R,T)

From section 3.6., equation 3.6.12, the inventory cost when the maximum re-order cover is fixed is

$$C = \frac{R_c}{T} + S + \frac{\sum_{n=1}^{\infty}\int_0^{M-R} G_8(M+y,T) * f^n(M-R-Y,DT)\,dY + G_8(M,T)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2T}}\right) + \sum_{n=2}^{\infty}\int_0^{M-R} n * f^{(n-1)}(M-R-y,DT)\,F\left(\frac{y-DT}{\sqrt{\sigma^2T}}\right)dy\right]}$$

9.5.1

Noting that M is always greater than R and the probability density function of $U(M) = \dfrac{\mu^v e^{-\mu M} M^{v-1}}{\Gamma(v)}$

9.5.2.

Hence averaging C over the states of M we obtain the inventory costs for variable M

$$C = \frac{R_c}{T} + S + \frac{\sum_{n=1}^{\infty}\int_M^{\infty} U(M)\int_0^{M-R} G_{18}(M+y,T) * f^n(M-R-y,DT)\,dy\,dM + \int_M^{\infty} U(M)G_8(M,T)\,dM}{T\left[\int_M^{\infty} U(M)\,F\left(\frac{M-R-DT}{\sqrt{\sigma^2T}}\right)dM + \sum_{n=2}^{\infty} n\int_M U(M) f^{n-1}(M-R-y,DT)\,F\left(\frac{y-DT}{\sqrt{\sigma^2T}}\right)dy\,dM\right]}$$

9.5.3.

Section 9.6. (Q,R) Exponential cost term.

The inventory costs for fixed batch quantity and exponential backorder cost is given in section 5.1. equation 5.1.18.

$$C = \frac{DS}{Q} + \frac{Qhc}{2} + hc\,(R-DL) + \frac{hc}{Q}\,B(Q,R) + \frac{D}{Q}G_{15}(Q,R) + \frac{Ds}{Q}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

where $B(Q,R) = \beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$ 9.6.1.

Also from equation 9.2.12, G_{54} (R,L) was defined as

$$G_{54}(R,L) = \int_0^\infty \frac{U(Q)}{Q}\,\beta\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) dQ \qquad 9.6.2.$$

and from 9.2.5., G_{53} (R,L) was defined as

$$G_{53}(R,L) = \int_0^\infty \frac{U(Q)}{Q}\,\alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) dQ \qquad 9.6.3.$$

The only additional integral needed to be able to define the inventory costs for random supply is

$$G_{64}(R,L) = \int_0^\infty \frac{U(Q)\ G_{15}(Q,R)}{Q}\ dQ \qquad 9.6.4.$$

From equation 5.1.10.

$$G_{15}(Q,R) = \frac{Db_1}{b_4}\exp -\left[-\frac{\sigma^2 b_4^2}{2D^2} + b_4 R - b_4 L\right] F\left(\frac{R - \frac{\sigma^2 b_4}{D} - DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \exp -\left[-\frac{\sigma^2 L b_4^2}{2D^2} + \frac{b_4 R}{D} - b_4 L\right] . \exp\left(\frac{b_4 Q}{D}\right) F\left(\frac{R+Q-\frac{\sigma^2 b_4}{D} - DL}{\sqrt{\sigma^2 L}}\right) - \frac{Db_1}{b_4}\left[F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. - F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right] \qquad 9.6.5.$$

Multiplying by U(Q) where $U(Q) = e^{-\mu Q}\, Q^{v-1} \mu^v / \Gamma(v)$ we have

$$\frac{U(Q)\ G_{15}(Q,R)}{Q} = \frac{\mu^v\, e^{-\mu Q}\, Db_1 *}{b_4 \Gamma(v)}\ Q^{v-2} \exp\left(\frac{\sigma^2 L b_4^2}{2D^2} - \frac{b_4 R}{D} + b_4 L\right) F\left(\frac{R - \frac{\sigma^2 b_4^2}{D} - DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{Db_1}{b_4}\,\frac{\mu^v}{\Gamma(v)}\ \exp\left(\frac{\sigma^2 b_4^2}{2D^2} - \frac{b_4 R}{D} + b_4 L\right) Q^{v-2}\ e^{-Q\left(\mu - \frac{b_4}{D}\right)} F\left(\frac{R+Q-\frac{\sigma^2 b}{D} - DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{Db_1}{b_4}\,\frac{\mu^v\, Q^{v-2}}{\Gamma(v)}\, e^{-\mu Q}\left[F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right] \qquad 9.6.6.$$

Hence

$$\int_0^{\infty} \frac{U(Q)\ G_{15}\ (Q,R)\ dQ}{Q}, \text{ applying 9.1.10 and 9.1.16}$$

$$G_{64}(R,L) = \frac{D\ b_1}{\Gamma(v)\ (v-1)b_4} \exp\left(\frac{\sigma^2 L b_4^2}{2D^2} - \frac{b_4 R}{D} + b_4 L\right) F\left(\frac{R-\sigma^2 b_4 - DL}{\sqrt{\sigma^2 L}}\right) - \frac{Db_1}{b_4} \mu^v \frac{1}{\Gamma(v)}$$

$$\exp\left[\left(\frac{\sigma^2 L b_4^2}{2D^2} - \frac{b_4 R}{D} + b_4 L\right) + \left(\mu - \frac{b_4}{D}\right)\left(R - \frac{\sigma^2 b_4}{D}\right) + \left(\mu - \frac{b_4}{D}\right) * \frac{\sigma^2 L}{2} - DL\right] \sum_{Z=1}^{v-1}$$

$$\sum_{i=0}^{\frac{v-Z-1}{2}} \frac{(v-2)!}{\left(\mu - \frac{b_4}{D}\right)^Z (v-1-Z)!} \binom{v-1-Z-i}{i} * \left(DL - R + \frac{\sigma^2 b_4}{D} - \left(\mu - \frac{b_4}{D}\right)\sigma^2 L\right)^{v-1-Z-2i} * \left(\frac{\sigma^2 L}{2}\right)^i$$

$$- \frac{Db_1 \mu}{(v-1)\ b_4} \cdot \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + \frac{Db_1}{b_4} * \frac{\mu^v}{\Gamma(v)} \exp\left[R\mu + \mu^2 \frac{\sigma^2 L}{2} - D\mu L\right] \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}}$$

$$\frac{(v-2)!}{\mu^Z (v-1-Z)!} \binom{v-1-Z-i}{i} \left(DL - R - \mu\sigma^2 L\right)^{v-1-Z-2i} * \left(\frac{\sigma^2 L}{2}\right)^i \qquad 9.6.7.$$

Hence applying all the above integrals and averaging 9.6.1. over the states of Q, we obtain the inventory costs when supply is random for exponential backorder costs.

$$C = \frac{DS\mu}{(v-1)} + \frac{vhc}{2} + hc(R-DL) + hc\left(\frac{\mu}{(v-1)}\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - G_{54}(R,L)\right) + DG_{64}(R,L) + Ds^*$$

$$\left(\frac{\mu}{(v-1)}\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - G_{53}\ (R,L)\right) \qquad 9.6.8.$$

Section 9.7. (Q,R,T) Exponential Costs

In this section the inventory cost for (Q,R,T) is derived by averaging over the states of Q, the inventory cost for fixed batch quantities, Q. From equation 5.2.20. the inventory cost for (Q,R,T) when the supply Q is fixed is

$$C = \frac{R_c}{T} + \frac{S.PoR}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right) + \frac{hc}{QT}\Big(G_3(R,T+L) - G_3(R,L) - G_3(R+Q,T+L)$$

$$+ G_3(R+Q,L)\Big) + \frac{1}{QT}\Big(G_{18}(R,T+L) - G_{18}(R,L) - G_{18}(R+Q,L+T) + G_{18}(R+Q,L)\Big) + \frac{s}{QT}$$

$$\Big(G_4(R,T+L) - G_4(R,L) - G_4(R+Q,L+T) + G_4(R+Q,L)\Big) \qquad 9.7.1.$$

The probability density function of $Q, U(Q) = \dfrac{e^{-\mu Q}\, Q^{v-1}\, \mu^v}{\Gamma(v)}$

From equation 9.3.3.

$$G_{61}(R,T) = \int_0^\infty \frac{U(Q)}{Q}\, G_3(R+Q,T)\, dQ \qquad 9.7.2.$$

From 9.3.5.

$$G_{63}(R,T) = \int_0^\infty \frac{U(Q)}{Q}\, G_4(R+Q,T)\, dQ \qquad 9.7.3.$$

From 9.3.20.

$$PoR_T = \int_0^\infty U(Q)\, PoR\, dQ.$$

Let $G_{83}(R,T) = \displaystyle\int_0^\infty \frac{U(Q)}{Q}\, G_{18}(R+Q,T)\, dQ \qquad 9.7.4.$

Applying 8.8.4 we have

$$G_{18}(R+Q,T) = \frac{2D^2 b_1}{(\sigma^2 b_4^3 + 2D^2 b_4^2)}\, e^{\left[T\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right) - \frac{b_4}{D} * (R+Q)\right]} F\left(\frac{R+Q-T\left(D+\sigma^2 \frac{b_4}{D}\right)}{\sqrt{\sigma^2 T}}\right)$$

$$- \frac{b_1}{b_4} F\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)\left[\left(T - \frac{R}{D} - \frac{\sigma^2}{2D^2} + \frac{1}{b_4}\right) - \frac{Q}{D}\right] - \frac{\sigma^4 b_1 b_4^2}{2D^2(\sigma^2 b_4^2 + 2D^2 b_4) b_4}\, e^{\frac{2(R+Q)}{\sigma^2}} *$$

$$F\left(\frac{R+Q+DL}{\sqrt{\sigma^2 T}}\right) - \frac{2\sqrt{\sigma^2 T}}{D * b_4}\, b_1\, g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.7.5.$$

Multiplying by $\frac{U(Q)}{Q}$ we have

$$\frac{G_{18}}{Q}(R+Q,T)\,U(Q) = \frac{2D^2 b_1 \mu^v Q^{v-2}}{(\sigma^2 b_4^3 + 2D^2 b_4^2)\Gamma(v)}\, e^{\left[T\left(\frac{\sigma^2 b_4 + 2D^2 b_4}{2D^2} - \frac{b_4 R}{DT}\right) - Q\left(\mu + \frac{b_4}{D}\right)\right]} *$$

$$F\left(\frac{R+Q-T\left(D+\sigma^2 \frac{b_4}{D}\right)}{\sqrt{\sigma^2 T}}\right) - \frac{\mu^v e^{-\mu Q} b_1}{b_4 \Gamma(v)} * \left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right)\left[\left(T - \frac{R}{D} - \frac{\sigma^2}{2D^2} + \frac{1}{b_4}\right) Q^{v-2} - \frac{Q^{v-1}}{D}\right]$$

$$- \frac{\sigma^4 b_1 b_4^2 \mu^v Q^{v-2} e^{\frac{2R}{\sigma^2}}}{2D^2(\sigma^2 b_4^2 + 2D^2 b_4) b_4} \cdot e^{-Q(2R/\sigma^2 + \mu)} * F\left(\frac{R+Q+DT}{\sqrt{\sigma^2 T}}\right) - \frac{2\sqrt{\sigma^2 T}\, b_1}{D b_4} *$$

$$\frac{\mu^{v} e^{-\mu Q} Q^{v-2}}{\Gamma(v)} * g\left(\frac{R+Q-DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.7.6.$$

Hence $\int_0^\infty \frac{U}{Q}(Q)\ G_{18}(R+Q,T)\ dQ$ applying 9.1.10., 9.1.16 and 9.1.23 we have

$$G_{83}(R,T) = \frac{2D^2 b_1 \mu^{v}\ e^{T\left(\frac{\sigma^2 b_4 + 2D^2 b_4}{2D^2} - \frac{b_4 R}{D}\right)}}{\left(\sigma^2 b_4^3 + 2D^2\ b_4^2\right)\Gamma(v)} \sum_{z=1}^{v-1} \sum_{i=0}^{\frac{v-1-z}{2}} \frac{(v-2)}{\left(\mu + \frac{b_4}{D}\right)^{z}(v-1-z)!}$$

$$\binom{v-1-z-i}{i}(DT-R-\mu\sigma^2 T)^{v-1-z-2i} \left(\frac{\sigma^2 T}{2}\right)^{i} * e^{\left[\left(\mu+\frac{b_4}{D}\right)\left(R+\left(\mu+\frac{b_4}{D}\right)\sigma^2 T - \left(D+\frac{b_4\sigma^2}{D}\right) * T\right)\right]}$$

$$e^{\left(R\mu+\mu^2\frac{\sigma^2}{2}T-D\mu T\right)} * \frac{\mu^{v} b_1}{b_4\Gamma(v)}\left[\left(T-\frac{R}{D} - \frac{\sigma^2}{2D^2} + \frac{1}{b_4}\right) \sum_{z=1}^{v-1} \sum_{i=0}^{\frac{v-1-z}{2}} \frac{(v-2)!}{\mu^{z}(v-1-z)!}\binom{v-1-z-i}{i}\right.$$

$$\left.(DT-R-\mu\sigma^2 T)^{v-1-2i} \left(\frac{\sigma^2 T}{2}\right)^{i} - \frac{1}{D}\sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\mu^{z}(v-z)!}\binom{v-z-i}{i} (DT-R-\mu\sigma^2 T)^{v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^{i}\right]$$

$$- \frac{\sigma^4 b_1 b_4^2\ \mu^{v}}{2D^2(\sigma^2 b_4^3 + 2D^2 b_4^2)\ \Gamma(v)}\ e^{\left(R+\frac{\mu^2\sigma^2}{2}T - D\mu T\right)} \sum_{z=1}^{v-1} \sum_{i=0}^{\frac{v-z-1}{2}} \frac{(v-2)!}{\left(\mu - \frac{2D}{\sigma^2}\right)^{z} * (v-z)!}$$

$$\binom{v-z-i}{i} * \left(DT-R-\mu\sigma^2 T\right)^{v-z-2i} \left(\frac{\sigma^2 T}{2}\right)^{i} - \frac{2\sigma^2 T b_1}{D\ b_4} \times \frac{\mu}{\Gamma(v)}\ e^{\left(R\mu+\mu^2\frac{\sigma^2}{2}T-D\mu T\right)}$$

$$\sum_{i=0}^{(v-2)/2} \binom{v-2-i}{i}(DT-R-\mu\sigma^2 T)^{v-z-2i}\left(\frac{\sigma^2 T}{2}\right)^{i} \qquad 9.7.7.$$

Hence the inventory cost for (Q,R,T) when the supply is random and lead time is constant and backorder cost is exponential, is

$$C = \frac{R_c}{T} + \frac{S.Pc R_T}{T} + hc\left(\frac{v}{2\mu} + R - DL - \frac{DT}{2}\right) + \frac{hc}{T}\left(\frac{\mu}{(v-1)} G_{53}(R,T+L) - \frac{\mu}{(v-1)} G_3\right.$$

$$\left.(R,L) - G_{61}\ (R,T+L) + G_{61}(R,L)\right) + \frac{1}{T}\left(\frac{\mu}{(v-1)} G_{18}(R,T+L) - \frac{\mu}{(v-1)} G_{18}(R,L) - G_{83}\right.$$

$$\left.(R,T+L) + G_{82}\ (R,L)\right) + \frac{s}{T}\left(\frac{\mu}{(v-1)} G_4(R,T+L) - \frac{\mu}{(v-1)} G_4(R,L) - G_{63}\ (R,T+L) + G_{63}\right.$$

$$\left.(R,L)\right) \qquad 9.7.8.$$

Section 9.8. (M,T) Exponential cost.

In section 5.3. the inventory cost for fixed maximum re-order cover was derived. In this section, the inventory cost for (M,T) when supply is random is derived. The inventory cost when the maximum re-order cover is fixed is given in equation 5.3.8.

$$C = \frac{R_c+S}{T} + hc\left(M-DL-\frac{DT}{2}\right)^{+} \frac{hc}{T} * (G_2(M,T+L) - G_2(M,L)) + \frac{1}{T}(G_{19}(M,T+L) - G_{19}(M,L))$$

$$+ \frac{s}{T}(R_0(M,L+T) - R_0(M,L)) \qquad 9.8.1.$$

The p.d.f of M, $U(M) = M^{v-1}\mu^{v} e^{-\mu M} / \Gamma(v)$ and also the following integrals have been given section 9.4. which are :

From equation 9.4.8.

$$G_{57}(T) = \int_0^\infty U(M)\, G_2(M,T)\, dM$$

From equation 9.4.10.

$$G_{58}(T) = \int_0^\infty U(M)\, R_0(M,T)\, dM.$$

The only additional integral needed to define the inventory costs for random supply is $G_{65}(T) = \int_0^\infty U(M)\, G_{19}(M,T)\, dM.$ 9.8.3.

From equation 5.3.9.

$$G_{19}(M,T) = \frac{2Db_1}{b_4(\sigma^2 b_4^2 + 2D^2 b_4)} * \exp\left[T\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right) - \frac{b_4 M}{D}\right] F\left(\frac{M - T(D + \frac{\sigma^2 b_4}{D})}{\sqrt{\sigma^2 T}}\right)$$

$$+ \frac{b_1}{b_4}\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^2 b_4^2\, b_1}{Db_4(\sigma^2 b_4^2 + 2D^2 b_4)} * e^{\frac{2DM}{\sigma^2}} * F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.8.4.$$

Multiplying by U(M) we have

$$U(M)\, G_{19}(M,T) = \frac{2Db_1 \mu^v M^{v-1}}{b_4(\sigma^2 b_4^2 + 2D^2 b_4)\Gamma(v)} * \exp\left[T\left(\frac{\sigma^2 b_4^2 + 2D^2 b_4}{2D^2}\right) - M\left(\frac{b_4}{D} + \mu\right)\right] *$$

$$F\left(\frac{M - T*\left(D + \frac{\sigma^2 b_4}{D}\right)}{\sqrt{\sigma^2 T}}\right) + \frac{b_1}{b_4}\,\frac{1}{\sqrt{\sigma^2 T}}\,\frac{\mu^v e^{-\mu M}}{\Gamma(v)}\left[M^{v} - DTM^{v-1}\right] g\left(\frac{M-DT}{\sqrt{\sigma^2 T}}\right)$$

$$\frac{\sigma^2 b_4^2\, b_1 \quad \mu^v}{Db_4\,(\sigma^2 b_4 + 2D^2 b_4)\,\Gamma(v)}\, M^{v-1} * e^{\frac{2DM}{\sigma^2}} * F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right) \qquad 9.8.5.$$

Hence

$\int_0^\infty U(M)\, G_{19}(M,T)\,dM$ applying 9.1.10., 9.1.16. and 9.1.23. we have

$$G_{65}(T) = \frac{2Db_1\,\mu^v e^{\left[\left(\frac{\sigma^2 b_4^2 T + 2D^2 b_4 T}{2D^2}\right) + \left(\mu + \frac{b_4}{D}\right)\left(\frac{\sigma^2 T}{2}\left(\mu + \frac{b_4}{D}\right) - DT\right)\right]}}{b_4 * \Gamma(v)\left(\sigma^2 * b_4^2 + 2D^2 * b_4\right)}$$

$$\sum_{z=1}^{v} \sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)!}{\left(\mu + \frac{b_4}{D}\right)^z (v-z)!} \binom{v-z-i}{i} * (DT-\mu\sigma^2 T)^{v-z-2i} * \left(\frac{\sigma^2 T}{2}\right)^i + \frac{b_1}{b_4}\,\frac{\mu^v}{\Gamma(v)}$$

$$e^{\left(\frac{\mu^2\sigma^2 T}{2} - DT\mu\right)}\left[\sum_{i=0}^{\frac{v}{2}} \binom{v-i}{i}\left(DT-\mu\sigma^2 T\right)^{v-2i} * \left(\frac{\sigma^2 T}{2}\right)^i - DT \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i}\left(DT-\mu\sigma^2 T\right)^{v-1-2i}\left(\frac{\sigma^2 T}{2}\right)^i\right]$$

$$\frac{-\sigma^2 b_1 \quad \mu^v}{D\left(\sigma^2 b_4 + 2D^2\right)\Gamma(v)}\, e^{\left(\frac{\mu^2\sigma^2 T}{2} - D\mu T\right)} \sum_{z=1}^{v}\sum_{i=0}^{\frac{v-z}{2}} \frac{(v-1)}{\left(\mu - \frac{2D}{\sigma^2}\right)^z (v-z)!}\binom{v-z-i}{i}$$

$$(DT-\mu\sigma^2 T)^{v-z-2i} \quad \left(\frac{\sigma^2 T}{2}\right)^i \qquad 9.8.6.$$

Hencethe inventory costs for (M,T) when the supply is random and lead time constant is

$$C = \frac{(R_c + S)}{T} + \frac{hc.v}{\mu} - hc\,\left(DL + \frac{DT}{2}\right) + hc\,(G_{57}(T+L) - G_{57}(L)) + \frac{1}{T}\,(G_{65}(T+L) - G_{65}(L)) + \frac{s}{T}\,(G_{59}(T+L) - G_{59}(L)) \qquad 9.8.7.$$

Section 9.9. (M,R,T)

The inventory costs when the cost of a backorder is an exponential function of the length of time of a backorder and the maximum re-order cover M does not vary is given in section 5.4., equation 5.4.11.

$$C = \frac{R_c}{T} + \frac{S + \sum_{n-1}^{\infty} \int_0^{M-R} G_{21}(R+y,T) f^n (M-R-y,DT)\, dy + G_{21}(M,T)}{T\left[F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty} \int_0^{M-R} n\, f^{n-1}(M-R-y,DT)\; F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right) dy \right]}$$ 9.8.1.

Hence averaging over the states of M, the inventory costs when M is a variable

$$C = \frac{R_c}{T} + \frac{S + \sum_{n=1}^{\infty} \int_M^{\infty} U(M) \int_0^{M-R} G_{21}(R+y,T) f^n (M-R-y,DT)\, dy + U(M)\, G_{21}(M,T)\, dM}{T\left[\int_M^{\infty} U(M)\, F\left(\frac{M-R-DT}{\sqrt{\sigma^2 T}}\right) + \sum_{n=2}^{\infty} \int_0^{M-R} n\, f^{n-1}(M-R-y,DT)\; F\left(\frac{y-DT}{\sqrt{\sigma^2 T}}\right) dy \right]}$$ 9.9.2.

