APPLICATIONS OF DIFFERENTIAL GEOMETRY TO HIGH SPIN FIELD THEORIES

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THESIS SUMMARY

The main aim of this thesis is to investigate the application of methods of differential geometry to the constraint analysis of relativistic high spin field theories. As a starting point the coordinate dependent descriptions of the Lagrangian and Dirac–Bergmann constraint algorithms are reviewed for general second order systems. These two algorithms are then respectively employed to analyse the constraint structure of the massive spin–1 Proca field from the Lagrangian and Hamiltonian viewpoints.

As an example of a coupled field theoretic system the constraint analysis of the massive Rarita–Schwinger spin–$\frac{3}{2}$ field coupled to an external electromagnetic field is then reviewed in terms of the coordinate dependent Dirac–Bergmann algorithm for first order systems. The standard Velo–Zwanziger and Johnson–Sudarshan inconsistencies that this coupled system seemingly suffers from are then discussed in light of this full constraint analysis and it is found that both these pathologies degenerate to a field–induced loss of degrees of freedom.

A description of the geometrical version of the Dirac–Bergmann algorithm developed by Gotay, Nester and Hinds begins the geometrical examination of high spin field theories. This geometric constraint algorithm is then applied to the free Proca field and to two Proca field couplings; the first of which is the minimal coupling to an external electromagnetic field whilst the second is the coupling to an external symmetric tensor field. The onset of acausality in this latter coupled case is then considered in relation to the geometric constraint algorithm.

KEYWORDS : Differential geometry, Constraint analysis, Relativistic high spin field theories, Geometric constraint algorithm, Acausality

DEDICATION

This thesis is dedicated to my loving parents.

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CHAPTER I

INTRODUCTION

The use of a Lagrangian or Hamiltonian function to describe finite dimensional classical mechanics in the unconstrained case is a standard procedure [1]. In general the Lagrangian L describing such systems depends on n generalized coordinates q_i, n corresponding velocities $\dot{q}_i$ and sometimes also on time. The matrix $W = \left[\frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j} \right]$ has maximal rank n for an unconstrained system and in these instances the Lagrangian is said to be regular. The Lagrangian and Hamiltonian formalisms are closely linked and given a particular Lagrangian the first step in the transition to the Hamiltonian side is to introduce the canonical momenta p_i . In the regular case, where the momenta are independent functions of the velocities, there exists a simple prescription to obtain the Hamiltonian. However many dynamical systems are encountered where W is of a singular nature, that is it is not of maximal rank, and this usually indicates that the system will possess an underlying constraint structure. Furthermore when $|W| = 0$ it is no longer possible to solve the canonical momenta uniquely for all the velocities and the transition to the Hamiltonian formalism is far more complicated in the singular case. It was the desire for a systematic means of investigation of constrained dynamical systems which ultimately resulted in the development of the Lagrangian constraint algorithm and the Dirac–Bergmann algorithm for constrained Lagrangian and Hamiltonian theories respectively.

The main theme of this thesis lies not so much with finite dimensional dynamical systems but more with field theories which are systems with an infinite number of degrees of freedom. By analogy with the finite case the usual starting point for the analysis of a field theory is that of a physically meaningful Lagrangian. Many field theories are described by singular Lagrangians and as before this tends to imply the possibility of the existence of constraints. Typically for high spin field theories with spins greater than $\frac{1}{2}$, it is found that the number of physical degrees of freedom is less than the number of components of the field object used to describe the field. This is readily seen in the case of the massive spin–1 Proca field which is usually described in terms of a four–vector, A_μ say. However, the Proca field is known to have three independent degrees of freedom and in order to eliminate the unwanted degree of freedom a constraint equation is required. The extension of the finite dimensional Lagrangian and Dirac–Bergmann constraint algorithms to the infinite dimensional case is relatively straightforward with only a few field theoretic anomalies appearing en route [2]. One of the main differences that does arise is that in the finite case the constraints are algebraic relations whereas they are generally partial differential equations in the field theoretic instance.

The constraint analysis of a free high spin field theory, that is one where there is no coupling to any other fields, is basically just a complexity problem which usually does not lead to any kind of inconsistency. As an illustration of this the free Proca field will be analysed via both the Lagrangian and Dirac–Bergmann algorithms. On the other hand it has long been known that many types of inconsistency can occur when high spin fields are coupled to themselves or to external fields and consequently there is great interest in studying coupled systems. Some of the most common inconsistencies are :-

i) algebraic inconsistencies between the equations of the analysis in the presence of an external field,

ii) change in the number of physical degrees of freedom,

iii) loss of constraints,

iv) acausal propagation of the physical degrees of freedom,

v) non–positive definiteness of the commutators or anticommutators on quantizing the theory.

More specifically Velo and Zwanziger [3] discovered at the classical level that the massive spin–1 Proca field coupled to an external symmetric tensor field propagated acausal modes for certain values of the external field. This pathology of acausality was also observed, again by Velo and Zwanziger [4], in the case when an external electromagnetic field is coupled to the massive spin-$\frac{3}{2}$ Rarita–Schwinger field. A few years before Velo and Zwanziger's work, Johnson and Sudarshan [5] found on quantizing the theory of this coupled Rarita–Schwinger system that the anticommutators were indefinite.

The Rarita–Schwinger field coupled to an external electromagnetic field is clearly a field theory which seems to suffer from some of the above inconsistencies under certain circumstances. In order to look at this coupled system more carefully it will be analysed using the Dirac–Bergmann algorithm. This coupled Rarita–Schwinger system is an example of a first order field theory and since the constraint algorithms referred to so far in this thesis only apply to second order systems then strictly speaking a first order version of the Dirac–Bergmann algorithm should be employed for this analysis. This issue of a specific first order formulation of the Lagrangian and Dirac–Bergmann constraint algorithms was recently tackled by Scherer [6]. First of all he presented a first order finite dimensional formulation of the constraint algorithms on the Lagrangian and Hamiltonian sides and then he generalized these ideas to cover the infinite dimensional case. The investigation of this coupled Rarita–Schwinger system will be such that it is essentially a detailed review of the constraint analysis of Hasumi, Endo and Kimura [7] carried out in the context of Scherer's first order Dirac–Bergmann algorithm. Since this Dirac–Bergmann algorithm is described in the explicitly time independent case then the external electromagnetic field is taken to be time independent so that a smooth application of the constraint algorithm can be effected.

One of the major difficulties in the analysis of the constraint structure of a coupled high spin field theory is that there is a large number of possible coordinate dependent formulations of the same theory. Some high spin field theories have however been developed which seem to avoid the aforementioned inconsistencies and these theories owe their 'success' in the main to their geometric nature. It therefore seems natural to reformulate the above coordinate based high spin inconsistencies in terms of a coordinate independent geometrical framework. In doing this a deeper insight into the problems associated with coupled high spin field theories will hopefully be gained.

The geometrical formulation of finite dimensional regular classical mechanics is now well documented [8]. In geometrical terms if a differentiable manifold Q represents the configuration space of a dynamical system then the tangent bundle TQ represents velocity phase space and the cotangent bundle T^*Q represents phase space. As a consequence of this Lagrangian and Hamiltonian mechanics are geometrically formulated on TQ and T^*Q respectively. The cotangent bundle T^*Q is an example of the important concept of a symplectic manifold, that is a manifold with a strongly nondegenerate closed 2–form ω defined on it. As already stated T^*Q physically represents phase space whereas the 2–form ω is basically a geometrical generalization of the Poisson bracket of Hamiltonian mechanics. Geometrically Hamilton's equations of motion in the unconstrained case are given by $i(X_H)\omega = dH$ where H is a real–valued function on T^*Q known as the Hamiltonian and X_H is a vector field called the Hamiltonian vector field. The dynamics of the system are determined by solving the geometric Hamilton equations for X_H which equivalently amounts to finding the integral curves of X_H. The strong nondegeneracy of the 2–form ω on T^*Q ensures that the linear map $\flat : T(T^*Q) \rightarrow T^*(T^*Q)$, defined by $\flat(X_H) = i(X_H)\omega$, is an isomorphism. Consequently the geometric Hamilton equations can always be solved uniquely for X_H, that is $X_H = \flat^{-1}(dH)$.

When considering a constrained system from a geometrical viewpoint the 2–form ω on some manifold M is no longer strongly nondegenerate but is instead either weakly nondegenerate or degenerate. If ω is weakly nondegenerate or degenerate then it is said to be presymplectic and correspondingly (M, ω) is then called a presymplectic manifold. An essential first step towards the goal of geometrizing the aforementioned coordinate based inconsistencies would be the development of a geometrical constraint algorithm. Such an initial step was taken when Gotay, Nester and Hinds [9] published a geometrical formulation of the Dirac–Bergmann algorithm which actually generalized and improved on the local coordinate dependent version of this constraint algorithm. Their geometric constraint algorithm gives the necessary and sufficient conditions for the solvability of generalized Hamilton–type equations of the form $i(X)\omega = \alpha$ on some presymplectic manifold (M, ω) where α is a closed 1–form called the Hamiltonian 1–form. The only prerequisite that is needed before this geometric algorithm can be set into action is that there should exist an underlying presymplectic manifold (M, ω). Once under way the geometric constraint algorithm is an iterative process which looks for solutions of a generalized Hamilton equation on successively smaller submanifolds of M with consistency of the solutions maintained at each iteration. Furthermore the geometric algorithm of Gotay, Nester and Hinds is field theory friendly since it is set in an infinite dimensional symplectic geometrical arena. As a consequence of this, this geometrical algorithm is the most practical means of geometrically analysing the constraint structure of a high spin field theory. In fact, as a demonstration of their algorithm, Gotay, Nester and Hinds geometrically investigated the case of the electromagnetic field. At a slightly later date, Gotay and Nester [10] generalized the geometric algorithm and they then geometrically examined the massive spin–1 Proca field with this generalized algorithm.

In order to make geometrical contact with the inconsistencies discussed earlier it will be necessary to geometrically investigate some coupled field theoretic systems. To this end two Proca field couplings will be analysed in this thesis via the Gotay–Nester–Hinds algorithm. Before this is undertaken this geometric algorithm will be applied to the free Proca field so as to gain some experience in using the algorithm. The two Proca field couplings that will be considered are the Proca field minimally coupled to an external electromagnetic field and the Proca field coupled to an external symmetric tensor field. In both these examples it is assumed that the relevant external field is time independent since the geometric constraint algorithm is only applicable to systems which do not display any explicit time dependence. From the coordinate dependent work of Velo and Zwanziger [3] it is known that the electromagnetic coupling leads to a consistent system of equations on the final constraint submanifold. However, as previously mentioned, the coupling of a symmetric tensor field to the Proca field exhibits acausality for certain values of the external field. This symmetric tensor field coupling is therefore of great interest in terms of studying how the acausality pathology is manifested geometrically.

In light of the above discussion, the thesis can conveniently be divided into two parts. The first part is concerned with the coordinate dependent approach to high spin field theories whilst the second part is devoted to the geometrical investigation of these theories. A more detailed description of the contents of each chapter will now be given.

Chapter II provides a brief summary of the mechanics of unconstrained dynamical systems from both the Lagrangian and Hamiltonian viewpoints. In the first instance only finite dimensional systems are considered but these ideas are then extended to the infinite dimensional case, thereby covering field theoretic systems.

The first section of chapter III deals with the analysis of finite dimensional constrained Lagrangian systems. Having examined these types of system a detailed review of the Dirac–Bergmann algorithm for finite dimensional constrained Hamiltonian systems is then given. In the final section of this chapter an indication of how these finite dimensional constraint algorithms can be generalized to the field theoretic case is described. For simplicity the Lagrangian and Dirac–Bergmann constraint algorithms are described for systems which are not explicitly time dependent. In preparing for the later geometrical approach to constrained high spin field theories, the massive spin–1 Proca field is investigated via the coordinate dependent Lagrangian and Dirac–Bergmann constraint algorithms.

The constraint algorithms of chapter III are for quite general second order systems. However numerous first order systems exist and it is the purpose of chapter IV to investigate the constraint structure of such systems. The format of chapter III is maintained in this chapter in that the constraint algorithms are reviewed first of all for finite dimensional Lagrangian and Hamiltonian systems and then these concepts are extended to the infinite dimensional case. As before only systems showing no explicit time dependence are considered. The last part of this chapter is devoted to the examination of the massive spin$-\frac{3}{2}$ Rarita–Schwinger field coupled to an external electromagnetic field via the first order Dirac–Bergmann constraint algorithm. To conclude with the implications of the constraint analysis of this coupled system are then discussed.

Chapter IV marks the end of the coordinate based approach to high spin field theories in this thesis. In chapter V a brief survey of some of the more important ideas of differential geometry is given. The main concepts are introduced under the general headings of differentiable manifolds, the tangent space at a point of a manifold, fibre bundles and differential forms and their properties.

Equipped with the differential geoemtric ideas presented in chapter V, chapter VI begins with a discussion on symplectic forms and symplectic manifolds. These two concepts are initially introduced in the most general of terms and then, more specifically, they are considered from the point of view of dynamical systems. The second section of this chapter covers the dynamically important notions of integral curves and the fibre derivative. Chapter VI is concluded by giving the geometrical version of time independent Hamiltonian mechanics in the regular case. In particular Hamilton's equations of motion and the Poisson bracket are put in a geometrical background.

Before giving an account of the geometrization of the time independent Dirac–Bergmann algorithm, chapter VII opens by introducing the idea of a presymplectic manifold. This is then followed by two slightly modified descriptions of the Gotay–Nester–Hinds geometric algorithm. The first of these descriptions is fairly abstract whilst the second one is more useful when it comes to applying the geometric algorithm to field theoretic examples. In the next section of chapter VII a physically meaningful classification scheme for the submanifolds generated by the geometric algorithm is reviewed. This chapter finishes by applying the more practical version of the geometric algorithm to the case of the massive spin–1 Proca field. By considering this example it is then possible to directly compare the geometric analysis of the Proca field with the corresponding coordinate based Dirac–Bergmann investigations given in chapter III.

The whole of chapter VIII is concerned with the application of the geometric algorithm to coupled field theoretic systems. In particular the cases of the Proca field minimally coupled to an external electromagnetic field and the Proca field coupled to an external symmetric tensor field are geometrically examined in this chapter. Chapter VIII culminates in a propagation analysis on the equations of motion and constraints of this symmetric tensor field coupling.

Chapter IX, the final chapter of this thesis, consists of a concluding discussion along with an indication of the possible directions the research project could follow in order to improve and expand on the work done so far.

The summation convention is assumed throughout this thesis unless it is specifically stated otherwise.

PART 1

THE COORDINATE DEPENDENT APPROACH TO HIGH SPIN FIELD THEORIES

CHAPTER II

A BRIEF REVIEW OF THE DYNAMICS OF UNCONSTRAINED SYSTEMS

There are many good texts covering in some detail the subject of classical mechanics, for example Goldstein [1]. This chapter is designed to set the scene for the forthcoming discussion of constrained dynamical systems, as well as being a precursor for the translation of the coordinate dependent description of dynamics into a geometrical framework.

A Lagrangian and Hamiltonian analysis of finite dimensional unconstrained systems

Consider a dynamical system described by the n independent generalized coordinates $q_1, \ldots, q_n$ and their respective velocities denoted by $\dot{q}_1, \ldots, \dot{q}_n$, where $\dot{q}_i = \frac{dq_i}{dt}$ for $i = 1, \ldots, n$. The dynamics of the system will be assumed to be derivable from an action A given by

$$A = \int_{t_1}^{t_2} L\left(q_i, \dot{q}_i, t\right) dt \qquad i = 1, \ldots, n \tag{2.1}$$

between the times t_1 and t_2 where $L = L\left(q_i, \dot{q}_i, t\right)$ is the Lagrangian function. The usual Hamilton's variational principle applied to (2.1) gives the familiar Euler–Lagrange equations, that is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \ldots, n. \tag{2.2}$$

Since L depends at most on first order derivatives it follows from (2.2) that the Euler–Lagrange equations will at most be of second order.

From (2.2) the equations for the accelerations are found to be

$$\frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j} \ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \, \partial q_j} \dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \, \partial t} - \frac{\partial L}{\partial q_i} = 0 \qquad i, j = 1, \ldots, n \qquad (2.3)$$

or equivalently

$$W_{ij}\left(q, \dot{q}, t\right) \ddot{q}_j = E_i\left(q, \dot{q}, t\right) \qquad i, j = 1, \ldots, n \qquad (2.4)$$

where

$$W_{ij}\left(q, \dot{q}, t\right) = \frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j} \qquad i, j = 1, \ldots, n \qquad (2.5)$$

and

$$E_i\left(q, \dot{q}, t\right) = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \, \partial q_j} \dot{q}_j - \frac{\partial^2 L}{\partial \dot{q}_i \, \partial t} \qquad i, j = 1, \ldots, n. \qquad (2.6)$$

If $|W| \neq 0$, where $W = \left[W_{ij}\right]$, then the Lagrangian is said to be regular or non–singular and it follows that (2.4) can be written as

$$\ddot{q}_i = Y_{ij}\left(q, \dot{q}, t\right) E_j \qquad i, j = 1, \ldots, n \qquad (2.7)$$

where $Y = \left[Y_{ij}\right] = W^{-1}$. Clearly all the equations given by (2.7) are of second order and the motion of the system is determined uniquely for all time by specifying 2n initial conditions, for example the n q_i's and the n $\dot{q}_i$'s at the initial time t_1.

Besides the Lagrangian description there is also the Hamiltonian formulation of dynamics. This formulation differs from the Lagrangian one in that now the motion of the system is described by a set of $2n$ independent first order equations. The usual starting point for making the transition from Lagrangian to Hamiltonian mechanics is to introduce generalized or canonical momenta, p_i, which are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\left(q, \dot{q}, t\right) \qquad i = 1, \ldots, n. \tag{2.8}$$

The quantities (q, p) are known as canonical variables. It should be noted that the space spanned by the q's is known as the configuration space, whereas $\left(q, \dot{q}\right)$ describe velocity phase space and (q, p) describe momentum phase space, which will be referred to merely as phase space. In addition $\left(q, \dot{q}, t\right)$ and (q, p, t) are respectively local coordinates for velocity and momentum state space.

Mathematically the objective in going from the Lagrangian to Hamiltonian formalism is to change the variables in the dynamical functions from $\left(q, \dot{q}, t\right)$ to (q, p, t). This involves trying to uniquely solve the equations defining the canonical momenta, that is (2.8), in terms of the velocities $\dot{q}_i$. This can only be done if the Jacobian J given by

$$J = \frac{\partial(p_i)}{\partial(\dot{q}_j)} \equiv \left\| \left[\frac{\partial p_i}{\partial \dot{q}_j} \right] \right\| = \left\| \left[\frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j} \right] \right\| \neq 0 \qquad i, j = 1, \ldots, n. \tag{2.9}$$

From (2.5) it is seen that this condition is equivalent to $|W| \neq 0$ and so it follows that there is a unique solution $\dot{q}_i = c_i\,(q, p, t)$ for (2.8) only when the Lagrangian is regular. Consequently when the Lagrangian is regular it is possible to switch between $\left(q, \dot{q}, t\right)$ and (q, p, t) in a one to one manner.

Consider now the function H, called the Hamiltonian function, defined by

$$H = p_i \dot{q}_i - L\left(q, \dot{q}, t\right) \qquad i = 1, \ldots, n \tag{2.10}$$

from which it follows that

$$dH = \dot{q}_i \, dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \qquad i = 1, \ldots, n \tag{2.11}$$

after use of (2.8). In the present analysis where the Lagrangian is regular then (2.11) suggests that the Hamiltonian H is a function of the q's, p's and t, that is

$$H = H(q, p, t). \tag{2.12}$$

Taking the total differential of (2.12) leads to

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt \qquad i = 1, \ldots, n \tag{2.13}$$

and since in the regular case the coordinates and momenta are independent then a comparison of (2.11) and (2.13) reveals that

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad i = 1, \ldots, n, \tag{2.14}$$

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, n, \tag{2.15}$$

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \tag{2.16}$$

By substituting (2.8) into (2.2) it is readily seen that

$$\frac{dp_i}{dt} = \dot{p}_i = \frac{\partial L}{\partial q_i} \qquad i = 1, \ldots, n \tag{2.17}$$

and as a consequence of (2.17) equation (2.15) now becomes

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, n. \tag{2.18}$$

Equations (2.14) and (2.18) are known as Hamilton's equations of motion and they represent the aforementioned set of 2n coupled first order differential equations. They will have a unique solution for all time if the initial values at time t_1 of the n q_i's and n p_i's are specified.

From (2.10) and (2.12) it follows that

$$L(q, \dot{q}, t) = p_i \dot{q}_i - H(q, p, t) \qquad i = 1, \ldots, n \qquad (2.19)$$

and it should be noted that Hamilton's equations of motion, like the Euler–Lagrange ones, can also be obtained from an action. In other words varying the action obtained by putting (2.19) into (2.1), that is

$$A = \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t)\right) dt \qquad i = 1, \ldots, n, \qquad (2.20)$$

as a functional of the q_i and p_i leads to Hamilton's equations.

By employing Hamilton's equations as given by (2.14) and (2.18), then the time development of a function $B = B(q, p, t)$ defined on momentum state space is given by

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \{B, H\} \qquad (2.21)$$

where

$$\{B, H\} = \frac{\partial B}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, n \qquad (2.22)$$

is the Poisson bracket of B with H. More generally the Poisson bracket of two differentiable functions, B and C, on momentum state space is defined to be

$$\{B(q, p, t), C(q, p, t)\} = \frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_i} \qquad i = 1, \ldots, n. \tag{2.23}$$

From (2.23) it can be verified that if B, C and D are functions on momentum state space and c_1 and c_2 are constants then the Poisson bracket has the following properties :-

i) It is antisymmetric, that is

$$\{B, C\} = -\{C, B\}. \tag{2.24a}$$

ii) It is linear in that

$$\left\{\left(c_1 B + c_2 C\right), D\right\} = c_1\{B, D\} + c_2\{C, D\}. \tag{2.24b}$$

iii)

$$\left\{c_1, B\right\} = 0. \tag{2.24c}$$

iv) It obeys the product rule

$$\{BC, D\} = B\{C, D\} + \{B, D\}C. \tag{2.24d}$$

v) It satisfies Jacobi's identity, that is

$$\{B, \{C, D\}\} + \{C, \{D, B\}\} + \{D, \{B, C\}\} = 0. \tag{2.24e}$$

In addition to the above there are also the so–called fundamental Poisson brackets of the q's and p's. They satisfy

$$\left\{q_i, q_j\right\} = 0 \qquad i, j = 1, \ldots, n, \tag{2.25}$$

$$\left\{p_i, p_j\right\} = 0 \qquad i, j = 1, \ldots, n, \tag{2.26}$$

$$\left\{q_i, p_j\right\} = \delta_{ij} \qquad i, j = 1, \ldots, n. \tag{2.27}$$

B Analysis of classical unconstrained field theories

Up to this point only regular systems with a finite number of degrees of freedom have been investigated. An outline of how this analysis can be extended so as to incorporate field theories, which are systems with an infinite number of degrees of freedom, will now be considered [2]. This is also a convenient opportunity to introduce some of the notation that will be used in describing any subsequent field theoretic examples. Only four–dimensional space–time will be considered and in terms of coordinates this is given by

$$x = x^{\mu} = \left(x^0, x^1, x^2, x^3\right) \equiv \left(x^0, \underline{x}\right) \qquad (2.28)$$

where $\underline{x} = \left(x^1, x^2, x^3\right)$ is a three–dimensional space vector and x^0 is assumed to play the role of the time parameter. The metric is taken to be

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1) \qquad \mu, \nu = 0, \ldots, 3. \qquad (2.29)$$

It was seen for a Lagrangian system with a finite number of degrees of freedom that the dynamics is described by the coordinates $q_i(t)$ for $i = 1, \ldots, n$. In the infinite dimensional case the dynamics is taken to be described by the N space–time dependent functions $Q_I(x)$ where I is some discrete index running from 1 to N. In essence the discrete label i of the $q_i(t)$ in the finite dimensional case has been replaced by the continuum label $\underline{x}$ and the additional discrete label I in the following manner

$$q_i(t) \equiv q(t, i) \rightarrow q_I(t, \underline{x}) \equiv Q_I(x). \qquad (2.30)$$

A field theoretic Lagrangian L is a functional in the N fields Q_I and their time derivatives which are denoted by $\dot{Q}_I$ where $\dot{Q}_I = \frac{\partial Q_I}{\partial x^0} \equiv \partial_0 Q_I$. If only first derivatives are admitted in the Lagrangian then

$$L = L[Q, \dot{Q}] \tag{2.31}$$

where the square brackets indicate functional dependence and Q stands for the array of N fields Q_I. Locally the Lagrangian given by (2.31) can be expressed in terms of a Lagrangian density $\mathcal{L}$ by

$$L = \int \mathcal{L}\, d^3\underline{x} \tag{2.32}$$

where $\mathcal{L}$ is a function of the Q and the derivatives $\frac{\partial Q}{\partial x^\mu} \equiv \partial_\mu Q$ for $\mu = 0, \ldots, 3$, that is $\mathcal{L} = \mathcal{L}\left(Q, \partial_\mu Q\right)$. The $d^3\underline{x}$ in (2.32) indicates that the integral is over all space.

The field theoretic analogue of the action A given by (2.1) is a functional of the Q and can be expressed as

$$A \equiv A[Q] = \int_{t_1}^{t_2} L\, dt = \int \mathcal{L}\, d^4x. \tag{2.33}$$

Variational considerations in relation to the action A given by (2.33) lead to the Euler–Lagrange field equations

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta \dot{Q}_I}\right) - \frac{\delta L}{\delta Q_I} = 0 \qquad I = 1, \ldots, N \tag{2.34}$$

where $\frac{\delta L}{\delta Q_I}$ is the functional derivative of L with respect to Q_I. Equivalently equation (2.34) can be expressed in terms of the Lagrangian density in which case the Euler–Lagrange field equations become

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu Q_I\right)}\right) - \frac{\partial \mathcal{L}}{\partial Q_I} = 0 \qquad \begin{matrix} \mu = 0, \ldots, 3 \\ I = 1, \ldots, N \end{matrix}. \tag{2.35}$$

In going over to the Hamiltonian formulation the canonical momenta, denoted by $\Pi^I(x)$, are defined to be

$$\Pi^I(x) = \frac{\delta L}{\delta \dot{Q}_I(x)} = \frac{\partial \mathcal{L}}{\partial \dot{Q}_I(x)} \qquad I = 1, \ldots, N \tag{2.36}$$

and the Hamiltonian H is given by

$$H = \int \left(\Pi^I(x)\, \dot{Q}_I(x) \right) d^3\underline{x} - L \qquad I = 1, \ldots, N. \tag{2.37}$$

In a manner similar to (2.32) this Hamiltonian can be written in terms of a Hamiltonian density $\mathcal{H}$, that is

$$H = \int \mathcal{H}\, d^3\underline{x} \tag{2.38}$$

where

$$\mathcal{H} = \Pi^I \dot{Q}_I - \mathcal{L} \qquad I = 1, \ldots, N. \tag{2.39}$$

The field theoretic versions of Hamilton's equations of motion, given by (2.14) and (2.18) in the regular finite case, are respectively

$$\dot{Q}_I(x) = \frac{\delta H}{\delta \Pi^I(x)} \qquad I = 1, \ldots, N \tag{2.40}$$

and

$$\dot{\Pi}^I(x) = -\frac{\delta H}{\delta Q_I(x)} \qquad I = 1, \ldots, N. \tag{2.41}$$

Consider now two phase space functionals, $B[Q, \Pi]$ and $C[Q, \Pi]$, which have no explicit time dependence in that they only depend on time through the Q and Π. The field theoretic Poisson bracket of B and C is defined to be

$$\{B, C\}_{x^0 = y^0} = \int \left(\frac{\delta B}{\delta Q_I(z)} \frac{\delta C}{\delta \Pi^I(z)} - \frac{\delta B}{\delta \Pi^I(z)} \frac{\delta C}{\delta Q_I(z)} \right) d^3\underline{z} \qquad I = 1, \ldots, N \qquad (2.42)$$

which is essentially a generalization of (2.23). It should be noted that the field theoretic Poisson bracket is only defined for equal times, that is $x^0 = y^0$. In terms of (2.42) equations (2.40) and (2.41) can be rewritten as

$$\dot{Q}_I(x) = \left\{ Q_I(x), H(x^0) \right\} \qquad I = 1, \ldots, N \qquad (2.43)$$

and

$$\dot{\Pi}^I(x) = \left\{ \Pi^I(x), H(x^0) \right\} \qquad I = 1, \ldots, N \qquad (2.44)$$

respectively. Furthermore, the field theoretic analogues of the fundamental Poisson brackets given by (2.25), (2.26) and (2.27) in the finite case are respectively

$$\left\{ Q_I(x), Q_J(y) \right\}_{x^0 = y^0} = 0 \qquad I, J = 1, \ldots, N, \qquad (2.45)$$

$$\left\{ \Pi^I(x), \Pi^J(y) \right\}_{x^0 = y^0} = 0 \qquad I, J = 1, \ldots, N, \qquad (2.46)$$

$$\left\{ Q_I(x), \Pi^J(y) \right\}_{x^0 = y^0} = \delta_I^J \, \delta^3(\underline{x} - \underline{y}) \qquad I, J = 1, \ldots, N, \qquad (2.47)$$

where $\delta^3(\underline{x} - \underline{y})$ is the three–dimensional Dirac delta function.

CHAPTER III

ANALYSIS OF DYNAMICAL SYSTEMS WITH CONSTRAINTS

In this chapter a detailed review of the Lagrangian and Dirac–Bergmann constraint algorithms is presented along similar lines to the treatment given by Sudarshan and Mukunda [11]. First of all the constraint analysis of Lagrangian systems whose Lagrangians are not regular is investigated in terms of the Euler–Lagrange equations as equations for the accelerations. Having examined these systems the transition from the Lagrangian to the Hamiltonian formalism in the case when the Lagrangian is again not regular is then considered. In essence the investigation of constrained Lagrangian or Hamiltonian systems is an involved iterative process and as a consequence of this only systems that are not explicitly time dependent will be analysed in order to simplify matters. The constraint analysis will be initially described for finite systems and then an outline of how these ideas can be extended to the field theoretic case will be discussed and illustrated by an example.

A Lagrangian description of finite dimensional constrained systems

In terms of the generalized coordinates q_i and their respective velocities $\dot{q}_i$ for $i = 1, \ldots, n$, encountered in chapter II, the starting point of this analysis is the Lagrangian L given by

$$L = L\left(q_i, \dot{q}_i\right) \qquad i = 1, \ldots, n. \tag{3.1}$$

Initially the q_i and $\dot{q}_i$ are all assumed to be independent of one another. From (2.3), bearing in mind that the system under consideration has no explicit time dependence, the Euler–Lagrange equations containing the accelerations can be written as

$$W_{ij}\left(q, \dot{q}\right) \ddot{q}_j = E_i\left(q, \dot{q}\right) \qquad i, j = 1, \ldots, n \tag{3.2}$$

where

$$W_{ij}(q, \dot{q}) = \frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j} \qquad i, j = 1, \ldots, n \tag{3.3}$$

and

$$E_i(q, \dot{q}) = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \, \partial q_j} \dot{q}_j \qquad i, j = 1, \ldots, n. \tag{3.4}$$

It was seen in chapter II that if $|W| \neq 0$ then the Lagrangian is regular; however in many cases it is found that $|W| = 0$ and this gives rise to the possible presence of constraints. When $|W| = 0$ it is not possible to solve for all the accelerations in the manner of (2.7) and the Lagrangian is said to be singular.

Clearly if $|W| = 0$ then not all the rows of W are independent and if R denotes the rank of W, which is calculated by treating the q_i and $\dot{q}_i$ as independent, then $R < n$. It then follows that W has $(n - R)$ zero eigenvalues and $(n - R)$ corresponding linearly independent left null eigenvectors, λ^a for $a = 1, \ldots, (n - R)$. It should be noted that the λ^a are linearly independent provided that, as previously mentioned, the q_i and $\dot{q}_i$ are treated as independent at this stage of the analysis. The λ^a satisfy the conditions

$$\lambda_i^a(q, \dot{q}) \, W_{ij}(q, \dot{q}) = 0 \qquad \begin{array}{l} a = 1, \ldots, (n - R) \\ i, j = 1, \ldots, n \end{array} \tag{3.5}$$

and as a consequence of this if the Euler–Lagrange equations given by (3.2) are contracted with the λ^a then

$$\lambda_i^a \, W_{ij} \, \ddot{q}_j = \lambda_i^a \, E_i = 0 \qquad \begin{array}{l} a = 1, \ldots, (n - R). \\ i, j = 1, \ldots, n \end{array} \tag{3.6}$$

The conditions

$$\lambda_i^a (q, \dot{q})\ E_i (q, \dot{q}) = 0 \qquad \begin{matrix} a = 1, \ldots, (n - R) \\ i = 1, \ldots, n \end{matrix} \qquad (3.7)$$

represent (n – R) relations between the q_i and $\dot{q}_i$. More generally any relations between the q_i and $\dot{q}_i$ that come out of the analysis are known as constraints in the Lagrangian sense. These constraints are a consequence of the Euler–Lagrange equations of motion and they place restrictions on the choice of the initial values of the q_i and $\dot{q}_i$.

The relations represented by (3.7) can behave in one of three possible ways and these will now be outlined :-

i) The first possibility is that some or maybe all of the relations may in fact be inconsistent. As a simple example to illustrate this, consider the Lagrangian given by

$$L (q, \dot{q}) = \dot{q} - q. \qquad (3.8)$$

Putting (3.8) into the Euler–Lagrange equations given by (3.2) leads to the nonsensical statement that $0 = 1$. From this point henceforth it will be assumed that the Lagrangian of the system under consideration is such that no inconsistencies occur at any stage of the analysis. When this is the case the Lagrangian is said to be admissible.

ii) The second possibility is that the Lagrangian is such that the (n – R) relations given by (3.7) are satisfied identically. In this case there are no real constraints and the motion can only be determined from the Euler–Lagrange equations (3.2). However, since the rank of W is R, which is less than n, then there are only R independent equations in (3.2). (3.2) can thus be used to express R of the accelerations in terms of the remaining (n – R) accelerations, all of the coordinates and all of the velocities. Without any loss of generality it is possible to solve for the first R accelerations and this leads to equations of the form

$$\ddot{q}_k = f_k\left(q_1, \ldots, q_R, \dot{q}_1, \ldots, \dot{q}_R\,;\, q_{R+1}, \ldots, q_n, \dot{q}_{R+1}, \ldots, \dot{q}_n, \ddot{q}_{R+1}, \ldots, \ddot{q}_n\right) \qquad k = 1, \ldots, R. \tag{3.9}$$

(3.9) represents R second order equations which may be solved by choosing an arbitrary set of functions of time for the coordinates $q_{R+1}, \ldots, q_n$ and then specifying a physically meaningful set of initial values at $t = 0$ for the $q_1, \ldots, q_R$ and $\dot{q}_1, \ldots, \dot{q}_R$. The $q_1, \ldots, q_R$ are then uniquely determined at any later time by (3.9).

It should be noted that the appearance of arbitrary functions of time in the general solution of the equations of motion is a characteristic feature of constrained systems. The initial conditions are not sufficient to obtain a unique solution for the system because different choices of the arbitrary functions give rise to different solutions.

iii) The third and most general possibility is that the relations given by (3.7) are neither identically fulfilled nor give rise to any inconsistencies. Remembering that the q_i and $\dot{q}_i$ are still taken to be independent, suppose that of these $(n - R)$ equations K of them are functionally independent, K_1 of them are functionally dependent and K_2 of them are identically satisfied. Clearly $K + K_1 + K_2 = (n - R)$. In order to distinguish these K independent Lagrangian constraints from the others suppose they are written in the form

$$C_r\left(q, \dot{q}\right) = 0 \qquad r = 1, \ldots, K \le (n - R). \tag{3.10}$$

Consider now a 2n–dimensional space, S say, defined by the independent coordinates q_i and the velocities $\dot{q}_i$ for $i = 1, \ldots, n$. The situation in light of the constraints (3.10) is one where the motion of the system is restricted to a $(2n - K)$–dimensional surface V in S. In the determination of the surface V it is important to ensure that functions of constraints, for example $\left(C_1\right)^2$, are not treated as independent constraints. To ensure that this problem does not arise the constraints C_r are chosen such that the $(K \times 2n)$ matrix

$$\begin{bmatrix} \dfrac{\partial C_r}{\partial q_i} & \dfrac{\partial C_r}{\partial \dot{q}_i} \end{bmatrix} \qquad \begin{array}{l} r = 1, \ldots, K \\ i = 1, \ldots, n \end{array} \tag{3.11}$$

has finite elements and is of maximal rank K. It should be noted that the q_i and $\dot{q}_i$ are initially treated as independent when evaluating the derivatives in (3.11) and only after this differentiation are they restricted to V.

The rank of W was originally evaluated in the space S where the q_i and $\dot{q}_i$ are independent. However it is now known that the q_i and $\dot{q}_i$ are not all independent since they satisfy (3.10). In view of this the rank of W must be re–evaluated with the q_i and $\dot{q}_i$ restricted to the surface V. When this is carried out it may be found that the rank of W decreases and consequently it may then be possible to find further left null eigenvectors which in turn may lead to more independent constraints between the q_i and $\dot{q}_i$. This means that the motion will then be restricted to a surface of lower dimension than V. This process may repeat itself until eventually, for a finite system, the rank of W no longer decreases.

The upshot of all this is that the motion is described by (3.2) and is restricted to a $(2n - K')$–dimensional surface V' which is defined by K' independent constraint equations

$$C_r(q, \dot{q}) = 0 \qquad r = 1, \ldots, K' \leq (n - R') \tag{3.12}$$

where R' is the rank of W on the surface V'. As a consequence of (3.12) it follows that

$$\lambda_i(q, \dot{q})\, E_i(q, \dot{q}) = 0 \qquad i = 1, \ldots, n \tag{3.13}$$

is automatically satisfied on V' for every left null eigenvector λ of W on V'.

Up to this point of the analysis only algebraic manipulations have been performed to arrive at the constraints (3.12). It is however a dynamical system that is being investigated and in principle it is possible that the constraints could change with time and this in turn would mean that the number of physical degrees of freedom of the system would also change with time. This would be an unacceptable state of affairs and so to prevent it from happening the constraints must be preserved in time. In other words a constraint valid at some initial time must remain valid for all subsequent times. This involves differentiating the constraints (3.12) with respect to time and this may ultimately lead to further independent constraints and the possibility of further independent equations for the accelerations after some algebraic manipulation.

The constraints arrived at in (3.12) can now, by algebraic manipulations, be split into a maximum number depending on the q_i alone and the remainder depending also on the $\dot{q}_i$ in a non–trivial manner. These constraints will be termed type A and type B respectively. In light of this separation (3.12) can equivalently be written as

$$C^A_s(q) = 0 \qquad s = 1, \ldots, K_1', \tag{3.14a}$$

$$C^B_t(q, \dot{q}) = 0 \qquad t = 1, \ldots, K_2' \tag{3.14b}$$

where it has been assumed that there are K_1' type A and K_2' type B constraints. Clearly $K_1' + K_2' = K'$. Obviously all the type A and B constraints are independent.

Consider now the conditions that the type A constraints are preserved in time. These are given by taking the time derivative of (3.14a) and this leads to

$$\frac{d}{dt}\left(C^A_s(q)\right) = \frac{\partial C^A_s(q)}{\partial q_i}\,\dot{q}_i = 0 \qquad \begin{array}{l} s = 1, \ldots, K_1' . \\ i = 1, \ldots, n \end{array} \tag{3.15}$$

By adding (3.15) to (3.14b) and considering the combined set (3.14) and (3.15) as a whole it may be possible to find further independent type A and B constraints after some algebraic manipulation. If further independent type A constraints are generated in this way then the time derivatives of these must be added to the existing constraint system and the above process must be repeated until after a finite number of repetitions the procedure terminates.

The final situation is then one where there are now K_1'' type A and K_2'' type B independent constraints, that is

$$C_s^A(q) = 0 \qquad s = 1, \ldots, K_1'' , \qquad (3.16a)$$

$$C_t^B(q, \dot{q}) = 0 \qquad t = 1, \ldots, K_2'' . \qquad (3.16b)$$

The time derivatives of the type A constraints (3.16a) automatically vanish as a consequence of (3.16). The constraint equations (3.16) define a surface V'' in S to which the motion is now restricted. In addition since (3.16a) does not generate any new type B constraints it follows that $K_2'' \geq K_1''$.

Now if the constraints (3.16a) are differentiated twice and the constraints (3.16b) differentiated once with respect to time then respectively, the following equations are obtained

$$\frac{d^2}{dt^2}\left(C_s^A(q)\right) = \frac{\partial^2 C_s^A(q)}{\partial q_i \, \partial q_j} \dot{q}_i \dot{q}_j + \frac{\partial C_s^A(q)}{\partial q_i} \ddot{q}_i = 0 \qquad s = 1, \ldots, K_1'', \quad i, j = 1, \ldots, n \qquad (3.17a)$$

$$\frac{d}{dt}\left(C_t^B(q, \dot{q})\right) = \frac{\partial C_t^B(q, \dot{q})}{\partial q_i} \dot{q}_i + \frac{\partial C_t^B(q, \dot{q})}{\partial \dot{q}_i} \ddot{q}_i = 0 \qquad t = 1, \ldots, K_2'' . \quad i = 1, \ldots, n \qquad (3.17b)$$

It is readily apparent that (3.17a) and (3.17b) contain accelerations and some of them may be further equations for the accelerations independent of the Euler–Lagrange equations (3.2). By considering equations (3.17) in conjunction with (3.2) then the system of equations for the accelerations given by

$$W'_{vj}\, \ddot{q}_j = E'_v \qquad \begin{array}{l} v = 1, \ldots, \left(n + K''_1 + K''_2\right) \\ j = 1, \ldots, n \end{array} \tag{3.18}$$

is obtained where $W' = \left[W'_{vj}\right]$ is an $\left(\left(n + K''_1 + K''_2\right) \times n\right)$ matrix given by

$$W' = \begin{bmatrix} W_{ij} \\ \dfrac{\partial C^A_s}{\partial q_j} \\ \dfrac{\partial C^B_t}{\partial \dot{q}_j} \end{bmatrix} \qquad \begin{array}{l} i, j = 1, \ldots, n \\ s = 1, \ldots, K''_1 \\ t = 1, \ldots, K''_2 \end{array} \tag{3.19}$$

and $E' = \left[E'_v\right]$ is an $\left(\left(n + K''_1 + K''_2\right) \times 1\right)$ column vector given by

$$E' = \begin{bmatrix} E_i \\ -\dfrac{\partial^2 C^A_s}{\partial q_i\, \partial q_j}\, \dot{q}_i\, \dot{q}_j \\ -\dfrac{\partial C^B_t}{\partial q_i}\, \dot{q}_i \end{bmatrix} \qquad \begin{array}{l} i, j = 1, \ldots, n \\ s = 1, \ldots, K''_1 \\ t = 1, \ldots, K''_2 \end{array} . \tag{3.20}$$

All the steps outlined previously that were applied to (3.2) must now be gone through again with respect to (3.18). In general W' will not be of maximal rank and further constraints independent of (3.16) may be found when (3.18) is contracted with the left null eigenvectors of W'. These new constraints may change the rank of W' and as before this may lead to more independent constraints and so on until the rank of W' no longer changes. Any new type A constraints could then lead to more type A and B constraints by a single differentiation with respect to time whereas differentiating the newly uncovered type A constraints twice and the type B constraints once with respect to time could lead to new equations for the accelerations and so on and so forth. For an admissible Lagrangian this iterative procedure will eventually terminate leaving the following situation.

There are R''' independent equations for the accelerations given by

$$W''_{pj}(q, \dot{q})\,\ddot{q}_j = E''_p(q, \dot{q}) \qquad \begin{matrix} p = 1, \ldots, R''' \leq n \\ j = 1, \ldots, n. \end{matrix} \tag{3.21}$$

The motion is restricted to a surface V''' in S defined by the K'''_1 independent type A and K'''_2 independent type B constraints given by

$$C^A_s(q) = 0 \qquad s = 1, \ldots, K'''_1, \tag{3.22a}$$

$$C^B_t(q, \dot{q}) = 0 \qquad t = 1, \ldots, K'''_2. \tag{3.22b}$$

The equations

$$\frac{d}{dt}\left(C^A_s(q)\right) = 0 \qquad s = 1, \ldots, K'''_1 \tag{3.23}$$

are algebraically derivable from (3.22) or in other words they are satisfied on V'''. In addition the equations given by

$$\frac{d^2}{dt^2}\left(C^A_s(q)\right) = 0 \qquad s = 1, \ldots, K'''_1, \tag{3.24a}$$

$$\frac{d}{dt}\left(C^B_t(q, \dot{q})\right) = 0 \qquad t = 1, \ldots, K'''_2 \tag{3.24b}$$

are algebraically derivable from (3.21) and (3.22). All this can be summed up by saying that any algebraic or derivative operation on (3.21) and (3.22) does not lead to any new independent constraints or equations for the accelerations. It should also be noted that $K_1''' \leq K_2''' \leq R''' \leq n$.

The constraint analysis is now finally complete and in order to round off the analysis of constrained Lagrangian systems the general form of the solutions to the above equations will now be investigated. Consider first of all the type A constraints given by (3.22a). Since these represent K_1''' independent equations it follows that the $\left(K_1''' \times n\right)$ matrix

$$\left[\frac{\partial C_s^A(q)}{\partial q_i}\right] \qquad \begin{array}{l} s = 1, \ldots, K_1''' \\ i = 1, \ldots, n \end{array} \tag{3.25}$$

has maximal rank K_1''' on V'''. Therefore it is in principle possible to rewrite the type B constraints such that the first K_1''' of them are of the form

$$\dot{q}_i \frac{\partial C_s^A}{\partial q_i} = 0 \qquad \begin{array}{l} s = 1, \ldots, K_1''' . \\ i = 1, \ldots, n \end{array} \tag{3.26}$$

Suppose that the remaining $\left(K_2''' - K_1'''\right)$ type B constraints are denoted by

$$D_u^B\left(q, \dot{q}\right) = 0 \qquad u = 1, \ldots, \left(K_2''' - K_1'''\right). \tag{3.27}$$

The type B constraints given by (3.22b) depend on the velocities in an essential way and so the $\left(K_2''' \times n\right)$ matrix

$$\left[\frac{\partial C_t^B(q,\dot{q})}{\partial \dot{q}_i}\right] \equiv \begin{bmatrix} \dfrac{\partial}{\partial \dot{q}_i}\left(\dot{q}_j \dfrac{\partial C_s^A}{\partial q_j}\right) \\ \dfrac{\partial}{\partial \dot{q}_i}\left(D_u^B\right) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial C_s^A}{\partial q_i} \\ \dfrac{\partial D_u^B}{\partial \dot{q}_i} \end{bmatrix}$$

$$\begin{aligned} s &= 1, \ldots, K_1''' \\ t &= 1, \ldots, K_2''' \\ u &= 1, \ldots, \left(K_2''' - K_1'''\right) \\ i, j &= 1, \ldots, n \end{aligned} \tag{3.28}$$

has maximal rank K_2''' on V'''. With this in mind it is, in principle, possible to arrange things such that the first K_2''' equations for the accelerations are in fact the time derivatives of (3.26) and (3.27), that is

$$\frac{\partial C_s^A}{\partial q_i}\ddot{q}_i + \frac{\partial^2 C_s^A}{\partial q_i \partial q_j}\dot{q}_i \dot{q}_j = 0 \qquad \begin{aligned} s &= 1, \ldots, K_1''' \\ i, j &= 1, \ldots, n \end{aligned} \tag{3.29}$$

and

$$\frac{\partial D_u^B}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial D_u^B}{\partial q_i}\dot{q}_i = 0 \qquad \begin{aligned} u &= 1, \ldots, \left(K_2''' - K_1'''\right) \\ i &= 1, \ldots, n \end{aligned} \tag{3.30}$$

respectively. The remaining $\left(R''' - K_2'''\right)$ equations for the accelerations will now be written as

$$W_{mj}'''(q, \dot{q})\,\ddot{q}_j = E_m'''(q, \dot{q}) \qquad \begin{aligned} m &= 1, \ldots, \left(R''' - K_2'''\right). \\ j &= 1, \ldots, n \end{aligned} \tag{3.31}$$

Thus all the information of the dynamical system is contained in (3.22a), (3.26), (3.27), (3.29), (3.30) and (3.31).

