'THE PREPARATION OF TAPES AND BOOKLETS SUITABLE FOR A REVISION COURSE IN PARTICLE DYNAMICS.'

BY KENNETH A.H. JACKSON.

FOR DEGREE M.PHIL

AT THE UNIVERSITY OF ASTON IN BIRMINGHAM.

Submitted in June 1980.

THE PREPARATION OF TAPES AND BOOKLETS SUITABLE FOR A REVISION COURSE IN PARTICLE DYNAMICS.

SUMMARY

The work examines the problems which University of Aston first year students have in their Dynamics Course and gives reasons for the choice of audio tapes and booklets in the form of a programmed learning course. It demonstrates the necessity for developing these programmed units of the course in specific stages. The student is guided so that he uses the learning hierarchy of dynamic theory, demonstration of the method and then given further guided practice leading to the completion of entire problems. Some conclusions arising from the whole work are noted, together with some favourable comments by students. The completed course booklets and tape scripts are given as appendices 1 to 10.

Submitted by Kenneth A.H. Jackson for the degree M. Phil, 1980.

PROGRAMMED LEARNING: PARTICLE DYNAMICS REVISION.

Dedication

To my mother,

who died at home during the course of my residence.
She would have been so proud.

Acknowledgments

I should like to thank the following people who allowed me to have the opportunity and support in undertaking the work.

1. My wife Beryl and family, who have been my mainstays for well beyond the actual academic year in residence.

2. The Newcastle upon Tyne Polytechnic Authorities who granted me full salary to undertake the project.

3. Professor D. E. Lawden of the Department of Mathematics for allowing the project to be undertaken.

4. My personal supervisor Mr. R. G. Clarke, whose very keen, close contact ensured the best quality of my work.

5. Mr. T. Hubsch, Head of the Modern Language Department, Newcastle Polytechnic, who allowed me considerable time using his tape recording facilities and in particular their chief technician Mr. D. Peirson.

LIST OF CONTENTS.

CHAPTER ONE

The Problem and Proposed Solution

1.1 The dynamics course concerning us at the University of Aston in Birmingham is a conventional one which first year students study, as part of the Honours Degree Course in Mathematics. For about two thirds of the time, they are concerned with Particle Dynamics, and the remaining third with Rigid Body Dynamics. The teaching method has been for Mr. R.J. Clarke (R.J.C.) to conduct a one hour class lecture, twice a week, together with regular set work. These periods cover sessions on theory and the working and explanation of examples. The set work is returned weekly with necessary comments and an appropriate score. R.J.C. is always available at the end of the lecture, or in his office, if students wish to consult him. Students respond to this, and readily ask for help, and some for extra work. This is given in the form of suggested examples from texts, but is inevitably limited in scope as the students have a wide range of academic and ethnic backgrounds.

1.2 Each year a small number of students are in difficulties for one of three main reasons. First, some of them take 'A' level courses where the dynamics teaching is weak. Second, some students have taken a statistics option in the context of an 'A' level course. This is becoming more popular and the number of these students will increase. Third, there are those who are weak in dynamics and need more time spent on this subject. Other students, not necessarily in difficulties, ask for some helpful revision material, especially towards the end of the course. It is the object of this project to design a course in Particle Dynamics which will fulfil the needs of all these students.

<u>1.3</u> There is no doubt that Dynamics is a difficult topic in mathematics, and if students are to be helped by this material, they have to be shown how to break it down into four essential stages:

i) They must be shown how to interpret a dynamics question into diagrammatic form showing relevant mathematical information. This gives a concrete source of information.

ii) They must then apply the relevant dynamics principles connecting the given material to produce equations.

iii) They must use their knowledge of pure mathematical techniques to manipulate these equations successfully.

iv) They must be able to translate the mathematics back into dynamic terms and make relevant deductions.

1.4 In the preparation of material for this purpose, the following assumptions have been made:

i) All the students have attended the series of lectures by R.J.C., and therefore have some prior knowledge of the material.

ii) They find the subject difficult and lack confidence in their own ability.

iii) Some would be of poor academic ability.

iv) A number of them would be potentially good, but have been hampered by their academic experiences.

v) They are willing to devote three or four hours to repairing their shortcomings in a particular topic.

With this in mind, I determined that a revision course would be of most use.

1.5 I envisage the material being used in three ways.

i) Where the lecturer discovers that a student is weak, and suggests he should use it.

ii) Where a weak student asks for help, and is directed to use it.

iii) Where the body of students decides to use the material for revision.

1.6 There are specific aims which need to be taken into account in the structure of the scheme:

i) To consolidate the students knowledge of the Aston University Particle Dynamics course.

ii) To explain how to translate questions into diagrammatic form.

iii) To explain to students the application of dynamical principles.

iv) To show them how to manipulate the Pure Mathematics and to make sure they do it correctly.

v) To show them how to reinterpret mathematical results into dynamical terms.

vi) To enable students to hear and read dynamics at the same time, to familiarize them with the terminology.

vii) The students must be able to work through the course without supervision, to save staff time.

viii) The material must be self-correcting, giving answers and solutions to problems where needed.

ix) To improve a student's confidence in the subject.

<u>1.7</u> These considerations all led me to the conclusion that the course must be presented in 'Programmed Learning' form. That is, the subject matter must be divided into sequential 'frames' of work, giving answers to previous frames. In this way, students may be shown how to perform processes, or check their own attempts minimizing errors. Specific aim 1.5(vi) suggests the use of an audio tape.

<u>1.8</u> My only previous experience in Programmed Learning (JACKSON 1966) had been in connection with booklet /machine presentation. I thought that this would be too

limited in scope; students rapidly become bored by booklets, and would respond more readily to a voice. I looked at three existing schemes to examine how the human voice had already been used in these circumstances.

The first of these was a cassette tape presentation in the Chemistry Department at the University of Aston, through the courtesy of Mr. P.D. Groves. These were obviously effective, but covered the complete presentation of a course. They were used in stead of lectures and referred to booklets, films, slides, models and the computer. These tapes were too wide in scope and not the answer to my problem.

The second scheme was a purchased set of cassette tape / booklets from the University of Edinburgh by Mr. J.W. Seare. I obtained these through the courtesy of Mr. W.O. Storer of the Mathematics Department at Aston. They concerned topics in pure mathematics, and were mainly student paced exercises; the students were given rules followed by a number of exercises. There seemed to be very little interaction between the voice and the booklet and thus it is not a particularly effective teaching aid.

The third scheme was an Open University tape / booklet on graph work by courtesy of Mr. R. Wilson, Crewe College, Alsager. This was most effective, but assumed no previous knowledge of the subject, which is of course the task of the Open University (MELTON 1977).

A common feature of these schemes was the use of a buzzer to indicate the command 'SWITCH OFF'. This seemed impersonal and remote. I had noticed that Postlethwaite in his 'Audio Tutorial Approach' used music for this purpose, but the type was not specified (POSTLETHWAITE 1972).

<u>1.9</u> Knowledge of these schemes confirmed my idea of using a system of audio tapes and booklets for self study, which are being tried out in some areas of science teaching (BRIDGE 1976). My proposal was therefore as follows:

To produce a series of linked audio tapes and booklets, based upon a 'large step' programmed learning approach. The work would be in addition to the lecture course and would enable the student to take it away and work through it at his own pace (vide MACK '77). The 'frames' would include those giving information, demonstrated examples, practice examples with assistance, and complete problems, each giving answers or solutions. The tape should control the steps in working through the booklets, giving helpful information and hints on working.

<u>1.10</u> There were four reasons why I wished to use the voice as a method of instruction:

i) Aural contact is more personal and friendly; a student might obey the suggestions of a human, rather than a booklet or buzzer.

ii) When listening to the voice, the student is in contact with an authoratitive guide on the subject. (POSTLETHWAITE)

iii) The voice is flexible and can be used to emphasize important stages.

iv) The student would become more familiar with the terminology of dynamics, hearing it used in the correct context.

1.11 Before the use of speech, the oldest form of instruction was by imitation, a laborious and unstructured process. Subsequent methods using the voice have been considered of great importance by many civilisations, not least among these being the Greeks. In their system of schooling, out of seven subjects studied, they had three concerned with speech (the trivium - grammar, rhetoric and dialectic). This had a profound influence on the succeeding civilisations in Europe and Britain (CLARKE '71).

The Old Testament contains many examples of exhortations by The Prophets, and later, during the Roman Civilisation, of Christ in His ministry - 'And He spake many things unto them in parables.' (Matt. III). Since then ministers and priests have used the sermon as a vehicle of teaching.

Early British scholars relied heavily on help from European centres of learning, and we hear of John (an abbott) of Salisbury, wandering in France during the years 1136 to '47, learning dialectic from Abelard at Chartres University. (HASKINS '63). This reliance on speech is also emphasized by the ancient university method of examination by disputation: 'A verbal battle in which the student pits his wits against the University Chancellor or his senior representative (HASKINS). This still remains in the form of oral examinations in connection with the submission of theses for higher degrees!

With the foundation of a national system of education in the mid-nineteenth century, there arose a need for short cuts, an aid to the unfortunate teachers in their over large classes. A whole series of rhymes were used, to be quoted when necessary, (not necessarily with understanding).

For example, in mathematics:

'Twelve and eight
make twenty straight.'

and: 'A pint of water,
weighs a pound and a quarter.' (Vide, my parents)

The acquisition of 'good' speech has always received close attention in schools, and even today, we have many separate 'O' level examinations in English 'grammar' and 'literature'.

The beginning of broadcasting in 1924 has led, among other matters to a tradition of 'talks' about a tremendous variety of topics, keeping people aware of current events, and widening their outlook. Since 1948, when 'School Broadcasts' began, the B.B.C. has produced vast amounts of helpful material, supplementing the teachers' voice. During the year 1974-5, 1004 hours of educational radio material were broadcast at all levels, from infant to Open University (ANNAN '76). Some 90% of schools are reported as using these facilities.

At a conference of members of the Institute of Mathematics and its applications (Education Section) on 26th November 1977 to consider teaching in Primary Schools, the President, Dame Katherine Ollerenshaw remarked that "Children need to speak more mathematics."

1.12 In Higher Education, lectures are still a very important vehicle for teaching, but we must be careful that the ritual does not dull their impact. We still have a useful tradition whereby a newly appointed professor delivers an inaugural lecture, which is of course authoritative, and at a high level. Another considerable method of teaching at a high level is by reading books. These need self discipline to be used effectively by the students, and since many books may be needed to understand a single new topic, they can be enormously time consuming. The tutorial is also an effective personal contact with students, often providing useful dialogues, but if large numbers are to work in this way, it is too demanding from the staffing point of view.

1.13 These audio tapes and booklets are presented as a help towards overcoming these difficulties, providing an aural contact, and structure review tasks, available on demand.

CHAPTER TWO

PROCEDURE AND ANALYSIS

2.1 Since I had little previous experience of tape recording, I thought it prudent to make a trial of the material and method. It was also thought that it would be wiser to try some topic other than Particle Dynamics in case matters went awry. A 24 page booklet and corresponding tape, with 33 instructions, concerning the Apollonius Theorem in geometry, was drafted and recorded. In this, the student was shown how to prove the theorem, asked to prove it himself, and then work through four examples. This was tried by R.J.C., and a final year student from the mathematics department. The student, who had not encountered the Apollonius Theorem before, used it successfully, and commented favourably on the self-pacing aspect. R.J.C. was also impressed by its teaching, but thought that the book was too detailed, and the voice overused. However, we were both of the opinion that similar material could be of considerable benefit to the first year students in dynamics.

2.2 As a result of this trial, the following decisions on format were made:

i) Each booklet should aim to include about 10 related examples covering some three to four hours work.

ii) Revision information would be included in a booklet, where appropriate.

iii) The work was to be divided into large blocks or 'frames'. Some of these were to be 'directed' i.e. demonstrated or assisted in some way. Others would be for individual working with an occasional helping hint (See Table 1

iv) To save paper, the separate frames would not be placed on successive pages to prevent cheating, but produced continuously, and delineated by heavy lines across the page. Students would be encouraged to use sheets of paper to cover material in advance of their working.

v) The tape should be used to give guidance through the booklet, explaining or expanding particular points in the work.

vi) A verbal 'switch off' would be used.

vii) A fairly brisk pace would be maintained with a gap of 5 seconds between instructions.

2.3. As few students would have used this type of material before, they would need some booklet instructions, when no spervision was available. They should be included in every booklet, so that a single topic could be used in isolation from the whole scheme, and not lengthy enough to be discouraging. The set, as given in Fig 1. p 31 has proved effective. To save space in the appendices, the instructions have only been placed in the front of Booklet 1 (See page 31).

Justifications for these instructions are given below:-

Instruction 1: We have a concise statement of the method of working.

Instruction 2: This is self-explanatory.

Instruction 3: This is an attempt to save 'cheating' in some form, though I do not think that this is important, provided the student has made some effort to answer the question himself.

Instruction 4: This was to save paper used in the booklet to minimize expenditure by the department.

Instruction 5: Students will need the material for their own revision, and it would be a waste of time to have to copy out questions and solutions.

Instruction 6: This is to let the students feel that they have complete freedom with the tape.

Instruction 7: This is obvious, but mistakes are possible with 10 packs of material.

2.4 Having read the syllabus in Particle Dynamics, to see how it should be broken down into the booklets, there appeared to be eight natural subdivisions of the topics. However, having looked at the range of problems which might occur under 'Motion in Two Dimensions', and 'Central Forces', it was decided that each of these warranted two booklets. Thus the total number is ten.

These are the booklet titles (a quick guide to content) and the parts of the syllabus covered within them;-

1. 'Uniform Motions and Newton's Laws' - Descriptive motion, uniform acceleration, Newton's Laws of Motion.
2. 'Power, Energy and Hooke's Law' - Conservation of energy, Conservative forces, Hooke's Law.
3. 'Momentum and Restitution' - Conservation of momentum, impulse and restitution.
4. 'Motion in Two Dimensions' - Perpendicular acceleration components, Projectiles above a horizontal plane and an inclined plane.
5. 'Extension of Motion in Two Dimensions' - More complex examples on impulse, restitution and Newton's Laws.
6. 'Variable Forces' - Motion under variable forces, projectiles with resistance.
7. 'Oscillation' - Simple harmonic motion, damped harmonic motion, forced oscillations.
8. 'Restricted Motion' - Circular motion, normal and tangential accelerations.
9. 'Central Forces' - Acceleration in polar coordinates, central forces and plane motion. Differential equation of the orbit.
10. 'Orbits' - Conics in polar form, inverse square law, periodic time, velocity in orbit.

2.5 I tried to prepare the first tape and booklet by my own efforts, based upon the syllabus for Booklet 1. The result was unusable for three main reasons. The material did not assume sufficient knowledge by the students, the questions were of the wrong type in that they failed to test the students sufficiently, and the overall standard was too low. As this had taken some 60 to 65 hours to write, I decided that it required more discussion with R.J.C. of the material at several stages in the production.

2.6 A more structured approach was detailed, which would allow any material to be discussed and commented upon at specific stages of development. These stages were to be enforced, with broad agreement before passing onto the next. It is worth listing these stages, as they allowed R.J.C. and myself to keep an accurate check on the subsequent material, sometimes enabling the simultaneous development of three booklets to proceed.

Stage 1. Agreement of outlines.

This was to ensure that we had the right number and type of problems, and an outline of the dynamical theory to be covered.

Stage 2. Stated examples.

This was to ensure that there was no overlap with problems already used by R.J.C., and that they were of the correct standard.

Stage 3. Revision statements and solutions.

That dynamic theory, which was intended for use as a revision guide, was written out as it would be appearing in the booklet. The problems were worked in similar ways to those used by R.J.C. with his students, and the solutions written out fully.

Stage 4. Draft Booklet and tapescript.

Using the Stage 3 materials, the theory and examples were converted into a programmed booklet and accompanying script. This was a very lengthy process, as the problem methods had to be demonstrated, practised and tried by the user. The problems had to be cut into workable pieces and the script varied for each function. The whole draft was then edited, and examined for the page layout, before making a fair copy for discussion with R.J.C.. Initially these processes took in the order of 110 to 120 hours, and even with practice the overall time was considerable.

Stage 5. Typing of the booklet.

This was undertaken by a secretary in the Mathematics Department to ensure clarity and the incorporation of all the necessary symbols. The diagrams were then drawn by myself and subsequently lettered in type.

Stage 6. Recording of the script.

A portable tape recorder was borrowed from the Communication Media, and to ensure a quiet background a room in the centre of the Modern Languages Dept. was used.

Stage 7. Testing the Material.

By allowing volunteer students to try it for themselves, I could assess any breakdown in the function of the material.

Stage 8. Modification of booklets or tapes.

This would allow for corrections as needed.

2.7 The selection of questions and revision materials used in the booklets are summarized in Table 1 (pp 24 - 28 This indicates how closely they are linked in each booklet, apart from Booklets 5 and 6. Booklet 5 is an extension of the work in No. 4, at a higher level and No. 6 is concerned almost purely with practice in integration, and uses the examples to demonstrate the methods.

2.8 The great length of time involved in stage 4 (para 2.6), may be explained by looking at some examples of the structure of the booklets, which had to be progressive. Where necessary, relevant revision information was given as a particular frame to be studied by a student. This gave him sufficient basic material to be able to start, by refreshing his memory. For example, in Booklet 7 'Oscillations', there are frames on S.H.M. (p 168), damped oscillations (p 180) and forced oscillations (p 185).

The student is required to examine many diagrams, in order to demonstrate their importance and use.

Most of these are given with a problem, and are to be studied as methods of recording information e.g. Booklet 4, example 2 (p 94), example 4 (p 98); Booklet 8, example 1 (p 199).

A student is also given practice in drawing his own diagrams, and these have to be checked against the answers, enabling him to insert omissions or correct mistakes e.g. Booklet 5, example 1 (p118); Booklet 7, example 3 (p 203).

Introductory examples in each booklet are demonstrated by using the combined voice and booklet to give the appropriate approach and solution e.g. Booklet 3 example 1 (p 72). When the student has absorbed the information and diagram, he listens to Instruction No. 5 (p 86) which gives the step by step processes, and then he has to look at the complete working in the next frame. Or, in a more complex example, he is given the equations of motion to check, and then the method of solution is given in the next frame. e.g. Booklet 9, example 1 (p 228), Instructions 2, 3, 4 (pp245-6).

Once the methods are demonstrated, the problems are worked increasingly by the student himself, and for this purpose, they are broken down into several stages. At each stage, however, the correct procedures and answers are shown e.g. Booklet 3, example 6 (p 78). Instructions 18, 19, 20, 21 (p89); Booklet 8, example 3 (p203) Instructions 9, 10, 11 (p221).

It is important that a student is aware of different dynamical solutions to some problems, and where possible, these are shown. e.g. Booklet 7, example 4, last frame (p176), Instruction 17 (p192); Booklet 8, example 7, last frame (p212) Instruction 24 (p224).

The student's attention is also drawn to alternative mathematical treatments in one or two problems, which allows for some degree of choice in his treatment of answers e.g. Booklet 8, example 6 (p208).

Problems containing the derivation of mathematical statements from the working often prove too difficult for the weaker student, and to alleviate this difficulty, this type of proof is demonstrated in a number of examples. e.g. Booklet 4, example 7 (p102); Booklet 9, example 2 (p230).

Finally, if the booklets are revising the work satisfactorily, a student should be able to solve complete questions himself. Such exercises are included, as the last example in a booklet, or after completion of a particular topic within it. The taped instructions give broad hints towards the solution, and the complete answer is given in the booklet, which is read through as a check. It is hoped that successes gained in this way will improve a student's self-confidence. e.g. Booklet 2, example 6 (p79), Instruction 17 (p88); Booklet 8, example 10 (p217), Instruction 35 (p226).

2.9 It was only possible to undertake a limited amount of testing by the students for whom they were intended. Three of the first year course, who performed badly in their Xmas test in Dynamics agreed to try out the tapes and booklets. In case of difficulty during the trial, they performed the work under supervision, but in fact this was only needed on one occasion. They were able to try the first three booklets, and one was able to continue to the fourth and fifth. They were questioned afterwards about the booklet contents and layout, and the tape contents, and the quality of the voice.

Identifying a few errors, the students thought the booklet layout and content satisfactory, necessitating few changes in that part of the material. However the portable tape recorder had produced some distortions in opening phrases, and loud clicks on switching 'on' or 'off'. The students were not unduly critical of these however, as the words were identifiable after rewinding, and they said the overall quality did not interfere with their concentration.

However, I thought that these distortions and clicks would produce growing irritation in someone unconnected with the development of the programme. To obviate this I have subsequently recorded all the material using a language laboratory in the Modern Languages Dept. of Newcastle upon Tyne Polytechnic, which has produced high quality tapes. Editing of errors was conducted easily, and I have deliberately left a few minor ones in, so that the student may feel that 'the voice' has human qualities.

2.10 On completion of the booklets and their recordings, Table 2 (p29) was compiled to allow for some comparisons to be made. This shows that despite the roughly constant number of questions in the first 8 booklets, there is a steady increase in the number of pages, frames, instructions, and the overall tape time. This indicates the increasing difficulty of the material making more steps necessary to help the student. The number of frames and instructions keep roughly in step as the majority of frames concluded with the instruction 'SWITCH ON'. The last two booklets on 'Orbits' involved questions with much longer written answers, and this was why the total number was reduced to 16. This is reflected by the shorter overall recording times. It must be emphasized that the 'running times' listed are purely an indication of the amount of tape used, and will bear little relation to the overall times taken by the student. He must take as long as he needs to complete each element of the course.

2.11 Looking at the overall project, the following conclusions may be drawn.

1. The devising of the separate stages in the development of the booklets/tapes have proved to be effective. It is only by developing the stages in this way that one can build up the material successfully in the least possible time.

2. Throughout this development there must be a very close

consultation with someone who teaches a group with whom one has no direct connection. This was possible at this particular time as I was able to attend the lectures for the first year students as part of my overall course programme, and these proved invaluable.

3. The students found the material most useful, and were loath to criticize it using a questionnaire. When asked in a more relaxed atmosphere they were more willing and the following points emerged:

i) They found the booklet instructions comprehensive.

ii) They all used the tape rewind facility a number of times.

iii) They liked the self-pacing with immediate availability of answers and methods. This is in agreement with November in his Manchester trials (NOVEMBER '78).

iv) They found most of the 'hints' on the tape to be useful in helping with problems.

v) The verbal command 'Switch off' was effective and not at all boring.

vi) My North East regional accent was not found to be obtrusive once the students had heard my voice several times.

vii) They did not favour using headphones, even though they studied in a building with a great deal of background traffic noise from a motorway.

4. It would appear from the trials already conducted, that the overall standard is correct, but this needs to be

verified with use. The students need to have some particular time limit within which to work so that the material is not underused. The grouping of the work into 10 booklets is only a convenience in allocating topics to common bases.

Some of these units appeared to need several hours to complete and thus it would be necessary to break them down further if the participant students are not to be discouraged. This could be carried out in three separate ways.

i) Rewrite the work into 30 booklets; but this would be too cumbersome.

ii) Include extra instructions on page 1 of each booklet advising the students about convenient stopping places.

iii) The method I favour would use the tape counter facility, so that students would be using this in addition to the other instructions. The tape could tell a student that a particular counter number was a convenient stopping place. In trying to start from a particular topic, the counter numbers could also be given on page 1.

<u>2.12</u> A natural sequel to this work would be to produce revision material for the Rigid Bodies Dynamics syllabus used in the same course.

2.13 The programme that has been produced here is not the only self instructional packaged material used at the University of Aston, but the provision of such is fairly sparse. Students using it will, I think, appreciate these remarks made by Professor Goldschmid of Lausanne: "It is likely that the trend towards individual instruction will increase and be intensified. If so, instruction in higher education will be profoundly altered and may enter a promising future." (GOLDSCHMID '76) There is of course the problem of time in which to do the tremendous amount of preparation required. This is perhaps where people like myself could devote a complete sabbatical year to form small teams devoted slowly to the work.

TABLE 1. SUMMARY OF THE CONTENTS OF THE BOOKLETS

Booklet 1. Uniform motion and Newton's Laws

Topics	No. of Revision Frames	No. of related questions
Uniform acceleration.	3	4
Vertical motion under gravity.	1	2
Newton's Laws of motion	3	2 on single pulleys 2 on multiple "

Booklet 2. Power, energy and Hooke's Law.

Topics	No. of Revision Frames	No. of related questions
Power and energy	2	2
K.E. and conservation	2	2 on energy on incline
Conservative forces	1	2 on energy and momentum.
Hooke's Law	1	2
Hooke's Law and P.E.	1	2

Booklet 3. Momentum and Restitution.

Topics	No. of revision frames	No. of related questions
Momentum, impulse	2	2 impulse along straight strings
Conservation of momentum	1	3
Coefficient of restitution	2	4

Booklet 4. Motion in two dimensions

Topics	No. of revision frames	No. of related questions
Vectors	1	1
Relative velocities	1	2
Projectiles above horizontal plane	2	3
Projectiles above inclined plane	1	2
Inclined impact	1	2
		1 projectile with impact

Booklet 5. Extension of Motion in Two Dimensions

Topics	No. of Revision frames	No. of related questions
Motion on inclined planes.	1	4
Impulse along inclined strings		3
Gun barrel inclined to ground.		1
Impact spheres on inclined paths		3

Booklet 6. Variable Forces

Topics	No. of Revision frames	No. of related questions
Horizontal motion		4
and resistance proportional kv^n		3
Vertical motion, resist kv^n		3

Booklet 7. Oscillations

Topics	No. of revision frames	No. of related questions
S.H.M.	1	1 along strings 2 using approx. 2 using solution of D.E.
Damped oscillations	1	3 (1 for each case)
Forced oscillations	1	1 stable, 1 unstable motion

Booklet 8. Restricted Motion

Tangential and normal acceleration	1	3 motion in circle 5 motion on smooth curve 1 motion on rough curve 1 single sided restriction

Booklet 9. Central Forces

Topics	No. of revision frames	No. of related questions
Polar coordinates	1	3 string through hole. 1 apse, elastic string. 2 apses 1 elliptical orbits
D.E. of an orbit		2 orbit from given force

Booklet 10. Orbits

Topics	No. of revision frames	No. of related questions
Orbits		3 more difficult forces
Polar coordinates conics	1	3 inverse square law
Velocity in elliptic orbit	1	1 change of velocity, orbit

TABLE 2. COMPARISON OF THE BOOKLET FORMATS

Booklet Number	No. of Questions	No. of Pages	No. of Frames	No. of tape Instructions	Tape runs (mins)
1	10	12	25	26	12
2	10	13	24	26	12
3	9	15	26	26	14
4	11	18	25	29	17
5	11	17	34	36	19
6	10	18	32	31	20
7	10	22	36	37	24
8	10	22	34	36	23
9	9	19	30	27	14
10	7	17	25	21	13

APPENDIX 1.

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

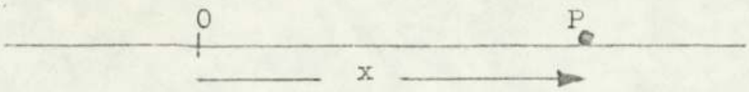
BOOKLET 1 to be used with TAPE 1

"Uniform Motion, and Newton's Laws"

Read the instructions on page 1 thoroughly.

PLEASE DO NOT OPEN THE BOOKLET ANY FURTHER YET.

1) The work in this booklet is divided into sections bounded by heavy lines across the page. You will be asked to work through these in sequence by instructions on the appropriate tape. Keep strictly to this sequence.
2) To save running a voiceless tape, you will be told to 'switch off' at the end of every instruction.
3. The maximum benefit from the work will be obtained by not looking ahead. Avoid this by covering the following work with a piece of paper at each heavy line.
4) You will need paper for your own working.
5) You may keep the booklet when you are finished as a permanent record of the work.
6) If you miss, misunderstand, or forget any instruction, stop the tape, and rewind it so that you are able to replay the part you need. You may do that as often as you like.
7) Check that you have the correct tape, insert it into the playing machine, and SWITCH ON.



The S.I. unit of length is the metre. (m)

When you are ready to go on, SWITCH ON the tape.

When you have written the two forms of velocity,

check your answer below.

$$\text{Velocity} = \frac{dx}{dt} = \dot{x}$$

The positive sense is the same as the positive sense of x.

The unit of velocity is metre per second. ($m\ s^{-1}$)

When you are ready, SWITCH ON the tape.

When you have written the three forms of acceleration,

check your answer below.

Acceleration $= \frac{d^2x}{dt^2} = \ddot{x} = v\frac{dv}{dx}$, and the positive sense of acceleration is the same as the positive sense of x and v.

The unit of acceleration is metre per second per second (ms^{-2}).

SWITCH ON

EXAMPLE 1.

A particle moves such that its distance from an origin is given by $x = t^3 - 6t^2 + 9t + 5$.

i) Find when the particle is at rest.

ii) What are their distances from the origin at these times?

iii) How far is the particle away from the origin when the acceleration is zero?

When you have finished this example, check your solution on the next page.

Solution to example 1.

i) velocity $= \frac{dx}{dt} = \dot{x} = 3t^2 - 12t + 9 = 3(t-3)(t-1)$

$\dot{x} = 0$ when $t = 3$ or 1.

ii) when $t = 1$, $x = 9$

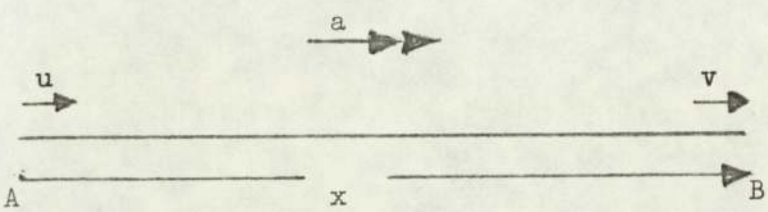
when $t = 3$, $x = 5$

iii) acceleration $= \frac{d^2x}{dt^2} = \ddot{x} = 6t - 12 = 6(t-2)$

$\ddot{x} = 0$ when $t = 2$, and $x = 7$

SWITCH ON

Consider the motion from A to B in time t



x = distance AB

a = acceleration

u = initial velocity

v = final velocity

$v = u+at$ $\qquad$ $x = ut + \frac{1}{2}at^2$

$v^2 = u^2+2ax$ $\qquad$ $x = vt - \frac{1}{2}at^2$

$$x = \left(\frac{u+v}{2}\right).t$$

You should learn and remember these equations.

SWITCH ON

EXAMPLE 2

i) Use $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$, to derive (a) $v = u+at$

(b) $x = ut + \frac{1}{2}at^2$

ii) Use $a = v\frac{dv}{dx}$ to derive $v^2 = u^2 + 2ax$

When you have finished,

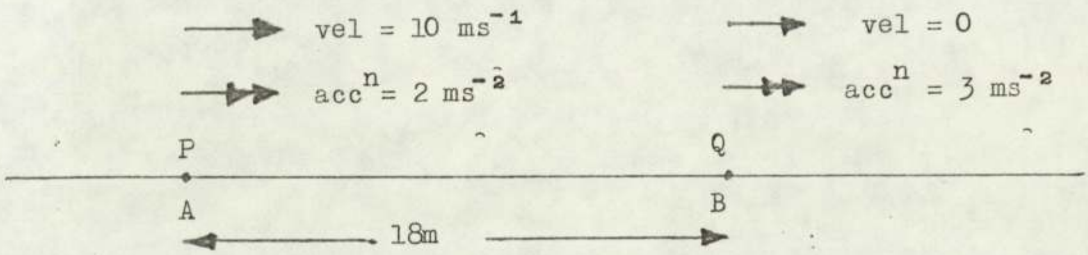
SWITCH ON.

EXAMPLE 3.

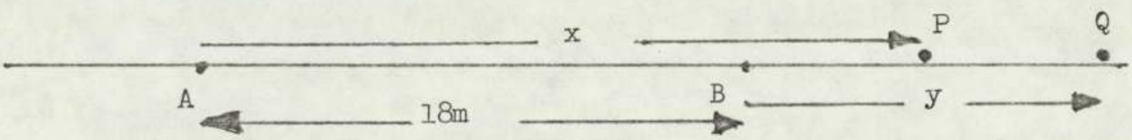
Two particles P and Q move in the same straight line, with Q initially 18m in front of P. Q starts from rest with an acceleration of $3ms^{-2}$, and P starts in pursuit with a velocity 10 ms^{-1} and an acceleration of 2 ms^{-2}. Prove that P will overtake Q after an interval of 2s, and that Q will in turn overtake P after a further interval of 16s.

When you have read this carefully, SWITCH ON

Initial conditions with P at A and Q at B.



At time t



When you understand these diagrams,

SWITCH ON

$$x = 10t + t^2 \quad \text{and } y = \frac{3}{2}t^2$$

For coincidence $y + 18 = x$

Hence $t^2 - 20t + 36 = 0 = (t-2)(t-18)$

i.e. $t = 2$ or 18

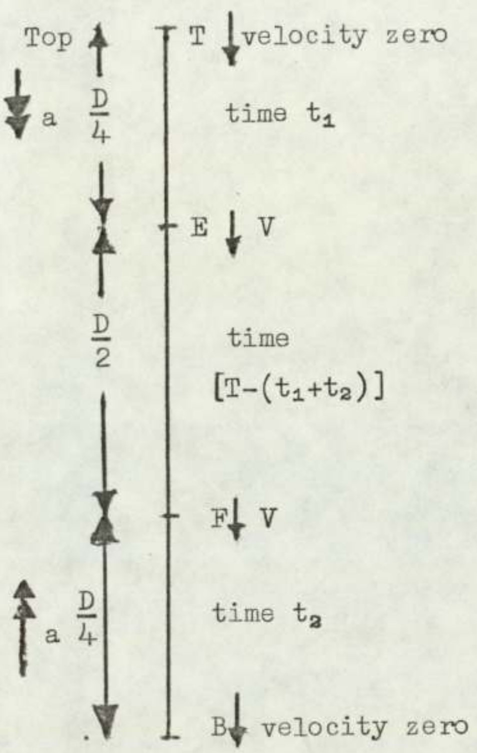
So P overtakes Q after 2 seconds, and 16 seconds afterwards, Q with its greater acceleration, overtakes P.

SWITCH ON

EXAMPLE 4.

A pit cage goes down a mine shaft of depth D, in time T. For the first quarter of the distance, the cage is accelerated uniformly, and during the last quarter, retarded uniformly. The acceleration and retardation are equal. Find the uniform speed of the cage whilst descending the centre portion of the shaft.

SWITCH ON



From T to E, use $v = u+at$;

$$V = 0 + at_1 \rightarrow V = at_1$$

From F to B, use $v = u+at$,

$$0 = V - at_2 \rightarrow V = at_2$$

Hence $t_1 = t_2$

Time from E to F is $(T-2t_1)$.

From T to E use $X = \left(\frac{u+V}{2}\right)t$

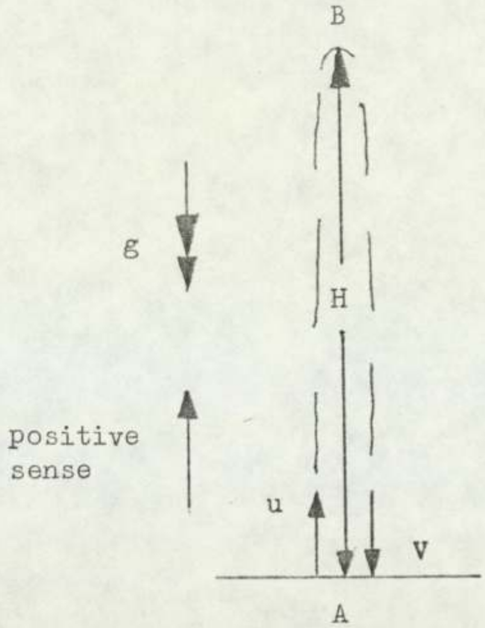
$$\frac{D}{4} = \frac{V}{2}t_1 \rightarrow t_1 = \frac{D}{2V}$$

From E to F, the speed is uniform with

$$\frac{D}{2} = V(T-2t_1)$$

Substitute for t_1, to give $V = \frac{3D}{2T}$

SWITCH ON



The motion takes place in the straight line AB, but the diagram conveniently shows the motion up and down.

Use $x = ut + \frac{1}{2}at^2$ from A back to A

$$0 = ut - \tfrac{1}{2}gt^2$$

Hence time of flight $= \dfrac{2u}{g}$

Use $v = u+at$ from A back to A

$$-V = u - g\left(\frac{2u}{g}\right) \rightarrow V = u$$

Use $v = u+at$ and $v^2=u^2+2ax$ from A to B

time to highest point $= \dfrac{u}{g}$ and $H = \dfrac{u^2}{2g}$.

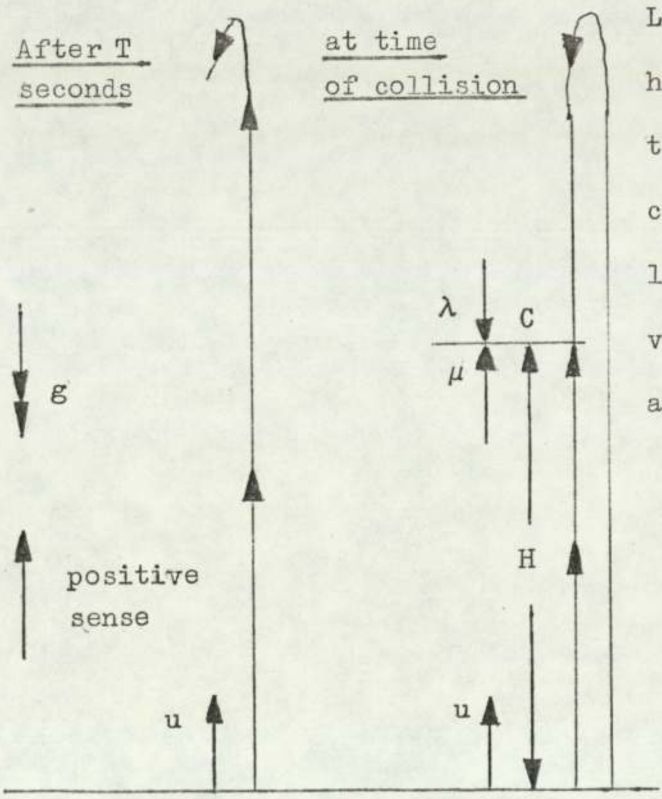
SWITCH ON

EXAMPLE 5.