CHAPTER 10

RANDOM SUPPLY AND CONTINUOUS LEAD TIMES

Introduction

In chapter 8 lead times were assumed to be continuous random variables and supply was not a distributed variable while in chapter 9 supply was a distributed variable and lead times were constant. In this chapter we combine the assumptions that supply is a random variable and lead times are continuous random variables. Both are assumed to be gamma distributed. The probability density function of lead time L, H(L)

$$H(L) = \frac{\exp -\alpha L \; L^{k-1} \alpha^{k}}{\Gamma(k)} \qquad L > 0$$

and the probability density function of supply Q, U(Q)

$$U(Q) = \frac{\exp -\mu Q \; Q^{v-1} \mu^{v}}{\Gamma(v)} \qquad Q > 0$$

We could either obtain the inventory costs for random suuply and continuous lead times by either averaging over the states of lead time, the inventory costs for random supply and constant lead times or by averaging over the states of batch quantity, the inventory costs for continuous lead times and fixed supply. We shall derive the inventory costs for random supply and continuous lead times by averaging over the states of lead times the inventory costs for random supply and constant lead times derived in chapter 9. This approach has been chosen because it is mathematically simpler than averaging over the states of supply the inventory costs for fixed supply and continuous lead times.

The costs equations are derived for two cases for which the cost of a backorder is a quadratic function and when it is an exponential function of the length of time of a backorder.

In sections 10.1,10.2,10.3, 10.4 we derive the cost equations for models (Q,R),(nQ,R,T),(M,T) and (M,R,T) respectively for the quadratic case. In sections 10.5,10.6,10.7, we derive the cost equations for the exponential case.

SECTION 10.1 (Q,R) QUADRATIC COST. RANDOM SUPPLY AND CONTINUOUS LEAD TIMES

The inventory costs for random supply and continuous lead times and when the cost is a quadratic function of the length of time of a backorder is given in equation 9.2.19.

From equation 9.2.19.

$$C=\frac{DS\ u}{(v-1)}+\frac{hc.v}{2u}+(R-DL)hc+Db_1\left(\frac{u}{(v-1)}\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-G_{53}(R,L)\right)+(b_2+hc)\left(\frac{u}{(v-1)}\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-G_{54}(R,L)\right)$$

$$+\frac{b_3}{D}\left(\frac{u}{(v-1)}\theta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-G_{55}(R,L)\right)$$

$$+Ds\left(\frac{u\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)}{(v-1)}-G_{53}(R,L)\right) \qquad 10.1.1.$$

Noting the following:

$$G_{29}(Z)=\int_0^\infty H(L)\alpha\left(\frac{Z-DL}{\sqrt{\sigma^2 L}}\right)dL$$, where $G_{29}(Z)$

is given in equation 8.3.20.

$$G_{30}(Z)=\int_0^\infty H(L)\beta\left(\frac{Z-DL}{\sqrt{\sigma^2 L}}\right)dL$$ where $G_{30}(Z)$

is given in 8.3.21. and

$$G_{31}(Z) = \int_0^{\infty} H(L) \left(\frac{Z-DL}{\sqrt{\sigma^2 L}} \right) dL \qquad \text{where } G_{31}(Z)$$

is given in 8.3.22.

The following integrals are needed in order to be able to define the costs when supply is random and lead time is continuous.

$$G_{66}(R) = \int_0^{\infty} H(L) G_{53}(R,L) \, dL \qquad 10.1.2.$$

$$G_{67}(R) = \int_0^{\infty} H(L) G_{54}(R,L) \, dL \qquad 10.1.3.$$

$$G_{68}(R) = \int_0^{\infty} H(L) G_{55}(R,L) \, dL \qquad 10.1.4.$$

From 9.2.8

$$G_{53}(R,L) = \frac{\sigma^2 L \, \mu^{v}}{\Gamma(v)} e^{\left(R\mu + \frac{u^2\sigma^2 L}{2} - DuL\right)} \sum_{i=0}^{\frac{v-2}{2}} \binom{v-2-i}{i} (DL-R-u\sigma^2 L)^{v-2-2i}$$

$v-2-2i$

$$\left(\frac{\sigma^2 L}{2}\right)^{i} - \frac{u^{v}}{\Gamma(v)} e^{\left(Ru + u^2 \frac{\sigma^2}{2} L - DuL\right)} (R-DL) \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \frac{(v-2)!}{u^{Z}(v-Z)!}$$

$$\binom{v-3-1-i}{i}$$

$$*(DL-R-u\sigma^2L)^{V-Z-1-2i}\left(\frac{\sigma^2L}{2}\right)^i - \frac{u^V}{\Gamma(V)}e^{(Ru+u^2\frac{\sigma^2}{2}L-DuL)}$$

$$\sum_{Z=1}^{V}\sum_{i=0}^{\frac{V-Z}{2}}\frac{(V-1)!}{\mu^Z(V-Z)!}\binom{V-Z-i}{i}^{V-Z-2i}\left(\frac{\sigma^2L}{2}\right)^i$$

10.1.5.

Simplifying

$$G_{53}(R,L)=\sigma^2u^VLe^{Ru}e^{L\left(Du-\frac{u^2\sigma^2}{2}\right)}\sum_{i=0}^{\frac{V-2}{2}}\sum_{w=0}^{V-2-2i}\binom{V-2-i}{i}(D-\mu\sigma^2)^w$$

$$(-R)^{V-2-2i-w}\left(\frac{\sigma^2}{2}\right)^i\binom{V-2-2i}{w}L^{w+i}$$

$$-\frac{\mu^V}{\Gamma(V)}e^{R\mu-L\left(D\mu-\frac{\mu^2\sigma^2}{2}\right)}(R-DL)\sum_{Z=1}^{V-1}\sum_{i=0}^{(V-Z-1)/2}\sum_{w=0}^{V-Z-2i-1}\frac{(V-2)!}{\mu^Z(V-Z)!}$$

$$\binom{V-Z-1-i}{i}*$$

$$\times\ (D-u\sigma^2)^w(-R)^{v-Z-1-2i-w}\left(\frac{\sigma^2}{2}\right)^i L^{w+i}\binom{v-Z-1-2i}{w}$$

$$-\frac{u^v}{\Gamma(v)}e^{Ru-L\left(\frac{u^2}{2}\sigma^2-Du\right)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\frac{(v-1)!}{u^Z(v-Z)!}$$

$$\binom{V-Z-i}{i}^{V-Z-2i}$$

$$\times(D-u\sigma^2)^w(-R)^{v-Z-1-2i-w}\left(\frac{\sigma^2}{2}\right)^i L^{w+i}\binom{v-z-2i}{w} \qquad 10.1.6.$$

Multiplying by H(L)

$$H(L)G_{53}(R,L)=\frac{\sigma^2u^v\alpha^k e^{Ru}}{\Gamma(v)\ (K)}e^{-L\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)}\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-2-2i}\binom{V-2-i}{i}$$

$$\times(D-u\sigma^2)^w(-R)^{v-2-2i-w}\left(\frac{\sigma^2}{2}\right)^i L^{w+i+k}\binom{v-2-2i}{w}$$

$$-\frac{u^v\alpha^k e^{Ru-L\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)}}{\Gamma(v)\ \Gamma(k)}\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-3}{2}-1}\sum_{w=0}^{v-z-1-2i}\frac{(V-2)!}{\mu^Z(V-Z)!}\binom{V-Z-1-2i}{i}$$

$$\left(RL^{w+i+k-1}-DL^{w+i+k}\right)(-R)^{v-Z-1-2i-w}\left(\frac{\sigma^2}{2}\right)^i\times(D-u\sigma^2)^w\binom{v-Z-1-2i}{w}$$

$$-\frac{\alpha^k u^v e^{Ru-L\left(\frac{u^2\sigma^2}{2}-Du+\alpha\right)}}{\Gamma(v)\,\Gamma(k)} \sum_{=1}^{V} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \frac{(v-1)!}{u^Z (v-Z)!}$$

$$\times \binom{v-Z-i}{i}^{v-Z-2i} (D-u\sigma^2)^w \times$$

$$(-R)^{v-Z-1-2i-w}\left(\frac{\sigma^2}{2}\right)^i L^{w+i+k-1} \binom{v-Z-2i}{w}$$

10.1.7.

Hence

$\int_0^\infty H(L)G_{53}(R,L)\, dL$ noting that

$$\int_0^\infty e^{-\theta L} L^y \, dL = \Gamma(Y+1)/\theta^{y+1}$$

10.1.8.

then

$$G_{56}(R) = \frac{\sigma^2 u^v \alpha^k}{\Gamma(v)\,\Gamma(k)} e^{Ru} \sum_{i=0}^{\frac{v-2}{2}} \sum_{w=0}^{v-2-2i} \binom{v-2-2i}{i} (D-\mu\sigma^2)^w$$

$$(-R)^{v-2-2i-w}\left(\frac{\sigma^2}{2}\right)^i \frac{\Gamma(w+i+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{w+i+k+1}} \times (D-u\sigma^2)^w \binom{v-2-2i}{w}$$

$$-\frac{u^v \alpha^k e^{Ru}}{\Gamma(v)\,\Gamma(k)} \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \sum_{w=0}^{v-Z-1-2i} \frac{(v-2)!}{u^Z (v-Z)!} \binom{v-Z-1-i}{i} (-R)^{v-Z-1-2i-w} \times \left(\frac{\sigma^2}{2}\right)^i$$

$$\left(\frac{R\,\Gamma(w+i+k)}{\left(D\mu-\frac{\mu^2\sigma^2}{2}+\alpha\right)^{w+i+k}} - \frac{D\,\Gamma(w+i+k+1)}{\left(D\mu-\frac{\mu^2\sigma^2}{2}+\alpha\right)^{w+i+k+1}}\right)\binom{v-Z-1-2i}{w}$$

$$\frac{-\alpha^{k} u^{v} e^{Ru}}{\Gamma(v)\,\Gamma(k)} \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \frac{(v-1)!}{u^{3}(v-Z)!} \binom{v-Z-i}{i}^{v-Z-2i} \left(D-u\sigma^{2}\right)^{w}$$

$$(-R)^{v-Z-1-2i-w}\left(\frac{\sigma^{2}}{2}\right)^{i} \frac{\Gamma(w+i+k)}{\left(Du-\frac{u^{2}}{2}\sigma^{2}+\alpha\right)^{w+i+k}} \binom{v-Z-2i}{w}$$

10.1.9

From 9.2.13

$$G_{54}(R,L)=\frac{u^{v}}{\Gamma(v)} e^{Ru-L\left(Du-\frac{u^{2}}{2}\sigma^{2}\right)}\Big[\left(\sigma^{2}L+R^{2}\mp DRL+D^{2}L^{2}\right)$$

$$\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \frac{(v-2)!}{u^{Z}(v-1-Z)!} \binom{v-Z-i-1}{i} \left(DL-R-u\sigma^{2}L\right)^{v-1-Z-2i} \left(\frac{\sigma^{2}}{2}L\right)^{2}$$

$$+2(R-DL)\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \binom{v-Z-i}{i} \frac{(v-1)!}{u^{Z}(v+1-3)!} \left(DL-R-u\sigma^{2}L\right)^{v-Z-2i} \left(\frac{\sigma^{2}L}{2}\right)^{i}$$

$$+\sum_{Z=1}^{v+1} \sum_{i=0}^{(v+1-Z)/2} \frac{v!}{\mu^{Z}(v+2-Z)!} \binom{v+1-Z-i}{i} \left(DL-M-\mu\sigma^{2}L\right)^{v+1-Z-2i} \left(\frac{\sigma^{2}L}{2}\right)^{i}\Big]$$

$$-\frac{\sigma^{2}Lu^{v}}{2\,\Gamma(v)} e^{Ru-L\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)} \Big[(R-DL)\sum_{i=0}^{\frac{v-2}{2}} \binom{v-2-i}{i} \left(DL-R-u\sigma^{2}L\right)^{v-2-2i}$$

$$\left(\frac{\sigma^{2}L}{2}\right)^{i}+\sum_{i=0}^{\frac{v-1}{2}} \binom{v-i-1}{i} \left(DL-R-u\sigma^{2}L\right)^{v-i-2i} \left(\frac{\sigma^{2}L}{2}\right)^{i}\Big]$$

10.1.10.

Simplifying

$$G_{54}(R,L)=\frac{u^{v}}{\Gamma(v)}\,e^{Ru-L\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)}\Bigg[\left(\sigma^{2}L+R^{2}-2RL+D^{2}L^{2}\right)\times$$

$$\sum_{Z=1}^{v-1}\ \sum_{i=0}^{\frac{v-Z-1}{2}}\ \sum_{w=0}^{v-1-Z-2i}\frac{(v-2)!}{u^{Z}(v-1-Z)!}\binom{v-Z-i-1}{i}\binom{v-1-Z-2i}{w}$$

$$*\quad (D-u\sigma^{2})^{w}(-R)^{v-1-Z-2i-w}\times\left(\frac{\sigma^{2}}{2}\right)^{i}L^{i+w+1}$$

$$+2(R-DL)\sum_{Z=1}^{v}\ \sum_{i=0}^{\frac{v-Z}{2}}\ \sum_{w=0}^{v-Z-2i}\binom{v-Z-i}{i}\binom{v-Z-2i}{w}\frac{(v-1)!}{u^{Z}(v+1-Z)!}$$

$$(D-u\sigma^{2})^{w}(-R)^{v-Z-2i-w}\times\left(\frac{\sigma^{2}}{2}\right)^{i}L^{i+w+1}$$

$$+\sum_{Z=1}^{v+1}\ \sum_{i=0}^{\frac{v-Z+1}{2}}\ \sum_{w=0}^{v+1-Z-2i}\binom{v+1-Z-i}{i}\binom{v+1-Z-2i}{w}*\frac{v!}{u^{Z}(v+2-Z)!}(D-u\sigma^{2})^{w}(-R)^{v+1-Z-2i-w}\left(\frac{\sigma^{2}L}{2}\right)^{i}L^{i+w+1}\Bigg]$$

$$-\frac{\sigma^{2}u^{v}e^{Ru-L\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)}}{2\,\Gamma(v)}\Bigg[(R-DL)\sum_{i=0}^{\frac{v-2}{2}}\ \sum_{w=0}^{v-2-2i}\binom{v-2-i}{i}\binom{v-2-2i}{w}$$

$$(D-u\sigma^{2})^{w}\,(-R)^{v-2-2i-w}\left(\frac{\sigma^{2}}{2}\right)^{i}L^{i+w+1}$$

$$+\sum_{i=0}^{\frac{v-1}{2}}\ \sum_{w=0}^{v-1-2i}\binom{v-i-1}{i}\binom{v-1-2i}{w}(D-u\sigma^{2})^{w}(-R)^{v-1-2i-w}\left(\frac{\sigma^{2}}{2}\right)^{i}L^{i+w+1}$$

10.1.11

Substituting into 10.1.3 and noting that

$$\int_0^\infty e^{-\theta L} L^y \, dy = \Gamma(Y+1)/\theta^{y+1} \qquad \text{we have}$$

$$G_{67}(R) = \frac{\alpha^k u^v e^{Ru}}{\Gamma(v)\Gamma(k)} \left(\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \sum_{w=0}^{v-1-Z-2i} \frac{(v-2)!}{u^Z (v-Z)!} \binom{v-Z-i-1}{i} \times \right.$$

$$\binom{v-1-Z-2i}{w} (D-u\sigma^2)^w (-R)^{v-1-Z-2i-w} \left(\frac{\sigma^2}{2}\right)^i \left(\frac{\Gamma(i+w+k)}{\left(Du-\frac{u^2}{2}\sigma^2+\alpha\right)^{i+w+k}} \right.$$

$$\times (1+R^2)$$

$$\left. + (\sigma^2-2R) \frac{\Gamma(i+w+k+1)}{\left(Du-\frac{u^2}{2}\sigma^2+\alpha\right)^{i+w+k+1}} + D^2 \frac{\Gamma(i+w+k+2)}{\left(Du-\frac{u^2}{2}\sigma^2+\alpha\right)^{i+w+k+2}} \right)$$

$$+2 \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \binom{v-Z-i}{i} \binom{v-Z-2i}{w} \frac{(v-1)!}{u^Z (v+1-Z)!} (D-u\sigma^2)^w$$

$$(-R)^{v-Z-2i-w} \left(\frac{\sigma^2}{2}\right)^i \times \left(R \frac{\Gamma(i+k+w)}{\left(Du-\frac{u^2}{2}\sigma^2+\alpha\right)^{i+k+w}} - D \frac{\Gamma(i+k+w+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+k+w+1}} \right)$$

$$+ \sum_{Z=1}^{v+1} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \binom{v-Z-i}{i} \binom{v-Z-2i}{w} \frac{v!}{u^Z (v+2-Z)!} (D-u\sigma^2)^w \times$$

$$\left. (-R)^{v-Z-2i} \left(\frac{\sigma^2}{2}\right)^i \frac{\Gamma(i+w+k)}{\left(Du-\frac{u^2}{2}\sigma^2+\alpha\right)^{i+w+k}} \right)$$

$$-\frac{\sigma^2 a^k u^v e^{Ru}}{2\Gamma(v)\Gamma(k)}\left[\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-2-2i}\binom{v-2-i}{i}\binom{v-2-2i}{w}(D-u\sigma^2)^w \ast\right.$$

$$(-R)^{v-2-2i-w}\left(\frac{\sigma^2}{2}\right)^i\left(\frac{R\,\Gamma(i+k+w)}{(Du-\frac{u^2\sigma^2}{2}+a)^{i+k+Z}}\right.$$

$$\left.-\frac{D\,\Gamma(i+k+w+1)}{(Du-\frac{u^2\sigma^2}{2}+a)^{i+k+w+1}}\right)+\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-2i}\binom{v-i-1}{i}\binom{v-1-2i}{w}\ast$$

$$\left.(D-u\sigma^2)^w(-R)^{v-1-2i-w}\left(\frac{\sigma^2}{2}\right)^i\frac{\Gamma(i+k+w)}{(Du-\frac{u^2\sigma^2}{2}+a)^{i+w+k}}\right]$$

10.1.12

From 9.2.18.

$$G_{55}(R,L)=\frac{u^v\sigma^2 L e^{Ru-L(Du-\frac{u^2\sigma^2}{2})}}{3\Gamma(v)}\left[\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}(DL-R-u\sigma^2 L)^{v-2i}\ast\right.$$

$$\left(\frac{\sigma^2}{2}L\right)^i+2(R-DL)\sum_{i=0}^{\frac{v-1}{2}}\binom{v-i-1}{i}\left(DL-R-u\sigma^2 L\right)^{v-2i-1}\left(\frac{\sigma^2}{2}L\right)^i$$

$$\left.+(R^2-2RDL+D^2L^2+2\sigma^2 L)\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i}\left(DL-R-u\sigma^2 L\right)^{v-2-2i}\left(\frac{\sigma^2}{2}L\right)^i\right]$$

$$-\frac{u^{v}e^{Ru-L\left(Du-\frac{u^2}{2}\sigma^2\right)}}{3\,\Gamma(v)}\left[\sum_{Z=1}^{v+2}\sum_{i=0}^{\frac{v+1}{2}}\frac{(v+1)!}{u^{Z}(v+Z-Z)!}(DL-R-u\sigma^2L)^{v+2-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}\right.$$

$$+3(R-DL)\sum_{Z=1}^{v+1}\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}\frac{v!}{u^{Z}(v+2-Z)!}(DL-R-u\sigma^2L)^{v-Z-2i}\left(\frac{\sigma^2}{2}L\right)^{i}$$

$$+3(R^2-2DRL+\sigma^2L)\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-1}{2}}\binom{v-i-1}{i}\frac{(v-1)!}{u^{Z}(v-Z)!}(DL-R-u\sigma^2L)^{v-Z-2i-1}*\left(\frac{\sigma^2}{2}L\right)^{i}$$

$$+(R^3-L(2DR^2-3\sigma^2R+DR^2)+L^2(3D^2R-3\sigma^2D)-D^3L^3)$$

$$\left.\times\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i}\frac{(v-1)!}{u^{Z}(v-Z-1)!}(DL-R-u\sigma^2L)^{v-2-Z-2i}\left(\frac{\sigma^2}{2}L\right)^{i}\right]$$

10.1.13

Simplifying

$$G_{55}(R,L)=\frac{u^{v}\sigma^2e^{Ru-L\left(Du-\frac{u^2\sigma^2}{2}\right)}}{3\,\Gamma(v)}\sum_{i=0}^{\frac{v}{2}}\sum_{w=0}^{v-2i}\binom{v-i}{i}\binom{v-2i}{w}(D-u\sigma^2)^{w}$$

$$(-R)^{v-2i-w}\left(\frac{\sigma^2}{2}\right)^{i}L^{i+w+1}$$

$$+2(R-DL)\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-2i-1}\binom{v-i-1}{i}\binom{v-2i-1}{w}(D-u\sigma^2)^{w}(-R)^{v-2i-1-w}\left(\frac{\sigma^2}{2}\right)^{i}$$

$$L^{i+w+1} + (R^2 - L(2RD - 2\sigma^2) + D^2L^2) \sum_{i=0}^{\frac{v-2}{2}} \sum_{w=0}^{v-2-2i} \binom{v-2-i}{i} \binom{v-2-2i}{w}$$

$$\times (D-u\sigma^2)^w (-R)^{v-2-2i-w} \left(\frac{\sigma^2}{2}\right)^i L^{i+w+1} \Bigg]$$

$$\frac{-u^v e^{Ru - L\left(Du - \sigma^2 \frac{u^2}{2}\right)}}{3\Gamma(v)} \Bigg[\sum_{Z=1}^{v+2} \sum_{i=0}^{\frac{v+1}{2}} \sum_{w=0}^{v+2-Z-2i} \binom{v+1-i}{i} \binom{v+2-Z-2i}{w}$$

$$\frac{(v+1)!}{u^Z (v+3-Z)!} \times (D-u\sigma^2)^w (-R)^{v+2-Z-2i} \left(\frac{\sigma^2}{2}\right)^i L^{i+w}$$

$$+3(R-DL) \sum_{Z=1}^{v+1} \sum_{i=0}^{\frac{v}{2}} \sum_{w=0}^{v-Z-2i} \binom{v-i}{i} \binom{v-Z-2i}{w} \frac{v!}{u^Z (v+2-Z)!}$$

$$(D-u\sigma^2)^w (-R)^{v-Z-2i-w} \times \left(\frac{\sigma^2}{2}\right)^i L^{i+w}$$

$$+3(R^2 - L(2DR - \sigma^2)) \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-1}{2}} \sum_{w=0}^{v-Z-2i-1} \binom{v-i-1}{i} \frac{(v-1)!}{u^Z (v-Z)!} \binom{v-Z-2i-1}{w}$$

$$(D-u\sigma^2)^w (-R)^{v-Z-2i-1-w} \left(\frac{\sigma^2}{2}\right)^i L^{i+w}$$

$$+ (R^3 - 3(DR^2 - \sigma^2 R)L + 3L^2(D^2R - \sigma^2 D) - D^3L^3)$$

$$\times \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-2}{2}} \sum_{w=0}^{v-2-Z-2i} \binom{v-2-i}{i} \binom{v-2-Z-2i}{w} \frac{(v-1)!}{u^{Z}(v-Z-1)!}$$

$$(D-u\sigma^2)^{w}(-R)^{v-2-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i} \times L^{i+w} \Bigg] \qquad 10.1.14$$