Now the type A constraints (3.22a) can be used to express K_1''' of the generalized coordinates in terms of the remaining $\left(n - K_1'''\right)$ coordinates. Without any loss of generality the type A constraints can be written as

$$q_a = g_a\left(q_\sigma\right) \qquad \begin{array}{l} a = 1, \ldots, K_1''' \\ \sigma = \left(K_1''' + 1\right), \ldots, n \end{array} \tag{3.32}$$

for some functions g_a. By using (3.32) in conjunction with (3.26) and (3.29) it is possible to remove these first K_1''' coordinates from the theory completely. This means that everything in the system is then expressible in terms of the $\left(n - K_1'''\right)$ coordinates q_σ, their velocities $\dot{q}_\sigma$ and their accelerations $\ddot{q}_\sigma$. In this way the type A constraints are essentially eliminated from the theory leaving a system with $\left(n - K_1'''\right)$ coordinates subject to $\left(K_2''' - K_1'''\right)$ type B constraints

$$D_u^B(q, \dot{q}) \equiv D_u^B\left(g_a(q_\sigma), q_\sigma, \dot{g}_a(q_\sigma, \dot{q}_\sigma), \dot{q}_\sigma\right) = F_u^B\left(q_\sigma, \dot{q}_\sigma\right) = 0$$

$$\begin{array}{l} u = 1, \ldots, \left(K_2''' - K_1'''\right) \\ a = 1, \ldots, K_1''' \\ \sigma = \left(K_1''' + 1\right), \ldots, n \end{array} \tag{3.33}$$

and a set of $\left(K_2''' - K_1'''\right) + \left(R''' - K_2'''\right) = \left(R''' - K_1'''\right)$ independent equations for the accelerations $\ddot{q}_\sigma$, that is

$$\frac{\partial F_u^B}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial F_u^B}{\partial q_\sigma}\dot{q}_\sigma = 0 \qquad \begin{array}{l} u = 1, \ldots, \left(K_2''' - K_1'''\right), \\ \sigma = \left(K_1''' + 1\right), \ldots, n \end{array} \tag{3.34a}$$

$$W''''_{m\sigma}\left(q_\rho, \dot{q}_\rho\right)\ddot{q}_\sigma = E''''_m\left(q_\rho, \dot{q}_\rho\right) \qquad \begin{array}{l} m = 1, \ldots, \left(R''' - K_2'''\right) \\ \sigma, \rho = \left(K_1''' + 1\right), \ldots, n \end{array} . \tag{3.34b}$$

It should be noted that (3.33) is merely (3.27) with everything expressed in terms of the q_σ whereas (3.34a) and (3.34b) represent (3.30) and (3.31) respectively with everything written in terms of the q_σ. The situation is now one such that the motion is constrained to a $\left(2n - 2K_1''' - \left(K_2''' - K_1'''\right)\right) = \left(2n - K_1''' - K_2'''\right)$ - dimensional surface in the $2\left(n - K_1'''\right)$ - dimensional space $\left\{q_\sigma, \dot{q}_\sigma\right\}$.

Now the equations of motion given by (3.34a) ensure that if the type B constraints in the form of (3.33) are satisfied at $t = 0$ then they will be satisfied for all subsequent times. From this it follows that the constraint equations (3.33) need only be used in restricting the permitted initial values and only the acceleration equations (3.34) need to be considered in order to determine the motion of the system.

The acceleration equations (3.34) represent $\left(R''' - K_1'''\right)$ equations in terms of the $\left(n - K_1'''\right)$ accelerations $\ddot{q}_\sigma$. Thus equations (3.34) make it possible to express $\left(R''' - K_1'''\right)$ accelerations $\ddot{q}_\alpha$ in terms of the remaining accelerations, $\ddot{q}_A$ say, in the following way

$$\ddot{q}_\alpha = h_\alpha\left(q_\beta, \dot{q}_\beta\, ;\, q_A, \dot{q}_A, \ddot{q}_A\right) \qquad \alpha, \beta = 1, \ldots, \left(R''' - K_1'''\right) \qquad (3.35)$$

where h_α are some functions and the number of q_A is $\left(n - K_1'''\right) - \left(R''' - K_1'''\right) = \left(n - R'''\right)$. In light of this separation of the accelerations it has correspondingly been assumed in (3.35) that the coordinates q_σ and the velocities $\dot{q}_\sigma$ have been separated into q_α's and q_A's and $\dot{q}_\alpha$'s and $\dot{q}_A$'s respectively. Since none of the type B constraints (3.33) can depend on the q_A and $\dot{q}_A$ alone it follows from the form of (3.35) that there are no equations of motion for the q_A. As a consequence of this the q_A remain arbitrary and the overall situation is similar to that described earlier in case ii). The solutions to (3.35) are obtained by choosing arbitrary functions of time, one for each of the q_A, and then by assigning initial values at $t = 0$ for the q_α and $\dot{q}_\alpha$ which are consistent with the type B constraints (3.33). The q_α are then uniquely determined for all time and they will always satisfy (3.33).

As a final comment it should be noted that since the type A constraints can be used to eliminate some of the coordinates, then the most general form of a constrained Lagrangian system is one in which only type B constraints are present.

B Dirac–Bergmann algorithm for finite dimensional constrained Hamiltonian systems

It was seen in chapter II that the transition from the Lagrangian to the Hamiltonian formalism is only straightforward when the Lagrangian is regular. In this case it is possible to uniquely express all the n velocities $\dot{q}_i$ in terms of the canonical momenta p_i , defined by (2.8), and the generalized coordinates q_i . However when the Lagrangian is singular it is no longer possible to solve the equations defining the canonical momenta, that is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) \qquad i = 1, \ldots, n, \tag{3.36}$$

for all the velocities uniquely. It should be noted that (3.36) is the explicitly time independent analogue of (2.8).

Now by using (3.36), equation (3.3) can be written in the form

$$W_{ij}(q, \dot{q}) = \frac{\partial p_i}{\partial \dot{q}_j} \qquad i, j = 1, \ldots, n. \tag{3.37}$$

Suppose for a singular system that the rank of W is R where $R < n$. Then it follows that there are only R independent momenta p_b and it is only possible to solve (3.36) for R of the velocities in terms of the q_i, the p_b and the remaining $(n-R)$ velocities $\dot{q}_B$. In other words

$$\dot{q}_\gamma = f_\gamma\left(q_j, p_b, \dot{q}_B\right) \qquad j = 1, \ldots, n \tag{3.38}$$

where without any loss of generality it may be assumed that γ runs from 1 to R and B from (R + 1) to n. The index b takes on R values out of 1, ... , n. Substituting (3.38) back into (3.36) gives

$$p_i = g_i\left(q_j, \dot{q}_\gamma, \dot{q}_B\right) = g_i\left(q_j, f_\gamma\left(q_j, p_b, \dot{q}_B\right), \dot{q}_B\right)$$

$$= h_i\left(q_j, p_b, \dot{q}_B\right) \qquad , \qquad i = 1, \ldots, n. \tag{3.39}$$

It immediately follows that for those R values that the index b assumes that $h_b \equiv p_b$ whilst the other $(n-R)$ functions of h must be independent of the $\dot{q}_B$ since if this is not true then it would be possible to solve for more of the velocities. In this latter case (3.39) reduces to the conditions

$$p_\varepsilon = h_\varepsilon\left(q_j, p_b\right) \qquad j = 1, \ldots, n \tag{3.40}$$

where ε assumes the $(n-R)$ values out of 1, ..., n that b does not. The $(n-R)$ relations between the q_j and p_b given by (3.40) are called primary constraints because the equations of motion are not used to derive them. The n relations (3.38) and (3.40) are completely equivalent to (3.36) and the functions f_γ and h_ε are determined in a unique way in (3.38) and (3.40).

Now it was seen in chapter II for regular Lagrangians that equations (3.36) mediated the transformation from velocity phase space $\left(q, \dot{q}\right)$ to phase space (q, p) in a one to one manner. Furthermore in the regular case the Hamiltonian H is a function of the q_i

and p_i obtained by substituting the solutions $\dot{q}_i = c_i\,(q, p)$ of (3.36) into the expression

$$H = p_i\,\dot{q}_i - L(q, \dot{q}) \qquad i = 1, \ldots, n. \qquad (3.41)$$

(3.41) is the explicitly time independent version of (2.10). In the singular case where the n q_i, the R p_b and the $(n - R)$ $\dot{q}_B$ are taken to be the 2n independent coordinates the Hamiltonian takes on the uniquely determined form

$$H = p_i\,\dot{q}_i - L(q, \dot{q}) = \widetilde{W}\left(q_i, p_b, \dot{q}_B\right) \qquad i = 1, \ldots, n. \qquad (3.42)$$

In (3.42) whenever the index i takes on one of the values of ε then p_ε has to be replaced by $h_\varepsilon\left(q_j, p_b\right)$ of (3.40) and whenever i assumes one of the values of γ then $\dot{q}_\gamma$ has to be replaced by $f_\gamma\left(q_j, p_b, \dot{q}_B\right)$ of (3.38).

On evaluating the partial derivatives of $\widetilde{W}$ with respect to the independent variables q_i, p_b and $\dot{q}_B$, the following results are obtained :-

$$\frac{\partial \widetilde{W}}{\partial q_i} = \frac{\partial h_\varepsilon}{\partial q_i}\,\dot{q}_\varepsilon + p_\gamma \frac{\partial f_\gamma}{\partial q_i} - \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial \dot{q}_\gamma}\frac{\partial f_\gamma}{\partial q_i}$$

$$= \frac{\partial h_\varepsilon}{\partial q_i}\,\dot{q}_\varepsilon - \frac{\partial L}{\partial q_i} \qquad i = 1, \ldots, n, \qquad (3.43)$$

$$\frac{\partial \widetilde{W}}{\partial p_b} = \dot{q}_b + p_\gamma \frac{\partial f_\gamma}{\partial p_b} - \frac{\partial L}{\partial \dot{q}_\gamma}\frac{\partial f_\gamma}{\partial p_b} + \dot{q}_\varepsilon \frac{\partial h_\varepsilon}{\partial p_b}$$

$$= \dot{q}_b + \dot{q}_\varepsilon \frac{\partial h_\varepsilon}{\partial p_b}, \qquad (3.44)$$

$$\frac{\partial \widetilde{W}}{\partial \dot{q}_B} = p_B + p_\gamma \frac{\partial f_\gamma}{\partial \dot{q}_B} - \frac{\partial L}{\partial \dot{q}_B} - \frac{\partial L}{\partial \dot{q}_\gamma}\frac{\partial f_\gamma}{\partial \dot{q}_B} = 0 \tag{3.45}$$

where (3.43), (3.44) and (3.45) have been simplified by making use of (3.36). From (3.45) it follows that the function $\widetilde{W}$, which was originally expressed in terms of the variables q_i, p_b and $\dot{q}_B$, is in fact not a function of the unsolved velocities $\dot{q}_B$. In light of this (3.42), (3.43) and (3.44) now respectively become

$$p_i \dot{q}_i - L(q, \dot{q}) = \widetilde{W}(q_i, p_b) \qquad i = 1, \dots, n, \tag{3.46}$$

$$-\frac{\partial L}{\partial q_i} = \frac{\partial \widetilde{W}}{\partial q_i} - \frac{\partial h_\varepsilon}{\partial q_i}\dot{q}_\varepsilon \qquad i = 1, \dots, n \tag{3.47}$$

and

$$\dot{q}_b = \frac{\partial \widetilde{W}}{\partial p_b} - \dot{q}_\varepsilon \frac{\partial h_\varepsilon}{\partial p_b} . \tag{3.48}$$

By comparing (3.48) with (3.38) it can be assumed without any loss of generality that like the γ the index b also runs from 1 to R and the index ε, like B, runs from (R + 1) to n. In this way the p_b can be taken to be the p_γ and the $\dot{q}_B$ to be the $\dot{q}_\varepsilon$. Now with this in mind and by making use of the Euler–Lagrange equations in the form of (2.17), that is

$$\frac{\partial L}{\partial q_i} = \dot{p}_i \qquad i = 1, \dots, n, \tag{3.49}$$

then (3.47) and (3.48) can be written in the Hamilton–like form

$$\dot{p}_i = -\frac{\partial \widetilde{W}}{\partial q_i} + \frac{\partial h_\varepsilon}{\partial q_i}\dot{q}_\varepsilon \qquad \begin{matrix} i = 1, \dots, n \\ \varepsilon = (R+1), \dots, n \end{matrix} , \tag{3.50}$$

$$\dot{q}_\gamma = \frac{\partial \widetilde{W}}{\partial p_\gamma} - \frac{\partial h_\varepsilon}{\partial p_\gamma}\dot{q}_\varepsilon \qquad \begin{array}{l}\gamma = 1, \dots, R \\ \varepsilon = (R+1), \dots, n\end{array} \quad . \tag{3.51}$$

(3.50) and (3.51) are analogous to the Hamilton equations of motion for regular systems given by (2.18) and (2.14) respectively. However in (3.50) and (3.51) there are extra terms on the right–hand side which are linear in the unsolved velocities $\dot{q}_\varepsilon$ and furthermore in the singular case there are only (n + R) equations compared to the 2n equations in the regular case.

Equations (3.50) and (3.51) in conjunction with the primary constraints (3.40) are the start of the constraint analysis in Hamiltonian form. At this stage of the analysis there are as many undetermined velocities $\dot{q}_\varepsilon$ as there are primary constraints, that is to say (n – R). These undetermined velocities correspond to the initially undetermined accelerations of the constrained Lagrangian analysis, but in the Hamiltonian case the $\dot{q}_\varepsilon$ must be retained as coordinates of the phase space. Now, in a manner similar to that seen in the Lagrangian case, demanding that the primary constraints are preserved in time could lead to new constraints on the coordinates and momenta as well as new equations relating some of the unsolved velocities to the coordinates and momenta. Eventually the situation is reached where there is a full set of constraints and a set of relations between the coordinates, the momenta and the unsolved velocities such that the time derivatives of the constraints do not lead to any new equations of either sort. At this point only a subset of the original (n – R) undetermined velocities will still be undetermined; the others can all be expressed in terms of the coordinates and the momenta. Those velocities that remain undetermined each give rise to one arbitrary function of time in the solutions of the Hamilton equations in direct analogy to the arbitrary functions corresponding to the undetermined accelerations in the Lagrangian analysis.

From (2.21) the time development of a function defined on phase space, B = B(q, p) say, in the regular case, where no explicit time dependence is assumed, is given by

$$\frac{dB}{dt} = \{B, H\} \tag{3.52}$$

where H is the Hamiltonian of the system. It should be noted that in the regular case the Poisson bracket in (3.52) is defined over all phase space. Unfortunately for singular systems, where constraints are present, the motion does not take place on full phase space (q, p). Instead the motion is restricted to a subspace of full phase space and the function $\widetilde{W}(q_i, p_\gamma)$ is only defined on this subspace. In order to overcome this problem a modification of the major objects of analytical dynamics, such as Poisson brackets and the Hamiltonian, is required. In essence the dynamical object under consideration is evaluated in the first instance as if the q_i and p_i are independent and only at the end of the construction are these variables restricted to the subspace defined by the constraint equations.

Ideally it would be nice to work entirely on the hypersurface defined by the constraints but in practice this is not always convenient. Consequently if the analysis is to be carried out in full phase space, then there must be some means of distinguishing between equations which are only true on the constraint hypersurface and those which are true to some extent off this hypersurface. In this context the concept of equations being true off the constraint hypersurface refers to the fact that the equations hold in some finite shell around the hypersurface. To this end the ideas of weak and strong equations will now be introduced.

Suppose that the constrained hypersurface in phase space is denoted by M and that $B = B(q, p)$ and $C = C(q, p)$ are two phase space functions defined in a finite neighbourhood of M. The values of B and C on M are obtained by replacing the variables p_ε by the functions $h_\varepsilon(q_i, p_\gamma)$ of (3.40). In other words the value of B on M is given by

$$B(q_j, p_j)|M = B(q_j, p_\gamma, h_\varepsilon(q_j, p_\gamma)) \,. \tag{3.53}$$

If after this replacement B and C become equal, that is if B and C are equal on M, then they are said to be weakly equal and this is denoted by

$$B(q_i, p_i) \approx C(q_i, p_i) \qquad i = 1, \ldots, n. \tag{3.54}$$

Consider now the 2n–dimensional gradients of B and C at each point in phase space which will respectively be denoted by ∇B and ∇C. These gradients are given by

$$\nabla B = \left(\frac{\partial B}{\partial q_i}, \frac{\partial B}{\partial p_i}\right) \qquad i = 1, \ldots, n, \tag{3.55a}$$

$$\nabla C = \left(\frac{\partial C}{\partial q_i}, \frac{\partial C}{\partial p_i}\right) \qquad i = 1, \ldots, n. \tag{3.55b}$$

Suppose that ∇B and ∇C are evaluated on M, that is the partial derivatives of these gradients are calculated by treating the q_i and p_i as initially independent and only then restricting them to M. If after this the gradients are equal on M and in addition B and C are equal on M, then B and C are said to be strongly equal. Strong equality between B and C is denoted by

$$B(q_i, p_i) \simeq C(q_i, p_i) \qquad i = 1, \ldots, n \tag{3.56}$$

and can be summarized by $B \simeq C$ if and only if $B \approx C$ and $\nabla B \approx \nabla C$. It goes without saying that since weak and strong equality depend on the underlying constraint hypersurface then the definition of equality will change if the constraint hypersurface changes.

Initially the constraint hypersurface M is defined by the primary constraints (3.40). Since the functions $h_\varepsilon = h_\varepsilon(q_j, p_\gamma)$ are well–defined it follows that the functions

$$\phi_\varepsilon = \phi_\varepsilon(q_j, p_j) = p_\varepsilon - h_\varepsilon(q_j, p_\gamma) \qquad \begin{aligned} j &= 1, \ldots, n \\ \gamma &= 1, \ldots, R \\ \varepsilon &= (R+1), \ldots, n \end{aligned} \tag{3.57}$$

are defined throughout the whole of phase space. The constraint hypersurface M can then be defined in terms of the weak equations

$$\phi_\varepsilon\left(q_j, p_j\right) \approx 0 \qquad\qquad \varepsilon = (R+1), \ldots, n. \tag{3.58}$$

It is apparent from (3.57) that the ϕ_ε do not vanish strongly because $\frac{\partial\phi_\varepsilon}{\partial p_\tau} = \delta_{\varepsilon\tau}$, where τ like ε runs from $(R + 1)$ to n, does not vanish on M.

Consider now the case that B and C are weakly, but not necessarily strongly equal. It is important to know to what extent this equality is effective off the constraint hypersurface. The answer to this lies in the following theorem which, together with its proof [11], are fundamental to the analysis of constraints in Hamiltonian form.

Theorem

If B and C are weakly equal on a constraint hypersurface defined by $\phi_\varepsilon \approx 0$ then the strong equality

$$B - \phi_\varepsilon \frac{\partial B}{\partial p_\varepsilon} \simeq C - \phi_\varepsilon \frac{\partial C}{\partial p_\varepsilon} \tag{3.59}$$

also holds.

Proof

On the constraint hypersurface only the q_j $(j = 1, \ldots, n)$ and the p_γ $(\gamma = 1, \ldots, R)$ are actually independent; the remaining p_ε are given by (3.40). Thus only the differentials dq_j and dp_γ are independent on the constraint hypersurface whereas the dp_ε are determined by

$$dp_\varepsilon = \frac{\partial h_\varepsilon}{\partial q_j}\, dq_j + \frac{\partial h_\varepsilon}{\partial p_\gamma}\, dp_\gamma . \tag{3.60}$$

Consequently on the constraint hypersurface $\frac{\partial B}{\partial p_\varepsilon}$ and $\frac{\partial C}{\partial p_\varepsilon}$ cannot strictly be evaluated. However by replacing the p_ε by the h_ε of (3.40) in B and C and remembering that $B \approx C$ then this leads to the following equations valid on the constraint hypersurface :-

$$\frac{\partial B}{\partial q_j} + \frac{\partial B}{\partial p_\varepsilon}\frac{\partial h_\varepsilon}{\partial q_j} = \frac{\partial C}{\partial q_j} + \frac{\partial C}{\partial p_\varepsilon}\frac{\partial h_\varepsilon}{\partial q_j} \qquad j = 1, \ldots, n, \tag{3.61a}$$

$$\frac{\partial B}{\partial p_\gamma} + \frac{\partial B}{\partial p_\varepsilon}\frac{\partial h_\varepsilon}{\partial p_\gamma} = \frac{\partial C}{\partial p_\gamma} + \frac{\partial C}{\partial p_\varepsilon}\frac{\partial h_\varepsilon}{\partial p_\gamma} \qquad \gamma = 1, \ldots, R. \tag{3.61b}$$

Now from (3.57)

$$\frac{\partial h_\varepsilon}{\partial q_j} = -\frac{\partial \phi_\varepsilon}{\partial q_j} \qquad \begin{matrix} \varepsilon = (R+1), \ldots, n, \\ j = 1, \ldots, n \end{matrix} \tag{3.62a}$$

$$\frac{\partial h_\varepsilon}{\partial p_\gamma} = -\frac{\partial \phi_\varepsilon}{\partial p_\gamma} \qquad \begin{matrix} \varepsilon = (R+1), \ldots, n. \\ \gamma = 1, \ldots, R \end{matrix} \tag{3.62b}$$

By using (3.62) the equations (3.61) can be expressed in terms of the ϕ_ε giving rise to the weak equations

$$\frac{\partial}{\partial q_j}\left(B - \phi_\varepsilon \frac{\partial B}{\partial p_\varepsilon}\right) \approx \frac{\partial}{\partial q_j}\left(C - \phi_\varepsilon \frac{\partial C}{\partial p_\varepsilon}\right) \qquad j = 1, \ldots, n, \tag{3.63a}$$

$$\frac{\partial}{\partial p_\gamma}\left(B - \phi_\varepsilon \frac{\partial B}{\partial p_\varepsilon}\right) \approx \frac{\partial}{\partial p_\gamma}\left(C - \phi_\varepsilon \frac{\partial C}{\partial p_\varepsilon}\right) \qquad \gamma = 1, \ldots, R \tag{3.63b}$$

where the term $\phi_\varepsilon \frac{\partial^2 B}{\partial q_j \, \partial p_\varepsilon}$, which should appear on the left–hand side of (3.63a), along with all similar terms, vanishes because $\phi_\varepsilon \approx 0$ on the constraint hypersurface.

In (3.63b) the range of γ is 1 to R. However if γ is replaced by an index τ lying in the range $(R + 1)$ to n then the weak equality is still maintained because each side of (3.63b) becomes zero due to the fact that $\frac{\partial \phi_\varepsilon}{\partial p_\tau} = \delta_{\varepsilon\tau}$.

In light of this the index γ can be extended to an index j running from 1 to n, that is (3.63b) can be written as

$$\frac{\partial}{\partial p_j}\left(B - \phi_\varepsilon \frac{\partial B}{\partial p_\varepsilon}\right) \approx \frac{\partial}{\partial p_j}\left(C - \phi_\varepsilon \frac{\partial C}{\partial p_\varepsilon}\right) \qquad j = 1, \ldots, n \tag{3.64}$$

where it is understood that (3.64) is trivially true for $j > R$.

Since it was originally assumed that $B \approx C$ then it is apparent that

$$B - \phi_\varepsilon \frac{\partial B}{\partial p_\varepsilon} \approx C - \phi_\varepsilon \frac{\partial C}{\partial p_\varepsilon} \tag{3.65}$$

and since (3.63a) and (3.64) show that the gradients of $\left(B - \phi_\varepsilon \frac{\partial B}{\partial p_\varepsilon}\right)$ and $\left(C - \phi_\varepsilon \frac{\partial C}{\partial p_\varepsilon}\right)$ are weakly equal it immediately follows that (3.59) holds.

QED

An important special case of the above theorem occurs if C is taken to be the zero function. Then if B vanishes weakly it is strongly equal to a linear combination of the weakly vanishing functions ϕ_ε which define the constraint hypersurface. In other words if $B \approx 0$ then

$$B \simeq \phi_\varepsilon \frac{\partial B}{\partial p_\varepsilon} \ . \tag{3.66}$$

The results generated by the above theorem will now be applied to the constrained Hamilton equations given by (3.50) and (3.51). The concept of weak equality can be used to replace the constrained Hamiltonian function $\widetilde{W}(q_j, p_\gamma)$ by any function $W(q_j, p_j)$, defined on full phase space, which is weakly equal to $\widetilde{W}$. Whereas $\widetilde{W}(q_j, p_\gamma)$ is independent of the p_ε there is no need for the new function W to be independent of these off the constraint hypersurface M. From (3.63a) and (3.64) it follows that since $\widetilde{W}$ is independent of the p_ε and $\widetilde{W} \approx W$ that

$$\frac{\partial \widetilde{W}}{\partial q_j} \approx \frac{\partial}{\partial q_j}\left(W - \phi_\varepsilon \frac{\partial W}{\partial p_\varepsilon}\right) \qquad \begin{array}{l} j = 1, \ldots, n \\ \varepsilon = (R+1), \ldots, n \end{array}, \tag{3.67}$$

$$\frac{\partial \widetilde{W}}{\partial p_j} \approx \frac{\partial}{\partial p_j}\left(W - \phi_\varepsilon \frac{\partial W}{\partial p_\varepsilon}\right) \qquad \begin{array}{l} j = 1, \ldots, n \\ \varepsilon = (R+1), \ldots, n \end{array}. \tag{3.68}$$

Now substituting (3.67) and (3.68) into (3.50) and (3.51) and at the same time replacing the h_ε by the ϕ_ε, as given by (3.62), leads to the following weak equations

$$\dot{p}_j \approx -\frac{\partial}{\partial q_j}\left(W - \phi_\varepsilon \frac{\partial W}{\partial p_\varepsilon}\right) - \frac{\partial \phi_\varepsilon}{\partial q_j}\dot{q}_\varepsilon \qquad \begin{array}{l} j = 1, \ldots, n \\ \varepsilon = (R+1), \ldots, n \end{array}, \tag{3.69}$$

$$\dot{q}_\gamma \approx \frac{\partial}{\partial p_\gamma}\left(W - \phi_\varepsilon \frac{\partial W}{\partial p_\varepsilon}\right) + \frac{\partial \phi_\varepsilon}{\partial p_\gamma}\dot{q}_\varepsilon \qquad \begin{array}{l} \gamma = 1, \ldots, R \\ \varepsilon = (R+1), \ldots, n \end{array}. \tag{3.70}$$

Just as was found for (3.63b) the range of the index γ in (3.70) can be extended to an index j which runs from 1 to n without destroying the weak equality. When $j = \tau > R$ both sides of (3.70) simply become $\dot{q}_\tau$ because $\frac{\partial \phi_\varepsilon}{\partial p_\tau} = \delta_{\varepsilon\tau}$. In view of this (3.70) can be rewritten as

$$\dot{q}_j \approx \frac{\partial}{\partial p_j}\left(W - \phi_\varepsilon \frac{\partial W}{\partial p_\varepsilon}\right) + \frac{\partial \phi_\varepsilon}{\partial p_j}\dot{q}_\varepsilon \qquad \begin{array}{l} j = 1, \ldots, n \\ \varepsilon = (R+1), \ldots, n \end{array}. \tag{3.71}$$

Suppose now that a function $H = H(q_j, p_j)$ is introduced where

$$H(q_j, p_j) = W(q_j, p_j) - \phi_\varepsilon \frac{\partial W}{\partial p_\varepsilon}. \tag{3.72}$$

This function H is characterized by the property of being strongly equal to $\widetilde{W}$, that is $H \simeq \widetilde{W}$, but it is otherwise arbitrary. In terms of the H given by (3.72) equations (3.71) and (3.69) can now be written respectively as

$$\dot{q}_j \approx \{q_j, H\} + \{q_j, \phi_\varepsilon\} \dot{q}_\varepsilon \qquad j = 1, \ldots, n, \qquad (3.73)$$

$$\dot{p}_j \approx \{p_j, H\} + \{p_j, \phi_\varepsilon\} \dot{q}_\varepsilon \qquad j = 1, \ldots, n. \qquad (3.74)$$

The { , } in (3.73) and (3.74) denotes the usual Poisson bracket defined over all phase space and in an analogous manner to (2.23) this is given by

$$\{B(q, p), C(q, p)\} = \frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_i} \qquad i = 1, \ldots, n. \qquad (3.75)$$

Equations (3.73) and (3.74) together with the primary constraints (3.58) are equivalent to the original Lagrangian system. The unsolved velocities $\dot{q}_\varepsilon$ are still present in (3.73) and (3.74) but they always appear multiplied by the weakly vanishing functions ϕ_ε . The Poisson brackets of the $\dot{q}_\varepsilon$ with any phase space function are to be regarded as undefined. It should be noted that (3.73) and (3.74) can also be written in the form

$$\dot{q}_j \approx \{q_j, H_P\} \qquad j = 1, \ldots, n, \qquad (3.76)$$

$$\dot{p}_j \approx \{p_j, H_P\} \qquad j = 1, \ldots, n \qquad (3.77)$$

where H_P is the primary Hamiltonian given by

$$H_P = H + \phi_\varepsilon \dot{q}_\varepsilon . \qquad (3.78)$$

The time preservation of the primary constraints (3.58) will now be considered where time differentiation is interpreted as the generalization of (3.52) suggested by (3.73) and (3.74). In other words if $B(q_j, p_j)$ is a function defined on full phase space then

$$\frac{dB}{dt} = \dot{B} \approx \{B, H\} + \{B, \phi_\varepsilon\} \dot{q}_\varepsilon . \qquad (3.79)$$

From (3.79) it then follows that the condition that the primary constraints (3.58) are preserved in time leads to the set of equations

$$\left\{ \phi_\varepsilon, H \right\} + \left\{ \phi_\varepsilon, \phi_\eta \right\} \dot{q}_\eta \approx 0 \qquad \varepsilon, \eta = (R+1), \ldots, n. \qquad (3.80)$$

(3.80) must now be examined for any new constraints or new information on the $\dot{q}_\eta$. The only place the $\dot{q}_\eta$ occur in (3.80) is linearly in the second term and consequently the determination of the $\dot{q}_\eta$ hinges on the nature of the matrix of constraints

$$P = \left[\left\{ \phi_\varepsilon, \phi_\eta \right\} \right]. \qquad (3.81)$$

It was seen at (2.24a) that the Poisson bracket is antisymmetric in its arguments and therefore it follows that the matrix P is also antisymmetric. There are two possibilities to now consider :-

i) The simplest possibility is that P is non–singular on the constraint hypersurface M, that is

$$|P| \neq 0, \qquad (3.82)$$

in which case all of the previously undetermined velocities get determined by (3.80). It should be noted that since a non–singular antisymmetric matrix must be of even order then from the form of P in (3.81) it follows that there must be an even number of primary constraints in this case. Suppose that $\left[(P^{-1})_{\xi\varepsilon} \right]$ denotes the weak inverse of P, that is

$$(P^{-1})_{\xi\varepsilon} \left\{ \phi_\varepsilon, \phi_\eta \right\} \approx \delta_{\xi\eta}, \qquad (3.83)$$

then from (3.83) and (3.80) it can be seen that

$$\dot{q}_\varepsilon \approx -(P^{-1})_{\varepsilon\eta} \left\{ \phi_\eta, H \right\} \qquad \varepsilon, \eta = (R+1), \ldots, n. \qquad (3.84)$$

Substituting (3.84) back into (3.79) leads to the general equation of motion

$$\frac{dB}{dt} \approx \{B, H\} - \left\{B, \phi_\varepsilon\right\} \left(P^{-1}\right)_{\varepsilon\eta} \left\{\phi_\eta, H\right\}. \tag{3.85}$$

(3.85) together with the primary constraints (3.58) completely determine the motion of the system. There are no unsolved velocities and the situation corresponds to the Lagrangian case in which no arbitrary functions of time occur in the solutions of the equations of motion. Initial conditions for the q_j and p_j can be specified arbitrarily at time $t = 0$ provided these conditions satisfy the primary constraints at $t = 0$. The q_j and p_j can then be determined for all subsequent times via the equations of motion (3.85). It should be noted that if the B in (3.85) is replaced by one of the ϕ_ε then (3.85) vanishes weakly showing the the constraints are preserved in time. Overall in this case there are restrictions on the initial conditions but no arbitrariness in the motion.

ii) The second and more general possibility is that the matrix P given by (3.81) is in fact singular on M, that is

$$|P| \approx 0. \tag{3.86}$$

It is now not possible to determine all of the unsolved velocities $\dot{q}_\varepsilon$ from (3.80). Suppose that the rank of P evaluated on M is R_P, in other words

$$\text{rank } P \approx R_P, \tag{3.87}$$

where clearly since there are only $(n - R)$ ϕ_ε then $R_P < (n - R)$. In this case it then follows that P must have $\left(n - R - R_P\right)$ zero eigenvalues and $\left(n - R - R_P\right)$ corresponding linearly independent left null eigenvectors $\lambda^b \left(q_j, p_j\right)$ which satisfy

$$\lambda^b_\varepsilon \left(q_j, p_j\right) \left\{\phi_\varepsilon, \phi_\eta\right\} \approx 0 \qquad b = 1, \ldots, \left(n - R - R_P\right). \tag{3.88}$$

Operating on (3.80) with the λ^b gives rise to

$$\lambda_\varepsilon^b \left\{ \phi_\varepsilon, H \right\} \approx 0 \qquad b = 1, \ldots, \left(n - R - R_P\right) \qquad (3.89)$$

after use of (3.88). Equations (3.89) are further conditions on the q_j and p_j. If (3.89) are already identically satisfied on M then no new constraints are generated. On the other hand if they are not satisfied on M then further constraints exist between the q_j and p_j and the motion of the system becomes restricted to a hypersurface of lower dimensionality, M′ say, than M. Suppose that out of (3.89) there are A constraint equations which are independent amongst themselves and of the ϕ_ε and let these be denoted by the functions $\chi_\theta\left(q_j, p_j\right) \approx 0$ for $\theta = 1$ to A. The new constraint hypersurface M′ is now defined by the $(n - R + A)$ weak equations

$$\phi_\varepsilon \approx 0 \qquad \varepsilon = (R + 1), \ldots, n, \qquad (3.90a)$$

$$\chi_\theta \approx 0 \qquad \theta = 1, \ldots, A \qquad (3.90b)$$

and is of dimensionality $2n-(n - R + A) = (n + R - A)$. It is important to remember that weak and strong equality are always defined relative to the current constraint hypersurface; at this present stage of the analysis the weak equality in (3.90) is therefore defined relative to M′. The constraints $\chi_\theta \approx 0$ have been obtained from the equations of motion and they are known as secondary constraints.

Now the rank of P must be re–computed, in light of the new constraints, on M′. It may be found that it is less than R_P ultimately giving rise to more conditions of the form of (3.89). These may in turn generate more independent secondary constraints which must be added to the existing $\chi_\theta \approx 0$ and as a consequence of this the constraint hypersurface will be further restricted, to M″ say. In a manner similar to that seen in the Lagrangian analysis this will then necessitate a further re–evaluation of the rank of

P. This process of having to re–compute the rank of P will eventually end when no new independent secondary constraints are uncovered.

The situation will then be such that the motion is restricted to a hypersurface M'' defined by (n – R) primary constraints and A' secondary constraints, that is

$$\phi_\varepsilon \approx 0 \qquad \varepsilon = (R+1), \ldots, n, \qquad (3.91a)$$

$$\chi_\theta \approx 0 \qquad \theta = 1, \ldots, A' \qquad (3.91b)$$

on M''. The rank of P will now be R_P' and A' will be such that $A' \le \left(n - R - R_P'\right)$. Additionally for every left null eigenvector λ of P satisfying

$$\lambda_\varepsilon \left\{ \phi_\varepsilon, \phi_\eta \right\} \approx 0, \qquad (3.92)$$

the condition

$$\lambda_\varepsilon \left\{ \phi_\varepsilon, H \right\} \approx 0 \qquad (3.93)$$

is obeyed on M''.

The A' secondary constraints arose from the requirement that the primary constraints be preserved in time. Now the secondary constraints must also be preserved in time. From (3.79) this will require an analysis of the equations

$$\left\{ \phi_\varepsilon, H \right\} + \left\{ \phi_\varepsilon, \phi_\eta \right\} \dot{q}_\eta \approx 0 \qquad \varepsilon = (R+1), \ldots, n, \qquad (3.94a)$$

$$\left\{ \chi_\theta, H \right\} + \left\{ \chi_\theta, \phi_\eta \right\} \dot{q}_\eta \approx 0 \qquad \theta = 1, \ldots, A' \qquad (3.94b)$$

in a manner analogous to the analysis of (3.80). The coefficient matrix of the $\dot{q}_\eta$ is now the $\left(\left(n - R + A'\right) \times (n - R)\right)$ rectangular matrix

$$E = \begin{bmatrix} \{\phi_\varepsilon, \phi_\eta\} \\ \{\chi_\theta, \phi_\eta\} \end{bmatrix} \qquad \begin{matrix} \varepsilon, \eta = (R+1), \ldots, n. \\ \theta = 1, \ldots, A' \end{matrix} \tag{3.95}$$

Each left null eigenvector of the matrix E has the form $(\lambda_\varepsilon, \lambda_\theta)$ and satisfies

$$(\lambda_\varepsilon, \lambda_\theta)\, E = \lambda_\varepsilon \{\phi_\varepsilon, \phi_\eta\} + \lambda_\theta \{\chi_\theta, \phi_\eta\} \approx 0. \tag{3.96}$$

Operating on equations (3.94) with each of the left null eigenvectors $(\lambda_\varepsilon, \lambda_\theta)$ yields a condition on the q_j and p_j of the form

$$\lambda_\varepsilon \{\phi_\varepsilon, H\} + \lambda_\theta \{\chi_\theta, H\} \approx 0 \tag{3.97}$$

after use of (3.96). Of course (3.97) will contain nothing new if all the λ_θ are zero since it will then simply reduce to (3.93). On the other hand if the λ_θ are not all zero then (3.97) is either identically satisfied on M'' or else it implies a new independent tertiary constraint. The time preservation of this new constraint will generate an additional equation containing the $\dot{q}_\varepsilon$ which results in an extension of (3.94). This process of obtaining new equations involving the $\dot{q}_\varepsilon$ and new constraints by considering the time preservation of existing constraints will eventually terminate.

At this point the situation is such that there are $(n - R)$ primary constraints and A'' l–ary constraints where, $l \geq 2$, which define a hypersurface M''', in other words

$$\phi_\varepsilon \approx 0 \qquad \varepsilon = (R+1), \ldots, n, \tag{3.98a}$$

$$\chi_\theta \approx 0 \qquad \theta = 1, \ldots, A'' \tag{3.98b}$$

where the weak equality now refers to M'''. The velocities $\dot{q}_\varepsilon$ satisfy the system of equations

$$\left\{ \phi_\varepsilon, H \right\} + \left\{ \phi_\varepsilon, \phi_\eta \right\} \dot{q}_\eta \approx 0 \qquad \varepsilon = (R+1), \ldots, n, \tag{3.99a}$$

$$\left\{ \chi_\theta, H \right\} + \left\{ \chi_\theta, \phi_\eta \right\} \dot{q}_\eta \approx 0 \qquad \theta = 1, \ldots, A''. \tag{3.99b}$$

In addition for every left null eigenvector $\left(\lambda_\varepsilon', \lambda_\theta'\right)$ of the $\left(\left(n - R + A''\right) \times (n - R)\right)$ matrix E' given by

$$E' = \begin{bmatrix} \left\{ \phi_\varepsilon, \phi_\eta \right\} \\ \left\{ \chi_\theta, \phi_\eta \right\} \end{bmatrix} \qquad \begin{array}{l} \varepsilon, \eta = (R+1), \ldots, n, \\ \theta = 1, \ldots, A'' \end{array} \tag{3.100}$$

then the following condition is satisfied

$$\lambda_\varepsilon' \left\{ \phi_\varepsilon, H \right\} + \lambda_\theta' \left\{ \chi_\theta, H \right\} \approx 0. \tag{3.101}$$

The constraint analysis is now finally over. The ϕ_ε and χ_θ form a complete set of constraints and no more equations for the $\dot{q}_\varepsilon$ can be generated.

The only outstanding task that now remains is to investigate to what extent equations (3.99) actually determine the $\dot{q}_\varepsilon$. The answer to this depends on the rank of the matrix E' in (3.100). Suppose the rank of E' when it is evaluated on M''' is given by

$$\text{rank}\, E' \approx R_{E'} \leq (n - R). \tag{3.102}$$

It therefore follows that equations (3.99) determine precisely $R_{E'}$ linearly independent combinations of the $\dot{q}_\varepsilon$ in terms of the q_i and p_i and this leaves $\left(n - R - R_{E'}\right)$ linear combinations completely arbitrary. It should be noted that if E' is of maximal rank, that is $(n - R)$, then there are no arbitrary linear combinations of the $\dot{q}_\varepsilon$ and in this case it is possible to fix all the $\dot{q}_\varepsilon$ as functions of the q_i and p_i. When E' is not of maximal rank then the problem is one of determining which linear combinations are determined and which are arbitrary.

This problem is approached by introducing the concept of first and second class constraints . This new classification of constraints is applicable to primary and l–ary constraints, where $l \geq 2$. A first class function is defined as one whose Poisson bracket with the complete set of constraints vanishes weakly. All other constraint functions are said to be second class.

The aim is now to divide all the constraint functions ϕ_ε and χ_θ into first and second class functions. This division will first of all be done for the primary constraints $\phi_\varepsilon \approx 0$. Since $R_{E'}$ is the rank of E' then this number represents the maximum number of linearly independent columns of E' and consequently there must be $\left(n - R - R_{E'}\right)$ independent relations between the columns of E'. Suppose such a set of relations is given by

$$\left\{ \phi_\varepsilon, \phi_\eta \right\} \Phi^l_\eta \approx 0 \qquad l = 1, \ldots, \left(n - R - R_{E'}\right), \qquad (3.103a)$$

$$\left\{ \chi_\theta, \phi_\eta \right\} \Phi^l_\eta \approx 0 \qquad l = 1, \ldots, \left(n - R - R_{E'}\right) \qquad (3.103b)$$

where the Φ^l_η are the coefficients in these relations. Due to the algebraic nature of the primary constraints then they could equally well be replaced by any $(n - R)$ linearly independent combinations of themselves without harming the theory. Taking (3.103) as a lead, suppose that the ϕ_ε are replaced by the $\left(n - R - R_{E'}\right)$ independent combinations

$$\phi_l = \Phi^l_\eta \, \phi_\eta \qquad l = 1, \ldots, \left(n - R - R_{E'}\right) \qquad (3.104)$$

along with $R_{E'}$ other independent combinations ϕ_m where m runs from 1 to $R_{E'}$. It then follows from (3.103) that the Poisson bracket of any of the ϕ_l with all other constraints vanishes weakly, that is

$$\left\{ \phi_l, \phi_{l'} \right\} \approx \left\{ \phi_l, \phi_m \right\} \approx \left\{ \phi_l, \chi_\theta \right\} \approx 0 \quad l' = 1, \ldots, \left(n - R - R_{E'}\right) . \quad (3.105)$$

The ϕ_m cannot possess this property for if they did this would imply further relations between the columns of E' in addition to (3.103). The ϕ_l are therefore first class constraints whereas the ϕ_m are second class constraints.

Consider now the constraints $\chi_\theta \approx 0$ which without any loss of generality can be replaced by the new set

$$\chi'_\theta = S_{\theta\kappa}\,\chi_\kappa + T_{\theta l}\,\phi_l + U_{\theta m}\,\phi_m \approx 0 \qquad (3.106)$$

where $S = \left[S_{\theta\kappa}\right]$ is a non–singular matrix and $T = \left[T_{\theta l}\right]$ and $U = \left[U_{\theta m}\right]$ are rectangular matrices. The matrices S, T and U are chosen so that as many possible independent combinations χ_L of the χ_θ which are first class can readily be constructed. This leaves a balance of second class constraints χ_M of the χ_θ. The χ_L satisfy the conditions

$$\left\{\chi_L, \chi_{L'}\right\} \approx \left\{\chi_L, \chi_M\right\} \approx \left\{\chi_L, \phi_l\right\} \approx \left\{\chi_L, \phi_m\right\} \approx 0. \qquad (3.107)$$

It follows from the above that no first class constraints can be constructed from the χ_M and ϕ_m alone.

Equations (3.99) can now be rewritten in terms of the full set of constraints ϕ_l, ϕ_m, χ_L and χ_M. First of all it should be noted that the $\dot{q}_\varepsilon$ always occur in the combination $\phi_\varepsilon\,\dot{q}_\varepsilon$ in (3.99) and this can be re–expressed as

$$\phi_\varepsilon\,\dot{q}_\varepsilon = \phi_l\,Q_l + \phi_m\,Q_m \qquad (3.108)$$

where the Q_l and Q_m are linearly independent combinations of the $\dot{q}_\varepsilon$. In light of this, equations (3.99) now become

$$\{\phi_l, H\} \approx 0, \tag{3.109a}$$
$$\{\chi_L, H\} \approx 0, \tag{3.109b}$$
$$\{\phi_m, H\} + \{\phi_m, \phi_{m'}\} Q_{m'} \approx 0, \tag{3.109c}$$
$$\{\chi_M, H\} + \{\chi_M, \phi_{m'}\} Q_{m'} \approx 0. \tag{3.109d}$$

The Q_l no longer appear in (3.109) because they are combined with first class constraints, ϕ_l, which have vanishing Poisson brackets with all other constraints. Equations (3.109) are completely equivalent to (3.99) and consequently no new equations for the Q_l and Q_m can be generated. It immediately follows that there are as many undetermined combinations of the velocities Q_l as there are primary first class constraints and further, each Q_l appears as an arbitrary function of time in the Hamiltonian equations of motion. The linear combinations of the velocities Q_m are determined by (3.109c) and (3.109d) whereas (3.109a) and (3.109b) merely express properties of the function H and the first class constraints ϕ_l and χ_L.