A particle is projected vertically upwards, and T seconds later, another is projected up with the same initial velocity.

i) Show that the particles meet after a further time $\left(\frac{u}{g} - \frac{T}{2}\right)$

ii) Show that at that time they will both have the same speed $\frac{1}{2}gT$.



Let the particles collide at height H after a further time t. Total time to the point of contact is (t+T) after the first launch. Use λ and μ for the velocities of the particles at the time of collision,

SWITCH ON.

Apply $x = ut + \frac{1}{2}at^2$ for both particles to the point of contact C. $u(t+T) - \frac{1}{2}g(t+T)^2 = H = ut - \frac{1}{2}gt^2$

SWITCH ON

$$-\lambda = u - g\left(\frac{u}{g} + \frac{T}{2}\right) \longrightarrow \lambda = \frac{1}{2}\,g\,T$$

$$\mu = u - g\left(\frac{u}{g} - \frac{T}{2}\right) \longrightarrow \mu = \frac{1}{2}\,g\,T$$

SWITCH ON

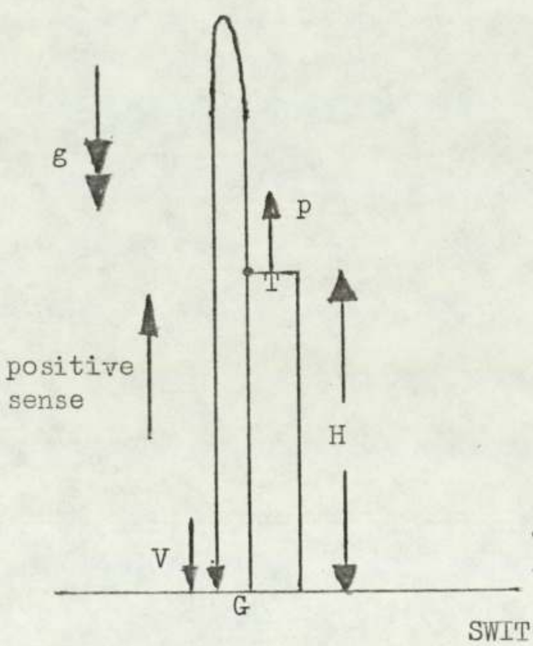
EXAMPLE 6.

A stone is thrown vertically upwards with a velocity of p from the top of a tower which has a height 'H'. Find

i) The velocity with which the stone reaches the ground.

ii) The time of flight.

Do not look at this solution until you have finished the example.



Consider a vertical axis with origin at T and positive sense upwards.

Apply $v^2 = u^2 + 2ax$ from T to G.

$V^2 = p^2 + 2(-g)(-H)$

$V = \sqrt{p^2 + 2gH}$

Apply $v = u + at$ from T to G.

$-V = p - gt$

$t = \frac{1}{g}\left[p + \sqrt{p^2 + 2gH}\right]$

SWITCH ON

Newton's First Law.

A body will remain at rest or move with uniform velocity in a straight line, until it is compelled to change its state by an external force.

When you have studied this,

SWITCH ON.

Newton's Second Law.

When a body is in motion the rate of change of momentum is proportional to the external force, and takes place in the same direction as the force.

Force $\propto$ Rateof change of momentum. When the mass is constant, this reduces to

$$F = m\frac{dv}{dt} = ma$$

with all the units in the S.I. system.

Force in newtons (N), mass in kilograms (kg)

Acceleration in metres per second per second (ms^{-2})

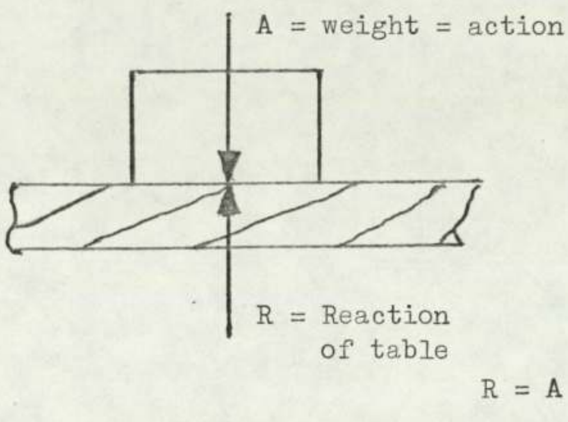
This equation is of fundamental importance in dynamics and is called the Equation of Motion.

SWITCH ON

Newton's Third Law.

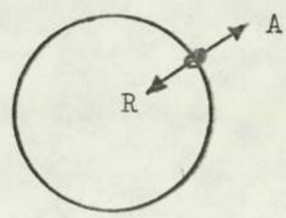
To every action, there is an equal and opposite reaction.

A block resting on a table



R = A

A bead moving on a circular wire.



A = action of the bead on the wire

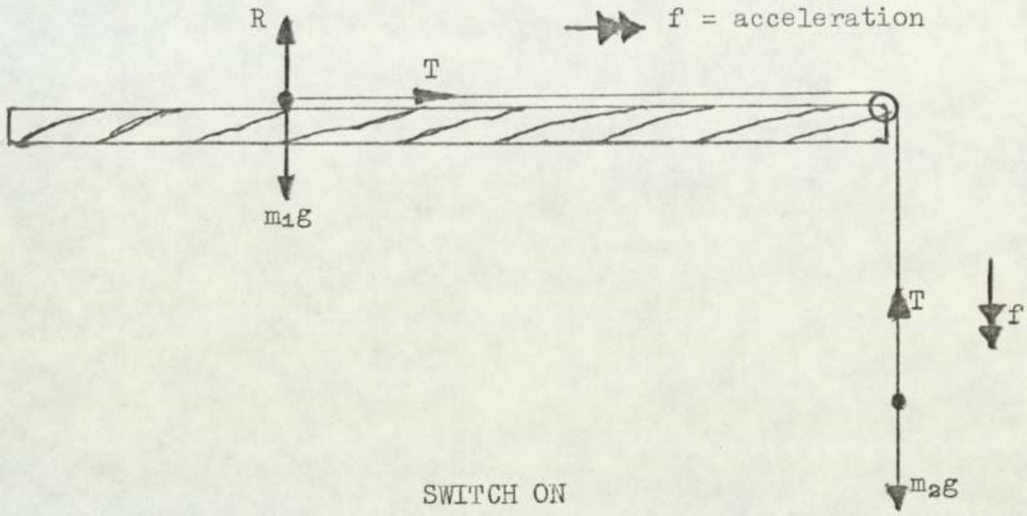
R = reaction of the wire on the bead

When you have studied this,

SWITCH ON

EXAMPLE 7.

Two masses m_1 and m_2 are connected by a light inextensible string. m_1 is placed on a smooth horizontal table and the string passes over a light pulley at the end of the table, and m_2 is hanging vertically. Find the acceleration of the particles when they are released from rest.



SWITCH ON

The two equations of motion are

$$m_1f = T \qquad \text{and} \qquad m_2f = m_2g - T$$

Notice that the positive sense of f defines the positive direction of acceleration and force.

Add the equations to eliminate T and give $f = \dfrac{m_2g}{(m_1 + m_2)}$

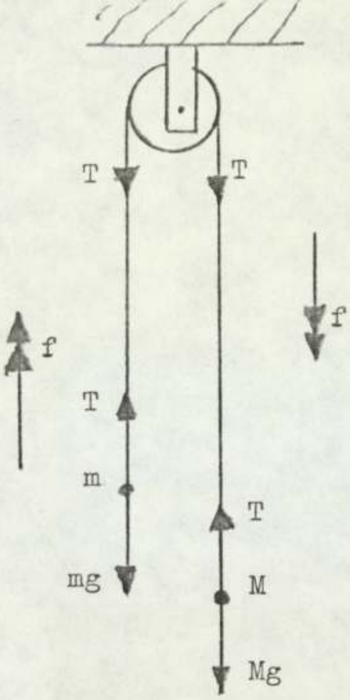
Since m_1 does not leave the table vertically, so $R = m_1g$

SWITCH ON

EXAMPLE 8.

Two masses m and M are suspended from the ends of a light string, which passes over a fixed smooth pulley. Show that, if $M > m$, the acceleration of each mass is $\dfrac{(M-m)g}{(M+m)}$. Also find the tension in the string.

Do not consult the solution until you have finished the question.



Let the common acceleration be f.

The equations of motion are

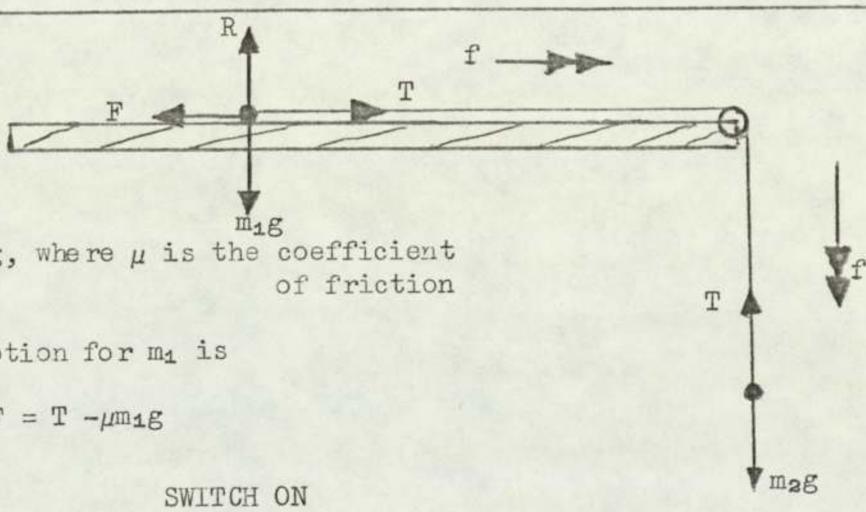
$$Mf = Mg - T$$

$$mf = T - mg$$

Add to eliminate T and substitute for f in the first equation to give:

$$T = \frac{2mMg}{(M+m)}$$

SWITCH ON



$F = \mu R = \mu\, m_1 g$, where μ is the coefficient of friction

Equation of motion for m_1 is

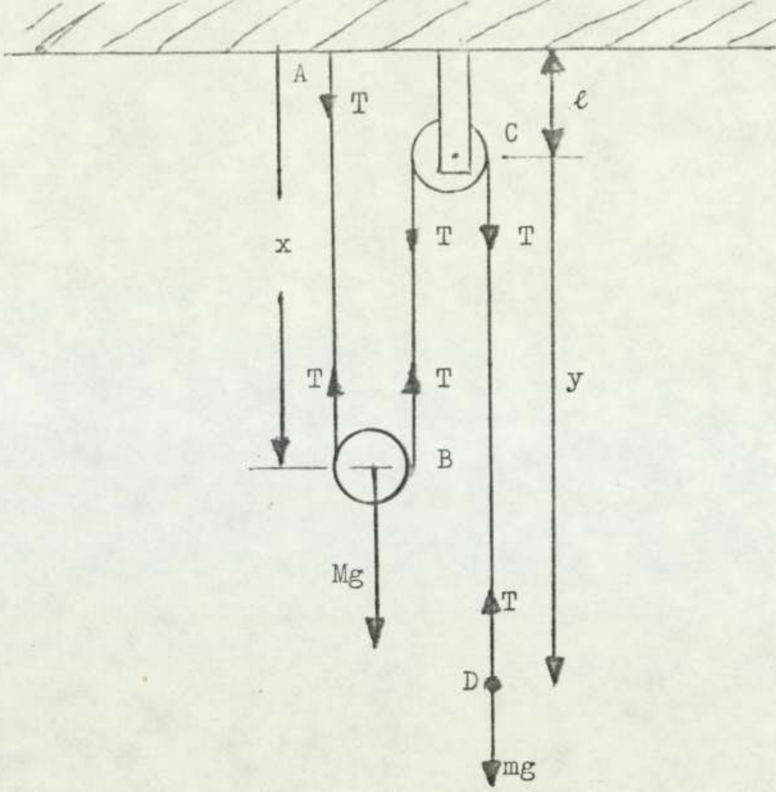
$$m_1 f = T - F = T - \mu m_1 g$$

SWITCH ON

EXAMPLE 9.

A light string ABCD has one end fixed at A, passes under a movable pulley of mass M at B, and over a fixed pulley at C. It carries a mass m at D. The parts of the string not passing over the pulleys are vertical. Show that the pulley at B descends with acceleration $\left(\frac{M - 2m}{M + 4m}\right)g$, if $M > 2m$.

The diagram is overleaf



When you have studied these forces,

SWITCH ON

The length of the string is given by

$x + (x - \ell) + y +$ distances round pulleys

i.e. $2x + y - \ell =$ a constant.

Differentiate wrt 't' twice to give, $2\ddot{x} + \ddot{y} = 0$

The equations of motion are

$M\ddot{x} = Mg - 2T$ and $m\ddot{y} = mg - T$

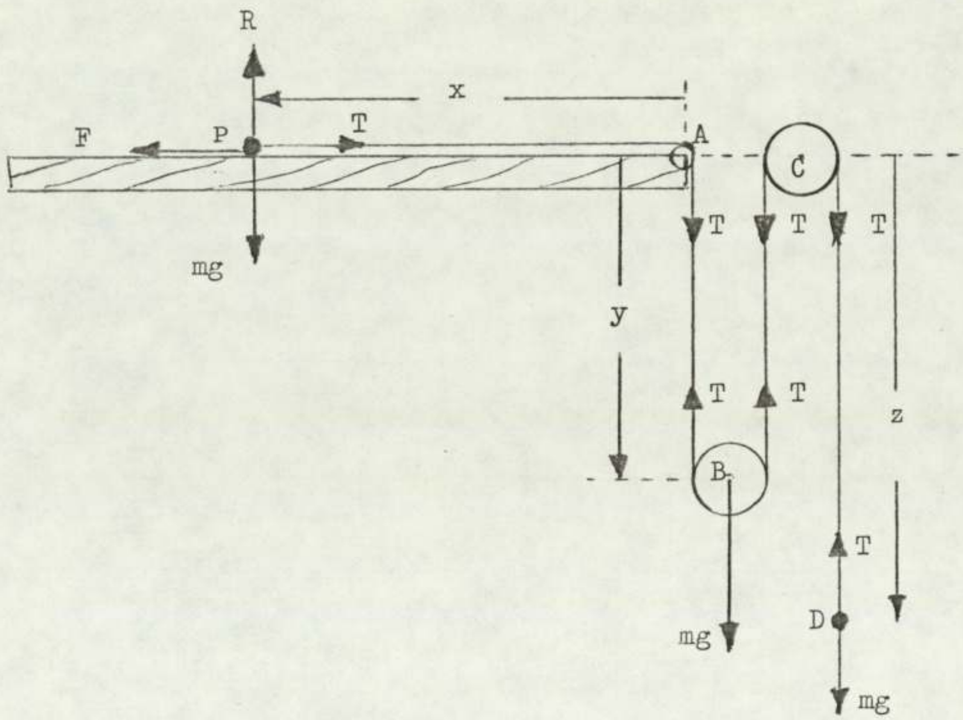
N.B. Always write mass × acceleration first and use this to specify the positive sense.

Eliminate T and $\ddot{y}$ to give $\ddot{x} = \dfrac{(M-2m)g}{(M+4m)}$

SWITCH ON.

EXAMPLE 10.

A particle P, of mass m rests on a rough horizontal table, with coefficient of friction μ, and is attached at one end to a light horizontal string, which passes over a smooth fixed pulley A at the edge of the table. The string then passes under a smooth movable pulley B of mass m, and over a smooth fixed pulley C, at the same level as A. The other end of the string is attached to a particle D, of mass m, which hangs vertically. All the portions of the string not in contact with the pulleys are horizontal or vertical. Assuming that motion takes place, find the tension in the string.



The equations of motion are:-

$$m\ddot{x} = \mu\, mg - T$$

$$m\ddot{y} = mg - 2T$$

$$m\ddot{z} = mg - T$$

Since the length of the string is constant

$$x + 2y + z = \text{constant.}$$

Continued overleaf- 42.

Hence $\ddot{x} + 2\ddot{y} + \ddot{z} = 0$

Substitute for $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ and show that

$$T = \frac{mg(\mu+3)}{6}$$

SWITCH ON.

SCRIPT FOR TAPE 1

UNIFORM MOTION AND NEWTON'S LAWS

1. This is the tape to be used with Booklet 1 of 'A Revision Course in Particle Dynamics' by Kenneth A. Jackson. It is concerned with motion under constant acceleration, and Newton's Laws.

A particle represents a theoretical body which is fundamental to dynamics. Its dimensions, though not zero, are sufficiently small for the internal structure to be unimportant. You can, then, conveniently locate its centre of mass at a point. Moreover, in later studies of a rigid body, the centre of a mass moves as a particle. In order to describe the motion of a particle, you need frames of reference, and the simplest of these occurs for motion in a straight line. Turn to page two in the booklet, where we shall consider this. SWITCH OFF.

2. This diagram shows the position of a particle P, measured in S.I. units, from a fixed origin O, with the arrow denoting the positive sense. The rate of change of position along this line is called the velocity. Write the two calculus notations which you know for this, and also indicate the positive sense of velocity. SWITCH OFF.

3. Usually, the velocity is also changing, and its rate of change is called acceleration. Write the

three calculus notations for acceleration, and also indicate the positive sense. SWITCH OFF.

4. When x is stated as a function of time, it can be differentiated to give the velocity and acceleration, enabling statements to be made about the motion. Example one is a straightforward question of this type. Read it carefully, and when you have completed this, check it overleaf. SWITCH OFF.

5. An important type of straight line motion occurs when the acceleration is constant. In this case, it is convenient to have standard equations ready for use. These are listed for you below. SWITCH OFF.

6. It is important that you understand how these equations have been derived, and example two revises this for you. These three equations are each obtained by direct integration, and the constants are found from the initial conditions. Remember, when you do this question that the velocity is u, when the time and distance are both zero. SWITCH OFF.

7. Example three illustrates the use of these constant acceleration equations. Read this carefully. SWITCH OFF.

8. Diagrams are very useful in dynamics. In this case, two are helpful, because you are told about initial conditions, and then questioned about a later part of the motion. Study the two diagrams below. SWITCH OFF.

9. The first diagram clearly shows the initial conditions and the second shows the position at time t. The particles will be coincident when x and (y+18) are the same. Use the standard equation $x = ut + \frac{1}{2}at^2$, and find expressions for x and y at time t. When you have done this, read the next section in the booklet. SWITCH OFF.

10. You should try an example for yourself. Read Number four carefully. SWITCH OFF.

11. First, draw a clear diagram, showing the distances involved, and introduce convenient symbols for velocities, accelerations and time. You will need to use the standard equation $v = u + at$, and the one using average velocity. When you have completed this, or if you run into difficulty, read the answer in the booklet. SWITCH OFF.

12. A particle thrown vertically upwards, also experiences uniform acceleration, which is called the acceleration due to gravity. It is always directed downwards and is given the symbol'g' for convenience. It is approximately $9.8ms^{-2}$. Consider the motion of a particle which is thrown vertically upwards with a velocity u, and returns to the same point. Find the time of flight, and as much information about the motion as possible.

Draw a diagram if you wish. When you have finished, check your results in the booklet. SWITCH OFF.

13. Example five is a more difficult one on vertical motion, and we shall work through part of this together. Read the question carefully, and study the diagram. SWITCH OFF.

14. The upwards and downwards paths of the particles have been separated for clarity. Using the information on the diagrams, a relationship between height and time can be found for the point C. Do this, and check your answer in the booklet. SWITCH OFF.

15. These two equations for H give a single equation for little t. Solve this equation and obtain your first answer. To answer the second part, use the standard equation $v = u + at$ for both particles, with the appropriate time. Check your answer afterwards in the booklet. SWITCH OFF.

16. Now try example six by yourself. Draw your diagram showing the height, velocities and acceleration. You will find it helpful to consider the top of the tower as the origin of the motion. When you have finished this, check your answers below. SWITCH OFF.

17. So far you have been studying velocities and accelerations without considering why these occur. The three laws of Isaac Newton explain this. The first law states that "A body will remain at rest, or move with uniform velocity in a straight line, until it is compelled to change its state by an external force." Read this

statement carefully in the booklet. SWITCH OFF.

18. To illustrate this first law, consider a ball which can remain stationary on level ground, until it is pushed along. Also, a billiard ball, once struck, will remain on a straight, steady course. An external force can change a state of motion, but the same force has different effects on bodies with different masses; that is the quantities of matter in the bodies. The second law provides a measure of the effect of a force, and states that "When a body is in motion, the rate of change of momentum is proportional to the external force, and takes place in the same direction as the force". Consider this statement carefully, in the booklet. SWITCH OFF.

19. Newton's Third Law states that "To every action there is an equal and opposite reaction". Read this statement carefully in the booklet, and study the illustrative sketches, which show how reactions can either prevent motion, or constrain it along a particular path. SWITCH OFF.

20. We shall now work together a problem involving the equations of motion. Read example seven carefully, and study the diagram. SWITCH OFF.

21. The diagram shows all the forces acting on the particle. Notice, that as the pulley is smooth, the string has a uniform tension throughout. The two particles will have the same acceleration, as the string is inextensible. Let this be f. The two equations of motion are written for you in the section below. Check these before

you use them. SWITCH OFF.

22. You should now attempt example eight yourself. Read the question carefully. Remember, you must draw a clear diagram showing all forces and masses; then find the equation of motion for each mass. Finally, use these equations to eliminate any unwanted quantities, to find whatever is required. SWITCH OFF.

23. So far in all this work, you have considered all motions to be without friction. You will remember that example seven concerned a smooth table. Suppose that the table were rough. Motion would only occur if the tension is greater than limiting friction. Assume that this is so, and examine the new equation of motion in the section below. SWITCH OFF.

24. Now consider example nine, which is more difficult. We shall work through this example, so read it carefully and study the diagram. SWITCH OFF.

25. The acceleration of the masses at B and D are no longer the same, because B is not attached to the string, but supported by it. To help in this situation, it is useful to introduce the lengths x and y into the diagram. Notice that x is the distance of B from a fixed level, and the pulley at B has an acceleration x downwards. The connection between these accelerations is explained in the booklet and the solution completed. Follow this carefully. SWITCH OFF.

26. You should now try example ten by yourself. Read it carefully, draw the diagram, mark in all the forces,

and introduce suitable displacements. Take care in the problem that you obtain the correct total length of string. Do not read the solution below before you have finished your own. SWITCH OFF.

27. This completes the work about motion under constant acceleration and Newton's laws. Please rewind the tape before you remove it from the machine.

Thank you!

APPENDIX 2.

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

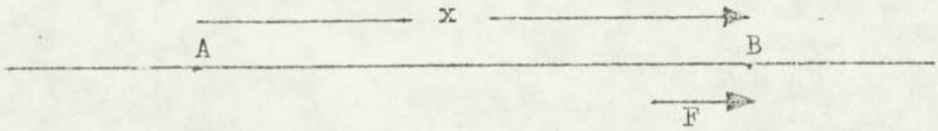
BOOKLET 2 to be used with TAPE 2

"Power,Energy and Hooke's Law"

Read the instructions on page 1 thoroughly.

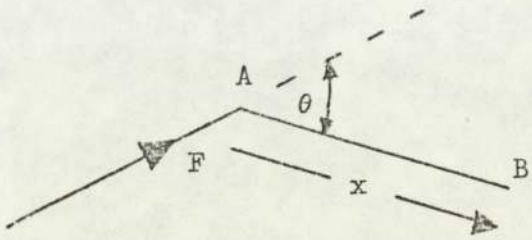
(Ommitted for convenience.)

When a force moves its point of application it is said to perform work



Work done from A to B = $F \times x$

If the displacement is not in the same line as the direction of the force.



W.D. = $(F \cos\theta)x$

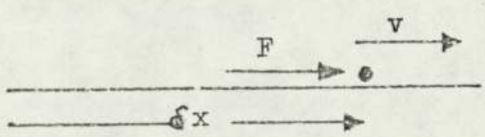
$F \cos\theta$ is called the component of the force along AB.

Unit of work 1JOULE = 1 newton metre 1J = Nm.

When you have studied this,

SWITCH ON

POWER is the rate of doing work.



$$\text{Power} = \lim_{\delta t \to 0} F . \frac{\delta x}{\delta t} = Fv$$

Unit of power 1 WATT = 1 Joule per second $1W = 1JS^{-1}$

For <u>constant velocity</u> ($v ms^{-1}$)

$$P = F\,v \text{ watts}$$

An accelerating body requires additional power to allow for the accelerating force.

For <u>constant acceleration</u> ($a\ ms^{-2}$)

$P = F\,v$, but $F = ma$ (Newton's equation of motion)

i.e. $P = (ma)\,v$

v is now changing, and the velocity must be taken at a particular time.

SWITCH ON

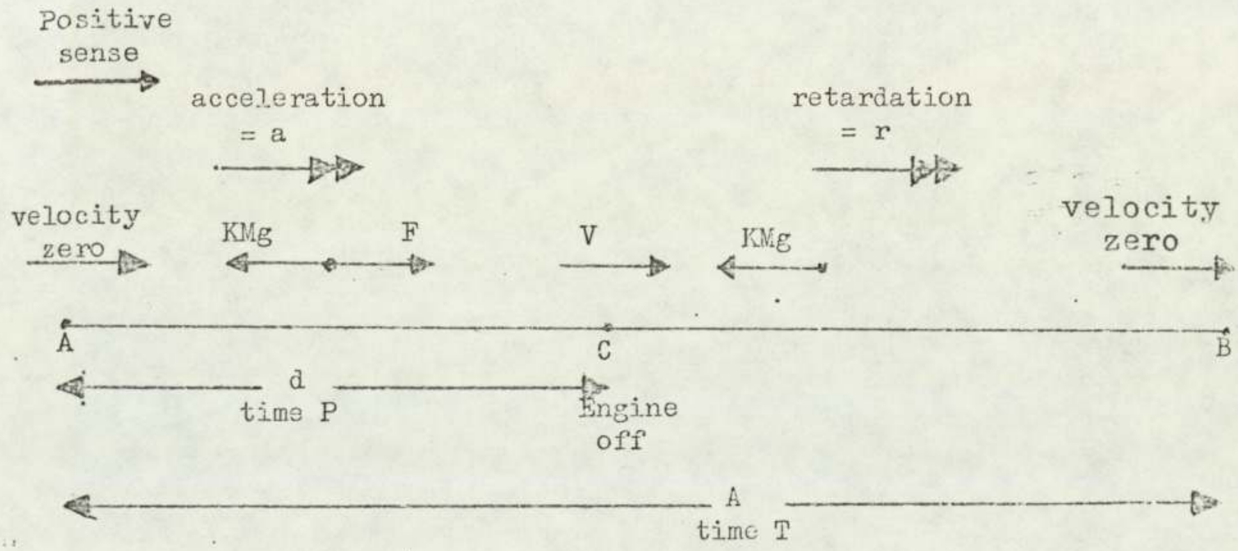
EXAMPLE 1.

A vehicle of mass M, starts from rest at A, and travels with uniform acceleration for a time P. The engine is then switched off and the vehicle comes to rest at B without the brakes being used. The distance from A to B is A, and the total time taken is T. The resistance due to friction is K times the weight of the vehicle. Prove that

$$(T-P) = \frac{2A}{KgT},$$

and the greatest rate of working of the motor during the journey is

$$\frac{4A^2M}{TP(T-P)}.$$



When you are ready to proceed,

SWITCH ON

Equations	A to C	C to B
$F = ma$,	$F - Kmg = Ma$,	$Mr = -KMg$
$x = \left(\frac{u+v}{2}\right)t$,	$d = \frac{VP}{2}$,	$A-d = \frac{V}{2}(T-P)$
$v = u+at$,	$V = aP$,	$0 = V+ r(T-P)$

SWITCH ON

From the second pair of equations, show that $V = \frac{2A}{T}$. Use this result, with the two equations involving r, to derive the first result.

SWITCH ON

Power $= F\,v = (Ma + KMg)v$

This will be maximum when v is maximum, i.e. at C.

So maximum power $= M(a+Kg)V$

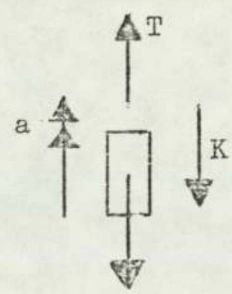
Substitute for a, V and Kg to derive the second result.

SWITCH ON

EXAMPLE 2.

A lift, of mass M, moves in a vertical shaft against a constant friction force K. Find the power of the motor which can pull it up at a steady speed U. If this power is constant show that the acceleration of the lift would be $\left(\frac{Mg+K}{M}\right)$ if the upward velocity were U/2.

Do not read the solution until you have finished.



Use 'F = ma', $Ma = T - Mg - K$

when $a = 0$ for steady speed, $T = Mg+K$

Power = 'Tv' i.e. Power $= (Mg+K)U$

When speed is U/2,

Power $= (Ma+Mg+K)U/2$

i.e. $(Mg+K)U = (Ma+Mg+K)U/2$

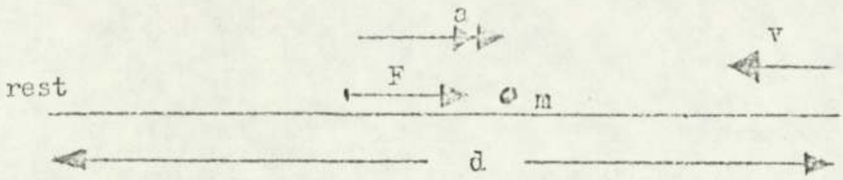
hence $a = \frac{(Mg+K)}{M}$

SWITCH ON

KINETIC ENERGY of a body is the energy possessed by virtue of its motion, and is measured by the amount of work it does in coming to rest against a resistance.

A particle moving with speed v has kinetic energy $\frac{1}{2}mv^2$. Unit is the joule, as for work. If this particle has mass m, and is brought to rest in a distance d, with retardation a, by a constant force F;

-continued overleaf-



then $F = ma$

Use $v^2 = u^2 + 2ax$, $\qquad 0 = v^2 - 2ad$

$$ad = \tfrac{1}{2}v^2$$

Work done by the particle is $Fd = mad$

$$= m(\tfrac{1}{2}v^2) = \tfrac{1}{2}mv^2$$

SWITCH ON

Conservation of Energy

The kinetic energy at the end of a change in motion is the sum of the initial kinetic energy and the work done *on* the particle.

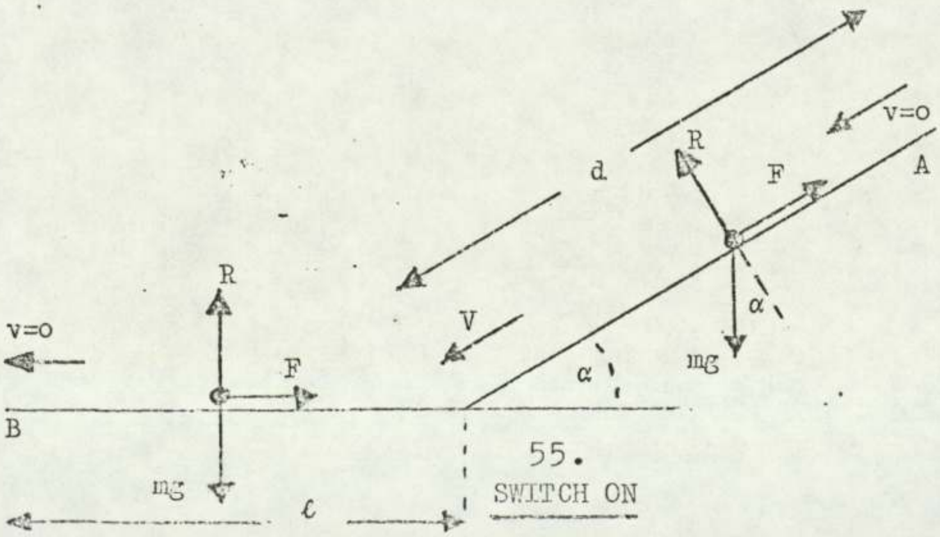
or: $\quad K.E.\big/_{END} = K.E.\big/_{BEGINNING} + \text{Work done}\big/_{\substack{EXTERNAL\\FORCES}}$

This must be remembered.

SWITCH ON

EXAMPLE 3.

The foot of a rough inclined plane, of inclination α, is joined to an equally rough horizontal surface. A particle is held at rest at a distance 'd' up the inclined plane, and then released. How far will it travel along the horizontal surface, if the coefficient of friction is μ? [Assume there is a smooth transition from one plane to the other.]



SWITCH ON

On the inclined plane,

resolved component of the weight $= mg\sin\alpha$

$R = mg\cos\alpha$

$F = \mu R = \mu\, mg\cos\alpha$

On the horizontal plane, $F = \mu\, mg$

Now $K.E./_{end} = K.E./_{begin} + W.D/_{forces}$

$0 = 0 + mg\sin\alpha . d - \mu\, mg\cos\alpha . d - \mu\, mg\ell$

by rearranging terms, $\ell = \dfrac{d(\sin\alpha - \mu\cos\alpha)}{\mu}$

SWITCH ON

At the foot of the incline, $K.E = \frac{1}{2}mV^2$

Use $K.E._{END} = K.E._{BEGIN} + W.D._{FORCES}$

i.e. $\frac{1}{2}mV^2 = 0 + mg\sin\alpha . d - \mu\, mg\cos\alpha . d.$

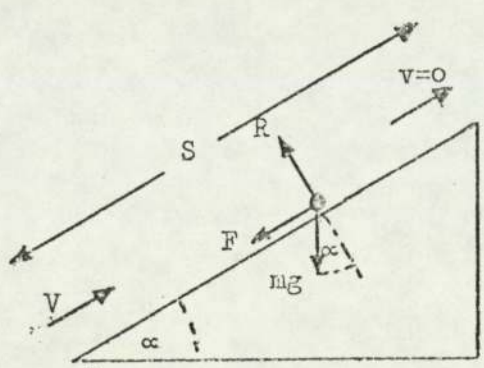
Hence $V = \sqrt{2gd(\sin\alpha - \mu\cos\alpha)}$

SWITCH ON

EXAMPLE 4.

A particle of mass m, is projected up a plane of inclination α, with a velocity V. How far does the particle move up the plane, if the coefficient of friction is μ?

Do not look at this solution until yours is complete.



$K.E./_{END} = K.E./_{BEGIN} + W.D./_{FORCES}$

$0 = \frac{1}{2}mV^2 - mg\sin\alpha . S - \mu\, mg\cos\alpha . S$

hence $S = \dfrac{V^2}{2g(\sin\alpha + \mu\cos\alpha)}.$

SWITCH ON

Conservative Forces.

If the work done in bringing a body from one position to another is independent of the path taken, then the force is said to be conservative. Gravity is such a force, but friction is not. The POTENTIAL ENERGY of a body, is the amount of work it can do in moving from its actual position to some convenient standard position.

For a particle 'm' at height 'h' above the earth's surface, it is the work it can do in falling that distance. This is taking the earth's surface as the zero or standard position.

P.E. = mgh Unit is also the joule.

The Principle of Conservation of Energy.

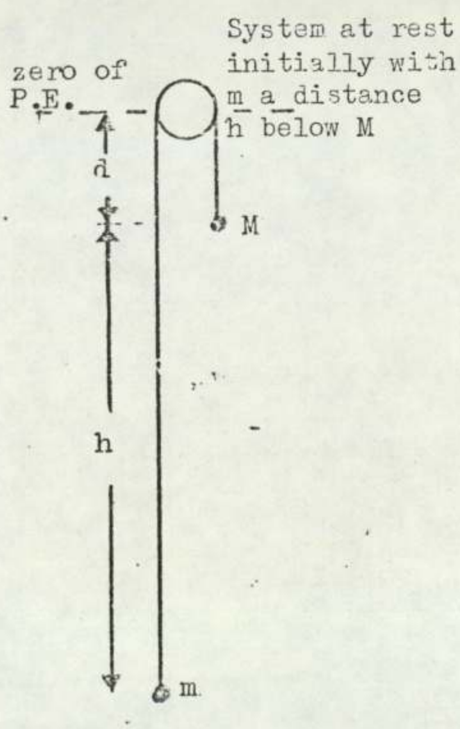
If a system of bodies is in motion under the action of a conservative system of forces, the sum of the kinetic and potential energies is constant.

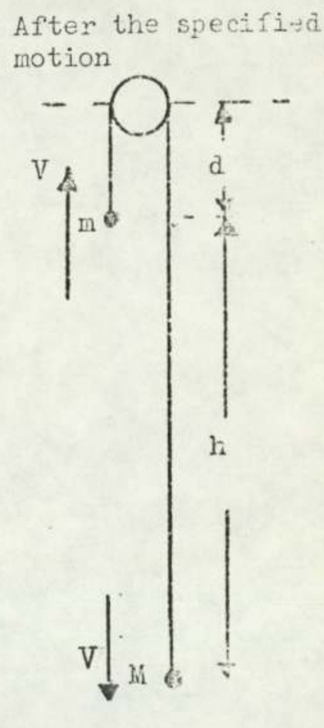
Briefly:- P.E. + K.E. = Constant.

SWITCH ON.

EXAMPLE 5.

