

## Vacuum Spacetimes of Embedding Class Two

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# Vacuum Spacetimes of Embedding Class Two

**Alan Barnes**

School of Engineering & Applied Science, Aston University, Birmingham B4 7ET, UK

E-mail: [barnesa@aston.ac.uk](mailto:barnesa@aston.ac.uk)

**Abstract.** Doubt is cast on the much quoted results of Yakupov that the torsion vector in embedding class two vacuum space-times is necessarily a gradient vector and that class 2 vacua of Petrov type III do not exist. The first result is equivalent to the fact that the two second fundamental forms associated with the embedding necessarily commute and has been assumed in most later investigations of class 2 vacuum space-times. Yakupov stated the result without proof, but hinted that it followed purely algebraically from his identity:  $R_{ijkl}C^{kl} = 0$  where  $C_{ij}$  is the commutator of the two second fundamental forms of the embedding.

From Yakupov's identity, it is shown that the only class two vacua with non-zero commutator  $C_{ij}$  must necessarily be of Petrov type III or N. Several examples are presented of non-commuting second fundamental forms that satisfy Yakupov's identity and the vacuum condition following from the Gauss equation; both Petrov type N and type III examples occur. Thus it appears unlikely that his results could follow purely algebraically. The results obtained so far do not constitute definite counter-examples to Yakupov's results as the non-commuting examples could turn out to be incompatible with the Codazzi and Ricci embedding equations. This question is currently being investigated.

## 1. Introduction

A spacetime is said to be of *embedding class N* if it can be (locally) isometrically embedded in a flat pseudo-Riemannian manifold of dimension  $4 + N$ , but in no flat manifold of *smaller* dimension. Thus a spacetime is of embedding class 2 if it can be minimally embedded in a flat space of signature  $(+1, -1, -1, -1, e_1, e_2)$  where  $e_i = \pm 1$ . It is well-known that there are no vacuum spacetimes of embedding class 1 and any spacetime is of embedding class 6 or less [1].

The embedding of a spacetime of class two is determined by two second fundamental forms  $\Omega_{ab}$  and  $\Lambda_{ab}$  plus a torsion vector  $t_a$  satisfying

$$\text{Gauss equations : } R^{ab}{}_{cd} = 2e_1\Omega_{[c}^a\Omega_{d]}^b + 2e_2\Lambda_{[c}^a\Lambda_{d]}^b \quad (1)$$

$$\text{Codazzi equations : } \Omega_{[b;c]}^a = -e_2\Lambda^a{}_{[b}t_{c]} \quad (2)$$

$$\Lambda_{[b;c]}^a = e_1\Omega^a{}_{[b}t_{c]} \quad (3)$$

$$\text{Ricci equations : } t_{[a;b]} = -\Omega^c{}_{[a}\Lambda_{b]c} \quad (4)$$

The two normals to the embedded spacetime in the enveloping six-dimensional manifold (and so the two second fundamental forms) are only determined up to a rotation and reflections if  $e_1e_2 = 1$  or up to a boost and reflections if  $e_1e_2 = -1$ . Thus, for  $e_1e_2 = 1$ :

$$\tilde{\Omega} = \pm(\cos\theta\Omega + \sin\theta\Lambda) \quad \tilde{\Lambda} = \pm(-\sin\theta\Omega + \cos\theta\Lambda) \quad (5)$$

whereas, for  $e_1 e_2 = -1$ :

$$\tilde{\Omega} = \pm(\cosh \theta \Omega + \sinh \theta \Lambda) \quad \tilde{\Lambda} = \pm(\sinh \theta \Omega + \cosh \theta \Lambda) \quad (6)$$

Under this change of normals the torsion vector transforms as  $\tilde{t}_a = t_a + \theta_{,a}$ . Thus, if the torsion vector is a gradient, (equivalently, from (4), if  $\Omega$  &  $\Lambda$  commute), the torsion vector may be set to zero and the two Codazzi equations (2, 3) decouple.

Yakupov [2] proved a useful identity for class 2 vacua:

$$R^{ab}{}_{cd} C^{cd} = 0 \quad \text{where} \quad C_{ab} = \Omega^e{}_{[a} \Lambda_{b]e}. \quad (7)$$

Note  $\mathbf{C}$  is the commutator of  $\Omega$  and  $\Lambda$ . Subsequently, he stated *without proof* two results [3]:

- For class 2 vacua, the commutator  $C_{ab}$  must vanish and the torsion vector  $t_a$  is a gradient.
- There are no class 2 vacua of Petrov type III.