In order to see how (3.109c) and (3.109d) determine the Q_m consider the square matrix Δ of Poisson brackets of the full set of second class constraints, that is

$$\Delta = \begin{bmatrix} \{\phi_m, \phi_{m'}\} & \{\phi_m, \chi_{M'}\} \\ \{\chi_M, \phi_{m'}\} & \{\chi_M, \chi_{M'}\} \end{bmatrix}. \tag{3.110}$$

Now Δ must be non–singular on the constraint hypersurface M''' for if it was singular then there would be at least one linear relation between the rows or columns of Δ. This in turn would imply the existence of a linear combination of second class constraints having weakly vanishing Poisson brackets with all the second class constraints. From the nature of the definition of a first class object this linear combination of second class constraints would also have weakly vanishing Poisson brackets with the first class constraints and from this it follows that this linear combination of second class constraints is in fact a first class quantity. This statement contradicts the original assumption that the set of first class constraints is maximal and so Δ cannot be singular on M‴. Since Δ is non–singular and antisymmetric then it follows that Δ must also be of even dimension.

If the inverse of Δ is given by

$$G = \Delta^{-1} = \begin{bmatrix} G_{mm'} & G_{mM'} \\ G_{Mm'} & G_{MM'} \end{bmatrix} \tag{3.111}$$

then the following conditions are satisfied

$$G_{mm'} \left\{ \phi_{m'}, \phi_{m''} \right\} + G_{mM'} \left\{ \chi_{M'}, \phi_{m''} \right\} \approx \delta_{mm''}, \tag{3.112a}$$
$$G_{mm'} \left\{ \phi_{m'}, \chi_{M''} \right\} + G_{mM'} \left\{ \chi_{M'}, \chi_{M''} \right\} \approx 0, \tag{3.112b}$$
$$G_{Mm'} \left\{ \phi_{m'}, \phi_{m''} \right\} + G_{MM'} \left\{ \chi_{M'}, \phi_{m''} \right\} \approx 0, \tag{3.112c}$$
$$G_{Mm'} \left\{ \phi_{m'}, \chi_{M''} \right\} + G_{MM'} \left\{ \chi_{M'}, \chi_{M''} \right\} \approx \delta_{MM''} . \tag{3.112d}$$

Combining (3.109c) and (3.109d) and making use of (3.112a) leads to the condition

$$Q_m \approx - G_{mm'} \left\{ \phi_{m'}, H \right\} - G_{mM'} \left\{ \chi_{M'}, H \right\} \tag{3.113}$$

whereas if (3.112c) is used instead of (3.112a) then the condition

$$G_{Mm'} \left\{ \phi_{m'}, H \right\} + G_{MM'} \left\{ \chi_{M'}, H \right\} \approx 0 \tag{3.114}$$

is obtained. (3.114) is a property of H and the combinations of second class constraints $\left(G_{Mm'}\, \phi_{m'} + G_{MM'}\, \chi_{M'} \right)$.

Substituting (3.113) into (3.79), after use of (3.108), gives rise to

$$\frac{dB}{dt} \approx \{B, H\} + \left\{B, \phi_l\right\} Q_l - \left\{B, \phi_m\right\} G_{mm'} \left\{ \phi_{m'}, H \right\}$$
$$- \left\{B, \phi_m\right\} G_{mM'} \left\{ \chi_{M'}, H \right\} . \tag{3.115}$$

Equation (3.115) unfortunately does not exhibit the second class constraint functions ϕ_m and χ_M symmetrically. However the expression

$$- \left\{B, \chi_M\right\} G_{Mm'} \left\{\phi_{m'}, H\right\} - \left\{B, \chi_M\right\} G_{MM'} \left\{\chi_{M'}, H\right\} \qquad (3.116)$$

is seen to vanish weakly in light of (3.114) and so (3.116) can be added to (3.115) to give

$$\begin{aligned} \frac{dB}{dt} \approx \{B, H\} &+ \left\{B, \phi_l\right\} Q_l - \left\{B, \phi_m\right\} G_{mm'} \left\{\phi_{m'}, H\right\} \\ &- \left\{B, \phi_m\right\} G_{mM'} \left\{\chi_{M'}, H\right\} - \left\{B, \chi_M\right\} G_{Mm'} \left\{\phi_{m'}, H\right\} \\ &- \left\{B, \chi_M\right\} G_{MM'} \left\{\chi_{M'}, H\right\} . \qquad (3.117) \end{aligned}$$

Suppose now that μ_v is used to denote the entire set of second class constraints ϕ_m and χ_M where the first $R_{E'}$ of the μ_v are the ϕ_m and the remainder are the χ_M. In addition let the matrix elements of Δ and G be denoted now by $\Delta_{vv'}$ and $G_{vv'}$ respectively. (3.117) can now be written in the more symmetrical form

$$\frac{dB}{dt} \approx \{B, H\} + \left\{B, \phi_l\right\} Q_l - \left\{B, \mu_v\right\} G_{vv'} \left\{\mu_{v'}, H\right\}. \qquad (3.118)$$

The final formalism for a singular Hamiltonian system thus consists of the general equation of motion (3.118), a set of first class constraints given by $\phi_l \approx 0$ and $\chi_L \approx 0$ and a set of combined second class constraints given by $\mu_v \approx 0$. The Hamiltonian $H(q_j, p_j)$ is determined unambiguously in full phase space because it is strongly equal to the known function $\widetilde{W}(q_j, p_\gamma)$. In addition H has the property that its Poisson brackets with the first class constraint functions ϕ_l and χ_L vanish weakly due to (3.109a) and (3.109b). This property of H combined with the way that the second class constraints appear in (3.118) ensures that all the constraints are preserved in time. Thus if the initial conditions on the q_j and p_j are specified at $t = 0$ such that they are consistent with the constraints and the Q_l are specified arbitrarily as functions of time then the equation of motion (3.118) can be solved for the q_j and p_j at any subsequent time. Since the constraints are satisfied for all time then the motion which started on the constraint hypersurface M''' will always remain on M'''. As is to be expected there is a great deal of similarity between this final situation and the one found in the corresponding Lagrangian analysis.

C Analysis of field theories with constraints

Up to this point the constraint algorithms described in sections A and B of this chapter only deal with dynamical systems with a finite number of degrees of freedom. The aim now is to give an indication of how these constraint algorithms, in both the Lagrangian and Hamiltonian cases, can be extended so that they also cover the field theoretic case. In essence the generalization to the field theoretic case is relatively straightforward although there are obviously some important differences which will now be examined [2].

Consider first of all the Lagrangian case. In terms of the Lagrangian density defined by (2.32) the field theoretic Euler–Lagrange equations are given by (2.35). Equations (2.35) can equivalently be written as

$$\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu Q_I)\partial Q_J}\left(\partial_\mu Q_J\right) + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu Q_I)\partial(\partial_\nu Q_J)}\left(\partial_\mu \partial_\nu Q_J\right) - \frac{\partial \mathcal{L}}{\partial Q_I} = 0 \qquad \begin{matrix}\mu, \nu = 0, \ldots, 3. \\ I, J = 1, \ldots, N\end{matrix} \tag{3.119}$$

Suppose that

$$\left(W_{IJ}\right)_{\mu\nu} = \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu Q_I)\partial(\partial_\nu Q_J)}\,, \tag{3.120}$$

then (3.119) becomes

$$(W_{IJ})_{\mu\nu}\left(\partial_\mu\,\partial_\nu\,Q_J\right) = \frac{\partial\mathcal{L}}{\partial Q_I} - \frac{\partial^2\mathcal{L}}{\partial\left(\partial_\mu\,Q_I\right)\partial Q_J}\left(\partial_\mu\,Q_J\right). \qquad (3.121)$$

The field theoretic analogue to (3.2) is given by expressing (3.121) in the form

$$\begin{aligned}(W_{IJ})_{00}\left(\partial_0\,\partial_0\,Q_J\right) &= \frac{\partial\mathcal{L}}{\partial Q_I} - \frac{\partial^2\mathcal{L}}{\partial\left(\partial_\mu\,Q_I\right)\partial Q_J}\left(\partial_\mu\,Q_J\right)\\ &\quad - (W_{IJ})_{0i}\left(\partial_0\,\partial_i\,Q_J\right) - (W_{IJ})_{i0}\left(\partial_i\,\partial_0\,Q_J\right)\\ &\quad - (W_{IJ})_{ij}\left(\partial_i\,\partial_j\,Q_J\right) \qquad i, j = 1, \ldots, 3 \end{aligned} \qquad (3.122)$$

and since x^0 is the time development parameter it is

$$(W_{IJ})_{00} = \frac{\partial^2\mathcal{L}}{\partial\left(\partial_0\,Q_I\right)\partial\left(\partial_0\,Q_J\right)} \qquad (3.123)$$

which determines whether the Lagrangian density, and consequently the Lagrangian, is singular or not. If

$$\left|\left[(W_{IJ})_{00}\right]\right| = 0 \qquad (3.124)$$

then not all the accelerations $\partial_0\,\partial_0\,Q_J$ in (3.122) can be determined and the possibility of constraints arises.

The remainder of the analysis now proceeds in a manner similar to that detailed in section A of this chapter. However spatial derivatives, that is derivatives with respect to the x^i where $i = 1$ to 3, will now occur in the constraints due to the nature of some of the terms on the right–hand side of (3.122). As a consequence of this, field theoretic constraints are in general no longer algebraic relations but differential equations instead.

It can be seen from the definition of the field theoretic canonical momenta $\Pi^I(x)$ given by (2.36) that

$$\frac{\partial \Pi^I}{\partial(\partial_0 Q_J)} = (W_{IJ})_{00} \qquad (3.125)$$

after comparison with (3.123). Consequently if

$$\left\| \left[\frac{\partial \Pi^I}{\partial(\partial_0 Q_J)} \right] \right\| = 0, \qquad (3.126)$$

in other words if (3.124) holds, then the system will once again be singular. In an analogous way to the finite Hamiltonian case, (3.126) indicates the presence of primary constraints and these may ultimately be deduced from (2.36). The primary constraints, along with any l–ary Hamiltonian constraints for $l \geq 2$ that may be uncovered, will in general in the field theoretic case be functionals of the variables Q_I and Π^I and their spatial derivatives. That is to say, suppose the primary constraints are denoted by $\phi^\varepsilon \approx 0$ then

$$\phi^\varepsilon = \phi^\varepsilon \left[Q, \Pi, (\partial_i Q), (\partial_i \Pi) \right] \qquad i = 1, \ldots, 3. \qquad (3.127)$$

Although the primary constraints in this instance are labelled by the finite index ε there are in fact an infinite number of them; that is one for each ε and each space point. In view of this sums that occurred in the finite dimensional analysis become integrals in the field theoretic analysis. In order to illustrate this last statement consider the primary Hamiltonian given by (3.78) in the finite case. In going to the infinite dimensional case the primary Hamiltonian H_P is given by

$$H_P = H + \int \left(u_\varepsilon(x)\, \phi^\varepsilon(x) \right) d^3\underline{x} \qquad (3.128)$$

where the $u_\varepsilon(x)$ are Lagrange multiplier functions. By analogy with (2.38) equation (3.128) can also be written as

$$H_P = \int \mathcal{H}_P \, d^3\underline{x} \qquad (3.129)$$

where $\mathcal{H}_P$ is the primary Hamiltonian density. A comparison of (3.78) and (3.128) indicates that the multipliers $u_\varepsilon(x)$ are now playing the role of the velocities $\dot{q}_\varepsilon$.

The field theoretic analogue of (3.79) for determining the time development of a functional B of phase space variables and their spatial derivatives is

$$\frac{\partial B}{\partial x^0} = \partial_0 B \approx \{B, H\} + \int \left(u_\varepsilon(y) \left\{ B(x), \phi^\varepsilon(y) \right\} \right) d^3\underline{y} \qquad (3.130)$$

where the field theoretic Poisson bracket in (3.130) is defined by (2.42). Equation (3.130) can equivalently be written in terms of the primary Hamiltonian H_P of (3.128) by

$$\partial_0 B \approx \left\{ B, H_P \right\}. \qquad (3.131)$$

From (3.130) it follows that the condition for the field theoretic primary constraints $\phi^\varepsilon \approx 0$ to be preserved in time is given by

$$0 \approx \left\{ \phi^\varepsilon(x), H \right\} + \int \left(u_\eta(y) \left\{ \phi^\varepsilon(x), \phi^\eta(y) \right\} \right) d^3\underline{y} \qquad (3.132)$$

where η assumes the same values as the index ε. By analogy with the finite case it is the nature of the matrix

$$\left[P_{\varepsilon\eta}\,(\underline{x}, \underline{y}) \right] = \left[\left\{ \phi^\varepsilon(x), \phi^\eta(y) \right\}_{x^0 = y^0} \right] \qquad (3.133)$$

in (3.132), which is continuous in x and y but discrete in ε and η, that now governs the determination of the multipliers u_η.

Now if the determinant of (3.133) does not vanish then it possesses an inverse $\left[P^{-1}_{\eta\xi}(y, z)\right]$ which must satisfy the conditions

$$\int \left(P_{\varepsilon\eta}(\underline{x}, \underline{y})\, P^{-1}_{\eta\xi}(y, z)\right) d^3\underline{y} = \int \left(P^{-1}_{\varepsilon\eta}(x, y)\ \ P_{\eta\xi}(\underline{y}, \underline{z})\right) d^3\underline{y}$$

$$= \delta_{\varepsilon\xi}\, \delta^3(\underline{x} - \underline{z}). \tag{3.134}$$

However this inverse, $\left[P^{-1}_{\eta\xi}(y, z)\right]$, is not necessarily unique since

$$\left[P^{-1}_{\eta\xi}(y, z) + \tilde{P}^{-1}_{\eta\xi}(y, z)\right] \tag{3.135}$$

would also be an inverse of (3.133) if $\left[\tilde{P}^{-1}_{\eta\xi}(y, z)\right]$ satisfied the condition

$$\int \left(P_{\varepsilon\eta}\ (\underline{x}, \underline{y})\ \ \tilde{P}^{-1}_{\eta\xi}(y, z)\right) d^3\underline{y} = 0. \tag{3.136}$$

This non–uniqueness in the inverse of (3.133) is a purely field theoretic phenomenon which comes about because the inverse of (3.133) must also be an inverse in the continuous labels x and y. There is no counterpart of this in the finite dimensional analysis.

If on the other hand the determinant of (3.133) vanishes, then by analogy with (3.88) in the finite case, the algorithm prescribes looking for eigenvectors λ^b of (3.133) with zero eigenvalues which satisfy

$$\int \left(\lambda_{\varepsilon}^{b}(x) \ P_{\varepsilon\eta}\,(\underline{x}, \underline{y})\right) d^3\underline{x} \ = \ 0. \tag{3.137}$$

However once again in the field theoretic case (3.137) is not sufficient by itself to determine the $\lambda_{\varepsilon}^{b}$ uniquely.

Suppose that the situation has now been reached where all the independent primary and l–ary constraints for $l \geq 2$ have been found and the task of finding the maximal number of first class constraints is being considered. It was seen in the finite case that the first class constraints could be found by algebraic manipulations. Unfortunately the situation is not as simple in the field theoretic case because now the constraints are differential equations and it is possible that a linear combination of constraints and their spatial derivatives could become a first class constraint.

Having succeeded in separating the constraints into a maximal number of first and second class constraints then the second class constraints can be used to define the matrix

$$\Delta \ = \ \left[\Delta_{\sigma\tau}\,(\underline{x}, \underline{y})\right] \ = \ \left[\left\{\mu^{\sigma}(x), \mu^{\tau}(y)\right\}\right] \tag{3.138}$$

where μ denotes the second class constraints. The Δ matrix in (3.138) is the analogue of the matrix given by (3.110) in the finite dimensional analysis. Unlike the finite case there is now no reason for Δ to be non–singular and as already seen even if it was non–singular its inverse will not necessarily be unique. Consequently in the field theoretic case it no longer means anything to say that the number of second class constraints is even.

In conjunction with the above field theoretic peculiarities a truly rigourous treatment of the field theoretic case requires a careful consideration of spatial boundary conditions as done, for example, in the paper by Steinhardt [12]. This comes about because the constraints found in the analysis are differential equations which according to (3.127) can depend on spatial derivatives. However in order to give a clear exposition of the constraint analysis in any subsequent field theoretic calculations this boundary condition problem will not be examined in any great detail.

So far the constraint algorithms have been described in only the most general of terms and now seems to be an ideal opportunity to demonstrate the practical application of these algorithms. To this end the specific example of the massive spin–1 Proca field will be considered. This field theoretic example will be analyzed in the first instance via the Lagrangian constraint algorithm and then via the Dirac–Bergmann constraint algorithm.

A Lagrangian L for the spin–1 Proca field A_μ is given by

$$L = \int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \right) d^3\underline{x} \qquad \mu, \nu = 0, \ldots, 3 \qquad (3.139)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (3.140)$$

and the metric convention (2.29) has been adopted. A comparison of (3.139) with (2.32) indicates that the Lagrangian density $\mathcal{L}$ in this example is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \qquad (3.141)$$

and this may equivalently be written as

$$\mathcal{L} = -\frac{1}{4} g^{\mu\gamma} g^{\nu\sigma} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\gamma A_\sigma - \partial_\sigma A_\gamma\right) + \frac{m^2}{2} g^{\mu\gamma} A_\mu A_\gamma$$

$$\mu, \nu, \gamma, \sigma = 0, \ldots, 3. \qquad (3.142)$$

Now from (3.142) it is found that

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\tau A_\beta\right)} = F^{\beta\tau} = \partial^\beta A^\tau - \partial^\tau A^\beta \qquad \beta, \tau = 0, \ldots, 3 \qquad (3.143)$$

and consequently

$$\frac{\partial^2 \mathcal{L}}{\partial\left(\partial_\lambda A_\alpha\right)\partial\left(\partial_\tau A_\beta\right)} = g^{\tau\alpha} g^{\beta\lambda} - g^{\alpha\beta} g^{\lambda\tau} \qquad \alpha, \beta, \lambda, \tau = 0, \ldots, 3. \qquad (3.144)$$

From a comparison of (3.144) and (3.120) it immediately follows that

$$\left(W_{\alpha\beta}\right)_{\lambda\tau} = g^{\tau\alpha} g^{\beta\lambda} - g^{\alpha\beta} g^{\lambda\tau} \qquad (3.145)$$

since in (3.144) it is the fields A_α which represent the Q_I of (3.120). Therefore

$$\left(W_{\alpha\beta}\right)_{00} = g^{0\alpha} g^{\beta 0} - g^{\alpha\beta} \qquad (3.146)$$

and in matrix form (3.146) is given by

$$\left[\left(W_{\alpha\beta}\right)_{00}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (3.147)$$

Clearly from (3.147) $\left|\left[\left(W_{\alpha\beta}\right)_{00}\right]\right| = 0$ and the rank of $\left[\left(W_{\alpha\beta}\right)_{00}\right]$ is 3. This signals the presence of constraints in the system and also indicates that not all of the accelerations are determined at this stage.

From (2.35) the Euler–Lagrange equations are given by

$$\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\lambda A_\alpha \right)} \right) - \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0 \quad . \tag{3.148}$$

and after making use of (3.143) and the fact that

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = m^2 A^\alpha, \tag{3.149}$$

the Euler–Lagrange equations (3.148) become

$$\partial_\lambda \partial^\alpha A^\lambda - \partial_\lambda \partial^\lambda A^\alpha - m^2 A^\alpha = 0 \qquad \alpha, \lambda = 0, \ldots, 3. \tag{3.150}$$

Now when α takes on the values $i = 1, \ldots, 3$ then (3.150) becomes

$$\partial_0 \partial^i A^0 + \partial_j \partial^i A^j - \partial_0 \partial^0 A^i - \partial_j \partial^j A^i - m^2 A^i = 0$$
$$i, j = 1, \ldots, 3. \tag{3.151}$$

Equations (3.151) contain the accelerations for each of the A^i, that is $\partial_0 \partial^0 A^i$, and so they represent equations of motion for the A^i.

On the other hand, when $\alpha = 0$ then (3.150) leads to the condition

$$\partial_i \partial^0 A^i - \partial_i \partial^i A^0 - m^2 A^0 = 0 \qquad i = 1, \ldots, 3. \tag{3.152}$$

Equation (3.152) does not contain any acceleration–type terms and so is in fact a constraint; it could be termed a Lagrangian primary constraint. It should also be noted that this constraint is a partial differential equation.

The constraint given by (3.152) could equally well have been obtained, in a manner more in keeping with the previous description of the Lagrangian constraint algorithm, if both sides of

$$
\begin{aligned}
(W_{\alpha\beta})_{00}\left(\partial_0\,\partial_0\,A_\beta\right) = \frac{\partial \mathcal{L}}{\partial A_\alpha} - \frac{\partial^2 \mathcal{L}}{\partial\left(\partial_\lambda\,A_\alpha\right)\partial A_\beta}\left(\partial_\lambda\,A_\beta\right) \\
- (W_{\alpha\beta})_{0i}\left(\partial_0\,\partial_i\,A_\beta\right) - (W_{\alpha\beta})_{i0}\left(\partial_i\,\partial_0\,A_\beta\right) \\
- (W_{\alpha\beta})_{ij}\left(\partial_i\,\partial_j\,A_\beta\right) \qquad (3.153)
\end{aligned}
$$

had been contracted with the components of $(1\ 0\ 0\ 0)$, which is a left null eigenvector of (3.147). Equation (3.153) is merely (3.122) rewritten for the case under consideration.

Since the elements of (3.147) are all constants it follows that its rank will not change in light of the constraint (3.152). Consequently the next step of the algorithm is to consider the time preservation of (3.152). The time preservation condition is found by differentiating (3.152) with respect to time which after some minor rearrangement leads to the condition

$$\partial_i\,\partial_0\,\partial^0\,A^i - \partial_0\,\partial_i\,\partial^i\,A^0 - m^2\,\partial_0\,A^0 = 0 \qquad i = 1, \ldots, 3. \qquad (3.154)$$

(3.154) clearly involves acceleration terms in the form $\partial_0\,\partial^0\left(\partial_i\,A^i\right)$ and so (3.154) must now be considered in conjunction with (3.151) as well as, if necessary, the constraint (3.152) to see if it leads to any new constraints or equations of motion. By substituting $\partial_0\,\partial^0\,A^i$ from (3.151) into (3.154) it is found that

$$m^2\left(\partial_i\,A^i + \partial_0\,A^0\right) = 0 \qquad i = 1, \ldots, 3 \qquad (3.155)$$

and since $m \neq 0$ it immediately follows that

$$\partial_i\,A^i + \partial_0\,A^0 = 0 \qquad i = 1, \ldots, 3. \qquad (3.156)$$

(3.156) is known as the Lorentz condition and it can also be written in the form

$$\partial_0\,A_0 - \partial_i\,A_i = 0 \qquad i = 1, \ldots, 3. \qquad (3.157)$$

Equation (3.156) is a new constraint which may be termed a Lagrangian secondary constraint. The requirement that this new constraint is preserved in time leads to the condition

$$\partial_0 \partial^0 A^0 + \partial^0 \partial_i A^i = 0 \qquad i = 1, \ldots, 3. \tag{3.158}$$

Now if (3.158) is substituted into (3.152) the equation

$$\partial_0 \partial^0 A^0 + \partial_i \partial^i A^0 + m^2 A^0 = 0 \qquad i = 1, \ldots, 3 \tag{3.159}$$

is obtained. (3.159) is an equation of motion for A^0 because it contains the acceleration of A^0, namely $\partial_0 \partial^0 A^0$. Equations (3.151) and (3.159) together determine all the accelerations of the field A_μ and so the Lagrangian constraint analysis is complete.

In passing it should be noted that if the constraint (3.156) is substituted into (3.151) then the equations of motion for the A^i simplify down to

$$\partial_0 \partial^0 A^i + \partial_j \partial^j A^i + m^2 A^i = 0 \qquad i, j = 1, \ldots, 3. \tag{3.160}$$

The constraint analysis of the Proca field will now be investigated via the Dirac–Bergmann algorithm and the results obtained will be compared with those of the Lagrangian constraint algorithm. The momenta conjugate to A_μ are according to (2.36) given by

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_\mu\right)} \qquad \mu = 0, \ldots, 3 \tag{3.161}$$

and after using (3.143) these momenta become

$$\Pi^{\mu} = F^{\mu 0} \qquad \mu = 0, \ldots, 3. \qquad (3.162)$$

Clearly from (3.162) when $\mu = i$, where i runs from 1 to 3, then

$$\Pi^{i} = F^{i0} = \partial^{i} A^{0} - \partial^{0} A^{i} = \partial_{0} A_{i} - \partial_{i} A_{0} \qquad i = 1, \ldots, 3 \qquad (3.163)$$

whereas when $\mu = 0$ then

$$\Pi^{0} = F^{00} = 0. \qquad (3.164)$$

Equation (3.163) contains the velocities $(\partial_0 A_i)$, whilst (3.164) expresses the fact that $(\partial_0 A_0)$ does not occur in the Lagrangian density given by (3.142). (3.164) is consequently a primary constraint which can be expressed in the form

$$\phi^{0} \equiv \Pi^{0} \approx 0. \qquad (3.165)$$

In reality there are an infinite number of these primary constraints; namely one for each space point.

A space–time decomposition of the Lagrangian density given by (3.141) leads to the equation

$$\mathcal{L} = \frac{1}{2} F^{i0} F^{i0} - \frac{1}{4} F_{ij} F_{ij} + \frac{m^2}{2} A_0 A_0 - \frac{m^2}{2} A_i A_i \qquad i, j = 1, \ldots, 3 \qquad (3.166)$$

and after substituting (3.163) into (3.166) this gives rise to

$$\mathcal{L} = \frac{1}{2} \Pi^{i} \Pi^{i} - \frac{1}{4} F_{ij} F_{ij} + \frac{m^2}{2} A_0 A_0 - \frac{m^2}{2} A_i A_i \qquad i, j = 1, \ldots, 3. \qquad (3.167)$$

From (2.39) the Hamiltonian density is defined to be

$$\mathcal{H} = \Pi^{\mu}\left(\partial_0 A_{\mu}\right) - \mathcal{L} \qquad \mu = 0, \ldots, 3 \tag{3.168}$$

and after use of (3.167) this simplifies to

$$\mathcal{H} = \Pi^i\left(\partial_0 A_i\right) - \frac{1}{2}\Pi^i\Pi^i + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}A_0A_0 + \frac{m^2}{2}A_iA_i\,. \tag{3.169}$$

However substituting the velocities $\left(\partial_0 A_i\right)$ from (3.163) into (3.169) leads to a Hamiltonian density of the form

$$\mathcal{H} = \frac{1}{2}\Pi^i\Pi^i + \Pi^i\left(\partial_i A_0\right) + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}A_0A_0 + \frac{m^2}{2}A_iA_i\,. \tag{3.170}$$

From a consideration of (3.128) and (3.129) it then follows that the primary Hamiltonian is given by

$$H_P = \int\left(\frac{1}{2}\Pi^i\Pi^i + \Pi^i\left(\partial_i A_0\right) + \frac{1}{4}F_{ij}F_{ij} - \frac{m^2}{2}A_0A_0 + \frac{m^2}{2}A_iA_i + u_0\,\Pi^0\right)d^3\underline{x} \tag{3.171}$$

where u_0 is a Lagrange multiplier function.

In this example the equal–time fundamental Poisson brackets, implied by (2.47), are

$$\left\{A_{\mu}(x), \Pi^{\nu}(y)\right\}_{x^0 = y^0} = \delta_{\mu}^{\nu}\,\delta^3(\underline{x} - \underline{y}) \tag{3.172}$$

and in accordance with (2.45) and (2.46) all the other combinations are zero.

The next stage of the algorithm is to ensure that the primary Hamiltonian constraint given by (3.165) is preserved in time. From a consideration of (3.131) this condition will hold if

$$0 \approx \{\phi^0, H_P\} = \{\Pi^0, H_P\} \tag{3.173}$$

and from the nature of the fundamental Poisson brackets (3.172) only the terms of H_P containing A_0 will be different from zero in (3.173). Therefore (3.173) may be rewritten as

$$0 \approx \left\{\Pi^0(x), \int \left(\Pi^i(\partial_i A_0) - \frac{m^2}{2} A_0 A_0\right) d^3\underline{y}\right\} \tag{3.174}$$

and after a partial integration (3.174) becomes

$$0 \approx \left\{\Pi^0(x), \int \left(\partial_i(\Pi^i A_0) - A_0(\partial_i \Pi^i) - \frac{m^2}{2} A_0 A_0\right) d^3\underline{y}\right\}. \tag{3.175}$$

The first term of the integral in (3.175) can be transformed into a surface integral by using Gauss' divergence theorem. In other words

$$\int \left(\partial_i(\Pi^i A_0)\right) d^3\underline{y} = \int \left(\Pi^i A_0\right) n_i \, dS = \int \left(\Pi^i A_0\right) dS_i \tag{3.176}$$

where the $n_i(y)$ are the components of the outward normal to the surface S. It was mentioned earlier that boundary conditions would only be treated at a formal level and in view of this it will be assumed that (3.176) vanishes at infinity. The time preservation condition for the primary constraint, that is (3.175), thus reduces further to

$$0 \approx \left\{\Pi^0(x), \int \left(- A_0(\partial_i \Pi^i) - \frac{m^2}{2} A_0 A_0\right) d^3\underline{y}\right\} \tag{3.177}$$

and this may equivalently be written as

$$0 \approx \int \left\{ \Pi^0(x), \left(- A_0 \left(\partial_i \Pi^i \right) - \frac{m^2}{2} A_0 A_0 \right)(y) \right\} d^3\underline{y} . \qquad (3.178)$$

After some minor manipulation (3.178) gives rise to the condition

$$\partial_i \Pi^i + m^2 A_0 \approx 0 \qquad i = 1, \ldots, 3. \qquad (3.179)$$

(3.179) is a secondary Hamiltonian constraint which may be denoted by

$$\chi^0 \equiv \partial_i \Pi^i + m^2 A_0 \approx 0. \qquad (3.180)$$

This secondary Hamiltonian constraint can readily be seen to be the weak equality version of the primary Lagrangian constraint (3.152) if the Π^i given by (3.163) are substituted into (3.179).

The time preservation of the secondary Hamiltonian constraint (3.180) must now be considered. Once again from (3.131) this is ensured provided that

$$0 \approx \left\{ \chi^0, H_P \right\} = \left\{ \left(\partial_i \Pi^i + m^2 A_0 \right), H_P \right\} \qquad (3.181)$$

after use of (3.180). The $\partial_i \Pi^i$ term of χ^0 only has non–vanishing Poisson brackets with the terms of H_P containing A_j, whereas the $m^2 A_0$ term only has non–vanishing Poisson brackets with the terms of H_P containing Π^0. In light of this (3.181) becomes

$$0 \approx \left\{ \left(\partial_i \Pi^i \right)(x), \int \left(\frac{1}{4} F_{jk} F_{jk} + \frac{m^2}{2} A_j A_j \right) d^3\underline{y} \right\} +$$
$$\left\{ m^2 A_0(x), \int \left(u_0(y) \Pi^0(y) \right) d^3\underline{y} \right\} \qquad (3.182)$$

and (3.182) may now be written as

$$0 \approx \frac{\partial}{\partial x^i}\left[\int \left\{\Pi^i(x), \left(\frac{1}{4} F_{jk}(y)\, F_{jk}(y) + \frac{m^2}{2} A_j(y)\, A_j(y)\right)\right\} d^3\underline{y} \right]$$

$$+ \int \left\{ m^2 A_0(x), u_0(y)\, \Pi^0(y)\right\} d^3\underline{y} \, . \tag{3.183}$$

Ultimately (3.183) reduces to the condition

$$m^2 u_0 - m^2 \partial_i A_i \approx 0 \tag{3.184}$$

and once again since $m \neq 0$ (3.184) becomes

$$u_0 - \partial_i A_i \approx 0. \tag{3.185}$$

The condition (3.185) determines the only Lagrange multiplier function u_0 and the Dirac–Bergmann constraint analysis is therefore complete.

In order to round off this constrained Hamiltonian analysis consider from (3.131) the time development of A_0, that is

$$\partial_0 A_0(x) \approx \left\{A_0(x), H_P\right\}. \tag{3.186}$$

Equation (3.186) simplifies to

$$\partial_0 A_0(x) \approx \int \left\{ A_0(x), u_0(y)\, \Pi^0(y)\right\} d^3\underline{y} = u_0(x) \tag{3.187}$$

and therefore as mentioned earlier the Lagrange multipliers, in this case there is only the u_0, take on the role of the velocities that could not originally be solved for; $\partial_0 A_0$ in this instance. On substituting (3.187) into (3.185) it is readily seen that (3.185) is the weak version of the secondary Lagrangian constraint given by (3.157).

Similarly from (3.131) the time development of the A_i is given by

$$\partial_0 A_i(x) \approx \{A_i(x), H_P\} \qquad i = 1, \ldots, 3 \tag{3.188}$$

and this can equivalently be written as

$$\partial_0 A_i(x) \approx \int \left\{A_i(x), \left(\frac{1}{2}\Pi^j \Pi^j + \Pi^j\left(\partial_j A_0\right)\right)(y)\right\} d^3\underline{y}\,. \tag{3.189}$$

After some manipulation (3.189) leads to the condition

$$\partial_0 A_i \approx \Pi^i + \partial_i A_0 \qquad i = 1, \ldots, 3. \tag{3.190}$$

Equation (3.190) is essentially just the weak version of (3.163).

On the other hand the time development of the Π^i from (3.131) is

$$\partial_0 \Pi^i(x) \approx \{\Pi^i(x), H_P\} \qquad i = 1, \ldots, 3 \tag{3.191}$$

and after substitution of the relevant terms of H_P, (3.191) becomes

$$\partial_0 \Pi^i(x) \approx \int \left\{\Pi^i(x), \left(\frac{1}{4} F_{jk} F_{jk} + \frac{m^2}{2} A_j A_j\right)(y)\right\} d^3\underline{y}\,. \tag{3.192}$$

After some calculation the upshot of (3.192) is

$$\partial_0 \Pi^i \approx \partial_j F_{ji} - m^2 A_i\,. \tag{3.193}$$

By substituting F_{ji} and Π^i, as given by (3.140) and (3.163) respectively, into (3.193) it can be seen that (3.193) is merely the weak version of the equations of motion (3.151) which were obtained in the Lagrangian analysis.

As a final point it is of interest to note, in light of the fundamental Poisson brackets, that

$$\left\{\phi^0(x), \phi^0(y)\right\} \equiv \left\{\Pi^0(x), \Pi^0(y)\right\} \approx 0, \qquad (3.194)$$

$$\left\{\chi^0(x), \chi^0(y)\right\} \equiv \left\{\left(\partial_i \Pi^i + m^2 A_0\right)(x), \left(\partial_j \Pi^j + m^2 A_0\right)(y)\right\} \approx 0 \qquad (3.195)$$

and

$$\left\{\phi^0(x), \chi^0(y)\right\} \equiv \left\{\Pi^0(x), \left(\partial_i \Pi^i + m^2 A_0\right)(y)\right\} \approx -m^2 \delta^3(\underline{x} - \underline{y}). \qquad (3.196)$$

Since $m \neq 0$ it follows that the right–hand side of (3.196) does not vanish in general. Consequently the primary and secondary Hamiltonian constraints, given by ϕ^0 and χ^0 respectively, are both second class because (3.196) shows that neither of them have weakly vanishing Poisson brackets with all the other constraints.

CHAPTER IV

CONSTRAINT ANALYSIS OF FIRST ORDER DYNAMICAL SYSTEMS

The constraint analysis examined in chapter III, via the Lagrangian and Dirac–Bergmann algorithms, was described for quite general second order systems. These are systems whose Euler–Lagrange equations are second order differential equations since they are equations containing acceleration terms. The special case of constrained first order systems, where the Euler–Lagrange equations are of first order, will now be considered from the point of view of the analysis given by Scherer [6]. In a manner in keeping with chapter III the constraint algorithms on the Lagrangian and Hamiltonian sides will first of all be reviewed for finite first order systems. The results of this finite dimensional analysis will then be extended to the field theoretic case. Finally a field theoretic example will be investigated via the first order Dirac–Bergmann algorithm in order to illustrate an application of this algorithm.

A Description of the Lagrangian and Dirac–Bergmann constraint algorithms for finite first order systems

Consider first of all the Lagrangian analysis where once again the system is described by the initially independent generalized coordinates q_i and their corresponding velocities $\dot{q}_i$, for $i = 1$ to n, in the 2n–dimensional space S. The most general Lagrangian L allowed in this analysis is one which is linear in the velocities. In other words

$$L = L(q_i, \dot{q}_i) = q_i \, A_{ij}(q) \, \dot{q}_j - H'(q) \qquad i, j = 1, \ldots, n \qquad (4.1)$$

where $A = [A_{ij}]$ represents an $(n \times n)$ matrix depending only on the q_i and H' is a function also depending only on the q_i. Once again no explicit time dependence has been assumed for this analysis.

Substituting (4.1) into (2.2) leads to Euler–Lagrange equations of the form

$$\left(A_{ij} - A_{ji} + q_l \frac{\partial A_{lj}}{\partial q_i} - q_l \frac{\partial A_{li}}{\partial q_j}\right) \dot{q}_j = \frac{\partial H'}{\partial q_i} \qquad i, j, l = 1, \ldots, n \tag{4.2}$$

and these may be rewritten as

$$W_{ij}(q)\ \dot{q}_j = E_i(q) \tag{4.3}$$

where clearly

$$W_{ij}(q) = A_{ij} - A_{ji} + q_l \frac{\partial A_{lj}}{\partial q_i} - q_l \frac{\partial A_{li}}{\partial q_j} \tag{4.4}$$

and

$$E_i(q) = \frac{\partial H'}{\partial q_i}\,. \tag{4.5}$$

Now if $|W| \neq 0$ then W possesses an inverse Y, say, and consequently (4.3) can be solved for all the velocities, that is

$$\dot{q}_i = Y_{ij}(q)\ E_j \qquad i, j = 1, \ldots, n. \tag{4.6}$$

On the other hand it is more likely that $|W| = 0$ in which case not all the velocities in (4.3) can be determined and this indicates the possible presence of constraints. The Lagrangian constraint algorithm now proceeds in a similar manner to that described in section A of chapter III.

Since W is singular then it follows that it has $(n - R)$ zero eigenvalues and $(n - R)$ corresponding linearly independent left null eigenvectors where R is the rank of W. If

(4.3) is then contracted with these left null eigenvectors this will result in a set of relations between the q_i. Of these relations only a subset of them may be functionally independent and these independent relations will be termed primary Lagrangian constraints. These constraints have the effect of restricting the space S to a constraint surface of lower dimensionality.

In light of these primary constraints the rank of W, which was originally evaluated assuming that the q_i were all independent, may now decrease when it is re–evaluated on this constraint surface. If this happens then the above procedure must be repeated until the rank of W does not change when it is computed on the current constraint surface. The upshot of all this is that there are now K independent primary Lagrangian constraints given by

$$C_r(q) = 0 \qquad\qquad r = 1, \ldots, K \qquad (4.7)$$

which define a constraint surface V in S.

The next stage of the algorithm is to ensure that the constraints given by (4.7) are preserved in time on V. This requirement leads to the conditions

$$\frac{d}{dt}\left(C_r(q)\right) = \frac{\partial C_r}{\partial q_j}\dot{q}_j = 0 \qquad\qquad \begin{matrix} r = 1, \ldots, K \\ j = 1, \ldots, n \end{matrix} \qquad (4.8)$$

and these are clearly a new set of equations for the velocities. (4.8) must now be considered in conjunction with (4.3) and this leads to the system of equations for the velocities given by

$$W'_{sj}(q)\ \dot{q}_j = E'_s(q) \qquad\qquad s = 1, \ldots, (n + K) \qquad (4.9)$$

where

$$W' = \left[W'_{sj}\right] = \begin{bmatrix} W_{ij} \\ \dfrac{\partial C_r}{\partial q_j} \end{bmatrix} \qquad \begin{array}{l} i, j = 1, \ldots, n \\ r = 1, \ldots, K \\ s = 1, \ldots, (n + K) \end{array} \tag{4.10}$$

and

$$E' = \left[E'_{s}\right] = \begin{bmatrix} E_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{array}{l} i = 1, \ldots, n \\ s = 1, \ldots, (n + K) \end{array} \tag{4.11}$$

It should be noted that W' is an $\big((n + K) \times n\big)$ matrix and E' is an $\big((n + K) \times 1\big)$ column vector which has K rows of zeros.

From here the analysis continues by investigating the rank of W' which in general will not be maximal. If the rank of W' is R' then W' will have $\big(n + K - R'\big)$ linearly independent left null eigenvectors. By contracting (4.9) with the left null eigenvectors of W' more relations between the q_i will be obtained. Of these a certain number of them may be functionally independent amongst themselves and the primary constraints given by (4.7). These new independent relations are secondary Lagrangian constraints and they restrict the constraint surface V to one of lower dimensionality.

As seen before the rank of W' may decrease when it is computed on this new constraint surface and the above process has to be repeated until the rank of W' no longer changes and all the K', say, secondary constraints

$$C'_t(q) = 0 \qquad t = 1, \ldots, K' \tag{4.12}$$

have been obtained. The constraint surface is now denoted by V'.

The secondary constraints (4.12) must now be preserved in time and this gives rise to the conditions

$$\frac{d}{dt}\left(C_t'(q)\right) = \frac{\partial C_t'}{\partial q_j}\dot{q}_j = 0 \qquad \begin{matrix} t = 1, \ldots, K'. \\ j = 1, \ldots, n \end{matrix} \qquad (4.13)$$

When equations (4.13) are considered in conjunction with (4.9) the following system of equations

$$W''_{uj}(q)\ \dot{q}_j = E''_u(q) \qquad u = 1, \ldots, (n + K + K') \qquad (4.14)$$

is obtained. In (4.14) $W'' = \left[W''_{uj}\right]$ is an $\left((n + K + K') \times n\right)$ matrix and $E'' = \left[E''_u\right]$ is an $\left((n + K + K') \times 1\right)$ column vector which are respectively given by

$$W'' = \begin{bmatrix} W'_{sj} \\ \dfrac{\partial C_t'}{\partial q_j} \end{bmatrix} \qquad \begin{matrix} j = 1, \ldots, n \\ s = 1, \ldots, (n + K) \\ t = 1, \ldots, K' \\ u = 1, \ldots, (n + K + K') \end{matrix} \qquad (4.15)$$

and

$$E'' = \begin{bmatrix} E'_s \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{matrix} s = 1, \ldots, (n + K) \\ u = 1, \ldots, (n + K + K') \end{matrix}\ . \qquad (4.16)$$

Equations (4.14), (4.15) and (4.16) have been obtained in a similar way to equations (4.9), (4.10) and (4.11) and the previous steps of the analysis must now be repeated until the following situation is reached.

There are K''' Lagrangian constraints given by

$$C_v'''(q) = 0 \qquad v = 1, \ldots, K''' \qquad (4.17)$$

and the velocities obey the equations

$$W'''_{wj}(q)\ \dot{q}_j = E'''_w(q) \qquad w = 1, \ldots, \left(n + K'''\right) \qquad (4.18)$$

where $W''' = \left[W'''_{wj}\right]$ is the $\left(\left(n + K'''\right) \times n\right)$ matrix given by

$$W''' = \begin{bmatrix} W_{ij} \\ \dfrac{\partial C'''_v}{\partial q_j} \end{bmatrix} \qquad \begin{array}{l} i, j = 1, \ldots, n \\ v = 1, \ldots, K''' \\ w = 1, \ldots, \left(n + K'''\right) \end{array} \qquad (4.19)$$

and $E''' = \left[E'''_w\right]$ is the $\left(\left(n + K'''\right) \times 1\right)$ column vector given by

$$E''' = \begin{bmatrix} E_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{array}{l} i = 1, \ldots, n \\ w = 1, \ldots, \left(n + K'''\right) \end{array} . \qquad (4.20)$$

Overall the motion is restricted to a surface V''' in S. The left null eigenvectors of W''' produce no new constraints which are independent of those given by (4.17). In addition if rank $W''' = R''' < n$ then R''' of the velocities can be determined whereas the remaining $\left(n - R'''\right)$ velocities appear as arbitrary functions of time in the solutions to the equations of motion.

In going over to the Dirac–Bergmann Hamiltonian analysis the canonical momenta, as given by (3.36), are found from (4.1) to be

$$p_i = q_j\ A_{ji} \qquad i, j = 1, \ldots, n. \qquad (4.21)$$

It was seen in the second order case described in section B of chapter III that the momenta are in general dependent on both the q_i and $\dot{q}_i$ and it was then possible to solve for some of the $\dot{q}_i$ in terms of the q_i , the independent momenta and the remaining $\dot{q}_i$. However since the momenta in (4.21) do not depend on the velocities in any way then they yield n primary Hamiltonian constraints which can be written as

$$\phi_i(q, p) = p_i - q_j A_{ji} \approx 0 \tag{4.22}$$

where $\approx$ denotes weak equality as described in chapter III.

Now from (3.42) the Hamiltonian is found to be

$$H(q, p) = H'(q) \tag{4.23}$$

after using (4.1). By analogy with (3.78) the primary Hamiltonian H_P is given by

$$H_P(q, p) = H'(q) + u_i \phi_i \tag{4.24}$$

where the Lagrange multipliers u_i are now playing the role of the velocities.

The time development of a function $B = B(q_i, p_i)$ defined on full phase space is by analogy with (3.79)

$$\frac{dB}{dt} = \dot{B} \approx \{B, H'\} + \{B, \phi_i\} u_i \qquad i = 1, \ldots, n \tag{4.25}$$

or equivalently in terms of H_P

$$\frac{dB}{dt} = \dot{B} \approx \{B, H_P\}. \tag{4.26}$$

In (4.25) and (4.26) the brackets denote the full phase space Poisson brackets as given by (3.75).

The next step in the algorithm is to demand that the primary Hamiltonian constraints (4.22) are preserved in time. From (4.25) this requirement is guaranteed if

$$0 \approx \{\phi_i, H'\} + \{\phi_i, \phi_j\}\, u_j \qquad i, j = 1, \ldots, n \qquad (4.27)$$

and after use of (3.75), (4.27) becomes

$$0 \approx -\frac{\partial H'}{\partial q_i} + \left(A_{ij} - A_{ji} + q_l \frac{\partial A_{lj}}{\partial q_i} - q_l \frac{\partial A_{li}}{\partial q_j}\right) u_j \qquad i, j, l = 1, \ldots, n. \qquad (4.28)$$

In view of (4.4) and (4.5) equation (4.28) can be rewritten as

$$0 \approx -E_i + W_{ij}\, u_j \qquad i, j = 1, \ldots, n. \qquad (4.29)$$

However since the primary constraints (4.22) only restrict the momenta, which do not occur in (4.29), then the equality in (4.29) may be taken to be strong and so (4.29) becomes

$$W_{ij}(q)\, u_j \simeq E_i(q) \qquad i, j = 1, \ldots, n. \qquad (4.30)$$

The underlying constraint structure of (4.30) can now be investigated in direct analogy to the Lagrangian constraint analysis of (4.3). The condition that (4.30) has solutions for the u_j when $|W| = 0$ leads to a set of secondary Hamiltonian constraints

$$\chi_\theta(q) \approx 0 \qquad \theta = 1, \ldots, K \qquad (4.31)$$

which are identical to the primary Lagrangian constraints (4.7).