Masses of m and M are attached to the ends of a light string which passes over a smooth pulley. Find the velocity of the mass M when it has fallen through a distance h.





$$(K.E. + P.E.)_{end} = (K.E.+P.E.)_{begin}$$

$$(\tfrac{1}{2}mV^2 + \tfrac{1}{2}MV^2) + [-mgd - Mg(d+h)] = 0 + [(-Mgd - mg(d+h)]$$

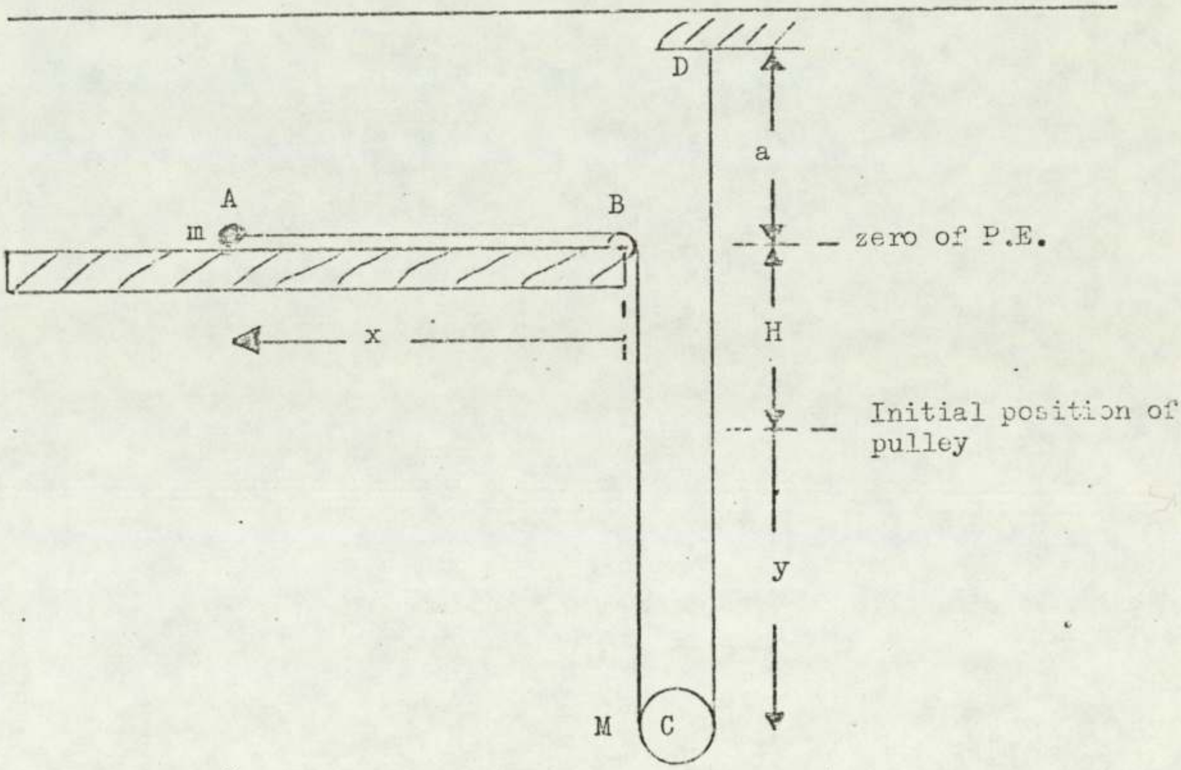
$$\text{hence } V = \sqrt{\frac{2gh(M-m)}{(M+m)}}$$

SWITCH ON

EXAMPLE 6.

A light string ABCD is fastened at A to a mass m, free to slide on a smooth horizontal table. It passes over a fixed smooth pulley at B at the edge of the table, under a movable smooth pulley C, of mass M, and is fixed D, vertically above C. All the sections of the string are either horizontal or vertical. The system is allowed to move under gravity. Show, that when the velocity of the pulley C is V, it must have fallen through a distance $h = \frac{(4m+M)V^2}{2Mg}$.

Do not look at the solution until you have completed the question.



$$(K.E.+P.E.)_{t=0} = (K.E.+P.E.)_t$$

$$0 - Mgh = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M\dot{y}^2 - Mg(H+y)$$

As the string is of constant length,

$$x + 2(H+y) + a = \text{Constant.}$$

-continued overleaf-

Diff. w.r.t. time to give $\dot{x} + 2\dot{y} = 0$

let $\dot{y} = V$ when $y = h$, $\dot{x} = -2V$.

Substitute in the energy equation,

$$-MgH = \tfrac{1}{2}m(-2V)^2 + \tfrac{1}{2}MV^2 - Mg(H+h).$$

hence h.

SWITCH ON

Hooke's Law.

The tension in an elastic string is proportional to its extension.

i.e. $T = \dfrac{\lambda x}{\ell}$

where λ is a constant for a particular string. It is called the Modulus of Elasticity and has units of force.

x is the extension beyond its natural length.

ℓ is the natural length.

N.B. 1. The law only applies if a string is not stretched beyond the elastic limit, after which there is permanent deformation.

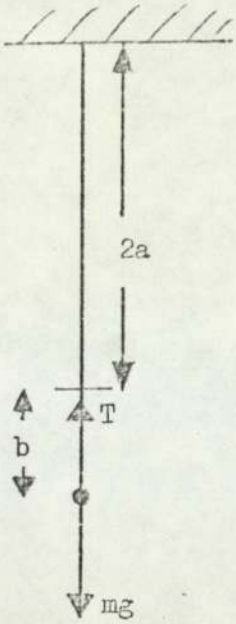
2. It also applies to springs in extension and compression, provided they have negligible weight.

SWITCH ON

EXAMPLE 7.

An elastic string of natural length 2a, is found to extend a distance 'b' when a particle of mass m is suspended from a free end. The particle is then removed and attached to the midpoint of the string, the ends of which are then tied to two points A and B in the same vertical line at a distance apart greater than 2a. Assuming that in the equilibrium position, the lower part of the string remains taut, show that the displacement of the particle is $\frac{b}{4}$ from the midpoint of AB.

DIAGRAM ON THE NEXT PAGE

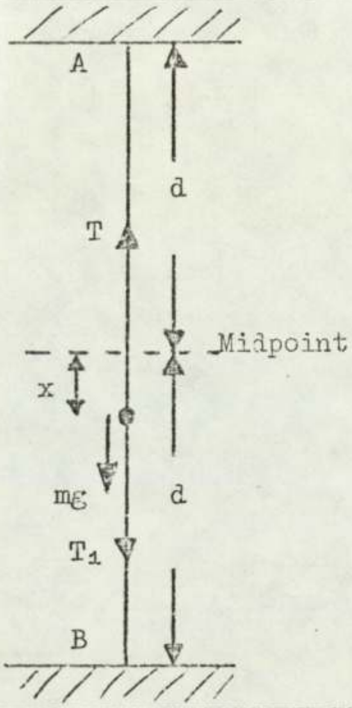


Use $T = \frac{\lambda x}{\ell}$

When the particle is hanging from one end, tension $T = mg = \frac{\lambda b}{2a}$.

$$\rightarrow \lambda = \frac{2mga}{b}$$

SWITCH ON



Let the displacement to the midpoint be x and AB = 2d.

Tension in the upper string $T = \frac{\lambda(d+x-a)}{a}$

For lower string $T_1 = \lambda\frac{(d - x - a)}{a}$

In equilibrium $T = mg + T_1$

Substitute for T, T_1, λ,

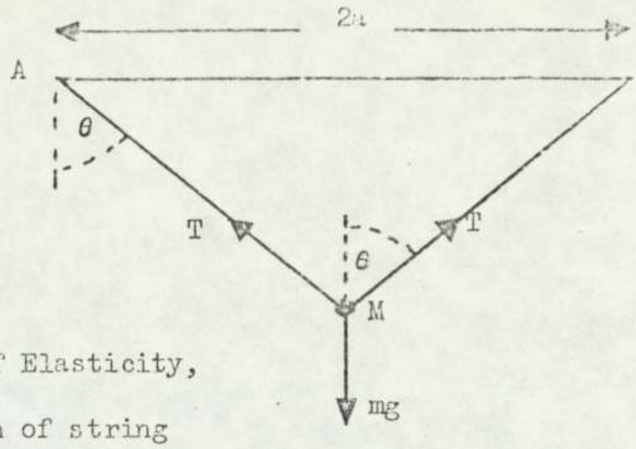
hence $x = \frac{b}{4}$ Check this yourself.

SWITCH ON

EXAMPLE 8.

The ends of an elastic string of natural length 2a are fixed in a horizontal line at a distance 2a apart. A particle of mass m is attached to the midpoint and rests in equilibrium. If each half of the string is inclined to the vertical at an angle θ, show that the modulus of elasticity is $\frac{mg}{2(\cot\theta - \cos\theta)}$.

(Do not read the solution overleaf until yours is complete).



If λ = Modulus of Elasticity,

and x = extension of string

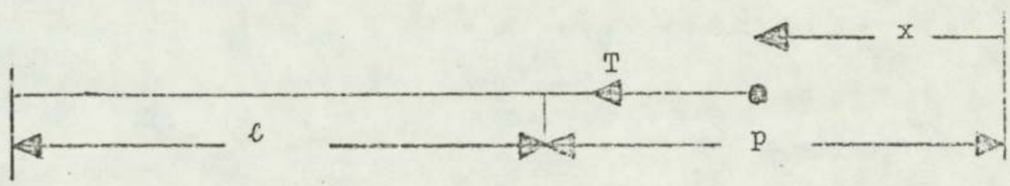
For equilibrium, $2T \cos\theta = mg$

where $T = \frac{\lambda x}{\ell} = \frac{\lambda(AM-a)}{a} = \frac{\lambda(a \operatorname{cosec}\theta - a)}{a}$

Substitute in above equation, hence λ

SWITCH ON

Consider a string, natural length ℓ, stretched a distance p. Work is done by the tension as it returns the string to its natural length.



$$\begin{matrix}\text{Potential} \\ \text{energy}\end{matrix} = \begin{matrix}\text{Total} \\ \text{work}\end{matrix} = \lim_{\delta x \to 0} \sum_{x=0}^{x=p} T.\delta x = \int_0^p T.dx = \frac{\lambda}{\ell}\int_0^p (p-x)dx$$

$$= \frac{\lambda}{\ell}\left[px - \frac{x^2}{2}\right]_0^p = \frac{\lambda}{\ell}\left(p^2 - \frac{p^2}{2}\right) = \frac{\lambda p^2}{2\ell}$$

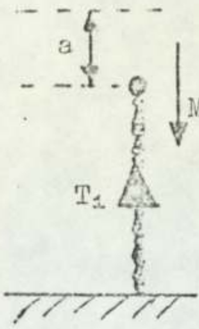
SWITCH ON

EXAMPLE 9.

A spring of negligible weight is compressed a distance 'a' by a mass M. Show that if the mass is allowed to fall on to the spring from a height $\frac{3}{2}a$ above it, the maximum compression of the spring in the motion which follows, is 3a.

THE DIAGRAMS ARE ON THE NEXT PAGE

First compression

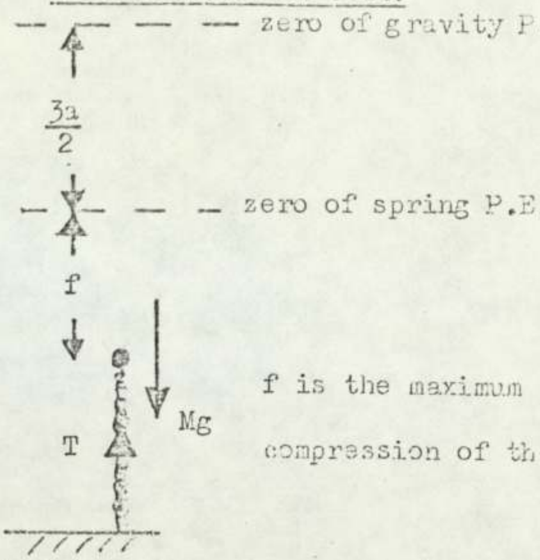


For equilibrium

$$Mg = T_1 = \frac{\lambda a}{\ell}$$

$$\lambda = \frac{Mg\ell}{a}$$

This diagram shows the position of instantaneous rest.



f is the maximum compression of the spring

When you have studied these,

SWITCH ON

$$(K.E.+P.E.)_{start} = (K.E.+ P.E.)_{full\ compression}$$

$$0 = 0 + \frac{\lambda f^2}{2\ell} - Mg\left(\frac{3a}{2} + f\right)$$

Substitute for λ, this reduces to

$$f^2 - 2af - 3a^2 = 0$$

$$(f - 3a)(f + a) = 0,$$ which is satisfied

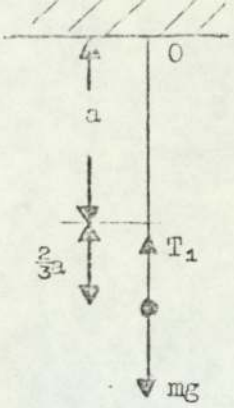
for $f = 3a$ or $- a$, but $f = - a$ has no physical meaning.

SWITCH ON

EXAMPLE 10.

A mass is suspended from a fixed point O by an elastic string of natural length a, and when the mass is hanging freely, the length of the string is 5a/3. Show, that if the mass is allowed to fall freely from rest at O, the greatest length of the string in the subsequent motion is 3a. Show also, that the speed with which the mass is moving when it is distance 2a from O, is $\sqrt{\frac{5ga}{2}}$.

(Do not read the solution overleaf until yours is complete).

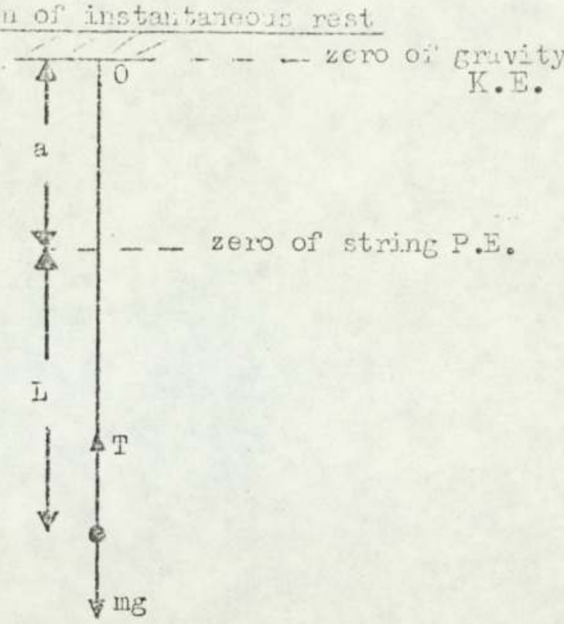


For equilibrium,

$$mg = T_1 = \frac{\lambda 2a}{a.3}$$

$$\lambda = \frac{3}{2}mg$$

Position of instantaneous rest



Let L = length of maximum extension.

Energy equation

$$(K.E.+P.E)_{START} = (K.E.+P.E.)_{\substack{MAX\\EXTENSION}}$$

$$0 = 0 + \frac{\lambda L^2}{2a} - mg(a+L)$$

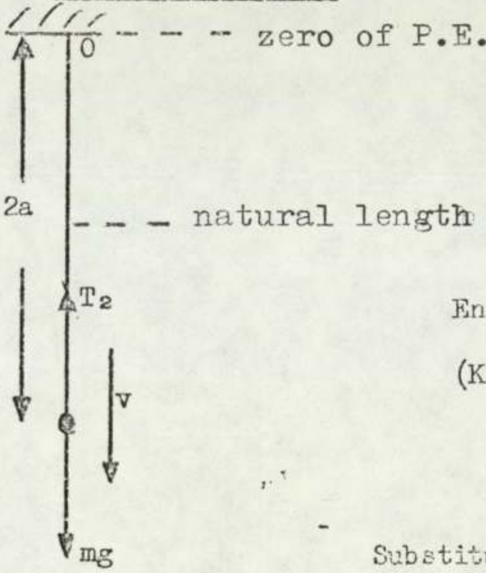
hence $$0 = 4a^2 + 4aL - 3L^2$$

$$0 = (2a - L)(2a + 3L)$$

which is satisfied when L = 2a, i.e. Total length = 3a.

($L = -\frac{2}{3}a$ has no physical meaning.)

String in Motion



Let v = velocity of mass when 2a from 0

Energy equation

$$(K.E.+ P.E.)_{START} = (K.E.+ P.E.)_{2a}$$

$$0 = \left(\tfrac{1}{2}mv^2 - mg.2a + \frac{\lambda a^2}{2a}\right)$$

Substitute for λ, gives v

SWITCH ON

63.

SCRIPT FOR TAPE 2

POWER, ENERGY AND HOOKE'S LAW

1. This is the tape to be used with Booklet 2 of 'A Revision Course in Particle Dynamics' by Kenneth A. Jackson. It is concerned with 'Power, Energy and Hooke's Law.

We know that a force may cause a body to move and when a force moves its point of application in this way, it is said to perform work. This depends upon the magnitude of the force and its displacement. Turn to page 2 in the booklet, where this is summarized for you. SWITCH OFF.

2. The rate at which a force does work is called power, which depends on the velocity. Look at the next section of the booklet, where this is explained. SWITCH OFF.

3. Read the first example carefully, and study the diagram, which summarizes the question. SWITCH OFF.

4. This example will involve equations for constant acceleration, and the equation of motion for the two parts of the journey. Look below at the next section to see how these are used. SWITCH OFF.

5. In the right hand column, observe that the retardation is negative, as the motion is opposed by KMg in the negative sense. Now follow the instructions below this.

SWITCH OFF.

6. The greatest power will be obtained in the first part of the journey as the engine only works from A to C. The remainder of the solution is in the next section. SWITCH OFF.

7. You should now attempt example two yourself. Read the question, and draw a diagram showing forces, velocities and acceleration. You will need to use the equation of motion, and that for power. When you have completed this, check the solution below the question. SWITCH OFF.

8. When a body is in motion, it can overcome resistances, thus doing work on them. This capability of working is called kinetic energy. Read the next section about this. SWITCH OFF.

9. Energy may be converted from one form to another, without loss, under certain conditions. This is conservation of energy and is of great importance in dynamics. An example of this occurs if a body, which has some kinetic energy, undergoes a change in velocity. The change in kinetic energy is then the work done. This is summarized in the booklet. SWITCH OFF.

10. Example three illustrates this conservation of energy. Read the question carefully and study the diagram. SWITCH OFF.

11. The diagram shows that the particle has no kinetic energy at the beginning or end of the motion, as it is at rest. It gains energy as the resolved part

of the weight pulls it down the slope to velocity V, doing positive work. Throughout the motion it is opposed by friction, which eventually destroys it, by doing negative work. The energy equation for this motion is given in the next section of the booklet. SWITCH OFF.

12. By using the energy equation for the appropriate point in the motion, calculate the velocity when the particle reaches the foot of the incline, yourself. Check this afterwards. SWITCH OFF.

13. Now try example four by yourself. Read the question carefully, draw a diagram, and only check your solution when you have finished. SWITCH OFF.

14. In example four, the motion up the plane was limited, as friction destroyed the energy. Even without friction, the particle comes to rest, but the kinetic energy is replaced by potential energy. Forces which are associated with potential energy are called conservative forces, and throughout the motion, the sum of the kinetic and potential energies is constant. Study this in the next section. SWITCH OFF.

15. We shall work through example five to show how this principle is used. Read the question, and study the diagrams. SWITCH OFF.

16. In a potential energy problem it is essential that a zero level for potential energy is selected, through some fixed point in the system. Masses placed above this level have a positive potential energy. We also need to show masses, distances, and velocities in

the diagram. The string tension is an internal force, and does no work. The smooth pulley will not cause loss of energy as it turns, and the light pulley has negligible mass. The problem is solved by using the conservation of energy equation at the beginning and end of the motion. Follow this in the next section. SWITCH OFF.

17. Work through example six yourself. Read it carefully, and draw a diagram indicating some convenient zero of potential energy level. Remember that the pulley arrangement will cause the masses to have different velocities. When you have have finished, check your working in the booklet. SWITCH OFF.

18. In the work so far, inextensible strings have been used but now the study must include the behaviour of elastic strings. The extension of these is governed by Hooke's Law, named after Robert Hooke, their discoverer. He was an active scientist who lived at the same time as Newton, and is credited with the application of springs to the balance wheels of watches. His law is stated in the booklet. SWITCH OFF.

19. We will work through example seven to demonstrate the application of the law. Read the question carefully and look at the first diagram. SWITCH OFF.

20. Notice in this question, that lamda is not given directly, but may be calculated from the initial equilibrium. This often occurs in spring problems. Look, now, at the lower diagram. Suspending the

particle from the mid-point is effectively cutting the string into two strings, of length 'a', but the modulus retains its value. Follow the remainder of the problem here. SWITCH OFF.

21. You should now answer example eight yourself. Read it, draw the diagram, and when the solution is complete, check it overleaf. SWITCH OFF.

22. When an elastic string is stretched, it stores energy, since it will jump back when released. This is a potential energy, as it is due to the position of the end. The formula for this potential energy is revised in the next section. SWITCH OFF.

23. Problems involving elastic strings and springs can often be solved by using energy considerations, as is shown in example nine. Read this carefully and study the diagrams. SWITCH OFF.

24. Notice in the second diagram that it is necessary to show two zero levels of potential energy. At the maximum displacement, the particle is instantaneously at rest, and all the kinetic energy becomes potential energy. The mass in this position has positive potential energy with respect to the spring, but negative for gravity. Because there is no kinetic energy, the problem reduces to a balance of these two potential energies. Look below at the energy equation for this. SWITCH OFF.

25. You should now attempt example ten by your own efforts. Again you will need to consider the position of instantaneous rest, and use the energy equation for

each part of the question. When you have finished this check your working on the last page of the booklet.

SWITCH OFF.

26. This completes the work on Power, energy and Hooke's Law. Please rewind the tape before you remove it from the machine. Thank you!

APPENDIX 3.

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

BOOKLET 3 to be used with TAPE 3

"Momentum and Restitution"

Read the instructions on page 1 thoroughly.

(Ommitted for convenienc.)

The MOMENTUM of a particle is the product of the mass of the particle and its velocity.

i.e. momentum = $m \times v$

The units are $Kg.ms^{-1}$, but have no special name.

NOTE. As this product has both magnitude and direction, momentum is a vector quantity.

When you have studied this,

SWITCH ON

For a single particle,

$$m\frac{dv}{dt} = F \quad \text{and } m\,dv = F\,dt$$

$$\int_{v_1}^{v_2} m.dv = m\,v_2 - m\,v_1 = \int_{t_1}^{t_2} F.dt.$$

Let $t_2 \to t_1$ with the velocity changing abruptly from v_1 to v_2, then,

$$\text{Change in momentum} = \lim_{t_2 \to t_1} \int_{t_1}^{t_2} F\,dt = \text{impulse}.$$

When you have studied this,

SWITCH ON

<u>The Principle of Conservation of Momentum.</u>

$$\begin{bmatrix}\text{Total momentum}\\ \text{before impact}\end{bmatrix} = \begin{bmatrix}\text{total momentum}\\ \text{after impact.}\end{bmatrix}$$

<u>Just before impact</u> <u>Just after impact</u>

i.e. $\left[m_1u_1 + m_2u_2\right] = \left[m_1v_1 + m_2v_2\right]$

When you are ready to proceed,

SWITCH ON

EXAMPLE 1.

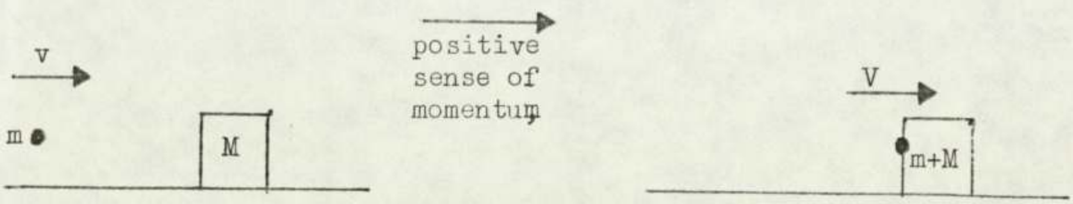
A particle of mass m, moving with a velocity v, strikes a stationary block of mass M, which is free to move in the direction of the particle, and is embedded in it.

i) Find the common velocity of the bodies.

ii) Show that the loss in kinetic energy is $\frac{Mmv^2}{2(M+m)}$

iii) Find the impulse exerted on the block by the particle.

Just before impact   Just after impact



Let V be the common velocity.

When you are ready,

SWITCH ON

---

i) Total momentum before impact = total momentum after

$$mv = (m+M)V$$

$$\text{hence,} \quad V = \frac{mv}{(m+M)}$$

ii) Loss in K.E. = $K.E._{START} - K.E._{END}$

$$= \tfrac{1}{2}mv^2 - \tfrac{1}{2}(m+M)V^2$$

Substitute for V to obtain the answer in terms of the given velocity v.

iii) Impulse on the block = change in momentum

$$= M(V-0)$$

$$= \frac{Mmv}{(m+M)}$$

When you have studied this,

SWITCH ON

EXAMPLE 2.

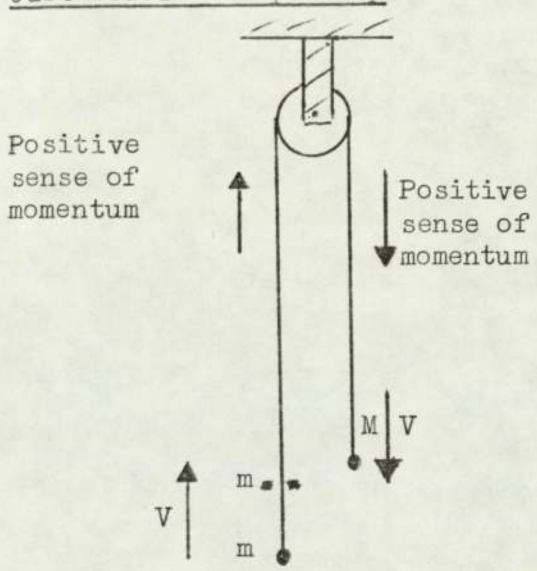
Two masses m and M, are attached to the ends of a light string, which passes over a fixed frictionless pulley. At the moment when the particles are moving with velocity V, the lighter mass, m, picks up a small ring, also of mass m. Find,

i) the velocity of the string just after this event ,

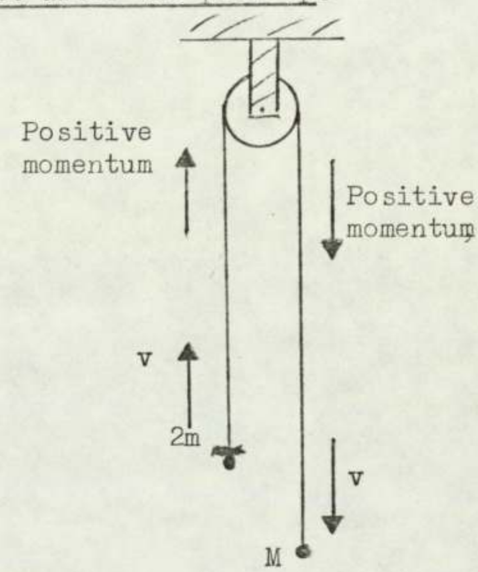
ii) the impulsive tension felt by the particle M.

When you have drawn the diagrams, check them below.

Just before the pick up.



Just after the pick up.



SWITCH ON

---

i) Use conservation of momentum,

then, $mV+MV = v(M+2m)$ i.e. $v = \frac{V(M+m)}{(M+2m)}$

ii) Impulse = change of momentum = $M(V-v) = -Mv - (-MV)$

Substitute for V in this, to obtain $\frac{mMV}{(2m+M)}$

SWITCH ON

Newton's Law of Restitution.

When two particles collide directly, the relative velocity after impact is in a fixed ratio to the relative velocity before impact.

i.e. $\frac{\text{velocity of separation}}{\text{velocity of approach}} = e = \text{constant}$

(Continued on the next page)

The constant, e, is called the COEFFICIENT OF RESTITUTION and it depends upon the materials of the two bodies. Being dimensionless, it has no units, and is such that,

$$0 < e < 1$$

with $e = 1$ called perfectly elastic

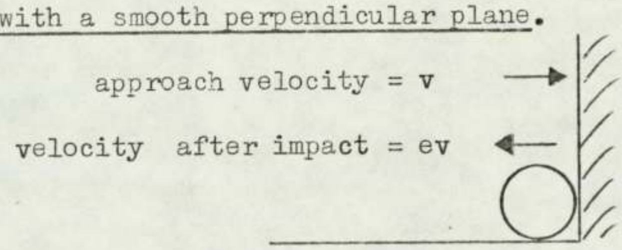
and $e = 0$ called inelastic.

$$\frac{\text{velocity of separation}}{\text{velocity of approach}} = \frac{v_2 - v_1}{u_1 - u_2} = e$$

where u_1 and u_2 are known velocities, and v_1 and v_2 have to be found.

SWITCH ON.

Collison with a smooth perpendicular plane.



Noticé that the speed after impact is less than the speed before, since $e < 1$.

SWITCH ON

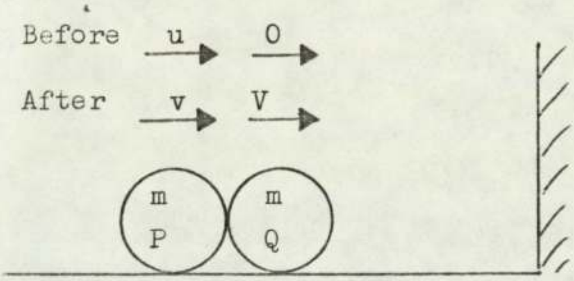
EXAMPLE 3.

The line joining the centres of two equal smooth balls P and Q, which lie on a smooth table, is perpendicular to a smooth vertical wall. The ball P, farthest from the wall, slides towards Q, which is at rest, with velocity u. After the impact, Q moves towards the wall. If e is the coefficient of restitution between the balls, and e_1 that between Q and the wall, find

i) the velocities of P and Q after their second collision

ii) the impulse exerted by P on Q, and the wall on Q.

1st collision of P and Q



$\longrightarrow$ positive sense.
Assume masses m
to find V and v

When you are ready to proceed,

SWITCH ON

---

i) Conservation of momentum,

$$mu = mv + mV \qquad \ldots \qquad 1$$

Law of restitution,

$$\frac{\text{velocity of separation}}{\text{velocity of approach}} = \frac{V-v}{u} = e$$

$$V - v = eu \qquad \ldots \qquad 2$$

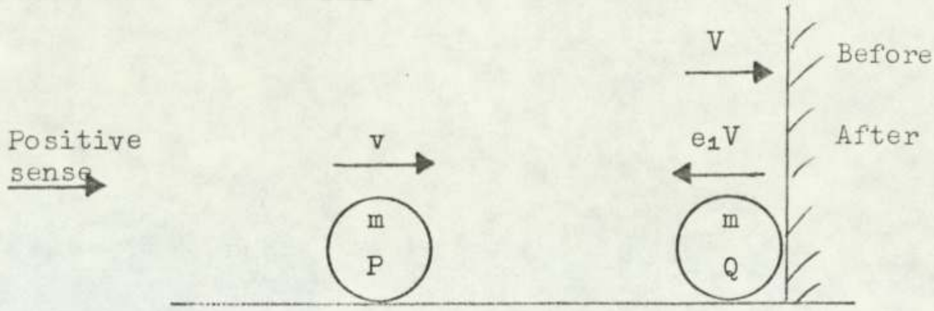
These equations give $V = \frac{1}{2}u(1+e) \qquad \ldots \qquad 3$

and $v = \frac{1}{2}u(1-e) \qquad \ldots \qquad 4$

As $e < 1$, V and v are both positive, and therefore, directed towards the wall, but $V > v$, since $(1+e) > (1-e)$, so Q will move away from P, and collide with the wall.

SWITCH ON

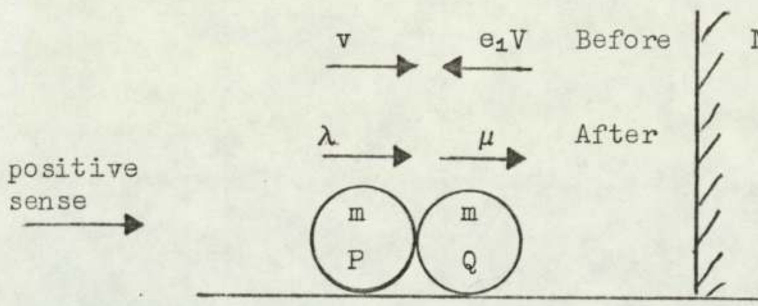
Collision of Q and the wall.



Q moves away from the wall, and must, therefore, have a second collision with P, which is still moving towards the wall.

SWITCH ON

Second collision of P and Q.



N.B. Initial velocity of Q is negative, and its momentum is negative.

Use λ and μ as velocities of P and Q after impact, respectively.

Conservation of momentum, $mv - me_1V = m\lambda + m\mu$

$$\text{i.e.} \quad (v-e_1V) = (\lambda+\mu) \qquad \ldots \qquad 5$$

Law of restitution

$$\frac{(\mu - \lambda)}{(v+e_1V)} = e \qquad \text{or } (\mu-\lambda) = (v+e_1V)e \qquad \ldots \qquad 6$$

Equation 5 + equation 6 will give μ in terms of u and V.

Use 3 and 4 to give

$\mu = \frac{1}{4}\, u(1-e-e_1+e_1e^2) = \frac{1}{4}u(1-e^2)(1-e_1)$

$e^2 < 1$ and $e_1 < 1$, so μ is positive. Q moves back towards the wall again.

Subtract 6 from 5 and obtain

$\lambda = \frac{1}{4}u[(1-e)^2 - e_1(1+e)^2]$

We do not know if $e_1(1+e)^2$ is greater or less than $(1-e)^2$, so that the direction of P cannot be determined, without a knowledge of the relative magnitude of e and e_1.

SWITCH ON.

3ii)

Impulse = Change in momentum

Impulse of P on Q = $m(V-0) = \frac{1}{2}mu(1+e)$

Impulse of wall on Q = $mV+me_1V = m e_1V - (-mV)$

$= mV(1+e_1) = \frac{1}{2}mu(1+e)(1+e_1).$

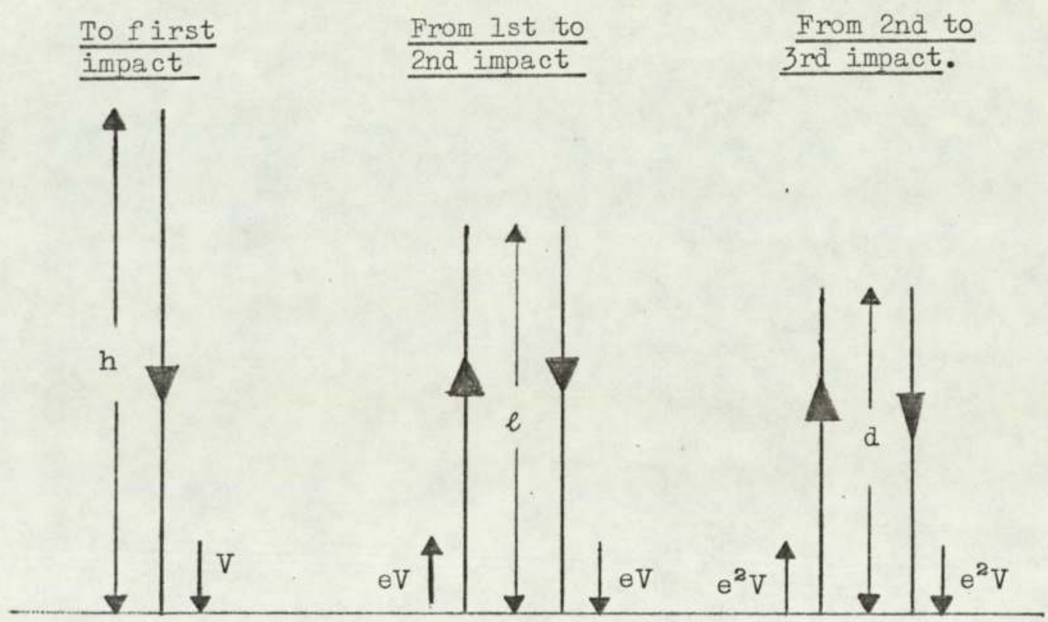
SWITCH ON

EXAMPLE 4.

A particle is dropped from a height h onto a smooth horizontal surface, and rebounds to a height ℓ above it. Find the coefficient of restitution, and show that the height of the next bounce will be ℓ^2/h.

Do not read this solution until yours is complete.

The motion is in one vertical line, but separate diagrams illustrate it more clearly.



Let V = initial velocity, then eV = 1st rebound velocity and e^2V = 2nd rebound velocity.

Use "$v^2 = u^2 + 2ax$" for first fall,

$V^2 = 2gh$ i.e. $V = \sqrt{2gh}$

Use "$v^2 = u^2 + 2ax$" for first rebound to greatest height,

$0 = (eV)^2 - 2g\ell$

hence $e = \frac{\sqrt{\ell}}{\sqrt{h}}$

Continued on next page

Use "$v^2 = u^2 + 2ax$" for second rebound to greatest height,

$$0 = (e^2V)^2 - 2gd$$

hence $d = \ell^2/h$

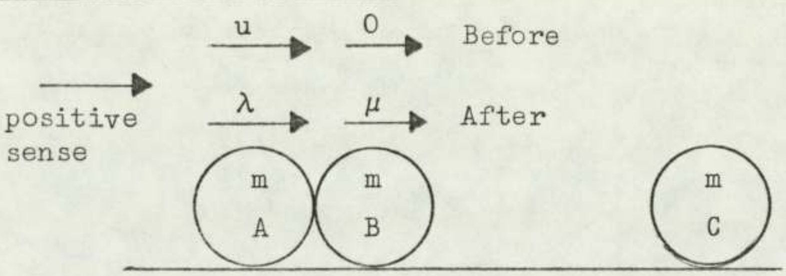
SWITCH ON

EXAMPLE 5.

Three spheres A,B and C of equal mass lie at rest in a straight line. If the sphere A is given a velocity u, towards B and C, show that the velocities of A,B and C after two impacts, are given by : $\frac{1}{2}(1-e)u$, $\frac{1}{4}(1-e^2)u$, $\frac{1}{4}(1+e)^2u$, respectively, where e is the coefficient of restitution at all impacts. Find also the impulse between the spheres at both impacts.