Substituting into 10.1.4 and noting that

$\int_0^{\infty} e^{-\theta L} L^{Y}\, dL = \Gamma(Y+1)/\theta^{Y+1}$ then we have

$$G_{67}(R) + \frac{u^{v}\alpha^{k}\sigma^{2}}{3\Gamma(v)\Gamma(k)} e^{Ru} \Bigg[\sum_{i=0}^{\frac{v}{2}} \sum_{w=0}^{v-2i} \binom{v-i}{i}\binom{v-2i}{w} (D-u\sigma^2)^{w}(-R)^{v-2i-w}$$

$$\left(\frac{\sigma^2}{2}\right)^{i} \frac{\Gamma(i+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k+1}}$$

$$+2\sum_{i=0}^{\frac{v-1}{2}} \sum_{w=0}^{v-2i-1} \binom{v-i-1}{i}\binom{v-2i-1}{w} (D-u\sigma^2)^{w} (-R)^{v-2i-1-w} \left(\frac{\sigma^2}{2}\right)^{i}$$

$$\left(\frac{R\,\Gamma(i+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k+1}} - \frac{D\,\Gamma(i+w+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}\right)^{i+w+k+2}} \right) + \sum_{i=0}^{\frac{v-2}{2}} \sum_{w=0}^{v-2-2i}$$

$$\binom{v-2-i}{i}\binom{v-2-2i}{w} (D-u\sigma^2)^{w}(-R)^{v-2-2i-w}\left(\frac{\sigma^2}{2}\right)^{i} \times \left(\frac{R^2\,\Gamma(i+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k+1}} \right.$$

$$\left. -(2DR-2\sigma^2)\frac{\Gamma(i+w+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k+2}} + D^2\frac{\Gamma(i+w+k+3)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+Z+k+3}} \right)$$

$$\frac{-u^{v}\alpha^{k}e^{Ru}}{3\,\Gamma(v)\,\Gamma(k)}\left[\sum_{Z=1}^{v+2}\sum_{i=0}^{\frac{v+1}{2}}\sum_{w=0}^{v+2-Z-2i}\binom{v+1-i}{i}\binom{v+2-Z-2i}{w}\frac{(v+1)!}{u^{Z}(v+3-Z)!}\right.$$

$$\times(D-u\sigma^{2})^{w}(-R)^{v+2-Z-2i}\left(\frac{\sigma^{2}}{2}\right)^{i}\frac{\Gamma(i+w+k)}{\left(Du-\frac{u^{2}\sigma^{2}}{2}+\alpha\right)^{i+w+k}}$$

$$+3\sum_{=1}^{v+1}\sum_{i=0}^{\frac{v}{2}}\sum_{w=0}^{v-Z-2i}\binom{v-i}{i}\binom{v-Z-2i}{w}\frac{v!}{u^{Z}(v+2-Z)!}(D-u\sigma^{2})^{w}(-R)^{v-\,-2i-w}\left(\frac{\sigma^{2}}{2}\right)^{i}$$

$$\left(\frac{R\,\Gamma(i+w+k)}{\left(Du-\frac{u^{2}\sigma^{2}}{2}+\alpha\right)^{i+w+k}}-\frac{D\,\Gamma(i+w+k+1)}{\left(Du-\frac{u^{2}\sigma^{2}}{2}+\alpha\right)^{i+w+k+1}}\right)$$

$$+3\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-Z-2i-1}\binom{v-i-1}{i}\frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-2i-1}{w}\left(D-u\sigma^{2}\right)^{w}$$

$$(-R)^{v-Z-2i-1-w}\left(\frac{\sigma^{2}}{2}\right)^{i}\left(\frac{R^{2}\,\Gamma(i+w+k)}{\left(Du-\frac{u^{2}\sigma^{2}}{2}+\alpha\right)^{i+w+k}}-\frac{(2DR-\sigma^{2})\,\Gamma(i+w+k+1)}{\left(Du-\frac{u^{2}\sigma^{2}}{2}+\alpha\right)^{i+w+k+1}}\right)$$

$$+\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-2-Z-2i}\binom{v-2-i}{i}\binom{v-2-Z-2i}{w}\frac{(v-1)!}{u^{Z}(v-Z-1)!}$$

$$(D-u\sigma^{2})^{w}(-R)^{v-2-Z-2i}\left(\frac{\sigma^{2}}{2}\right)^{i}$$

$$\times\left(\frac{R^{3}\,\Gamma(i+w+k)}{\left(Du-\frac{u^{2}\sigma^{2}}{2}+\alpha\right)^{i+w+k}}-\frac{3(DR^{2}-\sigma^{2}R)\,\Gamma(i+w+k+1)}{\left(Du-\frac{u^{2}\sigma^{2}}{2}+\alpha\right)^{i+w+k+1}}\right.$$

$$+\frac{3(D^2 R-\sigma^2 D)\,\Gamma(i+w+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k+2}} - \frac{D^3\,\Gamma(i+w+k+3)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k+3}}\Bigg)$$

10.1.15.

Hence the inventory costs for (ω, R) when supply is random and lead time is continuous is

$$C=\frac{DSu}{(v-1)}+\frac{hc.v}{2\mu}+\left(R-\frac{DK}{\alpha}\right)hc+Db_1\left(\frac{u}{(v-1)}\;G_{29}(R)\right.$$
$$\left.-G_{66}(R)\right)+(b_2+hc)\left(\frac{u}{(v-1)}\;G_{30}(R)-G_{67}(R)\right)$$
$$+\frac{b_3}{D}\left(\frac{u}{(v-1)}\;G_{31}(R)-\quad{}_{68}(R)\right)$$
$$+Ds\left(\frac{u}{(v-1)}\;G_{29}(R)\;-\;G_{55}(R)\right)$$

10.1.16.

SECTION 10.2 (Q, R, T)

The inventory costs for fixed supply of batch quantity when the cost of a backorder is a quadratic function of the length of time of a backorder from 9.3.21. substituting L_2+T for $T+L$ and L_1 for L.

$$C+\frac{R_c}{T}+\frac{S.PO R_T}{T}+hc\left(\frac{v}{2u}+R-DL-\frac{DT}{2}\right)$$

$$+\frac{b_1}{T}\left(\frac{u}{(v-1)}G_1(R,T+L_2)-\frac{G_1(R,L_1)u}{(v-1)}-G_{60}(R,T+L_2)+G_{60}(R,L_1)\right)$$

$$+\frac{(hc+b_2)}{T}\left(\frac{u}{(v-1)}G_3(R,T+L_2)-\frac{u}{(v-1)}G_3(R,L_1)-G_{61}(R,T+L_2)+G_{61}(R,L_1)\right)$$

$$+\frac{b_3}{T}\left(\frac{u}{(v-1)}G_{11}(R,T+L_2)-\frac{u}{(v-1)}G_{11}(R,L_1)-G_{62}(R,T+L_2)+G_{62}(R,L_1)\right)$$

$$+\frac{s}{T}\left(\frac{u}{(v-1)}G_4(R,T+L_2)-\frac{u}{(v-1)}G_4(R,L_1)-G_{63}(R,T+L_2)+G_{63}(R,L_1)\right)$$

10.2.1.

The probability density function of lead time $L, H(L)$

$$H(L)=\frac{L^{k-1}e^{-aL}a^k}{\Gamma(k)}$$

From 8.4.10 and 8.4.28

$$G_{32}(R)=\int_0^\infty H(L)G_1(R,L)\,dL \qquad 10.2.2.$$

$$G_{35}(R,T)=\int_0^\infty H(L)G_1(R,L+T)\,dL$$

from 8.4.11 and 8.4.27

$$G_{33}(R) = \int_0^\infty H(L) G_3(R,L)\, dL \qquad 10.2.3$$

$$G_{36}(R,T) + \int_0^\infty H(L) G_3(R+L+T)\, dL$$

from 8.4.12 and 8.4.28

$$G_{34}(R) = \int_0^\infty H(L) G_{11}(R,L)\, dL \qquad 10.2.4$$

$$G_{37}(R,T) = \int_0^\infty H(L) G_{11}(R,L+T)\, dL$$

$$G_{69}(R,T) = \int_0^\infty H(L) G_{60}(R,T+L)\, dL \qquad 10.2.5$$

$$G_{70}(R,T) = \int_0^\infty H(L) G_{61}(R,T+L)\, dL \qquad 10.2.6$$

$$G_{71}(R,T) = \int_0^\infty H(L) G_{62}(R,T+L)\, dL \qquad 10.2.7$$

$$G_{72}(R,T) = \int_0^\infty H(L) G_{63}(R,T+L)\, dL \qquad 10.2.8$$

Let $G_{73}(R,T) = \int_0^\infty H(L) G_4(R,T+L)\, dL$

From 9.3.9, substituting T+L for T

$$G_{60}(R,T+L) = \frac{\sigma^2 (T+L) u^v}{\Gamma(v)} e^{Ru - (T+L)\left(Du - \frac{u^2}{2}\sigma^2\right)} \Gamma\left(\sigma^2 (T+L) + \left(R+D\right)(T+L)^2\right)$$

$$\times \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \frac{(v-2)!}{u^{Z}(v-Z)!} \binom{v-Z-i-1}{i} \left(D(T+L)-R-u\sigma^{2}(T+L)\right)^{v-Z-2i-1}$$

$$\left(\frac{\sigma^2}{2}\right)^{i}(T+L)^{i}$$

$$-2\left(D(T+L)-R\right)\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{u^{Z}(v-Z)!} \binom{v-Z-i}{i} \left(D(T+L)-R-u\sigma^{2}(T+L)\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2}{2}\right)^{i}(T+L)^{i}$$

$$+\sum_{Z=1}^{v+1} \sum_{i=0}^{v+1-Z} \frac{v!}{u^{Z}(v+2-Z)!} \binom{v+1-Z-i}{i} D\left((T+L)-R-u\sigma^{2}(T+L)\right)^{v+1-Z-2i}$$

$$\left(\frac{\sigma^2}{2}\right)^{i}(L+T)^{i}$$

$$-\frac{\sigma^2(T+L)u^{v}e^{Ru-(T+L)(Du-\frac{u^2}{2}\sigma^2)}}{2\,\Gamma(v)}\left[(R-D(T+L)\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i}\right.$$

$$\times\left(D(T+L)-R-u^{2}\sigma^{2}(T+L)\right)^{v-2-2i}\left(\frac{\sigma^2}{2}\right)^{i}(T+L)^{i}$$

$$\left.+\sum_{i=0}^{\frac{v-1}{2}}\binom{v-i-1}{i}\left(D(T+L)-R-u\sigma^{2}(T+L)\right)^{v-2i-1}\left(\frac{\sigma^2}{2}\right)^{i}(T+L)^{i}\right]$$

10.2.9

Simplifying

$$G_{60}(R,T+L)=\frac{\sigma^2(T+L)}{\Gamma(v)}u^{v}e^{Ru-T(Du-\frac{u^2}{2}\sigma^2)}e^{-L(Du-\frac{u^2}{2}\sigma^2)}$$

$$\left[(\sigma^2T+R^2+L(\sigma^2-2DT)+D^2L^2)\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-Z-2i-1}\frac{(v-2)!}{u^{2}(v-Z)!}\right.$$

$$\binom{v-Z-i-1}{i}(DT-R-u\sigma^2T)^{v-Z-2i-1-w}\left(\frac{\sigma^2}{2}\right)^i(D-u\sigma^2)^w\sum_{j=0}^{i}\binom{i}{j}T^{i-j}L^{j+w}$$

$$-2((DT-R)+DL)\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j=0}^{i}\frac{(v-1)!}{u^Z(v-Z)!}\binom{v-Z-i}{i}$$

$$(DT-R-u\sigma^2T)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^i\binom{i}{j}T^{i-j}L^{j+w}\times(D-u\sigma^2)^w$$

$$+\sum_{Z=1}^{v+1}\sum_{i=0}^{v+1-Z}\sum_{w=0}^{v+1-Z-2i}\sum_{j=0}^{i}\frac{v!}{u^Z(v+2-Z)!}\binom{v+1-Z-i}{i}\binom{i}{j}$$

$$(DT-R-u\sigma^2T)^{v+1-Z-2i}\times(D-u\sigma^2)^w\left(\frac{\sigma^2}{2}\right)^i\;T^{i-j}\;L^{j+w}$$

$$-\frac{\sigma^2}{2}u^v\frac{e^{\left(Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)-L\left(Du-\frac{u^2\sigma^2}{2}\right)\right)}}{2\,\Gamma(v)}\Bigg[\left((RT-DT^2)+L(R-2DT)-DL^2\right)$$

$$\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-2-2i}\sum_{j=0}^{i}\binom{v-2-i}{i}\binom{i}{j}\binom{v-2-2i}{w}(DT-R-u\sigma^2T)^{v-2-2i}\left(\frac{\sigma^2}{2}\right)^i$$

$$(D-u\sigma^2)^w\;T^{i-j}\;L^{j+w}$$

$$+\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-2i-1}\sum_{j=0}^{i}\binom{v-i-1}{i}\binom{i}{j}\binom{v-2i-1}{w}(DT-R-u\sigma^2T)^{v-2i-1}\left(\frac{\sigma^2}{2}\right)^i$$

$$T^{i-j}\;(D-u\sigma^2)^w\;\;L^{j+w}\Bigg] \qquad 10.2.10.$$

Substituting into 10.2.5 and noting that

$$\int_0^{\infty} L^y e^{-\phi y}\,dy = \Gamma(y+1)/\phi^{y+1}$$

Then we have

$$G_{69}(R,T) = \frac{\sigma^2 u^v \alpha^k e^{Ru-T(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)\ \Gamma(k)} \left[\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \sum_{w=0}^{v-Z-2i-1} \sum_{j=0}^{i} * \right.$$

$$\binom{v-Z-i-1}{i} \frac{(v-2)!}{u^Z(v-Z)!}$$

$$\times (DT-R-u\sigma^2 T)^{v-Z-2i-1} (D-u\sigma^2)^w \left(\frac{\sigma^2}{2}\right)^i T^{i-j} \left((\sigma^2 T^2 + R^2 T) \frac{\Gamma(j+w+k)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{j+w+k}} \right.$$

$$+ (R^2 - 2\sigma^2 T - 2DT^2) \frac{\Gamma(j+w+k+1)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{j+w+k+1}} + (\sigma^2 2DT + D^2 T) \frac{\Gamma(j+w+k+2)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{j+w+k+2}}$$

$$\left. + D^2 \frac{\Gamma(j+w+k+3)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{j+w+k+3}} \right)$$

$$-2 \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \sum_{j=0}^{i} \frac{(v-1)!}{u^Z(v-Z)!} \binom{v-Z-i}{i} (DT-R-u\sigma^2 T)^{v-Z-2i} \quad *$$

$$\left(\frac{\sigma^2}{2}\right)^i (D-u\sigma^2)^w \binom{i}{j} T^{i-j} \left((DT^2-RT) \frac{\Gamma(j+w+k)}{(Du-u^2\sigma^2+\alpha)^{j+w+k}} \right.$$

$$\left. + \frac{(2DT-R)\,\Gamma(j+w+k+1)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{j+w+k+1}} + \frac{D\,\Gamma(j+w+k+2)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{j+w+k+2}} \right)$$

$$+\sum_{Z=1}^{v+1}\sum_{i=0}^{v+1-Z}\sum_{w=0}^{v+1-Z-2i}\sum_{j=0}^{i}\frac{v!}{u^{Z}(v+2-Z)!}\binom{v+1-Z-i}{j}\binom{i}{j} \quad *$$

$$(DT-R-u\sigma^2T)^{v+1-Z-2i}\left(D-u\sigma^2\right)^{w}\left(\frac{\sigma^2}{2}\right)^{i}$$

$$T^{i-j}\left(\frac{T\Gamma(j+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+w+k}}+\frac{\Gamma(j+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+w+k+1}}\right)\Bigg]$$

$$\frac{-\sigma^2u^ve^{Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)}a^k}{2\Gamma(v)\ \Gamma(k)}\Bigg[\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-2-2i}\sum_{j=0}^{i}{}^{v-2-i}\binom{i}{j}\binom{v-2-2i}{w}$$

$$x(DT-R-u\sigma^2T)^{v-2-2i}\left(\frac{\sigma^2}{2}\right)^{i}(D-u\sigma^2)^{w}T^{i-j}\left(\frac{(RT-DT^2)\Gamma(j+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}\right)^{j+w+k}}\right.$$

$$+\frac{(R-2DT)\Gamma(j+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}\right)^{j+w+k+1}}$$

$$\left.-\frac{D\Gamma(j+w+k+2)}{\left(Du-\frac{u^2\sigma^2}{j}+a\right)^{j+w+k+2}}\right)+\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-2i-1}\sum_{j=0}^{i}\binom{v-i-1}{i}\binom{i}{j}\binom{v-2i-1}{w} \quad *$$

$$(DT-R-u\sigma^2T)^{v-2i-1}\left(\frac{\sigma^2}{2}\right)^{i}(D-u\sigma^2)^{w}T^{i-j}\ \frac{\Gamma(i+k+w)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+w+k}}\Bigg]$$

10.2.11.

From 9.3.13

$$G_{61}(R,T+L)=\frac{u^v}{\Gamma(v)}e^{Ru-(T+L)\left(-\frac{u^2\sigma^2}{2}+Du\right)}\Bigg[\frac{(D^2(T+L)^3}{6}$$

$$-(T+L)^2\left(\frac{DR}{2}-\frac{\sigma^2}{4}\right)+(T+L)\left(\frac{R^2}{2}\right)+\left(-\frac{\sigma^4}{4D}R_3-\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}-\frac{\sigma^6}{8D^4}\right) \quad *$$

$$\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \frac{(v-2)!}{u^Z (v-1-Z)!} \binom{v-Z-1}{i} (DT-R-u\sigma^2 T+DL-u\sigma^2 L)^{v-1-Z-2i}$$

$$\times \frac{\sigma^2}{2}(T+L) + \left(-\frac{\sigma^4}{4D^3} - \frac{D(T+L)^2}{2} - \frac{R\sigma^2}{2D^2} + R(T+L) + \frac{R^2}{2D}\right)$$

$$\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{u^Z (v-Z)!} \binom{v-Z-i}{i} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i} \left(\frac{\sigma^2}{2}(T+L)\right)^i$$

$$+\frac{1}{6D} \sum_{Z=1}^{v+1} \sum_{i=0}^{\frac{v+1-Z}{2}} \frac{(v+1)!}{u^Z (v-Z+2)!} \binom{v+1-Z-i}{i} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v+1-Z-2i}$$

$$\left.\left(\frac{\sigma^2}{2}(T+L)\right)^i \right]$$

$$+\frac{\sigma^2(T+L)u^v e^{\left(Ru-(T+L)\left(-\frac{u^2\sigma^2}{2}+Du\right)\right)}}{(v)} \left[\left(\frac{D(T+L)^2}{6} - \frac{R(T+L)}{3} + \frac{R^2}{6D} + \frac{\sigma^2(T+L)}{12D}\right.\right.$$

$$\left.+\frac{\sigma^2 R}{4D^2} + \frac{\sigma^4}{4D^3}\right) \sum_{i=0}^{\frac{v-2}{2}} \binom{v-2-i}{i} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-2-2i} \left(\frac{\sigma^2(T+L)}{2}\right)^i$$

$$+\left(-\frac{(T+L)}{3} + \frac{R}{3D} + \frac{\sigma^2}{4D^2}\right) \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-1-2i} \left(\frac{\sigma^2(T+L)}{2}\right)^i$$

$$\left.+\frac{1}{6D} \sum_{i=0}^{v} \binom{v-i}{i} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-2i} \left(\frac{\sigma^2(T+L)}{2}\right)^i \right]$$

$$+\frac{\sigma^6}{8D^4}\frac{u^v e^{Ru-(L+T)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z}{2}}\frac{(v-2)!}{\left(u-\frac{2D}{\sigma^2}\right)^Z(v-1-Z)!}\binom{v-Z-i}{i}$$

$$\times\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i} \qquad 10.2.12.$$

Simplifying

$$G_{61}(R,T+L)=\frac{u^v e^{Ru-(L+T)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\left(\left(-\frac{\sigma^4 R}{4D^3}-\frac{\sigma^2 R^2}{4D^2}+\frac{R^3}{6D}-\frac{\sigma^6}{8D^4}+\frac{D^2T^3}{6}\right.\right.\right.$$

$$\left.-\frac{DRT^2}{2}+\frac{\sigma^2T^2}{4}+\frac{TR^2}{2}\right)+\left(\frac{D^2T^2}{2}-DRT+\frac{\sigma^2T}{2}+\frac{R^2}{2}\right)L+\left(\frac{D^2T}{2}-\frac{DR}{2}+\frac{\sigma^2}{4}\right)L^2+\frac{D^2}{6}L^3\Bigg) *$$

$$\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-1-Z-2i}\sum_{j=0}^{i}\frac{(v-2)!}{u^Z(v-1-Z)!}\binom{v-1-Z-2i}{w} *$$

$$(DT-R-u\sigma^2T)^{v-1-Z-2i-w}(D-u\sigma^2)^Z\left(\frac{\sigma^2}{2}\right)^i *$$

$$\binom{i}{j}T^{i-j}L^{j+w}\binom{v-Z-1}{i}$$

$$+\left(\left(-\frac{\sigma^4}{4D^3}-\frac{DT^2}{2}-\frac{R\sigma^2}{2D^2}+RT+\frac{R^2}{2D}\right)-L(DT-R)-\frac{D^2L^2}{2}\right) *$$

$$\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j=0}^{i}\frac{(v-1)!}{u^Z(v-Z)!}\binom{v-Z-i}{i}\binom{i}{j}\binom{v-Z-2i}{w} *$$

$$(DT-R-u\sigma^2T)^{v-Z-2i-w} *$$

$$\left(\frac{\sigma^2}{2}\right)^i (D-u\sigma^2)^w\ T^{i-j}L^{j+w}$$

$$+\frac{1}{6D}\sum_{Z=1}^{v+2}\ \sum_{i=0}^{\frac{v+2-Z}{2}}\ \sum_{w=0}^{v+2-Z-2i}\ \sum_{j=0}^{i}\frac{(v+1)!}{u^Z(v-Z+2)!}\binom{v+2-Z-i}{i}\binom{i}{j} *$$

$$\binom{v+2-Z-2i}{w}\left(\frac{\sigma^2}{2}\right)^i$$

$$\left.\times(DT-R-u\sigma^2 T)^{v+2-Z-2i-w}(D-u\sigma^2)^w\ T^{i-j}\ L^{j+w}\right]$$

$$+\frac{\sigma^2 u^v e^{Ru-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\left(\left(\frac{DT^3}{6}-\frac{RT^2}{3}+\frac{R^2T}{6D}+\frac{\sigma^2T^2}{12D}+\frac{\sigma^2RT}{4D^2}+\frac{\sigma^4T}{4D^3}\right)\right.\right.$$

$$+L\left(\frac{DT^2}{6}-\ \frac{2RT}{3}+\frac{R^2}{6D}+\frac{\sigma^2T}{12D}+\frac{\sigma^2R}{4D^2}+\frac{\sigma^4}{4D^3}+\frac{DT^2}{3}+\frac{\sigma^2}{12D}\right)$$

$$\left.+L^2\left(\frac{DT}{2}-\frac{R}{3}+\frac{\sigma^2}{12D}\right)+\frac{D}{6}\ L^3\right)\sum_{i=0}^{v-2}\ \sum_{w=0}^{v-2-2i}\ \sum_{j=0}^{i}\binom{v-2-i}{i}\binom{i}{j} *$$

$$(DT-R-u\sigma^2T)^{v-2-2i-w}\left(\frac{\sigma^2}{2}\right)^i(D-u\sigma^2)^w T^{i-j}L^{j+w}$$

$$+\left(\left(\frac{-T^2}{3}+\frac{RT}{3D}+\frac{\sigma^2T}{4D^2}\right)-L\left(\frac{2T}{3}-\frac{R}{3D}-\frac{\sigma^2}{4D^2}\right)\right)\sum_{i=0}^{\frac{v-1}{2}}\ \sum_{w=0}^{v-1-2i}\ \sum_{j}^{i} *$$

$$\binom{v-1-i}{i}\binom{i}{j}\left(DT-R-u\sigma^2T\right)^{v-1-2i-w}\left(\frac{\sigma^2}{2}\right)^i(D-u\sigma^2)^w T^{i-j}L^{j+w}$$

$$\left.+\frac{1}{6D}\sum_{i=0}^{\frac{v}{2}}\ \sum_{w=0}^{v-2i}\ \sum_{j}^{i}\binom{v-i}{i}\left(DT-R-\mu\sigma^2T\right)^{v-2i-w}\left(\frac{\sigma^2}{2}\right)^i\left(D-\mu\sigma^2\right)^w T^{i-j}L^{j+w}\right]$$

$$+\frac{\sigma^6}{8D^4}\frac{u^v}{\Gamma(v)}e^{Ru-(L+T)\left(Du-\frac{u^2\sigma^2}{2}\right)}\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j=0}^{i} *$$

$$\frac{(v-2)!}{\left(u-\frac{2D}{\sigma^2}\right)^Z(v-1-Z)!}\binom{v-Z-i}{i}\binom{i}{j}$$

$$x(DT-R-u\sigma^2T)^{v-Z-2i-w}\left(D-u\sigma^2\right)^w\left(\frac{\sigma^2}{2}\right)^i T^{i-j}L^{j+w}$$