The condition that the secondary constraints (4.31) are preserved in time must now be considered and this is guaranteed from (4.25) if

$$0 \approx \left\{\chi_\theta, H'\right\} + \left\{\chi_\theta, \phi_j\right\} u_j \qquad \begin{matrix} j = 1, \ldots, n \\ \theta = 1, \ldots, K \end{matrix} \qquad (4.32)$$

and this simplifies to

$$\frac{\partial \chi_\theta}{\partial q_j} u_j \approx 0 \qquad \begin{matrix} j = 1, \ldots, n \\ \theta = 1, \ldots, K \end{matrix} \quad . \qquad (4.33)$$

The time preservation conditions for the primary constraints (4.30) must now be considered in conjunction with the time preservation conditions for the secondary constraints (4.33) and this leads to the system of equations for the u_j

$$W'_{sj}(q)\ u_j \approx E'_s(q) \qquad s = 1, \ldots, (n + K) \quad (4.34)$$

where $W' = \left[W'_{sj}\right]$ and $E' = \left[E'_s\right]$ are given respectively by (4.10) and (4.11). The weak equality in (4.34) corresponds to the restriction of the motion to the constraint surface V in the Lagrangian analysis.

The Dirac–Bergmann constraint algorithm now continues in precisely the same way as the Lagrangian constraint algorithm except that time preservation is now ensured via

$$\left\{\text{constraint}, H_P\right\} \approx 0 \qquad (4.35)$$

and it is the multipliers u_i which are now to be determined rather than the velocities $\dot{q}_i$. The following situation is eventually reached.

There are n primary Hamiltonian constraints given by (4.22) and K''' l–ary Hamiltonian constraints

$$\chi'_{\varphi}(q) \approx 0 \qquad \varphi = 1, \ldots, K''' \qquad (4.36)$$

where $l \geq 2$. The Hamiltonian constraints given by (4.36) are just the Lagrangian constraints given by (4.17). In addition the u_j satisfy the conditions

$$W'''_{wj}(q)\, u_j \approx E'''_w(q) \qquad w = 1, \ldots, (n + K''') \qquad (4.37)$$

where $W''' = \left[W'''_{wj}\right]$ and $E''' = \left[E'''_w\right]$ are respectively given by (4.19) and (4.20). As in the Lagrangian case the conditions for the existence of solutions of u_j lead to no new constraints independent of those given by (4.36). Also if rank $W''' = R''' < n$ then, as seen in the Lagrangian constraint analysis, $(n - R''')$ of the u_j will remain completely arbitrary.

Finally, it is of interest to note that the Hamilton equations of motion are given by

$$\dot{q}_i \approx \left\{q_i, H_P\right\} = u_i \qquad i = 1, \ldots, n \qquad (4.38)$$

and

$$\dot{p}_i \approx \left\{p_i, H_P\right\} = u_j \frac{\partial}{\partial q_j}\left(q_k A_{ki}\right) \qquad i, j, k = 1, \ldots, n. \qquad (4.39)$$

(4.38) merely emphasizes the already known fact that the multipliers are taking on the role of the velocities in the Dirac–Bergmann algorithm.

B Analysis of field theoretic first order systems with constraints

An indication of how the last section can be generalized to the infinite dimensional case will now be discussed [6]. The most general field theoretic Lagrangian L that is admitted for the forthcoming analysis is

$$L = \int \left(Q_I \, A^{\mu}_{IJ}(Q) \left(\partial_\mu \, Q_J \right) - K(Q) \right) d^3\underline{x} \qquad \begin{matrix} \mu = 0, \ldots, 3 \\ I, J = 1, \ldots, N \end{matrix} \tag{4.40}$$

where the $A^{\mu} = \left[A^{\mu}_{IJ} \right]$ are matrices depending on the fields Q and K is a function depending also on the Q. Now from a consideration of (2.32) it follows from (4.40) that the Lagrangian density $\mathcal{L}$ of the system is

$$\mathcal{L} = \mathcal{L}\left(Q, \partial_\mu \, Q\right) = Q_I \, A^{\mu}_{IJ}(Q) \left(\partial_\mu \, Q_J \right) - K(Q). \tag{4.41}$$

By defining

$$H'\left(Q, \partial_k \, Q\right) = K(Q) - Q_I \, A^{k}_{IJ} \left(\partial_k \, Q_J \right) \qquad \begin{matrix} k = 1, \ldots, 3 \\ I, J = 1, \ldots, N \end{matrix} \tag{4.42}$$

it can readily be seen that (4.41) can then be re–expressed in the space–time decomposed form

$$\mathcal{L} = Q_I \, A^{0}_{IJ}(Q) \left(\partial_0 \, Q_J \right) - H'\left(Q, \partial_k \, Q\right). \tag{4.43}$$

The field theoretic Euler–Lagrange equations are found to be

$$\left(A^{0}_{IJ} - A^{0}_{JI} + Q_L \frac{\partial A^{0}_{LJ}}{\partial Q_I} - Q_L \frac{\partial A^{0}_{LI}}{\partial Q_J} \right) \left(\partial_0 \, Q_J \right) = \frac{\partial H'}{\partial Q_I} - \partial_k \left(\frac{\partial H'}{\partial \left(\partial_k \, Q_I \right)} \right) \qquad \begin{matrix} I, J, L = 1, \ldots, N \\ k = 1, \ldots, 3 \end{matrix} \tag{4.44}$$

after substitution of (4.43) into (2.35). (4.44) can be written in the more compact form

$$W_{IJ}(Q)\left(\partial_0 Q_J\right) = E_I(Q) \qquad I, J = 1, \ldots, N \qquad (4.45)$$

where obviously

$$W_{IJ}(Q) = A^0_{IJ} - A^0_{JI} + Q_L \frac{\partial A^0_{LJ}}{\partial Q_I} - Q_L \frac{\partial A^0_{LI}}{\partial Q_J} \qquad (4.46)$$

and

$$E_I(Q) = \frac{\partial H'}{\partial Q_I} - \partial_k \left(\frac{\partial H'}{\partial\left(\partial_k Q_I\right)} \right). \qquad (4.47)$$

(4.45) is the field theoretic analogue of (4.3) of the finite dimensional analysis except that now it is possible for spatial derivatives to occur in the subsequent analysis. The problem of spatial boundary conditions that ensues due to the presence of these spatial derivatives will be treated in the same way as it was in section C of chapter III; in other words it will not be treated too rigourously. With this borne in mind the Lagrangian constraint analysis proceeds from (4.45) essentially as it did from (4.3) in the finite case until a similar final situation is reached.

Consider now the Dirac–Bergmann analysis of (4.43). By putting (4.43) into (2.36) the field theoretic canonical momenta are found to be

$$\Pi^I = Q_J \, A^0_{JI}(Q) \qquad I, J = 1, \ldots, N. \qquad (4.48)$$

By analogy with (4.22) in the finite dimensional case, (4.48) yields the N primary Hamiltonian constraints

$$\phi^I = \Pi^I - Q_J \, A^0_{JI} \approx 0 \qquad I, J = 1, \ldots, N. \qquad (4.49)$$

From (2.39) the Hamiltonian density $\mathcal{H}$ is found in this case to be

$$\mathcal{H}(Q, \Pi) = H'(Q, \partial_k Q) \qquad k = 1, \ldots, 3 \tag{4.50}$$

and after consideration of (3.128) and (3.129) it follows from (4.50) that the primary Hamiltonian density is given by

$$\mathcal{H}_P(Q, \Pi) = H'(Q, \partial_k Q) + u_I \phi^I . \tag{4.51}$$

The u_I in (4.51) are multiplier functions which again assume the role of the velocities. Now from (3.129) it follows that the primary Hamiltonian H_P is given by

$$H_P = \int \left(H'(Q, \partial_k Q) + u_I \phi^I \right) d^3\underline{x} . \tag{4.52}$$

Furthermore it can be seen from (3.131) that the time development of a functional B of phase space variables and their spatial derivatives can be expressed by

$$\partial_0 B \approx \left\{ B(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \tag{4.53}$$

after a consideration of (3.129). The field theoretic Poisson brackets in (4.53) are defined by (2.42).

As in the finite case the conditions that the primary constraints (4.49) are preserved in time must now be considered. This is ensured from (4.53) if

$$0 \approx \left\{ \phi^I(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \tag{4.54}$$

or equivalently if

$$0 \approx \int \{ \phi^I(x), H'(y) \} \, d^3\underline{y} + \int \{ \phi^I(x), \phi^J(y) \} \, u_J(y) \, d^3\underline{y} . \tag{4.55}$$

After some manipulation (4.55) becomes

$$W_{IJ}(Q) \; u_J \simeq E_I(Q) \tag{4.56}$$

where W_{IJ} and E_I are respectively given by (4.46) and (4.47). The fact that the equality in (4.56) can be taken to be strong follows from an argument similar to the one given after (4.29) in the finite case.

Now, as was seen in the infinite dimensional Lagrangian constraint algorithm, the analysis proceeds from (4.56) just as it did from (4.30) in the finite dimensional case until once again a similar final situation is reached.

As an application of the field theoretic version of the Dirac–Bergmann algorithm for first order systems the case of a massive spin–$\frac{3}{2}$ particle coupled to an external electromagnetic field will now be considered. This example has the additional advantage in that it will also serve to show how the constraint algorithm can be used to handle coupled systems. The investigation of this coupled system will take its lead from the work done by Hasumi, Endo and Kimura [7] but will be such that it is a more detailed look at the constraint analysis of this system than the one presented by them.

A hermitian Lagrangian density $\mathcal{L}$ for the Rarita–Schwinger spin–$\frac{3}{2}$ field ψ_μ minimally coupled to an external electromagnetic field A_μ is given by

$$\mathcal{L} = \frac{1}{2}\,\varepsilon^{\lambda\rho\mu\nu}\,\overline{\psi}_\lambda\,\gamma_5\,\gamma_\mu\left(\partial_\nu\,\psi_\rho\right) - \frac{1}{2}\,\varepsilon^{\lambda\rho\mu\nu}\left(\partial_\nu\,\overline{\psi}_\lambda\right)\gamma_5\,\gamma_\mu\,\psi_\rho$$
$$+\ im\,\overline{\psi}_\lambda\,\sigma^{\lambda\rho}\,\psi_\rho - i\,e\,\varepsilon^{\lambda\rho\mu\nu}\,\overline{\psi}_\lambda\,\gamma_5\,\gamma_\mu\,\psi_\rho\,A_\nu$$
$$\lambda, \rho, \mu, \nu = 0, \ldots, 3. \qquad (4.57)$$

In this analysis the metric convention given by (2.29) has been adopted. In (4.57) the gamma matrices γ_μ satisfy the Clifford algebra

$$\gamma^\mu\,\gamma^\nu + \gamma^\nu\,\gamma^\mu = 2g^{\mu\nu} \qquad \mu, \nu = 0, \ldots, 3 \qquad (4.58)$$

and additionally

$$\gamma_5 = i\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3 \qquad (4.59)$$

and

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\,\gamma^\nu - \gamma^\nu\,\gamma^\mu\right) \qquad \mu, \nu = 0, \ldots, 3. \qquad (4.60)$$

Also in (4.57)

$$\overline{\psi}_\lambda = \psi_\lambda^\dagger\,\gamma_0 \qquad \lambda = 0, \ldots, 3 \qquad (4.61)$$

and the convention $\varepsilon^{0123} = -\varepsilon_{0123} = 1$ has been assumed.

At this point it is important to remember that the constraint algorithms have all been described for the explicitly time independent case. With this in mind the external electromagnetic field A_μ must be assumed to be time independent if the previously described Dirac–Bergmann constraint algorithm is to be successfully applied to this first order example. With this assumption it follows that A_μ depends only on the spatial coordinates x^i where $i = 1$ to 3.

Now a space–time decomposition of (4.57) leads to

$$\mathcal{L} = \frac{1}{2}\,\varepsilon^{\lambda\rho\mu 0}\,\overline{\psi}_\lambda\,\gamma_5\,\gamma_\mu\left(\partial_0\,\psi_\rho\right) + \frac{1}{2}\,\varepsilon^{\lambda\rho\mu k}\,\overline{\psi}_\lambda\,\gamma_5\,\gamma_\mu\left(\partial_k\,\psi_\rho\right)$$
$$-\frac{1}{2}\,\varepsilon^{\lambda\rho\mu 0}\left(\partial_0\,\overline{\psi}_\lambda\right)\gamma_5\,\gamma_\mu\,\psi_\rho - \frac{1}{2}\,\varepsilon^{\lambda\rho\mu k}\left(\partial_k\,\overline{\psi}_\lambda\right)\gamma_5\,\gamma_\mu\,\psi_\rho$$
$$+\,\mathrm{i}m\,\overline{\psi}_\lambda\,\sigma^{\lambda\rho}\,\psi_\rho - \mathrm{i}\,e\,\varepsilon^{\lambda\rho\mu\nu}\,\overline{\psi}_\lambda\,\gamma_5\,\gamma_\mu\,\psi_\rho\,A_\nu$$
$$\lambda,\rho,\mu,\nu = 0,\ldots,3\,. \qquad k = 1,\ldots,3 \qquad (4.62)$$

The momenta conjugate to $\left(\psi_\lambda\right)_a$ and $\left(\psi^\dagger_\lambda\right)_a$, where a denotes a bispinor index and runs from 1 to 4, are in accordance with (2.36) defined to be

$$\left(\Pi^\lambda\right)_a = \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_0\,\psi_\lambda\right)}\right)_a \qquad \lambda = 0,\ldots,3 \quad a = 1,\ldots,4 \qquad (4.63)$$

and

$$\left(\Pi^\lambda\right)^\dagger_a = \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_0\,\psi^\dagger_\lambda\right)}\right)_a \qquad \lambda = 0,\ldots,3 \quad a = 1,\ldots,4 \qquad (4.64)$$

respectively. By putting (4.62) into (4.63) and (4.64) it is found that

$$\Pi^0_a = 0 \qquad a = 1,\ldots,4, \qquad (4.65a)$$

$$\Pi^k_a = \frac{1}{2}\left(\varepsilon^{kij}\,\psi^\dagger_i\,\gamma_5\,\gamma_j\,\gamma_0\right)_a \qquad i,j,k = 1,\ldots,3, \quad a = 1,\ldots,4 \qquad (4.65b)$$

$$\left(\Pi^0_a\right)^\dagger = 0 \qquad a = 1,\ldots,4, \qquad (4.65c)$$

$$\left(\Pi^k_a\right)^\dagger = \frac{1}{2}\left(\varepsilon^{kij}\,\gamma_0\,\gamma_5\,\gamma_j\,\psi_i\right)_a \qquad i,j,k = 1,\ldots,3 \quad a = 1,\ldots,4 \qquad (4.65d)$$

and equation (4.49) then implies that (4.65) yields the following primary constraints

$$\phi^0_a \equiv \Pi^0_a \approx 0 \qquad a = 1, \ldots, 4, \tag{4.66a}$$

$$\phi^k_a \equiv \Pi^k_a - \frac{1}{2}\left(\varepsilon^{kij}\,\psi^\dagger_i\,\gamma_5\,\gamma_j\,\gamma_0\right)_a \approx 0 \qquad \begin{matrix} i, j, k = 1, \ldots, 3, \\ a = 1, \ldots, 4 \end{matrix} \tag{4.66b}$$

$$\left(\phi^0_a\right)^\dagger \equiv \left(\Pi^0_a\right)^\dagger \approx 0 \qquad a = 1, \ldots, 4, \tag{4.66c}$$

$$\left(\phi^k_a\right)^\dagger \equiv \left(\Pi^k_a\right)^\dagger - \frac{1}{2}\left(\varepsilon^{kij}\,\gamma_0\,\gamma_5\,\gamma_j\,\psi_i\right)_a \approx 0 \qquad \begin{matrix} i, j, k = 1, \ldots, 3. \\ a = 1, \ldots, 4 \end{matrix} \tag{4.66d}$$

The Hamiltonian density $\mathcal{H}$ is defined to be

$$\mathcal{H} = \Pi^\mu\left(\partial_0\,\psi_\mu\right) + \left(\partial_0\,\psi^\dagger_\mu\right)\left(\Pi^\mu\right)^\dagger - \mathcal{L} \qquad \mu = 0, \ldots, 3. \tag{4.67}$$

(4.67) is merely an extension of (2.39) which incorporates the conjugate fields $\psi^\dagger_\mu$.

After some manipulation (4.67) becomes

$$\begin{aligned} \mathcal{H} = &-\frac{1}{2}\,\varepsilon^{\lambda\rho\mu k}\,\overline{\psi}_\lambda\,\gamma_5\,\gamma_\mu\left(\partial_k\,\psi_\rho\right) + \frac{1}{2}\,\varepsilon^{\lambda\rho\mu k}\left(\partial_k\,\overline{\psi}_\lambda\right)\gamma_5\,\gamma_\mu\,\psi_\rho \\ &- im\,\overline{\psi}_\lambda\,\sigma^{\lambda\rho}\,\psi_\rho + i\,e\,\varepsilon^{\lambda\rho\mu\nu}\,\overline{\psi}_\lambda\,\gamma_5\,\gamma_\mu\,\psi_\rho\,A_\nu \\ &\qquad \lambda, \rho, \mu, \nu = 0, \ldots, 3 \\ &\qquad k = 1, \ldots, 3 \end{aligned} \tag{4.68}$$

and from (4.51) it follows that the primary Hamiltonian density $\mathcal{H}_P$ is given by

$$\mathcal{H}_P = \mathcal{H} + u_{\mu a}\,\phi^\mu_a + u^\dagger_{\mu a}\left(\phi^\mu_a\right)^\dagger \qquad \begin{matrix} \mu = 0, \ldots, 3 \\ a = 1, \ldots, 4 \end{matrix} \tag{4.69}$$

where $u_{\mu a}(x)$ and $u^\dagger_{\mu a}(x)$ are Lagrange multiplier functions which are as yet undetermined.

The fundamental equal–time Poisson brackets are from a consideration of (2.47) found to be

$$\left\{\psi_{\mu a}(x), \Pi^\nu_b(y)\right\} = \left\{\psi^\dagger_{\mu a}(x), \left(\Pi^\nu_b\right)^\dagger(y)\right\} = \delta^\nu_\mu\;\delta_{ab}\;\delta^3(\underline{x} - \underline{y}) \tag{4.70}$$

and from (2.45) and (2.46) all the other combinations are zero.

The next step of the analysis is to ensure that the primary Hamiltonian constraints (4.66) are preserved in time. Consider first of all the time preservation of (4.66c). From (4.54) this is guaranteed provided that

$$0 \approx \left\{ \left(\phi_a^0\right)^\dagger(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} = \left\{ \left(\Pi_a^0\right)^\dagger(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \qquad (4.71)$$

and from the form of the fundamental Poisson brackets (4.70) only the terms of $\mathcal{H}_P$ that contain $\psi_0^\dagger$ will make any contribution to (4.71). In view of this (4.71) can be written as

$$\begin{aligned} 0 \approx \Bigg\{ \left(\Pi_a^0\right)^\dagger(x), \int \Bigg(&- \frac{1}{2}\, \varepsilon^{0\rho\mu k} \left(\psi_{0b}\right)^\dagger \left(\gamma_0\, \gamma_5\, \gamma_\mu \left(\partial_k\, \psi_\rho\right)\right)_b \\ &+ \frac{1}{2}\, \varepsilon^{0\rho\mu k} \left(\partial_k\left(\psi_{0b}\right)^\dagger\right) \left(\gamma_0\, \gamma_5\, \gamma_\mu\, \psi_\rho\right)_b - im \left(\psi_{0b}\right)^\dagger \left(\gamma_0\, \sigma^{0\rho}\, \psi_\rho\right)_b \\ &+ ie\, \varepsilon^{0\rho\mu\nu} \left(\psi_{0b}\right)^\dagger \left(\gamma_0\, \gamma_5\, \gamma_\mu\, \psi_\rho\, A_\nu\right)_b \Bigg)(y)\, d^3\underline{y} \Bigg\} \end{aligned}$$

$$\begin{matrix} \rho, \mu, \nu = 0, \ldots, 3 \\ k = 1, \ldots, 3 \\ a, b = 1, \ldots, 4 \end{matrix} \qquad . \qquad (4.72)$$

A partial integration of the second term in the integral in (4.72) leads to

$$\begin{aligned} &\int \left(\frac{1}{2}\, \varepsilon^{0\rho\mu k} \left(\partial_k\left(\psi_{0b}\right)^\dagger\right) \left(\gamma_0\, \gamma_5\, \gamma_\mu\, \psi_\rho\right)_b\right)(y)\, d^3\underline{y} = \\ &\int \left(\frac{1}{2}\, \varepsilon^{0\rho\mu k}\, \partial_k\left(\left(\psi_{0b}\right)^\dagger\left(\gamma_0\, \gamma_5\, \gamma_\mu\, \psi_\rho\right)_b\right)\right)(y)\, d^3\underline{y} \; - \\ &\int \left(\frac{1}{2}\, \varepsilon^{0\rho\mu k} \left(\psi_{0b}\right)^\dagger\left(\gamma_0\, \gamma_5\, \gamma_\mu \left(\partial_k\, \psi_\rho\right)\right)_b\right)(y)\, d^3\underline{y} \qquad . \qquad (4.73) \end{aligned}$$

and the first integral on the right–hand side of (4.73) can be transformed into a surface integral by Gauss' divergence theorem in a manner similar to (3.176). If this surface integral is assumed to vanish at infinity then (4.72) now becomes

$$0 \approx \int \Big\{ \left(\Pi_a^0\right)^{\dagger}(x), \Big(-\varepsilon^{0\rho\mu k} \left(\psi_{0b}\right)^{\dagger} \left(\gamma_0\, \gamma_5\, \gamma_\mu \left(\partial_k\, \psi_\rho\right)\right)_b$$

$$- im \left(\psi_{0b}\right)^{\dagger} \left(\gamma_0\, \sigma^{0\rho}\, \psi_\rho\right)_b + ie\, \varepsilon^{0\rho\mu\nu} \left(\psi_{0b}\right)^{\dagger} \left(\gamma_0\, \gamma_5\, \gamma_\mu\, \psi_\rho\, A_\nu\right)_b \Big)(y) \Big\}$$

$$d^3\underline{y} \qquad (4.74)$$

and this simplifies to the condition

$$\sigma^{ij}\, D_i\, \psi_j + m\, \gamma^j\, \psi_j \approx 0 \qquad i, j = 1, \ldots, 3 \qquad (4.75)$$

where

$$D_i = \partial_i - ieA_i \qquad i = 1, \ldots, 3. \qquad (4.76)$$

(4.75) is a secondary Hamiltonian constraint which will be denoted by

$$\chi \equiv \sigma^{ij}\, D_i\, \psi_j + m\, \gamma^j\, \psi_j \approx 0. \qquad (4.77)$$

A similar time preservation analysis of the primary constraint (4.66a) leads to another secondary Hamiltonian constraint which is essentially the conjugate of χ, that is

$$\overline{\chi} \equiv \chi^{\dagger}\, \gamma_0 \equiv \left(D_i^*\, \overline{\psi}_j\right) \sigma^{ij} + m\, \overline{\psi}_j\, \gamma^j \approx 0 \qquad i, j = 1, \ldots, 3 \qquad (4.78)$$

where

$$D_i^* = \partial_i + ieA_i \qquad i = 1, \ldots, 3. \qquad (4.79)$$

Now the time preservation condition for the primary constraint (4.66d) is

$$0 \approx \left\{ \left(\phi_a^k\right)^\dagger(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\}$$

$$= \left\{ \left(\left(\Pi_a^k\right)^\dagger - \frac{1}{2} \left(\varepsilon^{kij}\, \gamma_0\, \gamma_5\, \gamma_j\, \psi_i \right)_a \right)(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \qquad (4.80)$$

after consideration of (4.54). Once again by taking the nature of the fundamental Poisson brackets (4.70) into account equation (4.80) becomes

$$0 \approx \left\{ \left(\Pi_a^k\right)^\dagger(x), \int \left(-\frac{1}{2}\, \varepsilon^{l\rho\mu i} \left(\psi_{lb}\right)^\dagger \left(\gamma_0\, \gamma_5\, \gamma_\mu \left(\partial_i\, \psi_\rho \right) \right)_b \right.\right.$$

$$+ \frac{1}{2}\, \varepsilon^{l\rho\mu i} \left(\partial_i \left(\psi_{lb}\right)^\dagger \right) \left(\gamma_0\, \gamma_5\, \gamma_\mu\, \psi_\rho \right)_b - im \left(\psi_{lb}\right)^\dagger \left(\gamma_0\, \sigma^{l\rho}\, \psi_\rho \right)_b$$

$$+ ie\, \varepsilon^{l\rho\mu\nu} \left(\psi_{lb}\right)^\dagger \left(\gamma_0\, \gamma_5\, \gamma_\mu\, \psi_\rho\, A_\nu \right)_b -$$

$$\left.\left. \frac{1}{2}\, u_{lb}\, \varepsilon^{lij} \left(\psi_{ic}\right)^\dagger \left(\gamma_5\, \gamma_j\, \gamma_0 \right)_{cb} \right)(y)\, d^3\underline{y} \right\}$$

$$- \frac{1}{2} \left\{ \left(\varepsilon^{kij} \left(\gamma_0\, \gamma_5\, \gamma_j \right)_{ac} \psi_{ic} \right)(x), \int \left(u_{lb}\, \Pi_b^l \right)(y)\, d^3\underline{y} \right\}$$

$$\rho, \mu, \nu = 0, \ldots, 3 \quad i, j, k, l = 1, \ldots, 3 \quad a, b, c = 1, \ldots, 4 \qquad (4.81)$$

and after a consideration of the spatial boundary conditions (4.81) eventually simplifies to

$$\sigma^{ki}\, u_i \approx \sigma^{ki}\, D_i\, \psi_0 - \varepsilon^{kij}\, \gamma_0\, \gamma_5\, \gamma_0\, D_j\, \psi_i - m\, \gamma^k\, \psi_0 -$$

$$im\, \gamma_0\, \sigma^{ki}\, \psi_i + ie\, \sigma^{ki}\, \psi_i\, A_0 \qquad i, j, k = 1, \ldots, 3. \qquad (4.82)$$

By a similar argument the time preservation requirement for the primary constraint (4.66b) leads to the conjugate of (4.82), that is to say an equation containing the $u_i^\dagger$.

It should be noted that (4.56) of the general analysis is in essence now represented by (4.82) in this example except that the equality is no longer taken to be strong due to the occurrence of the secondary Hamiltonian constraints given by (4.77) and (4.78).

(4.82) must now be investigated to see to what extent the u_i can be determined. By operating on both sides of (4.82) with

$$-\frac{i}{2}\left(\gamma^l\gamma^k - 2g^{lk}\right) \qquad l, k = 1, \ldots, 3 \qquad (4.83)$$

and making use of the fact that

$$-\frac{i}{2}\left(\gamma^l\gamma^k - 2g^{lk}\right)\sigma^{ki} = -g^{li} \qquad i, k, l = 1, \ldots, 3, \qquad (4.84)$$

it is found after some manipulation that

$$\begin{aligned} u_l \approx\ & D_l\,\psi_0 + i\,\varepsilon^{lij}\,\gamma_0\,\gamma_5\,\gamma_0\,D_j\,\psi_i + ie\,\psi_l\,A_0 - im\,\gamma_0\,\psi_l \\ & -\frac{i}{2}\,\gamma_0\,\gamma^l\,\sigma^{ij}\,D_j\,\psi_i - \frac{im}{2}\,\gamma^l\,\psi_0 \qquad i, j, l = 1, \ldots, 3. \qquad (4.85) \end{aligned}$$

Thus the multipliers u_l have now been determined and if a similar analysis is performed on the conjugate of (4.82) then this will lead to the determination of the $u_l^\dagger$.

The time preservation of the secondary Hamiltonian constraints given by (4.77) and (4.78) must now be examined. The condition that (4.77) is preserved in time is

$$0 \approx \left\{ \chi(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\}$$

$$= \left\{ \left(\sigma^{ij} D_i \psi_j + m\, \gamma^j \psi_j\right)(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \tag{4.86}$$

after use of (4.53). (4.86) is equivalent to

$$0 \approx \left\{ \left(\left(\sigma^{ij} D_i\right)_{ab} \psi_{jb}\right)(x), \int \left(u_{lc}\, \Pi^l_c\right)(y)\, d^3\underline{y} \right\}$$

$$+ \left\{ \left(m\left(\gamma^j\right)_{ab} \psi_{jb}\right)(x), \int \left(u_{lc}\, \Pi^l_c\right)(y)\, d^3\underline{y} \right\}$$

$$i, j, l = 1, \ldots, 3 \qquad a, b, c = 1, \ldots, 4 \tag{4.87}$$

and this ultimately becomes

$$\sigma^{ij} D_i\, u_j + m \gamma^j\, u_j \approx 0 \qquad i, j = 1, \ldots, 3. \tag{4.88}$$

Substituting for u_j, as given by (4.85), into (4.88) leads, after making use of the secondary constraint (4.77), to the condition

$$2R\gamma_0\, \psi_0 + \Gamma^k\, \psi_k \approx 0 \qquad k = 1, \ldots, 3 \tag{4.89}$$

where the matrix R is

$$R = \frac{1}{2}\left(1 - \left(\frac{e}{3m^2}\right) \sigma^{ij} F_{ij}\right) \qquad i, j = 1, \ldots, 3 \tag{4.90}$$

and the matrices Γ^k are

$$\Gamma^k = \left(\gamma^k + \left(\frac{ie}{3m^2}\right)\left(\gamma^i \gamma^k \gamma^0 \left(\partial_i A_0\right) - \gamma^0 \gamma^k \gamma^i \left(\partial_i A_0\right) + \gamma^i \gamma^k \gamma^j F_{ij}\right)\right) \qquad i, j, k = 1, \ldots, 3 \qquad (4.91)$$

and the F_{ij} in (4.90) and (4.91) are given by

$$F_{ij} = \partial_i A_j - \partial_j A_i \qquad i, j = 1, \ldots, 3. \qquad (4.92)$$

Equation (4.89) is a tertiary Hamiltonian constraint and it will be denoted by

$$\theta \equiv 2R\gamma_0 \psi_0 + \Gamma^k \psi_k \approx 0. \qquad (4.93)$$

The corresponding tertiary Hamiltonian constraint obtained by considering the time preservation of the secondary constraint (4.78) is

$$\bar{\theta} \equiv \theta^\dagger \gamma_0 \equiv 2\bar{\psi}_0 \gamma_0 R + \bar{\psi}_k \Gamma^k \approx 0. \qquad (4.94)$$

The remainder of the constraint analysis now hinges on the nature of the matrix R in (4.90), that is to say whether R is singular or non–singular. It is found that R satisfies the relation

$$R^2 - R + \frac{1}{4}\left(1 - \left(\frac{2e}{3m^2}\right)^2 N_i N_i\right) = 0 \qquad i = 1, \ldots, 3 \qquad (4.95)$$

where the N_i are the components of the magnetic field vector and are given by

$$N^i = \frac{1}{2} \varepsilon^{ijk} F_{jk} \qquad i, j, k = 1, \ldots, 3. \qquad (4.96)$$

As a consequence of (4.95) it follows that

$$|R| = \left(\frac{1}{4}\right)^2 \left(1 - \left(\frac{2e}{3m^2}\right)^2 N_i N_i\right)^2 . \qquad (4.97)$$

There are now two possibilities to consider :-

i) As the first case suppose that $|R| \neq 0$. The constraint analysis continues by demanding that the tertiary constraints given by (4.93) and (4.94) are preserved in time. For the constraint (4.93) this is guaranteed from (4.53) if

$$
\begin{aligned}
0 &\approx \left\{ \theta(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \\
&= \left\{ \left(2R\gamma_0\, \psi_0 \;+\; \Gamma^k\, \psi_k\right)(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\}
\end{aligned}
\tag{4.98}
$$

and this can be expressed as

$$
\begin{aligned}
0 &\approx \left\{ \left(\left(2R\gamma_0\right)_{ab}\, \psi_{0b}\right)(x), \int \left(u_{0c}\, \Pi^0_c\right)(y)\, d^3\underline{y} \right\} \\
&+ \left\{ \left(\left(\Gamma^k\right)_{ab}\, \psi_{kb}\right)(x), \int \left(u_{lc}\, \Pi^l_c\right)(y)\, d^3\underline{y} \right\} \\
&\qquad\qquad k,\, l = 1, \ldots, 3 \\
&\qquad\qquad a,\, b,\, c = 1, \ldots, 4\, .
\end{aligned}
\tag{4.99}
$$

After some manipulation (4.99) leads to the condition

$$
2R\gamma_0\, u_0 \;+\; \Gamma^k\, u_k \;\approx\; 0 \tag{4.100}
$$

and since $|R| \neq 0$ (4.100) can be used to determine the multipliers u_0, that is

$$
u_0 \;\approx\; -\frac{1}{2}\, \gamma_0\;\; R^{-1}\, \Gamma^k\, u_k \tag{4.101}
$$

where the u_k are given by (4.85). A corresponding analysis of the time preservation condition of (4.94) leads to the determination of the $u_0^\dagger$.

The constraint analysis is now complete because all of the multipliers have been determined and no more constraints are generated. By computing the Poisson brackets of all the possible combinations of the constraints it is found that none of the constraints have weakly vanishing Poisson brackets with all the other constraints. Consequently all the constraints are second class.

ii) The second case is when $|R| = 0$ or equivalently from (4.97)

$$1 - \left(\frac{2e}{3m^2}\right)^2 N_i N_i = 0. \tag{4.102}$$

The situation now becomes much more complicated and in order to simplify the subsequent calculations it will henceforth be assumed that $\partial_i A_0 = 0$ and $F_{ij} = \text{constant}$. With these last two assumptions in mind the external field is now a constant pure magnetic field and from (4.91) the Γ^k become

$$\Gamma^k = \left(\gamma^k + \left(\frac{ie}{3m^2}\right)\gamma^i \gamma^k \gamma^j F_{ij}\right). \tag{4.103}$$

At this point of the analysis it is convenient to introduce the matrix $\tilde{R}$ which is defined to be

$$\tilde{R} = I - R = \frac{1}{2}\left(1 + \left(\frac{e}{3m^2}\right)\sigma^{ij} F_{ij}\right). \tag{4.104}$$

When R is singular, that is when (4.102) holds, then (4.95) reduces to

$$R^2 = R \tag{4.105}$$

and from this it is trivial to deduce that

$$\tilde{R}^2 = \tilde{R} \tag{4.106}$$

and

$$R\tilde{R} = \tilde{R}R = 0. \tag{4.107}$$

The form of equations (4.105) and (4.106) indicate that the matrices R and $\tilde{R}$ are projection operators. In light of this the constraints (4.93) and (4.94) can be projected with these matrices and this gives rise to the conditions

$$R\theta \equiv 2R\gamma_0 \psi_0 + R\Gamma^k \psi_k \approx 0, \tag{4.108a}$$

$$\tilde{R}\theta \equiv \tilde{\theta} \equiv \tilde{R}\Gamma^k \psi_k \approx 0 \tag{4.108b}$$

and

$$\bar{\theta}R \equiv 2\bar{\psi}_0 \gamma_0 R + \bar{\psi}_k \Gamma^k R \approx 0, \tag{4.109a}$$

$$\bar{\theta}\tilde{R} \equiv \tilde{\bar{\theta}} \equiv \bar{\psi}_k \Gamma^k \tilde{R} \approx 0. \tag{4.109b}$$

Equations (4.108) are equivalent to (4.93) whereas equations (4.109) are equivalent to (4.94).

The constraint analysis now proceeds by investigating the consistency conditions for the time preservation of (4.108) and (4.109). Therefore consider first of all the time preservation of (4.108a) which is ensured from (4.53) provided

$$0 \approx \left\{ \big(R\theta\big)(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\}$$

$$= \left\{ \big(2R\gamma_0 \psi_0 + R\Gamma^k \psi_k\big)(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\}. \tag{4.110}$$

(4.110) is equivalent to

$$0 \approx \left\{ \left((2R\gamma_0)_{ab}\, \psi_{0b}\right)(x), \int \left(u_{0c}\, \Pi^0_c\right)(y)\; d^3\underline{y} \right\}$$

$$+ \left\{ \left((R\Gamma^k)_{ab}\, \psi_{kb}\right)(x), \int \left(u_{lc}\, \Pi^l_c\right)(y)\; d^3\underline{y} \right\}$$

$$k, l = 1, \ldots, 3 \qquad a, b, c = 1, \ldots, 4 \tag{4.111}$$

and this ultimately gives rise to the condition

$$2R\gamma_0\; u_0 + R\Gamma^k\; u_k \approx 0. \tag{4.112}$$

From (4.112) it is readily seen that

$$Ru_0 \approx -\frac{1}{2}\,\gamma_0\; R\Gamma^k\; u_k \tag{4.113}$$

where Γ^k and u_k are respectively given by (4.103) and (4.85). Thus the consistency condition of (4.108a) has determined Ru_0 and in a corresponding manner the time preservation of (4.109a) would lead to the determination of $u_0^\dagger R$.

On the other hand the time preservation condition of (4.108b) is from (4.53)

$$0 \approx \left\{ \tilde{\theta}(x), \int \mathcal{H}_P(y)\; d^3\underline{y} \right\} = \left\{ \left(\tilde{R}\Gamma^k\, \psi_k\right)(x), \int \mathcal{H}_P(y)\; d^3\underline{y} \right\} \tag{4.114}$$

which, after a consideration of the form of $\mathcal{H}_P$ as given by (4.69), can be written as

$$0 \approx \left\{ \left((R\Gamma^k)_{ab} \, \psi_{kb} \right)(x), \int \left(u_{lc} \, \Pi^l_c \right)(y) \, d^3\underline{y} \right\}$$

$$k, l = 1, \ldots, 3 \qquad a, b, c = 1, \ldots, 4 \ . \qquad (4.115)$$

(4.115) then gives rise to the condition

$$\tilde{R}\Gamma^k u_k \approx 0 \qquad (4.116)$$

which after making use of equations (4.104), (4.103) and (4.85) and the constraints (4.77) and (4.93) and performing some extensive rearrangement results in the equation

$$\tilde{R} \, \gamma_0 \, \psi_0 + \tilde{R}\Lambda^k \, \psi_k \approx 0 \qquad (4.117)$$

where

$$\Lambda^k = \gamma^k + \left(\frac{i}{2m}\right)\left(\frac{2e}{3m^2}\right)^2 F_{lm} \, D^l \, \gamma^m \, F_{ij} \, \gamma^i \, g^{jk}$$
$$+ \left(\frac{2}{m}\right)\left(\frac{2e}{3m^2}\right) F_{ij} \, D^i \, g^{jk} \quad i, j, k, l, m = 1, \ldots, 3. \quad (4.118)$$

(4.117) is a quaternary Hamiltonian constraint equation which will be denoted by

$$\xi \equiv \tilde{R} \, \gamma_0 \, \psi_0 + \tilde{R}\Lambda^k \, \psi_k \approx 0. \qquad (4.119)$$

In addition the demand that (4.109b) is preserved in time leads correspondingly to the quaternary Hamiltonian constraint

$$\xi^\dagger \equiv \psi_0^\dagger \, \gamma_0 \, \tilde{R} + \psi_k^\dagger \, (\Lambda^k)^\dagger \tilde{R} \approx 0 \qquad (4.120)$$

where the $(D^l)^\dagger$ in $(\Lambda^k)^\dagger$ are given by

$$(D^l)^\dagger = \partial^l + ie A^l . \qquad (4.121)$$

The new tier of quaternary constraints (4.119) and (4.120) must, like all the previously uncovered constraints, be preserved in time. The consistency condition of (4.119) is from (4.53)

$$\begin{aligned} 0 &\approx \left\{ \xi(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \\ &= \left\{ \left(\tilde{R}\gamma_0\, \psi_0 + \tilde{R}\Lambda^k\, \psi_k \right)(x), \int \mathcal{H}_P(y)\, d^3\underline{y} \right\} \end{aligned} \qquad (4.122)$$

and this ultimately becomes

$$\begin{aligned} 0 \approx &\left\{ \left(\left(\tilde{R}\gamma_0 \right)_{ab} \psi_{0b} \right)(x), \int \left(u_{0c}\, \Pi^0_c \right)(y)\, d^3\underline{y} \right\} \\ &+ \left\{ \left(\left(\tilde{R}\Lambda^k \right)_{ab} \psi_{kb} \right)(x), \int \left(u_{lc}\, \Pi^l_c \right)(y)\, d^3\underline{y} \right\} \\ &\qquad k, l = 1, \ldots, 3 \\ &\qquad a, b, c = 1, \ldots, 4 \;. \end{aligned} \qquad (4.123)$$

Equation (4.123) in turn reduces to

$$\tilde{R}\gamma_0\, u_0 + \tilde{R}\Lambda^k\, u_k \approx 0 \qquad (4.124)$$

from which it follows that

$$\tilde{R}u_0 \approx -\, \gamma_0\, \tilde{R}\Lambda^k\, u_k . \qquad (4.125)$$

In this way the consistency condition for (4.119) has determined $\tilde{R}u_0$ in terms of the u_k given by (4.85). In an analogous manner the time preservation condition of (4.120) would result in the determination of $u_0^\dagger \tilde{R}$.

The determination of u_0 is now effected by adding together (4.113) and (4.125), that is

$$u_0 \approx -\gamma_0 \left(\frac{1}{2} R \Gamma^k + \tilde{R} \Lambda^k \right) u_k \tag{4.126}$$

after noting from (4.104) that

$$R + \tilde{R} = I. \tag{4.127}$$

A corresponding combination of the expressions for $u_0^\dagger R$ and $u_0^\dagger \tilde{R}$ would clearly determine $u_0^\dagger$.

At this point the constraint analysis of the system terminates because all the multipliers have now been determined and no more constraints are generated. Furthermore, the new quaternary constraints do not have weakly vanishing Poisson brackets with all the other constraints and so, as in the case when $|R| \neq 0$, all the constraints are second class.

Prior to the investigations of Hasumi, Endo and Kimura [7], whose constraint analysis constitutes the basis of the detailed calculations that have just been considered, a lot of other work had been done on the Rarita–Schwinger field coupled to an external electromagnetic field. In particular, Johnson and Sudarshan [5] observed on quantizing the theory that the anticommutators of this coupled Rarita–Schwinger system were non–positive definite. Later Velo and Zwanziger [4] discovered, at the classical level, that for certain values of the external electromagnetic field the system either propagated acausal modes or did not propagate at all.

One of the main aims of the paper by Hasumi et al was to thoroughly analyse the Johnson–Sudarshan pathology in terms of the Dirac–Bergmann algorithm. In doing this they found that for certain values of the external electromagnetic field, given by (4.102), then the constraints (4.93) and (4.94) could not be used freely as Johnson and Sudarshan had correspondingly assumed in their analysis. Indeed it has just been explicitly demonstrated that when the condition (4.102) holds then a new tier of constraints, that is (4.119) and (4.120), is obtained and this indicates that the original Johnson–Sudarshan analysis was incomplete for these critical field values. This new hierarchy of constraints results in a change in the number of degrees of freedom of the system and the anticommutators of the theory in these cases are far more complicated than the ones derived by Johnson and Sudarshan.

Velo and Zwanziger's acausal modes of propagation were uncovered when the characteristic determinant of their 'true equation of motion' was analysed. This 'true equation of motion' was obtained by freely substituting their secondary constraint into their original equation of motion. At a first glance Velo and Zwanziger's free use of their secondary constraint seems very reminiscent of the Johnson–Sudarshan analysis which ultimately led to the non–positive definiteness of the anticommutators. Furthermore Velo and Zwanziger showed that the onset of acausality occurred when the magnitude of the external electromagentic field satisfied (4.102).

All of this would seem to suggest a common origin for the Johnson–Sudarshan and Velo–Zwanziger inconsistencies. This point was addressed by Kobayashi and Takahashi [13] who converted the Rarita–Schwinger field coupled to an external electromagnetic field into an equivalent constrained mechanical model. With the aid of this mechanical model they demonstrated that the Rarita–Schwinger paradoxes discussed above do indeed have a common origin. They identified this origin as being the non–

existence of a unique inverse of the matrix (4.90) when the external electromagnetic field satisfies the critical condition given by (4.102).

Now Hasumi et al showed that the Johnson–Sudarshan constraint analysis was incomplete and in light of the fact that the Johnson–Sudarshan and Velo–Zwanziger inconsistencies share a common origin, it analogously follows that the Velo–Zwanziger analysis must also be incomplete. The original Velo–Zwanziger treatment was carried out using a version of the Lagrangian constraint algorithm whereas the work of Hasumi et al and the detailed investigation given in this chapter were tackled via the Dirac–Bergmann algorithm. In a recent article Cox [14] completed the Velo–Zwanziger constraint analysis on the Lagrangian side and at the same time he also demonstrated the equivalence of his results to those obtained by Hasumi et al on the Hamiltonian side. Not surprisingly Cox uncovered a new level of constraints for those values of the external field satisfying (4.102) and this again signalled a change in the number of degrees of freedom of the system.

The upshot of all this is that for those critical values of the external field, given by (4.102), Hasumi et al and Cox have respectively shown that the Johnson–Sudarshan and Velo–Zwanziger diseases do not actually surface because the original constraint analyses which led to their discovery were incomplete. Instead it transpires in both cases that a new level of constraints is unearthed and consequently the Johnson–Sudarshan and Velo–Zwanziger inconsistencies are both seen to degenerate to a loss of degrees of freedom.

PART 2

THE GEOMETRICAL APPROACH TO HIGH SPIN FIELD THEORIES

CHAPTER V

A BRIEF REVIEW OF DIFFERENTIAL GEOMETRY

The aim of this chapter is to introduce some of the main concepts of differential geometry that will be required for the subsequent geometrical treatment of dynamical systems.

A Differentiable manifolds

Essentially a manifold is a generalization of the concept of a surface in Euclidean space. However a manifold is defined such that it is a space in its own right rather than being an embedding in some higher dimensional space. More precisely an n–dimensional topological manifold M is a Hausdorff topological space such that for every point $m \in M$ there exists an open set $U \subset M$ and a homeomorphic mapping φ from U to an open set $V \subset \mathbb{R}^n$. The pair (U, φ) is called a chart at m and U is known as the domain of the chart. A chart is also often referred to as a local coordinate system or a parameterization. The coordinates $(x^1, \ldots, x^n)$ of the image $\varphi(m) \in \mathbb{R}^n$ of the point $m \in M$ are called the local coordinates of m in the chart (U, φ). The above definition of a manifold basically states that M is a collection of points which locally, that is within a particular chart domain, look like $\mathbb{R}^n$.

Two charts on a manifold M, (U_1, φ_1) and (U_2, φ_2), are said to be compatible if for $U_1 \cap U_2 \neq \phi$, where ϕ denotes the empty set, then the sets $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are open subsets of $\mathbb{R}^n$ and the overlap maps $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are continuously differentiable to all orders, that is the overlap maps are C^∞. This is illustrated in figure 5.1.