Do not look at the solution below until you have completed your own.

Collision between A and B.



Conservation of momentum, $mu = m\lambda + m\mu$

Law of Restitution, $\dfrac{\text{velocity of separation}}{\text{velocity of approach}} = \dfrac{\mu-\lambda}{u-0} = e$

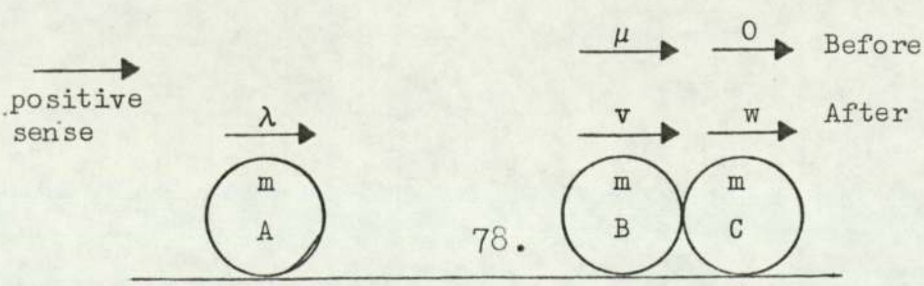
From these equations, $\lambda = \frac{1}{2}(1-e)u$; $\mu = \frac{1}{2}(1+e)u$

μ is obviously positive, and as $e < 1$, λ is also positive. Both spheres will proceed in the direction indicated by the arrows, but as $(1+e) > (1-e)$, so $\mu > \lambda$. B will move away from A and collide with C.

Impulse of A on B $= m\mu - 0$

$$= m\mu = \tfrac{1}{2}m(1+e)u.$$

Collision between B and C



(Continued on the next page)

Conservation of momentum, $m\mu = mv + mw$.

Law of restitution,

$$\frac{w - v}{\mu - 0} = e$$

hence $v = \frac{1}{4}(1-e^2)u$, and $w = \frac{1}{4}(1+e)^2u$.

As $1 > e^2$, both these velocities are positive and B and C move in the direction of the arrows, C moving away from B, as $(1+e)^2 > (1-e^2)$. Impulse between B and C = Change in momentum.

$$= mw = \frac{1}{4}mu(1+e)^2.$$

SWITCH ON

EXAMPLE 6.

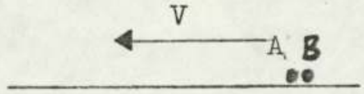
Two particles A and B, of masses m and 2m respectively, connected by an inextensible string of length a, are placed close together on a rough horizontal table. The coefficients of friction between the particles and the table are μ and $\frac{1}{2}\mu$ respectively. The particle A is projected along the table, away from B, with velocity V. Find

i) the common velocity just after the string becomes taut,

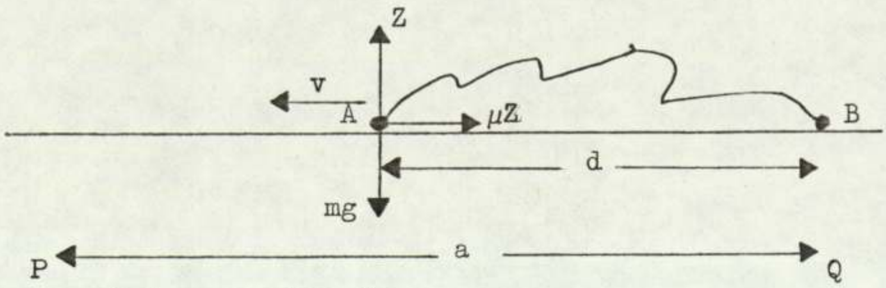
ii) the acceleration of each particle.

By considering the distances which each particle would move before coming to rest, show that B will overtake A if $V^2 > 20\mu ga$. It is given that $V > \sqrt{2\mu ga}$.

The start of the motion



While the string is slack.



Let $v = p$ when $d=a$

SWITCH ON

$(K.E.)_{END} = (K.E.)_{BEGINNING} + (\text{Work done by forces})$

$\frac{1}{2}mv^2 = \frac{1}{2}mV^2 - \mu mgd$

i.e. $v^2 = V^2 - 2\mu gd$

when $d = a$, so $v = p$

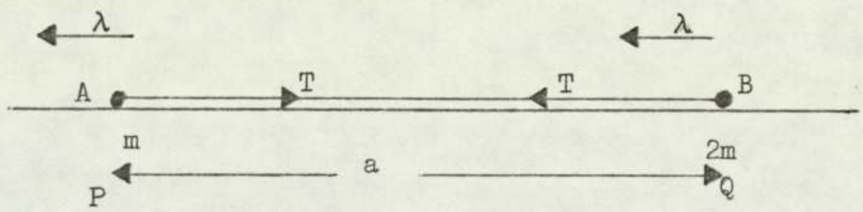
i.e. $p^2 = V^2 - 2\mu ga$

Given $V > \sqrt{(2\mu ga)}$, so p is positive.

i.e. $p = \sqrt{V^2 - 2\mu ga}$

At this point the string will become taut, and A will jerk B into motion via the string, causing them to have a common velocity, say λ.

At the instant of the jerk.



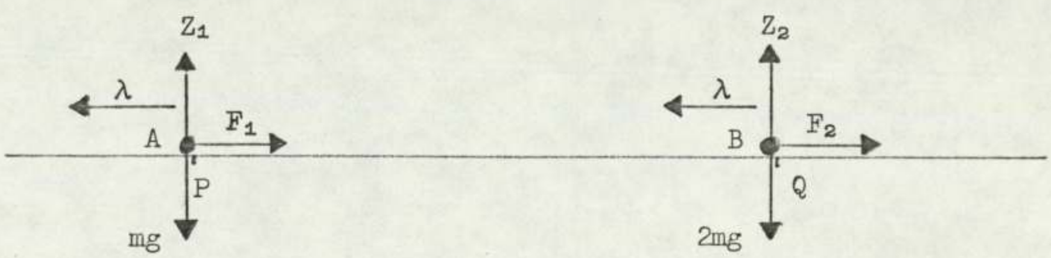
SWITCH ON

Conservation of momentum,

$$mp = (m+2m)\lambda$$

$$\frac{p}{3} = \lambda$$

Just past P and Q.



Let R_1 and R_2 be the respective retardations.

SWITCH ON

For A, $F_1 = \mu mg$ For B, $F_2 = \frac{1}{2}\mu(2mg) = \mu mg$

Use $F = ma$

$\mu mg = mR_1$ $\mu mg = 2mR_2$

$\mu g = R_1$ $\frac{1}{2}\mu g = R_2$

i.e. $R_1 > R_2$

(Continued on the next page)

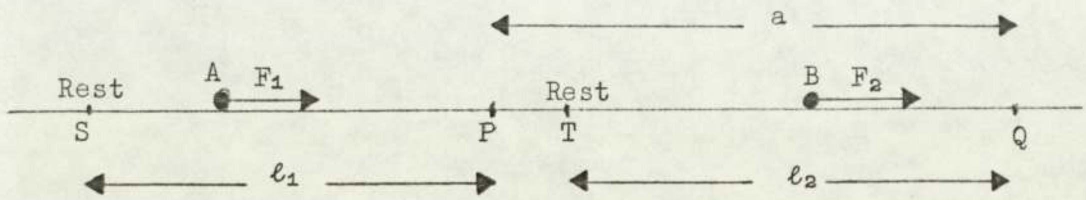
As the retardation of A is greater than that of B, B will approach A, and the string becomes slack.

The constant forces will allow B to continue to approach A until the K.E. is destroyed in both particles.

Suppose particle A comes to rest at S, after moving distance ℓ_1.

Suppose particle B comes to rest at T, after moving distance ℓ_2.

Remaining motion.



SWITCH ON

Conservation of Energy

For A,

$$0 = \tfrac{1}{2}m\lambda^2 - \mu mg\ell_1 \quad ; \qquad \ell_1 = \frac{\lambda^2}{2\mu g} \quad ;$$

For B,

$$0 = \tfrac{1}{2}(2m)\lambda^2 - \mu mg\ell_2 \qquad \ell_2 = \frac{\lambda^2}{\mu g}$$

B will overtake A if $\ell_2 > \ell_1 + a$

i.e. $\dfrac{\lambda^2}{\mu g} > \dfrac{\lambda^2}{2\mu g} + a$

Substitute for $\lambda^2 = \dfrac{p^2}{9} = \dfrac{(V^2 - 2\mu ga)}{9}$

hence $V^2 > 20\mu ga$.

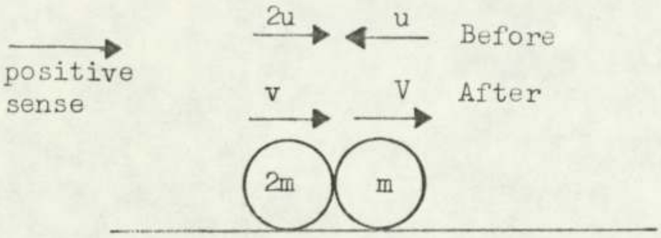
SWITCH ON

EXAMPLE 7.

Two particles of mass 2m, m, moving in opposite senses with speeds 2u, u respectively, collide. Half the kinetic energy is lost at the collison. Find the velocities of the particles after this, and the coefficient of restitution between them.

Do not look at this solution until you have completed your own.

The solution is on the next page.



Conservation of momentum,

$$2m(2u) - mu = 2mv + mV$$

$$3u = 2v + V \qquad \ldots \qquad 1$$

K.E. on impact $= \frac{1}{2}(2m)(2u)^2 + \frac{1}{2}mu^2 = \frac{9mu^2}{2}$

K.E. after impact $= \frac{1}{2}(2m)v^2 + \frac{1}{2}mV^2 = \frac{1}{2}\left(\frac{9mu^2}{2}\right)$

$$4v^2 + 2V^2 = 9u^2 \qquad \ldots \qquad 2$$

Solvd these for v,V, in terms of u,

to give $(2v-u)(2v-3u) = 0$

with roots $v = \frac{u}{2}$ or $\frac{3u}{2}$

when $v = \frac{u}{2}$, $V = 2u$

when $v = \frac{3u}{2}$, $V = 0$, the mass 2m cannot continue to the right if m comes to rest, so this answer is not physically possible, although algebraically correct. Thus $v = \frac{u}{2}$ and $V = 2u$, both in the original direction of the heavier mass.

$$e = \frac{\text{velocity of separation}}{\text{velocity of approach}} = \frac{V-v}{2u+u} = \frac{1}{2}$$

SWITCH ON

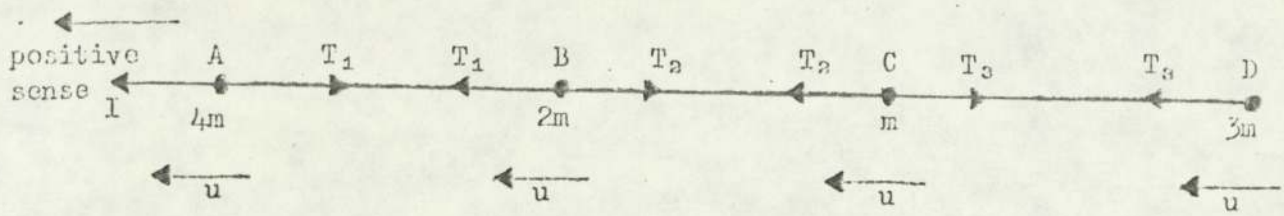
EXAMPLE 8.

Four particles A,B,C,D, are joined by light inextensible strings and lie in a straight line with the string just taut. The masses of A,B,C, and D are 4m, 2m, m and 3m respectively. If A is given an impulse I, away from B, find

i) the initial velocity of the particles,

ii) the impulsive tension which appears in the string BC.

(The diagram is on the next page)



All the particles will travel along DA with a common velocity: let this be u.

SWITCH ON

Impulse = change in momentum.

for A, $I - T_1 = 4mu$

" B, $T_1 - T_2 = 2mu$

" C, $T_2 - T_3 = mu$

" D, $T_3 = 3mu$

Adding all the equations to eliminate the impulsive tensions gives $10mu = I$, which is simply the impulse equation for the whole system.

Thus $u = \dfrac{I}{10m}$

T_2 can be found from the equations for C and D

i.e. $4mu = T_2 = \dfrac{2I}{5}$

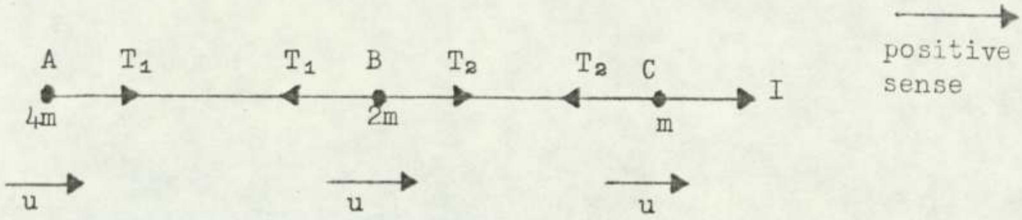
SWITCH ON

EXAMPLE 9.

Three particles A,B,C have masses 4m, 2m and m respectively and are joined by two light inextensible strings. They lie in a straight line and particle C is given a velocity u in the direction BC by an impulse I. Show that the ratio of the impulsive tensions in the strings is 2 : 3.

Do not look at this solution until you have finished your own.

(The solution is on the next page).



Impulse = change in momentum.

For A, $\quad T_1 = 4mu$

" B, $\quad T_2 - T_1 = 2mu$

" C, $\quad I - T_2 = mu$

Add to eliminate the impulsive tensions gives $I = 7mu$.

Hence $T_1 = \dfrac{4I}{7}$, $\quad T_2 = \dfrac{6I}{7}$, hence the ratio 2:3.

<u>SWITCH ON.</u>

SCRIPT FOR TAPE THREE

MOMENTUM, IMPULSE AND RESTITUTION

1. This is the tape to be used with Booklet 3 of 'A Revision Course in Particle Dynamics' by Kenneth A. Jackson. It is concerned with momentum, impulse and restitution.

Particles in motion have the ability to transfer this to others; that is, their velocities can be changed. This property is associated with momentum, which depends upon the mass and velocity of a particle. Turn to page 2 in the booklet, where this is defined for you. SWITCH OFF.

2. Particles moving in the same line often collide and by Newton's third law, have equal and opposite interactions at every collision. These interactions are called impulses, and they arise from very large forces acting for a very short time. The change in momentum for one particle, provides a measure of the impulse which acts on it. Look below at the next section about this. SWITCH OFF.

3. When two particles moving in the same straight line collide, there is an abrupt change in velocities, and equal and opposite impulses are felt by the particles. These cancel out, and the total momentum after the collision equals the total momentum before. This is called the conservation of momentum, which is stated in

the next section. SWITCH OFF.

4. Although there is conservation of momentum during an impact, there is always a loss of mechanical energy, and the energy equation cannot be used. Read the first example carefully and study the diagrams. We shall work through this example together. SWITCH OFF.

5. The particle propels the block forward, and they travel together with common velocity V. This direction is convenient for the positive sense of momentum, which is shown. The velocity is found from the conservation of momentum. The second part of the question involves the calculation of kinetic energy, before and after impact, and shows that this is not conserved. In the third part, the impulse on the block is the change in its momentum. Look below at the next section for the complete solution. SWITCH OFF.

6. You should now try example two yourself. Read it carefully, and draw the diagrams, for before and after impact, clearly showing the masses and velocities. When you have done this, switch on the tape again. SWITCH OFF.

7. The light string is considered to have negligible weight, and together with the frictionless pulley, have no effect on the motion. You see that one mass is rising, and the other falling, but because of the connecting string over the pulley, the momentum for each mass is positive in the direction of travel. The constant gravitational forces acting on the particles are finite,

and have no other effect at the instant of impulsive change in motion. Momentum is conserved at the impact. Complete your working, and check this below. SWITCH OFF.

8. If colliding particles are elastic, they rebound after an impact. This effect is called restitution. It is another topic which Isaac Newton studied in his researches, discovering a law which is stated in the next section of the booklet. SWITCH OFF.

9. Notice that after the impact there are two unknown velocities, v_1 and v_2. The conservation of momentum only gives one equation, and accordingly the Law of Restitution is needed to give the second equation. When collisions occur between particles and planes, the restitution law must be modified as explained in the next section. SWITCH OFF.

10. Example three covers both aspects of the Law of restitution. Read this, and study the diagram before we work through this question. SWITCH OFF.

11. All the motion takes place along the line perpendicular to the wall, as the smooth surfaces only produce impulses along this line. Notice in the diagram, that the masses and velocities before and after impact are shown, as well as a designated positive sense. We can now use the conservation of momentum and the Law of Restitution to find the unknown velocities, as is shown in the next section. SWITCH OFF.

12. In problems involving collision, it is important to interpret the results to ascertain direction and relative

velocities, as is shown here. The velocityof Q towards the wall is known. Find the velocity after this collision and then check this in the next section. SWITCH OFF.

13. In this second collision, P and Q are moving in opposite directions: thus the velocity and momentum of Q are in the negative sense. We still assume lamda and mu to be to the right, as is shown, and determine their senses from the equations. Follow carefully their determination in the next section. SWITCH OFF.

14. The second part of the question is a straightforward determination of the changes in momentum for each ball, in terms of the original velocity. Follow this in the next section. SWITCH OFF.

15. You should try example four for yourself, which is about restitution in vertical motion. It will be helpful to separate the parts of the vertical path into separate diagrams. Use the equation for free fall to find your incident velocity. On completion, check your solution in the next section. SWITCH OFF.

16. Read example five, which you should also attempt yourself. Remember to draw enough diagrams for the parts of the motion, and show all masses and velocities in them. When this is finished, check your working below. SWITCH OFF.

17. Example six is a slightly harder question which we will work through together. Read it through, and study carefully all the information on the diagram. SWITCH OFF.

18. The overall motion is in three sections, which must be considered separately. Study the projection of A first. The lower diagram shows it having travelled a distance d, against the friction forces, with the velocity v at this instant. This can be found by using the conservation of energy, and hence the velocity p when the string is just taut. Carry out this process and check your result in the next section. SWITCH OFF.

19. The jerk in the string, constitutes the second part of the motion. The impulsive tensions are internal to the system, and overall, the momentum is conserved. Use this principle to find the common velocity, lamda after the jerk. Check your answer in the next section. SWITCH OFF.

20. In the third part of the motion, the particles will come to rest as friction destroys their kinetic energy. Whether the retardations produced will allow B to overtake A, or not, depends upon their relative magnitudes. Use the information in this diagram, and the equation of motion to determine R_1 and R_2, and compare these. Check this in the next section. SWITCH OFF.

21. The distances travelled to rest, L_1 and L_2, can be found in terms of lamda, by using conservation of energy. Then form an inequality for the three distances concerned, to obtain the result. Assume that B may overtake A without interfering with its motion. SWITCH OFF.

22. You should try example seven yourself. Read it carefully, and insert symbols for the velocities after impact in your diagram. When you have finished, the solution may be checked on the next page. SWITCH OFF.

23. When spheres collide, the impulses are internal to the system. An external impulse occurs when a particle collides with a wall, and others can occur as hammer blows, kicks or jerks. The following example illustrates impulsive tensions in inextensible strings. Read this and study the diagram. SWITCH OFF.

24. The diagram shows the conditions at the moment of impulse, and there are different impulsive tensions in the string, of sizes T_1, T_2 and T_3. Since the strings are inextensible, the particles begin to move with the same velocity. Let this be u. The impulse and momentum change are to the left, in this problem, and this is a convenient direction for the positive sense. The tensions are found by writing the impulse equation for each particle separately. Follow the complete proof in the next section. SWITCH OFF.

25. Example nine is a similar question, which you should try yourself. Remember to have clear diagrams showing all masses, impulses, and velocities. Check your solution afterwards. SWITCH OFF.

26. This completes the work on 'Momentum, impulse and restitution.' Please rewind the tape before you remove it from the machine. Thank you!

APPENDIX 4

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

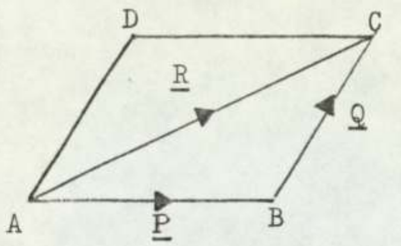
BOOKLET 4 to be used with TAPE 4

"Motion in Two Dimensions"

Read the instructions on page 1 thoroughly.

(Omitted for convenience)

Addition of vectors.



If AB represents vector $\underline{P}$, in magnitude and direction, and BC represents $\underline{Q}$, similarly, then the diagonal of the parallelogram ABCD represents their vector sum.

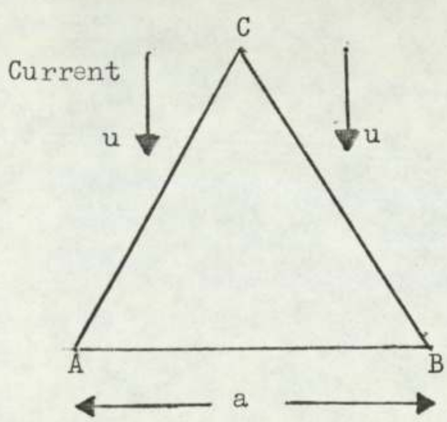
i.e. $\underline{P} + \underline{Q} = \underline{R}$

Thus AC represents $\underline{R}$, which is called the resultant of $\underline{P}$ and $\underline{Q}$.

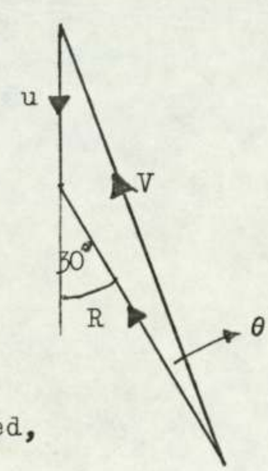
SWITCH ON

EXAMPLE 1.

Anequilateral triangular course, of side length 'a', is marked out by buoys in a broad straight reach of a river, the buoy C being upstream. A motor launch follows the course ABCA, where AB is perpendicular to the banks. If V is the speed of the launch in still water, and u is the speed of the current, show that while the launch is moving along BC, it is pointed at an angle θ to BC on the upstream side, where $\sin\theta = \frac{u}{2V}$. How long will it take the launch to travelalong BC?



Triangle of velocities, with R the resultant.

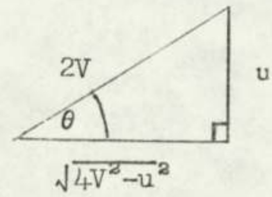


When you are ready to proceed,

SWITCH ON

By Sine Rule, $\frac{u}{\sin\theta} = \frac{V}{\sin 150^\circ}$, hence $\sin\theta = \frac{u}{2V}$

and the right angled triangle :



From the triangle of velocities,

$$R\cos 30^\circ = V\cos(30^\circ-\theta) - u,$$

hence $$R = \frac{\sqrt{4V^2 - u^2} - u\sqrt{3}}{2}$$

SWITCH ON

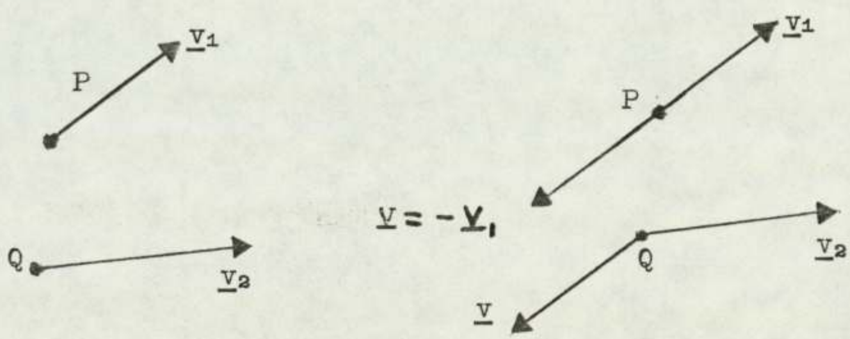
Time $t = \frac{a}{R} = \frac{a[u\sqrt{3} + \sqrt{4V^2-u^2}]}{2(V^2-u^2)}$

When you have checked this answer,

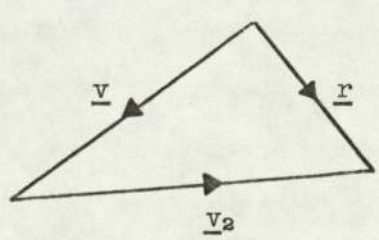
SWITCH ON

Relative Velocities.

To find the velocity of Q relative to P, reduce P to rest, by imposing a back velocity $\underline{v}_1$ on the system



For Q, now add $\underline{v}_2$ and $\underline{v}$ vectorially, to give the relative velocity $\underline{r}$.



$$\underline{r} = \underline{v} + \underline{v}_2$$

$$\underline{r} = \underline{v}_2 - \underline{v}_1$$

Notice that the actual velocity of Q is the vector sum.

(velocity of Q relative to P) + (velocity of P)

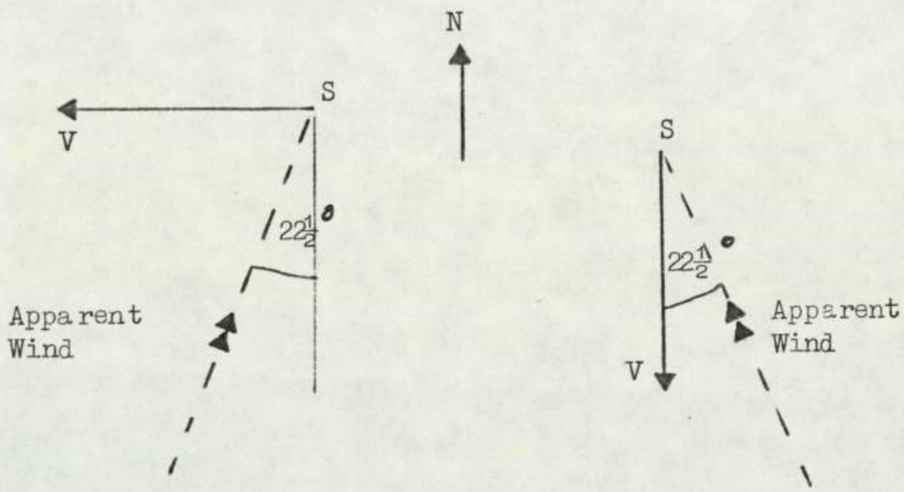
i.e. $(\underline{v}_2 - \underline{v}_1) + \underline{v}_1 = \underline{v}_2$

When you have studied this,

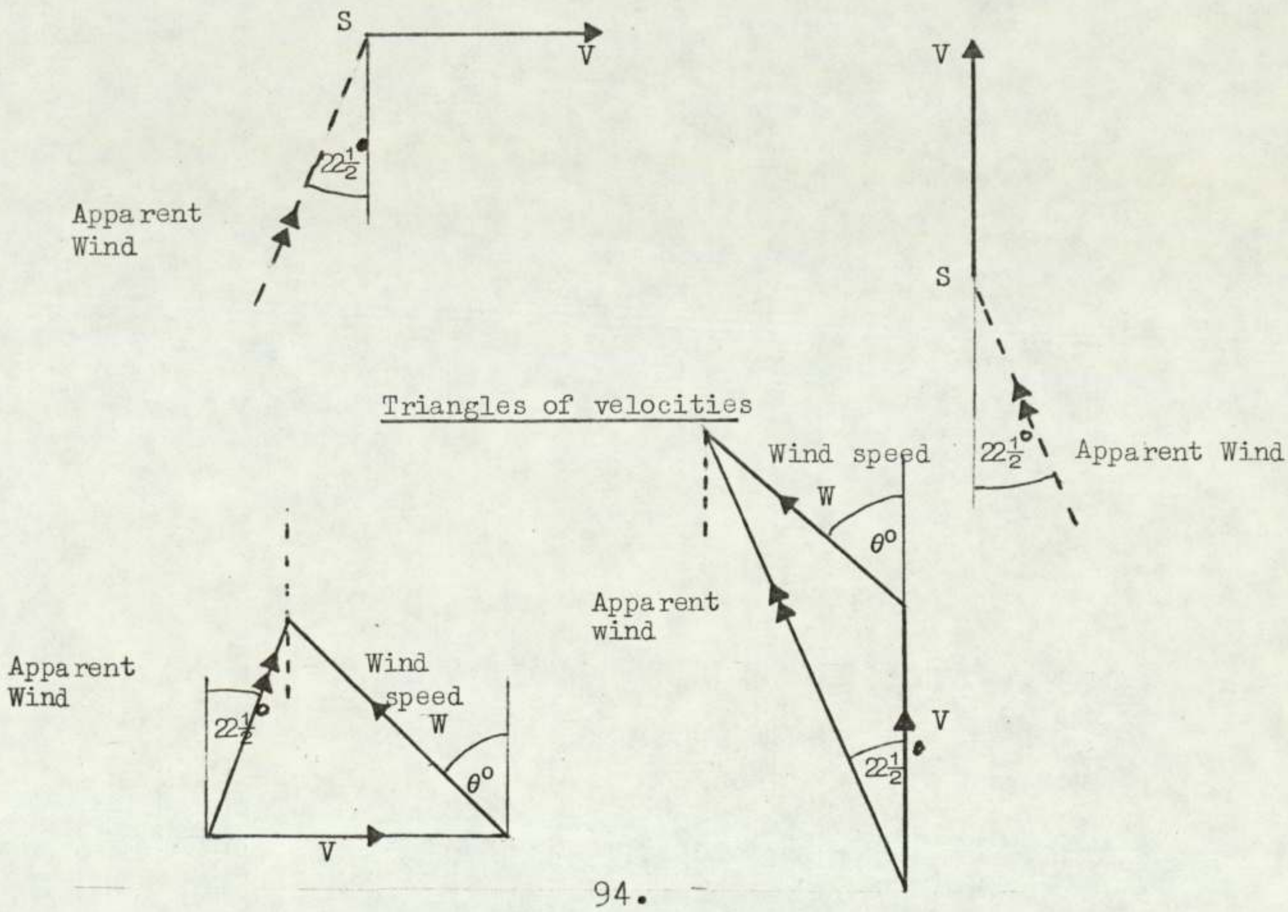
SWITCH ON.

EXAMPLE 2.

A ship is moving due West at V km per hour, and the wind appears to blow from $22\frac{1}{2}^\circ$ West of South. The ship then steams due south at the same speed, and the wind appears to blow from $22\frac{1}{2}^\circ$ East of south. Find the speed of the wind, and the true direction from which it blows, assuming they remain constant. What course must be steered, so that the ship may reach a port m km due west of its present position, and how long will this take?



When you understand these diagrams, SWITCH ON.



When you have studied these, SWITCH ON.

From left hand triangle, $\dfrac{W}{\sin 67\frac{1}{2}^\circ} = \dfrac{V}{\sin(\theta+22\frac{1}{2}^\circ)}$

From right hand triangle, $\dfrac{W}{\sin 22\frac{1}{2}^\circ} = \dfrac{V}{\sin(\theta-22\frac{1}{2}^\circ)}$

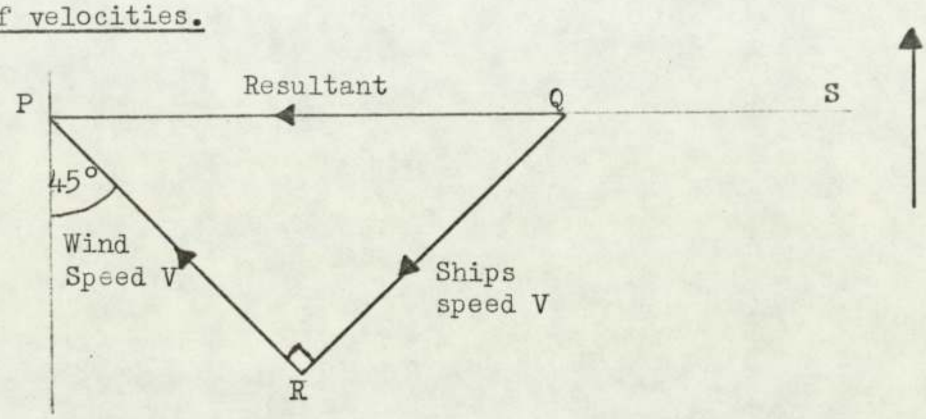
By rearrangement of these we obtain

$$W = \frac{V\sin 67\frac{1}{2}^\circ}{\sin(\theta+22\frac{1}{2}^\circ)} = \frac{V\sin 22\frac{1}{2}^\circ}{\sin(\theta-22\frac{1}{2}^\circ)}$$

After simplification, this reduces to $\tan\theta = 1$.

SWITCH ON

Triangle of velocities.



Stages in drawing this triangle are,

i) Draw the E-W line SP

ii) Mark in wind which blows towards P with speed V

iii) Hence QR for a meaningful triangle

As QR = V = PR so triangle PRQ is isosceles with $Q\hat{P}R = 45^\circ$, $P\hat{R}Q = 90^\circ$.

So QP is the resultant track and speed, which by Pythagoras $= V\sqrt{2}$. i.e. Time $= \dfrac{\text{dist}}{\text{speed}} = \dfrac{m}{V\sqrt{2}}$ hours

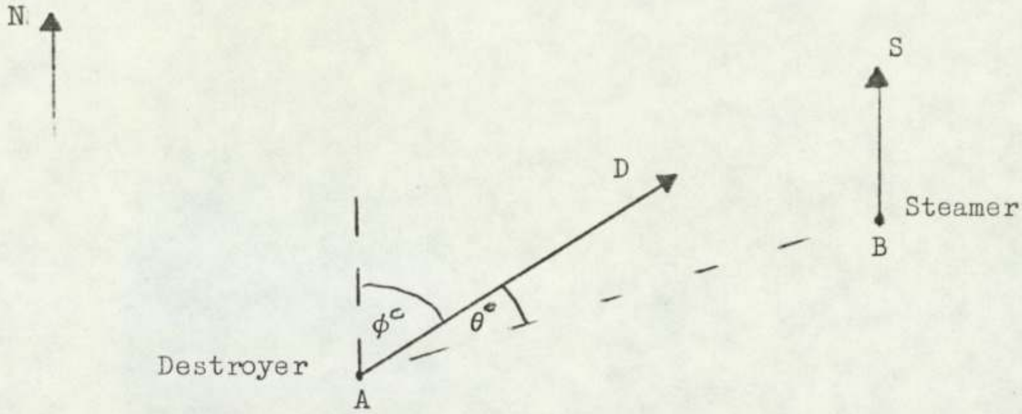
SWITCH ON

EXAMPLE 3.

A destroyer, steering Nϕ°E at D km hr^{-1}, observes at noon, a steamer which is steaming due north at S km hr^{-1}, and overtakes it at T minutes past noon. Find the bearing and distance of the steamer from the destroyer at noon.

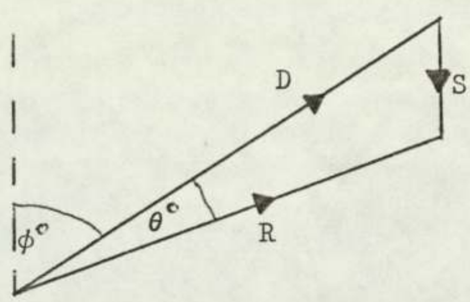
(The solution is on the next page)

Noon Position



Reduce the steamer to rest, so that the destroyer needs to move along AB.

Triangle of velocities



Notice that R is now the velocity of the destroyer relative to the steamer.

By the Sine Rule, $\dfrac{\sin\theta}{S} = \dfrac{\sin(\pi - \overline{\phi+\theta})}{D}$

hence, $\tan\theta = \dfrac{S\sin\phi}{(D-S\cos\phi)}$

From the diagram, $R\cos(\theta+\phi) = D\cos\phi - S$

and , $R = S\sqrt{S^2+D^2-2SD\cos\phi}$ (directly by cosine rule).

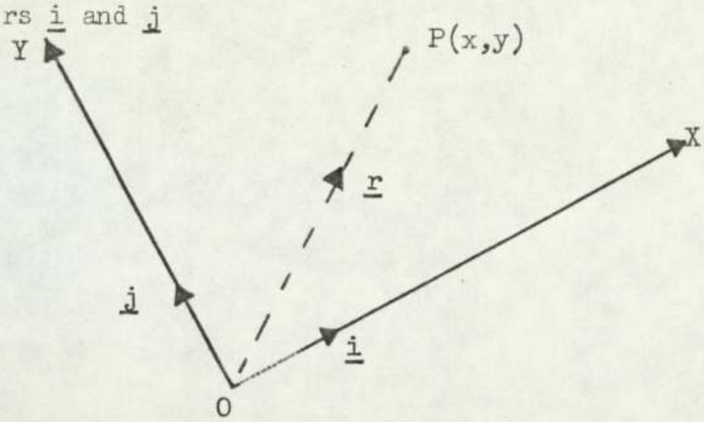
thus, distance $R\left(\dfrac{T}{60^c}\right) = \dfrac{ST}{60^c}\sqrt{S^2+D^2 - 2SD\cos\phi}$

The bearing is N,$(\phi+\theta)^c$E, where $\tan\theta$ is as above.

SWITCH ON.

Components in perpendicular directions.