10.2.13

Substituting into 10.2.6 and noting that

$$\int_0^\infty L^y e^{-\theta y}\,dy = \Gamma(y+1)/\theta^{y+1}$$

Then we have

$$G_{70}(R,T)=\frac{u^v \alpha^k e^{Ru-T(Du-u^2\sigma^2/2)}}{\Gamma(v)\;\Gamma(k)}\left[\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-1-Z-2i}\sum_{j=0}^{i} *\right.$$

$$\frac{(v-2)!}{u^Z(v-1-Z)!}\binom{v-1-Z-2i}{w} *$$

$$(DT-R-u\sigma^2T)^{v-1-Z-2i-w}(D-u\sigma^2)^w\left(\frac{\sigma^2}{2}\right)^i\binom{i}{j}T^{i-j}\binom{v-Z-i}{i} *$$

$$\left(\left(\left(\left(-\frac{\sigma^4R}{4D^3}-\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}-\frac{\sigma^6}{8D^4}+\frac{D^2T^3}{6}-\frac{DRT^2}{4}+\frac{\sigma^2T^2}{4}+TR^2\right)\frac{\Gamma(j+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{j+w+k}}\right.\right.\right.$$

$$+\left(\frac{D^2T^2}{2}-DRT+\frac{\sigma^2T}{2}+R^2\right)\frac{\Gamma(j+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{j+w+k+1}}+\left(\frac{D^2T}{2}-\frac{DR}{2}+\frac{\sigma^2}{4}\right)*$$

$$\frac{\Gamma(j+w+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{j+w+k+2}}$$

$$+\frac{D^2}{6}\left.\frac{\Gamma(j+w+k+3)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{j+w+k+3}}\right)$$

$$+\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j}^{i}\frac{(v-1)!}{u^Z(v-Z)!}\binom{v-Z-2i}{i}\binom{i}{j}\binom{v-Z-2i}{w}$$

$$\left(DT-R-u\sigma^2T\right)^{v-Z-2i-w}\left(\frac{\sigma^2}{2}\right)^i\left(D-u\sigma^2\right)^wT^{i-j}$$

$$\left(\left(-\frac{\sigma^4}{4D^3}-\frac{DT^2}{2}-\frac{R\sigma^2}{2D^2}+RT+\frac{R^2}{2D}\right)\frac{\Gamma(j+k+w)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{j+k+w}}-(DT-R)\frac{\Gamma(j+k+w+1)}{\left(D\mu-\frac{\mu^2\sigma^2}{2}+\alpha\right)^{j+k+w+1}}\right.$$

$$\left.-\frac{D^2}{2}\Gamma(j+k+w+2)\Big/\left(D\mu-\frac{\mu^2\sigma^2}{2}+\alpha\right)^{j+k+w+2}\right)$$

$$+\frac{1}{6D}\sum_{Z=1}^{V+1}\sum_{i=0}^{\frac{V+1-Z}{2}}\sum_{w=0}^{v+1-Z-2i}\sum_{j=0}^{i}\frac{(v+1)!}{u^Z(v+2-Z)!}\binom{v+1-Z-i}{i}\binom{i}{j}\binom{v+1-Z-2i}{w}$$

$$\left.\left(\frac{\sigma^2}{2}\right)^i\left(DT-R-u\sigma^2T\right)^{v+2-Z-2i-w}\left(D-u\sigma^2\right)^wT^{i-j}\frac{\Gamma(j+k+w)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{j+k+w}}\right]$$

$$+\frac{\sigma^2u^v\alpha^k e^{Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(Y)\,\Gamma(k)}\left[\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-2-2i}\sum_{j=0}^{i}\binom{v-2-i}{i}\binom{i}{j}\left(DT-R-u\sigma^2T\right)^{v-2-2i-}\right.$$

$$\left(\frac{\sigma^2}{2}\right)^i\left(D-u\sigma^2\right)^wT^{i-j}\left(\left(\frac{DT^3}{6}-\frac{RT^2}{3}+\frac{R^2T}{6D}+\frac{\sigma^2T^2}{12D}+\frac{\sigma^2RT}{4D^2}+\frac{\sigma^4T}{4D^3}\right)\right.$$

$$\frac{\Gamma(j+k+w)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{j+k+w}}$$

$$+\left(\frac{DT^2}{6}-\frac{2RT}{3}+\frac{R^2}{6D}+\frac{\sigma^2 T}{12D}+\frac{\sigma^2 R}{4D^2}+\frac{\sigma^4}{4D^3}+\frac{DT^2}{3}+\frac{\sigma^2}{12D}\right)\frac{\Gamma(j+k+w+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+k+w+1}}$$

$$+\left(\frac{DT}{2}-\frac{R}{3}+\frac{\sigma^2}{12D}\right)\frac{\Gamma(j+k+w+2)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+k+w+2}}+\frac{D}{6}\frac{\Gamma(j+k+w+3)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+k+w+2}}\Bigg)$$

$$+\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-2i-1}\sum_{j=0}^{i}{}^{v-i}\left(DT-R-u\sigma^2T\right)^{v-2i-w-1}\left(\frac{\sigma^2}{2}\right)^{w}T^{i-j}$$

$$\left(\left(-\frac{T^3}{3}+\frac{RT}{3D}+\frac{\sigma^2 T}{4D^2}\right)\right.$$

$$\frac{\Gamma(j+k+w)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+k+w}}-\left(\frac{2T}{3}-\frac{R}{3D}-\frac{\sigma^2}{4D^2}\right)\frac{\Gamma(j+k+w+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+k+w+1}}\Bigg)$$

$$+\frac{1}{6D}\sum_{i=0}^{\frac{v}{2}}\sum_{w=0}^{v-2i}\sum_{j=0}^{i}\binom{v-i}{i}\left(DT-R-u\sigma^2T\right)^{v-2i-w}\left(\frac{\sigma^2}{2}\right)^{i}\left(D-u\sigma^2\right)^{w}T^{i-j}$$

$$\left.\frac{\Gamma(j+k+w)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{j+k+w}}\right]$$

$$+\frac{\sigma^6 u^v a^k}{8D^4\,\Gamma(v)\,\Gamma(k)}e^{Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)}\left[\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j=0}^{i}\frac{(v-2)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-1-Z)!}\right.$$

$$\binom{v-Z-i}{i}\binom{i}{j}\left(DT-R-u\sigma^2 T\right)^{v-Z-2i-w}\left(\frac{\sigma^2}{2}\right)^{i} T^{i-j}\frac{\Gamma(i+j+w)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{k+j+w}}\Bigg]$$

10.2.14.

From 3.4.29

Substituting L+T for T, we have

$$G_4(R,T+L)=\left(\left(\frac{R-D(T+L)}{2D}\right)^2+\frac{\sigma^2 R}{2D^2}+\frac{\sigma^4}{2D^3}\right.\ F\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$+\sqrt{\frac{\sigma^2(T+L)}{2}}\left((T+L)-\frac{\sigma^2}{D^2}-\frac{R}{D}\right)g\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)-\frac{\sigma^4}{4D^3}e^{\frac{2DR}{\sigma^2}}\ F\left(\frac{R+D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

10.2.15

Multiplying by H(L) and Simplifying
we have

$$H(L)G_4(R,T+L)=\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\left[\frac{(R-DT)^2 L^{k-1}-L^k(R-DT)}{2D}+\frac{DL^{k+1}}{2}\right.$$

$$\left.+\frac{\sigma^4 R}{2D^2}L^{k-1}+\frac{\sigma^4 L^{k-1}}{2D^3}\right)F\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$+\sqrt{\frac{\sigma^2(T+L)}{2}}\quad\frac{e^{-\alpha L}\alpha^k}{\Gamma(k)}\left(\left(T-\frac{\sigma^2}{D^2}-\frac{R}{D}\right)L^{k-1}+L^k\right)g\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$-\frac{\sigma^4}{4D^3}\ F\left(\frac{R+D(T+L)}{\sigma^2(T+L)}\right)e\frac{^{-\alpha L}L^{k-1}\alpha^k}{\Gamma(k)}\ e^{\frac{2DR}{\sigma^2}}$$

10.2.16

Hence integrating $\int_0^\infty H(L)G_4(R,T+L)\ dL$ and applying 8.2.31, 8.2.34 and 8.2.35 we have

Let $\theta=2\alpha\sigma^2+D^2$

$$G_{73}(R,T) = \frac{\alpha^k e^{\frac{DR}{\sigma^2}} e^{\alpha T}}{2\sigma \Gamma(k)} \Bigg[\left(\frac{\sigma^4 R}{2D^2} + \frac{\sigma^4}{2D^3} \right) \frac{(R-DT)^2}{2D} \sum_{j=0}^{k-1} \sum_{Z=1}^{k-j}$$

$$\frac{(k-1-j)!}{\alpha^Z (k-j-Z)!} (-T)^j \binom{k-1}{j}$$

$$\left(2D\left(\frac{R}{\theta}\right)^{k-j-Z+\frac{1}{2}} K_{k-j-Z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-j-Z-\frac{1}{2}} K_{k-j-Z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \right)$$

$$-\left(\frac{R+DT}{D}\right) \sum_{j=0}^{k} \sum_{Z=1}^{k+j} \frac{(k-j)!}{\alpha^Z (k+1-j-Z)!} T^j \binom{k}{j}$$

$$\left(2D\left(\frac{R}{\theta}\right)^{k-j-Z+3/2} K_{k-j-Z+3/2}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-j-Z+\frac{1}{2}} K_{k-j-Z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \right)$$

$$+ \frac{\sigma^2 \alpha^k e^{\left(\frac{DR}{\sigma^2}+\alpha T\right)}}{2\sqrt{2\pi}\,\Gamma(k)} \Bigg[\left(T - \frac{\sigma^2}{D^2} - \frac{R}{D} \right) \sum_{j=0}^{k-1} T^j \binom{k-1}{j} \left(2\left(\frac{R}{\theta}\right)^{k-j+\frac{1}{2}} \times \right.$$

$$\left. K_{k-j+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \right) + 2 \sum_{j=0}^{k} T^j \binom{k}{j} \frac{R}{\cdot}^{k-j+3/2} \left(\frac{R}{\theta}\right)^{k-j+3/2} K_{k-j+3/2} \Bigg]$$

$$- \frac{\sigma^4}{4D^3} \frac{\alpha^k e^{\left(\alpha T + \frac{DR}{\sigma^2}\right)}}{\Gamma(k)} \Bigg[\sum_{j=0}^{k-1} \sum_{Z=1}^{k-j} T^j \binom{k-1}{j} \frac{(k-1-j)!}{\alpha^Z (k-j-Z)!} \left(-2D\left(\frac{R}{\theta}\right)^{k-j-Z+\frac{1}{2}} \right.$$

$$\left. K_{k-j-Z+\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) + 2R\left(\frac{R}{\theta}\right)^{k-j-Z-\frac{1}{2}} K_{k-j-Z-\frac{1}{2}}\left(\frac{R\theta}{\sigma^2}\right) \right) \Bigg]$$

10.2.17.

From 9.3.17, substituting L+T for L

$$G_{62}(R,T+L)= \frac{u^{v}e^{Ru-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\frac{1}{12D^3}\sum_{w=0}^{4}\sum_{Z=1}^{w-v-1}\right.$$

$$\sum_{i=0}^{\frac{w+v-Z}{2}}\binom{4}{w}R^{4-w}\binom{w+v-Z-i}{i}\frac{(w+v-2)!}{u^{Z}(w+v-1-Z)!}\Big(D(T+L)-R-u\sigma^2(T+L)\Big)^{w+v-Z-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^{i}$$

$$+\left(\frac{\sigma^2}{6D^4}-\frac{1}{3D^2}\right)\sum_{w=0}^{3}\sum_{Z=1}^{w+v-1}\sum_{i=0}^{\frac{w+v-Z}{2}}\binom{3}{w}R^{3-w}\binom{w+v-Z-i}{i}\frac{(w+v-2)!}{u^{Z}(w+v-1-Z)!}$$

$$-\Big(D(T+L)-R-u\sigma^2(T+L)\Big)^{w+v-Z-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^{i}$$

$$+\left(\frac{\sigma^4}{4D^5}+\frac{(T+L)^2}{2D}\right)\sum_{w=0}^{2}\sum_{i=0}^{w+v-1}\sum_{i=0}^{\frac{w+v-Z}{2}}\binom{2}{w}R^{2-w}\binom{w+v-Z-i}{i}\frac{(w+v-2)!}{u^{Z}(w+v-1-Z)!}$$

$$\Big(D(T+L)-R-u\sigma^2(T+L)\Big)^{w+v-\;-2i}\left(\sigma^2\frac{(T+L)}{2}\right)^{i}$$

$$+\left(\frac{\sigma^6}{4D^6}-\frac{\sigma^2T^2}{2D^2}-\frac{T^3}{3}\right)\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-i}{i}\Big(D(T+L)-R-u\sigma^2(T+L)\Big)^{v-Z-2i}$$

$$\left(\frac{\sigma^2}{2}(T+L)\right)^{i}+\left(\frac{\sigma^8}{8D^7}+\frac{\sigma^6}{4D^6}-\frac{\sigma^2T^2}{2D^2}-\frac{T^3}{3}+\frac{T^3\sigma^2}{3D}+\frac{T^4D}{12}\right)\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}$$

$$\left.\frac{(v-2)!}{u^{Z}(v-1-Z)!}\binom{v-Z-i-1}{i}\Big(D(T+L)-R-u\sigma^2(T+L)\Big)^{v-Z-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^{i}\right]$$

$$-\frac{2u^v\sigma^2 T}{\Gamma(v)}\ e^{Ru-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}\left[\left(\frac{\sigma^2 TR}{24D^3}+\frac{\sigma^4 T}{24D^4}-\frac{R^2 T}{8D^2}-\frac{\sigma^2 T^2}{8D^2}+\frac{T^2 R}{8D}-\frac{T^3}{24}+\frac{R^3}{24D^3}\right)\right.$$

$$\sum_{i=0}^{\frac{v-2}{2}}\binom{v-2-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-2-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^i$$

$$+\left(\frac{\sigma^2 T}{24D^3}-\frac{RT}{4D^2}+\frac{T^2}{8D}+\frac{R^2}{8D^3}+\frac{\sigma^2 R}{6D^4}+\frac{\sigma^2}{8D^5}\right)\sum_{i=0}^{\frac{v-1}{2}}\binom{v-1-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-1-2i}$$

$$\left(\frac{\sigma^2(T+L)}{2}\right)^i+\left(\frac{R}{8D^3}+\frac{\sigma^2}{12D^4}-\frac{T}{8D^2}\right)\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-2i}$$

$$\times\left(\frac{\sigma^2(T+L)}{2}\right)^i+\frac{1}{24D^3}\sum_{i=0}^{\frac{v+1}{2}}\binom{v+1-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v+1-2i}$$

$$\left.\left(\frac{\sigma^2(T+L)}{2}\right)^i\right]$$

$$-\frac{\sigma^8}{8D^7}\frac{u^v e^{Ru-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-1}{2}}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^Z(v-Z)!}\binom{v-1-i}{i}$$

$$\times\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-1-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^i \qquad 10.2.18.$$

Simplifying

$$G_{62}(R,T+L)=\frac{u^v e^{Ru-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\frac{1}{12D^3}\sum_{w=0}^{4}\sum_{Z=1}^{w+v-1}\right.$$

$$\sum_{i=0}^{\frac{w+v-Z}{2}} \sum_{f=0}^{w+v-Z-2i} \sum_{j=0}^{i} \binom{4}{w} R^{4-w} \binom{w+v-Z-i}{i} \frac{(w+v-2)!}{u^{Z}(w+v-1-Z)!}$$

$$(DT-R-u\sigma^2)^{w+v-Z-2i-f} \binom{w+v-z-2i}{f} (D-u\sigma^2)^f \binom{i}{j} \left(\frac{\sigma^2}{2}\right)^i$$

$$T^{i-j} L^{i+f}$$

$$+\left(\frac{\sigma^2}{6D^4} - \frac{1}{3D^2}\right) \sum_{w=0}^{3} \sum_{Z=1}^{w+v-1} \sum_{i=0}^{\frac{w+v-Z}{2}} \sum_{f=0}^{w+v-3-2i} \sum_{j}^{i} \binom{3}{w} R^{3-w} \binom{w+v-Z-i}{i}$$

$$\frac{(w+v-2)!}{u^{Z}(w+v-1-Z)!} \binom{i}{j} \binom{w+v-Z-2i}{f} \left(DT-R-u\sigma^2\right)^{w+v-Z-2i-f} (D-u\sigma^2)^f$$

$$\left(\frac{\sigma^2}{2}\right)^i T^{i-j} L^{i+f}$$

$$+\left(\frac{\sigma^4}{4D^5} + \frac{T^2}{2D} + \frac{TL}{D} + \frac{L^2}{2D}\right) \sum_{w=0}^{2} \sum_{Z=0}^{w+v-1} \sum_{i=0}^{\frac{w+v-Z}{2}} \sum_{f=0}^{w+v-Z-2i} \sum_{j=0}^{i} \binom{2}{w} R^{2-w}$$

$$\binom{w+v-Z-i}{i} \binom{w+v-Z-2i}{f} \binom{i}{j} \frac{(w+v-2)!}{u^{Z}(w+v-1-Z)!} \left(DT-R-u\sigma^2\right)^{w+v-z-2i-f}$$

$$(D-u\sigma^2)^f \left(\frac{\sigma^2}{2}\right)^i T^{i-j} L^{i+f}$$

$$+\left(\frac{\sigma^6}{4D^6} + \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3}\right) \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{f=0}^{v-Z-2i} \sum_{j=0}^{i} \frac{(v-1)!}{u^{Z}(v-Z)!} \binom{v-Z-i}{i} \binom{v-2i-Z}{f}$$

$$\binom{i}{j}\left(DT-R-u\sigma^2\right)^{v-Z-2i-f}(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i T^{i-j}L^{i+f}$$

$$+\left(\frac{\sigma^8}{8D^7}+\frac{\sigma^6}{4D^6}-\frac{\sigma^2T^2}{2D^2}-\frac{T^3}{3}+\frac{T^3\sigma^2}{3D}+\frac{DT^4}{12}\right)\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{f=0}^{v-Z-2i}\sum_{j=0}^{i}$$

$$\frac{(v-2)!}{u^Z(v-1-Z)!}\binom{v-Z-i-1}{i}\binom{i}{j}\binom{v-Z-2i}{f}\left(DT-R-u\sigma^2T\right)^{v-Z-2i-f}$$

$$\left.(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i L^{i+f}\right]$$

$$\frac{-2u^v\sigma^2T\; e^{Ru-(T+L)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)}$$

$$\left[\left(\frac{\sigma^2RT}{24D^3}+\frac{\sigma^4T}{24D^4}-\frac{R^2T}{8D^2}-\frac{\sigma^2T^2}{8D^2}+\frac{T^2R}{8D}-\frac{T^3}{24}+\frac{R^3}{24D^3}\right)\times\sum_{i=0}^{\frac{v-2}{2}}\sum_{f=0}^{v-2-2i}\sum_{j=0}^{i}\right.$$

$$\binom{v-2-i}{i}\binom{i}{j}\binom{v-2-2i}{f}\left(DT-R-u\sigma^2T\right)^{v-2-2i-f}(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i$$

$$T^{i-j}L^{i+f}$$

$$+\left(\frac{\sigma^2T}{24D^3}\frac{-RT}{4D^2}+\frac{T^2}{8D}+\frac{R^2}{8D^3}+\frac{\sigma^2R}{6D^4}-\frac{\sigma^2}{8D^5}\right)\sum_{i=0}^{\frac{v-1}{2}}\sum_{f=0}^{v-1-2i}\sum_{j=0}^{i}\binom{v-1-i}{i}\binom{i}{j}\binom{v-1-2i}{f}$$

$$\left(DT-R-u\sigma^2T\right)^{v-1-2i-f}(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i T^{i-j}L^{i+f}$$

$$+\left(\frac{R}{8D^3}+\frac{\sigma^2}{12D^4}\frac{-T}{8D^2}\right)\sum_{i=0}^{\frac{v}{2}}\sum_{f=0}^{v-1-2i}\sum_{j=0}^{i}\binom{v-i}{i}\binom{i}{j}\binom{v-2i}{f}$$

$$(DT-R-u\sigma^2T)^{v+1-2i-f}(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i \quad T^{i-j}L^{i+f}\Bigg]$$

$$-\frac{\sigma^8}{8D^7}\;\frac{u^v e^{Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\sum_{Z=1}^{v}\;\sum_{i=0}^{\frac{v-Z}{2}}\;\sum_{f=0}^{v-Z-2i}\;\sum_{j=0}^{i}\;\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^Z(v-Z)!}$$