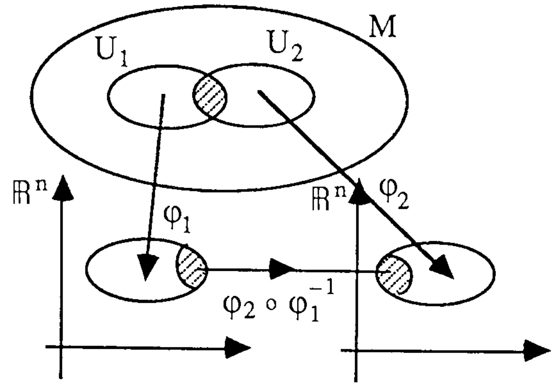

Figure 5.1 Compatibility of two charts on a manifold

This compatibility condition between charts ensures that there is a smooth transformation between the local coordinate system associated with the chart (U_1, φ_1) and the one associated with (U_2, φ_2) in the region covered by both coordinate systems.

An atlas on M is a set of compatible charts $\{(U_s, \varphi_s)\}$ of M such that the set of domains $\{U_s\}$ covers M, that is $M = \bigcup_s U_s$. M may however have many different atlases on it and so the concept of the equivalence of atlases is required. An atlas B_1 on M is equivalent to an atlas B_2 on M if every chart in B_1 is compatible with every chart in B_2 or alternatively the two atlases are equivalent if and only if $B_1 \cup B_2$ is also an atlas. This defines an equivalence relation on the set of all atlases for M and each equivalence class S of atlases is said to be a differentiable structure for M.

A C^∞ differentiable manifold is then defined to be the ordered pair (M, S) where S is a differentiable structure on M. In general a differentiable manifold is simply referred to as a manifold; in other words no reference is made to its associated differentiable structure.

A function g on a manifold M is a mapping $g : M \rightarrow \mathbb{R}$ such that a real number is assigned to each point $m \in M$. Suppose now that (U, φ) is a chart at $m \in M$ then it follows that $g \circ \varphi^{-1}$ is a mapping from an open set $V \subset \mathbb{R}^n$ into $\mathbb{R}$. The mapping $g \circ \varphi^{-1}$ represents the function g in the local chart (U, φ). Furthermore a function g is said to be C^∞ differentiable at m on a C^∞ differentiable manifold M if in a chart (U, φ) at m, the mapping $g \circ \varphi^{-1}$ is C^∞ differentiable at $\varphi(m)$. In view of this a function $g : M \to \mathbb{R}$ is said to be C^∞ on M if it is C^∞ differentiable at each point $m \in M$ and the space of C^∞ functions on M is denoted by either $C^\infty(M)$ or $F(M)$.

Consider now a mapping f between two manifolds M_1, of dimension n say, and M_2, of dimension p, that is $f : M_1 \to M_2$. f is said to be C^∞ differentiable at $m \in M_1$ if there is a chart (U_1, φ_1) at m and a chart (U_2, φ_2) at $f(m) \in M_2$ with $f(U_1) \subset U_2$ such that the map $\varphi_2 \circ f \circ \varphi_1^{-1}$ is C^∞ differentiable at $\varphi_1(m)$. In particular a mapping between manifolds $f : M_1 \to M_2$ is a C^∞ diffeomorphism if f is C^∞ differentiable and in addition f is a bijection and the map $f^{-1} : M_2 \to M_1$ is also C^∞ differentiable. Two manifolds M_1 and M_2 are said to be diffeomorphic if and only if there exists a diffeomorphism between them..

B The tangent space at a point of a manifold

The tangent space of a manifold M at a point $m \in M$ is denoted by T_mM and in essence it models the manifold at m. Put another way T_mM is a local linear approximation to M at m. There are several equivalent ways of defining the tangent space of a manifold and the one that will be adopted here is the so–called curves approach [8]. Basically this is a coordinate independent approach that makes use of the concept of a tangent vector as being the tangent at a point $m \in M$ to a curve on M passing through m.

A curve through $m \in M$ is a mapping $c : I \to M$ from an open interval $I \subset \mathbb{R}$ into M with $0 \in I$ and $c(0) = m$. The curve is said to be smooth if the map c is C^∞ differentiable.

Consider now two curves at $m \in M$, c_1 and c_2, and let (U, φ) be a chart on M such that $m \in U$. Then c_1 and c_2 are said to be tangential at $m \in U$ with respect to the chart (U, φ) if $\varphi \circ c_1$ and $\varphi \circ c_2$, whenever these maps make sense, are tangential at $\varphi(m)$. The tangency of $\varphi \circ c_1$ and $\varphi \circ c_2$ at $\varphi(m)$ is meaningful since this tangency refers to the usual tangency of curves in $\mathbb{R}^n$. All this is illustrated in figure 5.2.

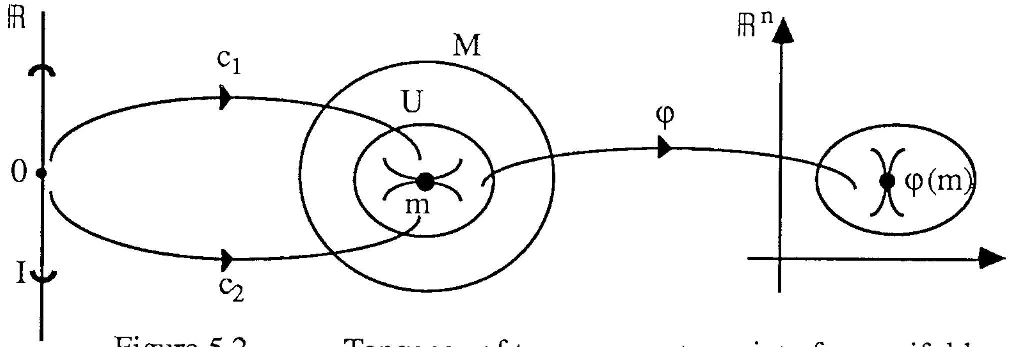

Figure 5.2 Tangency of two curves at a point of a manifold

It follows from the above that two curves are tangent with respect to the chart (U, φ) provided they have identical tangent vectors in (U, φ).

Suppose now that there are two charts (U_1, φ_1) and (U_2, φ_2) where $m \in U_1$ and $m \in U_2$. It can be shown [8] that two curves c_1 and c_2 are tangent at $m \in M$ with respect to (U_1, φ_1) if and only if they are tangent at $m \in M$ with respect to (U_2, φ_2). This ensures that the tangency of any curves at $m \in M$ is independent of the underlying chart as mentioned earlier.

The above idea of tangency at $m \in M$ is in fact an equivalence relation among the curves at m. The equivalence class of such curves at $m \in M$ will be denoted by $[c]_m$ where c is a representative of the class. All curves in a given equivalence class have the same tangent vector at m and so this tangent vector is defined by identifying it with the equivalence class of curves tangent at m.

In light of the above the tangent space T_mM of a manifold M at a point $m \in M$ is defined to be the set of all tangent vectors at m. In other words

$$T_mM = \{[c]_m : c \text{ is a curve at } m \in M\}. \tag{5.1}$$

It should be noted that the tangent space T_mM is a vector space isomorphic to $\mathbb{R}^n$ if M is an n–dimensional manifold. Since T_mM is a vector space it follows that it must have a basis. It can be shown [15] that if the coordinates of a neighbourhood of $m \in M$ are denoted by $(x^1, \ldots, x^n)$ then a basis for T_mM is given by $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$. This basis is known as the natural basis for T_mM.

The tangent bundle of M is the union of all its tangent spaces and it will be denoted by TM. Thus

$$TM = \bigcup_{m \in M} T_mM. \tag{5.2}$$

Consider now a mapping between an n–dimensional manifold M_1 and a p–dimensional manifold M_2, that is $f : M_1 \rightarrow M_2$ and suppose that f is C^∞ differentiable at $m \in M_1$. Then there is an associated linear mapping between the tangent spaces of M_1 and M_2 given by $f_* : T_mM_1 \rightarrow T_{f(m)}M_2$ such that a tangent vector in T_mM_1 is mapped into a tangent vector in $T_{f(m)}M_2$. If in terms of tangents to curves through m the equivalence class $[c]_m$ denotes a tangent vector $v_m \in T_mM_1$ then $f_* v_m \in T_{f(m)}M_2$ is given by

$$f_*([c]_m) = [f \circ c]_{f(m)}. \tag{5.3}$$

The mapping f_* is known as the push–forward map. Furthermore if there is a third manifold M_3 of dimension q and a C^∞ differentiable mapping between manifolds $h : M_2 \rightarrow M_3$ then

$$(h \circ f)_* = h_* \circ f_*. \tag{5.4}$$

C Fibre bundles

The idea of the tangent bundle of a manifold introduced in section B of this chapter is just a particular example of a more general geometric structure known as a fibre bundle. In general terms a bundle [16] is a triple (B, M, Π) where B is a topological space known as the bundle space, M is also a topological space known as the base space and $\Pi : B \rightarrow M$ is a continuous surjective map called the projection map. The inverse image $\Pi^{-1}(m)$ for $m \in M$ is termed the fibre at m and is denoted by F_m. Furthermore if for all $m \in M$, $\Pi^{-1}(m)$ is homeomorphic to some common space F then F is called the typical fibre and the bundle is said to be a fibre bundle. In the specific case when the space F is a vector space then the fibre bundle is termed a vector bundle.

A C^∞ differentiable bundle is then one which satisfies the above conditions except that B and M are now C^∞ differentiable manifolds and the projection map Π is also C^∞ differentiable.

For convenience a bundle (B, M, Π) is often just denoted by the bundle space B.

The tangent bundle TM will now be described in terms of the above fibre bundle language. TM can be given a fibre bundle structure and this consists of the base manifold M, the bundle manifold itself TM, which is represented by (m, v_m) for all $m \in M$ and all $v_m \in T_mM$, and a projection map $\tau_M : TM \rightarrow M$ which is such that

$$\tau_M(m, v_m) = m . \qquad (5.5)$$

The fibre at m is given by $\tau_M^{-1}(m)$ and in this case it is the tangent space at m, that is T_mM. In addition the typical fibre is $\mathbb{R}^n$. Now if M is an n–dimensional manifold then TM is a 2n–dimensional manifold. Suppose that in a chart (U, φ) the point

$m \in M$ has coordinates $(x^1, \ldots, x^n)$ and the components of v_m are $\left(v_m^1, \ldots, v_m^n\right)$ then the bundle TM has local coordinates $\left(x^1, \ldots, x^n, v_m^1, \ldots, v_m^n\right)$.

Consider now a general bundle (B, M, Π). A cross–section of this bundle is a mapping $\chi : M \to B$ with the property that

$$\Pi \circ \chi = id_M \tag{5.6}$$

where id_M is the identity on M. If χ is a C^∞ differentiable mapping then a vector field on M is defined to be a cross–section of the tangent bundle TM. In essence a vector field associates to each point $m \in M$ a tangent vector $v_m \in T_mM$ by the mapping $m \to (m, v_m)$. The set of all C^∞ cross–sections of TM is denoted by $\mathfrak{X}(M)$ and so a vector field on M is an element of $\mathfrak{X}(M)$.

The tangent bundle is not the only bundle structure that a differentiable manifold possesses. Before elaborating on this further consider first of all the following. If E is a vector space then its dual space, denoted by E^*, is the space of linear functionals from E to $\mathbb{R}$. Suppose that a basis in E is given by $\hat{e} = (e_1, \ldots, e_n)$, then a basis in E^*, given by $\hat{\alpha} = (\alpha_1, \ldots, \alpha_n)$, satisfies

$$e_i(\alpha_j) = \langle e_i | \alpha_j \rangle = \delta_{ij} \qquad i, j = 1, \ldots, n \tag{5.7}$$

where $\langle \, | \, \rangle$ denotes the natural pairing such that $E \times E^* \to \mathbb{R}$. Vectors in the space E are said to be contravariant whilst those in E^* are covariant.

Returning now to the consideration of other bundle structures on M suppose that the vector space E is in fact the tangent space to M at m, that is T_mM. The dual space of T_mM, T_m^*M, is called the cotangent space at $m \in M$ and its elements are called cotangent vectors or covariant vectors in contrast to elements of T_mM which are sometimes called contravariant vectors. It can be shown [15] that if $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ is the natural basis for T_mM in some neighbourhood of $m \in M$, then a natural basis for T^*_mM in this neighbourhood is given by $\left(dx^1, \ldots, dx^n\right)$ and from a comparison with (5.7) it immediately follows that

$$\left\langle \frac{\partial}{\partial x^i} \middle| dx^j \right\rangle = \delta^j_i \qquad i, j = 1, \ldots, n. \tag{5.8}$$

In direct analogy to the case of the tangent bundle the union of all cotangent spaces, that is $\bigcup_m T^*_mM$, can also be given a bundle structure and this is known as the cotangent bundle T^*M. The cotangent bundle consists of the triple $\left(T^*M, M, \tau^*_M\right)$ where T^*M is represented by $\left(m, \omega_m\right)$ for all $m \in M$ and all $\omega_m \in T^*_mM$ and τ^*_M is the projection map $\tau^*_M : T^*M \to M$ which is such that

$$\tau^*_M\left(m, \omega_m\right) = m. \tag{5.9}$$

A covariant vector field is a C^∞ cross–section of the cotangent bundle T^*M and it is often called a 1–form. In other words a 1–form on M is the assignment of a covariant vector $\omega_m \in T^*_mM$ at each point $m \in M$. The set of all C^∞ cross–sections of T^*M is denoted by $\mathcal{X}^*(M)$.

The above ideas behind the tangent and cotangent bundles can be generalized and this leads to the concept of tensor bundles on differentiable manifolds. A tensor of type (r, s) at a point m of a manifold M is given in terms of a multilinear mapping [16]

$$\left(\overset{r}{\times}\, T^*_mM\right) \times \left(\overset{s}{\times}\, T_mM\right) \to \mathbb{R}\,. \tag{5.10}$$

(5.10) indicates that the Cartesian product of $T^*_m M$ r times and $T_m M$ s times is mapped into the reals. The set of tensors of type (r, s) at $m \in M$ constitutes a tensor space of the manifold M at the point m, $T^{r,s}_m(M)$, which is said to be contravariant of order r and covariant of order s. As was seen in the tangent and cotangent cases it is now possible to take the manifold M together with the set of tensor spaces of type (r, s) suggested by (5.10) for all $m \in M$ and give this a bundle structure. The result of this is a tensor bundle of type (r, s) which is denoted by $T^{r,s}(M)$. It then follows, in direct analogy to the previous arguments, that a tensor field of type (r, s) on M is a C^∞ cross–section of $T^{r,s}(M)$ or equivalently a tensor field of type (r, s) is the assignment of a tensor of type (r, s) at each point $m \in M$. Finally it should be noted in passing that $T^{1,0}(M)$ and $T^{0,1}(M)$ can be identified with the tangent bundle TM and the cotangent bundle T*M respectively.

D Differential forms and their properties

Consider now the important case of the tensor space which is covariant of order k at a point m of the manifold M, that is $T^{0,k}_m(M)$. From (5.10)

$$T^{0,k}_m(M) : \left(\overset{k}{\times} T_m M\right) \to \mathbb{R} \tag{5.11}$$

and from these multilinear maps consider only the ones which are totally antisymmetric. These antisymmetric maps form a subspace $\Lambda^k_m(M)$ of $T^{0,k}_m(M)$ and its elements are known as exterior differential k–forms on M at a point m. As before the spaces of exterior differential k–forms at all points $m \in M$ can be collected together to form a bundle $\Lambda^k(M)$, a C^∞ cross–section of which is an exterior differential k–form field on M. The set of C^∞ cross–sections of $\Lambda^k(M)$ will be denoted by $\Omega^k(M)$. Exterior differential k–form fields are often simply referred to as k–forms.

Some elementary properties of k–forms will now be outlined. The sum of two k–forms is itself a k–form and the product of a k–form with a function is still a k–form. In addition if for a given k–form on M, $k > n$, where n is the dimension of M, then the k–form is identically zero [15].

It should be noted that the bundle of k–forms on M, that is $\Lambda^k(M)$, is more than just a vector space; it is in fact a module over $F(M) = C^\infty(M)$, the space of all C^∞ functions on M. In the simplest of terms a module over F(M) is essentially a generalization of a vector space in which the scalars are elements of F(M). Furthermore a C^∞ function on M can be viewed as a 0–form, that is to say an element of $\Omega^0(M)$, and in view of this it follows that $\Omega^0(M) = F(M)$. In addition it should be noted that $\Omega^1(M) = \mathfrak{X}^*(M)$.

Now between forms there is defined a multiplication known as the exterior or wedge product which is denoted by $\wedge$. If $\alpha, \eta \in \Omega^k(M)$, $\beta, \sigma \in \Omega^l(M)$, then the exterior product satisfies the following conditions :-

i)
$$\alpha \wedge \beta \in \Omega^{k+l}(M) \tag{5.12a}$$

and

$$\alpha \wedge \beta = (-1)^{kl}(\beta \wedge \alpha) . \tag{5.12b}$$

The condition given by (5.12b) shows that the exterior product is in general not commutative.

ii) The exterior product is associative, that is

$$(\alpha \wedge \beta) \wedge \tau = \alpha \wedge (\beta \wedge \tau) . \tag{5.12c}$$

iii) The exterior product is distributive in that

$$\alpha \wedge (\beta + \sigma) = (\alpha \wedge \beta) + (\alpha \wedge \sigma) \tag{5.12d}$$

and

$$(\alpha + \eta) \wedge \beta = (\alpha \wedge \beta) + (\eta \wedge \beta) . \tag{5.12e}$$

iv)

$$f \wedge \alpha = f\alpha \tag{5.12f}$$

and

$$f(\alpha \wedge \beta) = (f\alpha) \wedge \beta = \alpha \wedge (f\beta) . \tag{5.12g}$$

Consider now a local coordinate system $x = (x^1, \ldots, x^n)$, then a 1–form ω on M, that is $\omega \in \Omega^1(M)$, is given in terms of the natural basis for $T^*_m M$, $(dx^1, \ldots, dx^n)$, by

$$\omega = a_i(x)\, dx^i \qquad i = 1, \ldots, n \tag{5.13}$$

where the $a_i(x)$ are to be regarded as either 0–forms or equivalently C^∞ functions. More generally it can be shown [15] that in a chosen chart then a basis for k–forms on an n–dimensional manifold M is given by the $\binom{n}{k}$ independent k–forms

$$\left\{ dx^{I_1} \wedge \ldots \wedge dx^{I_k} : I_j = 1, \ldots, n \right\} \tag{5.14}$$

where the capital I's indicate ordered natural numbers, that is $I_j < I_{j+1}$. From this it follows that a 2–form σ can be written locally as

$$\sigma = \sum_{1 \le I_1 < I_2 \le n} a_{I_1 I_2}\, dx^{I_1} \wedge dx^{I_2} \equiv \frac{1}{2!}\, a_{ij}\, dx^i \wedge dx^j$$

$$i, j = 1, \ldots, n \qquad (5.15)$$

and locally a k–form α can be expressed as

$$\alpha = \sum_{1 \le I_1 < \ldots < I_k \le n} a_{I_1 \cdots I_k}\, dx^{I_1} \wedge \ldots \wedge dx^{I_k}$$

$$\equiv \frac{1}{k!}\, a_{i_1 \cdots i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \qquad i_j = 1, \ldots, n \qquad (5.16)$$

where the a_{ij} in (5.15) and the $a_{i_1 \cdots i_k}$ in (5.16) are totally antisymmetric. A k–form is C^∞ differentiable provided that all its strict components $a_{I_1 \cdots I_k}(x)$ are C^∞ functions of x.

Besides the exterior product between forms there is a unique operator d, known as the exterior derivative, which acts on forms. If $\alpha, \eta \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ and $f \in F(M)$ then the exterior derivative possesses the following properties :-

i) $$d\alpha \in \Omega^{k+1}(M). \qquad (5.17a)$$

Thus d maps a C^∞ differentiable k–form into a C^∞ differentiable (k + 1)–form.

ii) The exterior derivative is linear in that

$$d(\alpha + \eta) = d\alpha + d\eta \qquad (5.17b)$$

and if λ is a constant then

$$d(\lambda\alpha) = \lambda(d\alpha). \tag{5.17c}$$

iii) $$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta). \tag{5.17d}$$

iv) $$d^2 = 0. \tag{5.17e}$$

v) df is the ordinary differential of f.

vi) The mapping d is a local operation in that if α and η coincide on an open set $U \subset M$ then $d\alpha = d\eta$ on U. In other words the behaviour of α outside of U does not affect $d\alpha$ on U.

A k–form α is said to be exact if it is of the form $\alpha = d\xi$ for some $(k-1)$–form ξ and closed if $d\alpha = 0$. Clearly from (5.17e) every exact form is closed but the converse is not generally true. In fact only locally is it true to say that a closed form is exact and this is known as the Poincaré lemma [17].

In addition to the exterior derivative which from (5.17a) was seen to raise the degree of a form there exists a contraction operation which lowers the degree of a form. This contracted multiplication is known as the inner or interior product. Suppose that $X \in \mathfrak{X}(M)$ is a vector field on M and $\alpha \in \Omega^k(M)$ then the inner product of α by X is the differential form, $i_X\alpha$ or $i(X)\alpha$, of degree $(k-1)$ defined by

$$(i_X\,\alpha)(X_1, \ldots, X_{k-1}) = \alpha(X, X_1, \ldots, X_{k-1}). \tag{5.18}$$

(5.18) makes sense since a consideration of (5.11) suggests that a k–form can be defined as a functional on a set of k vector fields and likewise a $(k-1)$–form is a functional of $(k-1)$ vector fields. Now if $\alpha, \eta \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $\omega \in \Omega^1(M)$, $f \in F(M)$ and $X \in \mathfrak{X}(M)$ then the inner product satisfies the following :-

i) $$i(X)\alpha \in \Omega^{k-1}(M). \tag{5.19a}$$

This merely re–iterates the fact that the inner product takes a k–form into a (k – 1)–form.

ii) The inner product is linear in that

$$i(X)(\alpha + \eta) = i(X)\alpha + i(X)\eta \qquad (5.19b)$$

and if λ is a constant then

$$i(X)(\lambda\alpha) = \lambda(i(X)\alpha). \qquad (5.19c)$$

iii)
$$i(X)(\alpha \wedge \beta) = (i(X)\alpha) \wedge \beta + (-1)^k \alpha \wedge (i(X)\beta)\,. \qquad (5.19d)$$

iv)
$$(i(X))^2 = 0. \qquad (5.19e)$$

v)
$$i(X)f = 0. \qquad (5.19f)$$

vi)
$$i(X)\omega = \omega(X). \qquad (5.19g)$$

(5.19g) expresses the fact that the inner product takes a 1–form into a 0–form, that is a C^∞ function.

vii) The inner product is a local operation.

Consider now property vi) of the interior product in a local coordinate system $(x^1, \ldots, x^n)$ in the specific case when the 1–form ω is given by $\omega = df$ for some $f \in F(M)$. From property v) of the exterior derivative it follows that ω can locally be written as

$$\omega = df = \frac{\partial f}{\partial x^i}\, dx^i \qquad i = 1, \ldots, n. \qquad (5.20)$$

In terms of the natural basis for T_mM, $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$, the vector field X can be expressed as

$$X = X^j(x) \frac{\partial}{\partial x^j} \qquad j = 1, \ldots, n, \tag{5.21}$$

where the X^j are its components. From (5.19g) it follows that

$$i(X)\omega = i(X)df = df(X) = \langle X|df\rangle = X^i \frac{\partial f}{\partial x^i} \qquad i = 1, \ldots, n \tag{5.22}$$

after use of (5.8).

(5.22) is in fact the local version of an operation known as the Lie derivative of a function f with respect to the vector field X. From this it follows that if $f \in F(M)$ and $X \in \mathfrak{X}(M)$ then the Lie derivative of f with respect to X, which is denoted by L_Xf, is defined to be

$$L_Xf = i_X\, df. \tag{5.23}$$

The Lie derivative of a function is a local operation which is linear in that if $f, g \in F(M)$ and $X \in \mathfrak{X}(M)$ then

$$L_X(f+g) = L_Xf + L_Xg. \tag{5.24}$$

Furthermore it also satisfies the condition

$$L_X(fg) = (L_Xf)g + f(L_Xg)\,. \tag{5.25}$$

A consideration of (5.23) and (5.19g) reveals that the Lie derivative maps functions into functions.

The above concept of the Lie derivative of a function can readily be extended so that it acts on other geometric objects. Consider first of all the case of the Lie derivative acting on forms. Suppose $\alpha, \eta \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ and $X \in \mathfrak{X}(M)$ then the Lie derivative satisfies

$$L_X(\alpha + \eta) = L_X\alpha + L_X\eta \tag{5.26}$$

and

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta) \tag{5.27}$$

where (5.26) and (5.27) are the obvious generalizations of (5.24) and (5.25) respectively for forms. The Lie derivative also commutes with the exterior derivative, that is

$$L_X(d\alpha) = d(L_X\alpha)\,. \tag{5.28}$$

It is apparent that the Lie derivative maps k–forms into k–forms.

As a specific example to illustrate some of the properties of the Lie derivative suppose that $f, g \in F(M)$ and $X \in \mathfrak{X}(M)$ then the Lie derivative of the 1–form $\omega = f\,dg$ is found from (5.27) to be

$$L_X(f\,dg) = (i_X\,df)dg + fd(i_X\,dg) \tag{5.29}$$

after use of (5.23) and (5.28). By appealing to the properties of the exterior derivative and the inner product it is easy to show that

$$(i_X\,d + di_X)(f\,dg) = L_X\,(f\,dg). \tag{5.30}$$

(5.30) indicates that

$$L_X = i_X d + d i_X \tag{5.31}$$

for the arbitrary 1–form $\omega = f\, dg$. Furthermore with L_X as given by (5.31) it immediately follows that $L_X f$, for an arbitrary function f, reduces to the definition (5.23) after use of (5.19f). As a matter of fact (5.31) holds quite generally for higher degree forms and it is known as Cartan's identity. In light of (5.31) and (5.19e) it is straightforward enough to show that the Lie derivative also commutes with the inner product, that is

$$L_X i_X = i_X L_X . \tag{5.32}$$

The Lie derivative can also act on vector fields. In order to see this suppose that $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \Omega^k(M)$ then

$$L_X\big(i_Y\alpha\big) = i_{L_X Y}\alpha + i_Y\big(L_X\alpha\big) . \tag{5.33}$$

This result essentially follows from (5.25) if $i_Y\alpha$ is treated as a product, that is $i_Y\alpha = \alpha(Y)$. In the case when α is a 1–form given by $\alpha = df$, for some $f \in F(M)$, then (5.33) gives rise to

$$i_{L_X Y} df = L_X\big(i_Y\, df\big) - i_Y\big(L_X\, df\big) \tag{5.34}$$

and this can equivalently be written as

$$i_{L_X Y}\, df = \big(i_X\, d\, i_Y\, d - i_Y\, d i_X\, d\big)\, f \tag{5.35}$$

after making use of (5.23), (5.28), (5.31) and the fact that

$$i_X\, i_Y\, df = 0. \tag{5.36}$$

Now in a local coordinate system $\left(x^1, \ldots, x^n\right)$ the left–hand side of (5.35) is, by analogy with (5.22), given by

$$i_{L_X Y}\, df = \left(L_X Y\right)^i \frac{\partial f}{\partial x^i} \qquad i = 1, \ldots, n \tag{5.37}$$

whereas it can be shown that the right–hand side of (5.35) is locally given by

$$\left(i_X\, di_Y d - i_Y di_X d\right)f = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)(x) \frac{\partial f}{\partial x^i} \qquad i, j = 1, \ldots, n \tag{5.38}$$

after a consideration of the form of (5.21) and making use of (5.22). However it is easy to show locally that

$$[X, Y](x)f = (XY - YX)(x)f = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)(x) \frac{\partial f}{\partial x^i}$$
$$i, j = 1, \ldots, n. \tag{5.39}$$

Since f is arbitrary it follows from the local arguments given by (5.37), (5.38) and (5.39) that the Lie derivative of Y with respect to X is

$$L_X Y = [X, Y] \tag{5.40}$$

after using (5.35). The right–hand side of (5.40) is known as the Lie bracket of the vector fields X and Y. It should be noted that (5.40) was obtained by considering the arbitrary 1–form $\alpha = df$, but the result still holds for forms of higher degree.

Suppose now there is a mapping f between the n– and p–dimensional manifolds M_1 and M_2 respectively, that is $f : M_1 \rightarrow M_2$, which is C^∞ differentiable at $m \in M_1$. The map f induces a map f^*, called the pull–back map, on k–forms such that if

$\Lambda^k_m(M_1)$ and $\Lambda^k_{f(m)}(M_2)$ represent the space of k–forms at $m \in M_1$ and $f(m) \in M_2$ respectively, then

$$f^* : \Lambda^k_{f(m)}(M_2) \rightarrow \Lambda^k_m(M_1). \tag{5.41}$$

If $v_m \in T_m M_1$ then as seen in section B of this chapter $f_* v_m \in T_{f(m)} M_2$ and with this in mind the pull–back $(f^*\alpha)$ of the k–form α on M_2 is defined to be

$$(f^*\alpha)(m)\left(v_1, \ldots, v_k\right) = \alpha(f(m))\left(f_* v_1, \ldots, f_* v_k\right). \tag{5.42}$$

The k–form $(f^*\alpha)$ is also known as the form induced by f from α. Furthermore if $\alpha, \eta \in \Omega^k(M_2)$ and $\beta \in \Omega^l(M_2)$ then the pull–back map satisfies the following :-

i) $$f^*(\alpha + \eta) = f^*\alpha + f^*\eta . \tag{5.43a}$$

From this it follows that the pull–back map is linear. Also if λ is a constant then

$$f^*(\lambda\alpha) = \lambda(f^*\alpha). \tag{5.43b}$$

ii) $$f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta). \tag{5.43c}$$

iii) The exterior derivative d commutes with f^*, that is

$$f^*(d\alpha) = d(f^*\alpha). \tag{5.43d}$$

In the instance when there is a third manifold M_3 of dimension q, say, and a C^∞ differentiable mapping h where $h : M_2 \rightarrow M_3$ then the composite map

$(h \circ f) : M_1 \rightarrow M_3$ induces the map $(h \circ f)^*$ such that k–forms on M_3 are pulled back to M_1. In other words

$$(h \circ f)^* \; : \; \Lambda^k_{h(f(m))}(M_3) \; \rightarrow \; \Lambda^k_m(M_1) \tag{5.44}$$

and since the map f^* is given by (5.41) and

$$h^* \; : \; \Lambda^k_{h(f(m))}(M_3) \; \rightarrow \; \Lambda^k_{f(m)}(M_2) \tag{5.45}$$

it follows from (5.41), (5.44) and (5.45) that

$$(h \circ f)^* \; = \; f^* \circ h^* \, . \tag{5.46}$$

Additionally in the simple case when f is a C^∞ differentiable mapping $f : M_1 \rightarrow M_2$ and g is a C^∞ differentiable function on M_2, that is $g : M_2 \rightarrow \mathbb{R}$, then the pull–back of g under f is given by

$$f^*(g) \; = \; g \circ f. \tag{5.47}$$

CHAPTER VI

A REVIEW OF HAMILTONIAN MECHANICS IN GEOMETRICAL FORM

This chapter describes how the usual equations of finite dimensional Hamiltonian mechanics, as discussed in section A of chapter II, can be expressed in a coordinate independent geometrical framework. Only systems without any explicit time dependence will be considered in this analysis.

Suppose that the differentiable manifold Q represents the configuration space of some dynamical system with n degrees of freedom and that locally Q has coordinates given by q_i, for $i = 1$ to n. The tangent bundle of Q, TQ, can be identified with velocity phase space and local coordinates on TQ are given by $(q_i, \dot{q}_i)$ where here the $\dot{q}_i$ is merely a notation which does not necessarily imply an explicit time derivative. In view of this the geometrical formulation of Lagrangian mechanics essentially takes place on the manifold TQ. On the other hand the geometrical version of Hamiltonian mechanics takes place on the cotangent bundle of Q, T*Q, which is basically phase space with local coordinates given by (q_i, p_i).

A Symplectic manifolds

Geometrically the Poisson bracket of Hamiltonian mechanics is an example of a symplectic form on the cotangent bundle T*Q of the configuration space Q. In order to elaborate on the idea of a symplectic form consider a general manifold M. Then a symplectic form on M is a strongly nondegenerate closed 2–form ω on M. The term strongly nondegenerate will now be discussed.

Consider first of all a vector space E and a 2–form ω defined on it, that is $\omega : E \times E \rightarrow \mathbb{R}$. In this instance ω is said to be strongly nondegenerate if $\omega(X, Y) = 0$

for all $Y \in E$ implies $X = 0$. There are several different but equivalent ways of characterizing strong nondegeneracy [8] and two of these are given as follows :-

i) Suppose that $\hat{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a basis for E^*, the dual space of E, then in local coordinates the 2–form ω can be written as

$$\omega = \omega_{ij}\ \alpha_i \wedge \alpha_j \qquad i, j = 1, \ldots, n \qquad (6.1)$$

where the ω_{ij} are antisymmetric so that $\omega_{ij} = -\omega_{ji}$. The 2–form ω is then strongly nondegenerate if and only if $|[\omega_{ij}]| \neq 0$. Thus for strong nondegeneracy the antisymmetric matrix $[\omega_{ij}]$ must be non–singular and so it follows that $[\omega_{ij}]$, and consequently E and E^*, must be of even dimension.

ii) Consider the linear map $\flat : E \to E^*$ defined by

$$\langle Y | \flat(X) \rangle = \omega(X, Y) \qquad (6.2)$$

for $X, Y \in E$ where $\langle\, | \,\rangle$ again denotes the natural pairing encountered in section C of chapter V. Then ω is strongly nondegenerate if and only if the map $\flat$ is an isomorphism.

These concepts of strong nondegeneracy described above for the vector space E are readily carried over to a general manifold M. For instance the map $\flat : TM \to T^*M$, which from (6.2) can alternatively be defined as

$$\flat(X) = i(X)\omega \qquad (6.3)$$

where X is a vector field on M, can be thought of in terms of its action on each fibre of TM since a fibre of TM at some point $m \in M$ was seen in section C of chapter V to be the vector space T_mM. A 2–form ω on M is then said to be strongly nondegenerate if and only if $\flat$ is an isomorphism.

The usefulness of a strongly nondegenerate 2–form ω on M is that it can be used to link the spaces TM and T*M. This link is effected by the mapping $\flat$ described above which maps vector fields into 1–forms and the inverse of $\flat$, #, which maps 1–forms into vector fields. Suppose that X is a vector field on M and β is a 1–form on M, that is $X \in \mathfrak{X}(M)$ and $\beta \in \mathfrak{X}^*(M)$, then from (6.3)

$$\begin{aligned} \flat : \mathfrak{X}(M) &\rightarrow \mathfrak{X}^*(M) \\ X &\rightarrow \flat(X) \equiv X^{\flat} = i(X)\omega \end{aligned} \tag{6.4}$$

whereas the map # is defined by

$$\begin{aligned} \# : \mathfrak{X}^*(M) &\rightarrow \mathfrak{X}(M) \\ \beta &\rightarrow \#(\beta) \equiv \beta^{\#} = \flat^{-1}(\beta). \end{aligned} \tag{6.5}$$

The strong nondegeneracy of ω guarantees that $\flat$ is an isomorphism and that its inverse # uniquely exists.

In light of the above a symplectic manifold, denoted by (M, ω), is a manifold M with a symplectic form ω defined on it.

A very important symplectic manifold is the cotangent bundle T*M of some manifold M. Bearing dynamical considerations in mind consider now the specific case of the symplectic manifold T*Q, that is the cotangent bundle of some configuration space Q. It follows from the earlier discussion on strong nondegeneracy that T*Q is even dimensional. In terms of the local coordinates (q_i, p_i) on T*Q the canonical or Liouville 1–form θ is a 1–form on T*Q defined by

$$\theta = p_i \, dq_i \qquad i = 1, \ldots, n. \tag{6.6}$$

From the canonical 1–form (6.6) a symplectic form ω on T^*Q can be constructed in the following way

$$\omega = -d\theta = dq_i \wedge dp_i \qquad i = 1, \ldots, n \tag{6.7}$$

after making use of (5.17d) and (5.12b).

It should be remembered that a general symplectic form ω on a symplectic manifold (M, ω) would, in light of (5.15), normally look like

$$\omega = \omega_{ij} \, dx^i \wedge dx^j \qquad i, j = 1, \ldots, n \tag{6.8}$$

in the local coordinate system $(x^1, \ldots, x^n)$. However Darboux's theorem [15] states that for every point $m \in (M, \omega)$ it is always possible to replace the coordinates $(x^1, \ldots, x^n)$ by new coordinates, in this case $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, known as canonical coordinates such that ω can be written in the form given by (6.7).

B <u>Digression on integral curves and the fibre derivative</u>

First of all the dynamically important concept of an integral curve [16] will be introduced. It was seen in section C of chapter V that a vector field on a manifold M associates to each point $m \in M$ a tangent vector $v_m \in T_m M$. From a geometrical standpoint a vector field can be viewed as the right–hand side of a system of first order differential equations. In order to expand on this suppose that c is a smooth curve on M through $m \in M$, that is from section B of chapter V

$$\begin{aligned} c : I \subset \mathbb{R} &\rightarrow M \\ t &\rightarrow c(t) \end{aligned} \tag{6.9}$$

where $0 \in I$ and $c(0) = m$. In addition suppose that $X \in \mathfrak{X}(M)$ is a vector field on M. If now the tangent at each point $c(t)$ of the curve c is the vector $X(c(t))$, in other words the differential equation

$$\frac{dc(t)}{dt} = X(c(t)) \tag{6.10}$$

is satisfied for all $t \in I$, then the curve c is said to be an integral curve of X through $m \in M$. Equations of the type (6.10) are often encountered when dealing with dynamical systems as will be seen later.

Consider now a local coordinate system $\left(x^1, \ldots, x^n\right)$ on some neighbourhood of the point $m \in M$. Let the coordinates of the point $c(t)$ of the curve c in this neighbourhood be given by

$$c^i(t) = x^i \circ c(t) \qquad i = 1, \ldots, n \tag{6.11}$$

and let the components of the vector field X in this coordinate system be given by X^i. Then locally (6.10) becomes the set of first order differential equations

$$\frac{dc^i(t)}{dt} = X^i(c(t)) \qquad i = 1, \ldots, n \tag{6.12}$$

supplemented by the initial conditions

$$c^i(0) = x^i \circ c(0) = x^i(m) \qquad i = 1, \ldots, n. \tag{6.13}$$

(6.12) exhibits the first order differential equation nature of a vector field. The existence and uniqueness of solutions to (6.12) subject to the initial conditions (6.13) is encompassed in the standard theory of ordinary differential equations.

The ideas behind the fibre derivative map will now be described. In geometrical terms it has been seen that Lagrangian mechanics takes place on the tangent bundle TQ of some configuration space Q. From this it follows that a Lagrangian function L is a function on TQ, that is $L : TQ \rightarrow \mathbb{R}$. The background for the geometrical formulation of Hamiltonian mechanics is T^*Q and in general there is no canonical isomorphism between TQ and T^*Q. However given a Lagrangian L on TQ it is then possible to define a preferred map $\hat{F}L : TQ \rightarrow T^*Q$ which is known as the fibre derivative. The fibre derivative gets its name from the fact that it can be defined in terms of its action on a fibre. In order to see this suppose that $v_q, w_q \in T_qQ$ and $L_q : T_qQ \rightarrow \mathbb{R}$ where L_q is defined to be

$$L_q = L|T_qQ \qquad\qquad q \in Q, \tag{6.14}$$

that is L_q is the restriction of L to the fibre T_qQ over $q \in Q$. The fibre derivative $\hat{F}L_q$ of L_q is then defined to be

$$\langle w_q | \hat{F}L_q\,(v_q) \rangle = \frac{d}{dt}\, L_q\,(v_q + tw_q)\Big|_{t=0} \tag{6.15}$$

where $\langle\,|\,\rangle$ is again the natural pairing introduced in section C of chapter V. From the form of the left–hand side of (6.15) it can be seen that $\hat{F}L_q(v_q) \in T_q^*Q$ and consequently $\bigcup_q \hat{F}L_q(v_q)$ for all $q \in Q$ is part of T^*Q. In essence the $\langle w_q | \hat{F}L_q\,(v_q) \rangle$ of (6.15) is the derivative of L_q along the fibre T_qQ over $q \in Q$ in the direction of w_q.

In terms of the local coordinates $(q_i, \dot{q}_i)$ on TQ then the fibre derivative $\hat{F}L$ is such that

$$(q_i, \dot{q}_i) \rightarrow \hat{F}L(q_i, \dot{q}_i) = \left(q_i, \frac{\partial L}{\partial \dot{q}_i}\right) \qquad i = 1, \ldots, n. \tag{6.16}$$

The Lagrangian L is said to be regular if the fibre derivative $\hat{FL}$ is a local diffeomorphism and it can be shown [8] in terms of the local coordinates $(q_i, \dot{q}_i)$ that this condition is given by

$$\left\| \left[\frac{\partial^2 L}{\partial \dot{q}_i \, \partial \dot{q}_j} \right] \right\| \neq 0 \qquad\qquad i, j = 1, \ldots, n. \qquad (6.17)$$

Equation (6.17) is just the regularity condition encountered in section A of chapter II. In the stronger case when $\hat{FL}$ is a global diffeomorphism then L is said to be hyperregular. In the hyperregular case the Lagrangian formalism of mechanics on TQ is equivalent to the Hamiltonian formalism on T*Q.

C The geometrical formulation of Hamiltonian mechanics

With the help of the ideas introduced in sections A and B of this chapter it is now possible to give the geometrical description of Hamiltonian mechanics [8]. From a geometrical viewpoint there are two advantages in considering a dynamical system formulated on T*Q, that is in Hamiltonian form. The first of these is that vector fields can be associated with first order differential equations as seen in section B of this chapter. This ties in nicely with the fact that Hamilton's equations, as derived in section A of chapter II, are themselves a set of first order differential equations. The second advantage is that T*Q has a natural symplectic structure defined on it and this was outlined in section A of this chapter.

Consider now the Hamiltonian function H on $\hat{FL}(TQ) \subset T^*Q$, that is $H : T^*Q \rightarrow \mathbb{R}$, which is defined via

$$H \circ \hat{FL}(w) = \langle w | \hat{FL}(w) \rangle - L(w) \qquad (6.18)$$

where $w \in TQ$, L is the Lagrangian and $\hat{FL}$ is the fibre derivative defined by (6.15). Furthermore suppose that ω denotes a symplectic form on T*Q and that the condition

$$\omega\left(X_H, Y\right) = i(Y)\, dH = (dH) . Y \tag{6.19}$$

is satisfied. In (6.19) Y is an arbitrary vector field on T^*Q, X_H is a vector field on T^*Q known as the Hamiltonian vector field and H is the Hamiltonian given by (6.18). Equation (6.19) can alternatively be written in the form

$$i\left(X_H\right)\omega = \flat\left(X_H\right) = dH \tag{6.20}$$

from a consideration of (6.3). Since ω is strongly nondegenerate then the inverse of $\flat$, that is $\#$, uniquely exists. As a consequence of this the Hamiltonian vector field X_H is uniquely determined by (6.20), that is

$$X_H = \flat^{-1}(dH) = \#(dH) = (dH)^{\#} \tag{6.21}$$

after appealing to (6.5).

The connection between (6.20) and the usual Hamilton equations given by (2.14) and (2.18) will now be investigated. In terms of the local coordinates $\left(q_i, p_i\right)$ on T^*Q then $\left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right)$ represents a basis for vector fields on T^*Q. In view of this consider the Hamiltonian vector field on T^*Q, X_H, given by

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \qquad i = 1, \ldots, n \tag{6.22}$$

where the H in (6.22) is understood to be the Hamiltonian function in the local coordinate system. Equation (6.12) then indicates that $\left(q_i(t), p_i(t)\right)$ can only be an integral curve of X_H given by (6.22) if

$$\frac{dq_i}{dt} \equiv \dot{q}_i = \frac{\partial H}{\partial p_i} \qquad i = 1, \ldots, n \tag{6.23}$$

and

$$\frac{dp_i}{dt} \equiv \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, n. \qquad (6.24)$$

In other words, after comparing equations (6.23) and (6.24) with (2.14) and (2.18) respectively, it follows that $\left(q_i(t),\, p_i(t)\right)$ is an integral curve of (6.22) if and only if Hamilton's equations are satisfied.

Now from an argument similar to (5.22) it can be seen from (6.22) that

$$i(X_H)\, dq_i = \frac{\partial H}{\partial p_i} \qquad i = 1, \ldots, n \qquad (6.25)$$

and

$$i(X_H)\, dp_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, n. \qquad (6.26)$$

In the local coordinate system $\left(q_i,\, p_i\right)$ the symplectic form ω on T^*Q is given by (6.7) and so it follows that

$$\begin{aligned} i(X_H)\omega &= i(X_H)\left(dq_i \wedge dp_i\right) \\ &= \left(i(X_H)\, dq_i\right) \wedge dp_i - dq_i \wedge \left(i(X_H)\, dp_i\right) \qquad i = 1, \ldots, n \qquad (6.27) \end{aligned}$$

after using (5.19d). Substituting (6.25) and (6.26) into (6.27) leads to

$$i(X_H)\omega = \frac{\partial H}{\partial p_i}\, dp_i + \frac{\partial H}{\partial q_i}\, dq_i = dH \qquad i = 1, \ldots, n. \qquad (6.28)$$

(6.28) thus verifies (6.20) in terms of local coordinates provided the Hamiltonian vector field X_H is given in the form of (6.22). The upshot of the above local calculations is that equation (6.20), in conjunction with the vector field given by (6.22), is equivalent to the usual Hamilton equations (2.14) and (2.18). It should be noted however that the geometric formulation of Hamiltonian mechanics, that is (6.20), does not depend on any coordinate system and in addition it is also true globally.

The statement made at the start of section A of this chapter about the Poisson bracket of Hamiltonian mechanics being nothing more than a symplectic form on T^*Q can now be elaborated upon further. Suppose $B, C \in F(T^*Q)$ are C^∞ functions on T^*Q. Then by analogy with (6.20) the symplectic form ω on T^*Q associates the vector fields X_B and X_C to the functions B and C respectively via

$$i(X_B)\omega = dB, \tag{6.29a}$$

$$i(X_C)\omega = dC. \tag{6.29b}$$

Once again the strong nondegeneracy of ω ensures that X_B and X_C uniquely exist and by comparison with (6.21) it follows that

$$X_B = (dB)^\# , \tag{6.30a}$$

$$X_C = (dC)^\# . \tag{6.30b}$$

The Poisson bracket of B and C in terms of ω, X_B and X_C is then defined to be

$$\{B, C\} = \omega(X_B, X_C). \tag{6.31}$$

(6.31) can equivalently be written as

$$\{B, C\} = -i(X_B)\, i(X_C)\omega = i(X_C)\; i(X_B)\omega \tag{6.32}$$

and from (6.29) this simplifies down to

$$\{B, C\} = -i(X_B)\, dC = i(X_C)\, dB. \qquad (6.33)$$

By making use of the definition of the Lie derivative of a function, that is (5.23), it immediately follows that (6.33) can alternatively be expressed as

$$\{B, C\} = -L_{X_B} C = L_{X_C} B. \qquad (6.34)$$

The verification that (6.31) is equivalent to the usual Poisson bracket given by (3.75) will be demonstrated by applying local arguments to (6.33). Suppose in the local coordinate system (q_i, p_i) on T^*Q that dB is given by

$$dB = \frac{\partial B}{\partial q_i}\, dq_i + \frac{\partial B}{\partial p_i}\, dp_i \qquad i = 1, \ldots, n \qquad (6.35)$$

and by analogy with (6.22) that X_C is given by

$$X_C = \frac{\partial C}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial C}{\partial q_i}\frac{\partial}{\partial p_i} \qquad i = 1, \ldots, n. \qquad (6.36)$$

Then by considering a generalization of the local calculation (5.22) it follows from (6.33) that

$$\begin{aligned} \{B, C\} &= i(X_C)\, dB = \langle X_C | dB \rangle \\ &= \left\langle \left(\frac{\partial C}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial C}{\partial q_i}\frac{\partial}{\partial p_i}\right) \middle| \left(\frac{\partial B}{\partial q_i}\, dq_i + \frac{\partial B}{\partial p_i}\, dp_i\right) \right\rangle \\ &= \frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_i} \qquad i = 1, \ldots, n \qquad (6.37) \end{aligned}$$

after substituting for (6.35) and (6.36). Equation (6.37) is indeed the same as the expression for the Poisson bracket given by (3.75). It should be noted that the geometrical version of the Poisson bracket, that is (6.31), is of course coordinate independent.