Let OX and OY be any convenient perpendicular directions with unit vectors $\underline{i}$ and $\underline{j}$



Particle P has coordinates (x,y) and position vector $\overline{OP}$.

$$\overline{OP} = \underline{r} = \underline{i}x + \underline{j}y$$

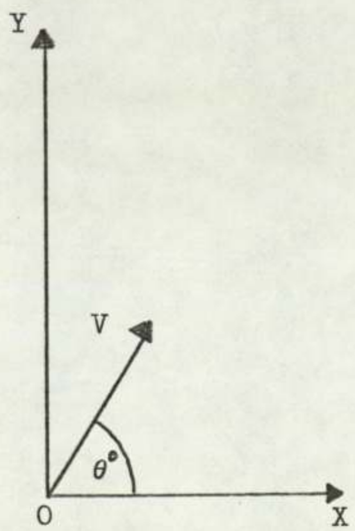
Velocity of P $\quad \underline{v} = \dot{\underline{r}} = \underline{i}\dot{x} + \underline{j}\dot{y}$

Acceleration of P $\quad \underline{a} = \ddot{\underline{r}} = \underline{i}\ddot{x} + \underline{j}\ddot{y}$

When you have studied these equations

SWITCH ON

PROJECTILES



If the particle is projected at velocity V from 0 in a direction making $\theta°$ with the ground.

For vertical motion

$\ddot{y} = -g$, one integration gives,

$\dot{y} = V\sin\theta - gt$, a second integration gives

$y = V\sin\theta . t - \frac{1}{2}gt^2$

For horizontal motion

$\ddot{x} = 0$, one integration gives,

$\dot{x} = V\cos\theta$, a second integration giv

$x = V\cos\theta . t$

Put y = 0 to obtain time of flight, $\quad T = \frac{2V\sin\theta}{g}$

Due to symmetry of vertical motion, maximum height is gained in time $\frac{1}{2}T$, $\quad h = \frac{V^2\sin^2\theta}{2g}$

Horizontal range = time of flight × horizontal velocity, $R = \frac{V^2\sin 2\theta}{g}$

(Continued on the next page)

Maximum range for a given velocity is when $\sin 2\theta = 1$

i.e. $\theta = 45°$.

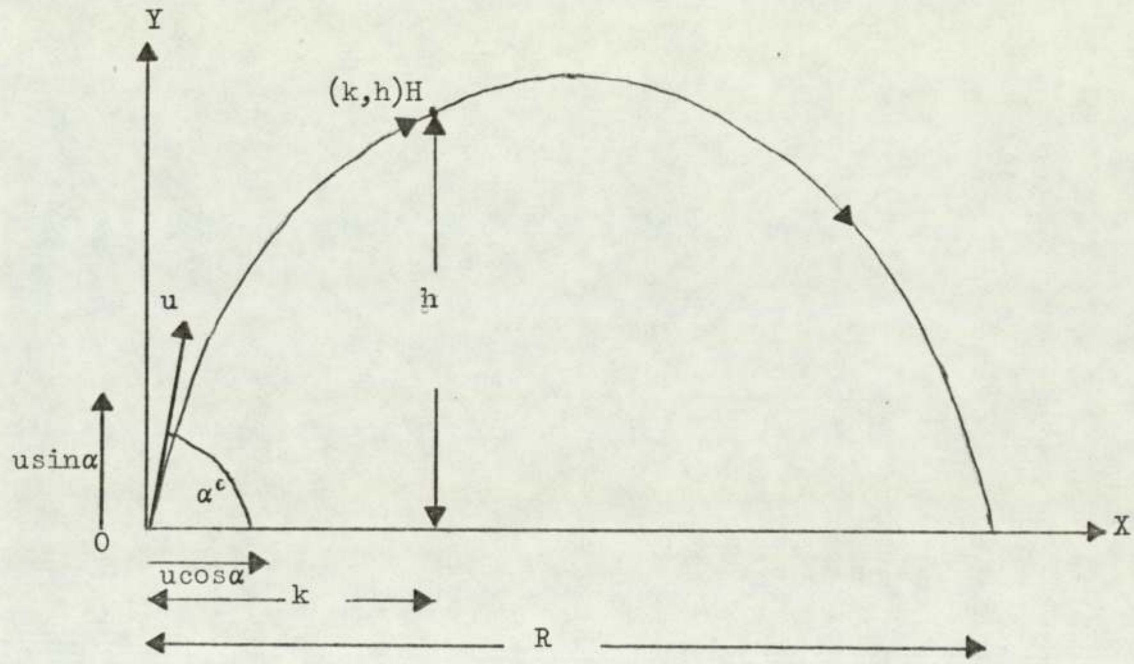
Any given range can be obtained with angles of projection θ and $\left(\frac{\pi}{2} - \theta\right)$, because $\sin 2\theta = \sin 2\left(\frac{\pi}{2} - \theta\right)$.

When you have studied these,

SWITCH ON

EXAMPLE 4.

A body is projected so that on its upward path it passes through a point, distant k horizontally and h vertically from the point of projection. Show that, if R is the range on the horizontal plane through the point of projection, the angle of projection is, $\tan^{-1}\left(\frac{Rh}{k(R-k)}\right)$



When you are ready to proceed,

SWITCH ON

Horizontal velocity = $u\cos\alpha$

$\therefore$ Time of flight to H $= \frac{\text{horizontal distance}}{\text{horizontal velocity}} = \frac{k}{u\cos\alpha}$

Use '$x = ut + \frac{1}{2}at^2$', vertically from O to H,

(Continued on the next page)

then $h = u\sin\alpha\left(\frac{k}{u\cos\alpha}\right) - \frac{g}{2}\left(\frac{k}{u\cos\alpha}\right)^2$

$$h = k\tan\alpha - \frac{gk^2}{2u^2\cos^2\alpha}$$

Use Range $'R = \frac{u^2\sin 2\alpha}{g}' = \frac{2u^2\sin\alpha.\cos\alpha}{g}$

Rewrite this as $\frac{Rg}{2\sin\alpha} = u^2\cos\alpha$, i.e. $\frac{Rg}{2\tan\alpha} = u^2\cos^2\alpha$

Substitute in the equation for h, gives the result.

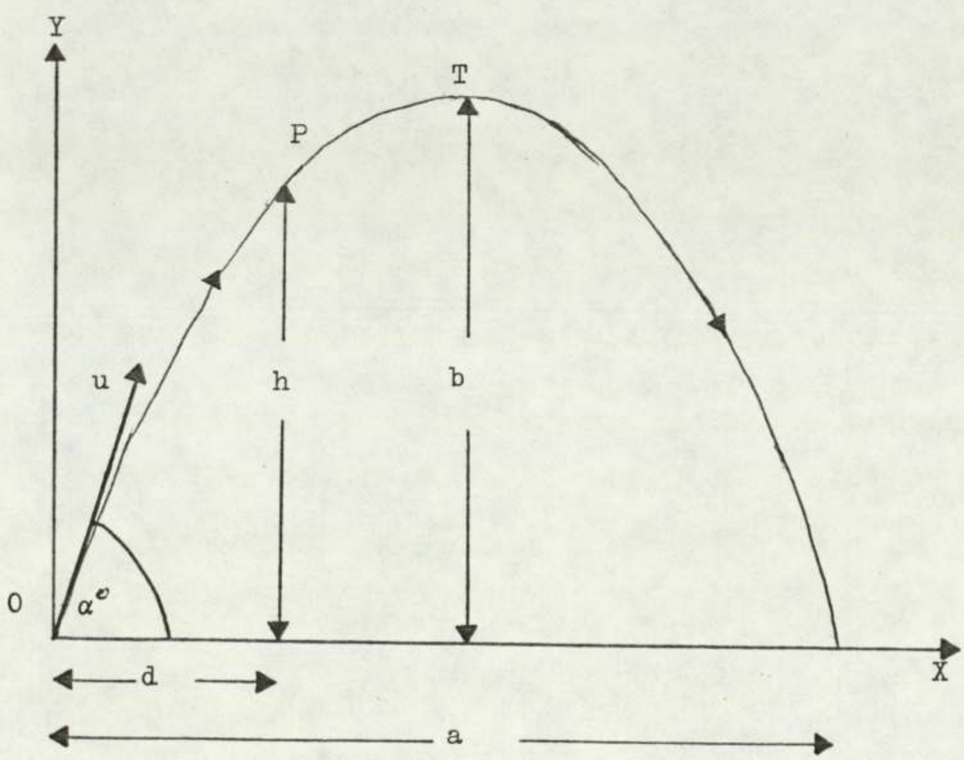
When you have checked this working,

SWITCH ON

EXAMPLE 5.

A particle projected from a point O, meets the horizontal plane through the point of projection after describing a horizontal distance 'a', and in the course of its trajectory attains a greatest height 'b' above the plane of projection. Find the velocity of projection in terms of a and b. Show, that when it has described a horizontal distance d, it has attained a height, $\frac{4bd(a-d)}{a^2}$

Do not read this solution until you have finished your own.



(Continued on the next page)

Use , '$\text{range} = \frac{2u^2 \sin\alpha \cos\alpha}{g}$' $= a$ ... (1)

and 'maximum height $= \frac{u^2 \sin^2\alpha}{2g}$' $= b$,

The second equation gives vertical component of velocity, $u\sin\alpha = \sqrt{2gb}$

Divide (1) by this equation gives $u\cos\alpha = \frac{a}{2}\sqrt{\frac{g}{2b}}$

hence $u^2 = 2gb + \frac{a^2}{4}\left(\frac{g}{2b}\right)$, and $u = \frac{\sqrt{g}}{2}\sqrt{\frac{a^2+16b^2}{2b}}$

Time to travel d horizontally $= \dfrac{d}{\frac{a}{2}\sqrt{\frac{g}{2b}}} = \frac{2d}{a}\sqrt{\frac{2b}{g}}$

Use '$x = ut + \frac{1}{2}at^2$' vertically from 0 to P.

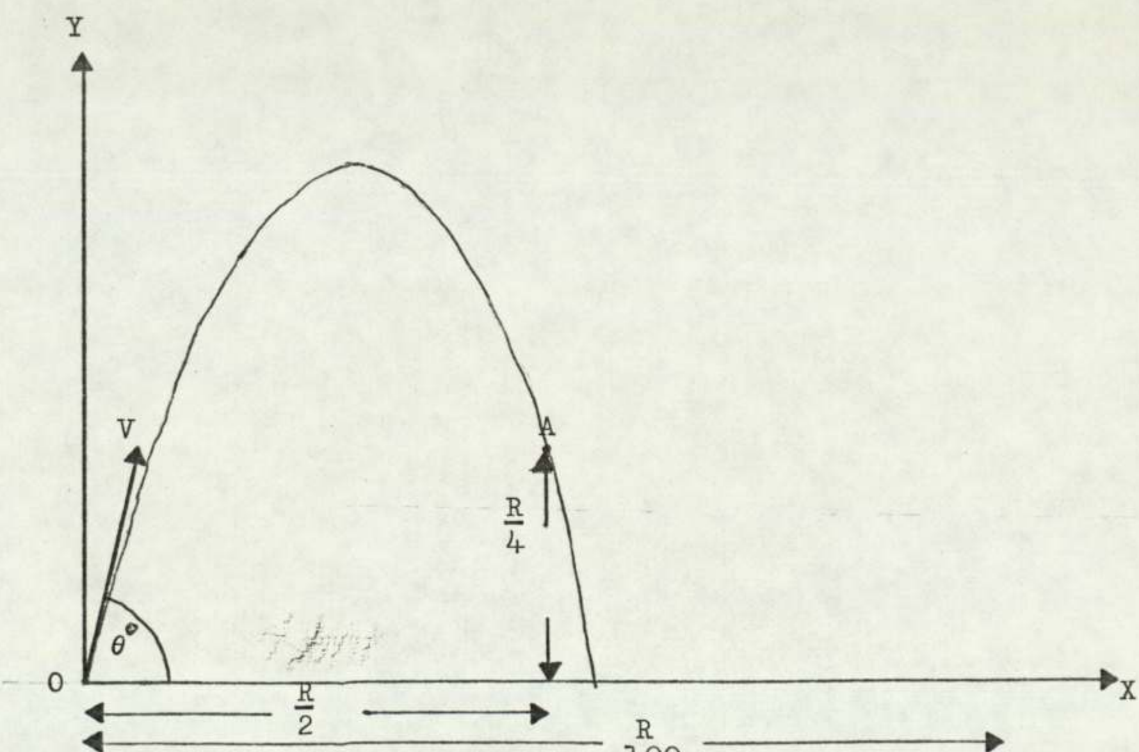
Simplify $h = \sqrt{2gb}\left(\frac{2d}{a}\sqrt{\frac{2b}{g}}\right) - \frac{g}{2}\left(\frac{2d}{a}\sqrt{\frac{2b}{g}}\right)^2$ for the answer.

SWITCH ON

EXAMPLE 6.

Show that, if R be the maximum horizontal range for a given velocity of projection, a particle can be projected with the same velocity to pass through a point, whose horizontal and vertical distances from the point of projection, are $\frac{R}{2}$ and $\frac{R}{4}$ respectively, provided that the tangent of the angle of projection is either 1 or 3, and that in the second case the horizontal range is $\frac{3}{5}$R.

Do not read the soltion yet!



(Continued on the next page.)

Range = $\dfrac{'V^2\sin2\theta'}{g}$, is maximum when $\sin2\theta = 1$,

i.e. $\theta = 45^\circ$ and $R = \dfrac{V^2}{g}$

Assume angle of projection θ, particle has to pass through A.

Time to A = $\dfrac{\text{distance}}{\text{horizontal velocity}} = \dfrac{\frac{1}{2}R}{V\cos\theta} = \dfrac{V}{2g\cos\theta}$

For vertical motion through A, use '$x = ut + \frac{1}{2}at^2$.'

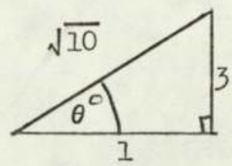
i.e. $\dfrac{R}{4} = \dfrac{V^2}{4g} = V\sin\theta\left(\dfrac{V}{2g\cos\theta}\right) - \dfrac{g}{2}\left(\dfrac{V}{2g\cos\theta}\right)^2$

which simplifies to $\tan^2\theta - 4\tan\theta + 3 = 0$.

$$(\tan\theta - 1)(\tan\theta - 3) = 0.$$

$\therefore$ $\tan\theta = 1$ or 3, the particle passes through A.

When $\tan\theta = 3$ we have the triangle

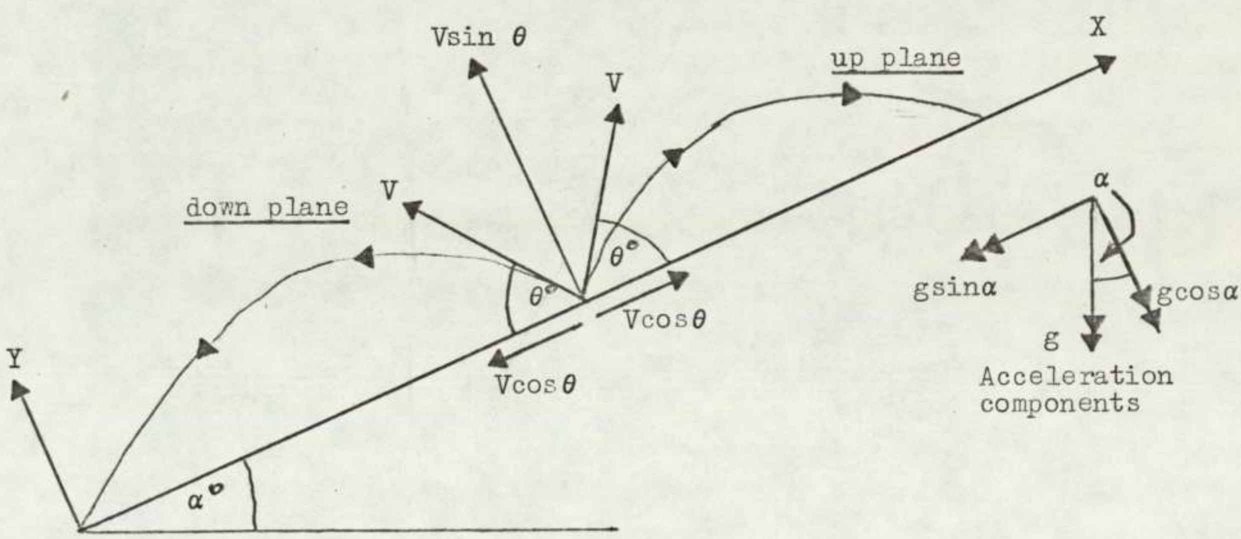


and $\sin\theta = \dfrac{3}{\sqrt{10}}$, $\cos\theta = \dfrac{1}{\sqrt{10}}$

Hence range = $\dfrac{V^2\sin2\theta}{g} = \dfrac{3}{5}\dfrac{V^2}{g} = \dfrac{3}{5}R$.

SWITCH ON

Projectiles above an inclined plane.



For motion perpendicular to the plane.

$\ddot{Y} = -g\cos\alpha$ integrate once with respect to time,

$\dot{Y} = V\sin\theta - g\cos\alpha(t)$ integrate again

$Y = V\sin\theta(t) - \frac{g}{2}\cos\alpha(t)^2$

(continued on the next page)

For motion up the plane, i.e. parallel to it, gravity retards.

$$\ddot{X} = -g\sin\alpha \qquad \text{integrate once,}$$

$$\dot{X} = V\cos\theta - g\sin\alpha(t) \qquad \text{integrate again,}$$

$$X = V\cos\theta(t) - \frac{g}{2}\sin\alpha(t)^2$$

For motion down the plane, parallel to it, gravity assists.

$$\ddot{X} = g\sin\alpha \quad \text{, gives}$$

$$\dot{X} = V\cos\theta + g\sin\alpha(t)$$

and $$X = V\cos\theta(t) + \frac{g}{2}\sin\alpha(t)^2$$

For time of flight, put $Y = 0$, gives $T = \dfrac{2V\sin\theta}{g\cos\alpha}$

Range, in each case, is found from the equation for X.

i.e. $R_{UP} = \dfrac{2V^2\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}$, $R_{DOWN} = \dfrac{2V^2\sin\theta\cos(\theta-\alpha)}{g\cos^2\alpha}$

When you have studied this,

SWITCH ON

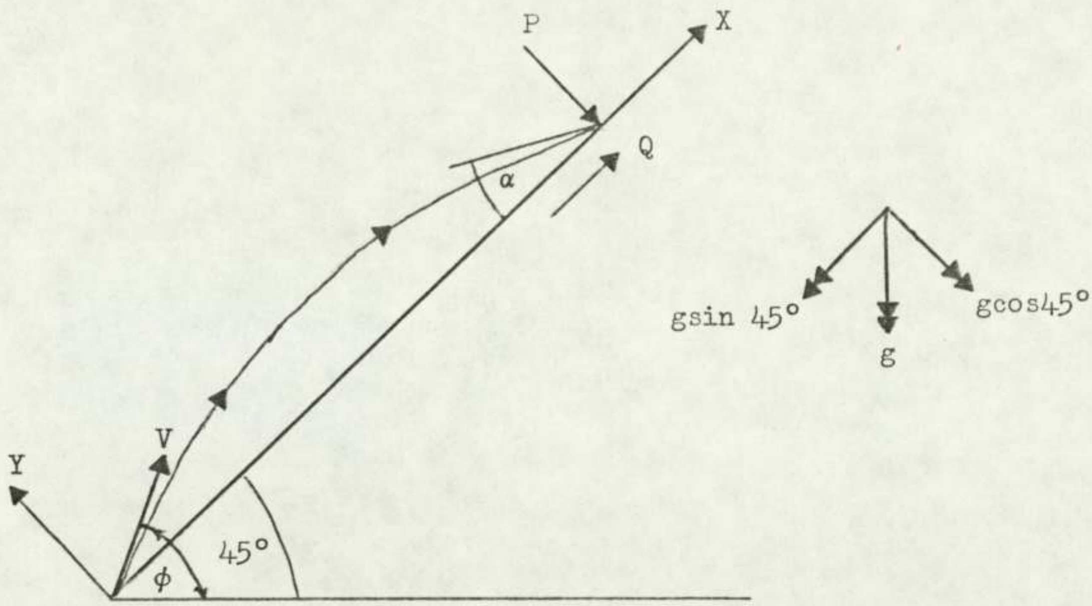
EXAMPLE 7.

A particle is projected at an angle ϕ to the horizontal on a plane inclined at 45° to the horizontal. Its path is in a vertical plane containing the line of greatest slope. Prove that the angle at which it meets the plane is α, where

$$\tan\alpha = \frac{(\tan\phi - 1)}{(3 - \tan\phi)}$$

At what angle must the particle be projected, so that it is travelling horizontally at the instant when it meets the plane again?

(The diagram is overleaf.)



When you are ready, SWITCH ON

At the point of impact, $\tan\alpha = \frac{P}{Q}$

Use time of flight = $\frac{'2V\sin\theta'}{g\cos\alpha}$ i.e. $\frac{2V\sin(\phi-45^\circ)}{g\cos45^\circ}$

For P (directed towards the plane),

use 'v = u+at' perpendicular to plane,

i.e. $-P = V\sin(\phi - 45^\circ) - g\cos45^\circ . \frac{2V\sin(\phi-45^\circ)}{g\cos45^\circ}$

and $P = V\sin(\phi-45^\circ)$

For Q, use 'v = u+at' parallel to the plane.

i.e. $Q = V\cos(\phi-45^\circ) - g\sin45^\circ . \frac{2V\sin(\phi-45^\circ)}{g\cos45^\circ}$

Hence the ratio $\frac{P}{Q}$, which simplifies to the result.

When you have checked this working,

SWITCH ON.

$\tan\alpha$ must be $\tan45^\circ = 1 = \frac{(\tan\phi-1)}{(3-\tan\phi)}$

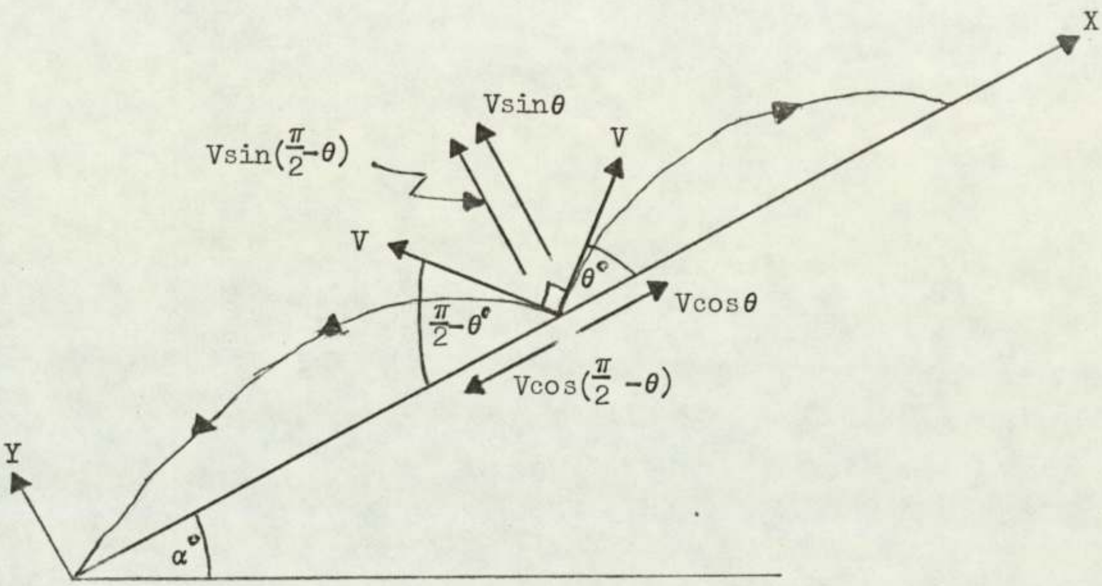
hence $\tan\phi = 2$.

SWITCH ON.

EXAMPLE 8.

From a point on the side of a flat hillside, two particles are projected in the vertical plane through the line of greatest slope, with equal velocities but in directions at right angles to each other. Show that the difference in their range, does not depend upon the angle of projection.

Do not look at this solution until yours is complete.



Use range up the plane '$R_U = \dfrac{2V^2\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}$'

and range down '$R_D = \dfrac{2V^2\sin\theta\cos(\theta-\alpha)}{g\cos^2\alpha}$'

But angle of projection down the plane $= \left(\frac{\pi}{2} - \theta\right)$

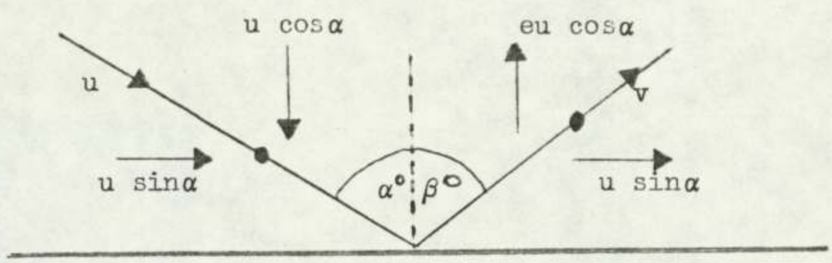
$$R_D = \frac{2V^2\sin(\frac{\pi}{2} - \theta)\cos(\frac{\pi}{2} - \theta - \alpha)}{g\cos^2\alpha}$$

Difference in ranges $= R_D - R_U$, which on simplication gives

$\dfrac{2V^2\sin\alpha}{g\cos^2\alpha}$, which is independent of θ, the angle of projection.

SWITCH ON.

Inclined impact with a smooth plane.



The component of velocity parallel to a smooth plane is unchanged after an impact.

The normal component of velocity is reduced by e, as is shown above.

The velocity after impact is v, where

$v^2 = (eu\cos\alpha)^2 + (u\sin\alpha)^2$; $v = u\sqrt{(e^2\cos^2\alpha + \sin^2\alpha)}$.

If β is measured with respect to the normal then

$$\tan\beta = \frac{u\sin\alpha}{eu\cos\alpha} = \frac{1}{e}\tan\alpha.$$

When you have studied this,

SWITCH ON.

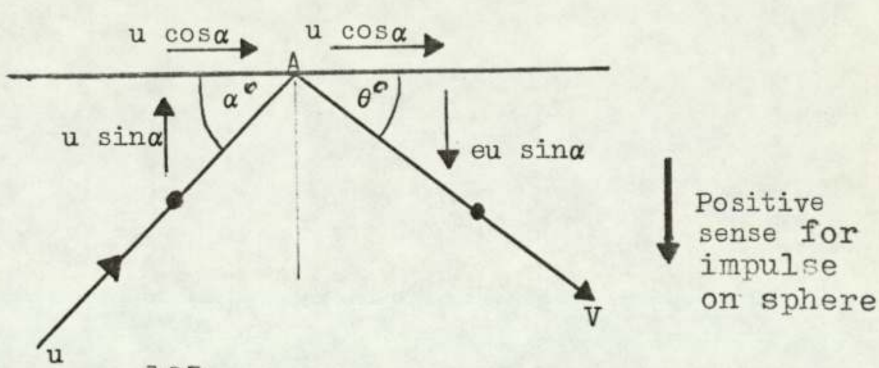
EXAMPLE 9.

A sphere of mass m, moving with a velocity u, impinges on a fixed smooth plane, the direction of motion making an angle α with the plane. If e is the coefficient of restitution between the sphere and the plane, find:

i) The magnitude and direction of the velocity of the sphere after impact.

ii) The impulse on the sphere at the impact.

iii) The loss of K.E.



105.

When you are ready SWITCH ON.

i) Normal component of velocity at A, before impact = $u\sin\alpha$

Component after impact = $eu\ \sin\alpha$

Component along the plane remains $u\ \cos\alpha$

$$\therefore\ V^2 = (eu\ \sin\alpha)^2 + (u\cos\alpha)^2$$

and $$V = u\sqrt{e^2\sin^2\alpha+\cos^2\alpha}\ \text{(magnitude)}$$

$$\tan\theta = \frac{eu\sin\alpha}{u\cos\alpha} = e\tan\alpha \qquad \text{(direction)}$$

ii) Impulse = change in momentum

$$I = meu\sin\alpha - (-mu\sin\alpha) = mu\sin\alpha(1+e).$$

iii) Loss of K.E. = $K.E._{START} - K.E._{END}$

$$= \tfrac{1}{2}mu^2 - \tfrac{1}{2}m\ [u^2(e^2\sin^2\alpha+\cos^2\alpha)].$$

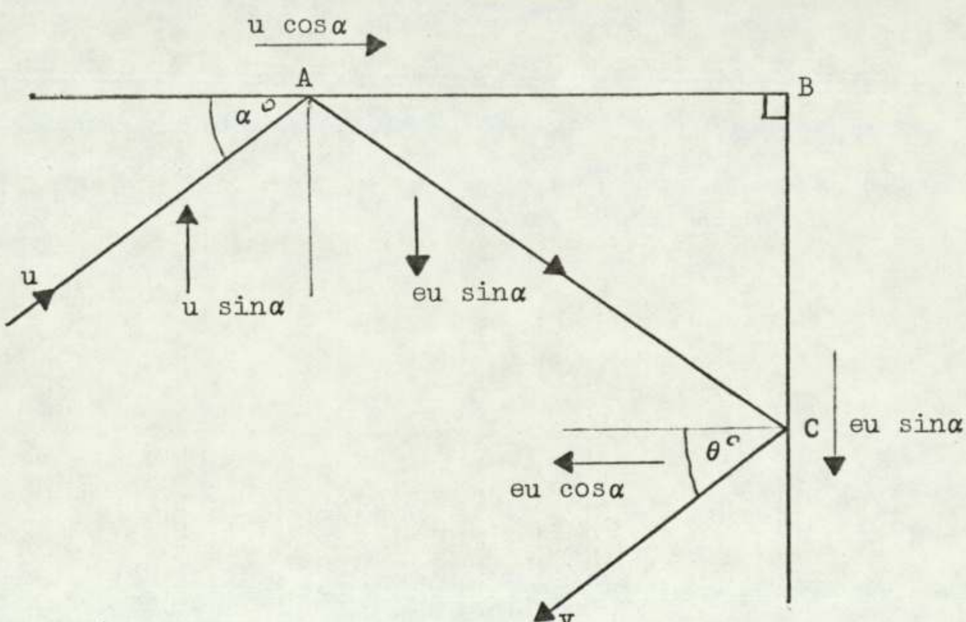
$$= \tfrac{1}{2}mu^2\sin^2\alpha(1-e^2).$$

SWITCH ON.

EXAMPLE 10.

A particle is free to move on a smooth horizontal plane, which is one of three mutually perpendicular planes, joined to form an internal corner. Show that if the particle is projected in a direction towards (but not directly into) the corner, it will emerge in a direction parallel to the ingoing one. Assume a common coefficient of restitution for the impacts.

Do not read this solution until you have tried your own.



(Continued on the next page).

There will be two impacts at A and C. If α is the initial inclination to AB, then the components of velocity will change as shown in the diagram. If θ is the emergent angle, measured to the normal at C,

$$\text{then } \tan\theta = \frac{\text{velocity component along BC}}{\text{velocity component perpendicular to BC}}$$

$$= \frac{eu\sin\alpha}{eu\cos\alpha} = \tan\alpha.$$

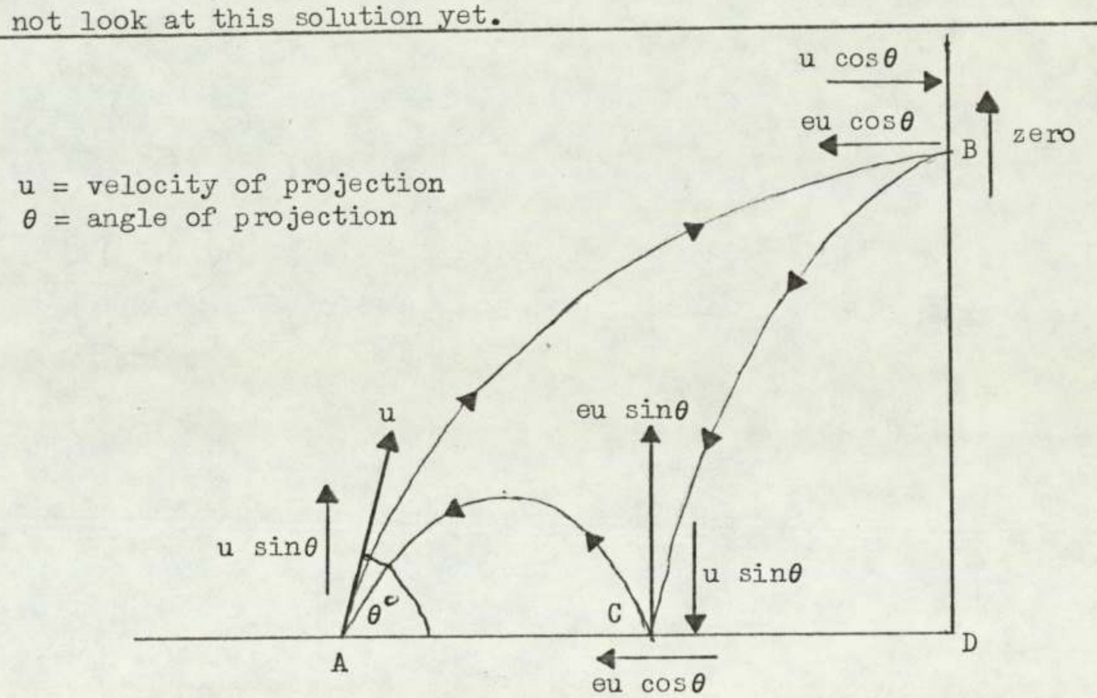
i.e. velocities of u and v are parallel

SWITCH ON.

EXAMPLE 11.

A ball is projected from a point on smooth level ground. It strikes a wall normally and returns to the starting point after bouncing once on the ground. Show that the coefficient of restitution is $\frac{1}{2}$ for both impacts.

Do not look at this solution yet.



Normal impact at B, time of flight $= \dfrac{u\sin\theta}{g}$

$$AD = \tfrac{1}{2}\text{ range} = \frac{u^2\sin\theta.\cos\theta}{g}$$

Since there is no change in the vertical motion at B,

(Continued on the next page)

the time of flight from B to C = time from A to B.

Due to normal impact at B, horizontal velocity = $eu\cos\theta$

Then CD = (velocity)(time) = $\dfrac{eu^2\sin\theta.\cos\theta}{g}$

Vertical component of velocity at C = $u\sin\theta$

After impact this is $eu\ \sin\theta$ upwards

Horizontal component is still $eu\ \cos\theta$

Then range CA = $2e^2u^2\sin\theta.\cos\theta/g = \dfrac{2(eu\sin\theta)(eu\cos\theta)}{g}$

Now AD = AC + CD, substitute and simplify, gives

$2e^2+e-1=0=(2e-1)(e+1)$, $e = \frac{1}{2}$ or -1.

As e cannot have a value -1, it must be $\frac{1}{2}$.

SWITCH ON.

SCRIPT FOR TAPE 4

MOTION IN TWO DIMENSIONS

1. This is the tape to be used with Booklet 4 of 'A Revision Course in Particle Dynamics' by Kenneth Jackson. It is concerned with motion in two dimensions.

All motion which has been studied by you so far, has been linear. Because of this, addition of acceleratin or displacement has been straightforward summing of these. You must now consider two dimensional motion, for which the vector quantities of velocity and acceleration, add according to the parallelogram law. Look on page two, where this is revised for you. SWITCH OFF.

2. In practice, the parallelogram is usually replaced by the vector triangle ABC. Notice that arrows on P and Q follow round, and that the arrow on R is in opposition. As these figures are scale drawings of the vectors, they may be used for calculation of angles and lengths directly. The first question is about triangles of velocities, and we shall work through this together. Read this question carefully, and look at the diagrams. SWITCH OFF.

3. The left hand diagram shows the course and current flow. To travel along BC, the launch must be steered upstream to allow for the current. The triangle of velocities on the right, is drawn with the current

and resultant vectors parallel to their required directions. As the velocities of u and v are known, they are drawn to the same scale, and shown as following vectors. Then complete the triangle, giving R in opposition, to the same scale, and θ as the required angle of deflection. Check this method for drawing the triangle, and then use the sine rule on it, to verify the answer for sin θ. Check this afterwards overleaf. SWITCH OFF.

4. As you see, we need to calculate R to determine the time along BC. To avoid using the cosine rule, we have used the sum of the projections of the vectors onto the line of the current, giving a vector result. Notice that R involves a difference of square root quantities which must be rationalized by multiplication, before we can divide by it. Follow the last step in the next section. SWITCH OFF.

5. As well as needing to compound velocities, we are often required to find the relative velocity of one body with respect to another. For parallel velocities, this is a straightforward subtraction, but for non linear velocities we need a vector difference. This is shown in the next section of the booklet. SWITCH OFF.

6. The second example is about relative velocities and we shall work through the first part of this together. Read the question carefully and study the diagrams. SWITCH OFF.

7. The diagrams represent the ships velocities, showing the apparent winds. As these are directions

relative to the ship, the ship must, in each case, be reduced to rest, by reversing its velocity. This enables the appropriate triangles of velocities to be drawn, as is shown in the next section. SWITCH OFF.

8. The left hand diagrams show the apparent wind at $22\frac{1}{2}$ degrees to the North South line. In the triangle the vectors have been arranged so that the sequence of arrows gives a summation of V and W, with the apparent wind as the resultant. This gives a wind from between east and south, which is labelled θ to the north line. The same process has been carried out for the other triangle. We may apply the Sine Rule to both of these. Follow this in the next section. SWITCH OFF.

9. Did you notice that 45 degrees satisfies the equation by inspection? Don't worry if you did not. This means that the wind blows from the south east. Find the wind speed by substitution in one of the equations above, and show that it equals the ship's speed. When you have done this, switch on the tape again. SWITCH OFF.

10. For the second part of the question, to follow a track west, the ship must sail so that it is blown onto the direction required. Construct the triangle of velocity in this way, and look closely at the angles and sides. Complete this yourself, and then check it in the next section. SWITCH OFF.

11. Example three is about the relative motion of a destroyer and a steamer. Read the question carefully and attempt the question yourself. You are advised to

draw the noon positions, and solve the problem by reducing the steamer to rest. When you have finished, check your solution on the next page. SWITCH OFF.

12. In two dimensional motion it is often best to work in terms of perpendicular components which are completely independent. The vector equations for this are listed in the next section. SWITCH OFF.