$$\binom{v-1-i}{i}\left(DT-R-u\sigma^2T\right)^{v-1-2i-f}\left(D-u\sigma^2\right)^f\binom{i}{j}\binom{v-1-2i}{f}$$

$$T^{i-j}\left(\frac{\sigma^2}{2}\right)^i L^{i+f} \qquad 10.2.19.$$

Substituting into 10.2.7 and noting that

$$\int_0^\infty e^{-\theta L}L^y\,dL=\Gamma(y+1)/\theta^{y+1}$$

$$G_{71}(R,T)=\frac{u^v e^{Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)}\alpha^k}{\Gamma(v)\,\Gamma(k)}\Bigg[\frac{1}{12D^3}\sum_{w=0}^{4}\;\sum_{Z=1}^{w+v-1}\;\sum_{i=0}^{\frac{w+v-Z}{2}}$$

$$\sum_{f=0}^{w+v-Z-2i}\;\sum_{j=0}^{i}\;\binom{4}{w}R^{4-w}\binom{w+v-Z-i}{i}\binom{w+v-Z-2i}{f}\binom{i}{j}$$

$$\frac{(w+v-2)!}{u^Z(w+v-1-Z)!}\,(DT-R-u\sigma^2)^{w+v-Z-2i-f}(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i\;T^{i-j}$$

$$\frac{\Gamma(i+f+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+f+k}}$$

$$+\left(\frac{\sigma^2}{6D^4}-\frac{1}{3D^2}\right)\sum_{w=0}^{3}\sum_{Z=1}^{w+v-1}\sum_{i=0}^{\frac{w+v-Z}{2}}\sum_{f=0}^{w+v-Z-2i}\sum_{j=0}^{i}\binom{3}{w}R^{3-w}$$

$$\binom{w+v-3-i}{i}\frac{(w+v-2)!}{u^Z(w+v-1-Z)!}\binom{i}{j}\binom{w+v-Z-2i}{f}\left(DT-R-u\sigma^2T\right)^{w+v-Z-2i-f}$$

$$(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i T^{i-j}\ \frac{\Gamma(i+f+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+f+k}}+\sum_{w=0}^{2}\sum_{Z=0}^{w+v-1}\sum_{i=0}^{\frac{w+v-Z}{2}}$$

$$\sum_{f=0}^{w+v-Z-2i}\sum_{j=0}^{i}\binom{2}{w}R^{2-w}\binom{w+v-Z-i}{i}\binom{w+v-Z-2i}{f}\binom{i}{j}\frac{(w+v-2)!}{u^Z(w+v-1-Z)!}$$

$$(DT-R-u\sigma^2)^{w+v-Z-2i-f}(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i T^{i-j}\left(\left(\frac{\sigma^4}{4D^5}+\frac{T^2}{2D}\right)\right.$$

$$\left.\times\frac{\Gamma(i+f+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+f+k}}+\frac{T}{D}\frac{\Gamma(i+f+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+f+k+1}}+\frac{1}{2D}\frac{\Gamma(i+f+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+f+k+2}}\right)$$

$$+\left(\frac{\sigma^6}{4D^6}-\frac{\sigma^2T^2}{2D^2}-\frac{T^3}{3}\right)\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{f=0}^{v-Z-2i}\sum_{j=0}^{i}\frac{(v-1)!}{u^Z(v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-2i-Z}{f}\binom{i}{j}\left(DT-R-u\sigma^2\right)^{v-\ \ -2i-f}(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i$$

$$T^{i-j}\ \frac{\Gamma(i+f+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+f+k}}$$

$$+\left(\frac{\sigma^8}{8D^7}+\frac{\sigma^6}{4D^6}-\frac{\sigma^2T^2}{2D^2}-\frac{T^3}{3}+\frac{T^3\sigma^2}{3D}+\frac{DT^4}{12}\right)\sum_{Z=1}^{v-1}\sum_{i=o}^{\frac{v-Z-1}{2}}\sum_{f=o}^{v-Z-2i}\sum_{j=o}^{i}$$

$$\frac{(v-2)!}{u^Z(v-1-Z)!}\binom{v-Z-i-1}{i}\binom{i}{j}\binom{v-Z-2i}{f}\left(DT-R-u\sigma^2T\right)^{v-Z-2i-f}$$

$$(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i\frac{\Gamma(i+f+k)}{(Du-\frac{u^2\sigma^2}{2}+a)^{i+f+k}}\Bigg]$$

$$\frac{-2u^va^k\sigma^2Te^{Ru-T(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)\,\Gamma(k)}\left[\left(\frac{\sigma^2RT}{24D^3}+\frac{\sigma^3T}{24D^4}-\frac{R^2T}{8D^2}-\frac{\sigma^2T^2}{8D^2}+\frac{T^2R}{8D}-\frac{T^3}{24}\right.\right.$$

$$\left.+\frac{R^3}{24D^3}\right)\sum_{i=o}^{\frac{v-2}{2}}\sum_{f=o}^{v-2-2i}\sum_{j=o}^{i}\binom{v-2-i}{i}\binom{i}{j}\binom{v-2-2i}{f}\left(DT-R-u\sigma^2T\right)^{v-2-2i-f}$$

$$(D-u\sigma^2)^f\left(\frac{\sigma^2}{2}\right)^i T^{i-j}\frac{\Gamma(i+f+k)}{(Du-\frac{u^2\sigma^2}{2}+a)^{i+f+k}}$$

$$+\left(\frac{\sigma^2T}{24D^3}-\frac{RT}{4D^2}+\frac{T^2}{8D}+\frac{R^2}{8D^3}+\frac{\sigma^2R}{6D^4}-\frac{\sigma^2}{8D^5}\right)\sum_{i=o}^{\frac{v-1}{2}}\sum_{f=o}^{v-1-2i}\sum_{j=o}^{i}\binom{v-1-i}{i}\binom{i}{j}\binom{v-1-2i}{f}$$

$$\left(DT-R-u\sigma^2T\right)^{v-1-2i-f}\left(D-u\sigma^2\right)^f\left(\frac{\sigma^2}{2}\right)^iT^{i-j}\frac{\Gamma(i+f+k)}{(Du-\frac{u^2\sigma^2}{2}+a)^{i+f+k}}$$

10.2.20.

From 9.3.18 substituting L+T for T

$$G_{63}(R,T+L) = \frac{u^v e^{Ru-(T+L)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)} \left[\left(\frac{\sigma^4}{2D^3} + \frac{\sigma^2}{2D^2} + \left(\frac{R-D(L+T)}{2D}\right)^2\right)\right.$$

$$\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \binom{v-1-Z-i}{i} \frac{(v-2)!}{u^Z(v-1-Z)!} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i-1}$$

$$\left(\frac{\sigma^2}{2}(T+L)\right)^i + \left(\frac{\sigma^2}{2D^2} + \frac{(R-D(L+T)}{D}\right) \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{u^Z(v-Z)!} \binom{v-Z-i}{i}$$

$$\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i} \left(\frac{\sigma^2(T+L)}{2}\right)^i$$

$$+\frac{1}{2D} \sum_{Z=1}^{\frac{v+1}{2}} \sum_{i=0}^{\frac{v+1-Z}{2}} \binom{v+1-Z-i}{i} \frac{v!}{u\ (v+1-Z)!} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v+1-Z-2i}$$

$$\left.\left(\frac{\sigma^2(T+L)}{2}\right)^i\right] + \frac{\sigma^2}{2}(T+L) \frac{u^v e^{Ru-(T+L)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)}$$

$$\left[\left(T+L-\frac{\sigma^2}{D^2}-\frac{R}{D}\right) \sum_{i=0}^{\frac{v-2}{2}} \binom{v-2-i}{i} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-2-2i} \left(\frac{\sigma^2}{2}(T+L)\right)^i\right.$$

$$\left.-\frac{1}{D} \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i} \left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-1-2i} \left(\frac{\sigma^2}{2}(T+L)\right)^i\right]$$

$$-\frac{\sigma^4}{4D^3} \frac{u^v}{\Gamma(v)} e^{Ru-(T+L)(Du-\frac{u^2\sigma^2}{2})} \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{(v-Z)!} \left(u-\frac{2D}{\sigma^2}\right)^Z$$

$$\times \binom{v-Z-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i}\left(\frac{\sigma^2}{2}(T+L)\right)^{i}$$

Simplifying we have

$$G_{63}(R,T+L)=\frac{u^{v}e^{Ru-(T+L)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)}\left[\left(\left(\frac{\sigma^4}{2D^3}+\frac{\sigma^2}{2D^2}+\frac{(R^2-DT)^2}{2D}\right)\right.\right.$$

$$\left.-2L\;\frac{(DR-D^2T)}{2D}+\frac{DL^2}{2}\right)\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-1-Z-2i}\sum_{j=0}^{i}\binom{v-1-Z-i}{i}$$

$$\binom{v-1-Z-2i}{w}\binom{i}{j}\frac{(v-2)!}{u^{Z}(v-1-Z)!}\left(DT-R-u\sigma^2T\right)^{v-1-Z-2i-w}$$

$$\left(D-u\sigma^2\right)^{w}\left(\frac{\sigma^2}{2}\right)^{i}\;T^{i-j}L^{i+w}$$

$$+\left(\frac{\sigma^2}{2D^2}+\frac{(R-DT)}{D}-L\right)\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j=0}^{i}\frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-i}{i}\left(\frac{\sigma^2}{2}\right)^{i}$$

$$\binom{v-Z-2i}{w}\binom{i}{j}\left(DT-R-u\sigma^2T\right)^{v-Z-2i-w}(D-u\sigma^2)^{w}T^{i-j}L^{i+w}$$

$$+\frac{1}{2D}\sum_{Z=1}^{\frac{v+1}{2}}\sum_{i=0}^{\frac{v+1-Z}{2}}\sum_{w=0}^{v+1-Z-2i}\sum_{j=0}^{i}\binom{v+1-Z-i}{i}\binom{i}{j}\binom{v+1-Z-2i}{w}$$

$$\left.\frac{v!}{u^{Z}(v+1-Z)!}\times(DT-R-u\sigma^2T)^{v+1-Z-2i-w}(D-u\sigma^2)^{w}\left(\frac{\sigma^2}{2}\right)^{i}T^{i-j}L^{i+w}\right]$$

$$+\frac{\sigma^2}{2}\;\frac{u^{v}e^{Ru-(T+L)(Du-\frac{u^2\sigma^2}{2})}}{(v)}\left[\left(\left(T^2-\frac{\sigma^2T}{D^2}-\frac{RT}{D}\right)+L\left(2T-\frac{\sigma^2}{D^2}-\frac{R}{D}\right)+L^2\right)\right.$$

$$\sum_{i=0}^{\frac{v-2}{2}} \sum_{w=0}^{v-2-2i} \sum_{j=0}^{i} \binom{v-2-i}{i}\binom{v-2-2i}{w}\binom{i}{j}\left(DT-R-u\sigma^2 T\right)^{v-2-2i-w} *$$

$$(D-u\sigma^2)^w T^{i-j}\left(\frac{\sigma^2}{2}\right)^i L^{i+w}$$

$$-\frac{(T+L)}{D} \sum_{i=0}^{\frac{v-1}{2}} \sum_{w=0}^{v-1-2i} \sum_{j=0}^{i} \binom{v-1-i}{i}\binom{v-1-2i}{w}\binom{i}{j}(DT-R-u\sigma^2 T)^{v-1-2i-w}$$

$$\left.(D-u\sigma^2)^w \left(\frac{\sigma^2}{2}\right)^i T^{i-j} L^{i+w}\right]$$

$$-\frac{\sigma^4}{4D^3} \frac{u^v}{\Gamma(v)} e^{Ru-(T+L)(Du-\frac{u^2\sigma^2}{2})} \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \sum_{j=0}^{i}$$

$$* \frac{(v-1)!}{(v-Z)!\left(u-\frac{2D}{\sigma^2}\right)^Z} \binom{v-Z-i}{i} \left(DT-R-u\sigma^2 T\right)^{v-Z-2i-w} (D-u\sigma^2)^w \left(\frac{\sigma^2}{2}\right)^i *$$

$$T^{i-j} L^{i+w}$$

Substituting into 10.2.8 and noting that $H(L)= e^{-\alpha L} L^k / \Gamma(k)$

and $\int_0^\infty e^{-\theta L} L^y \, dy = \Gamma(y+1)/\theta^{y+1}$ 10.2.22

Then

$$G_{72}(R,T) = \frac{u^v \alpha^k e^{Ru-T(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)\Gamma(k)} \left[\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \sum_{w=0}^{v-1-Z-2i} \sum_{j=0}^{i} * \right.$$

$$\binom{v-1-Z-i}{i}\binom{v-1-Z-2i}{w}\binom{i}{j}\frac{(v-2)!}{u^{Z}(v-1-Z)!}(DT-R-u\sigma^2T)^{v-1-Z-2i-w}$$

$$(D-u\sigma^2)^{w}\left(\frac{\sigma^2}{2}\right)^{i}T^{i-j}\left(\left(\frac{\sigma^4}{2D^3}+\frac{\sigma^2}{2D^2}+\frac{(R-DT)^2}{2D}\right)\frac{\Gamma(i+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k}}\right)$$

$$+\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j=0}^{i}\frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-i}{i}\left(\frac{\sigma^2}{2}\right)^{i}$$

$$\binom{v-Z-2i}{w}\binom{i}{j}(DT-R-u\sigma^2T)^{v-Z-2i-w}(D-u\sigma^2)^{w}T^{i-j}\left(\left(\frac{\sigma^2}{2D^2}+\frac{R}{D}-T\right)\right.$$

$$\left.\times\frac{\Gamma(i+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k}}-\frac{\Gamma(i+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{i+w+k+1}}\right)$$

$$+\frac{1}{2D}\sum_{Z=1}^{\frac{v+1}{2}}\sum_{i=0}^{\frac{v+1-Z}{2}}\sum_{w=0}^{v+1-Z-2i}\sum_{j=0}^{i}\binom{v+1-Z-i}{i}\binom{i}{j}\binom{v+1-Z-2i}{w}$$

$$\frac{v!}{u^{Z}(v+1-Z)!}\times(DT-R-u\sigma^2T)^{v+1-Z-2i-w}(D-u\sigma^2)^{w}\left(\frac{\sigma^2}{2}\right)^{i}T^{i-j}$$

$$\left.\times\frac{\Gamma(i+w+k)}{\left(D\mu-\frac{\mu^2\sigma^2}{2}+\alpha\right)^{i+w+k}}\right]$$

$$+\frac{\sigma^2}{2}\;\frac{a^k u^v e^{Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)\;\Gamma(k)}\left[\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-2-2i}\sum_{j=0}^{i}\binom{v-2-i}{i}\binom{v-2-2i}{w}\right.$$

$$\binom{i}{j}\left(DT-R-u\sigma^2T\right)^{v-2-2i-w}\left(D-u\sigma^2\right)^{w}T^{i-j}\left(\frac{\sigma^2}{2}\right)^{i}\left(\left(T^2-\frac{\sigma^2}{D^2}T-\frac{RT}{D}\right)\right.$$

$$\times\frac{\Gamma(i+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+w+k}}+\left(2T-\frac{\sigma^2}{D^2}-\frac{R}{D}\right)\frac{\Gamma(i+w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+w+k+1}}$$

$$\left.+\frac{\Gamma(i+w+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+w+k+2}}\right)$$

$$-\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-2i}\sum_{j=0}^{i}\binom{v-1-i}{i}\binom{v-1-2i}{w}\binom{i}{j}\left(DT-R-u\sigma^2T\right)^{v-1-2i-w}$$

$$\left.\left(D-u\sigma^2\right)^{w}\left(\frac{\sigma^2}{2}\right)^{i}T^{i-j}\left(\frac{T}{D}\;\frac{\Gamma(i+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+w+k}}-\frac{\Gamma(i+w+k+1)}{D\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+w+k+1}}\right)\right]$$

$$-\frac{\sigma^4}{4D^3}\;\frac{u^v a^k}{\Gamma(v)\;\Gamma(k)}\;e^{Ru-T\left(Du-\frac{u^2\sigma^2}{2}\right)}\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-Z-2i-1}\sum_{j=0}^{i}$$

$$\frac{(v-2)!}{(v-Z-1)!\left(u-\frac{2D}{\sigma^2}\right)^{Z}}\binom{v-Z-i}{i}\left(DT-R-u\sigma^2T\right)^{v-Z-2i-w-1}\left(D-u\sigma^2\right)^{w}$$

$$\left(\frac{\sigma^2}{2}\right)^{i}T^{i-j}\;\frac{\Gamma(i+w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{i+w+k}}$$

10.2.23

Let $G_{73}(R)= \int_0^{\infty} H(L)G_{60}(R,L)\ dL$

Let $G_{74}(R)= \int_0^{\infty} H(L)G_{61}(R,L)\ dL$

Let $G_{75}(R)= \int_0^{\infty} H(L)G_{62}(R,L)\ dL$

Let $G_{76}(R)= \int_0^{\infty} H(L)G_{63}(R,L)\ dL$

Hence from 10.2.5 to 10.2.8, by setting T equal

to 0 in the expressions.

$G_{73}(R)=G_{69}(R,0)$

$G_{71}(R)=G_{70}(R,0)$

$G_{72}(R)=G_{71}(R,0)$

$G_{74}(R)=G_{72}(R,0)$

Hence the inventory costs for (nQ, R, T) when the supply is gamma distributed and lead time is continuous and the cost of a backorder is a quadratic function of the length of time of a backorder is obtained by integrating C of 10.2.1 over the states of L_1 & L_2) $= \int_0^{\infty} H(L_1)H(L_2)\ C\ dL_1 dL_2$

Hence applying the above integrals

$$C=\frac{Rc}{T}+\frac{S.PoR_T}{T}+hc\left(\frac{V}{2u}+R-\frac{Dk}{a}-\frac{DT}{2}\right)$$

$$+\frac{b_1}{T}\left(\frac{u}{(v-1)}G_{35}(R,T)-\frac{u}{(v-1)}G_{32}(R)-G_{69}(R,T)+G_{69}(R,0)\right)$$

$$+\frac{(hc+b_2)}{T}\left(\frac{u}{(v-1)}G_{36}(R,T)-\frac{u}{(v-1)}G_{33}(R)-G_{70}(R,T)+G_{70}(R,0)\right)$$

$$+\frac{b_3}{T}\left(\frac{u}{(v-1)}G_{37}(R,T)-G_{34}(R)\frac{u}{(v-1)}-G_{71}(R,T)+G_{71}(R,0)\right)$$

$$+\frac{s}{T}\left(\frac{u}{(v-1)}G_{73}(R,T)-\frac{u}{(v-1)}G_{73}(R,0)-G_{72}(R,T)+G_{72}(R,0)\right)$$

10.2.24.

SECTION 10.3 (M.T)

The inventory costs for random supply and fixed lead times is given in Chapter 9 Section 4 equation 14. From 9.4.14.

$$C=\frac{Rc+S}{T}+\frac{hc.v}{u}-hc\frac{(DL+DT)}{2}+\frac{b_1}{T}\left(G_{56}(T+L_2)-G_{56}(L_1)\right)$$

$$+\frac{(b_2+hc)}{T}\left(G_{57}(T+L_2)-G_{57}(L_1)\right)+\frac{b_3}{T}\left(G_{58}(T+L_2)-G_{58}(L_1)\right)$$

$$+\frac{s}{T}\left(G_{59}(T+L_2)-G_{59}(L_1)\right) \qquad 10.3.1.$$

The inventory costs for random supply and continuous lead times are obtained by averaging the inventory costs for random supply and fixed lead times, quoted above, over the states of the lead times.

The probability density function of lead time

$$H(L)=\alpha^k e^{-\alpha L} L^{k-1} / \Gamma(k)$$

The following integrals are necessary for the derivation of the inventory costs.

$$G_{75}(T)=\int_0^\infty H(L)G_{56}(T+L)\ dL \qquad 10.3.2.$$

$$G_{76}(T)=\int_0^\infty H(L)G_{57}(T+L)\ dL \qquad 10.3.3$$

$$G_{77}(T)=\int_0^\infty H(L)G_{58}(T+L)\ dL \qquad 10.3.4.$$

$$G_{78}(T)= \int_{0}^{\infty} H(L)G_{59}(T+L)\; dL \qquad 10.3.5.$$

From 9.4.6. substituting L+T for T

$$G_{56}(T+L)= \frac{\sigma^2(T+L)u^v e^{-(L+T)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)} \sum_{i=0}^{\frac{v-1}{2}} \binom{v-1-i}{i}$$

$$(DT-u\sigma^2 T)^{v-1-2i} \left(\frac{\sigma^2(T+L)}{2}\right)^i$$

$$+\frac{u^v e^{-(L+T)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)} \Bigg[D(T+L)\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{u^Z(v-Z)!}$$

$$\binom{v-Z-i}{i}\Big(D(T+L)-u\sigma^2(T+L)\Big)^{v-Z-2i} \times \left(\frac{\sigma^2(T+L)}{2}\right)^i - \sum_{Z=1}^{v+1} \sum_{i=0}^{\frac{v+1-Z}{2}}$$

$$\frac{v!}{u^Z(v+1-Z)!}\binom{v+1-Z-i}{i}\left(DT-u\sigma^2 T\right)^{v+1-Z-2i} \left(\frac{\sigma^2(T+L)}{2}\right)^i \Bigg]$$

$$10.3.6.$$

Simplifying

$$G_{56}(T+L)\; \frac{\sigma^2 u^v e^{-(L+T)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)}(T+L) \sum_{i=0}^{\frac{v-1}{2}} \sum_{w=0}^{v-1-i}$$

$$\binom{v-1-i}{i}\binom{v-1-i}{w}\left(D-u\sigma^2\right)^{v-1-2i}\left(\frac{\sigma^2}{2}\right)^i T^{v-1-i-w} \quad L^w$$

$$+\frac{u^v e^{-(L+T)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)} \Bigg[D(T+L)\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-i} \frac{(v-1)!}{u^Z(v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-Z-2i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^i T^{v-Z-i} \quad L^w$$

$$-\sum_{Z=1}^{v+1}\sum_{i=0}^{\frac{v+1-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{v!}{u^Z(v+1-Z)!}\binom{v+1-Z-i}{i}\binom{v-Z-2i}{w} *$$

$$(D-u\sigma^2)^{v-Z-i}\left(\frac{\sigma^2}{2}\right)^i \quad T^{v-Z-i-w} \qquad 10.3.7.$$

Substituting into 10.3.2 and noting that

$H(L)=\alpha^k e^{-\alpha L} L^{k-1}/\Gamma(k)$ and

$\int_0^\infty e^{-\theta L} L^y \, dy = \Gamma(Y+1)/\theta^{y+1}$

Then $$G_{75}(T)=\frac{\sigma^2 u^v \alpha^k e^{T(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)\,\Gamma(k)}\left[\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-2i}\sum_{j=0}^{i+w}\right.$$

$$* \binom{v-1-i}{i}\binom{v-1-i}{w}\left(D-u\sigma^2\right)^{v-1-2i}\left(\frac{\sigma^2}{2}\right)^i T^{v-Z-i-w}\left(T\frac{\Gamma(w+k)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{w+k}}\right.$$

$$\left.+\frac{\Gamma(w+k+1)}{(Du-u^2\sigma^2+\alpha)^{w+k+1}}\right)$$

$$+\frac{u^v e^{-T(Du-\frac{u^2\sigma^2}{2})}\alpha^k}{\Gamma(v)\,\Gamma(k)}\left[\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-z-2i}\sum_{j=0}^{i+w}\frac{(v-1)!}{u^Z(v-Z)!}\right. *$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-i}\left(\frac{\sigma^2}{2}\right) T^{v-z-i-w}\left(DT\frac{\Gamma(w+k)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{w+k}}\right.$$

$$\left.+D\frac{\Gamma(w+k+1)}{(Du-\frac{u^2\sigma^2}{2}+\alpha)^{w+k+1}}\right)$$

$$-\sum_{z=1}^{v+1}\ \sum_{i=0}^{\frac{v+1-z}{2}}\ \sum_{w=0}^{v-z-i}\frac{v!}{u^{z}(v+1-z)!}\binom{v+1-z-i}{i}\binom{v-z-2i}{w}\times$$

$$(D-u\sigma^2)^{v-z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-z-i-w}\left[\frac{\Gamma(k+w)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{k+w}}\right]$$

10.3.8.