CHAPTER VII

THE GEOMETRICAL FORMULATION OF THE DIRAC–BERGMANN ALGORITHM

In chapter VI it was seen how the equations of finite dimensional regular Hamiltonian dynamics in the explicitly time independent case could be translated into a coordinate independent geometrical formalism. In this chapter the geometrical reformulation of the Dirac–Bergmann constraint algorithm for explicitly time independent singular systems, which was described in a coordinate framework in chapter III, will be reviewed in terms of the Gotay–Nester–Hinds algorithm [9]. This geometrical constraint algorithm basically globalizes the local Dirac–Bergmann theory of constraints. Furthermore the Gotay–Nester–Hinds algorithm is expressed in the language of infinite dimensional symplectic geometry and this thus ensures that it is of direct applicability to field theoretic problems.

At this point it should be remembered that the basic concepts of differential geometry introduced in chapter V were only discussed in the finite dimensional case, that is for manifolds modelled on $\mathbb{R}^n$. However the majority of these differential geometric ideas are directly carried over to the infinite dimensional case [15] provided the manifolds under consideration are Banach manifolds. In the infinite dimensional case a Banach manifold is a manifold which is modelled on an infinite dimensional Banach space. With this in mind it will be assumed for the forthcoming analysis of the geometric constraint algorithm that all manifolds are C^∞ differentiable Banach manifolds and that a submanifold of a Banach manifold is itself a Banach manifold.

Having given the details of the geometric constraint algorithm the remainder of this chapter will be devoted to the geometrical investigation of the massive spin–1 Proca field via this constraint algorithm.

A The Gotay–Nester–Hinds algorithm

Before going on to describe this algorithm the idea of a presymplectic manifold will first of all be introduced. Consider a Banach manifold M and let ω be a closed 2–form on M. Then by analogy with (6.3) in the finite dimensional case there is a linear map $\flat : TM \to T^*M$ defined by

$$\flat(X) \equiv X^\flat = i(X)\omega \qquad (7.1)$$

where X is a vector field on M. There are now three possibilities to consider about the map $\flat$:-

i) The first of these is that the map $\flat$ is an isomorphism. In this case, by analogy with chapter VI, the 2–form ω on M is said to be strongly nondegenerate and (M, ω) is called a symplectic manifold.

ii) The second possibility occurs when $\flat$ is injective but not surjective in which case ω is said to be weakly nondegenerate and (M, ω) is then a weakly symplectic manifold.

iii) The final and most general possibility is that $\flat$ is neither injective nor surjective and when this happens ω is said to be degenerate.

For the sake of simplicity when ω is either a weakly nondegenerate or degenerate 2–form then it will henceforth be referred to as presymplectic. Correspondingly when ω is presymplectic then (M, ω) is a presymplectic manifold.

Furthermore if N is a submanifold of a presymplectic manifold (M, ω) with inclusion map j where $j : N \to M$ then N is said to be a constraint submanifold and (M, ω, N) is called a canonical system.

The starting point of the geometric constraint algorithm is the generalized Hamiltonian–type equation given by

$$i(X)\omega_1 = \alpha_1 \tag{7.2}$$

where α_1 is a closed 1–form on the presymplectic manifold (M_1, ω_1) called the Hamiltonian 1–form and X is a vector field on M_1. The Hamiltonian 1–form α_1 gets its name from the fact that locally it is always possible to find a Hamiltonian H_1 on M_1 such that $\alpha_1 = dH_1$. When $\alpha_1 = dH_1$ then it is readily seen that (7.2) corresponds in essence to (6.20).

The geometric constraint algorithm gives necessary and sufficient conditions for the solvability of X in (7.2). In accordance with the coordinate based Dirac–Bergmann algorithm the geometric constraint algorithm is an iterative procedure which looks for solutions of (7.2) on successively smaller manifolds with consistency of the solutions maintained at each stage. The formalism is quite general in that it only requires the existence of a presymplectic manifold for its execution.

Now if α_1 is in the range of the map $\flat : TM_1 \to T^*M_1$ defined by (7.1) then the Hamilton equations, (7.2), have consistent solutions and can be solved for X giving

$$X = \flat^{-1}(\alpha_1) \tag{7.3}$$

in a manner analogous to (6.21). However α_1 will in general not be in the range of $\flat$ but instead there may exist points of M_1, which are assumed to form a submanifold M_2 of M_1, such that $\alpha_1|M_2$ is in the range of $\flat|M_2$. The situation is then one of trying to solve (7.2) restricted to M_2, that is

$$\big(i(X)\omega_1 - \alpha_1\big)|M_2 \equiv \big(i(X)\omega_1 - \alpha_1\big) \circ j_2 = 0 \tag{7.4}$$

where $j_2 : M_2 \rightarrow M_1$ is the inclusion map. (7.4) possesses solutions but for them to be physically meaningful the motion of the system must be constrained to the submanifold M_2. This will be ensured provided the vector field X is 'tangent' to M_2 in which case the motion will not be able to evolve off M_2 into an unphysical domain. The above 'tangent' condition is guaranteed if $X \in \underline{T_{M_2} M_2}$ where in this instance

$$\underline{T_{M_2} M_2} = (j_2)_* \left(T_{M_2} M_2\right) \tag{7.5}$$

and $T_{M_2} M_2$ represents the assignment of a tangent vector in $T_m M_2$ at each $m \in M_2$, that is $T_{M_2} M_2 \in \mathfrak{X}(M_2)$.

This tangency demand may not necessarily be satisfied and then solutions of (7.2) restricted to M_3, where M_3 is a submanifold of M_2, must be considered. The submanifold M_3 is defined to be

$$M_3 = \left\{ m \in M_2 : \alpha_1(m) \in \underline{\left(T_{M_2} M_2\right)^\flat} \right\} \tag{7.6}$$

where the term $\underline{\left(T_{M_2} M_2\right)^\flat}$ in (7.6) is interpreted as meaning $\flat \, \underline{\left(T_{M_2} M_2\right)}$. Since $T_{M_2} M_2 \in \mathfrak{X}(M_2)$ it follows from (7.5) that $\underline{T_{M_2} M_2} \subset \mathfrak{X}(M_1)$. Now it was seen in section A of chapter VI that the map $\flat$ takes vector fields to 1–forms and so in the present case

$$\flat : \mathfrak{X}(M_1) \rightarrow \mathfrak{X}^*(M_1) . \tag{7.7}$$

In view of (7.7) it can be seen that $\underline{\left(T_{M_2} M_2\right)^\flat} \subset \mathfrak{X}^*(M_1)$.

As before it must now be insisted that all solutions of (7.2) restricted to M_3 are in fact 'tangent' to M_3. This in general will necessitate further restrictions and consequently a further submanifold $M_4 \subset M_3$ will have to be considered and so on.

Proceeding with the algorithm in the manner described above leads to the generation of a sequence of submanifolds

$$M_1 \xleftarrow{j_2} M_2 \xleftarrow{j_3} M_3 \xleftarrow{j_4} \ldots \xleftarrow{j_l} M_l \xleftarrow{j_{l+1}} M_{l+1} \xleftarrow{j_{l+2}} \ldots \tag{7.8}$$

with their respective inclusion maps such that

$$M_1 \supset M_2 \supset M_3 \supset \ldots M_l \supset M_{l+1} \supset \ldots \tag{7.9}$$

where

$$M_{l+1} = \left\{ m \in M_l : \alpha_1(m) \in \underline{\left(T_{M_l} M_l\right)^{\flat}} \right\} \tag{7.10}$$

for $l \geq 1$. The term $\underline{\left(T_{M_l} M_l\right)^{\flat}}$ in (7.10) is here taken to mean $\flat\left(\left(k_l\right)_* \left(T_{M_l} M_l\right)\right)$ where for $l \geq 2$ the map k_l is given by

$$k_l = j_2 \circ j_3 \circ \ldots \circ j_l, \tag{7.11}$$

that is it is the composite of successive inclusion maps between the submanifolds of the sequence given by (7.8). Clearly from (7.11) $k_l : M_l \rightarrow M_1$. In the case when $l = 1$ then k_l is taken to be the identity map on M_1, that is $k_1 : M_1 \rightarrow M_1$. By an argument paralleling the one given after (7.6) it is readily seen that $\underline{\left(T_{M_l} M_l\right)^{\flat}} \subset \mathfrak{X}^*(M_1)$.

In the infinite dimensional case the sequence of submanifolds given by (7.9) may not terminate at all or it may terminate in one of the three possible ways outlined below :-

i) The situation $l = K$ is reached such that $M_K = \phi$, where ϕ is the empty set. This case implies that the generalized Hamilton equations given by (7.2) have no solutions and consequently there is no dynamics.

ii) The situation $l = K$ is reached such that $M_K \neq \phi$ but $\dim M_K = 0$. This results in a constraint submanifold consisting of isolated points and as in i) above there is no dynamics.

iii) There exists a K such that $M_K = M_{K+1}$ with $\dim M_K \neq 0$. This case arises when there is a final constraint submanifold M_K which is non–trivial and on which there are consistent equations of the form

$$\big(i(X)\omega_1 - \alpha_1\big)|M_K = 0 \tag{7.12}$$

with X 'tangent' to M_K. The statement that X is 'tangent' to M_K here means that $X \in \underline{T_{M_K} M_K} = (k_K)_*(T_{M_K} M_K)$ where k_K is given by (7.11) in the case when $l = K$. This final constraint submanifold M_K corresponds to the constraint submanifold N mentioned earlier.

On the other hand if the sequence of submanifolds does not terminate then the final constraint submanifold may be viewed as the intersection M_∞ of all the submanifolds M_l. The situation then reduces to one of the three cases described above depending on whether $M_\infty = \phi$, $\dim M_\infty = 0$ or $\dim M_\infty \neq 0$.

The only systems of interest are those that terminate in the manner of iii). In these cases, due to the nature of the constraint algorithm, there is at least one solution to the Hamilton equations guaranteed and in addition this solution is 'tangent' to M_K. Furthermore it should be noted that any solution X of (7.12) is not unique since any vector field Y on M_1 satisfying

$$i(Y)\omega_1|M_K = 0 \qquad (7.13)$$

which is also in $\underline{(T_{M_K} M_K)}$ can be added to X and this is still a solution of (7.12). In other words $\left(X + \ker \omega_1 \cap \underline{(T_{M_K} M_K)}\right)$ is also a solution of (7.12) where $\ker \omega_1$ is defined to be

$$\ker \omega_1 = \left\{ Y \in T_{M_1} M_1 : i(Y)\omega_1 = 0 \right\}. \qquad (7.14)$$

In addition the final constraint submanifold M_K is maximal in the sense that if P is any other constraint submanifold on which equations (7.2) are satisfied then $P \subset M_K$.

B An alternative formulation of the geometric constraint algorithm

The description of the Gotay–Nester–Hinds algorithm given in the last section is too abstract in that it is of little value when it comes to the explicit determination of the submanifolds M_l. It is therefore necessary to obtain an alternative expression for the submanifolds M_l which is equivalent to (7.10) but is of greater practical use.

Before this redefining of the M_l can be given some new definitions are required. Once again it will be assumed that N is a constraint submanifold of the presymplectic manifold (M, ω) with inclusion map $j : N \to M$. For each $p \in N$ then the symplectic complement, $(T_pN)^{\perp}$, of $j_*(T_pN) = \underline{(T_pN)}$ in T_pM is defined to be

$$(T_pN)^{\perp} = \left\{ z \in T_pM : \omega|N(x, z) = 0 \;\; \forall \; x \in \underline{(T_pN)} \right\} \qquad (7.15)$$

where T_pM denotes the tangent space of M restricted to the points $p \in N$ and $\omega|N$ is the restriction of ω to the points of N. $(TN)^{\perp}$ is then given by

$$(TN)^{\perp} = \bigcup_{p \in N} (T_pN)^{\perp}. \qquad (7.16)$$

An equivalent but sometimes more convenient definition of $(TN)^{\perp}$ is from [9] given by

$$(TN)^{\perp} = \bigcup_{p \in N} \left\{ z \in T_pM : j^*\left[i(z)\omega\right] = 0 \right\}. \tag{7.17}$$

Returning now to the problem of finding a more practical definition of M_1, consider first of all the submanifold M_2. From (7.10) M_2 is formally characterized by

$$M_2 = \left\{ m \in M_1 : \alpha_1(m) \in \left(T_{M_1} M_1\right)^{\flat} \right\} \tag{7.18}$$

and this may equivalently be expressed as

$$M_2 = \left\{ m \in M_1 : \langle Z|\alpha_1\rangle (m) = 0 \quad \forall \ Z \in \left(TM_1\right)^{\perp} \right\} \tag{7.19}$$

where as before $\langle \, | \, \rangle$ denotes a natural pairing which in this case is such that $TM_1 \times T^*M_1 \rightarrow \mathbb{R}$. The consistency conditions $\left\langle (TM_1)^{\perp} | \alpha_1 \right\rangle = 0$ in (7.19) correspond to the secondary constraints of the Dirac–Bergmann algorithm of chapter III and M_2 is known as the secondary constraint submanifold.

The situation is now one of solving (7.4) with the demand that X is 'tangent' to M_2 and this in turn may lead to further consistency conditions. Suppose that $W \in (TM_2)^{\perp}$ is an arbitrary element of $(TM_2)^{\perp}$ then it can be shown [9] that consistency with (7.4) implies

$$\langle W|\alpha_1\rangle \circ j_2 = 0. \tag{7.20}$$

(7.20) may not always hold and consequently (7.2) must be restricted to those points of M_2 where

$$\left\langle (TM_2)^{\perp} | \alpha_1 \right\rangle = 0. \tag{7.21}$$

Continuing in this way, as was seen in section A of this chapter, the algorithm generates a sequence of submanifolds

$$M_1 \xleftarrow{j_2} M_2 \xleftarrow{j_3} M_3 \xleftarrow{j_4} \ldots \tag{7.22}$$

where now for $l \geq 1$ the M_{l+1} are defined by

$$M_{l+1} = \left\{ m \in M_l : \langle Z | \alpha_1 \rangle (m) = 0 \quad \forall \; Z \in \left(TM_l \right)^{\perp} \right\} \tag{7.23}$$

and from (7.17)

$$\left(TM_l \right)^{\perp} = \bigcup_{m \in M_l} \left\{ z \in T_m \, M_l : \left(k_l \right)^* \left[i(z) \omega_1 \right] = 0 \right\}. \tag{7.24}$$

The k_l in (7.24) are given by (7.11) for $l \geq 2$ and k_1 is once again the identity map on M_1. The constraint functions on M_{l-1} which define M_l are a globalization of the l–ary constraints of the Dirac–Bergmann algorithm and these take the form

$$\left\langle \left(TM_{l-1} \right)^{\perp} \middle| \alpha_1 \right\rangle = 0 \tag{7.25}$$

for $l \geq 2$.

As before the geometric constraint algorithm may never terminate or it may terminate in one of the three possible ways as outlined in section A of this chapter. The fact that there always exists at least one solution X to (7.12) in case iii) follows from the general theorem [9] :-

Theorem

The canonical equations given by (7.12) possess solutions 'tangent' to M_K if and only if

$$<(TM_K)^{\perp}|\alpha_1> = 0. \tag{7.26}$$

It should be noted that this theorem is independent of the geometric constraint algorithm. In fact if N is any constraint submanifold of a presymplectic manifold (M, ω) then the equations

$$(i(X)\omega - \alpha)|N = 0 \tag{7.27}$$

have solutions 'tangent' to N if and only if

$$<(TN)^{\perp}|\alpha> = 0. \tag{7.28}$$

The final situation obtained from the geometric constraint algorithm is the canonical system (M_1, ω_1, M_K) with the equations of motion given by (7.12).

C A classification scheme for submanifolds of presymplectic manifolds

It would be useful to have a physically meaningful classification scheme for submanifolds of presymplectic manifolds. Such a classification scheme for symplectic manifolds is discussed in an article by Menzio and Tulczyjew [18]. The generalization of this scheme to the presymplectic case will now be reviewed.

Suppose that (M, ω, N) is a canonical system with inclusion $j : N \rightarrow M$. Now for each $p \in N \subset M$ consider the spaces $j_*(T_pN) = \underline{T_pN} \subset T_pM$ and $(T_pN)^{\perp}$, as given by (7.15). The constraint submanifold is then said to be :-

i) isotropic if for all $p \in N$ $\underline{T_pN} \subset (T_pN)^{\perp}$,

ii) coisotropic or first class if for all $p \in N$ $\underline{T_pN} \supset (T_pN)^{\perp}$,

iii) weakly symplectic or second class if for all $p \in N$ $\underline{T_pN} \cap (T_pN)^{\perp} = \{0\}$,

iv) Lagrangian if for all $p \in N$ $\underline{T_pN} = (T_pN)^{\perp}$.

If N does not fit into any of the above four categories then it is said to be a mixed constraint submanifold.

The importance of this classification scheme is in its application to the final constraint submanifold M_K of the geometric constraint algorithm. In this way the nature of the dynamical system to which (7.12) corresponds can be determined.

D The geometric analysis of the Proca field

In order to illustrate an application of the Gotay–Nester–Hinds algorithm the massive spin–1 Proca field will now be investigated. In choosing this example it will be possible to compare the results of this geometric analysis directly with those of the coordinate dependent Dirac–Bergmann analysis of the Proca field given in section C of chapter III.

The starting point of this geometric constraint algorithm is taken to be the Lagrangian L given by (3.139) where once again the metric convention (2.29) is adopted. A space–time decomposition of (3.139) leads to

$$
\begin{aligned}
L[A, \dot{A}] = \frac{1}{2}\int \Big(& \dot{A}_i \dot{A}_i - 2\dot{A}_i (\partial_i A_0) + (\partial_i A_0)(\partial_i A_0) \\
& - (\partial_i A_j)(\partial_i A_j) + (\partial_i A_j)(\partial_j A_i) + m^2 A_0^2 \\
& - m^2 A_i A_i \Big) \, d^3\underline{x} \qquad\qquad i, j = 1, \ldots, 3 \qquad (7.29)
\end{aligned}
$$

where $\dot{A}_i = \partial_0 A_i$.

For the subsequent analysis all manifolds are assumed to be suitably well–behaved C^∞ Banach manifolds. The configuration space Q in this field theoretic example is taken to be $A_\mu = (A_0, A_i)$ whereas velocity phasespace TQ is the manifold parameterized by $(A_\mu, \dot{A}_\nu)$ where $\mu, \nu = 0, \ldots, 3$.

To apply the Gotay–Nester–Hinds algorithm it is necessary to go over to the Hamiltonian side. It was seen in section B of chapter VI that this transition is effected by the fibre derivative map $\hat{FL}$. In this present case the fibre derivative is defined by

$$\hat{FL}\left(A_\mu, \dot{A}_\nu\right) . \left(\dot{A}_\rho, \dot{B}_\sigma\right) = \dot{D}L\left(A_\mu, \dot{A}_\nu\right) . \dot{B}_\sigma \qquad \mu, \nu, \rho, \sigma = 0, \ldots, 3 \tag{7.30}$$

where $\dot{D}L\left(A_\mu, \dot{A}_\nu\right) . \dot{B}_\sigma$ denotes the Frechét derivative along the fibre parameterized by $\dot{B}_\sigma$. In terms of local coordinates $\dot{D}L\left(A_\mu, \dot{A}_\nu\right)$ is given by $\frac{\delta L}{\delta \dot{A}_\nu}$. All this is essentially a generalization of (6.16) where the q_i and $\dot{q}_i$ have respectively been replaced by A_μ and $\dot{A}_\mu$. Therefore it follows from the Lagrangian given by (7.29), that

$$\begin{aligned} \dot{D}L\left(A_\mu, \dot{A}_\nu\right) . \dot{B}_\sigma &\equiv \frac{\delta L}{\delta \dot{A}_i} \bullet \dot{B}_i + \frac{\delta L}{\delta \dot{A}_0} \bullet \dot{B}_0 \\ &= \int \left(\dot{A}_i \, \dot{B}_i - \left(\partial_i A_0\right) \dot{B}_i \right) d^3\underline{x} \qquad i = 1, \ldots, 3, \end{aligned} \tag{7.31}$$

where the dark dot, •, in (7.31) represents a 'scalar product' over spatial coordinates.

It was seen in section B of chapter VI that the fibre derivative $\hat{FL} : TQ \rightarrow T^*Q$ is only a global diffeomorphism if L is hyperregular. However, in general and certainly in this case, the fibre derivative map is only a map into T^*Q. More specifically $\hat{FL}$ maps TQ into $M_1 = \hat{FL}(TQ) \subset T^*Q$ where M_1 is taken to be a submanifold of T^*Q.

If T^*Q is parameterized by $\left(A_\rho, \Pi^\sigma\right)$ then the natural pairing $\langle \mid \rangle : TQ \times T^*Q \rightarrow \mathbb{R}$ is defined to be

$$\left\langle \left(A_\mu, \dot{A}_\nu\right) \middle| \left(A_\rho, \Pi^\sigma\right) \right\rangle = \langle (A, \dot{A}) | (A, \Pi) \rangle \sim \langle \dot{A} | \Pi \rangle$$

$$= \int \left(\dot{A}_i \, \Pi^i + \dot{A}_0 \, \Pi^0 \right) d^3\underline{x} \qquad i = 1, \ldots, 3. \tag{7.32}$$

A consideration of the form of equations (7.30), (7.31) and (7.32) then indicates that

$$\hat{FL}\left(A_\mu, \dot{A}_\nu\right) = \left(A_\mu, \dot{A}_i - \partial_i A_0\right) \tag{7.33}$$

and this in turn suggests that the canonical momenta should be defined by

$$\Pi^i = \dot{A}_i - \partial_i A_0 \qquad i = 1, \ldots, 3. \tag{7.34}$$

Since Π^0 does not appear in the fibre derivative this means that (7.33) can be written as

$$\hat{FL}\left(A_\mu, \dot{A}_\nu\right) = \left(A_\mu, 0, \Pi^i\right) \tag{7.35}$$

in light of (7.34). The fact that Π^0 does not appear in (7.35) is expressed in terms of local coordinates by

$$\Pi^0 = \frac{\delta L}{\delta \dot{A}_0} = 0 \tag{7.36}$$

which, in accord with (3.164), means that $\dot{A}_0$ does not occur in the Lagrangian given by (7.29).

From these considerations it follows that (7.36) is a primary constraint. Furthermore the submanifold $M_1 \subset T^*Q$ that TQ is mapped into by $\hat{FL}$, is the primary constraint submanifold which is locally characterized by (7.36).

Now as seen in section A of chapter VI, T^*Q always carries a natural symplectic structure irrespective of whether the Lagrangian is regular or singular. Suppose Ω is a symplectic form on T^*Q and that $j_1 : M_1 \to T^*Q$ is the inclusion map. Then the 2–form ω_1 on M_1 obtained by pulling Ω back with j_1, that is

$$\omega_1 = (j_1)^*\Omega, \tag{7.37}$$

is in general presymplectic. The geometric constraint algorithm given in section B of this chapter can now be applied to the presymplectic manifold (M_1, ω_1).

Consider now the symplectic form Ω on T^*Q. In this field theoretic case Ω is given by

$$\Omega(v \oplus \sigma, w \oplus \tau) = \langle v|\tau\rangle - \langle w|\sigma\rangle$$
$$= \int \left(v_i\, \tau^i + v_0\, \tau^0 - w_i\, \sigma^i - w_0\, \sigma^0\right) d^3\underline{x} \qquad i = 1, \ldots, 3 \tag{7.38}$$

after consideration of (7.32). In (7.38) $(v \oplus \sigma)$ and $(w \oplus \tau)$ are tangent vectors to T^*Q. The notation $(v \oplus \sigma)$ will now be elaborated on.

Locally a chart on a manifold M can be considered as a representation of M around some point by a subspace of a vector space E. Let $U \subset E$ be the domain of some chart on M. In terms of this chart then $TU = U \times E$ and $T^*U = U \times E^*$ are respectively a chart on TM and T^*M where E^* is the dual space of E. In light of this and a consideration of the argument after (5.5), a point $m \in TM$ has the local representation $m = (x, u)$ where $x \in U$ and $u \in E$. Correspondingly a point $m' \in T^*M$ has the local representation $m' = (x, \lambda)$ where again $x \in U$ and now $\lambda \in E^*$. In terms of this notation a chart on $T(T^*M)$ is given by

$$T(T^*U) = (U \times E^*) \oplus (E \times E^*). \tag{7.39}$$

From (7.39) the local representation of a tangent vector to T^*M, that is $z \in T_{m'}(T^*M)$ say, is given by

$$z = (x, \lambda) \oplus (w, \tau) \tag{7.40}$$

where $w \in E$ and $\tau \in E^*$. In (7.40) the (x, λ) represents the base point on T^*M of the tangent vector, that is m', whereas (w, τ) represents the tangent vector on T^*M. A shorthand notation for (7.40), which highlights the vector part of the tangent vector, is given by

$$z = w \oplus \tau \tag{7.41}$$

where the base point (x, λ) has been suppressed.

Bearing the above in mind it follows that $(v \oplus \sigma)$ and $(w \oplus \tau)$ in (7.38) are given in full by

$$(v \oplus \sigma) = (A, \Pi) \oplus (v, \sigma), \tag{7.42a}$$

$$(w \oplus \tau) = (A', \Pi') \oplus (w, \tau) \tag{7.42b}$$

where (A, Π) and (A', Π') are the base points on T^*Q of these tangent vectors.

The first step of the geometric constraint algorithm proper is to determine $(TM_1)^{\perp}$. From a consideration of (7.15) for each $m_1 \in M_1$, $(T_{m_1} M_1)^{\perp}$ is the set of all tangent vectors on M_1 which annihilate all other tangent vectors on M_1, that is

$$(T_{m_1} M_1)^{\perp} = \left\{ z \in T_{m_1} M_1 : \omega_1(x, z) = 0 \;\; \forall \; x \in T_{m_1} M_1 \right\}. \tag{7.43}$$

Suppose $x = (A, \Pi) \oplus (v, \sigma) \sim (v \oplus \sigma) \in T_{m_1} M_1$ is an arbitrary tangent vector on M_1 and $z = (A, \Pi) \oplus (w, \tau) \sim (w \oplus \tau) \in T_{m_1} M_1$ is a typical tangent vector on M_1 where $m_1 = (A, \Pi) \in M_1$. Then the term $\omega_1(x, z) = 0$ in (7.43) is given by

$$\begin{aligned} 0 &= \omega_1(x, z) = \omega_1(v \oplus \sigma, w \oplus \tau) \sim \Omega(v \oplus \sigma, w \oplus \tau) \\ &= \int \left(v_i \tau^i + v_0 \tau^0 - w_i \sigma^i - w_0 \sigma^0\right) d^3\underline{x} \end{aligned} \tag{7.44}$$

after use of (7.38). Now since $x = (A, \Pi) \oplus (v, \sigma) \in T_{m_1} M_1$ where $(A, \Pi) \in M_1$ it follows from a consideration of (7.36) and (7.39) that $\Pi^0 = 0$ and $\sigma^0 = 0$. By a similar argument for $z = (A, \Pi) \oplus (w, \tau) \in T_{m_1} M_1$ it is found that $\Pi^0 = 0$ and $\tau^0 = 0$. In view of the fact that $\sigma^0 = \tau^0 = 0$ then (7.44) becomes

$$\int \left(v_i \tau^i - w_i \sigma^i\right) d^3\underline{x} = 0 \qquad i = 1, \ldots, 3. \tag{7.45}$$

However since the tangent vector x is arbitrary it follows that v_i and σ^i are arbitrary and so (7.45) can only hold if $\tau^i = w_i = 0$ for each $i = 1$ to 3. Thus in this way, only w_0 of z is left undetermined and this is arbitrary. Consequently the most general form of $z = (A, \Pi) \oplus (w, \tau) \sim (w \oplus \tau) \in (T_{m_1} M_1)^{\perp}$ is given by

$$z = (w \oplus \tau) = (w_0, 0, 0, 0) \oplus (0, 0, 0, 0) \equiv (w_0, \underline{0}) \oplus (0, \underline{0}) \tag{7.46}$$

where w_0 is arbitrary.

Now given a Lagrangian L on TQ then the fibre derivative $\hat{FL}$ induces a Hamiltonian H on T^*Q in the manner of (6.18). In this particular case $\hat{FL}$ induces a Hamiltonian H_1 on M_1 and from a consideration of (6.18) this is given by

$$H_1 = \langle (A, \dot{A}) | \hat{F}L(A, \dot{A}) \rangle - L[A, \dot{A}] \tag{7.47}$$

and after using (7.29), (7.32) and (7.33) this becomes

$$H_1 = \frac{1}{2} \int \Big(\dot{A}_i \dot{A}_i - (\partial_i A_0)(\partial_i A_0) + (\partial_i A_j)(\partial_i A_j) - (\partial_i A_j)(\partial_j A_i) - m^2 A_0^2 + m^2 A_i A_i \Big) d^3\underline{x} \qquad i, j = 1, \ldots, 3. \tag{7.48}$$

By putting the $\dot{A}_i$ from (7.34) into (7.48) it is found that

$$H_1[A, \Pi] = \frac{1}{2} \int \Big(\Pi^i \Pi^i + 2\Pi^i(\partial_i A_0) + (\partial_i A_j)(\partial_i A_j) - (\partial_i A_j)(\partial_j A_i) - m^2 A_0^2 + m^2 A_i A_i \Big) d^3\underline{x} \qquad i, j = 1, \ldots, 3. \tag{7.49}$$

The next stage of the geometric constraint algorithm is to ensure that the primary constraint (7.36) is preserved in time. From the algorithm this is guaranteed provided the consistency condition

$$\langle (TM_1)^{\perp} | dH_1 \rangle = 0 \tag{7.50}$$

is satisfied. It should be noted that in this analysis the Hamiltonian 1–form α_1 has been replaced by the specific closed 1–form dH_1 .

For an arbitrary tangent vector z on M_1 given by $z = (A, \Pi) \oplus (w, \tau) \sim (w \oplus \tau) \in T_{m_1} M_1$, where $(A, \Pi) \in M_1$, then

$$dH_1(A, \Pi) . (w \oplus \tau) = i(w \oplus \tau)\, dH_1(A, \Pi) \sim \langle T_{m_1} M_1 | dH_1(m_1) \rangle$$

$$= \tau^0 \bullet \frac{\delta H_1}{\delta \Pi^0} + \tau^l \bullet \frac{\delta H_1}{\delta \Pi^l} + w_0 \bullet \frac{\delta H_1}{\delta A_0} + w_l \bullet \frac{\delta H_1}{\delta A_l} \qquad l = 1, \ldots, 3. \tag{7.51}$$

In essence (7.51) is a generalization of (5.22). Furthermore, as seen earlier, the dot in the term $\tau^0 \bullet \frac{\delta H_1}{\delta \Pi^0}$, for example, represents a 'scalar product' over spatial coordinates.

Now as seen before for $z = (A, \Pi) \oplus (w, \tau) \in T_{m_1}M_1$ where $(A, \Pi) \in M_1$, then $\Pi^0 = \tau^0 = 0$. In addition by taking account of the relevant terms of (7.49) it is found that

$$\tau^l \bullet \frac{\delta H_1}{\delta \Pi^l} = \tau^l \bullet \left[\frac{\delta}{\delta \Pi^l} \left(\frac{1}{2} \int \left(\Pi^i \Pi^i + 2\Pi^i(\partial_i A_0) \right) d^3\underline{x} \right) \right]$$

$$= \int \left(\tau^i [\Pi^i + \partial_i A_0] \right) d^3\underline{x} \qquad i, l = 1, \ldots, 3. \tag{7.52}$$

Also

$$w_0 \bullet \frac{\delta H_1}{\delta A_0} = w_0 \bullet \left[\frac{\delta}{\delta A_0} \left(\int \left(\Pi^i (\partial_i A_0) - \frac{m^2}{2} A_0^2 \right) d^3\underline{x} \right) \right]$$

$$= w_0 \bullet \left[\frac{\delta}{\delta A_0} \left(\int \left(\partial_i (\Pi^i A_0) - (\partial_i \Pi^i) A_0 - \frac{m^2}{2} A_0^2 \right) d^3\underline{x} \right) \right] \tag{7.53}$$

after a partial integration. The integral of the first term on the right–hand side of (7.53) can be transformed into a surface integral via Gauss' divergence theorem in the same way as in (3.176). As in chapter III boundary conditions will again only be treated at a formal level. In view of this the surface integral mentioned above will be assumed to vanish at infinity in accord with the argument given after (3.176). (7.53) now becomes

$$w_0 \cdot \frac{\delta H_1}{\delta A_0} = w_0 \cdot \left[\frac{\delta}{\delta A_0} \left(\int \left(-\left(\partial_i \Pi^i\right) A_0 - \frac{m^2}{2} A_0^2 \right) d^3\underline{x} \right) \right]$$

$$= \int \left(w_0\left[-\partial_i \Pi^i - m^2 A_0 \right] \right) d^3\underline{x} \qquad i = 1, \ldots, 3. \qquad (7.54)$$

Furthermore

$$w_l \cdot \frac{\delta H_1}{\delta A_l} = w_l \cdot \left[\frac{\delta}{\delta A_l} \left(\frac{1}{2} \int \left(\left(\partial_i A_j\right)\left(\partial_i A_j\right) - \left(\partial_i A_j\right)\left(\partial_j A_i\right) \right.\right.\right.$$

$$\left.\left.\left. + m^2 A_i A_i \right) d^3\underline{x} \right) \right] \qquad i, j, l = 1, \ldots, 3. \qquad (7.55)$$

(7.55) leads, after some manipulation and consideration of spatial boundary conditions, to

$$w_l \cdot \frac{\delta H_1}{\delta A_l} = \int \left(w_i \left[\partial_j \partial_i A_j - \partial_j \partial_j A_i + m^2 A_i \right] \right) d^3\underline{x}$$

$$i, j, l = 1, \ldots, 3. \qquad (7.56)$$

Hence by putting $\tau^0 = 0$ and equations (7.52), (7.54) and (7.56) into (7.51) it is found that

$$dH_1(A, \Pi) . (w \oplus \tau) = \int \left(\tau^i\left[\Pi^i + \partial_i A_0 \right] + w_0\left[-\partial_i \Pi^i - m^2 A_0 \right] \right.$$

$$\left. + w_i \left[\partial_j \partial_i A_j - \partial_j \partial_j A_i + m^2 A_i \right] \right) d^3\underline{x}$$

$$i, j = 1, \ldots, 3. \qquad (7.57)$$

(7.57) is true for any tangent vector on M_1. Suppose then that $z \in \left(T_{m_1} M_1\right)^{\perp}$. Substituting $z = \left(w \oplus \tau\right) \in \left(T_{m_1} M_1\right)^{\perp}$, as given by (7.46), into (7.57) gives rise to

$$dH_1(A, \Pi) \,.\, \left(w \oplus \tau\right) \sim \left\langle \left(T_{m_1} M_1\right)^{\perp} \middle| dH_1(m_1) \right\rangle$$

$$= \int \left(w_0\left[-\partial_i \Pi^i - m^2 A_0\right]\right) d^3\underline{x} \qquad i = 1, \ldots, 3. \qquad (7.58)$$

It follows from (7.50) and (7.58) that the condition for the time preservation of the primary constraint (7.36) is given by

$$\int \left(w_0\left[-\partial_i \Pi^i - m^2 A_0\right]\right) d^3\underline{x} = 0 \qquad i = 1, \ldots, 3. \qquad (7.59)$$

Now since w_0 is arbitrary then (7.59) immediately implies that

$$\partial_i \Pi^i + m^2 A_0 = 0 \qquad i = 1, \ldots, 3. \qquad (7.60)$$

(7.60) is a secondary Hamiltonian constraint. This secondary constraint restricts the motion of the system to a submanifold $M_2 \subset M_1$ which is determined by (7.60). Clearly, as is to be expected, (7.60) is equivalent to the secondary Hamiltonian constraint (3.180) to within a weak equality.

Continuing with the geometric constraint algorithm it is now necessary to ensure that the secondary constraint (7.60) is preserved in time. This is assured provided the consistency condition

$$\left\langle (TM_2)^{\perp} \middle| dH_1 \right\rangle = 0 \qquad (7.61)$$

holds. It should be noted that (7.61) is just (7.21) with the α_1 replaced by dH_1.

Consequently it is now necessary to find $(TM_2)^{\perp}$. From (7.15) for each $m_2 \in M_2$ then $(T_{m_2} M_2)^{\perp}$ is the set of all vectors in the tangent space of M_1 restricted to M_2 which annihilate all vectors in $(j_2)_*(T_{m_2} M_2) = \underline{T_{m_2} M_2}$, where $j_2 : M_2 \to M_1$ is the inclusion map. In other words

$$(T_{m_2} M_2)^{\perp} = \left\{ z \in T_{m_2} M_1 : \omega_1 | M_2(x, z) = 0 \quad \forall\ x \in \underline{T_{m_2} M_2} \right\}. \qquad (7.62)$$

Suppose $z = (A, \Pi) \oplus (w,\tau) \sim (w \oplus \tau) \in T_{m_2} M_1$ is a typical tangent vector on M_1 restricted to M_2 and that $x = (A, \Pi) \oplus (v,\sigma) \sim (v \oplus \sigma) \in \underline{T_{m_2} M_2}$ is an arbitrary vector, where $(A, \Pi) \in M_2$. Now for $z = (A, \Pi) \oplus (w, \tau) \in T_{m_2} M_1$ where $(A, \Pi) \in M_2 \subset M_1$ it is found as before that $\Pi^0 = \tau^0 = 0$. On the other hand for $x = (A, \Pi) \oplus (v, \sigma) \in \underline{T_{m_2} M_2}$ where $(A, \Pi) \in M_2$ it follows from (7.36) and (7.60) that $\Pi^0 = 0$ and $\partial_i \Pi^i + m^2 A_0 = 0$ and consequently $\sigma^0 = 0$ and $\partial_i \sigma^i + m^2 v_0 = 0$. Therefore the term $\omega_1 | M_2(x, z) = 0$ in (7.62) is given by

$$\begin{aligned} 0 = \omega_1 | M_2(x, z) &= \omega_1 | M_2(v \oplus \sigma, w \oplus \tau) \sim \Omega(v \oplus \sigma, w \oplus \tau) \\ &= \int (v_i \tau^i - w_i \sigma^i)\, d^3\underline{x} \qquad i = 1, \ldots, 3 \end{aligned} \qquad (7.63)$$

after appealing to (7.38) and taking into account the fact that $\tau^0 = \sigma^0 = 0$. Now the v_i are arbitrary and so are the σ^i since they satisfy $\partial_i \sigma^i + m^2 v_0 = 0$ for arbitrary v_0. Consequently (7.63) can only hold if $\tau^i = w_i = 0$ for each $i = 1$ to 3. As in the evaluation of $(T_{m_1} M_1)^{\perp}$ the w_0 of z is undetermined and so remains arbitrary. Therefore

$$(T_{m_2} M_2)^{\perp} = \left\{ (w_0, \underline{0}) \oplus (0, \underline{0}) \in T_{m_2} M_1 : w_0 \text{ is arbitrary} \right\}. \qquad (7.64)$$

Substituting $z = (w \oplus \tau) \in (T_{m_2} M_2)^{\perp}$, as given by (7.64), into (7.57) yields

$$dH_1(A, \Pi) . (w \oplus \tau) \sim \langle (T_{m_2} M_2)^{\perp} | dH_1(m_2) \rangle$$

$$= \int \left(w_0[-\partial_i \Pi^i - m^2 A_0] \right) d^3\underline{x} = 0 \qquad i = 1, \ldots, 3, \qquad (7.65)$$

after using (7.61), as the time preservation condition of the secondary Hamiltonian constraint (7.60). However on M_2, $\partial_i \Pi^i + m^2 A_0 = 0$ and so (7.65) is automatically satisfied.

The geometric constraint analysis of the Proca field is now complete. In order to round off this geometric analysis the equations of motion of the system will now be examined.

Since $\langle (TM_2)^{\perp} | dH_1 \rangle = 0$ then it follows from the comments made after the theorem in section B of this chapter that there is at least one solution to the Hamilton equations

$$i(X)\omega_1 = dH_1 , \qquad (7.66)$$

evaluated on M_2, which is 'tangent' to M_2. In essence (7.66) is just (7.2) with $\alpha_1 = dH_1$. Now for $(A, \Pi) \in M_2$ then the statement that the Hamilton equations (7.66) are evaluated on M_2 can be written as

$$i(X)\omega_1 | (A, \Pi) = dH_1(A, \Pi) . \qquad (7.67)$$

Furthermore for X to be 'tangent' to M_2 then $X \in \underline{T_{M_2} M_2}$.

Suppose now that the vector field X is obtained by assigning a tangent vector of the form $x = (A, \Pi) \oplus (a, \xi) \sim (a \oplus \xi) \in \underline{T_{m_2} M_2}$ at each $m_2 = (A, \Pi) \in M_2$.

For vectors like x it can be seen from previous, similar considerations that $\Pi^0 = \xi^0 = 0$, $\partial_i \Pi^i + m^2 A_0 = 0$ and $\partial_i \xi^i + m^2 a_0 = 0$. In addition suppose that Z is an arbitrary vector field generated by the assignment of an arbitrary tangent vector of the form $z = (A, \Pi) \oplus (b, \eta) \sim (b \oplus \eta) \in T_{m_2} M_1$ at each point $m_2 = (A, \Pi) \in M_2$. Similar arguments to those used before indicate that $\Pi^0 = \eta^0 = 0$ for vectors of the form of z.

In terms of the arbitrary vector field Z, the equations of motion (7.66) can be written as

$$i(Z)\, i(X)\omega_1 = i(Z)\, dH_1 \tag{7.68}$$

and by analogy with (6.19), (7.68) can equivalently be expressed as

$$\omega_1(X, Z) = (dH_1) \, . \, Z \, . \tag{7.69}$$

In terms of the vector fields X and Z defined above then it follows from a consideration of (7.67) and (7.69) that the equations of motion on M_2 are

$$\omega_1(a \oplus \xi, b \oplus \eta)|(A, \Pi) = dH_1(A, \Pi) \, . \, (b \oplus \eta) \, . \tag{7.70}$$

By following a similar approach to that which led to (7.63) it is found that the left–hand side of (7.70) is given by

$$\omega_1(a \oplus \xi, b \oplus \eta)|(A, \Pi) = \int (a_i\, \eta^i - b_i\, \xi^i)\, d^3\underline{x} \qquad i = 1, \ldots, 3. \tag{7.71}$$

On the other hand the right–hand side of (7.70) is given by

$$dH_1(A, \Pi) \, . \, (b \oplus \eta) = \int \big(\eta^i[\Pi^i + \partial_i A_0] + b_i[\partial_j \partial_i A_j - \partial_j \partial_j A_i + m^2 A_i]\big)\, d^3\underline{x} \qquad i, j = 1, \ldots, 3. \tag{7.72}$$

(7.72) is basically just (7.57) after noting that $\partial_i \Pi^i + m^2 A_0 = 0$ on M_2 and that the τ^i and w_i have respectively been replaced by the η^i and b_i. On equating (7.71) and (7.72), it follows that (7.70) gives rise to

$$\int (a_i \eta^i - b_i \xi^i)\, d^3\underline{x} = \int \Big(\eta^i[\Pi^i + \partial_i A_0] + b_i[\partial_j \partial_i A_j - \partial_j \partial_j A_i + m^2 A_i]\Big)\, d^3\underline{x} \qquad i, j = 1, \ldots, 3. \qquad (7.73)$$

By equating the coefficients of the arbitrary η^i and b_i in (7.73) it is found that

$$a_i = \Pi^i + \partial_i A_0 \qquad i = 1, \ldots, 3 \qquad (7.74)$$

and

$$\xi^i = -\partial_j \partial_i A_j + \partial_j \partial_j A_i - m^2 A_i \qquad i, j = 1, \ldots, 3. \qquad (7.75)$$

Now for a vector of the form $x \sim (a \oplus \xi) \in \underline{T_{m_2} M_2}$ it is known that

$$\partial_i \xi^i + m^2 a_0 = 0 \qquad i = 1, \ldots, 3 \qquad (7.76)$$

and so on substituting (7.75) into (7.76) it is found that

$$m^2 a_0 - m^2 \partial_i A_i = 0 \qquad i = 1, \ldots, 3. \qquad (7.77)$$

Due to the fact that $m \neq 0$, then (7.77) becomes

$$a_0 = \partial_i A_i \qquad i = 1, \ldots, 3. \qquad (7.78)$$

Thus the vector field $X \sim x = (a \oplus \xi)$ is from (7.74), (7.75) and (7.78) given by

$$\begin{aligned} X \sim x &= \left((a_0, a_i) \oplus (\xi^0, \xi^i)\right) \\ &= \left((\partial_i A_i, \Pi^i + \partial_i A_0) \oplus (0, \partial_j \partial_j A_i - \partial_j \partial_i A_j - m^2 A_i)\right) \end{aligned} \quad (7.79)$$

where $\xi^0 = 0$ because $x \in \underline{T_{m_2} M_2}$.

The vector field X, given by (7.79), together with the primary constraint (7.36) and the secondary constraint (7.60) constitute the end product of the geometric analysis of the Proca field. Furthermore the final constraint submanifold is M_2 .

Now from the discussion after (7.40) it follows that the (a, ξ) in the representation of the vector field $X \sim x = (A, \Pi) \oplus (a, \xi)$ is in the tangent space and consequently it is found that

$$a_0 = \partial_0 A_0 \equiv \dot{A}_0 , \quad (7.80a)$$

$$a_i = \partial_0 A_i \equiv \dot{A}_i \qquad i = 1,\ldots, 3, \quad (7.80b)$$

$$\xi^0 = \partial_0 \Pi^0 \equiv \dot{\Pi}^0 , \quad (7.80c)$$

$$\xi^i = \partial_0 \Pi^i \equiv \dot{\Pi}^i \qquad i = 1,\ldots, 3. \quad (7.80d)$$

(7.80) comes about essentially as a generalization of the arguments leading to (6.23) and (6.24). In this case the A and Π are the field theoretic versions of the q and p respectively and $X \sim (a \oplus \xi)$ is a generalization of $X_H = \left(\frac{\partial H}{\partial p_i} \oplus \left(-\frac{\partial H}{\partial q_i}\right)\right)$, as given by (6.22).