For a summary of Yakupov's work in English, see [4]. This summary, and Yakupov himself, give a few hints, but no details, regarding the proof of these two results. As far as one can ascertain the proof was purely algebraic following from the Gauss equations (1), vacuum conditions (8) and the identity (7). Most subsequent work (e.g. [5] & [6]) has assumed  $C_{ab} = 0$  and, in particular, van den Bergh [6] confirmed the non-existence class two vacua of Petrov type III subject to this assumption.

## 2. Embedding Class Two Vacua

Contracting the Gauss equation (1) the vacuum condition becomes:

$$e_1(\Omega_c^a \Omega_b^c - \omega \Omega_b^a) + e_2(\Lambda_c^a \Lambda_b^c - \lambda \Lambda_b^a) = 0 \quad (8)$$

where  $\omega$  and  $\lambda$  are the traces of  $\Omega$  and  $\Lambda$  respectively. Let us call  $\Omega^2 - \omega \Omega$  and  $\Lambda^2 - \lambda \Lambda$  the *Ricci squares* of  $\Omega$  and  $\Lambda$  respectively. Thus, the Ricci squares of  $\Omega$  and  $\Lambda$  differ by at most a sign and so *a fortiori* have the same Segré type. Moreover, from (8), it follows that the Ricci square  $\Omega^2 - \omega \Omega$  and  $\Lambda$  commute and so do  $\Lambda^2 - \lambda \Lambda$  and  $\Omega$ . However, this does not imply that  $\Omega$  and  $\Lambda$  themselves commute. As will be seen below, it is easy to find many examples where the two second fundamental forms  $\Omega$  and  $\Lambda$  satisfy the vacuum conditions, but do not commute.

All of these examples were found to satisfy Yakupov's identity automatically. At first this was surprising as the published proof [2] used the Gauss, Codazzi and Ricci equations plus the Ricci identities. However, it was quickly realised that the identity can be proved using *only* the Gauss equations (1) and vacuum conditions (8). As the first step in his proof, Yakupov derived the following equations from the Codazzi equations (2, 3) using the Ricci identities:

$$\Omega_{a[b;cd]} = \Omega_{[b}^e R_{cd]ea} = 2e_2 t_{[c;d} \Lambda_{b]a} \quad \Lambda_{a[b;cd]} = \Lambda_{[b}^e R_{cd]ea} = -2e_1 t_{[c;d} \Omega_{b]a} \quad (9)$$

However, the second equality in each of these equations (with  $t_{[a;b]}$  replaced by the commutator  $C_{ab}$ ) can be derived by substituting for  $R_{abcd}$  using the Gauss equation (1) in the expressions  $\Omega_{[b}^e R_{cd]ea}$  and  $\Lambda_{[b}^e R_{cd]ea}$ . The rest of the proof uses only the Gauss and vacuum equations and follows identical lines to Yakupov's (with  $t_{[a;b]}$  replaced throughout by the commutator  $C_{ab}$ ). Thus (7) is a *purely algebraic constraint* and Yakupov's results [3] cannot follow purely algebraically; if these potential counter-examples are to be excluded, it is necessary to consider *integrability conditions* derived from the Codazzi and Ricci equations.

From the identity (7) it follows that  $C_{ab}$ , if non-zero, is an eigenbivector of the Riemann tensor with *zero* eigenvalue. This immediately excludes Petrov types II and D. Furthermore Brans [7] proved that Petrov type I vacuum spaces with a zero eigenvalue do not exist. His proof used the Newman-Penrose formalism and made extensive use of the Bianchi identities. Thus, *the only class two vacua with non-zero commutator  $C_{ab}$  must have Petrov type N or III.*

### 3. Construction of Non-Commuting $\Omega$ and $\Lambda$

If the Ricci square of  $\Omega$  (and so that of  $\Lambda$ ) is non-degenerate, then so are  $\Omega$  and  $\Lambda$  and, moreover, they have the same invariant subspace structure and so commute. Thus it is only necessary to consider cases where the Ricci squares are degenerate, but where  $\Omega$  and  $\Lambda$  are not (and cases where they are less degenerate than their Ricci squares). Furthermore  $\Omega$  and  $\Lambda$  must have different invariant subspace structures if they are not to commute. The cases where the Ricci square is degenerate are summarised below.