From 9.4.9. substituting T+L for T

$$G_{57}(T+L)=\frac{u^{v}e^{-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\left(\frac{\sigma^4}{4D^3}+\frac{D(T+L)^2}{2}+\frac{\sigma^2}{2D^2}-(T+L)\right)\right.$$

$$\sum_{z=1}^{v}\ \sum_{i=0}^{\frac{v-z}{2}}\frac{(v-1)!}{u^{z}(v-z)!}\binom{v-z-i}{i}\Big(D(T+L)-u\sigma^2(T+L)\Big)^{v-z-2i}$$

$$\left(\frac{\sigma^2(T+L)}{2}\right)^{i}+\frac{1}{2D}\sum_{z=1}^{v+2}\ \sum_{=0}^{\frac{v+2-}{2}}\frac{(v+1)!}{u^{z}(v+2-z)!}\binom{v+2-z-i}{i}$$

$$\left.\Big(D(T+L)-u\sigma^2(T+L)\Big)^{v+2-z-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^{i}\right]$$

$$+\frac{\sigma^2}{2}\frac{(T+L)u^{v}e^{-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\left(T-\frac{\sigma^2}{D^2}\right)\sum_{i=0}^{\frac{v-1}{2}}\binom{v-1-i}{i}\right.$$

$$\Big(D(T+L)-u\sigma^2(T+L)\Big)^{v-1-2i}\times\left(\frac{\sigma^2(T+L)}{2}\right)^{i}-\frac{1}{D}\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}$$

$$\left.\Big(D(T+L)-u\sigma^2(T+L)\Big)^{v-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^{i}\right]-\frac{u^{v}e^{-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{4D^3\,\Gamma(v)}$$

$$\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!} \binom{v-Z-i}{i} \left(D(T+L)-u\sigma^2(T+L)\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2(T+L)}{2}\right)^{i} \qquad 10.3.9.$$

Simplifying

$$G_{57}(T+L)=\frac{u^{v}e^{-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\left(\left(\frac{\sigma^4}{4D^3}+\frac{DT^2}{2}+\frac{\sigma^2}{2D^2}-T\right)+L(DT-1)+\frac{DL^2}{2}\right)\times\right.$$

$$\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-3-i}\frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2}{2}\right)^{i}T^{v-Z-i-w}L^{w}$$

$$+\frac{1}{2D}\sum_{z=1}^{v+1}\sum_{i=0}^{\frac{v+1-z}{2}}\sum_{w=0}^{v+1-Z-i}\frac{(v)!}{u^{Z}(v+2-Z)!}\binom{v+1-Z-i}{i}\binom{v+1-Z-i}{w}$$

$$\left.(D-u\sigma^2)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v+1-Z-i-w}L^{w}\right]$$

$$+\frac{\sigma^2}{2}\,\frac{u^{v}e^{-(T+L)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[(T+L)\left(T-\frac{\sigma^2}{D^2}\right)\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-i}\right.$$

$$\binom{v-1-i}{i}\binom{v-1-i}{w}(D-u\sigma^2)^{v-1-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-1-i-w}L^{w}$$

$$-\frac{(T+L)}{D}\sum_{i=0}^{\frac{v}{2}}\sum_{w=0}^{v-i}\binom{v-i}{i}\binom{v-i}{w}(D-u\sigma^2)^{v-i-w}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-i-w}L^{w}\Bigg]$$

$$\frac{-u^{v}e^{-(T+L)(Du-\frac{u^2\sigma^2}{2})}}{4D^3\Gamma(v)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}T^{v-Z-i-w}L^{w} \qquad 10.3.10$$

Substituting into 10.3.3 and noting that

$$H(L)=\frac{a^{-aL}L^{k-1}a^{k}}{\Gamma(k)}$$

and $\int_0^{\infty}e^{-aL}L^{y}\,dY=\Gamma(y+1)/a^{y+1}$

then

$$G_{76}(T)=\frac{u^{v}a^{k}e^{-T(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)\Gamma(k)}\left[\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{u^{Z}(v-Z)!}\right.$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-Z-i-w}$$

$$\left(\left(\frac{\sigma^4}{4D^3}+\frac{DT^2}{2}+\frac{\sigma^2}{2D^2}-T\right)\frac{\Gamma(w+K)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k}}\right.$$

$$\left.+(DT-1)\frac{\Gamma(w+k+1)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k+1}}+\frac{D}{2}\frac{\Gamma(w+k+2)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k+2}}\right)$$

$$+\frac{1}{2D}\sum_{Z=1}^{v+1}\sum_{i=0}^{\frac{v+1-Z}{2}}\sum_{w=0}^{v+1-Z-i}\frac{(v)!}{u^{Z}(v+1-Z)!}\binom{v+1-Z-i}{i}\binom{v+1-Z-i}{w}$$

$$(D-u\sigma^2)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v+1-Z-i-w}\left[\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{w+k}}\right]$$

$$+\frac{\sigma^2}{2}\,u^{v}e^{-T\left(Du-\frac{u^2\sigma^2}{2}\right)}\alpha^{k}\left[\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-i}\binom{v-1-i}{i}\binom{v-1-i}{w}\right.$$

$$(D-u\sigma^2)^{v-1-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-1-i-w}\left(\left(T^2-\frac{\sigma^2 T}{D^2}\right)\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{w+k}}\right.$$

$$\left.+\left(T-\frac{\sigma^2}{D^2}\right)\frac{\Gamma(w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{w+k+1}}\right)-\frac{1}{D}\sum_{i=0}^{\frac{v}{2}}\sum_{w=0}^{v-i}\binom{v-i}{i}\binom{v-i}{w}$$

$$(D-u\sigma^2)^{v-i-w}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-i-w}\left(\frac{T\,\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{w+k}}\right.$$

$$\left.\left.+\frac{\Gamma(w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{w+k+1}}\right)\right]$$

$$-\frac{u^{v}\alpha^{k}e^{-T\left(Du-\frac{u^2\sigma^2}{2}\right)}}{4D^3\,\Gamma(v)\,\Gamma(k)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}(D-u\sigma^2)^{v-Z-2i}T^{v-Z-i-w}\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+\alpha\right)^{w+k}}$$

10.3.11

From 9.14.12 substituting T+L for T

$$G_{58}(T+L) = -\frac{u^v}{\Gamma(v)} e^{\left(\frac{u^2\sigma^2}{2} - Du\right)(T+L)} \left[\frac{1}{3D^3} \sum_{Z=1}^{3+v} \sum_{i=0}^{\frac{v+3-Z}{2}} \frac{(v+2)!}{u^Z (v+3-Z)!} \binom{v+3-Z-i}{i} \left(D-u\sigma^2\right)^{v+3-Z-2i} (T+L)^{v+3-Z-2i} \left(\frac{\sigma^2 (T+L)}{2}\right)^i \right.$$

$$+ \left(\frac{\sigma^2}{2D^4} - \frac{(T+L)}{D^2}\right) \sum_{Z=1}^{2+v} \sum_{i=0}^{\frac{v+2-Z}{2}} \frac{(v+1)!}{u^Z (v+2-Z)!} \binom{v+2-Z-i}{i} (D-u\sigma^2)^{v+2-Z-2i} (T+L)^{v+2-Z-2i} \left(\frac{\sigma^2 (T+L)}{2}\right)^i$$

$$+ \left(\frac{\sigma^4}{2D^5} + \frac{(T+L)^2}{D}\right) \sum_{Z=1}^{v+1} \sum_{i=0}^{\frac{v+1-Z}{2}} \frac{v!}{u^Z (v+1-Z)!} \binom{v+1-Z-i}{i} \left(D-u\sigma^2\right)^{v+1-Z-2i} (T+L)^{v+1-Z-2i} \left(\frac{\sigma^2 (T+L)}{2}\right)^i$$

$$+ \left(\frac{\sigma^6}{4D^6} - \frac{\sigma^2 (T+L)^2}{2D^2} - \frac{(T+L)^3}{3}\right) \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{u^Z (v-Z)!} \binom{v-Z\cdot i}{i} \left. (D-u\sigma^2)^{v-Z-2i} (T+L)^{v-Z-2i} \left(\frac{\sigma^2 (T+L)}{2}\right)^i \right]$$

$$+ \frac{u^v e^{-(L+T)\left(Du - \frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)} \left[\left(-\frac{2}{3} \frac{\sigma^2 (T+L)^2}{D^3} + \frac{\sigma^2 (T+L)}{3D^3} + \frac{\sigma^4 (T+L)}{2D^4}\right) \right.$$

$$\sum_{i=0}^{\frac{v}{2}} \binom{v-i}{i} (D-u\sigma^2)^{v-2i} (T+L)^{v-2i} \left(\frac{\sigma^2 (T+L)}{2}\right)^i$$

$$+\left(\frac{\sigma^2}{3D}(T+L)^3+\frac{\sigma^4(T+L)^2}{6D^3}+\frac{8\sigma^6(T+L)}{D^5}\right)\sum_{i=0}^{\frac{v-1}{2}} v-1-i\,(D-u\sigma^2)^{v-2i}$$

$$(T+L)^{v-2i}\left(\frac{\sigma^2}{2}(T+L)\right)^{i}\Bigg]$$

$$+\frac{\sigma^6}{4D^6}\frac{e^{-(T+L)(Du-\frac{u^2\sigma^2}{2})}}{\Gamma(v)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}\binom{v-Z-i}{i}$$

$$(D-u\sigma^2)^{v-Z-2i}(T+L)^{v-Z-2i}\left(\frac{\sigma^2}{2}(T+L)\right)^{i}$$

10.3.12

Simplifying

$$G_{58}(T+L) = \frac{u^{v}}{\Gamma(v)} e^{-(T+L)\left(Du - \frac{u^2\sigma^2}{2}\right)} \Bigg[\frac{1}{3D^3} \sum_{Z=1}^{v+3} \sum_{i=0}^{\frac{v+3-Z}{2}} \sum_{w=0}^{v-Z+3-i}$$

$$\frac{(v+2)!}{u^{Z}(v+3-Z)!} \binom{v+3-Z-i}{i} \binom{v+3-Z-i}{w} (D-u\sigma^2)^{v+3-Z-2i} \left(\frac{\sigma^2}{2}\right)^{i}$$

$$T^{v+3-Z-i-w} L^{w} + \left(\frac{\sigma^2}{2D^4} - \frac{T}{D^2} - \frac{L}{D^2}\right) \sum_{Z=1}^{2+v} \sum_{i=0}^{\frac{v+2-Z}{2}} \sum_{w=0}^{v+2-Z-i} \frac{(v+1)!}{u^{Z}(v+2-Z)!}$$

$$\binom{v+2-Z-i}{i} \binom{v+2-Z-i}{w} \left(D-u\sigma^2\right)^{v+2-Z-i} \left(\frac{\sigma^2}{2}\right)^{i} T^{v+2-Z-i-w} L^{w}$$

$$+ \left(\frac{\sigma^4}{2D^5} + \frac{T^2}{D} + \frac{2TL}{D} + \frac{L^2}{D}\right) \sum_{Z=1}^{v+1} \sum_{i=0}^{\frac{v+1-Z}{2}} \sum_{w=0}^{v+1-Z-i} \frac{v!}{u^{Z}(v+1-Z)!}$$

$$\binom{v+1-Z-i}{i} \binom{v+1-Z-i}{w} \left(D-u\sigma^2\right)^{v+1-Z-2i} \left(\frac{\sigma^2}{2}\right)^{i} T^{v+1-Z-i-w} L^{w}$$

$$+ \left(\left(\frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} - \frac{T^3}{3} \right) - L\left(\frac{\sigma^2 T}{D^2} + \frac{T^2}{3}\right) - L^2\left(\frac{T}{3} + \frac{\sigma^6}{2D^2}\right) - \frac{L^3}{3} \right)$$

$$\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-i} \frac{(v-1)!}{u^{Z}(v-Z)!} \binom{v-Z-i}{i} \left(D-u\sigma^2\right)^{v-Z-2i} \left(\frac{\sigma^2}{2}\right)^{i}$$

$$\binom{v-Z-i}{w} T^{v-Z-i-w} L^{w} \Bigg]$$

$$+\frac{u^{v}e^{-(L+T)\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)}}{\Gamma(v)}\Bigg[\left(\left(-\frac{2}{3}\,\frac{\sigma^{2}T^{2}}{D^{2}}+\frac{\sigma^{2}T}{3D^{3}}+\frac{\sigma^{4}T}{2D^{4}}\right)\right.$$

$$\left.+L\left(-\frac{4T\sigma^{2}}{3D^{2}}+\frac{\sigma^{2}}{3D^{3}}+\frac{\sigma^{4}}{2D^{4}}\right)-\frac{2\sigma^{2}L^{2}}{3D^{2}}\right)\quad \ast$$

$$\sum_{i=0}^{\frac{v}{2}}\ \sum_{w=0}^{v-i}\binom{v-i}{i}\binom{v-i}{w}\left(D-u\sigma^{2}\right)^{v-2i}\,T^{v-i-w}\left(\frac{\sigma^{2}}{2}\right)^{i}L^{w}$$

$$+\left(\left(\frac{\sigma^{2}T^{3}}{3D}+\frac{\sigma^{4}T^{2}}{6D^{3}}+\frac{8\sigma^{6}T}{D^{5}}\right)+L\left(\frac{\sigma^{2}T^{2}}{D}+\frac{2\sigma^{4}T}{6D^{3}}+\frac{8\sigma^{6}}{D^{5}}\right)\right.$$

$$\left.+L^{2}\left(\frac{\sigma^{2}T}{D}+\frac{\sigma^{4}}{6D^{3}}\right)+\sigma^{2}L^{3}\right)\sum_{i=0}^{\frac{v-1}{2}}\ \sum_{w=0}^{v-i}\binom{v-1-i}{i}\binom{v-1-i}{w}\quad \ast$$

$$\left(D-u\sigma^{2}\right)^{v-2i}\left(\frac{\sigma^{2}}{2}\right)^{i}\,T^{v-i-w}L^{w}\Bigg]$$

$$+\frac{\sigma^{6}}{4D^{6}}\,\frac{u^{v}e^{-(T+L)\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)}}{\Gamma(v)}\sum_{Z=1}^{v}\ \sum_{i=0}^{\frac{v-Z}{2}}\ \sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^{2}}\right)^{Z}(v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^{2}\right)^{v-\ \ -2i}\left(\frac{\sigma^{2}}{2}\right)^{i}T^{v-Z-i-w}L^{w}$$

10.3.13.

Substituting into 10.3.4 and noting that

$$H(L)=\frac{e^{-aL}L^{k-1}a^{k}}{\Gamma(k)}$$

and $\int_0^{\infty} e^{-aL}L^{y}\,dL = \Gamma(y+1)/a^{Y+L}$

Then

$$G_{77}(T)= \frac{-u^{v}}{\Gamma(v)}\frac{a^{k}}{\Gamma(k)}e^{-T\left(Du-\frac{u^2\sigma^2}{2}\right)}\left[\frac{1}{3D^3}\sum_{Z=1}^{v+3}\sum_{i=0}^{\frac{v+3-Z}{2}}\sum_{w=0}^{v-Z+3-i}\right.$$

$$\frac{(v+2)!}{u^{Z}(v+3-z)!}\binom{v+3-Z-i}{i}\binom{v+3-Z-i}{w}\left(D-u\sigma^2\right)^{v+3-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i} \qquad *$$

$$T^{v+3-Z-i-w}\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}$$

$$+\sum_{Z=1}^{2+v}\sum_{i=0}^{\frac{v+2-Z}{2}}\sum_{w=0}^{v+2-Z-i}\frac{(v+1)!}{u^{Z}(v+2-Z)!}\binom{v+2-Z-i}{i}\binom{v+2-Z-i}{w}$$

$$(D-u\sigma^2)^{v+2-Z-i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v+2-Z-i-w}\left(\left(\frac{\sigma^2}{2D^4}-\frac{T}{D^2}\right)\frac{\Gamma(w+k)}{(Du-u^2\sigma^2+a)^{w+k}}\right.$$

$$\left.-\frac{\Gamma(w+k+1)}{D^2\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+1}}\right)+\sum_{Z=1}^{v+1}\sum_{i=0}^{\frac{v+1-Z}{2}}\sum_{w=0}^{v+1-Z-i}\frac{v!}{u^{Z}(v+1-Z)!}$$

$$\binom{v+1-i-Z}{i}\binom{v+1-Z-i}{w}\left(D-u\sigma^2\right)^{v+1-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i} T^{v+1-Z-i-w}$$

$$\left(\left(\frac{\sigma^4}{2D^5}+\frac{T^2}{D}\right)\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}+\frac{2T}{D}\frac{\Gamma(w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+1}}\right.$$

$$\left.+\frac{1}{D}\frac{\Gamma(w+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+2}}\right)$$

$$+\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{u^Z(v-Z)!}\binom{v-Z-i}{i}\left(D-u\sigma^2\right)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}\binom{v-Z-i}{w}$$

$$T^{v-Z-i-w}\left(\left(\frac{\sigma^6}{4D^6}-\frac{\sigma^2T^2}{2D^2}-\frac{T^3}{3}\right)\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}\right.$$

$$-\left(\frac{\sigma^2T}{D^2}+\frac{T^2}{3}\right)\frac{\Gamma(w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+1}}-\left(\frac{T}{3}+\frac{\sigma^2}{2D^2}\right)\frac{\Gamma(w+k+2)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+2}}$$

$$\left.\left.-\frac{1}{3}\frac{\Gamma(w+k+3)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+3}}\right)\right]$$

$$+\frac{u^v a^k e^{-T\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)\,\Gamma(k)}\left[\sum_{i=0}^{\frac{v}{2}}\sum_{w=0}^{v-i}\binom{v-i}{i}\binom{v-i}{w}\left(D-u\sigma^2\right)^{v-2i}\right.$$

$$T^{v-i-w}\left(\frac{\sigma^2}{2}\right)^{i}\left(\left(-\frac{2}{3}\frac{\sigma^2T^2}{D^2}+\frac{\sigma^2T}{3D^3}+\frac{\sigma^4T}{2D^4}\right)\frac{\Gamma(w+k)}{\left(D-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}\right.$$

$$+\left(-\frac{4}{3}\frac{T\sigma^2}{D^2}+\frac{\sigma^2}{3D^3}+\frac{\sigma^4}{2D^4}\right)\frac{\Gamma(w+k+1)}{(Du-u^2\sigma^2+a)^{w+k+1}}-\frac{2\sigma^2}{3D^2}\left.\frac{\Gamma(w+k+2)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k+2}}\right]$$

$$+\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-i}\binom{v-1-i}{i}\binom{v-1-i}{w}\left(D-u\sigma^2\right)^{v-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-i-w}$$

$$\left(\left(\frac{\sigma^2T^3}{3D}+\frac{\sigma^4T^2}{6D^3}+\frac{8\sigma^5T}{D^5}\right)\frac{\Gamma(w+k)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k}}\right.$$

$$+\left(\frac{\sigma^2T^2}{D}+\frac{\sigma^4T}{3D^3}+\frac{8\sigma^6}{D^5}\right)\frac{\Gamma(w+k+1)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k+1}}$$

$$+\left(\frac{\sigma^2T}{D}+\frac{\sigma^6}{6D^3}\right)\frac{\Gamma(w+k+2)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k+2}}+\frac{\sigma^2}{3D}\left.\frac{\Gamma(w+k+3)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k+3}}\right]$$

$$+\frac{\sigma^5}{4D^5}\frac{a^k u^v}{\Gamma(k)\Gamma(v)}e^{-T(Du-\frac{u^2\sigma^2}{2})}\sum_{Z=1}^{v}\sum_{=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}$$

$$\times\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-Z-i-w}\frac{\Gamma(w+k)}{(Du-\frac{u^2\sigma^2}{2}+a)^{w+k}}$$

10.3.14.

From 9.4.13. substituting T+L for T

$$G_{59}(T+L)=\frac{u^{v}}{\Gamma(v)}e^{-(L+T)\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)}\left[\left(T+L-\frac{\sigma^{2}}{2D^{2}}\right)\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\right.$$

$$\frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-i}{i}\left(D-u\sigma^{2}\right)^{v-Z-2i}(T+L)^{v-Z-2i}$$

$$\left(\frac{\sigma^{2}}{2}(T+L)\right)^{i}$$

$$-\frac{1}{D}\sum_{Z=1}^{v+1}\sum_{i=0}^{\frac{v+1-Z}{2}}\binom{v+1-Z-i}{i}\left(D-u\sigma^{2}\right)^{v+1-Z-2i}(T+L)^{v+1-Z-2i}$$

$$\left.\left(\frac{\sigma^{2}}{2}(T+L)\right)^{i}\frac{v!}{u^{Z}(v+1-Z)!}\right]$$

$$+\frac{\sigma^{2}u^{v}e^{-(L+T)\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)}}{D\,\Gamma(v)}(L+T)\sum_{i=0}^{\frac{v-1}{2}}\binom{v-1-i}{i}\left(D-u\sigma^{2}\right)^{v-1-2i}$$

$$(T+L)^{v-1-2i}\left(\frac{\sigma^{2}}{2}(T+L)\right)^{i}$$

$$+\frac{u^{v}\sigma^{2}}{2D^{2}}e^{-(L+T)\left(Du-\frac{u^{2}\sigma^{2}}{2}\right)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^{2}}\right)^{Z}(v-Z)!}$$

$$\binom{v-Z-i}{i}\left(D-u\sigma^{2}\right)^{v-Z-2i}(T+L)^{v-Z-2i}\left(\frac{\sigma^{2}}{2}\right)^{i}(T+L)^{i}$$

10.3.15

Simplifying

$$G_{59}(T+L)=\frac{u^{v}e^{-(L+T)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\left[\left(\left(T-\frac{\sigma^2}{2D^2}\right)+L\right)\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\right.$$

$$\frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-i}{w}(D-u\sigma^2)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-Z-i-w}L^{w}$$

$$-\frac{1}{D}\sum_{Z=1}^{v+1}\sum_{i=0}^{\frac{v+1-Z}{2}}\sum_{w=0}^{v+1-Z-i}\frac{v!}{u^{Z}(v+1-Z)!}\binom{v+1-Z-i}{i}\binom{v+1-Z-i}{w}$$

$$\left.(D-u\sigma^2)^{v+1-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v+1-Z-i-w}L^{w}\right]$$

$$+\frac{\sigma^2u^{v}e^{-(L+T)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{D\,\Gamma(v)}(L+T)\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-i}\binom{v-1-i}{i}\binom{v-1-i}{w}$$

$$(D-u\sigma^2)^{v-1-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-1-i-w}L^{w}$$

$$+\frac{u^{v}\sigma^2}{2D^2}\;\frac{e^{-(L+T)\left(Du-\frac{u^2\sigma^2}{2}\right)}}{\Gamma(v)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}(D-u\sigma^2)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i}T^{v-Z-i-w}L^{w}$$

10.3.16.