A comparison of (7.79) and (7.80) thus reveals that the equations of motion of the Proca system are

$$\dot{A}_0 = \partial_i A_i \qquad i = 1,\ldots, 3, \quad (7.81)$$

$$\dot{A}_i = \Pi^i + \partial_i A_0 \qquad i = 1,\ldots, 3, \quad (7.82)$$

$$\dot{\Pi}^0 = 0, \tag{7.83}$$

$$\dot{\Pi}^i = \partial_j \partial_j A_i - \partial_j \partial_i A_j - m^2 A_i \qquad i, j = 1,\ldots, 3. \tag{7.84}$$

These equations of motion will now be compared with the ones obtained in the coordinate dependent analysis given in section C of chapter III. By putting (3.187) into (3.185) it is readily seen, to within the weak equality, that the resulting condition is equivalent to (7.81). Clearly (3.190) is just the weak version of (7.82). Furthermore by substituting F_{ji}, as given by (3.140), into (3.193) it immediately follows that the resulting equation is equivalent to (7.84). Therefore, as was to be expected, the equations of motion obtained via the geometric analysis are the same as those obtained from the corresponding coordinate dependent approach.

Finally, the nature of the Proca system can be investigated by applying the classification scheme, outlined in section C of this chapter, to the canonical system $\left(M_1, \omega_1, M_2\right)$ with inclusion $j_2 : M_2 \rightarrow M_1$. In order to apply the classification scheme in this case it is necessary to consider the spaces $\underline{T_{m_2} M_2} = \left(j_2\right)_*\left(T_{m_2} M_2\right)$ and $\left(T_{m_2} M_2\right)^{\perp}$ at each $m_2 \in M_2$. Suppose then for the sake of argument that $z = \left(A, \Pi\right) \oplus \left(w, \tau\right) \sim \left(w \oplus \tau\right) \in \left(T_{m_2} M_2\right)^{\perp}$ where $m_2 = \left(A, \Pi\right) \in M_2$. The most general form of $z \in \left(T_{m_2} M_2\right)^{\perp}$ is given by (7.64). On the other hand if $z = \left(A, \Pi\right) \oplus \left(w, \tau\right) \sim \left(w \oplus \tau\right) \in \underline{\left(T_{m_2} M_2\right)}$ then as seen previously for $m_2 = \left(A, \Pi\right) \in M_2$, $\Pi^0 = \tau^0 = 0$, $\partial_i \Pi^i + m^2 A_0 = 0$ and $\partial_i \tau^i + m^2 w_0 = 0$. However for the condition $\partial_i \tau^i + m^2 w_0 = 0$ to hold when z is also an element of $\left(T_{m_2} M_2\right)^{\perp}$, then it follows from (7.64) that w_0 must be equal to zero. In view of this it is found that

$$\underline{\left(T_{m_2} M_2\right)} \cap \left(T_{m_2} M_2\right)^{\perp} = \{0\} \tag{7.85}$$

and this is true for all $m_2 \in M_2$. From classification iii) of section C of this chapter it follows immediately that the final constraint submanifold M_2 is in fact weakly symplectic or second class. Consequently the Proca canonical system (M_1, ω_1, M_2) is also weakly symplectic or second class. This is in accord with the coordinate based analysis given in section C of chapter III where the constraints were found to be second class after a consideration of (3.194), (3.195) and (3.196).

CHAPTER VIII

THE APPLICATION OF THE GEOMETRIC CONSTRAINT ALGORITHM TO COUPLED SYSTEMS

The geometrical analysis of a free field was investigated in section D of chapter VII where the Proca field was the field theoretic system under consideration. The purpose of this chapter is to apply the geometric constraint algorithm to two coupled field theoretic systems. The first of these systems will be the Proca field minimally coupled to an external electromagnetic field whilst the second system will be that of the Proca field coupled to an external symmetric tensor field. These two examples are quite distinct in that the couplings are introduced into the Lagrangian in two very different ways. In the first example the external electromagnetic field is incorporated into the Lagrangian in the kinetic or derivative terms whereas the symmetric tensor field in the second example is incorporated in the potential or non–derivative part of the Lagrangian. Furthermore it is known from the work of Velo and Zwanziger [3] that the minimal coupling of the electromagnetic field to the Proca field leads to causal propagation whilst the corresponding symmetric tensor field coupling, under some circumstances, leads to acausal propagation.

More generally it is important to note that the geometric constraint algorithm only applies to systems which do not display any explicit time dependence. As a consequence of this, in order to employ the geometric constraint algorithm as it stands, it is necessary to assume that all external fields are time independent. However, of course, the external fields can still be assumed to be dependent on spatial coordinates without affecting the geometrical theory in any way.

The geometrical treatment of the aforementioned coupled Proca systems will be tackled using a fairly straightforward generalization of the techniques employed in section D of chapter VII.

A <u>The geometrical investigation of the Proca field minimally coupled to an external electromagnetic field</u>

A Lagrangian L for the Proca field B_μ minimally coupled to an external electromagnetic field A_μ is given by

$$L = \int \left(-\frac{1}{2} \left(G_{\mu\nu}\right)^{\dagger} \left(G^{\mu\nu}\right) + m^2 \left(B_\mu\right)^{\dagger} \left(B^\mu\right)\right) d^3\underline{x}$$

$$\mu, \nu = 0, \ldots, 3 \qquad (8.1)$$

where

$$G_{\mu\nu} = D_\mu B_\nu - D_\nu B_\mu \qquad \mu, \nu = 0, \ldots, 3. \qquad (8.2)$$

In (8.2) D_μ is the 'derivative' given by

$$D_\mu = \partial_\mu - ie\, A_\mu \qquad \mu = 0, \ldots, 3 \qquad (8.3)$$

where A_μ is dependent only on spatial coordinates, that is $A_\mu = A_\mu\left(x^i\right)$ where $i = 1$ to 3. As in all previous field theoretic calculations the metric convention is again taken to be given by (2.29).

The Lagrangian (8.1), which forms the starting point of this analysis, is essentially the same as the one given in [3]. Additionally in order that the charged Proca field B_μ, in the presence of an external electromagnetic field A_μ, obeys the principle of conservation of electric charge then the Lagrangian describing the system should be invariant under global gauge transformations of the form [19]

$$B_\mu \to B'_\mu = e^{i\chi} B_\mu \qquad \mu = 0, \ldots, 3, \tag{8.4a}$$

$$B^*_\mu \to \left(B^*_\mu\right)' = e^{-i\chi} B^*_\mu \qquad \mu = 0, \ldots, 3, \tag{8.4b}$$

where χ is an arbitrary space–time independent real parameter. It is for this reason that the B_μ in (8.1) must be complex.

A space–time decomposition of (8.1) gives

$$\begin{aligned} L\left[B, B^*, \dot{B}, \dot{B}^*\right] = \int \Big(& \dot{B}^*_i \dot{B}_i - ie A_0 \dot{B}^*_i B_i - \dot{B}^*_i (\partial_i B_0) + \\ & ie A_i \dot{B}^*_i B_0 + ie A_0 B^*_i \dot{B}_i + e^2 A_0^2 B^*_i B_i - \\ & ie A_0 B^*_i (\partial_i B_0) - e^2 A_0 A_i B^*_i B_0 - \left(\partial_i B^*_0\right) \dot{B}_i \\ & + ie A_0 \left(\partial_i B^*_0\right) B_i + \left(\partial_i B^*_0\right) (\partial_i B_0) - \\ & ie A_i \left(\partial_i B^*_0\right) B_0 - ie A_i B^*_0 \dot{B}_i - e^2 A_0 A_i B^*_0 B_i \\ & + ie A_i B^*_0 (\partial_i B_0) + e^2 A_i A_i B^*_0 B_0 - \left(\partial_i B^*_j\right) (\partial_i B_j) \\ & + \left(\partial_i B^*_j\right) (\partial_j B_i) - e^2 A_i A_i B^*_j B_j + e^2 A_i A_j B^*_j B_i \\ & - ie A_i B^*_j (\partial_i B_j) + ie A_i B^*_j (\partial_j B_i) - \\ & ie A_j \left(\partial_i B^*_j\right) B_i + ie A_i \left(\partial_i B^*_j\right) B_j + \\ & m^2 B^*_0 B_0 - m^2 B^*_i B_i \Big) d^3\underline{x} \qquad i, j = 1, \ldots, 3 \end{aligned} \tag{8.5}$$

where $\dot{B}_i = \partial_0 B_i$. The $*$ in (8.5) in this instance refers to the usual operation of complex conjugation.

The geometric analysis now proceeds from (8.5) essentially in the same way as the free Proca field treatment given in section D of chapter VII. The only slight difference is that there are now twice as many field variables due to the presence of the complex conjugate variables B_0^* and B_i^*, where i = 1 to 3. As in the free Proca case, all the manifolds in this coupled investigation will be assumed to be well–behaved C^∞ Banach manifolds. The configuration space Q is now given by $(B, B^*) \equiv \left(B_\mu, B_\nu^*\right) = \left(B_0, B_i, B_0^*, B_i^*\right)$. Velocity phase space TQ, on the other hand, is now parameterized by $\left(B, B^*, \dot{B}, \dot{B}^*\right) \equiv \left(B_\mu, B_\nu^*, \dot{B}_\alpha, \dot{B}_\beta^*\right)$.

The fibre derivative $\hat{F}L$ is defined by

$$\hat{F}L\left(B, B^*, \dot{B}, \dot{B}^*\right) . \left(B, B^*, \dot{C}, \dot{C}^*\right) = \dot{D}L\left(B, B^*, \dot{B}, \dot{B}^*\right) . \left(\dot{C}, \dot{C}^*\right) \quad (8.6)$$

where, by analogy with (7.30), $\dot{D}L\left(B, B^*, \dot{B}, \dot{B}^*\right) . \left(\dot{C}, \dot{C}^*\right)$ is the Frechét derivative along the fibre parameterized by $\left(\dot{C}, \dot{C}^*\right)$. In a local coordinate system $\dot{D}L\left(B, B^*, \dot{B}, \dot{B}^*\right)$ is represented by $\left(\frac{\delta L}{\delta \dot{B}}, \frac{\delta L}{\delta \dot{B}^*}\right)$ and so it can be seen from the Lagrangian given by (8.5), that

$$\dot{D}L\left(B, B^*, \dot{B}, \dot{B}^*\right) . \left(\dot{C}, \dot{C}^*\right) \equiv \frac{\delta L}{\delta \dot{B}_i} \cdot \dot{C}_i + \frac{\delta L}{\delta \dot{B}_0} \cdot \dot{C}_0 + \frac{\delta L}{\delta \dot{B}_i^*} \cdot \dot{C}_i^* + \frac{\delta L}{\delta \dot{B}_0^*} \cdot \dot{C}_0^*$$

$$= \int \left(\left(\dot{B}_i^* + ie\, A_0\, B_i^* - \partial_i\, B_0^* - ie\, A_i\, B_0^*\right) \dot{C}_i + \left(\dot{B}_i - ie\, A_0\, B_i - \partial_i\, B_0 + ie\, A_i\, B_0\right) \dot{C}_i^*\right) d^3\underline{x}$$

$$i = 1, \ldots, 3. \quad (8.7)$$

If phase space T*Q is parameterized by $\left(B, B^*, \Pi, \Pi^*\right) \equiv \left(B_\mu, B_\nu^*, \Pi^\alpha, (\Pi^\beta)^*\right)$ then the natural pairing $\langle \mid \rangle : TQ \times T^*Q \to \mathbb{R}$ is now defined to be

$$\left\langle (B, B^*, \dot{B}, \dot{B}^*) \middle| (B, B^*, \Pi, \Pi^*) \right\rangle \sim \left\langle (\dot{B}, \dot{B}^*) \middle| (\Pi, \Pi^*) \right\rangle$$

$$= \int \left(\dot{B}_i \, \Pi^i + \dot{B}_0 \, \Pi^0 + \dot{B}_i^* (\Pi^i)^* + \dot{B}_0^* (\Pi^0)^* \right) d^3\underline{x}$$

$$i = 1, \ldots, 3. \qquad (8.8)$$

From the definition (8.8) in conjunction with (8.6) and (8.7), it is found that

$$\hat{FL}(B, B^*, \dot{B}, \dot{B}^*) = \Big(B, B^*, \dot{B}_i^* + ie\, A_0\, B_i^* - \partial_i\, B_0^* - ie\, A_i\, B_0^* ,$$

$$\dot{B}_i - ie\, A_0\, B_i - \partial_i\, B_0 + ie\, A_i\, B_0 \Big). \qquad (8.9)$$

The form of (8.9) suggests that the canonical momenta are given by

$$\Pi^i = \dot{B}_i^* + ie\, A_0\, B_i^* - \partial_i\, B_0^* - ie\, A_i\, B_0^* \qquad i = 1, \ldots, 3, \qquad (8.10a)$$

$$(\Pi^i)^* = \dot{B}_i - ie\, A_0\, B_i - \partial_i\, B_0 + ie\, A_i\, B_0 \qquad i = 1, \ldots, 3. \qquad (8.10b)$$

(8.9) can equivalently be written as

$$\hat{FL}(B, B^*, \dot{B}, \dot{B}^*) = \left(B, B^*, 0, \Pi^i, 0, (\Pi^i)^*\right) \qquad (8.11)$$

after making use of (8.10). From (8.11) it can be seen that

$$\Pi^0 = 0, \qquad (8.12a)$$

$$(\Pi^0)^* = 0. \qquad (8.12b)$$

Equations (8.12) are primary constraints and they characterize the primary constraint submanifold $M_1 \subset T^*Q$ which $\hat{FL}$ maps TQ into.

Suppose now that Ω denotes a symplectic form on T^*Q and that $j_1 : M_1 \to T^*Q$ is an inclusion map. Then the 2–form ω_1 induced on M_1 by the pullback of Ω on T^*Q

is given by (7.37). As before the geometric constraint algorithm can now be applied to the resulting presymplectic manifold $\left(M_1, \omega_1\right)$.

In the present case the symplectic form Ω on T^*Q is given by

$$\Omega\left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right), \left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right)$$

$$= <\left(v, v^*\right)|\left(\tau, \tau^*\right)> - <\left(w, w^*\right)|\left(\sigma, \sigma^*\right)>$$

$$= \int \Big(v_i\, \tau^i + v_0\, \tau^0 + \overset{*}{v_i}\left(\tau^i\right)^* + \overset{*}{v_0}\left(\tau^0\right)^* - w_i\, \sigma^i$$

$$- w_0\, \sigma^0 - \overset{*}{w_i}\left(\sigma^i\right)^* - \overset{*}{w_0}\left(\sigma^0\right)^*\Big) d^3\underline{x} \qquad i = 1, \ldots, 3 \qquad (8.13)$$

after appealing to (8.8). In (8.13) $\left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right)\right)$ and $\left(\left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right)$ are tangent vectors to T^*Q and they are basically the generalization of $\left(v \oplus \sigma\right)$ and $\left(w \oplus \tau\right)$ encountered in section D of chapter VII. By analogy with (7.42), $\left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right)\right)$ and $\left(\left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right)$ are a shorthand notation for

$$\left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right)\right) = \left(B, B^*, \Pi, \Pi^*\right) \oplus \left(v, v^*, \sigma, \sigma^*\right), \qquad (8.14a)$$

$$\left(\left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right) = \left(B', \left(B'\right)^*, \Pi', \left(\Pi'\right)^*\right) \oplus \left(w, w^*, \tau, \tau^*\right) \quad (8.14b)$$

where $\left(B, B^*, \Pi, \Pi^*\right)$ and $\left(B', \left(B'\right)^*, \Pi', \left(\Pi'\right)^*\right)$ are the base points on T^*Q of these tangent vectors.

The determination of $\left(TM_1\right)^\perp$ will now be considered. For each point $m_1 \in M_1$, $\left(T_{m_1} M_1\right)^\perp$ is given by (7.43). Suppose then that $x = \left(B, B^*, \Pi, \Pi^*\right) \oplus \left(v, v^*, \sigma, \sigma^*\right) \sim \left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right)\right) \in T_{m_1} M_1$ is an arbitrary tangent vector on M_1 and $z = \left(B, B^*, \Pi, \Pi^*\right) \oplus \left(w, w^*, \tau, \tau^*\right) \sim \left(\left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right) \in T_{m_1} M_1$ is a typical tangent vector on M_1, where $m_1 = \left(B, B^*, \Pi, \Pi^*\right) \in M_1$. It then follows that

$$0 = \omega_1(x, z) = \omega_1\left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right), \left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right)$$

$$\sim \Omega\left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right), \left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right)$$

$$= \int \Big(v_i\, \tau^i + v_0\, \tau^0 + \overset{*}{v}_i\left(\tau^i\right)^* + \overset{*}{v}_0\left(\tau^0\right)^* - w_i\, \sigma^i$$

$$- w_0\, \sigma^0 - \overset{*}{w}_i\left(\sigma^i\right)^* - \overset{*}{w}_0\left(\sigma^0\right)^* \Big) d^3\underline{x} \qquad (8.15)$$

after use of (8.13). Now with x given in the form above where $m_1 = \left(B, B^*, \Pi, \Pi^*\right) \in M_1$, it follows from (8.12) and a generalization of the argument given after (7.44) that $\Pi^0 = \left(\Pi^0\right)^* = 0$ and $\sigma^0 = \left(\sigma^0\right)^* = 0$. Similarly for the tangent vector z, it is found that $\Pi^0 = \left(\Pi^0\right)^* = 0$ and $\tau^0 = \left(\tau^0\right)^* = 0$. Since $\sigma^0 = \left(\sigma^0\right)^* = \tau^0 = \left(\tau^0\right)^* = 0$ then (8.15) reduces to

$$\int \left(v_i\, \tau^i + \overset{*}{v}_i\left(\tau^i\right)^* - w_i\, \sigma^i - \overset{*}{w}_i\left(\sigma^i\right)^* \right) d^3\underline{x} = 0$$

$$i = 1, \ldots, 3. \qquad (8.16)$$

In view of the fact that the tangent vector x is arbitrary, then the v_i, $\overset{*}{v}_i$, σ^i and $\left(\sigma^i\right)^*$ must themselves be arbitrary and so (8.16) can only be satisfied if $\tau^i = \left(\tau^i\right)^* = w_i = \overset{*}{w}_i = 0$ for each $i = 1$ to 3. As a consequence of all these considerations it can be seen that only the w_0 and $\overset{*}{w}_0$ of the tangent vector z remain undetermined and so the most general form of $z = \left(\left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right) \in \left(T_{m_1} M_1\right)^{\perp}$ is given by

$$z = \left(\left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right) = \left(\left(w_0, \underline{0}, \overset{*}{w}_0, \underline{0}\right) \oplus \left(0, \underline{0}, 0, \underline{0}\right)\right) \qquad (8.17)$$

where w_0 and $\overset{*}{w}_0$ are arbitrary. (8.17) is clearly just the generalized version of (7.46).

An extension of (7.47) indicates that the Hamiltonian H_1 induced on M_1 by the fibre derivative $\hat{FL}$ is in this case given by

$$H_1 = \langle (B, B^*, \dot{B}, \dot{B}^*) | \hat{FL}(B, B^*, \dot{B}, \dot{B}^*) \rangle - L[B, B^*, \dot{B}, \dot{B}^*]. \quad (8.18)$$

From a consideration of (8.5), (8.8) and (8.9), it is found that (8.18) gives rise to

$$H_1 = \int \Big(\dot{B}_i^* \dot{B}_i - e^2 A_0^2 B_i^* B_i + ie A_0 B_i^* (\partial_i B_0) + e^2 A_0 A_i B_i^* B_0 -$$
$$ie A_0 \left(\partial_i B_0^*\right) B_i - \left(\partial_i B_0^*\right)(\partial_i B_0) + ie A_i \left(\partial_i B_0^*\right) B_0 +$$
$$e^2 A_0 A_i B_0^* B_i - ie A_i B_0^* (\partial_i B_0) - e^2 A_i A_i B_0^* B_0 + \left(\partial_i B_j^*\right)(\partial_i B_j)$$
$$- \left(\partial_i B_j^*\right)(\partial_j B_i) + e^2 A_i A_i B_j^* B_j - e^2 A_i A_j B_j^* B_i +$$
$$ie A_i B_j^* (\partial_i B_j) - ie A_i B_j^* (\partial_j B_i) + ie A_j B_i \left(\partial_i B_j^*\right)$$
$$- ie A_i B_j \left(\partial_i B_j^*\right) - m^2 B_0^* B_0 + m^2 B_i^* B_i \Big) d^3\underline{x}$$
$$i, j = 1, \ldots, 3. \quad (8.19)$$

If the $\dot{B}_i^*$ and $\dot{B}_i$, given respectively by (8.10a) and (8.10b), are now substituted into (8.19) then

$$H_1[B, B^*, \Pi, \Pi^*] = \int \Big((\Pi^i)^* \Pi^i - ie A_0 B_i^* (\Pi^i)^* + \left(\partial_i B_0^*\right)(\Pi^i)^* +$$
$$ie A_i B_0^* (\Pi^i)^* + ie A_0 B_i \Pi^i + (\partial_i B_0) \Pi^i -$$
$$ie A_i B_0 \Pi^i + \left(\partial_i B_j^*\right)(\partial_i B_j) - \left(\partial_i B_j^*\right)(\partial_j B_i) +$$
$$e^2 A_i A_i B_j^* B_j - e^2 A_i A_j B_j^* B_i + ie A_i B_j^* (\partial_i B_j)$$
$$- ie A_i B_j^* (\partial_j B_i) + ie A_j B_i \left(\partial_i B_j^*\right) -$$
$$ie A_i B_j \left(\partial_i B_j^*\right) - m^2 B_0^* B_0 + m^2 B_i^* B_i \Big) d^3\underline{x}$$
$$i, j = 1, \ldots, 3. \quad (8.20)$$

The time preservation of the primary constraints (8.12) is guaranteed provided the consistency condition (7.50) is satisfied. Suppose that $z = (B, B^*, \Pi, \Pi^*) \oplus (w, w^*, \tau, \tau^*) \sim ((w, w^*) \oplus (\tau, \tau^*)) \in T_{m_1} M_1$ is an arbitrary tangent vector on M_1, where $(B, B^*, \Pi, \Pi^*) \in M_1$, then

$$
\begin{aligned}
& dH_1(B, B^*, \Pi, \Pi^*) . ((w, w^*) \oplus (\tau, \tau^*)) \\
&= i((w, w^*) \oplus (\tau, \tau^*))\, dH_1(B, B^*, \Pi, \Pi^*) \sim \langle T_{m_1} M_1 | dH_1(m_1) \rangle \\
&= \tau^0 \bullet \frac{\delta H_1}{\delta \Pi^0} + \tau^l \bullet \frac{\delta H_1}{\delta \Pi^l} + (\tau^0)^* \bullet \frac{\delta H_1}{\delta (\Pi^0)^*} + (\tau^l)^* \bullet \frac{\delta H_1}{\delta (\Pi^l)^*} \\
&\quad + w_0 \bullet \frac{\delta H_1}{\delta B_0} + w_l \bullet \frac{\delta H_1}{\delta B_l} + w_0^* \bullet \frac{\delta H_1}{\delta B_0^*} + w_l^* \bullet \frac{\delta H_1}{\delta B_l^*} \qquad l = 1, \ldots, 3. \qquad (8.21)
\end{aligned}
$$

Now for the tangent vector $z \in T_{m_1} M_1$, where $m_1 = (B, B^*, \Pi, \Pi^*) \in M_1$, it has been seen before that $\Pi^0 = (\Pi^0)^* = 0$ and $\tau^0 = (\tau^0)^* = 0$. Furthermore it is found from (8.20) that

$$
\begin{aligned}
\tau^l \bullet \frac{\delta H_1}{\delta \Pi^l} &= \tau^l \bullet \left[\frac{\delta}{\delta \Pi^l} \left(\int \left((\Pi^i)^* \, \Pi^i + ie\, A_0\, B_i\, \Pi^i + (\partial_i B_0) \Pi^i \right.\right.\right. \\
&\qquad \left.\left.\left. - ie\, A_i\, B_0\, \Pi^i \right) d^3\underline{x} \right) \right] \\
&= \int \left(\tau^i \left[(\Pi^i)^* + ie\, A_0\, B_i + \partial_i B_0 - ie\, A_i\, B_0 \right] \right) d^3\underline{x} \\
&\qquad\qquad i, l = 1, \ldots, 3. \qquad (8.22)
\end{aligned}
$$

From similar considerations it follows that

$$
(\tau^l)^* \bullet \frac{\delta H_1}{\delta (\Pi^l)^*} = \int \left((\tau^i)^* \left[\Pi^i - ie\, A_0\, B_i^* + \partial_i B_0^* + ie\, A_i\, B_0^* \right] \right) d^3\underline{x} \qquad i, l = 1, \ldots, 3. \qquad (8.23)
$$

In addition

$$w_0 \bullet \frac{\delta H_1}{\delta B_0} = w_0 \bullet \left[\frac{\delta}{\delta B_0} \left(\int \left((\partial_i B_0)\, \Pi^i - ie\, A_i\, B_0\, \Pi^i - m^2\, B_0^*\, B_0 \right) d^3\underline{x} \right) \right]$$

$$= \int \left(w_0 \left[-\partial_i\, \Pi^i - ie\, A_i\, \Pi^i - m^2\, B_0^* \right] \right) d^3\underline{x}$$

$$i = 1, \ldots, 3 \qquad (8.24)$$

after a partial integration and a consideration of the spatial boundary conditions similar to that which led to (7.54). A corresponding argument indicates that

$$w_0^* \bullet \frac{\delta H_1}{\delta B_0^*} = \int \left(w_0^* \left[-\partial_i\, (\Pi^i)^* + ie\, A_i\, (\Pi^i)^* - m^2\, B_0 \right] \right) d^3\underline{x}$$

$$i = 1, \ldots, 3. \qquad (8.25)$$

Also

$$w_1 \bullet \frac{\delta H_1}{\delta B_1} = w_1 \bullet \left[\frac{\delta}{\delta B_1} \left(\int \left(ie\, A_0\, B_i\, \Pi^i + \left(\partial_i\, B_j^*\right)\left(\partial_i\, B_j\right) - \left(\partial_i\, B_j^*\right)\left(\partial_j\, B_i\right) + e^2\, A_i\, A_i\, B_j^*\, B_j - e^2\, A_i\, A_j\, B_j^*\, B_i + ie\, A_i\, B_j^*\, \left(\partial_i\, B_j\right) - ie\, A_i\, B_j^*\, \left(\partial_j\, B_i\right) + ie\, A_j\, B_i \left(\partial_i\, B_j^*\right) - ie\, A_i\, B_j \left(\partial_i\, B_j^*\right) + m^2\, B_i^*\, B_i \right) d^3\underline{x} \right) \right]$$

$$= \int \left(w_i \left[ie\, A_0\, \Pi^i - \partial_j \partial_j B_i^* + \partial_j \partial_i B_j^* + e^2 A_j A_j B_i^* \right.\right.$$

$$- e^2 A_i A_j B_j^* - ie\, \partial_j \left(A_j B_i^* \right) + ie\, \partial_j \left(A_i B_j^* \right) +$$

$$\left.\left. ie\, A_j \left(\partial_i B_j^* \right) - ie\, A_j \left(\partial_j B_i^* \right) + m^2 B_i^* \right] \right) d^3\underline{x}$$

$$i, j, l = 1, \ldots, 3 \qquad (8.26)$$

after some partial integrations and the usual consideration of spatial boundary conditions. Similarly

$$w_l^* \bullet \frac{\delta H_1}{\delta B_l^*} = \int \left(w_i^* \left[-ie\, A_0 \left(\Pi^i \right)^* - \partial_j \partial_j B_i + \partial_j \partial_i B_j + e^2 A_j A_j B_i \right.\right.$$

$$- e^2 A_i A_j B_j + ie\, \partial_j \left(A_j B_i \right) - ie\, \partial_j \left(A_i B_j \right) -$$

$$\left.\left. ie\, A_j \left(\partial_i B_j \right) + ie\, A_j \left(\partial_j B_i \right) + m^2 B_i \right] \right) d^3\underline{x}$$

$$i, j, l = 1, \ldots, 3. \qquad (8.27)$$

Thus by substituting $\tau^0 = \left(\tau^0\right)^* = 0$ and equations (8.22), (8.23), (8.24), (8.25), (8.26) and (8.27) into (8.21) it is found that

$$dH_1(B, B^*, \Pi, \Pi^*) . ((w, w^*) \oplus (\tau, \tau^*))$$

$$= \int \Big(\tau^i[(\Pi^i)^* + ie A_0 B_i + \partial_i B_0 - ie A_i B_0] + (\tau^i)^*\Big[\Pi^i - ie A_0 B_i^* + \partial_i B_0^* + ie A_i B_0^*\Big] + w_0\Big[-\partial_i \Pi^i - ie A_i \Pi^i - m^2 B_0^*\Big] + w_0^*\Big[-\partial_i (\Pi^i)^* + ie A_i (\Pi^i)^* - m^2 B_0\Big] + w_i\Big[ie A_0 \Pi^i - \partial_j \partial_j B_i^* + \partial_j \partial_i B_j^* + e^2 A_j A_j B_i^* - e^2 A_i A_j B_j^* - ie \partial_j (A_j B_i^*) + ie \partial_j (A_i B_j^*) + ie A_j (\partial_i B_j^*) - ie A_j (\partial_j B_i^*) + m^2 B_i^*\Big] + w_i^* [-ie A_0 (\Pi^i)^* - \partial_j \partial_j B_i + \partial_j \partial_i B_j + e^2 A_j A_j B_i - e^2 A_i A_j B_j + ie \partial_j (A_j B_i) - ie \partial_j (A_i B_j) - ie A_j (\partial_i B_j) + ie A_j (\partial_j B_i) + m^2 B_i] \Big) d^3\underline{x} \qquad i, j = 1, \ldots, 3. \qquad (8.28)$$

Suppose now that $z \in (T_{m_1} M_1)^\perp$. Making the substitution $z = ((w, w^*) \oplus (\tau, \tau^*)) \in (T_{m_1} M_1)^\perp$, as given by (8.17), in (8.28) leads to the condition

$$dH_1(B, B^*, \Pi, \Pi^*) . ((w, w^*) \oplus (\tau, \tau^*)) \sim \langle (T_{m_1} M_1)^\perp | dH_1(m_1) \rangle$$

$$= \int \Big(w_0\Big[-\partial_i \Pi^i - ie A_i \Pi^i - m^2 B_0^*\Big] + w_0^* [-\partial_i (\Pi^i)^* + ie A_i (\Pi^i)^* - m^2 B_0] \Big) d^3\underline{x} \qquad i = 1, \ldots, 3. \qquad (8.29)$$

A consideration of (7.50) and (8.29) indicates that the primary constraints (8.12) are preserved in time if

$$\int \Big(w_0\Big[-\partial_i \Pi^i - ie A_i \Pi^i - m^2 B_0^*\Big] + w_0^* [-\partial_i (\Pi^i)^* + ie A_i (\Pi^i)^* - m^2 B_0] \Big) d^3\underline{x} = 0 \qquad i = 1, \ldots, 3 \qquad (8.30)$$

and since w_0 and w_0^* are arbitrary, it follows from (8.30) that

$$-\partial_i \Pi^i - ie\, A_i \Pi^i - m^2 B_0^* = 0 \qquad i = 1, \ldots, 3, \qquad (8.31a)$$

$$-\partial_i \left(\Pi^i\right)^* + ie\, A_i \left(\Pi^i\right)^* - m^2 B_0 = 0 \qquad i = 1, \ldots, 3. \qquad (8.31b)$$

Equations (8.31) are secondary Hamiltonian constraints which define a new submanifold $M_2 \subset M_1$ to which the motion is now restricted.

Now the demand that the secondary constraints (8.31) are preserved in time is ensured provided the consistency condition (7.61) is obeyed. Consequently the determination of $\left(TM_2\right)^{\perp}$ must now be considered. For each $m_2 \in M_2$, $\left(T_{m_2} M_2\right)^{\perp}$ is given by (7.62) where, as before, $\left(j_2\right)_* \left(T_{m_2} M_2\right) = \underline{\left(T_{m_2} M_2\right)}$ and $j_2 : M_2 \to M_1$ is the inclusion map. Suppose then that $z = \left(B, B^*, \Pi, \Pi^*\right) \oplus \left(w, w^*, \tau, \tau^*\right) \sim \left(\left(w, w^*\right) \oplus \left(\tau, \tau^*\right)\right) \in T_{m_2} M_1$ is a typical tangent vector on M_1 restricted to M_2 and that $x = \left(B, B^*, \Pi, \Pi^*\right) \oplus \left(v, v^*, \sigma, \sigma^*\right) \sim \left(\left(v, v^*\right) \oplus \left(\sigma, \sigma^*\right)\right) \in \underline{T_{m_2} M_2}$ is an arbitrary vector, where $\left(B, B^*, \Pi, \Pi^*\right) \in M_2$. Now the above $z \in T_{m_2} M_1$, where $\left(B, B^*, \Pi, \Pi^*\right) \in M_2 \subset M_1$, is such that $\Pi^0 = \left(\Pi^0\right)^* = 0$ and $\tau^0 = \left(\tau^0\right)^* = 0$. For the vector $x \in \underline{T_{m_2} M_2}$, where $\left(B, B^*, \Pi, \Pi^*\right) \in M_2$, it follows from (8.12) and (8.31) that $\Pi^0 = \left(\Pi^0\right)^* = 0$, $-\partial_i \Pi^i - ie\, A_i \Pi^i - m^2 B_0^* = 0$ and $-\partial_i\left(\Pi^i\right)^* + ie\, A_i\left(\Pi^i\right)^* - m^2 B_0 = 0$ and consequently $\sigma^0 = \left(\sigma^0\right)^* = 0$, $-\partial_i \sigma^i - ie\, A_i \sigma^i - m^2 v_0^* = 0$ and $-\partial_i\left(\sigma^i\right)^* + ie\, A_i\left(\sigma^i\right)^* - m^2 v_0 = 0$. Therefore it follows that

$$0 = \omega_1 | M_2(x, z) = \omega_1 | M_2\left((v, v^*) \oplus (\sigma, \sigma^*), (w, w^*) \oplus (\tau, \tau^*)\right)$$
$$\sim \Omega\left((v, v^*) \oplus (\sigma, \sigma^*), (w, w^*) \oplus (\tau, \tau^*)\right)$$
$$= \int \left(v_i \tau^i + v_i^*(\tau^i)^* - w_i \sigma^i - w_i^*(\sigma^i)^*\right) d^3\underline{x} \qquad (8.32)$$

after substituting $\tau^0 = (\tau^0)^* = \sigma^0 = (\sigma^0)^* = 0$ into (8.13). Now the v_i and v_i^* are arbitrary and so are the σ^i and $(\sigma^i)^*$ since they respectively satisfy the conditions $-\partial_i \sigma^i - ie\, A_i \sigma^i - m^2 v_0^* = 0$ and $-\partial_i (\sigma^i)^* + ie\, A_i (\sigma^i)^* - m^2 v_0 = 0$ for arbitrary v_0^* and v_0. In view of this (8.32) can only be satisfied if $\tau^i = (\tau^i)^* = w_i = w_i^* = 0$ for each $i = 1$ to 3. In this investigation of $(T_{m_2} M_2)^\perp$ the w_0 and w_0^* are not determined and so they remain arbitrary. In light of the above

$$(T_{m_2} M_2)^\perp = \Big\{ \left(w_0, \underline{0}, w_0^*, \underline{0}\right) \oplus \left(0, \underline{0}, 0, \underline{0}\right) \in T_{m_2} M_1 : w_0 \text{ and } w_0^* \text{ are arbitrary} \Big\} . \qquad (8.33)$$

On substituting $z = \left((w, w^*) \oplus (\tau, \tau^*)\right) \in (T_{m_2} M_2)^\perp$, as given by (8.33), into (8.28) it is found that

$$dH_1(B, B^*, \Pi, \Pi^*) . \left((w, w^*) \oplus (\tau, \tau^*)\right) \sim \left< (T_{m_2} M_2)^\perp | dH_1(m_2) \right>$$
$$= \int \Big(w_0\left[-\partial_i \Pi^i - ie\, A_i \Pi^i - m^2 B_0^*\right] + w_0^* \left[-\partial_i(\Pi^i)^* + ie\, A_i (\Pi^i)^* - m^2 B_0\right] \Big) d^3\underline{x} \qquad i = 1, \ldots, 3. \qquad (8.34)$$

On remembering that $-\partial_i \Pi^i - ie\, A_i \Pi^i - m^2 B_0^* = 0$ and $-\partial_i(\Pi^i)^* + ie\, A_i(\Pi^i)^* - m^2 B_0 = 0$ on M_2 it can be seen that (8.34) automatically satisfies (7.61), the time preservation condition of the secondary constraints (8.31).

At this point the geometric constraint analysis of the Proca field minimally coupled to an external electromagnetic field terminates. However, in order to give a full geometric picture the equations of motion of this coupled system will now be considered.

First of all suppose that the vector field X is such that it is composed of a vector of the form $x = (B, B^*, \Pi, \Pi^*) \oplus (a, a^*, \xi, \xi^*) \sim ((a, a^*) \oplus (\xi, \xi^*)) \in \underline{T_{m_2} M_2}$ at each point $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$. From similar arguments to those considered before it can be seen that $\Pi^0 = (\Pi^0)^* = \xi^0 = (\xi^0)^* = 0$, $-\partial_i \Pi^i - ie A_i \Pi^i - m^2 B_0^* = 0$, $-\partial_i (\Pi^i)^* + ie A_i(\Pi^i)^* - m^2 B_0 = 0$, $-\partial_i \xi^i - ie A_i \xi^i - m^2 a_0^* = 0$ and $-\partial_i(\xi^i)^* + ie A_i(\xi^i)^* - m^2 a_0 = 0$ for vectors like x.

Furthermore suppose Z is an arbitrary vector field obtained by the assignment of an arbitrary vector of the form $z = (B, B^*, \Pi, \Pi^*) \oplus (b, b^*, \eta, \eta^*) \sim ((b, b^*) \oplus (\eta, \eta^*)) \in T_{m_2} M_1$ at each point $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$. For vectors of the form of z then $\Pi^0 = (\Pi^0)^* = \eta^0 = (\eta^0)^* = 0$.

With the vector fields X and Z given above it follows, by analogy with the arguments leading to (7.70), that the equations of motion on M_2 are in this case given by

$$\omega_1 ((a, a^*) \oplus (\xi, \xi^*), (b, b^*) \oplus (\eta, \eta^*))|(B, B^*, \Pi, \Pi^*)$$

$$= dH_1(B, B^*, \Pi, \Pi^*) . ((b, b^*) \oplus (\eta, \eta^*)) . \tag{8.35}$$

(8.35) can be written as

$$\int \left(a_i\, \eta^i + a_i^*(\eta^i)^* - b_i\, \xi^i - b_i^*(\xi^i)^*\right) d^3\underline{x}$$

$$= \int \Big(\eta^i \left[(\Pi^i)^* + ie\, A_0\, B_i + \partial_i\, B_0 - ie\, A_i\, B_0 \right] + (\eta^i)^* \Big[\Pi^i$$
$$- ie\, A_0\, B_i^* + \partial_i\, B_0^* + ie\, A_i\, B_0^* \Big] + b_i \Big[ie\, A_0\, \Pi^i - \partial_j\, \partial_j\, B_i^*$$
$$+ \partial_j \partial_i\, B_j^* + e^2\, A_j\, A_j\, B_i^* - e^2\, A_i\, A_j\, B_j^* - ie\, \partial_j \left(A_j\, B_i^*\right) +$$
$$ie\, \partial_j \left(A_i\, B_j^*\right) + ie\, A_j \left(\partial_i\, B_j^*\right) - ie\, A_j \left(\partial_j\, B_i^*\right) + m^2\, B_i^* \Big]$$
$$+ b_i^* \big[-ie\, A_0 (\Pi^i)^* - \partial_j\, \partial_j\, B_i + \partial_j\, \partial_i\, B_j + e^2\, A_j\, A_j\, B_i -$$
$$e^2\, A_i\, A_j\, B_j + ie\, \partial_j \left(A_j\, B_i\right) - ie\, \partial_j \left(A_i\, B_j\right) - ie\, A_j \left(\partial_i\, B_j\right)$$
$$+ ie\, A_j \left(\partial_j\, B_i\right) + m^2\, B_i \big] \Big) d^3\underline{x} \qquad i, j = 1, \ldots, 3 \qquad (8.36)$$

since the left–hand side of (8.35) is derived from a similar argument to that which led to (8.32) and the right–hand side of (8.35) basically originates from (8.28) after remembering that $-\partial_i\, \Pi^i - ie\, A_i\, \Pi^i - m^2\, B_0^* = 0$ and $-\partial_i\, (\Pi^i)^* + ie\, A_i (\Pi^i)^* - m^2\, B_0 = 0$ on M_2 .

On equating the coefficients of the arbitrary b_i , b_i^* , η^i and $(\eta^i)^*$ in (8.36), it is found that

$$a_i = (\Pi^i)^* + ie\, A_0\, B_i + \partial_i\, B_0 - ie\, A_i\, B_0 \qquad i = 1, \ldots, 3, \qquad (8.37a)$$

$$a_i^* = \Pi^i - ie\, A_0\, B_i^* + \partial_i\, B_0^* + ie\, A_i\, B_0^* \qquad i = 1, \ldots, 3 \qquad (8.37b)$$

and

$$\xi^i = -ie\, A_0\, \Pi^i + \partial_j \partial_j\, B_i^* - \partial_j \partial_i\, B_j^* - e^2\, A_j\, A_j\, B_i^* +$$
$$e^2\, A_i\, A_j\, B_j^* + ie\, \partial_j \left(A_j\, B_i^*\right) - ie\, \partial_j \left(A_i\, B_j^*\right) -$$
$$ie\, A_j \left(\partial_i\, B_j^*\right) + ie\, A_j \left(\partial_j\, B_i^*\right) - m^2\, B_i^* \qquad i, j = 1, \ldots, 3, \qquad (8.38a)$$

$$(\xi^i)^* = ie\, A_0(\Pi^i)^* + \partial_j \partial_j B_i - \partial_j \partial_i B_j - e^2 A_j A_j B_i +$$
$$e^2 A_i A_j B_j - ie\, \partial_j (A_j B_i) + ie\, \partial_j (A_i B_j) +$$
$$ie\, A_j (\partial_i B_j) - ie\, A_j (\partial_j B_i) - m^2 B_i \qquad i, j = 1, \ldots, 3. \qquad (8.38b)$$

Now for a vector of the form of $x \sim ((a, a^*) \oplus (\xi, \xi^*)) \in \underline{T_{m_2} M_2}$ it is known that

$$-\partial_i \xi^i - ie\, A_i \xi^i - m^2 a_0^* = 0 \qquad i = 1, \ldots, 3, \qquad (8.39a)$$

$$-\partial_i(\xi^i)^* + ie\, A_i(\xi^i)^* - m^2 a_0 = 0 \qquad i = 1, \ldots, 3. \qquad (8.39b)$$

Therefore on putting (8.38a) into (8.39a) it is found, after using (8.31a), that

$$m^2 a_0^* - ie(\partial_i A_0)\Pi^i + ie\, m^2 A_0 B_0^* + e^2 F_{ij} A_i B_j^* -$$
$$ie\, F_{ij}\left(\partial_i B_j^*\right) - m^2 \partial_i B_i^* - ie\, m^2 A_i B_i^* = 0 \qquad i, j = 1, \ldots, 3 \qquad (8.40)$$

where the F_{ij} are given by (4.92). Since $m \neq 0$ then (8.40) can be written as

$$a_0^* = ie\, m^{-2} (\partial_i A_0)\Pi^i - ie\, A_0 B_0^* - e^2 m^{-2} F_{ij} A_i B_j^* +$$
$$ie\, m^{-2} F_{ij}\left(\partial_i B_j^*\right) + \partial_i B_i^* + ie\, A_i B_i^* \qquad i, j = 1, \ldots, 3. \qquad (8.41)$$

Correspondingly if (8.38b) is substituted into (8.39b), it is found that

$$a_0 = -ie\, m^{-2} (\partial_i A_0)(\Pi^i)^* + ie\, A_0 B_0 - e^2 m^{-2} F_{ij} A_i B_j$$
$$-ie\, m^{-2} F_{ij} (\partial_i B_j) + \partial_i B_i - ie\, A_i B_i \qquad i, j = 1, \ldots, 3 \qquad (8.42)$$

after using (8.31b).

The vector field $X \sim x = ((a,a^*)\oplus(\xi,\xi^*)) = \left(\left(a_0,a_i,a_0^*,a_i^*\right) \oplus \left(\xi^0,\xi^i,(\xi^0)^*,(\xi^i)^*\right)\right)$ is therefore fully determined in light of (8.37), (8.38), (8.41) and (8.42) and the fact that $\xi^0 = (\xi^0)^* = 0$. This vector field X, the primary constraints (8.12) and the secondary constraints (8.31) are the final results of the geometric constraint analysis. The final constraint submanifold is therefore M_2.