If a second fundamental form has Segré type  $[2, 1, 1]$  with eigenvalues  $\lambda$ ,  $\mu$  and  $\nu$  respectively, its Ricci square has degenerate Segré type:

$$\begin{aligned} [(1, 1), 1, 1] & \quad \text{if } \mu + \nu = 0 \\ [2, (1, 1)] & \quad \text{if } \lambda = 0 \\ [(1, 1), (1, 1)] & \quad \text{if } \lambda = \mu + \nu = 0 \end{aligned}$$

If it has Segré type  $[3, 1]$  with eigenvalues  $\lambda$  and  $\mu$  respectively, its Ricci square has degenerate Segré type:

$$\begin{aligned} [(2, 1), 1] & \quad \text{if } \lambda + \mu = 0 \\ [(3, 1)] & \quad \text{if } \lambda = 0 \end{aligned}$$

If it has Segré type  $[1, 1, 1, 1]$  with eigenvalues  $\lambda$ ,  $\mu$ ,  $\nu$  and  $\sigma$  respectively, its Ricci square has degenerate Segré type:

$$\begin{aligned} [(1, 1), 1, 1] & \quad \text{if } \nu + \sigma = 0 \\ [1, 1, (1, 1)] & \quad \text{if } \lambda + \mu = 0 \\ [(1, 1), (1, 1)] & \quad \text{if } \lambda + \mu = \nu + \sigma = 0 \end{aligned}$$

If it has Segré type  $[Z, \bar{Z}, 1, 1]$  with eigenvalues  $\lambda$ ,  $\bar{\lambda}$ ,  $\mu$  and  $\nu$  respectively, its Ricci square has degenerate Segré type:

$$\begin{aligned} [(1, 1), 1, 1] & \quad \text{if } \mu + \nu = 0 \\ [Z, \bar{Z}, (1, 1)] & \quad \text{if } \lambda + \bar{\lambda} = 0 \\ [(1, 1), (1, 1)] & \quad \text{if } \lambda + \bar{\lambda} = \mu + \nu = 0 \end{aligned}$$

There are at least 18<sup>1</sup> possible non-commuting combinations of Segré types compatible with the vacuum conditions (8). Most of these lead to Riemann tensors of Petrov type O (flat space) or type I (which are excluded by Brans' theorem). However, examples exist where the Riemann tensor has Petrov type III and N respectively; these are presented in the next two sections.

### 4. A Type III Example

The second fundamental forms  $\Omega$  and  $\Lambda$  both have Segré type  $[211]$  with Ricci squares of type  $[(21)1]$  (or both  $[(11)11]$  with Ricci squares  $[(111)1]$  if  $\beta = 0$ ) and take the form:

$$\Omega_{ab} = \beta \ell_a \ell_b + \lambda (\ell_a n_b + n_a \ell_b + y_a y_b) - \mu x_a x_b \quad \Lambda_{ab} = \beta \ell_a \ell_b + \lambda (\ell_a \tilde{n}_b + \tilde{n}_a \ell_b + y_a y_b) - \mu \tilde{x}_a \tilde{x}_b$$

where  $(\ell^a, n^a, x^a, y^a)$  is a half-null tetrad and  $\tilde{n}^a$  and  $\tilde{x}^a$  are related to  $(\ell^a, n^a, x^a)$  by a null rotation about  $\ell^a$  leaving  $y^a$  fixed:

$$\tilde{x}^a = x^a + \alpha \ell^a \quad \tilde{n}^a = n^a + \alpha x^a + \frac{1}{2} \alpha^2 \ell^a$$

To satisfy (8) the normals to the spacetime must have opposite causal characters (i.e.  $e_1 = -e_2$ ). The commutator is given by  $C_{ab} = 2\alpha(\lambda - \mu)^2 \ell_{[a} x_{b]}$  and the Riemann tensor has the form:

$$R^{ab}{}_{cd} = 4e_1 \alpha \lambda (\lambda - \mu) \left( \alpha (\ell^{[a} x^{b]}) \ell_{[c} x_{d]} - \ell^{[a} y^{b]} \ell_{[c} y_{d]} + \ell^{[a} x^{b]} \ell_{[c} n_{d]} + \ell^{[a} n^{b]} \ell_{[c} x_{d]} - \ell^{[a} y^{b]} x_{[c} y_{d]} - x^{[a} y^{b]} \ell_{[c} y_{d]} \right)$$

The Riemann tensor clearly has Petrov type III with  $\ell^a$  being the repeated principal null vector, since  $R^{ab}{}_{c[d} \ell^c \ell_{e]} = 0$ , but  $R^{ab}{}_{cd} \ell^c \neq 0$ .