13. The velocities and displacements resulting from these, affect bodies simultaneously, producing curved trajectories. The study of particles projected at angles less than 90 degrees to the ground, is revised for you in the next section. SWITCH OFF.

14. Before you leave this, let me point out that not all of this needs to be remembered. Memorize though, the methods and steps which lead towards the results. In examinations, you should be able to deduce everything you require, but to save time and space, I shall be quoting from these, and you may do so if you wish. We shall work through question four together, so read this and study the diagram. SWITCH OFF.

15. We must assume an angle of projection, alpha and later, find an expression for its tangent. The assumption enables us to write horizontal and vertical components of velocity. The particle at the point H must have taken the same time vertically as horizontally from the origin. This can be used with the range to obtain the relation required. Follow this carefully in the solution, which is complete in the next section. SWITCH OFF.

16. The next question is similar to this, and is for your own working. Read it with care and draw a diagram showing the distances. You may use the quoted expressions for maximum heights and range, which will help you find velocity components. Check your solution afterwards in the next section. SWITCH OFF.

17. You should now attempt question six yourself. When you have read this carefully, and drawn the diagram, remember that the maximum range will be a function of velocity only. When you have completed this, check your solution in the next section. SWITCH OFF.

18. We must now extend the study, to involve projectiles over inclined planes. In this case, the most convenient perpendicular axes are along, and perpendicular to, the plane. This is summarized for you in the next section. Again, note the methods for deriving results. I shall also quote these as required. Read this section very carefully. SWITCH OFF.

19. Now read question seven, which we shall work through together, and study this diagram. SWITCH OFF.

20. For this question, you must remember that the tangent of the angle of inclination of a curve at any point, is given by the ratio of the perpendicular and parallel velocities; that is, in the notation used here, P divided by Q. These are the velocities at the instant of impact, in this case, so they can be obtained by using the time of flight in the equations. Follow this in the section below. SWITCH OFF.

21. For the projectile to land horizontally, the angle alpha must be 45 degrees. You are now able to complete the remainder of the question yourself. Do this and check your answer in the next section. SWITCH OFF.

22. You should now read example 8, which is for your own working. Your diagram should show all angles, distances and velocities, and then ask yourself what these perpendicular angles of projection tell you. When you have finished, check your method of working in the next section. SWITCH OFF.

23. You are now in a position to study the impact of particles when they are no longer travelling normally towards a plane. With the more complicated paths involved it is often convenient to treat the particle as a point on the path, rather than a sphere of significant size. This type of impact is revised for you in the next section. SWITCH OFF.

24. Do not try and remember these formulae, but remember that you are compounding perpendicular velocities. Example 9 is about this type of impact, and again, we shall work through it. Read it, and study the diagram. SWITCH OFF.

25. The angles alpha and theta are marked, as the question refers to the plane for measurement. It also helps to show the normal and parallel components of velocities, before and after impact. In part one, the magnitude and direction are found in the usual way. For the second part, the change in momentum gives the impulse on the sphere. The loss in kinetic energy is a straight-

forward difference between the energy at the beginning and end of the motion. Follow this working in the next section. SWITCH OFF.

26. Example 10 is one for you to try yourself. Read it carefully and in your diagram show the perpendicular components of velocity and two reference angles. When it is complete, check your working in the next section. SWITCH OFF.

27. You should now try the last example. Read this carefully, and draw a clear diagram showing the component velocities throughout the motion, which will help you to find ranges within it. Again, check your working afterwards. SWITCH OFF

28. This completes the work about motion in two dimensions. Would you please rewind the tape before you remove it from the machine. Thank you!

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

BOOKLET 5 to be used with TAPE 5

"Extension of Motion in Two Dimensions"

Read the instructions on page 1 thoroughly.

(Omitted for convenience.)

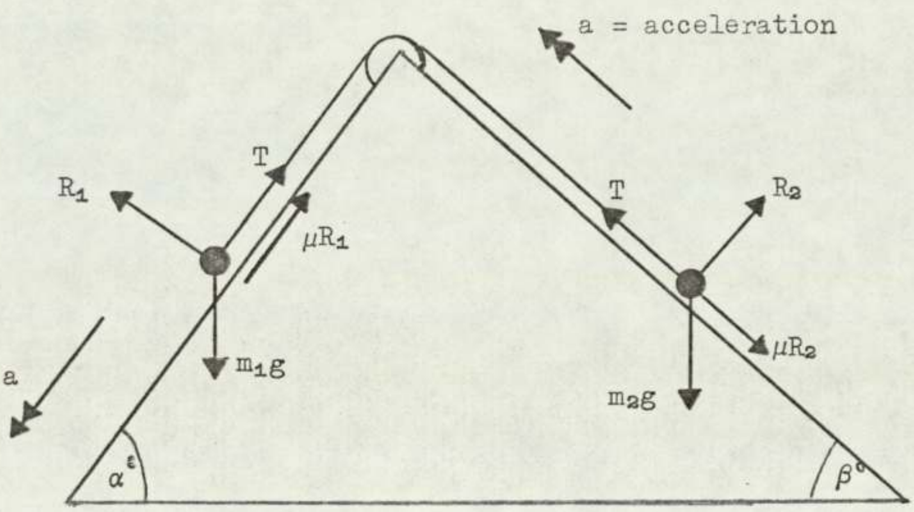
EXAMPLE 1.

Two particles, of masses m_1 and m_2 ($m_1 > m_2$), rest on the rough faces of a double inclined wedge with angles α and β to the horizontal respectively, ($\alpha > \beta$), and are connected by a light inextensible string, passing over a smooth pulley at the vertex of the plane. If the faces of the wedge are equally rough, with a coefficient of friction μ, find

i) the common acceleration of the particles,

ii) the tension in the string.

Complete your own diagram before checking below.



If you have omitted anything, insert it into your diagram and,

SWITCH ON

Use 'F = ma' for m_1, along the plane,

$$m_1 a = m_1 g \sin\alpha - T - F_1$$

$$\text{with } F_1 = \mu R_1 = \mu m_1 g \cos\alpha$$

and similarly for m_2,

$$m_2 a = T - m_2 g \sin\beta - F_2$$

$$\text{with } F_2 = \mu R_2 = \mu m_2 g \cos\beta$$

When you agree with this,

118.

SWITCH ON

i) Add the equations, giving

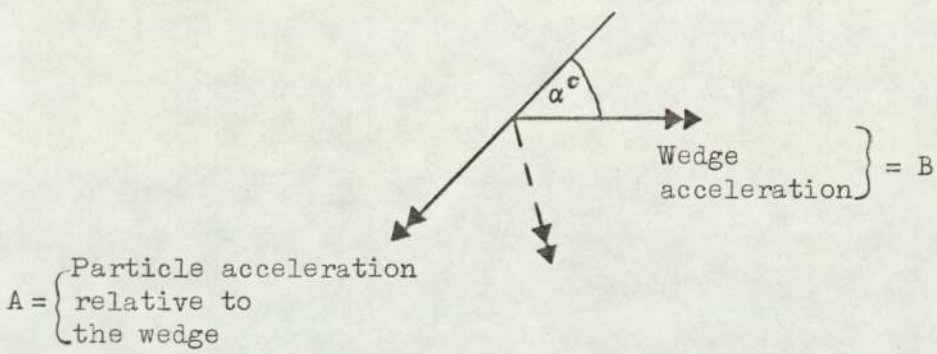
$$a = \frac{g[m_1\sin\alpha - m_2\sin\beta - \mu(m_1\cos\alpha + m_2\cos\beta)]}{(m_1 + m_2)}$$

ii) Substitute for 'a' in either equation to give T

$$T = m_1m_2 \frac{g[\sin\alpha + \sin\beta - \mu(\cos\alpha - \cos\beta)]}{(m_1 + m_2)}$$

SWITCH ON

The actual acceleration of a particle sliding on a moving wedge is the vectorial addition of the acceleration of the wedge plus the acceleration of the particle relative to the wedge.



Acceleration down the face = A - B $\cos\alpha$

Acceleration perpendicular to the face = B $\sin\alpha$

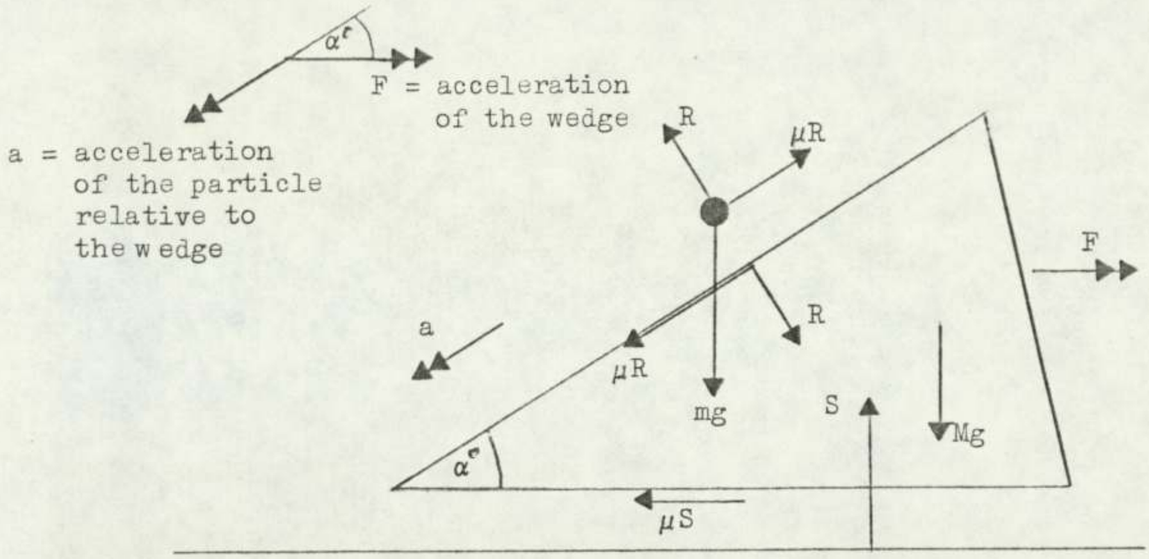
When you have studied this,

SWITCH ON

EXAMPLE 2.

A particle, of mass m, slides down the face of a rough wedge of mass M and slope α on a rough horizontal table. Find the acceleration of the wedge, if they both start to move. Assume that the coefficient of friction is the same for both surfaces.

(The diagram is on the next page)



When you have studied the forces and accelerations,

SWITCH ON

When you have identified

3 forces acting on the particle,

5 forces acting on the wedge.

SWITCH ON

Use 'F = ma' for the particle,

down the plane, $\quad m(a - F\cos\alpha) = mg\ \sin\alpha - \mu R \quad \ldots \quad (1)$

perpendicular $\quad mF\sin\alpha = mg\ \cos\alpha - R \quad \ldots \quad (2)$

for the wedge horizontally,

$$MF = R\ \sin\alpha - \mu R\ \cos\alpha - \mu S \quad \ldots \quad (3)$$

Vertically there is equilibrium and

$$S = R\cos\alpha + \mu R\ \sin\alpha + Mg \quad \ldots \quad (4)$$

When you have checked these,

SWITCH ON

From (3) and (4)

$$MF = R(\sin\alpha - 2\mu\ \cos\alpha - \mu^2\sin\alpha) - \mu Mg$$

$$F = \frac{g[m\cos\alpha\{\sin\alpha(1-\mu^2)-2\mu\cos\alpha\}-\mu M]}{[M+m\sin\alpha\{\sin\alpha(1-\mu^2)-2\mu\ \cos\alpha\}]}$$

SWITCH ON

EXAMPLE 3.

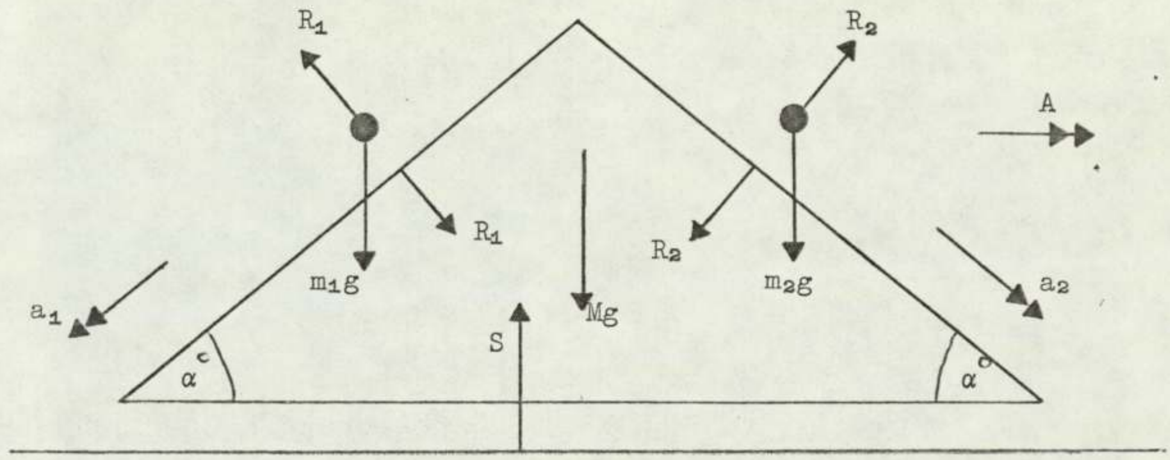
Two particles, of masses m_1 and m_2 ($m_1 > m_2$), are placed one on each of the smooth inclined faces of an isosceles wedge of mass M, and base angles α, which is free to move on a smooth horizontal plane. Show that when the particles are released from rest, the acceleration of the wedge is

$$\frac{(m_1-m_2)g\ \sin\alpha.\cos\alpha}{M + (m_1+m_2)\sin^2\alpha}$$

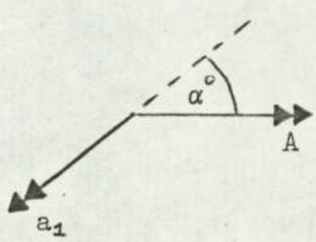
Find, also, the reaction of the wedge on the particle m_1.

When you have drawn your diagrams, check them below.

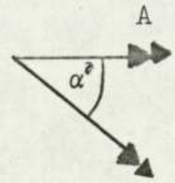
Assume that the wedge accelerates to the right.



Actual acceleration of m_1



Actual acceleration of m_2



SWITCH ON

121.

Use 'F = ma'

For the wedge horizontally, $MA = (R_1-R_2)\sin\alpha$ ... (1)

There is vertical equilibrium, $S = Mg+(R_1+R_2)\cos\alpha$ ... (2)

For m_1 along the plane,

$m_1(a_1-A\cos\alpha) = m_1 g\sin\alpha$ ... (3)

perpendicular to the plane,

$m_1 A\sin\alpha = m_1 g\cos\alpha - R_1$ ... (4)

For m_2 along the plane,

$m_2(a_2 + A\cos\alpha) = m_2 g\sin\alpha$ ... (5)

perpendicular to the plane,

$m_2 A\sin\alpha = R_2 - m_2 g\cos\alpha$ ... (6)

When you have checked these,

SWITCH ON

Use (4) and (6) in (1) to show that

$$A = \frac{(m_1-m_2)g\sin\alpha.\cos\alpha}{M + (m_1+m_2)\sin^2\alpha}$$

Substitute A into (4) to obtain $R_1 = \dfrac{m_1 g(M+2m_2\sin^2\alpha)\cos\alpha}{M+(m_1+m_2)\sin^2\alpha}$

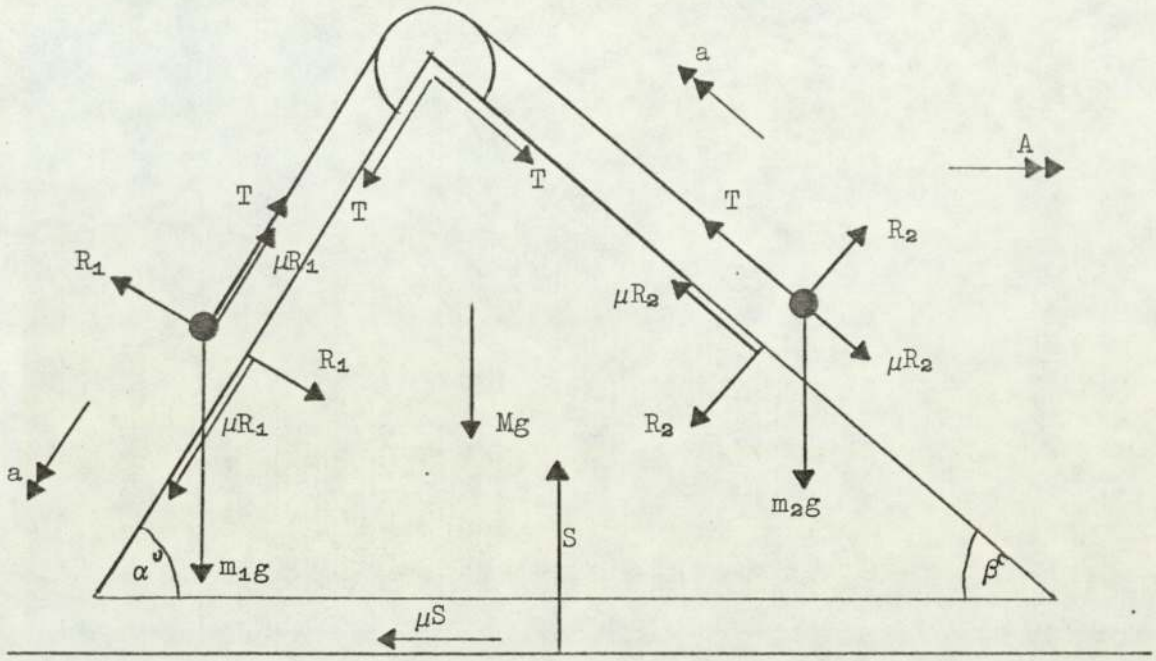
SWITCH ON

EXAMPLE 4.

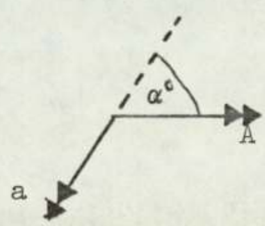
Two particles, of masses m_1 and m_2 ($m_1 > m_2$), rest on the rough faces of a double inclined wedge, with angles α and β to the horizontal respectively, ($\alpha > \beta$), and are connected by a light inextensible string, passing over a smooth pulley at the vertex of the plane. If the faces and base are equally rough, with a coefficient of friction μ, consider the effect on the motion if the wedge also moves on a rough horizontal plane. [This is a generalisation of EXAMPLE 1.]

When you have finished, check your diagrams on the next page.

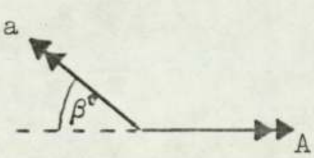
Assume that the acceleration of the wedge is A, and that the acceleration of the particles relative to the wedge is a.



Actual acceleration of m_1



Actual acceleration of m_2



When you have identified 9 forces acting on the wedge,

SWITCH ON

For the wedge there is vertical equilibrium,

$$R_1(\cos\alpha+\mu\sin\alpha)+T(\sin\alpha+\sin\beta)+R_2\cos\beta+Mg = S+\mu R_2\sin\beta \quad \ldots \quad (1)$$

'F = ma' horizontally

$$MA = R_1(\sin\alpha-\mu\cos\alpha)+T(\cos\beta-\cos\alpha)- R_2(\mu\cos\beta+\sin\beta)-\mu S \quad \ldots \quad (2)$$

For m_1 along the plane,

$$m_1(a - A\cos\alpha) = m_1g \sin\alpha - T-\mu R_1 \quad \ldots \quad (3)$$

perpendicular to the plane,

$$m_1A \sin\alpha = m_1g \cos\alpha-R_1 \quad \ldots \quad (4)$$

(Continued on next page)

For m_2 along the plane,

$$m_2(a-A\cos\beta) = T-m_2g\sin\beta - \mu R_2 \qquad \ldots \qquad (5)$$

perpendicular to the plane,

$$m_2A\sin\beta = R_2- m_2g\cos\beta \qquad \ldots \qquad (6)$$

When you have checked these equations,

SWITCH ON

Stages required to find A.

1. Combine (1) and (2) to eliminate S, and collect terms in R_1, R_2, and T.
2. Eliminate a from (3) and (5), which gives an expression for T in terms of A, R_1, and R_2.
3. Substitute for T in stage 1. Collect terms in R_1, R_2, and A.
4. Use (4) and (6) to substitute for R_1 and R_2 in stage 3. Hence A.

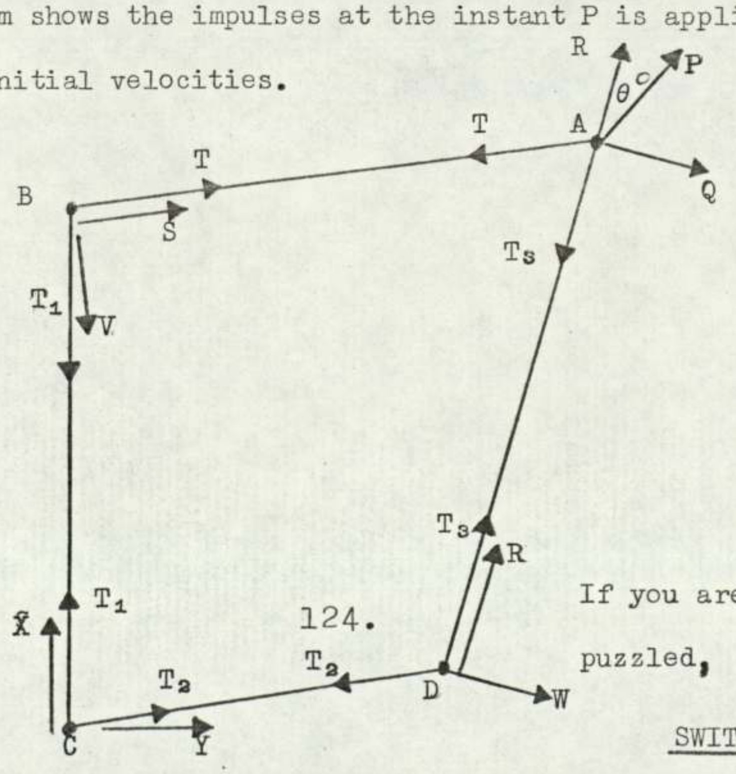
When you have agreed with this,

SWITCH ON

EXAMPLE 5.

Four equal particles, of mass m, at the corners of a square, are connected by light inextensible strings forming the sides of a square. If one particle receives a blow P along the diagonal outwards, show that its initial velocity is P/2m, and find the initial velocities of the other particles.

The diagram shows the impulses at the instant P is applied and also the initial velocities.

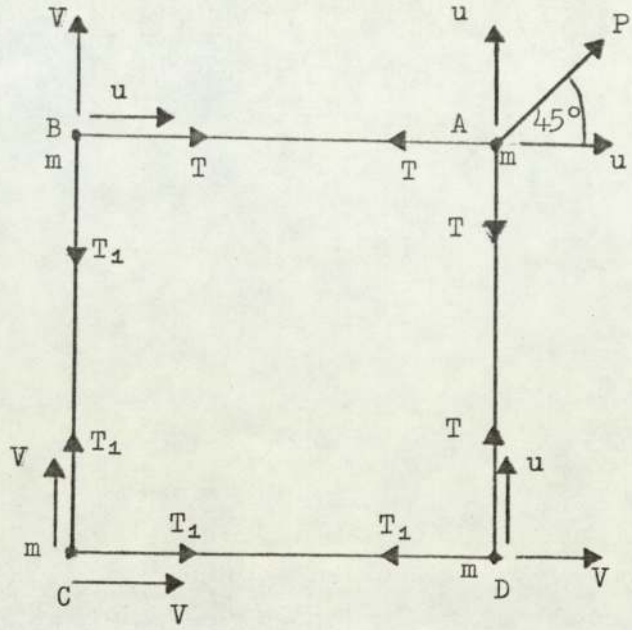


124.

If you are completely puzzled,

SWITCH ON

This diagram shows the impulses at the instant P is applied and also the initial velocities. Notice the great simplification provided by the regularity of the square.



When you have studied all the details,

SWITCH ON

Impulse = change in momentum

For A, along BA, $P\cos 45° - T = mu$ ... (1)

For B, along BA, $T = mu$ ... (2)

along CB, $mV = -T_1$ ... (3)

For C, along CB, $T_1 = mV$ ... (4)

When you have checked these,

SWITCH ON

Add equations (1) and (2) to give $u = \dfrac{P}{2m\sqrt{2}}$

Resultant velocity of A $= \sqrt{u^2+u^2} = \dfrac{P}{2m}$ along the diagonal.

Add equations (3) and (4) to give $V = 0$

Velocity of B and D is $\dfrac{P}{2m\sqrt{2}}$ directed towards A.

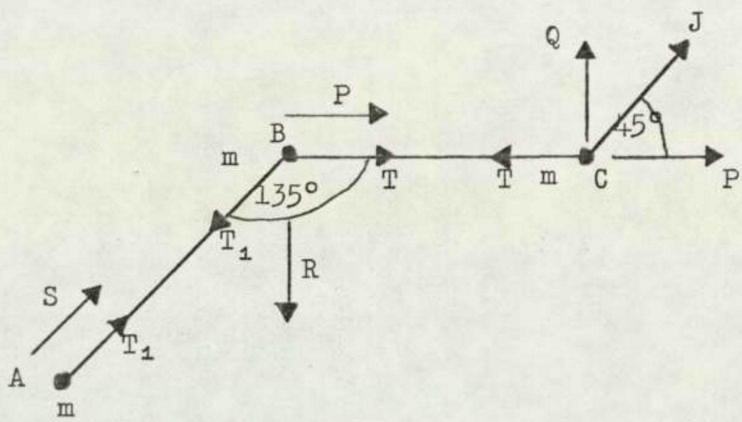
The velocity of C is zero.

SWITCH ON.

EXAMPLE 6.

Three particles A, B and C, each of mass m, lie at rest on a smooth horizontal table. Light inextensible strings connect A to B, and B to C. The strings are just taut with $A\hat{B}C = 135^\circ$, and an impulse J is applied to C, in the direction parallel to AB. Prove that A begins to move with speed J/7m, and find the impulsive tension in the string BC.

The diagram shows the impulse at the instant J is applied and also the initial velocities.



When you have studied these velocities,

SWITCH ON

Impulse = change in momentum.

For particle	C, along BC,	$J\cos 45^\circ - T = mP$	...	(1)
" "	B, along BC,	$T - T_1\cos 45^\circ = mP$	...	(2)
	perpendicular to BC	$T_1\cos 45^\circ = mR$	...	(3)
" "	A along AB	$T_1 = mS$	...	(4)

SWITCH ON

At the moment of impact, because AB does not stretch,

Velocity of A along AB = velocity of B along AB

i.e. $S = P\cos 45^\circ - R\cos 45^\circ$

or $S = (P-R)\frac{\sqrt{2}}{2}$ ... (5)

We begin solving overleaf.

Rewrite (5) as $mS = m\,P\frac{\sqrt{2}}{2} - mR\frac{\sqrt{2}}{2}$

Substitute from (3) in this,

$$mS = mP\frac{\sqrt{2}}{2} - \cancel{\frac{\sqrt{2}}{2}}\left(T_1\frac{\sqrt{2}}{2}\right) = mP\frac{\sqrt{2}}{2} - \frac{T_1}{2}$$

SWITCH ON

Add (1) and (2)

$$J\frac{\sqrt{2}}{2} - T_1\frac{\sqrt{2}}{2} = 2mP$$

$$\frac{\sqrt{2}}{4}(J-T_1) = mP \quad \text{Substitute above} \quad \dots \quad (6)$$

$$mS = \frac{\sqrt{2}}{4}(J-T_1)\frac{\sqrt{2}}{2} - \frac{T_1}{2}$$

$$mS = \frac{J}{4} - T_1\left(\tfrac{1}{4}+\tfrac{1}{2}\right)$$

SWITCH ON

Substitute from (4)

$$mS = \frac{J}{4} - \frac{3mS}{4} \qquad \text{i.e. } S = \frac{J}{7m} \text{ as required.}$$

also $mS = T_1 = \frac{J}{7}$

Substitute in (2) $\quad T = \frac{\sqrt{2}}{2}\left(\frac{J}{7}\right) + mP$

Substitute in (6) $\quad mP = \frac{\sqrt{2}}{4}\left(J - \frac{J}{7}\right)$

Hence $T = \frac{2\sqrt{2}J}{7}$ = tension in s tring BC.

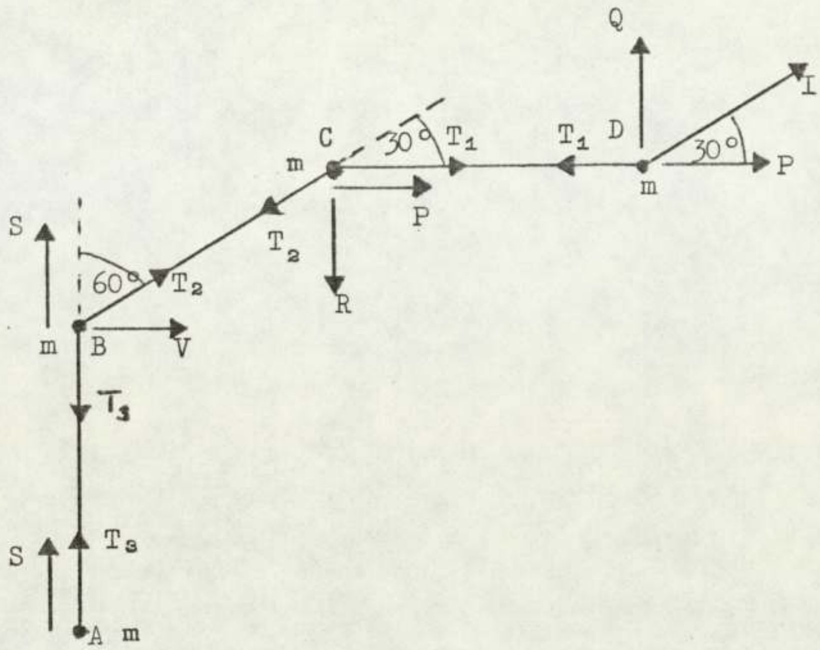
SWITCH ON

EXAMPLE 7.

Four equal particles A,B,C,D, each of mass m, are connected by equal light inextensible strings. They lie at rest on a smooth horizontal table with $A\hat{B}C = 120°$, $B\hat{C}D = 150°$. An impulse I is applied to D at an angle of 30° to CD produced, and directed away from A. Find the initial velocity of A.

When you have drawn the diagram and put in your component velocities and impulsive tensions, check overleaf.

The diagram shows the impulses at the instant that I is applied and also the initial velocities.



SWITCH ON

Impulse = change in momentum.

For D, along CD, $I\cos 30° - T_1 = mP$ ... (1)

C, along CD, $T_1 - T_2\cos 30° = mP$ ... (2)

perpendicular to CD, $T_2\cos 60° = mR$ ... (3)

B, along AB, $T_2\cos 60° - T_3 = mS$ ... (4)

perpendicular to AB, $T_2\cos 30° = mV$ ... (5)

A, along AB, $T_3 = mS$ ... (6)

Velocity of B along BC = velocity of C along BC

$S\cos 60° + V\cos 30° = P\cos 30° - R\cos 60°$ ... (7)

Add (1) and (2) to eliminate T_1 $\quad \frac{\sqrt{3}}{2}(I-T_2) = 2mP$

Subst. in (7) $\frac{1}{2}mS + mV\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{4}(I-T_2) - \frac{1}{2}mR$

Use (3) and (5) $\frac{1}{2}mS + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} T_2 = \frac{3}{8}(I-T_2) - \frac{1}{2}.\frac{1}{2}T_2$

$$4mS + 11\,T_2 = 3I$$

Add (4) and (6) $\quad \frac{1}{2}T_2 = 2mS$

Subst. above to give $S = \dfrac{I}{16m}$.

128.

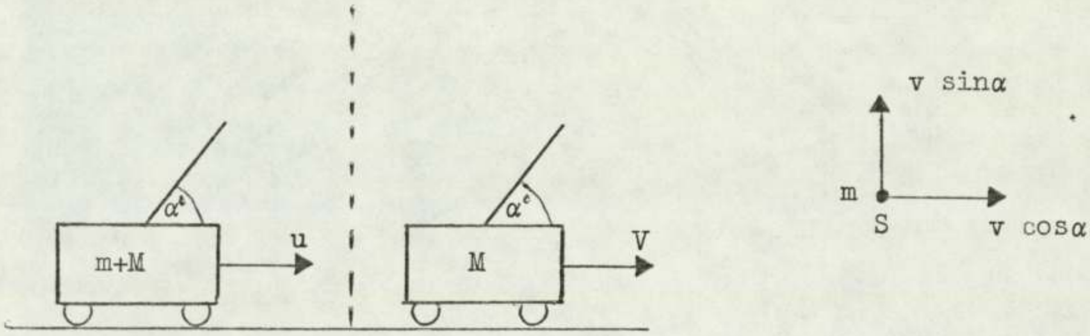
SWITCH ON.

EXAMPLE 8.

A gun is mounted on a railway truck which is free to run without friction on a straight horizontal railway track. The gun and truck together of mass M, are moving along the track with velocity u, when a shell of mass m (not included in M) is fired from the gun with muzzle velocity v, relative to the gun. If the gun barrel and the track lie in the same vertical plane, and the former is inclined at an angle α to the direction in which the gun is moving, show that the shell has a horizontal range.

$$\frac{2v\sin\alpha}{g}\left(u + \frac{Mv\cos\alpha}{(M+m)}\right)$$

Before firing | After firing | Velocity of the shell relative to the gun.



When you are ready to start

SWITCH ON

Total horizontal momentum before firing = (m+M)u

Horizontal momentum of gun and truck after firing = MV

Horizontal momentum of shell after firing = $m(V+v\cos\alpha)$

Horizontal momentum is conserved,

$$(m+M)u = MV + m(V+v\cos\alpha)$$

$$\text{i.e. } V = \frac{(m+M)u - mv\cos\alpha}{(m+M)}$$

SWITCH ON

$$\text{Time of flight} = \frac{2v\sin\alpha}{g}$$

Horizontal shell velocity after firing is $(V+v\cos\alpha)$

Substitute for V and simplifying gives $u + \frac{Mv\cos\alpha}{(M+m)}$

Range = (horizontal velocity × time) and hence result.

SWITCH ON.

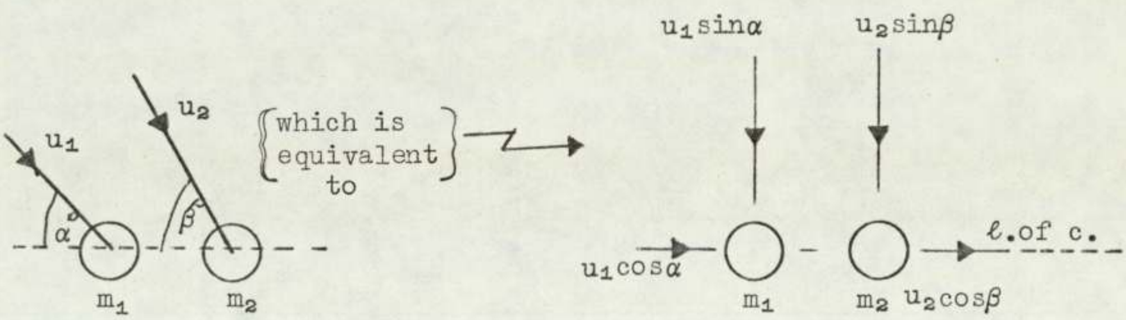
EXAMPLE 9.

Two spheres, of masses m_1 and m_2, respectively, have initial velocities u_1 and u_2, inclined at angles α and β, in the same sense, to the line of centres (ℓ.of c), when they collide. Show that the new velocity of m_2 along the line of centres is given by,

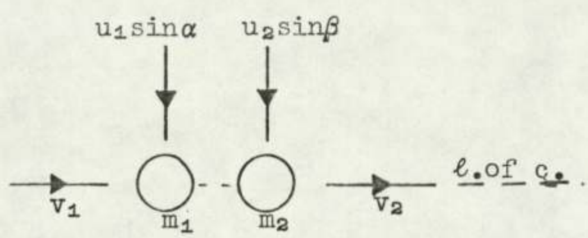
$$v_2 = \frac{m_1u_1(1+e)\cos\alpha + u_2(m_2-em_1)\cos\beta}{(m_1+m_2)}$$

where e is the coefficient of restitution.

Just before impact



Just after impact.



When you have studied these,

SWITCH ON.

Conservation of momentum along the line of centres,

$$m_1u_1\cos\alpha + m_2u_2\cos\beta = m_1v_1 + m_2v_2$$

Law of restitution along the line of centres,

$$v_2 - v_1 = e(u_1\cos\alpha - u_2\cos\beta)$$

When you have checked these,

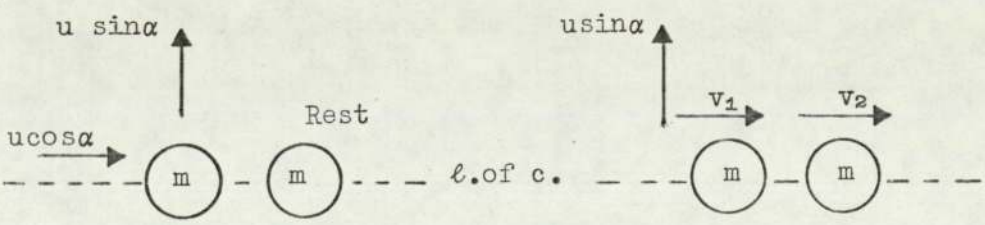
SWITCH ON.

EXAMPLE 10.

A smooth sphere of mass m, impinges on an equal sphere at rest. Before impact the first sphere was moving in a direction making an angle α with the line of centres at the moment of impact. Show that the direction of the first sphere is turned through an angle β, where $\tan\beta = \dfrac{(1+e)\tan\alpha}{1-e+2\tan^2\alpha}$

Do not read the solution below until you have found the velocity components of the first sphere after impact.