By analogy with (7.80) it is found for $X \sim x = (B, B^*, \Pi, \Pi^*) \oplus (a, a^*, \xi, \xi^*)$ that

$$a_0 = \partial_0 B_0 \equiv \dot{B}_0 , \tag{8.43a}$$

$$a_0^* = \partial_0 B_0^* \equiv \dot{B}_0^* , \tag{8.43b}$$

$$a_i = \partial_0 B_i \equiv \dot{B}_i \qquad i = 1, \ldots, 3, \tag{8.43c}$$

$$a_i^* = \partial_0 B_i^* \equiv \dot{B}_i^* \qquad i = 1, \ldots, 3, \tag{8.43d}$$

$$\xi^0 = \partial_0 \Pi^0 \equiv \dot{\Pi}^0 , \tag{8.43e}$$

$$(\xi^0)^* = \partial_0 ((\Pi^0)^*) \equiv (\dot{\Pi}^0)^* , \tag{8.43f}$$

$$\xi^i = \partial_0 \Pi^i \equiv \dot{\Pi}^i \qquad i = 1, \ldots, 3, \tag{8.43g}$$

$$(\xi^i)^* = \partial_0 ((\Pi^i)^*) \equiv (\dot{\Pi}^i)^* \qquad i = 1, \ldots, 3. \tag{8.43h}$$

In view of (8.43), (8.37), (8.38), (8.41) and (8.42) and the fact that $\xi^0 = (\xi^0)^* = 0$, it follows that the equations of motion of this system can be written as

$$\dot{B}_0 = -\mathrm{ie}\, m^{-2}(\partial_i A_0)(\Pi^i)^* + \mathrm{ie}\, A_0 B_0 - e^2 m^{-2} F_{ij} A_i B_j - \mathrm{ie}\, m^{-2} F_{ij} (\partial_i B_j) + \partial_i B_i - \mathrm{ie}\, A_i B_i \qquad i, j = 1, \ldots, 3, \tag{8.44}$$

$$\dot{B}_0^* = \mathrm{ie}\, m^{-2}(\partial_i A_0)\Pi^i - \mathrm{ie}\, A_0 B_0^* - e^2 m^{-2} F_{ij} A_i B_j^* + \mathrm{ie}\, m^{-2} F_{ij} \left(\partial_i B_j^*\right) + \partial_i B_i^* + \mathrm{ie}\, A_i B_i^* \qquad i, j = 1, \ldots, 3, \tag{8.45}$$

$$\dot{B}_i = (\Pi^i)^* + \mathrm{ie}\, A_0 B_i + \partial_i B_0 - \mathrm{ie}\, A_i B_0 \qquad i = 1, \ldots, 3, \tag{8.46}$$

$$\dot{B}_i^* = \Pi^i - ie A_0 B_i^* + \partial_i B_0^* + ie A_i B_0^* \qquad i = 1, \ldots, 3, \qquad (8.47)$$

$$\dot{\Pi}^0 = 0, \qquad (8.48)$$

$$(\dot{\Pi}^0)^* = 0, \qquad (8.49)$$

$$\dot{\Pi}^i = -ie A_0 \Pi^i + \partial_j \partial_j B_i^* - \partial_j \partial_i B_j^* - e^2 A_j A_j B_i^* + e^2 A_i A_j B_j^*$$
$$+ ie\, \partial_j \left(A_j B_i^*\right) - ie\, \partial_j \left(A_i B_j^*\right) - ie\, A_j \left(\partial_i B_j^*\right) +$$
$$ie\, A_j \left(\partial_j B_i^*\right) - m^2 B_i^* \qquad i, j = 1, \ldots, 3, \qquad (8.50)$$

$$(\dot{\Pi}^i)^* = ie A_0 (\Pi^i)^* + \partial_j \partial_j B_i - \partial_j \partial_i B_j - e^2 A_j A_j B_i +$$
$$e^2 A_i A_j B_j - ie\, \partial_j \left(A_j B_i\right) + ie\, \partial_j \left(A_i B_j\right) + ie\, A_j \left(\partial_i B_j\right)$$
$$- ie\, A_j \left(\partial_j B_i\right) - m^2 B_i \qquad i, j = 1, \ldots, 3. \qquad (8.51)$$

To conclude with, the nature of the Proca system minimally coupled to an external electromagnetic field will be investigated. In a manner analogous to the free Proca case, the classification scheme in section C of chapter VII will be applied to the canonical system (M_1, ω_1, M_2) with inclusion $j_2 : M_2 \to M_1$. This necessitates a consideration of the spaces $\underline{T_{m_2} M_2} = (j_2)_* \left(T_{m_2} M_2\right)$ and $\left(T_{m_2} M_2\right)^{\perp}$ at each $m_2 \in M_2$. Suppose then that $z = (B, B^*, \Pi, \Pi^*) \oplus (w, w^*, \tau, \tau^*) \sim ((w, w^*) \oplus (\tau, \tau^*)) \in \left(T_{m_2} M_2\right)^{\perp}$ where $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$. The most general form of $z \in \left(T_{m_2} M_2\right)^{\perp}$ is given by (8.33). However if

$$z = (B, B^*, \Pi, \Pi^*) \oplus (w, w^*, \tau, \tau^*) \sim ((w, w^*) \oplus (\tau, \tau^*)) \in \underline{\left(T_{m_2} M_2\right)}$$

then as before for $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$, $\Pi^0 = (\Pi^0)^* = \tau^0 = (\tau^0)^* = 0$, $-\partial_i \Pi^i - ie A_i \Pi^i - m^2 B_0^* = 0$, $-\partial_i (\Pi^i)^* + ie A_i (\Pi^i)^* - m^2 B_0 = 0$, $-\partial_i \tau^i - ie A_i \tau^i - m^2 w_0^* = 0$ and $-\partial_i (\tau^i)^* + ie A_i (\tau^i)^* - m^2 w_0 = 0$. For the conditions $-\partial_i \tau^i - ie A_i \tau^i - m^2 w_0^* = 0$ and $-\partial_i (\tau^i)^* + ie A_i (\tau^i)^* - m^2 w_0 = 0$ to hold when z is also an element of $\left(T_{m_2} M_2\right)^{\perp}$, it follows from (8.33) that $w_0 = w_0^* = 0$. Therefore the condition (7.85) holds for each $m_2 \in M_2$. Consequently it is found from the classification scheme in section C of chapter VII that this coupled system is second class just like the free Proca case.

B The geometrical investigation of the Proca field coupled to an external symmetric tensor field

In order to keep the notation in line with that employed in section A of this chapter, the Lagrangian L for the Proca field B_μ coupled to an external symmetric tensor field $T^{\mu\nu}$ is taken to be

$$L = \int \left(-\frac{1}{2} (H_{\mu\nu})^\dagger (H^{\mu\nu}) + m^2 (B_\mu)^\dagger B^\mu + \lambda (B_\mu)^\dagger T^{\mu\nu} B_\nu \right) d^3\underline{x} \qquad \mu, \nu = 0, \ldots, 3 \qquad (8.52)$$

where

$$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad \mu, \nu = 0, \ldots, 3 \qquad (8.53)$$

and λ is a coupling constant. Since the tensor field $T^{\mu\nu}$ is symmetric then

$$T^{\mu\nu} = T^{\nu\mu} \qquad \mu, \nu = 0, \ldots, 3 \qquad (8.54)$$

and in addition $T^{\mu\nu}$ is assumed to be dependent only on spatial coordinates, that is $T^{\mu\nu} = T^{\mu\nu}(x^i)$ where $i = 1$ to 3.

It should be noted that the Lagrangian (8.52) is basically the one adopted in [3]. Once again the metric convention given by (2.29) is assumed throughout this analysis.

A space–time decomposition of (8.52) leads to

$$L[B, B^*, \dot{B}, \dot{B}^*] = \int \Big(\dot{B}_i^* \dot{B}_i - \dot{B}_i^*(\partial_i B_0) - \left(\partial_i B_0^*\right) \dot{B}_i +$$

$$\left(\partial_i B_0^*\right)(\partial_i B_0) - \left(\partial_i B_j^*\right)(\partial_i B_j) + \left(\partial_i B_j^*\right)(\partial_j B_i)$$

$$+ m^2 B_0^* B_0 - m^2 B_i^* B_i + \lambda B_0^* T_{00} B_0 -$$

$$\lambda B_0^* T_{0i} B_i - \lambda B_i^* T_{i0} B_0 + \lambda B_i^* T_{ij} B_j\Big) d^3\underline{x}$$

$$i, j = 1, \ldots, 3 \qquad (8.55)$$

where as before $\dot{B}_i = \partial_0 B_i$.

The analysis now closely parallels that given in section A of this chapter. Configuration space Q is given by (B, B^*) whilst velocity phase space TQ is parameterized by $\left(B, B^*, \dot{B}, \dot{B}^*\right)$. The fibre derivative $\hat{F}L$ is defined by (8.6) and once again $\dot{D}L\left(B, B^*, \dot{B}, \dot{B}^*\right)$ is locally represented by $\left(\frac{\delta L}{\delta \dot{B}}, \frac{\delta L}{\delta \dot{B}^*}\right)$. In view of this, it follows from (8.55) that

$$\dot{D}L\left(B, B^*, \dot{B}, \dot{B}^*\right) . \left(\dot{C}, \dot{C}^*\right) \equiv \frac{\delta L}{\delta \dot{B}_i} \cdot \dot{C}_i + \frac{\delta L}{\delta \dot{B}_0} \cdot \dot{C}_0 + \frac{\delta L}{\delta \dot{B}_i^*} \cdot \dot{C}_i^* + \frac{\delta L}{\delta \dot{B}_0^*} \cdot \dot{C}_0^*$$

$$= \int \left(\left(\dot{B}_i^* - \partial_i B_0^*\right) \dot{C}_i + \left(\dot{B}_i - \partial_i B_0\right) \dot{C}_i^*\right) d^3\underline{x}$$

$$i = 1, \ldots, 3. \qquad (8.56)$$

With phase space T*Q parameterized by $\left(B, B^*, \Pi, \Pi^*\right)$, then the natural pairing $\langle \mid \rangle : TQ \times T^*Q \to \mathbb{R}$ is defined by (8.8). Following a similar argument to that leading to (8.11), it is found in this case that

$$\hat{F}L\left(B, B^*, \dot{B}, \dot{B}^*\right) = \left(B, B^*, 0, \dot{B}_i^* - \partial_i B_0^*, 0, \dot{B}_i - \partial_i B_0\right) \qquad (8.57)$$

where the canonical momenta are

$$\Pi^i = \dot{B}_i^* - \partial_i B_0^* \qquad i = 1, \ldots, 3, \qquad (8.58a)$$

$$\left(\Pi^i\right)^* = \dot{B}_i - \partial_i B_0 \qquad i = 1, \ldots, 3. \qquad (8.58b)$$

In addition it can be seen from (8.57) that $\Pi^0 = \left(\Pi^0\right)^* = 0$. In other words equations (8.12) hold in this case also. Thus the primary constraints of the current analysis, that is (8.12), are just the same as the ones obtained in the case of the Proca field minimally coupled to an external electromagnetic field. As before these primary constraints define a submanifold $M_1 \subset T^*Q$ which $\hat{FL}$ takes TQ into.

The starting point of the geometric constraint algorithm is the presymplectic manifold $\left(M_1, \omega_1\right)$ where, as in section A of this chapter, ω_1 is the presymplectic form induced on M_1 by pulling Ω on T^*Q back with the inclusion map $j_1 : M_1 \rightarrow T^*Q$.

As in the previous geometric calculations the next step of the analysis is to find $\left(TM_1\right)^{\perp}$. Precisely as in the determination of $\left(T_{m_1} M_1\right)^{\perp}$ in the investigation of the Proca field minimally coupled to an external electromagnetic field, it is found in this case that the most general form of $\left(T_{m_1} M_1\right)^{\perp}$ is also given by (8.17).

The Hamiltonian H_1 induced on M_1 can now be found in an analogous manner to (8.19) by considering (8.55), (8.8) and (8.57) in conjunction with (8.18) and this leads to

$$H_1 = \int \Big(\dot{B}_i^* \dot{B}_i - \left(\partial_i B_0^*\right)\left(\partial_i B_0\right) + \left(\partial_i B_j^*\right)\left(\partial_i B_j\right) - \left(\partial_i B_j^*\right)\left(\partial_j B_i\right) - m^2 B_0^* B_0 + m^2 B_i^* B_i - \lambda B_0^* T_{00} B_0 + \lambda B_0^* T_{0i} B_i + \lambda B_i^* T_{i0} B_0 - \lambda B_i^* T_{ij} B_j \Big) d^3\underline{x}$$

$$i, j = 1, \ldots, 3. \qquad (8.59)$$

Eliminating the $\dot{B}_i^*$ and $\dot{B}_i$ in (8.59), by using (8.58a) and (8.58b), gives rise to

$$H_1\left[B, B^*, \Pi, \Pi^*\right] = \int \Big((\Pi^i)^* \, \Pi^i + (\Pi^i)^* \left(\partial_i B_0^*\right) + \Pi^i \left(\partial_i B_0\right)$$
$$+ \left(\partial_i B_j^*\right)\left(\partial_i B_j\right) - \left(\partial_i B_j^*\right)\left(\partial_j B_i\right) - m^2 B_0^* B_0$$
$$+ m^2 B_i^* B_i - \lambda B_0^* T_{00} B_0 + \lambda B_0^* T_{0i} B_i +$$
$$\lambda B_i^* T_{i0} B_0 - \lambda B_i^* T_{ij} B_j \Big) d^3\underline{x}$$
$$i, j = 1, \ldots, 3. \qquad (8.60)$$

The primary constraints (8.12) will be preserved in time if (7.50) is satisfied. Now for an arbitrary tangent vector on M_1 given by $z = (B, B^*, \Pi, \Pi^*) \oplus (w, w^*, \tau, \tau^*) \sim ((w, w^*) \oplus (\tau, \tau^*)) \in T_{m_1} M_1$, where $(B, B^*, \Pi, \Pi^*) \in M_1$, then $dH_1 \; (B, B^*, \Pi, \Pi^*) \, . \, ((w, w^*) \oplus (\tau, \tau^*))$ is given by (8.21). In view of this consider first of all the term $\tau^l \bullet \frac{\delta H_1}{\delta \Pi^l}$. From (8.60) it is found that

$$\tau^l \bullet \frac{\delta H_1}{\delta \Pi^l} = \tau^l \bullet \left[\frac{\delta}{\delta \Pi^l} \left(\int \left((\Pi^i)^* \, \Pi^i + \Pi^i (\partial_i B_0) \right) d^3\underline{x} \right) \right]$$
$$= \int \left(\tau^i \left[(\Pi^i)^* + \partial_i B_0 \right] \right) d^3\underline{x} \qquad i, l = 1, \ldots, 3 \qquad (8.61)$$

and from similar considerations

$$(\tau^l)^* \bullet \frac{\delta H_1}{\delta (\Pi^l)^*} = \int \left((\tau^i)^* \left[\Pi^i + \partial_i B_0^* \right] \right) d^3\underline{x} \qquad i, l = 1, \ldots, 3. \quad (8.62)$$

Additionally

$$w_0 \bullet \frac{\delta H_1}{\delta B_0} = w_0 \bullet \left[\frac{\delta}{\delta B_0} \left(\int \left(\Pi^i (\partial_i B_0) - m^2 B_0^* B_0 - \lambda B_0^* T_{00} B_0 + \lambda B_i^* T_{i0} B_0 \right) d^3\underline{x} \right) \right]$$

$$= \int \left(w_0 \left[-\partial_i \Pi^i - m^2 B_0^* - \lambda T_{00} B_0^* + \lambda T_{i0} B_i^* \right] \right) d^3\underline{x} \qquad i = 1, \ldots, 3 \qquad (8.63)$$

after a partial integration and the usual consideration of spatial boundary conditions. Correspondingly

$$w_0^* \bullet \frac{\delta H_1}{\delta B_0^*} = \int \left(w_0^* \left[-\partial_i (\Pi^i)^* - m^2 B_0 - \lambda T_{00} B_0 + \lambda T_{i0} B_i \right] \right) d^3\underline{x} \qquad i = 1, \ldots, 3. \qquad (8.64)$$

Furthermore

$$w_l \bullet \frac{\delta H_1}{\delta B_l} = w_l \bullet \left[\frac{\delta}{\delta B_l} \left(\int \left(\left(\partial_i B_j^* \right)(\partial_i B_j) - \left(\partial_i B_j^* \right)(\partial_j B_i) + m^2 B_i^* B_i + \lambda B_0^* T_{0i} B_i - \lambda B_i^* T_{ij} B_j \right) d^3\underline{x} \right) \right]$$

$$= \int \left(w_i \left[-\partial_j \partial_j B_i^* + \partial_j \partial_i B_j^* + m^2 B_i^* + \lambda T_{0i} B_0^* - \lambda T_{ji} B_j^* \right] \right) d^3\underline{x} \qquad i, j, l = 1, \ldots, 3 \qquad (8.65)$$

after some partial integrations and the usual consideration of spatial boundary terms. A similar procedure reveals that

$$w_l^* \cdot \frac{\delta H_1}{\delta B_l^*} = \int \Big(w_i^* \left[-\partial_j \partial_j B_i + \partial_j \partial_i B_j + m^2 B_i + \lambda T_{0i} B_0 \right.$$

$$\left. - \lambda T_{ji} B_j \right] \Big) d^3\underline{x} \qquad i, j, l = 1, \ldots, 3. \qquad (8.66)$$

On substituting (8.61), (8.62), (8.63), (8.64), (8.65) and (8.66) into (8.21) and using the fact that $\tau^0 = (\tau^0)^* = 0$ since $z = ((w, w^*) \oplus (\tau, \tau^*)) \in T_{m_1} M_1$, it is found

that

$$dH_1(B, B^*, \Pi, \Pi^*) . ((w, w^*) \oplus (\tau, \tau^*)) \sim \langle T_{m_1} M_1 | dH_1(m_1) \rangle$$

$$= \int \Big(\tau^i \left[(\Pi^i)^* + \partial_i B_0 \right] + (\tau^i)^* \left[\Pi^i + \partial_i B_0^* \right] +$$

$$w_0 \left[-\partial_i \Pi^i - m^2 B_0^* - \lambda T_{00} B_0^* + \lambda T_{i0} B_i^* \right] +$$

$$w_0^* \left[-\partial_i (\Pi^i)^* - m^2 B_0 - \lambda T_{00} B_0 + \lambda T_{i0} B_i \right] +$$

$$w_i \left[-\partial_j \partial_j B_i^* + \partial_j \partial_i B_j^* + m^2 B_i^* + \lambda T_{0i} B_0^* - \lambda T_{ji} B_j^* \right]$$

$$+ w_i^* \left[-\partial_j \partial_j B_i + \partial_j \partial_i B_j + m^2 B_i + \lambda T_{0i} B_0 - \right.$$

$$\left. \lambda T_{ji} B_j \right] \Big) d^3\underline{x} \qquad i, j = 1, \ldots, 3. \qquad (8.67)$$

If now $z \in (T_{m_1} M_1)^\perp$, as given by (8.17), is put into (8.67) then, by an analogous argument to the one leading to (8.30), the condition

$$\int \Big(w_0 \left[-\partial_i \Pi^i - m^2 B_0^* - \lambda T_{00} B_0^* + \lambda T_{i0} B_i^* \right] + w_0^* \left[-\partial_i (\Pi^i)^* \right.$$

$$\left. - m^2 B_0 - \lambda T_{00} B_0 + \lambda T_{i0} B_i \right] \Big) d^3\underline{x} = 0 \qquad i = 1, \ldots, 3 \qquad (8.68)$$

is obtained. (8.68) is the time preservation condition for the primary constraints (8.12) and since w_0 and w_0^* are arbitrary, it follows that

$$-\partial_i \Pi^i - m^2 B_0^* - \lambda T_{00} B_0^* + \lambda T_{i0} B_i^* = 0 \qquad i = 1, \ldots, 3, \qquad (8.69a)$$

$$-\partial_i(\Pi^i)^* - m^2 B_0 - \lambda T_{00} B_0 + \lambda T_{i0} B_i = 0 \qquad i = 1, \ldots, 3. \qquad (8.69b)$$

Equations (8.69) are secondary Hamiltonian constraints which restrict the motion of the system to a submanifold $M_2 \subset M_1$.

The time preservation of the secondary constraints (8.69) is ensured provided the consistency condition (7.61) holds. In view of this it is necessary to find $(TM_2)^\perp$. As before for each $m_2 \in M_2$, $(T_{m_2} M_2)^\perp$ is given by (7.62). Following the path taken in determining $(T_{m_2} M_2)^\perp$ in section A of this chapter, it will again be assumed that $z = (B, B^*, \Pi, \Pi^*) \oplus (w, w^*, \tau, \tau^*) \sim ((w, w^*) \oplus (\tau, \tau^*)) \in T_{m_2} M_1$ is a typical vector and that $x = (B, B^*, \Pi, \Pi^*) \oplus (v, v^*, \sigma, \sigma^*) \sim ((v, v^*) \oplus (\sigma, \sigma^*)) \in \underline{T_{m_2} M_2}$ is an arbitrary vector, where $(B, B^*, \Pi, \Pi^*) \in M_2$. As seen before for the $z \in T_{m_2} M_1$ given above, then $\Pi^0 = (\Pi^0)^* = \tau^0 = (\tau^0)^* = 0$. On the other hand for the vector $x \in \underline{T_{m_2} M_2}$, where $(B, B^*, \Pi, \Pi^*) \in M_2$, it follows from (8.12) and (8.69) that $\Pi^0 = (\Pi^0)^* = 0$, $-\partial_i \Pi^i - m^2 \overset{*}{B}_0 - \lambda T_{00} \overset{*}{B}_0 + \lambda T_{i0} \overset{*}{B}_i = 0$ and $-\partial_i(\Pi^i)^* - m^2 B_0 - \lambda T_{00} B_0 + \lambda T_{i0} B_i = 0$ and furthermore that $\sigma^0 = (\sigma^0)^* = 0$, $-\partial_i \sigma^i - m^2 \overset{*}{v}_0 - \lambda T_{00} \overset{*}{v}_0 + \lambda T_{i0} \overset{*}{v}_i = 0$ and $-\partial_i(\sigma^i)^* - m^2 v_0 - \lambda T_{00} v_0 + \lambda T_{i0} v_i = 0$. Then in light of the above, it can be seen from (8.32) that

$$0 = \omega_1 | M_2(x, z) = \int \left(v_i \tau^i + \overset{*}{v}_i (\tau^i)^* - w_i \sigma^i - \overset{*}{w}_i (\sigma^i)^* \right) d^3\underline{x}. \quad (8.70)$$

Now the v_i and $\overset{*}{v}_i$ along with the σ^i and $(\sigma^i)^*$ are arbitrary since they satisfy $-\partial_i \sigma^i - m^2 \overset{*}{v}_0 - \lambda T_{00} \overset{*}{v}_0 + \lambda T_{i0} \overset{*}{v}_i = 0$ and $-\partial_i(\sigma^i)^* - m^2 v_0 - \lambda T_{00} v_0 + \lambda T_{i0} v_i = 0$ for arbitrary v_0 and $\overset{*}{v}_0$. Consequently (8.70) can only be satisfied if $\tau^i = (\tau^i)^* = w_i = \overset{*}{w}_i = 0$ for each $i = 1$ to 3. In this determination of $(T_{m_2} M_2)^{\perp}$ the w_0 and w_0^* are undetermined and so they remain arbitrary. As a result of these considerations it is found that the most general form of $(T_{m_2} M_2)^{\perp}$ is given by (8.33); in other words it is the same as the one uncovered in the analysis of the Proca field minimally coupled to an external electromagnetic field.

On substituting $z = ((w, w^*) \oplus (\tau, \tau^*)) \in (T_{m_2} M_2)^{\perp}$, as given by (8.33), into (8.67) it is found, by analogy with the derivation of (8.34), that

$$\left\langle (T_{m_2} M_2)^{\perp} \middle| dH_1(m_2) \right\rangle = \int \Big(w_0\left[-\partial_i \Pi^i - m^2 B_0^* - \lambda\, T_{00}\, B_0^* + \lambda\, T_{i0}\, B_i^*\right] + w_0^* \left[-\partial_i(\Pi^i)^* - m^2 B_0 - \lambda\, T_{00}\, B_0 + \lambda\, T_{i0}\, B_i\right] \Big) d^3\underline{x} \qquad i = 1, \ldots, 3. \tag{8.71}$$

On M_2, however, it is known that $-\partial_i \Pi^i - m^2 B_0^* - \lambda\, T_{00}\, B_0^* + \lambda\, T_{i0}\, B_i^* = 0$ and $-\partial_i(\Pi^i)^* - m^2 B_0 - \lambda\, T_{00}\, B_0 + \lambda\, T_{i0}\, B_i = 0$ and consequently (8.71) automatically vanishes thus satisfying (7.61), the condition that the secondary constraints (8.69) are preserved in time.

As before, this signifies the end of the geometric constraint algorithm for the Proca field coupled to an external symmetric tensor field. To complete the geometric investigation of this coupled system the equations of motion will now be analysed.

As in section A of this chapter suppose that the vector field X is such that $X \sim x = (B, B^*, \Pi, \Pi^*) \oplus (a, a^*, \xi, \xi^*) \in \underline{T_{m_2} M_2}$ for each $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$. In this case $\Pi^0 = (\Pi^0)^* = 0$, $-\partial_i \Pi^i - m^2 B_0^* - \lambda\, T_{00}\, B_0^* + \lambda\, T_{i0}\, B_i^* = 0$ and

$-\partial_i(\Pi^i)^* - m^2 B_0 - \lambda T_{00} B_0 + \lambda T_{i0} B_i = 0$ and furthermore $\xi^0 = (\xi^0)^* = 0$,

$-\partial_i \xi^i - m^2 \overset{*}{a}_0 - \lambda T_{00} \overset{*}{a}_0 + \lambda T_{i0} \overset{*}{a}_i = 0$ and

$-\partial_i(\xi^i)^* - m^2 a_0 - \lambda T_{00} a_0 + \lambda T_{i0} a_i = 0$. In addition suppose that Z is an arbitrary vector field given by $Z \sim z = (B, B^*, \Pi, \Pi^*) \oplus (b, b^*, \eta, \eta^*) \in T_{m_2} M_1$ for each $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$. For this vector field $\Pi^0 = (\Pi^0)^* = \eta^0 = (\eta^0)^* = 0$.

The equations of motion of this system on M_2 are given by (8.35). In this case (8.35) can be written as

$$\int \left(a_i \eta^i + \overset{*}{a}_i (\eta^i)^* - b_i \xi^i - \overset{*}{b}_i (\xi^i)^*\right) d^3\underline{x}$$

$$= \int \Big(\eta^i \left[(\Pi^i)^* + \partial_i B_0\right] + (\eta^i)^* \left[\Pi^i + \partial_i \overset{*}{B}_0\right] +$$

$$b_i\left[-\partial_j \partial_j \overset{*}{B}_i + \partial_j \partial_i \overset{*}{B}_j + m^2 \overset{*}{B}_i + \lambda T_{0i} \overset{*}{B}_0 - \lambda T_{ji} \overset{*}{B}_j\right]$$

$$+ \overset{*}{b}_i \left[-\partial_j \partial_j B_i + \partial_j \partial_i B_j + m^2 B_i + \lambda T_{0i} B_0 - \right.$$

$$\left. \lambda T_{ji} B_j\right] \Big) d^3\underline{x} \qquad i, j = 1, \ldots, 3 \qquad (8.72)$$

because the left–hand side of (8.35) is deduced from similar reasoning to that which resulted in (8.70) and the right–hand side of (8.35) essentially comes from (8.67) after using the fact that $-\partial_i \Pi^i - m^2 \overset{*}{B}_0 - \lambda T_{00} \overset{*}{B}_0 + \lambda T_{i0} \overset{*}{B}_i = 0$ and $-\partial_i(\Pi^i)^* - m^2 B_0 - \lambda T_{00} B_0 + \lambda T_{i0} B_i = 0$ on M_2.

On equating the coefficients of the arbitrary b_i, $\overset{*}{b}_i$, η^i and $(\eta^i)^*$ in (8.72), it is found that

$$a_i = (\Pi^i)^* + \partial_i B_0 \qquad i = 1, \ldots, 3, \qquad (8.73a)$$

$$a_i^* = \Pi^i + \partial_i B_0^* \qquad i = 1, \ldots, 3 \qquad (8.73b)$$

and

$$\xi^i = \partial_j \partial_j B_i^* - \partial_j \partial_i B_j^* - m^2 B_i^* - \lambda T_{0i} B_0^* + \lambda T_{ji} B_j^*$$
$$i, j = 1, \ldots, 3, \qquad (8.74a)$$

$$(\xi^i)^* = \partial_j \partial_j B_i - \partial_j \partial_i B_j - m^2 B_i - \lambda T_{0i} B_0 + \lambda T_{ji} B_j$$
$$i, j = 1, \ldots, 3. \qquad (8.74b)$$

Now it is known for a vector of the form of $x \sim \big((a, a^*) \oplus (\xi, \xi^*)\big) \in \underline{T_{m_2} M_2}$ that

$$-\partial_i \xi^i - m^2 a_0^* - \lambda T_{00} a_0^* + \lambda T_{i0} a_i^* = 0 \qquad i = 1, \ldots, 3, \qquad (8.75a)$$

$$-\partial_i (\xi^i)^* - m^2 a_0 - \lambda T_{00} a_0 + \lambda T_{i0} a_i = 0 \qquad i = 1, \ldots, 3. \qquad (8.75b)$$

On putting (8.73b) and (8.74a) into (8.75a) it is found that

$$m^2\left(\partial_i B_i^*\right) + \lambda(\partial_i T_{i0})B_0^* + 2\lambda T_{i0}\left(\partial_i B_0^*\right) - \lambda (\partial_i T_{ij})B_j^* -$$
$$\lambda T_{ij}\left(\partial_i B_j^*\right) + \lambda T_{i0} \Pi^i - \left(m^2 + \lambda T_{00}\right)a_0^* = 0$$
$$i, j = 1, \ldots, 3 \qquad (8.76)$$

after using (8.54). Since $m \neq 0$ then (8.76) can alternatively be written as

$$\left(1 + \lambda m^{-2} T_{00}\right)a_0^* = \partial_i B_i^* + \lambda m^{-2}(\partial_i T_{i0})B_0^* + 2\lambda m^{-2} T_{i0}\left(\partial_i B_0^*\right)$$
$$- \lambda m^{-2} (\partial_i T_{ij})B_j^* - \lambda m^{-2} T_{ij} \left(\partial_i B_j^*\right) +$$
$$\lambda m^{-2} T_{i0} \Pi^i \qquad i, j = 1, \ldots, 3. \qquad (8.77)$$

In a corresponding manner if (8.73a) and (8.74b) are put in (8.75b), it is found that

$$\left(1 + \lambda m^{-2}\, T_{00}\right)a_0 = \partial_i B_i + \lambda m^{-2}\left(\partial_i T_{i0}\right)B_0 + 2\lambda m^{-2}\, T_{i0}\left(\partial_i B_0\right) - \lambda m^{-2}\left(\partial_i T_{ij}\right)B_j - \lambda m^{-2}\, T_{ij}\left(\partial_i B_j\right) + \lambda m^{-2}\, T_{i0}\left(\Pi^i\right)^* \qquad i, j = 1, \ldots, 3. \qquad (8.78)$$

From a consideration of equations (8.43) along with (8.73), (8.74), (8.77) and (8.78), it can be seen that the equations of motion of this system are given by

$$\left(1 + \lambda m^{-2}\, T_{00}\right)\dot{B}_0 = \partial_i B_i + \lambda m^{-2}\left(\partial_i T_{i0}\right)B_0 + 2\lambda m^{-2}\, T_{i0}\left(\partial_i B_0\right) - \lambda m^{-2}\left(\partial_i T_{ij}\right)B_j - \lambda m^{-2}\, T_{ij}\left(\partial_i B_j\right) + \lambda m^{-2}\, T_{i0}\left(\Pi^i\right)^* \qquad i, j = 1, \ldots, 3, \qquad (8.79)$$

$$\left(1 + \lambda m^{-2}\, T_{00}\right)\dot{B}_0^* = \partial_i B_i^* + \lambda m^{-2}\left(\partial_i T_{i0}\right)B_0^* + 2\lambda m^{-2}\, T_{i0}\left(\partial_i B_0^*\right) - \lambda m^{-2}\left(\partial_i T_{ij}\right)B_j^* - \lambda m^{-2}\, T_{ij}\left(\partial_i B_j^*\right) + \lambda m^{-2}\, T_{i0}\, \Pi^i \qquad i, j = 1, \ldots, 3, \qquad (8.80)$$

$$\dot{B}_i = \left(\Pi^i\right)^* + \partial_i B_0 \qquad i = 1, \ldots, 3, \qquad (8.81)$$

$$\dot{B}_i^* = \Pi^i + \partial_i B_0^* \qquad i = 1, \ldots, 3, \qquad (8.82)$$

$$\dot{\Pi}^0 = 0, \qquad (8.83)$$

$$\left(\dot{\Pi}^0\right)^* = 0, \qquad (8.84)$$

$$\dot{\Pi}^i = \partial_j \partial_j B_i^* - \partial_j \partial_i B_j^* - m^2 B_i^* - \lambda\, T_{0i}\, B_0^* + \lambda\, T_{ji}\, B_j^* \qquad i, j = 1, \ldots, 3, \qquad (8.85)$$

$$\left(\dot{\Pi}^i\right)^* = \partial_j \partial_j B_i - \partial_j \partial_i B_j - m^2 B_i - \lambda\, T_{0i}\, B_0 + \lambda\, T_{ji}\, B_j \qquad i, j = 1, \ldots, 3. \qquad (8.86)$$

It should be noted that (8.83) and (8.84) come about because $\xi^0 = (\xi^0)^* = 0$ for $x = ((a, a^*) \oplus (\xi, \xi^*)) \in \underline{T_{m_2} M_2}$.

Equations (8.79), (8.80), (8.81), (8.82), (8.83), (8.84), (8.85) and (8.86) together with the primary constraints (8.12) and the secondary constraints (8.69) are the final set of equations obtained from the geometric analysis. As in the case of the Proca field minimally coupled to an external electromagnetic field, the final constraint submanifold is M_2.

As in section A of this chapter, the nature of the Proca system coupled to an external symmetric tensor field can be ascertained by considering the spaces $\underline{T_{m_2} M_2}$ and $(T_{m_2} M_2)^\perp$ at each $m_2 \in M_2$. Suppose then that
$z = (B, B^*, \Pi, \Pi^*) \oplus (w, w^*, \tau, \tau^*) \sim ((w, w^*) \oplus (\tau, \tau^*)) \in (T_{m_2} M_2)^\perp$
where $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$. It has already been seen that the most general form of $z \in (T_{m_2} M_2)^\perp$ is given by (8.33). On the other hand if
$z = (B, B^*, \Pi, \Pi^*) \oplus (w, w^*, \tau, \tau^*) \sim ((w, w^*) \oplus (\tau, \tau^*)) \in \underline{(T_{m_2} M_2)}$
then for $m_2 = (B, B^*, \Pi, \Pi^*) \in M_2$, $\Pi^0 = (\Pi^0)^* = \tau^0 = (\tau^0)^* = 0$,

$-\partial_i \Pi^i - (m^2 + \lambda T_{00}) \overset{*}{B_0} + \lambda T_{i0} \overset{*}{B_i} = 0,$

$-\partial_i (\Pi^i)^* - (m^2 + \lambda T_{00}) B_0 + \lambda T_{i0} B_i = 0,$

$-\partial_i \tau^i - (m^2 + \lambda T_{00}) \overset{*}{w_0} + \lambda T_{i0} \overset{*}{w_i} = 0$ and

$-\partial_i (\tau^i)^* - (m^2 + \lambda T_{00}) w_0 + \lambda T_{i0} w_i = 0$. At this point it will be assumed that $(m^2 + \lambda T_{00}) \neq 0$ so that equations of motion for B_0 and $\overset{*}{B_0}$ exist and are given by (8.79) and (8.80) respectively. With this assumption it can readily be seen that the conditions $-\partial_i \tau^i - (m^2 + \lambda T_{00}) \overset{*}{w_0} + \lambda T_{i0} \overset{*}{w_i} = 0$ and $-\partial_i (\tau^i)^* - (m^2 + \lambda T_{00}) w_0 + \lambda T_{i0} w_i = 0$ will only hold when z is also an element of $(T_{m_2} M_2)^\perp$, as given by (8.33), if $w_0 = \overset{*}{w_0} = 0$. Therefore, as seen in section A of this chapter, (7.85) holds for all $m_2 \in M_2$ provided the assumption

$\left(m^2 + \lambda\, T_{00}\right) \neq 0$ is satisfied. It then follows from the classification scheme given in section C of chapter VII that the Proca system coupled to an external symmetric tensor field is second class when $\left(m^2 + \lambda\, T_{00}\right) \neq 0$.

The acausal propagation, mentioned at the start of this chapter, that can sometimes occur when a symmetric tensor field is coupled to the Proca field will now be investigated. It can readily be seen that equations (8.80), (8.82), (8.84) and (8.86) are just the complex conjugate equations of (8.79), (8.81), (8.83) and (8.85) respectively. In view of this only equations (8.79), (8.81) and (8.86) and the secondary constraint (8.69b) will be considered for the propagation analysis. From this point onwards it will be assumed that only the T^{00} component of $T^{\mu\nu}$ is different from zero. This assumption greatly simplifies the analysis whilst still allowing the acausal nature of the system to be exhibited. Since $T_{0i} = T_{ij} = 0$ it follows that (8.79), (8.86) and (8.69b) now respectively become

$$\left(1 + \lambda m^{-2}\, T_{00}\right)\dot{B}_0 = \partial_i\, B_i \qquad i = 1, \ldots, 3, \qquad (8.87)$$

$$\left(\dot{\Pi}^i\right)^* = \partial_j\, \partial_j\, B_i - \partial_j\, \partial_i\, B_j - m^2\, B_i \qquad i, j = 1, \ldots, 3 \qquad (8.88)$$

and

$$-\partial_i\left(\Pi^i\right)^* - \left(m^2 + \lambda\, T_{00}\right) B_0 = 0 \qquad i = 1, \ldots, 3. \qquad (8.89)$$

By taking the time derivative of (8.87) and making use of (8.81) and (8.89), then this gives rise to the condition

$$\ddot{B}_0 - \left(1 + \lambda m^{-2}\, T_{00}\right)^{-1} \partial_i\, \partial_i\, B_0 + m^2\, B_0 = 0 \qquad i = 1, \ldots, 3 \qquad (8.90)$$

after some manipulation. It should be noted that in the derivation of (8.90) it has been assumed that $\left(1 + \lambda m^{-2}\, T_{00}\right) \neq 0$. (8.90) is essentially a 'wave equation' for B_0.

On the other hand the time derivative of (8.81), after using (8.87) and (8.88), leads to a 'wave equation' for the B_i, that is

$$\ddot{B}_i - \partial_j \partial_j B_i + \partial_i \left(\left(\frac{\lambda m^{-2} T_{00}}{1 + \lambda m^{-2} T_{00}} \right) \partial_j B_j \right) + m^2 B_i = 0$$

$$i, j = 1, \ldots, 3, \qquad (8.91)$$

where once again it has been assumed that $\left(1 + \lambda m^{-2} T_{00}\right) \neq 0$.

In deriving the 'wave equations' (8.90) and (8.91) the equations of motion obtained from the geometric analysis have been converted into a second order system of equations. From this second order formalism it is then possible to complete the causality analysis along the lines of Velo and Zwanziger [3]. In fact if $\nu = 0$ and $\nu = i$ are substituted into Velo and Zwanziger's final equation of motion then it is found that the 'wave equations' for B_0 and B_i are respectively obtained providing all the assumptions made en route to (8.90) and (8.91) are taken into account. In other words, (8.90) and (8.91) together are equivalent to Velo and Zwanziger's final equation of motion and consequently the propagation analysis then proceeds precisely as it does in [3]. The upshot is that the Proca field coupled to an external symmetric tensor field propagates acausal modes for those values of the external field satisfying $-1 < \lambda\, m^{-2}\, T_{00} < 0$.

CHAPTER IX

CONCLUSIONS

The overall objective of this thesis was to cast the long–established coordinate dependent approach to high spin field theories into a coordinate independent geometrical context. The motivation behind this was to try to gain a deeper insight into the various inconsistencies that plague high spin field theories coupled to themselves or to external fields. The Gotay–Nester–Hinds constraint algorithm [9] seems to be the most appropriate means of geometrically analysing high spin field theories from the practical point of view and consequently this algorithm has been the basis for all the geometrical calculations of this thesis.

Initially, in order to gain some orientation in the area of high spin field theories, the coordinate dependent versions of the Lagrangian and Dirac–Bergmann constraint algorithms were applied to the free massive spin–1 Proca field. This was more than just an exercise in using these constraint algorithms because the results of the Dirac–Bergmann analysis were later to be compared directly with the corresponding results obtained from the geometric investigation of the Proca field.

However, the main concern of this thesis was the investigation of coupled, rather than free, field theoretic systems. The first steps along this path were taken when the first order version of the coordinate dependent Dirac–Bergmann algorithm was applied to the massive Rarita–Schwinger field coupled to an external electromagnetic field. Since this Dirac–Bergmann algorithm was described for explicitly time independent systems in the thesis, the electromagnetic field was assumed to be time independent for ease of application of the theory. The approach adopted for the analysis of this coupled system was such that it was basically a detailed re–working of the constraint analysis of Hasumi, Endo and Kimura [7]. Hasumi et al's results had shown that the constraint analysis of Johnson and Sudarshan [5] was incomplete for certain critical values of the external electromagnetic field. It was for precisely these critical field values that Johnson and Sudarshan had observed that the anticommutators of their quantized theory were non–positive definite. In completing the constraint analysis of this coupled system on the Hamiltonian side, Hasumi et al found a new tier of constraints appearing at the critical field condition and as a consequence of this they effectively demonstrated that the Johnson–Sudarshan pathology was pre–empted by a loss of degrees of freedom. In a corresponding analysis on the Lagrangian side, paralleling that of Hasumi et al, Cox [14] has shown that the Velo–Zwanziger [4] acausality inconsistency similarly degenerates to a loss of degrees of freedom.

This degeneration of the Johnson–Sudarshan and Velo–Zwanziger diseases to a loss of degrees of freedom is very interesting. It in effect indicates that the type of inconsistency that troubles the Rarita–Schwinger field coupled to an external electromagnetic field is more restricted than was originally thought. Of course this coupled system still suffers from a loss of degrees of freedom pathology but the full constraint analysis has given a deeper understanding of the inconsistency problem.

It was the desire to express the Johnson–Sudarshan and Velo–Zwanziger inconsistencies in a geometrical formalism that initially generated the interest in geometrizing coupled high spin field theories. Before the daunting task of geometrizing this coupled Rarita–Schwinger theory could seriously be contemplated, the Gotay–Nester–Hinds geometric algorithm was applied to two Proca field couplings in order to gain a flavour of the geometrical approach to coupled high spin field theories. The first of the couplings that was considered was the minimal coupling to an external electromagnetic field, whereas the second one was the coupling to an external symmetric tensor field. The geometric constraint algorithm in its current formulation does not handle explicitly time dependent systems and so to facilitate the direct use of the algorithm the above external fields were assumed to be time independent.

The aforementioned coupled Proca systems in the time dependent case have already been analysed via a coordinate dependent Lagrangian constraint algorithm by Velo and Zwanziger [3]. Their investigations revealed that the Proca field minimally coupled to an external electromagnetic field gave rise to a completely consistent system of field equations. On the other hand the symmetric tensor field coupling was found, under certain circumstances, to either propagate acausal modes or not to propagate at all. Consequently it was of great interest to discover how the acausality in the symmetric tensor coupling was manifested geometrically.

It was found that the acausality inherent in the coupled symmetric tensor case only really surfaced in the latter stages of the overall analysis after the constraint equations and equations of motion had been uncovered by the geometric constraint algorithm. These equations of motion were seen to be first order in time derivatives but the equations for the time derivatives of the canonical momenta, Π^i and $(\Pi^i)^*$, were noted for containing second order spatial derivatives. This meant that the theory was not a first order formalism in the true sense. The propagation analysis of a system with this kind of formulation is not encompassed by standard theory. Before a causality analysis was performed on these equations of motion it was first of all assumed, mainly for the sake of simplicity, that only the T^{00} component of the symmetric tensor field $T^{\mu\nu}$ was non–zero. The equations of motion, with the aid of the constraints, were then transformed into second order 'wave equations' for the fields B_0 and B_i by eliminating the canonical momenta $(\Pi^i)^*$. The resulting 'wave equations' were found to be equivalent to Velo and Zwanziger's final equation of motion and the subsequent propagation analysis then proceeded precisely as it had done in Velo and Zwanziger's paper [3].

The symmetric tensor field coupling to the Proca field represents one of the simplest coupled high spin field theories exhibiting the acausality pathology. Part of the simplicity of this coupled theory lies in the fact that the acausality only becomes apparent when all the equations of motion and constraints have been determined. Unfortunately being then forced to convert these equations of motion to a second order formalism has meant that the causality analysis of the system has not strictly speaking been cast into a geometrical background. Furthermore unlike the more complicated Rarita–Schwinger field coupled to an external electromagetic field where it has been shown by Cox [14] that the acausality degenerates to a loss of degrees of freedom, there seems to be no pre–emption of the acausality of this symmetric tensor coupled system. Indeed the final constraint submanifold of this coupled Proca field, on which the propagation analysis is performed, must in some way decompose into three distinct regions where each of these regions possesses a different propagation property depending on the strength of the symmetric tensor field. More specifically one of these regions must represent the case of no propagation whatsoever, another must represent causal propagation and the final case must cover acausal propagation. The causality analysis of the theory found that the values of the external field governing acausal propagation were $-1 < \lambda\, m^{-2}\, T_{00} < 0$.

The decomposition of the final constraint submanifold in the symmetric tensor coupling gives rise to some important issues. All of the geometrical analysis in this thesis has essentially been carried out at the formal level in that it was the determination of the constraints and the equations of motion and their subsequent analysis that was of primary importance. The actual nature of the constraint submanifolds obtained at each stage of the geometric constraint algorithm was not really considered in any depth. For a fully detailed and rigourous geometrical interpretation of a field theory it would be necessary to carefully consider the partial differential equations that characterize a particular submanifold, together with any associated boundary conditions. In the case of the Proca and symmetric tensor coupling it seems that a thorough investigation of all the constraint submanifolds of the system is not crucial since the value of the external field only appears to affect the final constraint submanifold. However in the case of the Rarita–Schwinger field coupled to an external electromagnetic field it is known from coordinate based arguments that the value of the external field can radically alter the course of the analysis. From the geometrical viewpoint this would mean that a careful treatment of each of the constraint submanifolds of the theory would be required in order to successfully analyse the pre–emption of the Johnson–Sudarshan and Velo–Zwanziger inconsistencies by a loss of degrees of freedom.

The geometric constraint analysis of the free Proca field and the two coupled Proca cases via the Gotay–Nester–Hinds algorithm was seen to be a lengthy and cumbersome process. Gotay and Nester [10] have taken some steps to try to improve on the original geometric constraint algorithm by developing a generalized version of the Gotay–Nester–Hinds algorithm and they then applied this generalized algorithm to the free Proca case. For further research it would be interesting to discover whether or not this generalized constraint algorithm has any real calculational advantages over the Gotay–Nester–Hinds algorithm. Obvious candidates for this type of investigation are the free Rarita–Schwinger field and the Rarita–Schwinger field coupled to an external electromagnetic field. However, some preliminary calculations revealed that the Gotay–Nester–Hinds algorithm, and consequently probably the generalized algorithm as well, did not seem very suited to dealing with the free massive spin–$\frac{3}{2}$ Rarita–Schwinger field. The reasons for this are not clear but may revolve around the fact that the Rarita–Schwinger system is described by a Lagrangian which is first order in velocities whereas the free and coupled Proca Lagrangians are in essence second order in the velocities. Recently Cariñena, López and Rañada [20] have translated Scherer's [6] analysis of finite dimensional first order constrained systems into a geometrical arena. By extending Cariñena et al's work to the infinite dimensional case it should then, at least in principle, be possible to geometrically probe the free Rarita–Schwinger field and ultimately the case of this field coupled to an external electromagnetic field.

As a final point the extension of the Gotay–Nester–Hinds and generalized geometric constraint algorithms such that they would be applicable to time dependent systems is a further refinement of the theory that could be looked into. A description of the geometric approach to time dependent finite dimensional regular Hamiltonian systems is presented in Abraham and Marsden [8]. The dynamics in such instances no longer takes place on phase space given by T^*Q, but rather on momentum state space given by

$T^*Q \times \mathbb{R}$, where time is the parameter on $\mathbb{R}$. Since T^*Q is even dimensional then the manifold $T^*Q \times \mathbb{R}$ must be of odd dimension and consequently it cannot be symplectic. The generalization of these finite dimensional time dependent concepts to the infinite dimensional singular case would form the basis for a possible time dependent formulation of the geometric constraint algorithms.

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