<sup>1</sup> The precise number depends on whether degenerate Segré types are counted separately and whether use is made of the freedom (5, 6) in  $\Omega$  and  $\Lambda$  to show certain combinations of different Segré types are, in fact, equivalent

## 5. A Type N Example

The second fundamental forms  $\Omega$  and  $\Lambda$  have Segré types [211] and [(11)11] respectively with Ricci squares of type [(11)(11)] and take the form:

$$\Omega_{ab} = \beta \ell_a \ell_b - \lambda(x_a x_b - y_a y_b) \quad \Lambda_{ab} = -\mu \tilde{x}_a \tilde{x}_b - \nu \tilde{y}_a \tilde{y}_b$$

where  $(\ell^a, n^a, x^a, y^a)$  is a half-null tetrad and  $(\tilde{x}^a, \tilde{y}^a)$  is an orthonormal dyad given by:

$$\tilde{x}^a = \cos \theta x^a + \sin \theta y^a \quad \tilde{y}^a = -\sin \theta x^a + \cos \theta y^a$$

The vacuum conditions (8) are satisfied provided that  $e_1 \lambda^2 = e_2 \mu \nu$ . The commutator is given by  $C_{ab} = 2\lambda(\mu - \nu) \sin 2\theta x_{[a} y_{b]}$ . The Riemann tensor has the form:

$$R^{ab}{}_{cd} = -4e_1 \beta \lambda \left( \ell^{[a} x^{b]} \ell_{[c} x_{d]} - \ell^{[a} y^{b]} \ell_{[c} y_{d]} \right)$$

This clearly has Petrov type N with principal null vector  $\ell^a$  since  $R^{ab}{}_{cd} \ell^c = 0$ .

## 6. Conclusions

The examples presented in the previous two sections are potential counter-examples to Yakupov's result [3] that the commutator of the two second fundamental forms in a vacuum spacetime of embedding class two must vanish. In fact, after allowing for the freedom in the choice of the normals to the embedded spacetime (5, 6), they are essentially the only possible counter-examples.

The results obtained so far do not constitute definite counter-examples to Yakupov's results as they could turn out to be incompatible with the integrability conditions arising from the Codazzi and Ricci embedding equations (2-4). A preliminary investigation of these equations for the type III spacetimes reveals that they must be of Kundt's class [8] with the null vector  $\ell^a$  being twist-free, shear-free and geodesic and having zero expansion (i.e. the Newman-Penrose quantities  $\rho$ ,  $\sigma$  and  $\kappa$  all vanish), in agreement with a theorem of Collinson [9].

Similar techniques to those in §2-3 may be employed to investigate Einstein spaces (i.e.  $R_{ab} = \kappa g_{ab}$ ) of embedding class two. In this case Yakupov's identity takes the form

$$C^{ab}{}_{cd} C^{cd} = \frac{4}{3} \kappa C_{ab} \quad (10)$$

Here, again, it appears that there exist many examples in which the commutator of the second fundamental forms is non-zero contrary to Yakupov's assertion [3]. All Petrov types seem to be possible except types III, N and O which are immediately excluded by (10).

A full investigation of the integrability conditions for both the vacuum and the Einstein space cases is in progress and will be presented elsewhere.

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## References

- [1] Eisenhart L P 1949 *Riemannian Geometry* (Princeton, Princeton Univ. Press) pp 187-200
- [2] Yakupov M Sh 1968 On Einstein spaces of second embedding class *Sov. Phys., Doklady* **13**, 585-7
- [3] Yakupov M Sh 1973 Einstein spaces of class 2 (in Russian) *Grav. Teor. Otnos., Univ. Kazan* **9** 109-14
- [4] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 *Exact Solutions to Einstein's Field Equations (2nd ed.)* (Cambridge UK, Cambridge Univ. Press) chapter 37, pp 598-9
- [5] Hodgkinson D E 1984 Empty space-times of embedding class 2 *Gen. Rel. Grav.* **16** 569-87
- [6] Van den Bergh N 1996 Lorentz- and hyperrotation-invariant classification of symmetric tensors and the embedding class 2 problem *Class. Quantum Grav.* **13** 2817-37
- [7] Brans C H 1975 Some restrictions on algebraically general vacuum metrics *J. Math. Phys.* **16** 1008-10
- [8] Kundt W 1961 The plane-fronted gravitational waves *Z. Phys.* **163** 77-86 (see also chapter 31 of [4]).
- [9] Collinson C D 1962 Empty space-times of embedding class two *J. Math. Phys.* **7** 608-11