Just before impact   Just after impact



Conservation of momentum along the line of centres,

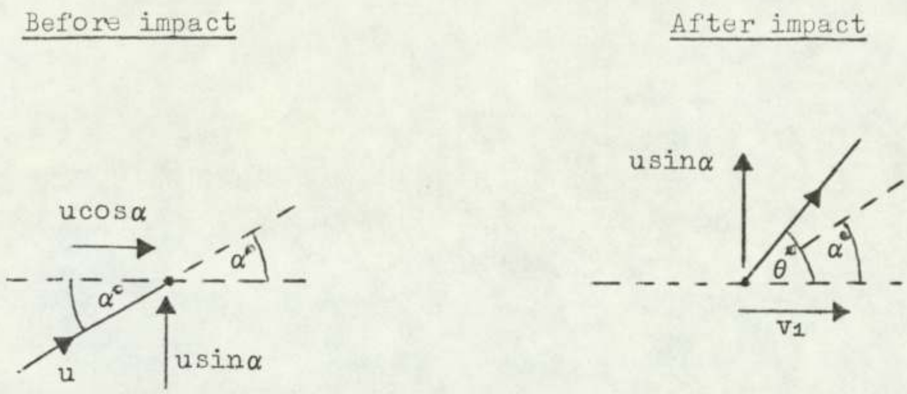
$$mu\cos\alpha = mv_2 + mv_1$$

Law of restitution along the line of centres,

$$v_2 - v_1 = eu\cos\alpha.$$

Eliminate v_2 to give $v_1 = \frac{1}{2}u\cos\alpha(1-e)$

SWITCH ON.



From the right hand diagram, $\tan\theta = \dfrac{u\sin\alpha}{v_1} = \dfrac{2\tan\alpha}{(1-e)}$

Angle of turn $= \beta = \theta - \alpha$

Hence result.

SWITCH ON.

EXAMPLE 11.

In a certain game a ball is rolled along a horizontal plane with velocity V, until it strikes an inclined plane, from which it rebounds. The object of the game is to make the ball, after rebounding, fall into a hole in the inclined plane. If θ be the inclination of the plane, e the coefficient of restitution between the ball and the plane, and if the hole be a distance d away from the junction of the planes, show that the ball will enter the hole if,

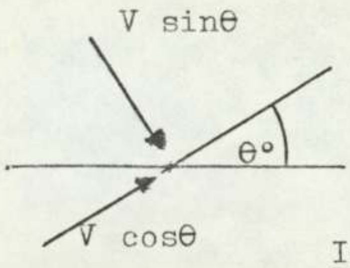
$$V = \sqrt{\frac{dg\ \mathrm{cosec}\,\theta}{2e(1 - e\tan^2\theta)}}$$

When you have drawn your diagram,

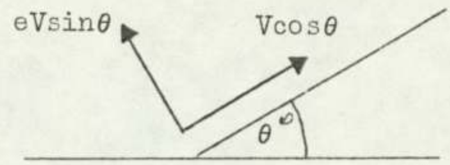
SWITCH ON.

The particle

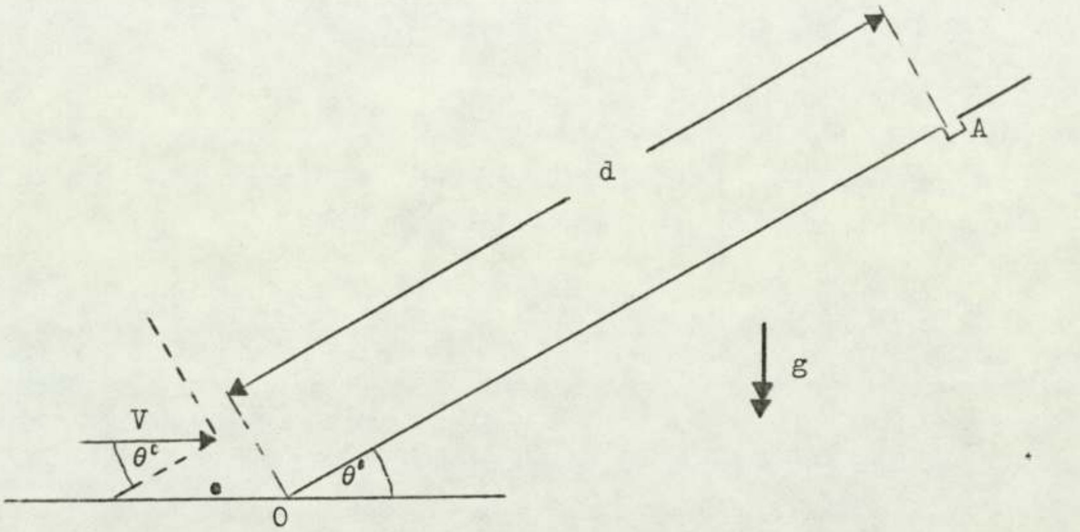
Just before impact



Just after impact.



Inclined plane



The perpendicular velocity $V\sin\theta$ will be reversed and reduced by a factor e, as shown. Use this velocity to find the time of flight from the standard formula for an inclined plane.

Use time of flight = $\dfrac{\text{"}2V\sin\theta\text{"}}{g\cos\alpha}$ i.e. $T = \dfrac{2(eV\sin\theta)}{g\cos\theta}$

For OA, use "$x = ut + \frac{a}{2}t^2$" parallel to the plane,

$$\text{i.e.}\quad d = V\cos\theta . T - \frac{g}{2}\sin\theta (T)^2$$
$$= V\cos\theta . \frac{2eV\sin\theta}{g\cos\theta} - \frac{g\sin\theta}{2}\left(\frac{2eV\sin\theta}{g\cos\theta}\right)^2$$

This gives the result on simplification.

SWITCH ON.

APPENDIX 5 CONTINUED

SCRIPT FOR TAPE 5

EXTENSION OF MOTION IN TWO DIMENSIONS

1. This is the tape to be used with Booklet 5 of 'A Revision Course in Particle Dynamics' by Kenneth Jackson. This is an extion of the work concerning motion in two dimensions, with more difficult examples. We shall, of course, work through some of these together. Part of the difficulty lies in the complexity of the problems which should be read with very close attention. Here, more than ever, clear diagrams are vital, to show exactly which forces are applicable to the motion.

The first example concerns connected particles on inclined planes, and we shall work through part of this together. Read it and draw your diagram showing all forces. Remember the friction will be acting on both faces. When you have finished, check your diagram below. SWITCH OFF.

2. If you missed any features on this diagram, try reading the question again, after you have drawn a diagram. Using F = ma, write down the equations of motion for the particles, and afterwards check them below. SWITCH OFF.

3. To find the acceleration we need to eliminate the tension which can be done by addition. To find the tension, substitute in either of the equations. Find a

and T, and then check these on the next page. SWITCH OFF.

4. Another factor arises when a particle slides down a moving wedge, as it has two contributions to its accleration, one down the plane, and the acceleration of the wedge itself. This is revised for you in the next section. SWITCH OFF.

5. The actual acceleration is the vector sum of A and B, and the most convenient perpendicular component directions are along, and perpendicular to the plane. We shall need these components in example 2, which we shall work through together. Read this and study the forces and accelerations in the diagram. SWITCH OFF.

6. Remember that in this complicated situation; we separate the moving parts, so that we may show more clearly their interactions. So we show an upward force, R, exerted by the plane on the particle, and an equal force downwards, exerted by the particle on the wedge. Similarly with the friction force. To check this, look below in the next section. SWITCH OFF.

7. We are now able to write the equations of motion using these groups of forces. Write these yourself for the particle down the plane, and perpendicular to it, and for the wedge, horizontally and vertically. Check these afterwards in the next section. SWITCH OFF.

8. You are now able to find F, by eliminating S and R between the equations 2, 3 and 4. Carry out this process, and then check below. SWITCH OFF.

9. These equations could, of course, be used

further to find R, or the acceleration of the particle by substitution. The next example, number 3, is mainly for your own working and concerns two particles on a moving wedge. Read this carefully, and draw a diagram, and acceleration vector diagrams, similar to those in the two previous examples. SWITCH OFF.

10. As all the forces and accelerations are now specified, we can write all the equations of motion. Do this and check them afterwards on the next page. SWITCH OFF.

11. You should now find A by substituting for R1 and R2 into the first equation, and then R1 by further substitution. Check your answer for this below. SWITCH OFF.

12. The next question, 4, is even more complex, and we shall look at the preliminary stages together. Read it and draw your diagrams very carefully. Remember there is a tension in the string, and this affects the wedge at the pulley. When you have included all the forces and accelerations, check on the next page. SWITCH OFF.

13. The action of the string through the pulley is equivalent to an added force T, parallel to each face, as shown, and these will have components horizontally and vertically. Assuming the accelerations as shown, write the six equations of motion you would need to solve this problem, and check then below, afterwards. SWITCH OFF.

14. As the algebra involved would be lengthy, and the essential mechanics has now been covered, I shall not ask you to find the acceleration of the wedge. However, you should be able to list the steps you would need. Do this, and then see if we agree in the next section. SWITCH OFF.

15. Some involved algebra also often occurs in the next type of example about impulses in strings connecting a number of particles together. Read example 5 and study the diagram carefully. SWITCH OFF.

16. This is obviously not the diagram, but it is drawn as a contrast to example 5. ABCD is an irregular figure, and P at some angle θ to DA. To allow for the varied directions of impulsive tensions and velocities, we must insert a number of perpendicular components, along and perpendicular to the strings. Only A and D have a common velocity R, because particles at the end of a straight string, must have the same component along it. Now look at the correct diagram on the next page. SWITCH OFF.

17. Because P acts along the diagonal, we have equal impulsive tensions and velocities on each side of it. Look at particle A, which has equal component velocities along BA and DA. B has a tension acting along BA, so also has a velocity component u, as the string does not stretch. The other component is denoted by V, which is also communicated through the string to particle C. Similarly for ADC. Because of the symmetry, we only need two impulsive

tensions instead of four. We only need to write the impulse equations for A, B and C, for perpendicular directions. These are written in the next section of the booklet; verify them carefully. SWITCH OFF.

18. By using equations 1 and 2, you can find u, and then equations 3 and 4 for V. Complete the question and check your answers in the next section. SWITCH OFF.

19. The symmetry made this a straightforward question, but notice that even though we inserted an unneeded velocity component, the algebra showed this to be zer. Now read example 6, which we shall analyse together, and look particularly at the component velocities shown in the diagram. SWITCH OFF.

20. Notice that the particles B and C have a common velocity P along the string BC, and as A is the last particle in the system, we can insert a single velocity S, along AB. We can now write the impulse equations, by resolving along, and perpendicular to the strings. Check these in the next section. SWITCH OFF.

21. There are 5 unknowns in these four equations, and so we need another relation. This is found from the string AB, which remains taut, and this type of equation is often used as a basis for substitution from the others. Follow this in the booklet, again. SWITCH OFF.

22. As the first substitution gives a term in T1, so we need to substitute for P in the same terms. This can be done by eliminating T from equations 1 and 2. Follow this in the next section. SWITCH OFF.

23. As we are finding S, we need to substitute for T1 in terms of this, using equation 4. To find the other tension we need to substitute back in equations 2 and 6. Finish this process, and check afterwards, below. SWITCH OFF.

24. You will notice in this question, that component of velocity Q, was not used at all. However, it could be found directly from the impulse J if required. Now attempt the next question yourself. Read it, and draw your diagram showing the impulsive tensions and component velocities. When you have done this, check these on the next page. SWITCH OFF.

25. The components shown, are not the only choice, but they are very convenient. Notice the repeated components, P and S. Remember that string BC remains taut and inextensible. When you have finished the working, check this in the section below. SWITCH OFF.

26. Another type of two dimensional example involves the firing of a gun, which depends upon the conservation of momentum, as in the next question. Read example 8 carefully, and study the diagrams. SWITCH OFF.

27. This problem depends upon the conservation of horizontal momentum, as there is no external impulse in this direction. The two diagrams help us to compare the momentum before and after, firing. In the right hand diagram, the shell and gun have a common horizontal velocity V, but because of the explosion, the shell is moving forward with an additional velocity, and its

actual velocity is the sum of these. Use this to write the conservation equatioj, and evaluate V. Check this below. SWITCH OFF.

28. The vertical motion is not affected by horizontal variations, and this enables you to obtain the time of flight of the shell. Remembering that the range is the product of the actual horizontal velocity and the time, you are now able to complete the question yourself. When you have done this, check in the next section. SWITCH OFF.

29. The next example concerns spheres colliding obliquely. Read this carefully, and study the diagram which shows that only the velocity components along the line of centres, change at the impact. SWITCH OFF.

30. Notice particularly, that there is no velocity change perpendicular to the line of centres. To find v1 and v2, apply the conservation and momentum and restitution laws along the line of centres. Follow this carefully in the next section. SWITCH OFF.

31. We are now able to eliminate v1between these equations. Do this yourself, to obtain the stated result, and then switch on the tape again. SWITCH OFF.

32. The next example is for your own working, and should be broken into two stages. Read example 10 carefully and draw your diagrams to show the motion before and after impact. Then find the two component velocities of the first sphere after impact, and check your working below. SWITCH OFF.

33. You should now draw a diagram showing the angles to the line of centres, and the component velocities, before and after impact. You should then be able to complete the question yourself, and check it afterwards. SWITCH OFF.

34. You should now attempt the last question. Read it carefully, and when you have drawn a diagram, switch on the tape again. SWITCH OFF.

35. A ball travelling along the horizontal plane with velocity V, can be considered to have two components of velocity, one perpendicular to the inclined plane, and the other directed up the plane. Use these components in your solution, and when you have finished, check your working overleaf. SWITCH OFF.

36. This completes the work on extended motion in two dimensions. Would you please rewind the tape before you remove it from the machine. Thank you!

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

BOOKLET 6 to be used with TAPE 6

"Variable Forces"

Read the instructions on page 1 thoroughly.

(Omitted for convenience.)

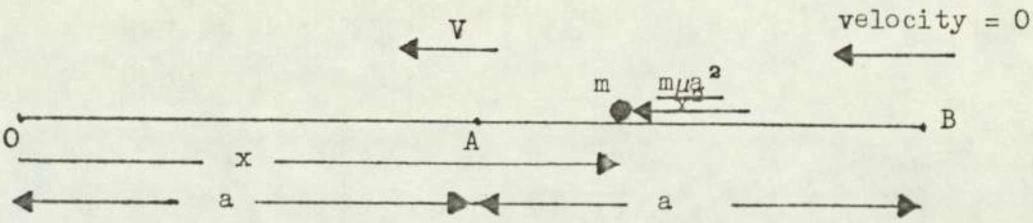
EXAMPLE 1.

A particle of mass m is free to move in a straight line, under the action of force F(x), which is always directed towards the origin.

If $F(x) = \frac{m\mu a^2}{x^2}$ for $x \geq a$

$= \frac{m\mu x}{a}$ for $x < a$

Prove that the particle will reach the origin with velocity $\sqrt{2\mu a}$, when it starts from rest at x = 2a. (μ and a are both constants).



When you have studied the first section of the motion,

SWITCH ON.

Use 'F = ma' from B to A.

$$m\ddot{x} = -\frac{m\mu a^2}{x^2}$$

$$\frac{v\,dv}{dx} = -\frac{\mu a^2}{x^2}, \text{ giving } v\,dv = -\frac{\mu a^2}{x^2}\,dx$$

$$\frac{v^2}{2} = \frac{\mu a^2}{x} + C$$

As v = 0 when x = 2a, $\quad C = \frac{-\mu a}{2}$

$$\frac{v^2}{2} = \frac{\mu a^2}{x} - \frac{\mu a}{2}$$

At A, x = a and $\dot{x} = v = -V$

Hence $V^2 = \mu a$ with $V = \sqrt{\mu a}$

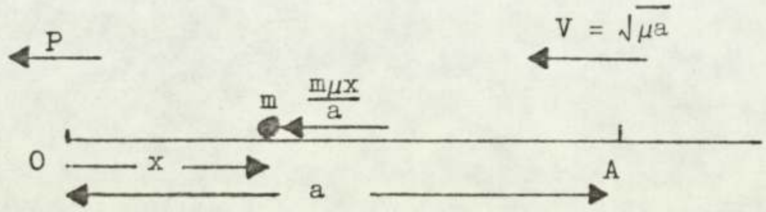
SWITCH ON

$$\int_{o}^{-V} v\,dv = -\mu a^2 \int_{2a}^{a} \frac{dx}{x^2}$$

$$\left[\frac{v^2}{2}\right]_{o}^{-V} = \mu a^2 \left[\frac{1}{x}\right]_{2a}^{a}$$

$$\frac{V^2}{2} = \mu a^2 \left[\frac{1}{a} - \frac{1}{2a}\right] = \frac{\mu a}{2} \qquad V = \sqrt{\mu a}$$

SWITCH ON.



$$\frac{mdv}{dx} = -\frac{m\mu x}{a} \quad , \quad vdv = -\frac{\mu}{a}\, x\, dx$$

$$\int_{-V}^{-P} v\, dv = -\frac{\mu}{a} \int_{a}^{o} x\, dx.$$

$$\frac{1}{2}\left[P^2 - \mu a\right] = -\frac{\mu}{2a}\left[0 - a^2\right] = \frac{\mu a}{2}$$

and $P^2 = 2\mu a$ with $P = \sqrt{2\mu a}$.

SWITCH ON.

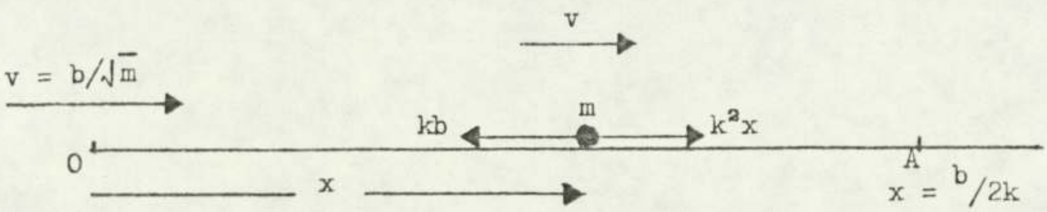
EXAMPLE 2.

A particle of mass m, moves along a straight line away from the origin O, under the action of a force k^2x, where x is the distance from O, against a constant resistance kb. If the velocity at O was ${}^{b}/\sqrt{m}$, find

i) the velocity as a function of distance,

ii) the time it takes to travel a distance ${}^{b}/2k$ from O.

The diagram is on the next page.



When you are ready,

SWITCH ON.

Use 'F = ma' from O to A.

$$m\ddot{x} = k^2x - kb \qquad \ldots \qquad (1)$$

$$\frac{v\,dv}{dx} = \frac{k^2}{m}x - \frac{kb}{m}$$

$$\frac{v^2}{2} = \frac{k^2x^2}{2m} - \frac{kbx}{m} + C$$

when $x = 0$, $v = b/\sqrt{m}$ $\qquad \therefore\ C = \dfrac{b^2}{2m}$

Hence $\dfrac{v^2}{2} = \dfrac{k^2x^2 - 2kbx + b^2}{2m}$

i.e. $v^2 = \frac{1}{m}(kx - b)^2$, which can also be written $= \frac{1}{m}(b-kx)^2$

So $v = 0$ when $x = \frac{b}{k}$, and this displacement also gives zero acceleration (see (1)). Hence the particle comes to permanent rest at $x = \frac{b}{k}$. So throughout the motion $x \leqslant \frac{b}{k}$, and the better form for v^2 is $v^2 = \frac{1}{m}(b-kx)^2$ i.e. $v = \dfrac{b-kx}{\sqrt{m}}$.

SWITCH ON.

---

i.e. $v = \dfrac{dx}{dt} = \dfrac{b-kx}{\sqrt{m}}$

$$\frac{dx}{b-kx} = \frac{dt}{\sqrt{m}}$$

$$\frac{1}{\sqrt{m}}\int_0^T dt = \int_0^{b/2k} \frac{dx}{(b-kx)} = -\frac{1}{k}\Big[\ell n(b-kx)\Big]_0^{b/2k}$$

and $T = \dfrac{\sqrt{m}}{k}\,\ell n(2)$

SWITCH ON.

EXAMPLE 3.

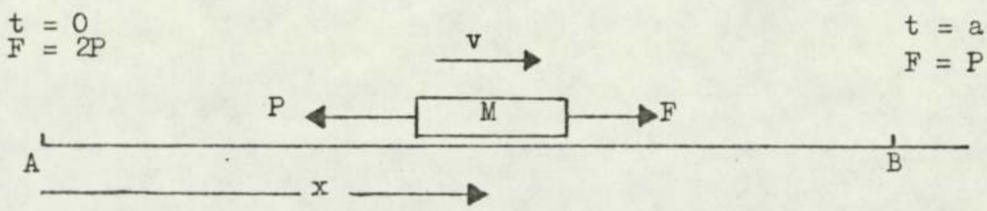
A car, of mass M, starts from rest and moves in a straight line against a constant resistance P. The motive force decreases linearly from 2P, initially, to P at the end of 'a' seconds. At time t, with $t < a$, find,

i) the motive force,

ii) the velocity,

iii) the power developed by the engine, and show that the maximum power is $\dfrac{16aP^2}{27M}$

Do not read this solution until you have finished your own.



i) Motive force $= \left(2P - \dfrac{Pt}{a}\right)$ after t seconds.

ii) Use 'F = ma' from A to B.

$$M\ddot{x} = \left(2P - \frac{Pt}{a}\right) - P$$

$$\frac{dv}{dt} = \frac{P}{M}(1 - {}^{t}/a)$$

$$v = \frac{P}{M}(t - {}^{t^2}/2a) + C$$

when $t = 0$, $v = 0$ $\therefore$ $C = 0$

$$v = \frac{Pt}{M}(1 - {}^{t}/2a)$$

iii) H = Power = Force × velocity

$$= \frac{2P^2t}{M}(1 - {}^{t}/2a)^2$$

Continued on next page.

For maximum power $\frac{dH}{dt} = 0$

$$\frac{dH}{dt} = \frac{2P^2}{M}\left(1 - \frac{t}{2a}\right)\left(1 - \frac{3t}{2a}\right)$$

$$\frac{dH}{dt} = 0 \text{ when } t = 2a \text{ or } t = \frac{2a}{3}$$

$t = 2a$ is outside the permitted range,

so $t = \frac{2}{3}a$ should give $\frac{d^2H}{dt^2}$ negative.

$$\frac{d^2H}{dt^2} = \frac{2P^2}{M}\left[-\frac{1}{2a}\left(1 - \frac{3t}{2a}\right) - \frac{3}{2a}\left(1 - \frac{t}{2a}\right)\right]$$

when $t = \frac{2a}{3}$ $\quad \frac{d^2H}{dt^2} = \frac{2P^2}{M}\left[-\frac{3}{2a}(1 - \frac{1}{3})\right]$ i.e. negative

Hence $H_{MAX} = \frac{16aP^2}{27M}$

SWITCH ON.

EXAMPLE 4.

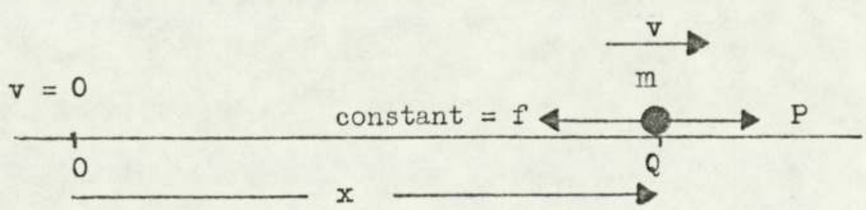
An engine of mass m, works at a constant power h, and moves in a straight line against a resistance f. Prove that if f is constant, the time taken to generate a velocity V, from rest, is

$$\frac{mh}{f^2}\,\ell n\left(\frac{h}{h - Vf}\right) - \frac{mV}{f}$$

and that if f is proportional to v, the time is

$$\frac{mV}{2f_1}\,\ell n\left(\frac{h}{h - Vf_1}\right)$$

where f_1 is the value of f, when v = V



SWITCH ON.

$$\text{Power} = h = Pv \quad \text{i.e. } P = \frac{h}{v}$$

Use 'F = ma' at Q

$$m\ddot{x} = \frac{h}{v} - f$$

$$m\,\frac{dv}{dt} = \frac{h-fv}{v}$$

$$\frac{v\,dv}{h - fv} = \frac{dt}{m}$$

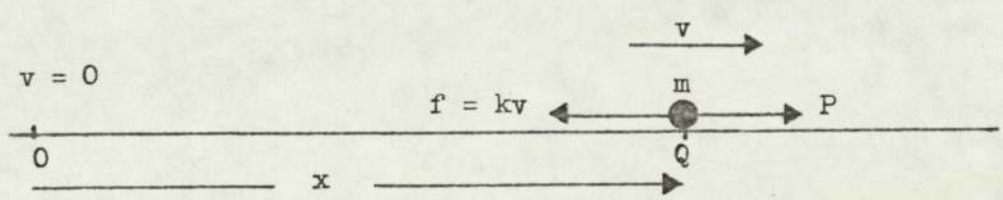
Now $\frac{v}{h-fv} = -\frac{1}{f}\left(\frac{h-fv-h}{h-fv}\right) = -\frac{1}{f}\left(1 - \frac{h}{h-fv}\right)$

Hence $\frac{dt}{m} = -\frac{1}{f}\left(1 - \frac{h}{h-fv}\right)dv$

$$\frac{1}{m}\int_0^T dt = -\frac{1}{f}\int_0^V \left(1 - \frac{h}{h-fv}\right)dv$$

$$\text{So } T = \frac{mh}{f^2}\,\ln\left(\frac{h}{h-fV}\right) - \frac{mV}{f}$$

SWITCH ON.



f = kv, where k is the constant of proportionality.

Use 'F = ma' at Q

$$m\ddot{x} = m\,\frac{dv}{dt} = \frac{h}{v} - kv = \frac{h-kv^2}{v}$$

$$\frac{v\,dv}{h-kv^2} = \frac{dt}{m}$$

$$\int_0^T dt = \int_0^V \frac{mv\,dv}{h-kv^2} = \frac{m}{2k}\,\ln\left(\frac{h}{h-kV^2}\right)$$

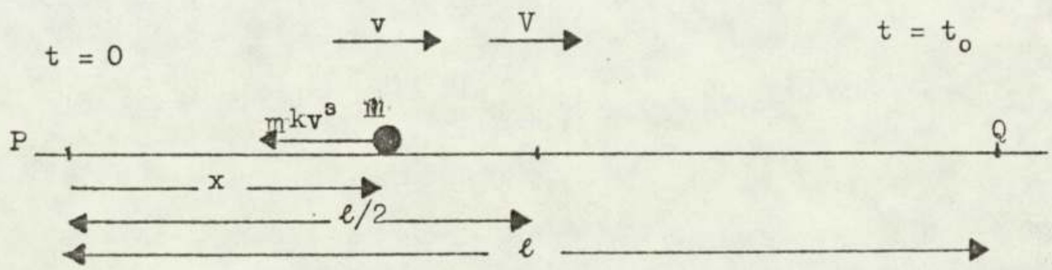
Use $f_1 = kV$ to eliminate k.

SWITCH ON.

EXAMPLE 5.

A particle is projected from point P towards Q, along a horizontal straight line, in a medium in which the resistance varies as the cube of the velocity. The particle takes time t_o to travel from P to Q, which is a distance ℓ. Prove that the velocity at the middle point of PQ is ℓ/t_o.

Do not look at the given solution until you have completed your own.



Let the initial velocity at P be u.

Use 'F = ma' from P to Q.

$$\frac{v\,dv}{dx} = \ddot{x} = -kv^3$$

$$-\frac{dv}{v^2} = k\,dx$$

$$\frac{1}{v} + C = kx$$

when $x = 0$, $v = u$, i.e. $C = -\frac{1}{u}$

i.e. $kx = \frac{1}{v} - \frac{1}{u}$

$$v = \frac{u}{ukx + 1} \qquad \ldots \quad (1)$$

when $x = \frac{\ell}{2}$, $V = \frac{2u}{uk\ell + 2} \qquad \ldots \quad (2)$

From (1) $v = \frac{dx}{dt} = \frac{u}{ukx+1}$

$$\int_0^{\ell} (ukx+1)dx = u\int_0^{t_o} dt.$$

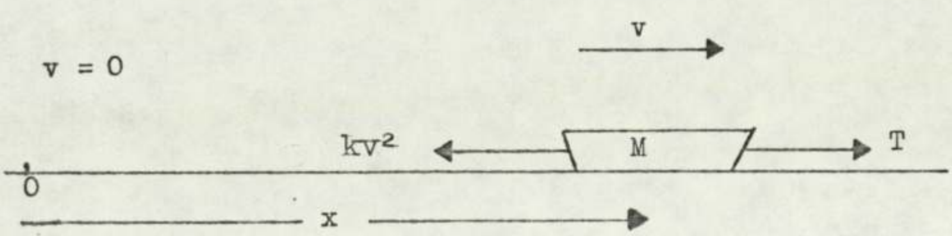
$$\frac{uk\ell^2}{2} + \ell = ut_o = \frac{\ell}{2}(uk\ell+2)$$

$$\frac{\ell}{t_o} = \frac{2u}{uk\ell+2} = V \text{ from } (2).$$

SWITCH ON.

EXAMPLE 6.

The power required to propel a steamer of mass M at the maximum speed V, is H. If the resistance is proportional to the square of the speed, and the engine exerts a constant thrust at all speeds, show that the time taken from rest, to acquire a velocity p is $\frac{MV^2}{2H}\,\ell n\left(\frac{V+p}{V-p}\right)$



Let the engine thrust at all speeds be T. This can be calculated from the power relationship when the speed is maximum.

Power = (thrust)(velocity)

H = TV i.e. $T = \frac{H}{V}$

When you have absorbed this information,

SWITCH ON.

For zero acceleration

$$kV^2 = T = \frac{H}{V} \qquad \text{i.e. } k = \frac{H}{V^3}$$

Use 'F = ma' to give

$$M\ddot{x} = M\frac{dv}{dt} = T - kv^2 = k(V^2 - v^2)$$

$$\frac{k\,dt}{M} = \frac{dv}{V^2-v^2} = \frac{1}{2V}\left[\frac{1}{V+v} + \frac{1}{V-v}\right] dv.$$

Let the steamer reach the velocity p in time t_1.

$$\frac{kt_1}{M} = \frac{1}{2V}\left[\ell n \frac{(V+v)}{(V-v)}\right]_0^p = \frac{1}{2V}\,\ell n\left(\frac{V+p}{V-p}\right)$$

Eliminate k, to obtain the given result.

SWITCH ON.

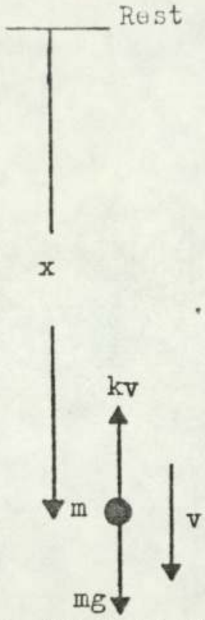
EXAMPLE 7.

A particle of mass m, falls from rest under gravity in a medium whose resistance is proportional to the velocity. If V is the terminal velocity in the medium, show that,

i) the particle is moving with velocity V/2 after time $\frac{V}{g}\,\ell n\, 2$

ii) the distance moved in this time is $V^2(2\ell n 2-1)/2g$.

The diagram is overleaf.



'F = ma' gives

$m\ddot{x} = mg - kv$

when $\ddot{x} = 0$, for terminal velocity.

then $v = V = \frac{mg}{k}$

When you have studied this,

SWITCH ON.

Use 'F = ma' for the motion.

$$\frac{m\,dv}{dt} = mg - kv = mg - \frac{mg}{V}v = \frac{mg}{V}(V-v)$$

$$\frac{dv}{(V-v)} = \frac{g}{V}\,dt$$

$$\int_0^{V/2} \frac{dv}{(V-v)} = \frac{g}{V}\int_0^T dt = -\Big[\ell n(V-v)\Big]_0^{V/2}$$

$$\frac{g}{V}T = \ell n\left(\frac{V}{V-V/2}\right) \quad \text{i.e. } T = \frac{V}{g}\,\ell n(2)$$

SWITCH ON.

Use $mv\frac{dv}{dx} = \frac{mg}{V}(V-v)$

$$\frac{v.dv}{(V-v)} = \frac{g}{V}.dx$$

Now $\frac{v}{(V-v)} = -\left(\frac{V-v-V}{V-v}\right) = -\left(1 - \frac{V}{(V-v)}\right)$

Hence $-\int_0^{V/2}\left(1 - \frac{V}{(V-v)}\right)dv = \frac{g}{V}\int_0^D dx$

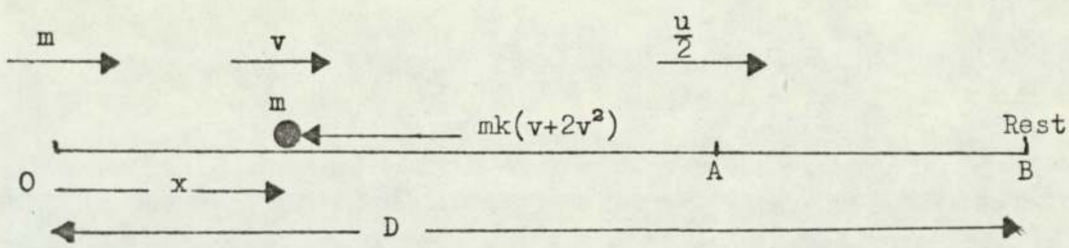
and $D = V^2(2\,\ell n2 - 1)/2g.$

SWITCH ON.

EXAMPLE 8.

A particle of mass m moves horizontally through a medium which offers a resistance of $mk(v+2v^2)$, where k is constant. If this is the only force acting, and the initial velocity is u, show that the particle will come to rest after travelling a distance $\frac{1}{2k}\ \ell n(1+2u)$, and that the velocity will be reduced to $^u/2$, after a time $\frac{1}{k}\ \ell n\left(\frac{2(u+1)}{2u+1}\right)$.

Do not look at the solution yet.



Use 'F = ma' from O to B.

$$m\ddot{x} = -mk(v+2v^2) = mv\frac{dv}{dx}$$

$$\int_u^o \frac{dv}{1+2v} = -k\int_o^D dx$$

$$-kD = \tfrac{1}{2}\Big[\ell n(1+2v)\Big]_u^o = \tfrac{1}{2}\Big[-\ell n(1+2u)\Big]$$

hence D.

Use 'F = ma' again.

$$\frac{dv}{dt} = -k(v+2v^2)$$

$$\int_u^{u/2} \frac{dv}{v(1+2v)} = -k\int_o^T dt$$

$$\text{L.H.S.} = \int_u^{u/2}\frac{dv}{v} - \int_u^{u/2}\frac{2dv}{1+2v} = \left[\ell n\left(\frac{v}{1+2v}\right)\right]_u^{\frac{u}{2}}$$

hence T.

153.

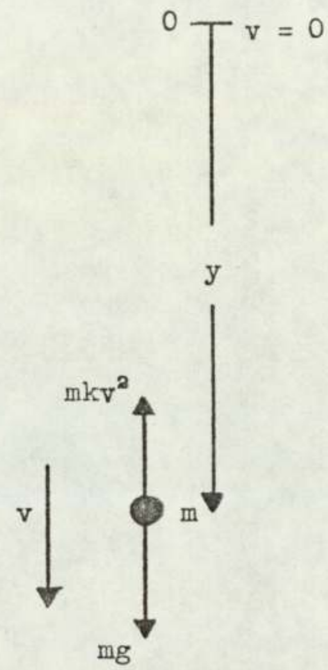
SWITCH ON.

EXAMPLE 9.

A particle of mass m, drops from rest under gravity and is acted upon by a resistance of mkv^2 where v is the velocity and k is a constant. Prove that the distance fallen in time t is

$$\frac{1}{k}\ln\left[\cosh t\sqrt{gK}\right]$$

If k is small, this formula can be simplified by using the Maclaurin expansion for $\ln(\cosh x)$. Show that, in this case, the distance fallen is approximately $\frac{gt^2}{2}\left(1 - \frac{kgt^2}{6}\right)$



When you are ready to proceed, SWITCH ON.

Use 'F = ma' for the descent from 0.

$$m\ddot{y} = m\frac{dv}{dt} = mg - mkv^2$$

$$\frac{dv}{g-kv^2} = dt.$$

Let the terminal velocity be V, then $g - kV^2 = 0$

i.e. $V^2 = \frac{g}{k}$ hence $dt = \frac{dv}{k(V^2-v^2)}$

$$t = \frac{1}{kV}\tanh^{-1}\left(\frac{v}{V}\right) + C$$

when t = 0, v = 0 hence C = 0.

SWITCH ON.

$$kVt = \tanh^{-1}\left(\frac{v}{V}\right)$$

$$\frac{v}{V} = \tanh(kVt)$$

$$v = \frac{dy}{dt} = V\tanh(kVt)$$

$$y = \frac{1}{k}\ln\cosh(kVt) + K$$

when $t = 0$, $y = 0$, hence $K = 0$

Substitute for V, $y = \frac{1}{k}\ln\left[\cosh(t\sqrt{gk})\right]$

SWITCH ON.

Maclaurin expansion

$$\text{let } f(x) = \ln(\cosh x)$$

$$f(x) = f(0) + x f^{i}(0) + \frac{x^2}{2!}f^{ii}(0) + \frac{x^3}{3!}f^{iii}(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

By repeated differentiation

$$f^{i}(x) = \tanh x$$

$$f^{ii}(x) = \operatorname{sech}^2 x$$

$$f^{iii}(x) = -2\operatorname{sech}^2 x\,.\tanh x$$

$$f^{iv}(x) = 2\operatorname{sech}^2 x[2\tanh^2 x - \operatorname{sech}^2 x]$$

$$f(x) = \frac{x^2}{2}\left(1 - \frac{x^2}{6}\right)$$

$$\text{Hence } y \simeq \frac{gt^2}{2}\left(1 - \frac{kgt^2}{6}\right)$$

SWITCH ON.