Substituting into 10.3.5 and noting that

$H(L)=e^{-\alpha L}L^{k-1}\alpha^{k}/\Gamma(k)$ and

$$\int_0^{\infty}e^{-\alpha L}L^{y}\,dL=\Gamma(y+1)/\alpha^{y+1}$$

Then

$$G_{78}(T) = \frac{u^v e^{-T\left(Du - \frac{u^2\sigma^2}{2}\right)} a^k}{\Gamma(v)\,\Gamma(k)} \Bigg[\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-i} \frac{(v-1)!}{u^Z (v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}(D-u\sigma^2)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i} T^{v-Z-i-w}\left(\left(T-\frac{\sigma^2}{2D^2}\right) \times\right.$$

$$\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}} \quad \left.\frac{\Gamma(w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+1}}\right)$$

$$-\frac{1}{D}\sum_{Z=1}^{v+1}\sum_{i=0}^{\frac{v+1-Z}{2}}\sum_{w=0}^{v+1-Z-i}\frac{v!}{u^Z(v+1-Z)!}\binom{v+1-Z-i}{i}\binom{v+1-Z-i}{w} *$$

$$(D-u\sigma^2)^{v+1-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i} T^{v+1-Z-i-w}\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}\Bigg]$$

$$+\frac{\sigma^2 u^v a^k}{D\,\Gamma(v)\,\Gamma(k)} e^{-T\left(Du-\frac{u^2\sigma^2}{2}\right)}\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-i}\binom{v-1-i}{i}\binom{v-1-i}{w} *$$

$$(D-u\sigma^2)^{v-1-2i}\left(\frac{\sigma^2}{2}\right)^{i} T^{v-1-i-w}\left(T\frac{\Gamma(w+k)}{(Du-u^2\sigma^2+a)^{w+k}}\right.$$

$$\left.+\frac{\Gamma(w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k+1}}\right)+\frac{\sigma^2 a^k u^v e^{-T\left(Du-\frac{u^2\sigma^2}{2}\right)}}{2D^2\,\Gamma(v)\,\Gamma(k)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}$$

$$\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-i}{w}(D-u\sigma^2)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i} *$$

$$T^{v-Z-i-w}\frac{\Gamma(k+w)}{(Du-u^2\sigma^2+a)^{k+w}}$$

10.3.17.

Hence averaging the inventory costs for random supply and fixed lead times, 10.3.1, over the states of lead times and applying the above integrals we have the costs for (M,T) when supply is random and lead time is continuous are given by

$$C=\frac{Rc+S}{T}+\frac{hcv}{u}-hc\left(\frac{Dk}{\alpha}+\frac{DT}{2}\right)+\frac{b_1}{T}\left(G_{75}(T)-G_{75}(0)\right)+\frac{(b_2+hc)}{T}\left(G_{76}(T)-G_{76}(0)\right)$$

$$+\frac{b_3}{T}\left(G_{77}(T)-G_{77}(0)\right)+\frac{s}{T}\left(G_{78}(T)-G_{78}(0)\right) \qquad 10.3.18$$

SECTION 10.4 (M,R,T)

In section 6 of Chapter 8 we derived the inventory cost for fixed maximum order cover and continuous lead times.

In equation 8.6.2 we defined

$G_{44}(R+Y,T)$ as

$$G_{44}(R+Y,T)=\int_0^\infty\int_0^\infty H(L_1)H(L_2)\Big(G_{14}(R+Y.T)-s\big(R_0(R+Y,T+L_2)-R_0(R+Y,L_1)\big)\Big)\,dL_1 dL_2 \qquad 10.4.1$$

where $G_{44}(R+Y,T)$ is the cost of carrying inventory and backorders when the inventory level is $R+Y$, from t to $t+T$.

Let $$G_{80}(R+Y,T)=G_{44}(R+Y,T)+\int_0^\infty\int_0^\infty H(L_1)H(L_2) * s\big(R_0(R+Y,T+L_2)-R_0(R+Y,L_1)\big)dL_1\, dL_2 \qquad 10.4.2$$

where $G_{80}(R+Y,T)$ includes the cost of a stockout dependent upon the number of stockouts only.

Let $$G_{81}(R+Y,T)=\int_0^\infty H(L)R_0(R+Y,T+L)\, dL \qquad 10.4.3$$

From 9.4.13. substituting $R+Y$, for M and $T+L$ for T

$$R_0(R+Y,T+L)= \left(T+L-\frac{R+Y}{D} - \frac{\sigma^2}{2D^2}\right)F\left(\frac{R+Y-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$+\frac{\sqrt{\sigma^2(T+L)}}{D}\, g\left(\frac{R+Y-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)+\frac{\sigma^2}{2D^2}\, e^{\frac{2D(R+Y)}{\sigma^2}}\, F\left(\frac{R+Y+D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

10.4.4

Multiplying by $H(L)$ where $H(L)=\frac{e^{-\alpha L}L^{k-1}\alpha^k}{\Gamma(k)}$

$$H(L)\ R_0(R+Y,T+L)=\frac{\alpha^k e^{-\alpha L}}{\Gamma(k)}\left(\left(T-\left(\frac{R+Y}{D}\right)-\frac{\sigma^2}{2D^2}\right)L^{k-1}+L^k\right)$$

$$*F\left(\frac{R+Y-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)+\frac{\sqrt{\sigma^2(T+L)}}{D}\ \frac{\alpha^k L^{k-1}e^{-\alpha L}}{\Gamma(k)}\ g\left(\frac{R+Y-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$+\frac{\sigma^2}{2D^2}*\frac{e^{\frac{2D(R+Y)}{\sigma^2}}e^{-\alpha L}L^{k-1}\alpha^k}{\Gamma(k)}F\left(\frac{R+Y+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

10.4.5

Hence integrating $\int_0^\infty H(L)R_0(R+Y,\ T+L)\ dL$

and applying 8.2.31, and 8.2.34 and 8.2.35 we have

Let $\theta^2 = 2\alpha\sigma^2+D^2$

$$\frac{e^{\frac{D(R+Y)}{\sigma^2}}\alpha^k e^{\alpha T}}{2\sigma\Gamma(k)}\left[\left(T-\left(\frac{R+Y}{D}\right)-\frac{\sigma^2}{2D^2}\right)\sum_{j=0}^{k-1}\sum_{=1}^{k-j} T^j\binom{k-1}{j}\frac{(k-1-j)!}{\alpha^Z(k-j-Z)!}\right.$$

$$\left(2D\left(\frac{R+Y}{\theta}\right)^{k-j-Z+\frac{1}{2}} K_{k-j-Z+\frac{1}{2}}\left(\frac{(R+Y)\theta}{\sigma^2}\right) + 2(R+Y)\left(\frac{R+Y}{\theta}\right)^{k-j-Z-\frac{1}{2}} * K_{k-j-Z-\frac{1}{2}}\left(\frac{\theta(R+Y)}{\sigma^2}\right)\right)$$

$$+\sum_{j=0}^{k}\sum_{Z=1}^{k+1-j} T^j \binom{k}{j} \frac{(k-j)!}{a^Z(k+1-j-Z)!}\left(2D\left(\frac{R+Y}{\theta}\right)^{k-j-Z+3/2} * K_{k-j-\ +3/2}\left(\frac{\theta(R+Y)}{\sigma^2}\right) + 2(R+Y)\left(\frac{R+Y}{\theta}\right)^{k-j-Z+\frac{1}{2}} K_{k-j-Z+\frac{1}{2}}\left(\frac{\theta(R+Y)}{\sigma^2}\right)\right)$$

$$+2\,\frac{a^k}{D\sqrt{2\pi}}\, e^{aT+D\frac{(R-y)}{\sigma^2}} \sum_{j=0}^{k-1} T^j \binom{k-1}{j}\left(\frac{R+Y}{\theta}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}\left(\frac{(R+Y)\theta}{\sigma^2}\right)$$

$$+\frac{\sigma^2}{2D^2}\,\frac{a^k e^{2T+D\frac{(R+Y)}{\sigma^2}}}{\sqrt{2\pi}\,\Gamma(k)} \sum_{j=0}^{k-1}\sum_{Z=1}^{k-j} T^j \binom{k-i}{j}\frac{(k-j)!}{a^Z(k+1-j-Z)!} *$$

$$\left(2D\left(\frac{R+Y}{\theta}\right)^{k-j-Z+\frac{1}{2}} K_{k-j-Z+\frac{1}{2}}\left(\frac{\theta(R+Y)}{\sigma^2}\right) + 2(R+Y)\left(\frac{R+Y}{\theta}\right)^{k-j-Z-\frac{1}{2}} * K_{k-j-Z-\frac{1}{2}}\left(\frac{\theta(R+Y)}{\sigma^2}\right)\right) \qquad 10.4.6$$

Hence we can obtain $G_{80}(R+Y,T)=G_{44}(R+Y,T)+G_{81}(R+Y,T)$, since we have both G_{44} and G_{81}

Hence from equation 9.5.3 substituting $G_{80}(R+Y,T)$ for $G_8(R+Y,T)$ we obtain the inventory costs for continuous lead time and random supply.

$$C = \frac{Rc}{T} + S + \frac{\sum_{n=1}^{\infty} \int_{R}^{\infty} U(M) \int_{0}^{M-R} G_{80}(R+Y, T) \times f^{n}(M-R-Y, DT)\, dY\, dM + \int_{R}^{\infty} U(M) G_{8}(M, T)\, dY}{T \left(\int_{R}^{\infty} U(M) F\left(\frac{M-R-DT}{\sqrt{\sigma^{2} T}}\right) dM + \sum_{n=2}^{\infty} n \int_{R}^{\infty} U(M) f^{n-1}(M-R-Y, DT) \cdot F\left(\frac{Y-DT}{\sqrt{\sigma^{2} T}}\right) dY\, DM \right)} \qquad 10.4.7$$

SECTION 10.5 (Q,R) RANDOM SUPPLY EXPONENTIAL COST TERM

In Section 9.6 we developed the inventory costs when the supply was random and the lead times were constant. From equation 9.6.8. the inventory cost C is

$$C=\frac{Dsu}{(v-1)}+\frac{vhc}{2u}+hc(R-DL)+hc\left(\frac{u}{(v-1)}\beta\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left.-G_{54}(R,L)\right)+D\ G_{64}(R,L)+Ds\left(\frac{u}{(v-1)}\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left.-G_{53}(R,L)\right). \qquad 10.5.1$$

The inventory costs when the supply is random and the lead times are continuous are now obtained by averaging the inventory cost for constant lead times above over the states of the lead times.

The probability density function of lead time L

$$H(L) = \frac{e^{-\alpha L}L^{k-1}\alpha^k}{\Gamma(k)} \qquad 10.5.2$$

From 8.3.21

$$G_{30}(Z)=\int_0^\infty H(L)\ \beta\left(\frac{Z-DL}{\sqrt{\sigma^2 L}}\right) dL \qquad 10.5.3$$

From 8.3.22

$$G_{29}(Z)=\int_0^\infty H(L)\,\alpha\left(\frac{Z-DL}{\sqrt{\sigma^2 L}}\right) dL \qquad 10.5.4$$

From 10.1.2

$$G_{66}(Z)=\int_0^{\infty} H(L)G_{53}(Z,L)\ dL \qquad 10.5.5$$

From 10.1.3

$$G_{67}(Z)=\int_0^{\infty} H(L)G_{54}(Z,L)\ dL \qquad 10.5.6$$

The following additional integrals are needed for the derivation of the inventory cost.

$$G_{82}(R)=\int_0^{\infty} H(L)G_{64}(R,L)\ dL \qquad 10.5.7$$

From 9.6.7

$$G_{64}(R,L)=\frac{Dub_1}{(v-1)b_4\Gamma(v)}\exp\left(\frac{\sigma^2Lb^2{}_4}{2D^2}-\frac{b_4R+b_4L}{D}\right)F\left(\frac{\frac{R-\sigma^2b_4-DL}{D}}{\sqrt{\sigma^2L}}\right)$$

$$-\frac{Db_1}{b_4}\frac{u^v}{\Gamma(v)}\exp\left[\left(\frac{\sigma^2Lb^2{}_4}{2D^2}-\frac{b_4R+b_4L}{D}\right)+\left(\frac{u-\sigma^2b_4}{D}\right)+\left(\frac{u-b_4}{D}\right)\frac{\sigma^2L-DL}{2}\right]$$

$$\times\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\frac{(v-2)!}{\left(\frac{u-b_4}{D}\right)^Z(v-1-Z)!}\binom{v-1-Z-i}{i}\left(DL-R+\frac{\sigma^2b_4}{D}-\left(\frac{u-b_4}{D}\right)\right.$$

$$\left.\times\sigma^2L\right)^{v-1-Z-2i}\left(\frac{\sigma^2}{2}L\right)^i-\frac{Db_1u}{(v-1)b_4}F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{Db_1}{b_4}\frac{u^v}{\Gamma(v)}\exp\left[Ru+\frac{u^2\sigma^2}{2}L-DuL\right]\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\frac{(v-2)!}{u^Z(v-1-z)!}.$$

$$\binom{v-1-Z-i}{i}\left(DL-R-u\sigma^2 L\right)^{v-1-Z-2i}\left(\frac{\sigma^2}{2}L\right)^{i}$$

10.5.8

Multiplying by H(L) and Simplifying we have

$$H(L)G_{64}(R,L)=\frac{Dub_1 a^k L^{k-1}}{\Gamma(k)(v-1)b_4\Gamma(v)}\ \exp\ [-L(a-\frac{\sigma^2 b_4^2}{2D^2}-b_4)-\frac{b_4 R}{D}$$

$$\times\ F\left(\frac{\frac{R-\sigma^2 b_4}{D}-DL}{\sqrt{\sigma^2 L}}\right)$$

$$\frac{-Db_1 u^v a^k}{b_4\Gamma(v)\Gamma(k)}\ \exp\left[-L(a-\frac{\sigma^2 b^2{}_4}{2D^2}-b_4-u+\frac{b_4}{D}+D)+\left(u-\frac{\sigma^2 b_4}{D}-\frac{b_4 R}{D}\right)\right] \times$$

$$\sum_{z=1}^{v-1}\sum_{i=0}^{\frac{v-z-1}{2}}\sum_{w=0}^{v-1-Z-2i}\frac{(v-2)!}{\left(u-\frac{b_4}{D}\right)^{Z}(v-1-Z)!}\binom{v-1-Z-i}{i}\binom{v-1-Z-2i}{w} \times$$

$$\left(\frac{\sigma^2}{2}\right)^{i}\left(-R+\frac{\sigma^2 b_4}{D}\right)^{v-1-Z-2i-w}\left(D-\left(u+\frac{b_4}{D}\right)\sigma^2\right)^{w} L^{i+w+k-1}$$

$$\frac{-Db_1 a^k u e^{-aL}L^{k-1}}{(v-1)b_4\Gamma(k)}\ F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\frac{Db_1}{b_4}\frac{u^v}{\Gamma(v)}\frac{a^k}{\Gamma(k)}\ \exp\ [\ -L(a+Du-\frac{u^2\sigma^2}{2})\ +Ru]\ \sum_{z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-1-Z-2i}$$

$$\times\ \frac{(v-2)!}{u^{Z}(v-1-Z)!}\binom{v-1-Z-i}{i}\binom{v-1-Z-2i}{w}(-R)^{v-1-Z-2i-w}(D-u\sigma^2)^{w} \times$$

$$\left(\frac{\sigma^2}{2}\right)^{i} L^{i+w+k-1}$$

10.5.9

Integrating and applying 8.2.18 we have and

let $\theta^2 = 2a\sigma^2 + D^2$, then we have

$$G_{82}(R) = \frac{Dub_1 \; a^k}{2\sigma\,(v-1)b_4\Gamma(k)} \exp\left[\frac{(-b_4R)}{D} + \frac{D}{\sigma^2}\left(R - \frac{\sigma^2 b_4}{D}\right)\right]$$

$$\sum_{Z=1}^{k} \frac{(k-1)!}{a^Z(k-Z)!}\left[2D\left(\frac{R-\frac{\sigma^2 b_4}{D}}{\theta}\right)^{k-Z+\frac{1}{2}} K_{k-Z+\frac{1}{2}}\left(\theta\left(\frac{R-\frac{\sigma^2 b_4}{D}}{\sigma^2}\right)\right) + 2\left(R-\frac{\sigma^2 b_4}{D}\right)\left(\frac{R-\frac{\sigma^2 b_4}{D}}{\theta}\right)^{k-Z-\frac{1}{2}} \times K_{k-Z-\frac{1}{2}}\left(\theta\left(R-\frac{\sigma^2 b_4}{D}\right)/\sigma^2\right)\right)$$

$$\frac{-Db_1u^v a^k \exp\left[\frac{u-\sigma^2 b_4}{D} - \frac{b_4 R}{D}\right]}{b_4\Gamma(v)\Gamma(k)} \sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-1-Z-2i}$$

$$* \frac{(v-2)!}{\left(\frac{u-b_4}{D}\right)^Z(v-1-Z)!}\binom{v-1-Z-i}{i}\binom{v-1-Z-2i}{w}\left(\frac{\sigma^2}{2}\right)^i\left(\frac{-R+\sigma^2 b_4}{D}\right)^{v-1-Z-2i-w} *$$

$$\left(\frac{D-\sigma^2 u-\sigma^2 b_4}{D}\right)^w \frac{\Gamma(i+w+k)}{\left(\frac{a-\sigma^2 b_4^2}{2D^2} - b_4 - u + \frac{b_4+D}{D}\right)^{i+w+k}}$$

$$\frac{-Db_1a^k u}{(v-1)b_4\Gamma(k)} \sum_{Z=1}^{k}\frac{(k-1)!}{a^Z(k-z)!}\left[2D\left(\frac{R}{\theta}\right)^{k-Z+\frac{1}{2}} K_{k-\;+\frac{1}{2}}\left(\frac{\theta R}{\sigma^2}\right)\right.$$

$$\left. + 2R\left(\frac{R}{\theta}\right)^{k-Z-\frac{1}{2}} K_{k-Z-\frac{1}{2}}\left(\frac{\theta R}{\sigma^2}\right)\right]$$

$$\frac{+Db_1u^v a^k}{b_4\Gamma(v)\Gamma(k)} e^{Ru} \sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}\sum_{w=0}^{v-1-Z-2i} \frac{(v-2)!}{u^Z(v-1-z)!}\binom{v-1-Z-i}{i} *$$

$$\binom{v-1-\quad-2i}{w}(-R)^{v-1-Z-2i-w}\left(D-u\sigma^2\right)^w\left(\frac{\sigma^2}{2}\right)^i \frac{\Gamma(i+w+k)}{\left(\alpha+Du-\frac{u^2\sigma^2}{2}\right)^{i+w+k}}$$

10.5.10.

Applying the above integrals, the inventory costs when supply is random and lead time is continuous is

$$C=\frac{DSu+vhc}{(v-1)2u}+hc\left(R-\frac{Dk}{\alpha}\right)+hc\left(\frac{u}{(v-1)}G_{30}(R)\right.$$

$$\left.-G_{67}(R)\right)+DG_{82}(R)+Ds\left(\frac{u}{(v-1)}G_{29}(R)\right.$$

$$\left.-G_{66}(R)\right)$$

10.5.11

SECTION 10.6 (nQ,R,T) EXPONENTIAL COST

The inventory cost when the supply is random and lead time is constant has been developed in Chapter 9 Section 7. for exponential backorder cost. From equation 9.7.8 the cost is

$$C = \frac{c}{T} + \frac{S.Po}{T} \, T + hc\left(\frac{v}{2u} + R - DL - \frac{DT}{2}\right)$$

$$+\frac{hc}{T}\left(\frac{u}{(v-1)} G_3(R,T+L) - \frac{u}{(v-1)} G_3(R,L) - G_{61}(R,T+L) + G_{61}(R,L)\right)$$

$$+\frac{1}{T}\left(\frac{u}{(v-1)} G_{18}(R,T+L) - \frac{u}{(v-1)} G_{18}(R,L) - G_{83}(R,T+L) + G_{83}(R,L)\right)$$

$$+\frac{s}{T}\left(\frac{u}{(v-1)} G_4(R,T+L) - \frac{u}{(v-1)} G_4(R,L) - G_{63}(R,T+L) + G_{63}(R,L)\right) \qquad 10.6.1$$

The inventory costs when the supply is random and lead time is continuous are now obtained by averaging the inventory costs for fixed lead time over the states of the lead times. The probability density function of lead time $H(L) = \frac{e^{-aL} L^{k-1} a^k}{\Gamma(k)}$

The following integrals are necessary for the derivation of the inventory cost for continuous lead times.

From 8.4.11 and 8.4.27.

$$G_{33}(R) = \int_0^{\infty} H(L) G_3(R,L)\, dL \qquad 10.6.2$$

$$G_{36}(R,T)= \int_0^\infty H(L)G_3(R,L+T)\, dL \qquad 10.6.3$$

$$G_{70}(R,T)= \int_0^\infty H(L)G_{61}(R,T+L)\, dL \qquad 10.6.4$$

From 8.8.2

$$G_{47}(R)= \int_0^\infty H(L)G_{18}(R,L)\, dL \qquad 10.6.5$$

$$G_{48}(R,T)= \int_0^\infty H(L)G_{18}(R,L+T)\, dL \qquad 10.6.6$$

From 10.2.8

$$G_{73}(R,T)= \int_0^\infty H(L)G_4'(R,T+L)\, dL \qquad 10.6.7$$

$$G_{72}(R,T)= \int_0^\infty H(L)G_{63}(R,T+L)\, dL \qquad 10.6.8$$

Let $G_{84}(R,T)= \int_0^\infty H(L)G_{83}(R,T+L)\, dL$ 10.6.9

From 9.7.7 substituting T+L for T

$$G_{83}(R,T+L)=\frac{2D^2b_1u^ve^{(T+L)\left(\frac{\sigma^2b_4+2D^2b_4}{2D^2}-\frac{b_4R}{D}\right)}}{\left(\sigma^2b^3{}_4+2D^2b^2{}_4\right)\Gamma(v)}\sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-1-}{2}}$$

$$\frac{(v-2)!}{\left(u+\frac{b_4}{D}\right)^Z(v-1-Z)!}\binom{v-1-Z-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-1-Z-2i}$$

$$\times\left(\frac{\sigma^2}{2}(T+L)\right)^{i} \times e^{\left(\frac{u+b_4}{D}\right)} \left(R+\left(\frac{u+b_4}{D}\right)\sigma^2(T+L)-\left(\frac{D+\sigma^2 b_4}{D}\right)(T+L)\right)$$

$$\frac{-e^{\left[\left(Ru+\frac{u^2\sigma^2}{2}(T+L)-Du(T+L)\right]\right.} u^v b_1}{b_4\Gamma(v)} \left[\left(T+L-\frac{R}{D}-\frac{\sigma^2}{2D^2}+\frac{1}{b_4}\right)\right.$$

$$\sum_{Z=1}^{V-1}\sum_{i=0}^{\frac{V-1-Z}{2}} \frac{(v-2)!}{u^Z(v-1-Z)!}\binom{v-Z-i-1}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i-1}\left(\frac{\sigma^2}{2}(T+L)\right)^i$$

$$-\frac{1}{D}\sum_{Z=1}^{V}\sum_{i=0}^{\frac{V-Z}{2}}\frac{(V-1)!}{u^Z(V-Z)!}\binom{V-Z-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{V-Z-2i}\left.\left(\frac{\sigma^2}{2}(T+L)\right)^i\right]$$

$$\frac{-\sigma^4 b_1 b_4^2 u^v}{2D^2\left(\sigma^2{}_4+2D^2b^2{}_4\right)\Gamma(v)} e^{\left(Ru-\frac{u^2\sigma^2}{2}(T+L)-Du(T+L)\right)} \sum_{Z=1}^{v-1}\sum_{i=0}^{\frac{v-Z-1}{2}}$$

$$\frac{(v-2)!}{\left(u-\frac{2D}{\sigma^2}\right)^Z(v-z)!}\binom{v-Z-i}{i}\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2(T+L)}{2}\right)^{i} \quad \frac{-2\sigma^2(T+L)b_1u^v}{D\sigma_4\Gamma(v)} e^{\left(Ru+(T+L)\left(\frac{u^2\sigma^2}{2}-Du\right)\right)} \sum_{i=0}^{\frac{v-2}{2}}$$

$$\binom{v-2-i}{i} \times\left(D(T+L)-R-u\sigma^2(T+L)\right)^{v-Z-2i}$$

10.6.10

Simplifying

$$G_{83}(R, T+L) = \frac{2D^2 b_1 u^v \alpha^k \exp\left[T\left(\frac{\sigma^2 b_4 + 2D^2 b_4}{2D^2} - \frac{b_4 R}{D}\right)\right]}{\left(\sigma^2 b^3{}_4 + 2D^2 b^2{}_4\right) \Gamma(v)\,\Gamma(k)}$$

$$\times \exp\left[\left(u + \frac{b_4}{D}\right)\left(R + \sigma^2 T\right) - T\left(D + \frac{\sigma^2 b_4}{D}\right)\right]$$

$$\times \exp\left[-L\left(\frac{b_4 R}{D} - \frac{\sigma^2 b_4}{2D^2} - b_4 - \sigma^2\left(u + \frac{b_4}{D}\right)^2 + \left(D + \frac{\sigma^2 b_4}{D}\right)\right)\right]$$

$$\times \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-1-Z}{2}} \sum_{w=0}^{v-1-Z-2i} \sum_{j=0}^{i+w} \frac{(v-2)!}{\left(u + \frac{b_4}{D}\right)^Z (v-1-Z)!} \binom{v-1-Z-i}{i}$$

$$\binom{v-1-Z-2i}{w} (-R)^{v-1-Z-2i-w} \left(D - u\sigma^2\right)^w \left(\frac{\sigma^2}{2}\right)^i \binom{i+w}{j} T^{i+w-j} L^j$$

$$-\frac{\exp\left[\left(Ru + \frac{u^2\sigma^2}{2}T - DuT\right) - L\left(Du - \frac{u^2\sigma^2}{2}\right)\right] u^v b_1 \alpha^k}{b_4\,\Gamma(v)\,\Gamma(k)}$$

$$\times \left[\left(\left(T - \frac{R}{D} - \frac{\sigma^2}{2D^2} + \frac{1}{b_4}\right) + L\right)\right] \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-1-Z}{2}} \sum_{w=0}^{v-1-Z-2i} \sum_{j=0}^{i+w} \frac{(v-2)!}{u^Z (v-1-Z)!}$$

$$\left(\frac{\sigma^2}{2}\right)^i \binom{v-1-Z-i}{i} \binom{v-1-Z-2i}{w} \binom{i+w}{j} \left(D - u\sigma^2\right)^w T^{i+w-j} L^j (-R)^{v-1-Z-2i-w}$$

$$-\frac{1}{D} \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \sum_{j=0}^{i+w} \frac{(v-1)!}{u^Z (v-Z)!} \binom{v-Z-i}{i} \binom{v-Z-2i}{w} \binom{i+w}{j}$$

$$\left.\left(\frac{\sigma^2}{2}\right)^i \left(D - u\sigma^2\right)^w (-R)^{v-Z-2i-w} T^{i+w-j} L^j \right]$$

$$-\frac{\sigma^4 b_1 b_4^2 u^v \exp[Ru+\frac{u^2\sigma^2}{2}T-DuT]\ \exp[-L(Du-\frac{u^2\sigma^2}{2})]}{2D^2(\sigma^2 b^3{}_4+2D^2 b^2{}_4)\ \Gamma(v)}$$

$$*\sum_{Z=1}^{v-1}\ \sum_{i=0}^{\frac{v-Z-1}{2}}\ \sum_{w=0}^{v-Z-2i-w}\ \sum_{j=0}^{i+w} \frac{(v-2)!}{\left(u-\frac{2D}{\sigma^2}\right)^Z (v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-2i}{w}$$

$$\binom{i+w}{j}(D-u\sigma^2)^w(-R)^{v-Z-2i-w}\left(\frac{\sigma^2}{2}\right)^i T^{i+w+j}L^j$$

$$-\frac{2\sigma^2(T+L)b_1 u^v \exp[Ru+\frac{u^2\sigma^2}{2}T-DuT]\ \exp[-L(Du-\frac{u^2\sigma^2}{2})]}{Db_4\,\Gamma(v)}$$

$$\sum_{i=0}^{\frac{v-2}{2}}\ \sum_{w=0}^{v-Z-2i}\ \sum_{j=0}^{i+w}\binom{v-2-i}{i}\binom{v-Z-2i}{w}\left(\frac{\sigma^2}{2}\right)^i (D-u\sigma^2)^w$$

$$(-R)^{v-Z-2i-w}T^{i+w-j}L^j \qquad 10.6.11$$

Multiplying by $H(L)$, where $H(L) = e^{-\alpha L}L^{k-1}\alpha^k/\Gamma(k)$

and integrating and noting that

$$\int_0^\infty e^{-\alpha L}L^y dL = \Gamma(y+1)/\alpha^{y+1}$$

we have

$$G_{84}(R,T)=\frac{2D^2 b_1\alpha^k u^v \exp\left[T\left(\frac{\sigma^2 b_4}{2D^2}+b_4-\frac{b_4 R}{D}\right)+\left(\frac{u+b_4}{D}\right)\left(R+\sigma^2 T\right)\right.}{(\sigma^2 b^3{}_4+2D^2 b^2{}_4)\ \Gamma(v)\,\Gamma(k)}$$

$$\left.-T\left(\frac{D+\sigma^2 b_4}{D}\right)\right]$$

$$\times \sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-1-Z}{2}} \sum_{w=0}^{v-1-Z-2i} \sum_{j=0}^{i+w} \frac{(v-2)!}{\left(\frac{u+b_4}{D}\right)^{Z}(v-1-Z)!} \binom{v-1-Z-i}{i}$$

$$\binom{v-1-Z-2i}{w}\left(\frac{\sigma^2}{2}\right)^{i} \frac{(-R)^{v-1-Z-2i-w}(D-u\sigma^2)^{w}\binom{i+w}{j} T^{i+w-j}\Gamma(j+k)}{\left(b_4 R-\frac{\sigma^2 b_4}{2D^2}-b_4\sigma^2(u+b_4)^2+\left(\frac{D+\sigma^2 b_4}{D}\right)\right)^{j+k}}$$

$$\frac{-a^k u^v b_1 \exp\left[Ru+\frac{u^2\sigma^2}{2}T-DuT\right]}{b_4\Gamma(v)\Gamma(k)}\left[\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-1-z}{2}} \sum_{w=0}^{v-1-Z-2i} \sum_{j=0}^{i+w}\right.$$

$$\frac{(v-2)!}{u^{Z}(v-1-Z)!}\left(\frac{\sigma^2}{2}\right)^{i}\binom{v-1-Z-i}{i}\binom{v-1-Z-2i}{w}\binom{i+w}{j}(D-u\sigma^2)^{w} T^{i+w-j}$$

$$(-R)^{v-1-Z-2i-w}\left(\frac{\left(T-\frac{R}{D}-\frac{\sigma^2}{2D^2}+\frac{1}{b_4}\right)\Gamma(j+k)}{\left(Du-\frac{u^2\sigma^2}{2}\right)^{j+k}}+\frac{\Gamma(j+k+1)}{\left(Du-\frac{\sigma^2u^2}{2}\right)^{j+k+1}}\right)$$

$$-\frac{1}{D}\sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \sum_{w=0}^{v-Z-2i} \sum_{j=0}^{i+w} \frac{(v-1)!}{u^{Z}(v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-2i}{w}\binom{i+w}{j}$$

$$\left.\left(\frac{\sigma^2}{2}\right)^{i}(D-u\sigma^2)^{w}(-R)^{v-Z-2i-w}T^{i+w-j}\frac{\Gamma(j+k)}{\left(Du-\frac{u^2\sigma^2}{2}\right)^{j+k}}\right]$$

$$\frac{-a^k\sigma^4 b_1 b_4^2 u^v \exp\left[Ru+\frac{u^2\sigma^2}{2}T-DuT\right]}{2D^2(\sigma^2 b_4^3+2D^2 b_4^2)\,\Gamma(v)\,\Gamma(k)}\left[\sum_{Z=1}^{v-1} \sum_{i=0}^{\frac{v-Z-1}{2}} \sum_{w=0}^{v-Z-2i} \sum_{j=0}^{i+w}\right.$$

$$\frac{(v-2)!}{\left(\frac{u-2D}{\sigma^2}\right)^{Z}(v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-2i}{w}\binom{i+w}{j}(D-u\sigma^2)^{w}\left(\frac{\sigma^2}{2}\right)^{i} \times$$

$$(-R)^{v-Z-2i-w}T^{i+w-j}\left[\frac{\Gamma(j+k)}{\left(Du-\frac{u^2\sigma^2}{2}\right)^{j+k}}\right]$$

$$\frac{-2a^{k}\sigma^2 b_1 u^{v}\exp[Ru+\frac{u^2\sigma^2}{2}T-DuT]}{\Gamma(v)\,\Gamma(k)\quad Db_4}\sum_{i=0}^{\frac{v-2}{2}}\sum_{w=0}^{v-Z-2i}\sum_{j=0}^{i+w}\binom{v-2-i}{i}$$

$$\binom{v-Z-2i}{w}\left(\frac{\sigma^2}{2}\right)^{i}(D-u\sigma^2)^{w}(-R)^{v-Z-2i-w}T^{i+w-j}\frac{\Gamma(j+k)}{(Du-\frac{u^2\sigma^2}{2})^{j+k}}$$

10.6.12

Hence applying the above integrals, the inventory costs when the supply is random and lead time is continuous and the cost of a backorder is exponential is $C=\frac{Rc}{T}+\frac{S.PoR_T}{T}+hc\left(\frac{v}{2u}+R-\frac{DK}{a}-\frac{DT}{2}\right)$

$$+\frac{hc}{T}\left(\frac{u}{\Gamma(v-1)}G_{36}(R,T)-G_{33}(R)-G_{70}(R,T)+G_{70}(R,o)\right)$$

$$+\frac{1}{T}\left(\frac{u}{\Gamma(v-1)}G_{48}(R,T)-G_{47}(R)-G_{84}(R,T)+G_{84}(R,o)\right)$$

$$+\frac{s}{T}\left(\frac{u}{\Gamma(v-1)}G_{73}(R,T)-\frac{u}{\Gamma(v-1)}G_{73}(R,o)-G_{72}(R,T)+G_{72}(R,o)\right)$$

10.6.13.

SECTION 10.7 (M,T) EXPONENTIAL COST TERMS.

In this section, the inventory cost when the supply is random and the lead times are continuous is now derived. In Section 9.8 the inventory cost when the supply was random and the lead times were constant was derived.

From equation 9.8.7 the inventory cost, substituting $T+L_2$ for $T+L$ and L_1 for L is

$$C=\left(\frac{Rc+S}{T}\right)+\frac{hcv}{u}-hc\left(DL_1+\frac{DT}{2}\right)+hc\left(G_{57}(T+L_2)\right.$$

$$-G_{57}(L)+\frac{1}{T}\left(G_{65}(T+L)-G_{65}(L)\right)+$$

$$\frac{S}{T}\left(G_{59}(T+L)-G_{59}(L)\right) \qquad 10.7.1$$

The inventory cost when the lead times are continuous is obtained by averaging the inventory costs C above above over the states of the lead times. The probability density function of lead times $H(L)=\frac{e^{-aL}L^{k-1}a^{k}}{\Gamma(k)}$

The following integrals are necessary for deriving the inventory costs when the lead times are continuous.

From 10.3.3

$$G_{76}(T)=\int_0^{\infty}H(L)G_{57}(T+L)\,dL \qquad 10.7.2$$

From 10.3.5

$$G_{78}(T)=\int_0^\infty H(L)G_{59}(T+L)\,dL \qquad 10.7.3$$

Let $$G_{85}(T)=\int_0^\infty H(L)G_{65}(T+L)\,dL \qquad 10.7.4$$

From 9.8.6, substituting T+L for T

$$G_{65}(T+L)=2Db_1u^v e^{\left[\left(\frac{\sigma^2 b_4T+b_4T}{2D^2}\right)+\left(\frac{u+b_4}{D}\right)\left(\frac{\sigma^2}{2}T\left(\frac{u+b_4}{D}\right)-DT\right)\right]}$$

$$\frac{\times\exp\left[-L\left(\frac{u+b_4}{D}\right)\left(D-\frac{\sigma^2}{2}\left(\frac{u+b_4}{D}\right)\right)-\left(\frac{\sigma^2b^2{}_4+b_4}{2D^2}\right)\right]}{b_4\Gamma(v)\;(\sigma^2b^2{}_4+2D^2b_4)}$$

$$\times\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-2}{2}}\frac{(v-1)!}{\left(\frac{u+b_4}{D}\right)^Z(v-Z)!}\binom{v-Z-i}{i}\left(D(T+L)-u\sigma^2(T+L)\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2(T+L)}{2}\right)^i+\frac{b_1}{b_4}\frac{u^v}{\Gamma(v)}\;e\;p\left(\frac{u^2\sigma^2}{2}T-DuT\right)\;\exp\left(-L\left(Du-\frac{u^2\sigma^2}{2}\right)\right)$$

$$\left[\sum_{i=0}^{\frac{v}{2}}\binom{v-i}{i}\times\left(D(T+L)-u\sigma^2(T+L)\right)^{v-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^i\right.$$

$$\left.-D(T+L)\sum_{i=0}^{\frac{v-1}{2}}\binom{v-1-i}{i}\left(D(T+L)-u\sigma^2(T+L)\right)^{v-1-2i}\left(\frac{\sigma^2(T+L)}{2}\right)^i\right]$$

$$\frac{-\sigma^2 b_1 u^v}{b\left(\sigma^2 b_4+2D^2\right)} \frac{\exp\left(\frac{u^2\sigma^2}{2}T-DuT\right)}{\Gamma(v)} \times \exp\left(-L\left(Du-\frac{u^2\sigma^2}{2}\right)\right)$$

$$\times \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-Z}{2}} \frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^Z (v-Z)!} \binom{v-Z-i}{i} \left(D(T+L)-u\sigma^2(T+L)\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2}{2}(T+L)\right)^i \qquad 10.7.5$$

Simplifying

$$G_{65}(T+L)=2Db_1u^v e^T\left[\left(\frac{\sigma^2 b_4+b_4}{2D^2}\right)+\left(\frac{u+b_4}{D}\right) \sigma^2\left(\frac{u+b_4}{D}\right)-D\right)\Bigg] \times$$

$$\frac{\exp\left[-L\left(\frac{u+b_4}{D}\right)\left(D-\sigma^2\left(\frac{u+b_4}{D}\right) -\left(\frac{\sigma^2 b^2{}_4+b_4}{2D^2}\right)\right)\right]}{b_4(\sigma^2 b^2{}_4+2D^2 b_4)\ \Gamma(v)}$$

$$\times \sum_{Z=1}^{v} \sum_{i=0}^{\frac{v-2}{2}} \sum_{w=0}^{v-Z-i} \frac{(v-1)!}{\left(\frac{u+b_4}{D}\right)^Z (v-Z)!} \binom{v-Z-i}{i}\binom{v-Z-i}{w}$$

$$(D-u\sigma^2)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^i T^{v-Z-i-w} L^w$$

$$+\frac{b_1}{b_4} \frac{u^v}{\Gamma(v)} \exp\left[\frac{u^2\sigma^2}{2}T-DuT\right] \exp -L\left(Du-\frac{u^2\sigma^2}{2}\right)\left[\sum_{i=0}^{\frac{v}{2}} \sum_{w=0}^{v-i} \binom{v-i}{i}\right.$$

$$\binom{v-i}{w}\left(D-u\sigma^2\right)^{v-2i}\left(\frac{\sigma^2}{2}\right)^i T^{v-i-w} L^w$$

$$-D(T+L)\sum_{i=0}^{\frac{Y-1}{2}}\sum_{w=0}^{Y-1-i}\binom{v-1-i}{i}\binom{v-1-i}{w}\left(D-u\sigma^2\right)^{v-1-i}\left(\frac{\sigma^2}{2}\right)^{i}$$

$$\left. T^{v-i-w}L^{w}\right]$$

$$\frac{-\sigma^2 b_1 u^v}{D(\sigma^2 b_4+2D^2)}\quad\frac{\exp\left(\frac{u^2\sigma^2}{2}T-DuT\right)\ \exp-L\left(Du-\frac{u^2\sigma^2}{2}\right)}{\Gamma(v)}$$

$$*\sum_{Z=1}^{V}\sum_{i=0}^{\frac{v-}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2}{2}\right)^{i}T^{v-Z-i-w}L^{w}\qquad 10.7.6$$

Multiplying by $H(L)$, where $H(L)=\frac{e^{-aL}L^{k-1}a^{k}}{\Gamma(k)}$

Integrating and noting that

$\int_0^\infty e^{-aL}L^{Y}dL=\Gamma(Y+1)/a^{Y+1}$ we have

$$G_{85}(T)=\frac{2Db_1u^ve^T\left(\frac{\sigma^2b_4+b_4}{D}\right)+\left(u+\frac{b_4}{D}\right)\left(\sigma^2\left(u+\frac{b_4}{D}\right)-D\right)\Big]a^k}{b_4(\sigma^2b^2{}_4+2D^2b_4)\ \Gamma(v)\ \Gamma(k)}$$

$$\times\sum_{Z=1}^{V}\sum_{i=0}^{\frac{Y-2}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u+\frac{b_4}{D}\right)^{Z}(v-Z)!}\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}$$

$$\left(\frac{\sigma^2}{2}\right)^{i} T^{v-Z-i-w} \frac{\Gamma(w+k)}{\left[\frac{(u+b_4)}{D}\left(D-\frac{u\sigma^2}{2}-\frac{b_4\sigma^2}{2D}\right)-\frac{\sigma^2 b_4^2}{2D^2}-b_4+a\right]^{w+k}}$$

$$+\frac{b_1}{b_4}\frac{u^v a^k}{\Gamma(v)\Gamma(k)} \exp\left[\frac{u^2\sigma^2}{2}T-DuT\right]\left[\sum_{i=0}^{\frac{v}{2}}\sum_{w=0}^{v-i}\binom{v-i}{i}\binom{v-i}{w}\right.$$

$$(D-u\sigma^2)^{v-2i}\left(\frac{\sigma^2}{2}\right)^{i} T^{v-i-w} \frac{\Gamma(w+k)}{(Du-u^2\sigma^2+a)^{w+k}}$$

$$-D\sum_{i=0}^{\frac{v-1}{2}}\sum_{w=0}^{v-1-i}\binom{v-1-i}{i}\binom{v-1-i}{w}(D-u\sigma^2)^{v-1-i}\left(\frac{\sigma^2}{2}\right)^{i} T^{v-i-w}$$

$$\left.\left(\frac{T\,\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}+\frac{\Gamma(w+k+1)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}\right)\right]$$

$$\frac{-\sigma^2 b_1 u^v a^k}{D(\sigma^2 b_4+2D^2)} \frac{\exp\left(\frac{u^2\sigma^2}{2}T-DuT\right)}{\Gamma(v)\Gamma(k)}\sum_{Z=1}^{v}\sum_{i=0}^{\frac{v-Z}{2}}\sum_{w=0}^{v-Z-i}\frac{(v-1)!}{\left(u-\frac{2D}{\sigma^2}\right)^{Z}(v-Z)!}$$

$$\binom{v-Z-i}{i}\binom{v-Z-i}{w}\left(D-u\sigma^2\right)^{v-Z-2i}\left(\frac{\sigma^2}{2}\right)^{i} T^{v-Z-i-w}\frac{\Gamma(w+k)}{\left(Du-\frac{u^2\sigma^2}{2}+a\right)^{w+k}}$$

10.7.7

Hence applying the above integrals the inventory cost when the supply is random, and the lead times are continuous and the cost of a backorder is an exponential function of the length of time of a backorder is

$$C=\left(\frac{R_c+S}{T}\right)+\frac{hc.V}{u}+hc\left(\frac{Dk}{\alpha}+\frac{DT}{2}\right)+hc\left(G_{76}(T)-G_{76}(O)\right)$$

$$+\frac{1}{T}\left(G_{85}(T)-G_{85}(O)\right)+\frac{s}{T}\left(G_{78}(T)-G_{78}(O)\right)$$

10.7.8

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