EXAMPLE 10.

A particle moving in a vertical plane under gravity, was projected with velocity V, at an inclination α to the horizontal in a uniform medium in which the resistance varies as the velocity. If k is the resistance per unit mass, when the body is moving with unit velocity, show that,

i) The altitude is maximum at time

$$\frac{1}{k}\ \ln\left(1 + \frac{kV\sin\alpha}{g}\right)$$

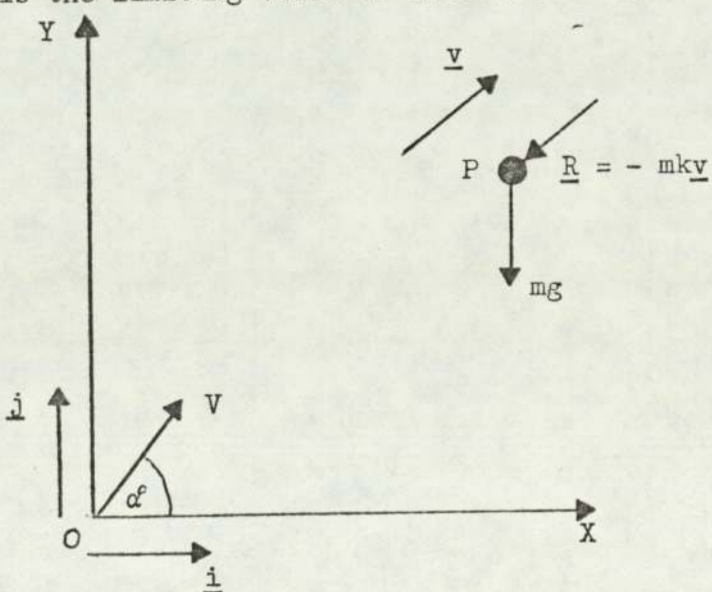
ii) maximum height = $H = \dfrac{V\sin\alpha}{k} - \dfrac{g}{k^2}\ \ln\left[1 + \dfrac{kV\sin\alpha}{g}\right]$

iii) $\lim\limits_{k \to 0} H = \dfrac{V^2\sin^2\alpha}{2g}$

iv) the horizontal distance covered at this time is

$$\frac{V^2\sin\alpha\ \cos\alpha}{g + kV\ \sin\alpha}$$

v) What is the limiting value of this as $k \to 0$?



Particle at P(x,y) at time t, with $\overline{OP} = \underline{r} = \underline{i}x + \underline{j}y$

When you are ready, SWITCH ON.

From the diagram, $m\ddot{\underline{r}} = -mg\underline{j} - mk\underline{v}$

$$\underline{i}\ddot{x} + \underline{j}\ddot{y} = -g\underline{j} - k(\underline{i}\dot{x} + \underline{j}\dot{y})$$

equating $\underline{j}$ components, $\ddot{y} + k\dot{y} = -g$ ... (1)

equating $\underline{i}$ components, $\ddot{x} + k\dot{x} = 0$ ... (2)

SWITCH ON.

Rewrite (1) as $\ddot{y} = -(g+k\dot{y})$ with $\dot{y} = v$

$$\frac{dv}{dt} = -(g+kv)$$

$$-\frac{dv}{(g+kv)} = dt$$

$$t = \frac{-1}{k}\left[\ell n(g+kv)\right] + C$$

When $t = 0$, $v = V\sin\alpha$ $\quad C = \frac{1}{k}\ell n(g+kV\sin\alpha)$

thus $t = \frac{1}{k}\ell n\left(\frac{g+kV\sin\alpha}{g+kv}\right)$ ... (3)

At the maximum height, $v = 0$ and $t = T$

$$\text{i.e. } T = \frac{1}{k}\ell n\left(1 + \frac{kV\sin\alpha}{g}\right)$$

SWITCH ON.

Rewrite (3) as $\frac{dy}{dt} = v = \frac{(g+kV\sin\alpha)e^{-kt} - g}{k}$

$$\int_o^H dy = \frac{1}{k}\int_o^T (g+kV\sin\alpha)e^{-kt}dt - \frac{g}{k}\int_o^T dt$$

$$H = \frac{1}{k^2}(g+kV\sin\alpha)(1-e^{-kT}) - \frac{gT}{k}$$

Substitute for T and e^{-kT} to give

$$H = \frac{V\sin\alpha}{k} - \frac{g}{k^2}\ell n\left[1 + \frac{kV\sin\alpha}{g}\right]$$

SWITCH ON.

$$\lim_{k \to o}\left\{\frac{V\sin\alpha}{k} - \frac{g}{k^2}\ell n\left[1 + \frac{kV\sin\alpha}{g}\right]\right\}$$

$$\lim_{k \to o}\left\{\frac{V\sin\alpha}{k} - \frac{g}{k^2}\left(\frac{kV\sin\alpha}{g} - \frac{k^2V^2\sin^2\alpha}{2g^2} + \frac{k^3V^3\sin^3\alpha}{3g^3} + \dots\right)\right\}$$

$$\lim_{k \to o}\left\{\frac{V\sin\alpha}{k} - \frac{V\sin\alpha}{k} + \frac{V^2\sin^2\alpha}{2g} - \frac{kV^3\sin^3\alpha}{3g^2} + \dots\right\}$$

$$\lim_{k \to o} H = \frac{V^2\sin^2\alpha}{2g}$$

This is the result for a projectile in a non resisting medium

SWITCH ON.

$$\ddot{x} + k\dot{x} = 0 \text{ can be written as}$$

$$D(D+k)x = 0 \text{ where } D \equiv \frac{d}{dt}$$

and $x = Ae^{-kt} + B$

when $t = 0$, $x = 0$, $A+B = 0$, $B = - A$

Hence $x = A(e^{-kt} - 1)$

To find A, use $\frac{dx}{dt} = V \cos\alpha$ at $t = 0$

$$\frac{dx}{dt} = - Ake^{-kt}$$

thus $A = - \frac{V\cos\alpha}{k}$

$$x = \frac{V\cos\alpha(1-e^{-kt})}{k}$$

SWITCH ON

Substitute for e^{-kT}, which you have already found, to give the distance

$$\frac{V^2\sin\alpha.\cos\alpha}{g + kV\sin\alpha}$$

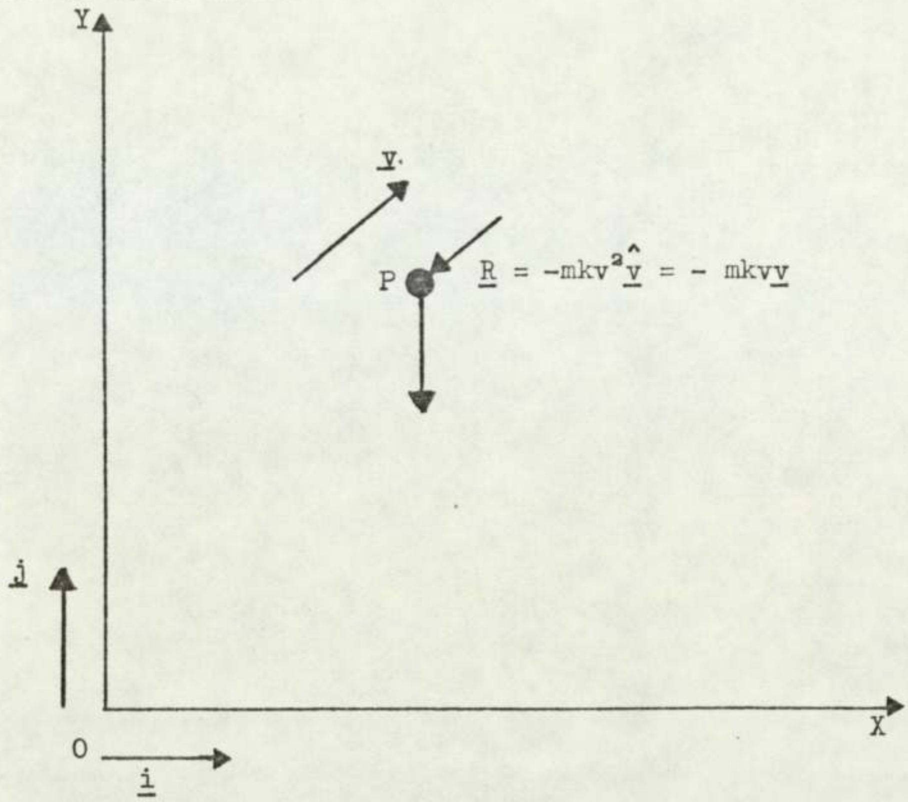
The denominator of this is g as $k \to o$.

The distance is then $\frac{V^2\sin\alpha.\cos\alpha}{g}$,

which is half the horizontal range without resistance.

SWITCH ON.

Projectile with resistance proportional to kv^2



A particle at P(x,y) at time t with $\overline{OP} = \underline{r} = \underline{i}x + \underline{j}y$

$$m\ddot{\underline{r}} = -mg\underline{j} - mkv\underline{v}$$

$$\underline{i}\,\ddot{x} + \underline{j}\ddot{y} = -g\underline{j} - kv(\underline{i}\dot{x} + \underline{j}\dot{y})$$

with $$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

Compare the coefficients of $\underline{i}$ and $\underline{j}$ to give

$$\ddot{x} = -k\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\ddot{y} = -g - k\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}$$

SWITCH ON.

SCRIPT FOR TAPE 6

VARIABLE FORCES

1. This is the tape to be used with Booklet 6 of 'A Revision Course in Particle Dynamics' by Kenneth Jackson, and is concerned with Variable Forces. The problems will provide you with much useful revision of calculus, as you will need to integrate the equation of motion to obtain other equations. Reading questions carefully will help you to decide which forms of the acceleration you require. Read through the first question and study the diagram. We shall work through this example together. SWITCH OFF.

2. As the particle moves to the origin under two separate forces, the calculations must be separated. Consider the motion from B to A. Using F = Ma, with the acceleration in the form Vdy by dx gives a connection between velocity and displacement. This may be integrated using a constant of integration, to find the velocity at A. Carry out this process, and check this below. SWITCH OFF.

3. The introduction of the constant of integration can be avoided by integrating between limits, the velocity changing from 0 to -V, as x changes from 2a to a. Do not be confused by the negative sign on the velocity, which simply indicates that it is directed towards O. Follow

this alternative integration in the next section, carefully. SWITCH OFF.

4. Remember that the limits for the displacement integration must correspond to the position of the particle on the line. That is, the particle is distant 2a from the origin when it is at rest, and this becomes the lower limit of the integral. To finish this question yourself, you will need another diagram. Draw this, using P for the velocity of the particle at O. Use a definite integral to find this, and check your working in the next section. SWITCH OFF.

5. Integration between limits is usually shorter, and should be used wherever convenient. Now read the second question, which introduces a resistance as well as a force. Study the diagram carefully, noting particularly the directions of the forces. SWITCH OFF

6. The diagram shows the resistance in opposition to the driving force, and this must be allowed for in the equation of motion. Notice that part one again requires a velocity/displacement relationship, after integration. As the question asks for velocity as a function of position, it is better to use a constant in the integration. You should now answer part one of this question, and when you have finished, check your answer. SWITCH OFF.

7. For the second part of the question, remember that velocity is dx/dt. Separate the variables and complete the question by using definite integration. Check this below afterwards. SWITCH OFF.

8. The next question also involves a resistance against a driving force, and should be tried by your own efforts. Read it carefully, and summarize the given information on a diagram. Notice that the question is asking for velocity as a function of time, which means that the acceleration should be taken as dv/dt. Remember that power equals force x velocity. Check your answer afterwards in the next section. SWITCH OFF.

9. As well as having constant resistances, some motions involve a resistance proportional to velocity. Read example four, which contrasts the difference between constant and variable resistance. Study the diagram for the first part carefully. SWITCH OFF.

10. In this motion with constant power, remember that it is the product of the thrust P, and velocity which is constant. As we require the time for a given velocity, V, we may use a definite integral. Complete the first part of the question, and then check it overleaf. SWITCH OFF.

11. For the second part of the question, we have the resistance proportional to the velocity. Let the constant of proportionality be small k. Notice that this does not appear in the answer and must be eliminated. Now complete the question, using a definite integral, and then check your working. SWITCH OFF.

12. Now read example 5, which is concerned with resistive force only, and draw a detailed diagram. You will first need the acceleration in the velocity displace ment form, and then the velocity as dx/dt. You can then

combine the two equations to find the given relationship. When you have finished, check your working below. SWITCH OFF.

14. Notice that when the steamer has maximum speed the acceleration is zero, and the thrust is equalled by the resistance. This gives a relationship for the constant of proportionality in the resistance kv^2. Solve the problem by using the acceleration as dv/dt, and check your working afterwards. SWITCH OFF.

15. Falling bodies in a resistive medium also reach a maximum speed when the acceleration is zero. This maximum speed is called the Terminal Velocity. Example 7. is concerned with this, and we shall work through part of the question together. Read it, and look at the diagram closely. SWITCH OFF.

16. In this example, we can simplify the work of integration by using the terminal velocity V, to substitute for k in the equation of motion. The resulting expression is then easily integrated between limits. Follow this working of part one carefully, in the next section. SWITCH OFF.

17. Notice in this section how compact the working was. You should now attempt the second part yourself. Start with the same equation of motion, but remember that you are looking for a distance and velocity relation. Again, integrate between limits. Check your working below when you have finished. SWITCH OFF.

18. Attempt question 8 by yourself. It is about

combined resistance but do not let this confuse you. The first definite integral is a straightforward velocity/ distance one, and the second will require the use of partial fractions. When you have finished, check your working below. SWITCH OFF.

19. Question 9 concerns a vertical motion with the resistance proportional to the square of the velocity. Read it carefully and study the diagram. SWITCH OFF.

20. Write the equation for the acceleration, remembering that weight is a vertical force. This mass must also have a terminal velocity, which you should use for convenience. Us dv/dt, and then separate the variables ready for integration. When you have recognised the type of integral, adjust the constants as needed, integrate, and find the constant of integration. Check your solution in the section below. SWITCH OFF.

21. It only remains to obtain the distance, by integrating the velocity. Rearrange your previous answer to give the velocity and you should then be able to complete the first section of the question. Check this afterwards, overleaf. SWITCH OFF.

22. Applied mathematics is always requiring special results and techniques from mathematical methods, and in this case we need the Maclaurin Expansion for log cosh x. Derive the first two non-zero terms in this series, and you will be able to finish the question. When you have done this successfully, switch on the tape again. Otherwise, you had better trace your error in the next section

of the booklet. SWITCH OFF.

23. Notice that this last result has two terms, the second one being a multiple of k. Make k = 0, and we have $y = \frac{1}{2}gt^2$, which agrees with the expected result for free fall under gravity.

The tenth question extends this work on resisted motion into two dimensions, and it is convenient to examine the motion in vectors. Read the question carefully, and note the vector directions in the diagram. SWITCH OFF.

24. We can now write the vector equation of motion, and express it in component form, which gives two ordinary differential equations. Look at the next section, where these equations are obtained. SWITCH OFF.

25. To obtain the time to the greatest height, we must use equation 1 for vertical motion. An expression for velocity and time may be found, and we can then use the fact that the vertical velocity is zero at the maximum height. Do this for yourself, and then check your working in the next section. SWITCH OFF.

26. Rearrange equation 3 to give v as the subject. A further integration between limits will give the maximum value of y, using the value of large T already found. Check this afterwards, below. SWITCH OFF.

27. It is always useful in applied mathematics, to consider limiting cases, as this helps to reveal possible errors in the results for more complicated motions. In this case, the log series will be useful. SWITCH OFF.

28. For the final part of this question, we shall have to use equation 2 for horizontal motion. This could be solved by using integral calculus, but it is more quickly dealt with, by treating it as a differential equation. Use the D operator, and when you have recognized the form of the solution, evaluate the two constants, using your initial conditions. Check your solution in the next section afterwards. SWITCH OFF.

29. This is, of course, the value of x for any t, and you require the value for large T. For the limiting value of the expression, consider the effect on the whole denominator, of making k small. Check your working afterwards. SWITCH OFF.

30. It is interesting to consider the same approach with the resistance proportional to the square of the velocity. Study this problem which is illustrated on the next page. SWITCH OFF.

31. These look a particularly nasty pair of equations don't they! In fact, little progress can be made with them, and even if we eliminate the surd, it is not very useful analytically. This is one of those occasions when numerical methods are needed.

This is the ned of the work on 'Variable Forces' Would you please rewind the tape before you remove it from the machine. Thank you!

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

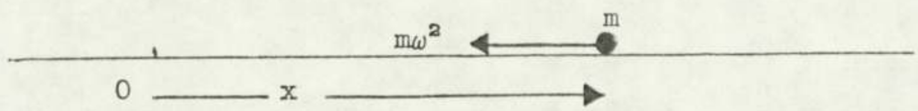
BOOKLET 7 to be used with TAPE 7

"Oscillations"

Read the instructions on page 1 thoroughly.

(Omitted for convenience)

Simple Harmonic Motion.



The equation of motion is $\frac{d^2x}{dt^2} = -\omega^2 x$

The solution of this differential equation is

$$x = A\cos\omega t + B\sin\omega t = C\cos(\omega t+\alpha),$$

where the constants A and B, or C and α are evaluated from known conditions, usually initial conditions.

Period of oscillations $= \frac{2\pi}{\omega}$

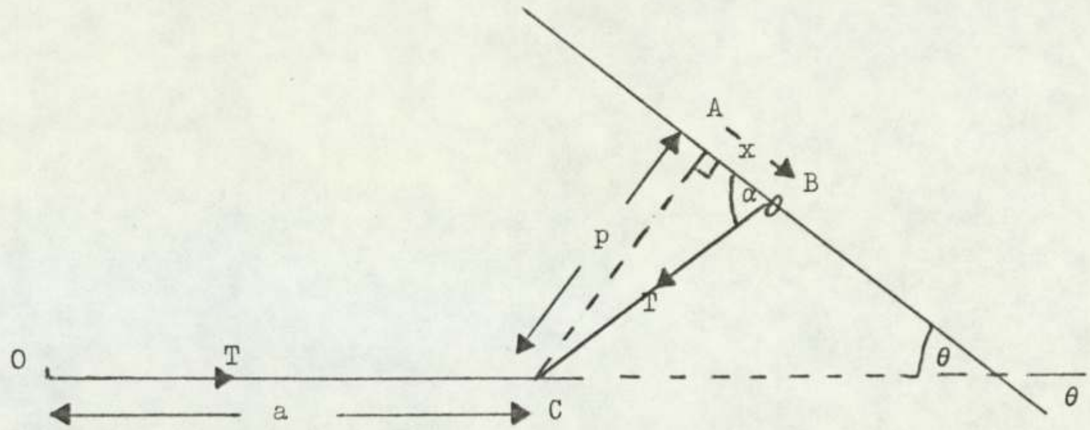
When you are ready,

SWITCH ON

EXAMPLE 1.

An elastic string is fixed at one end to a point O on a horizontal table. It passes through a fixed smooth ring C on the table, which is at a distance 'a' from O. The other end is fixed to a ring B, of mass m, which is free to slide along a smooth horizontal wire on the same level as the table. The inclination of the wire is θ to OC produced, and its perpendicular distance from C is p. If the unstretched length of the string is 'a', find the period of small oscillations of the ring about its equilibrium position and show that it is independent of p and θ. (The modulus of elasticity of the string is λ).

(The diagram is on the next page.)



When you have studied this,

SWITCH ON.

Extension = BC = x secα.

Use '$T = \frac{\lambda e}{\ell}$' , then $T = \frac{\lambda x \sec\alpha}{a}$

Use 'F = ma'

$$m\frac{d^2x}{dt^2} = -T\cos\alpha$$

$$\ddot{x} = \frac{-\lambda x \sec\alpha}{ma}.\cos\alpha = -\left(\frac{\lambda}{ma}\right)x.$$

This is S.H.M. of period $2\pi\sqrt{\frac{ma}{\lambda}}$.

This is independent of p or θ, i.e. the period of oscillation is not affected by the distance, or orientation of the wire from C.

SWITCH ON.

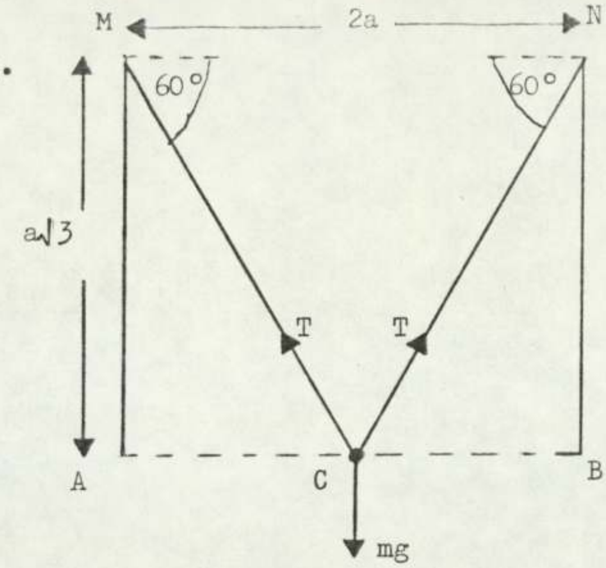
EXAMPLE 2.

A particle of mass m is supported at C by two elastic strings of modulus $\frac{2}{3}$ mg, which pass over smooth pegs M,N, and are attached at their other ends to two fixed points A,B, vertically below M,N and on the same level as C. MN is horizontal and of length 2a, CM and CN are inclined at 30° to the vertical and the whole figure is symmetrical about the vertical through C. If the particle is slightly disturbed in the direction of the vertical, prove that the period of oscillation will be approximately,

$$4\pi\sqrt{\frac{a(6-\sqrt{3})}{11g}}$$

(The diagram is overleaf.)

Equilibrium.



$\lambda = \frac{2}{3}mg$

ΔCMN is equilateral

When you have studied this,

SWITCH ON.

Let L = unstretched length of each string

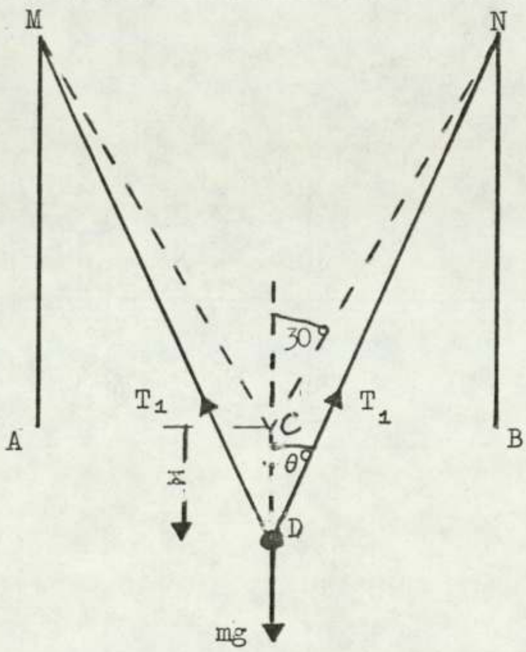
e = extension of each string

From equilibrium, $2T\cos 30^\circ = mg = 2(\frac{2}{3}mg)\frac{e}{L}\cdot\frac{\sqrt{3}}{2}$

And $(L+e) = CN+NB = a(2+\sqrt{3})$

hence $L = 2a$ and $e = a\sqrt{3}$.

SWITCH ON.



When you have checked this diagram,

SWITCH ON.

Using 'F = ma'

$$m\ddot{x} = mg - 2T_1\cos\theta$$

(Continued on the next page.)

By Cosine Rule.

$$DN^2 = x^2 + (2a)^2 - 2.x.2a \cos 150°$$

$$DN = \sqrt{x^2 + 4a^2 + 2\sqrt{3}ax} = \sqrt{4a^2 + 2\sqrt{3}ax} \quad \text{(approx.)}$$

Stretch of each string $= DN + BN - 2a$

$$= [\sqrt{4a^2+2\sqrt{3}ax} - a(2-\sqrt{3})]$$

$$\cos\theta = \frac{DC+NB}{DN} = \frac{(x + a\sqrt{3})}{\sqrt{4a^2+2\sqrt{3}\,ax}}$$

Substitute in the equation of motion

$$m\ddot{x} = mg - 2.\tfrac{2}{3}\,mg\ \frac{[\sqrt{4a^2+2\sqrt{3}ax}-a(2-\sqrt{3})]}{2a}\ \frac{(x+a\sqrt{3})}{\sqrt{4a^2+2\sqrt{3}ax}}$$

$$\ddot{x} = g\left[1 - \tfrac{2}{3}\left(\frac{(x+a\sqrt{3})}{a} - \frac{(2-\sqrt{3})(x+\sqrt{3}a)}{\sqrt{4a^2+2\sqrt{3}ax}}\right)\right]$$

$$\text{Now } \frac{1}{\sqrt{4a^2+2\sqrt{3}ax}} = \frac{1}{2a}\left(1 + \frac{\sqrt{3}x}{2a}\right)^{-\frac{1}{2}} = \frac{1}{2a}\left(1 - \frac{\sqrt{3}x}{4a}\right)$$

by a binomial approximation, neglecting second and higher powers of x.

SWITCH ON.

$$\ddot{x} = g\left\{1 - \tfrac{2}{3}\left[\frac{(x+a\sqrt{3})}{a} - \frac{(2-\sqrt{3})(x+\sqrt{3}a)(4a-\sqrt{3}x)}{8a^2}\right]\right\}$$

$$= \frac{-g(6+\sqrt{3})x}{12a}$$

$$= \frac{-x\ g(6+\sqrt{3})(6-\sqrt{3})}{12a(6-\sqrt{3})}$$

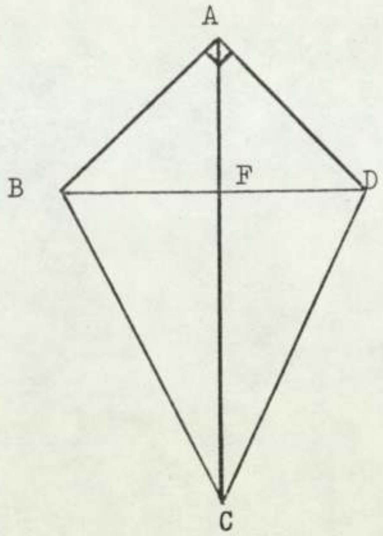
$$= -x\ \frac{g\ \ 11}{4a(6-\sqrt{3})}$$

hence period $= 2\pi\sqrt{\dfrac{4a(6-\sqrt{3})}{11g}}$

$$= 4\pi\sqrt{\frac{a(6-\sqrt{3})}{11g}}$$

SWITCH ON.

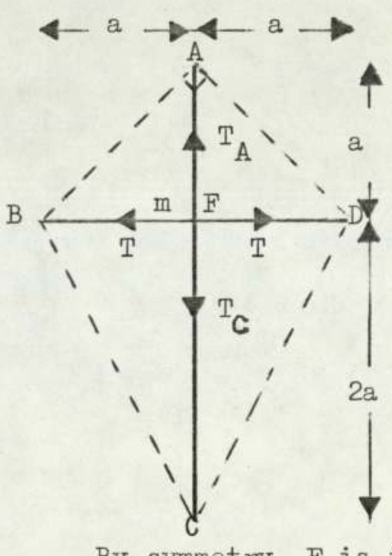
EXAMPLE 3.



A,B, C,D are four points on a smooth horizontal table. $B\hat{A}D = 90°$. $AF = a = BF = FD$. $FC = 2a$. Light elastic strings, each of natural length $\ell(< a,)$ are fastened to A,B and D. A string of length 2ℓ is attached to C. All the other ends of the strings are fastened to a particle of mass m. Confirm that F is the position of equilibrium, and show that the period of small oscillation, if the particle is displaced a small distance along the diagonal AC, and released, is $2\pi\sqrt{\dfrac{2a\ell}{\lambda(7a-4\ell)}}$. The strings do not become slack at any time in the motion, and the common modulus of elasticity is λ.

Do not look below until your two diagrams are completed.

Equilibrium Position.



Use '$T = \dfrac{\lambda e}{\ell}$'

For AF, $T_A = \dfrac{\lambda(a-\ell)}{\ell}$

For FC, $T_C = \dfrac{\lambda(2a-2\ell)}{2\ell}$

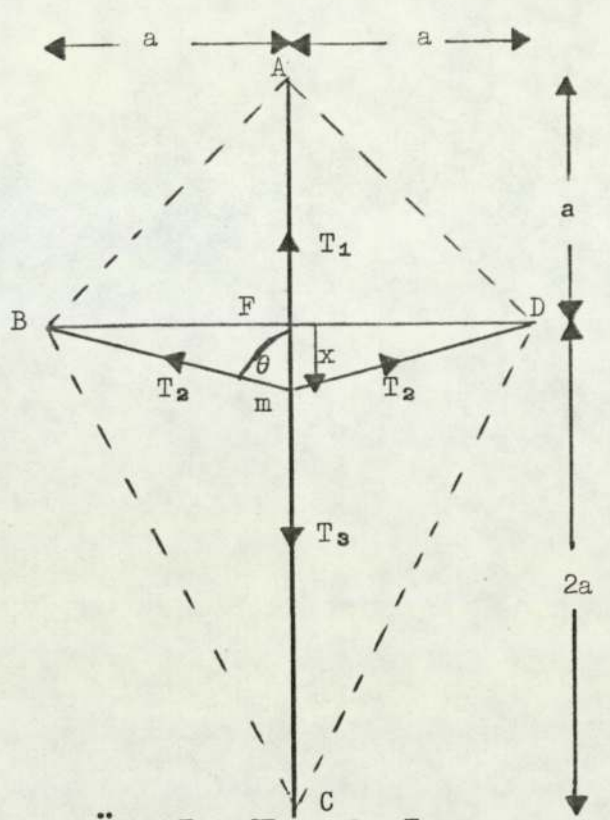
$= T_A$

i.e. F is equilibrium along AC.

By symmetry, F is in equilibrium along B D.

(Continued on the next page.)

Small displacement at time t.



The tensions on both sides of the diagonal AC are equal because of symmetry, and there is no resultant force perpendicular to AC

Use 'F = ma' along AC

$$m\ddot{x} = T_3 - 2T_2\cos\theta - T_1$$

SWITCH ON.

$$\ddot{x} = \frac{\lambda(2a-2\ell-x)}{2\ell} - \frac{2\lambda(\sqrt{a^2+x^2}-\ell)}{\ell}\cdot\frac{x}{\sqrt{a^2+x^2}} - \frac{\lambda(a+x-\ell)}{\ell}$$

$$= \frac{\lambda}{\ell}\left(-\frac{7x}{2} + \frac{2x\ell}{\sqrt{a^2+x^2}}\right) = \frac{\lambda}{\ell}\left(-\frac{7x}{2} + \frac{2\ell x}{a}\right) \text{ approx.}$$

$$= -x\lambda\left(\frac{7a-4\ell}{2a\ell}\right)$$ N.B. $7a - 4\ell = 7(a - \frac{4}{7}\ell)$, which is positive, as $a > \ell$.

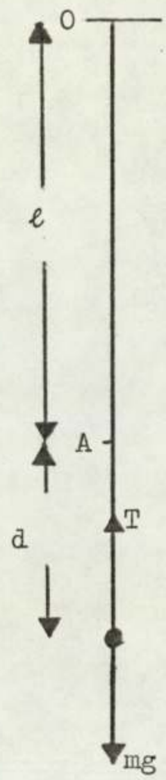
This is S.H.M, with period, $2\pi\sqrt{\dfrac{2a\ell}{\lambda(7a-4\ell)}}$

SWITCH ON.

EXAMPLE 4.

One end of an elastic string is attached to a fixed point, and the other to a particle which hangs in equilibrium, and causes an extension 'd' in the string. The particle is given a vertical velocity $2\sqrt{gd}$ upwards. Prove that the string becomes slack after a time $\frac{\pi}{6}\sqrt{\frac{d}{g}}$, and the particle first reaches its lowest point after a total time $\sqrt{\frac{d}{g}}\left(\frac{5\pi}{6} + 2\sqrt{3}\right)$.

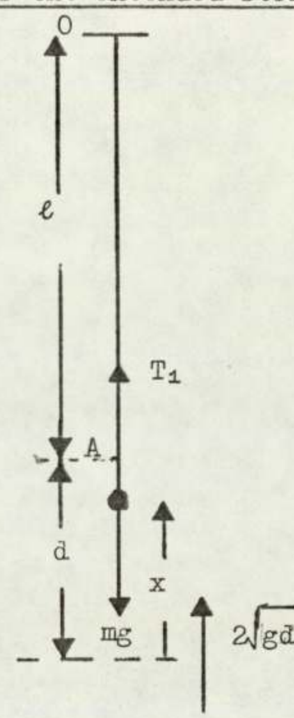
Equilibrium Position



$$mg = T = \frac{\lambda d}{\ell}$$

$$\frac{mg\ell}{d} = \lambda$$

General Displacement for the extended string.



When you understand these diagrams,

SWITCH ON.

$$m\ddot{x} = T_1 - mg$$

$$= \frac{mg\ell}{d} \cdot \frac{(d-x)}{\ell} - mg$$

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{d}\right)x$$

hence, $x = A\cos\sqrt{\frac{g}{d}}\,.t + B\sin\sqrt{\frac{g}{d}}\,.t$

174.

SWITCH ON.

When $t = 0$, $x = 0$, hence $A = 0$

$$\text{i.e. } x = B \sin\sqrt{\frac{g}{d}}.t$$

$$\frac{dx}{dt} = B\sqrt{\frac{g}{d}}.\cos\sqrt{\frac{g}{d}}.t$$

When $t = 0$, $\frac{dx}{dt} = 2\sqrt{gd}$, Hence $B = 2d$

and $x = 2d \sin\sqrt{\frac{g}{d}}.t$

SWITCH ON.

The string becomes slack at A.

When $x = d$, let the time be t_1

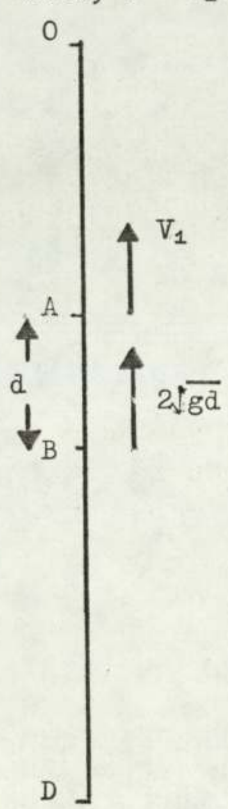
$$d = 2d \sin\sqrt{\frac{g}{d}}.t_1$$

i.e. $\sin\sqrt{\frac{g}{d}}.t_1 = \frac{1}{2} = \sin\frac{\pi}{6}$, Hence $t_1 = \frac{\pi}{6}\sqrt{\frac{d}{g}}$.

SWITCH ON.

$$\frac{dx}{dt} = 2\sqrt{dg}\,.\cos\sqrt{\frac{g}{d}}.t$$

At A, $t = t_1$ and $V_1 = 2\sqrt{dg}\,\cos\frac{\pi}{6} = \sqrt{3gd}$



Hence, there is some remaining energy, and the particle will rise in the air as the string becomes slack. It will continue rising, until its speed is zero, and then fall back to A. The string begins to stretch, and eventually brings the particle momentarily, to rest at its lowest point, D.

Total time = (B to A)+(free flight)+(A to B)+(B to D)

= 2(B to A)+(free flight)+ (B to D).

When you are ready to continue,
175.
SWITCH ON.

Time of free flight $= \dfrac{\text{"}2V\text{"}}{g} = \dfrac{2\sqrt{3gd}}{g} = 2\sqrt{3}\dfrac{\sqrt{d}}{\sqrt{g}}$

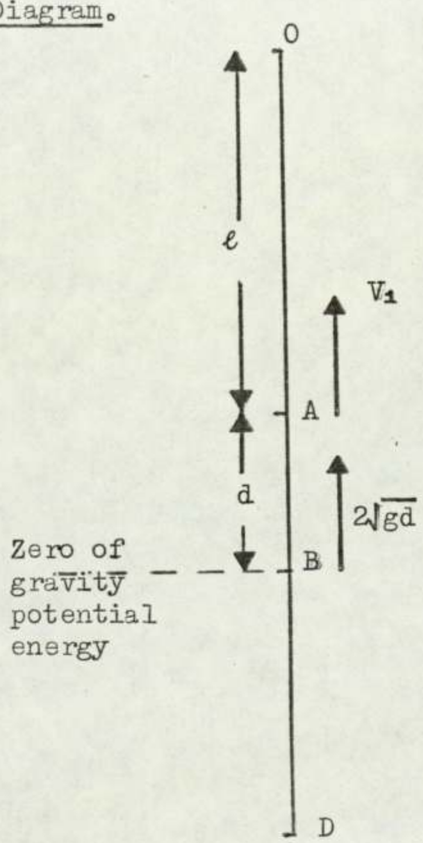
Time from B to D $= \frac{1}{4}$ (period)

$$\text{Total time} = 2\left(\frac{\pi}{6}\frac{\sqrt{d}}{\sqrt{g}}\right) + 2\sqrt{3}\frac{\sqrt{d}}{\sqrt{g}} + \frac{\pi}{2}\frac{\sqrt{d}}{\sqrt{g}}$$

$$= \frac{\sqrt{d}}{\sqrt{g}}\left(\frac{5\pi}{6} + 2\sqrt{3}\right)$$

SWITCH ON.

P.E. Diagram.



Let the horizontal through B, be the zero of gravity potential energy.

The P.E. will consist of two parts, that due to gravity, and that of the string i.e. $P.E_G$ and $P.E_S$ respectively.

$$\left(P.E_S + P.E_G + K.E.\right)_B = \left(P.E_S + P.E_G + K.E.\right)_A$$

$$\left(\frac{\frac{1}{2}mg\ell}{d}\cdot\frac{d^2}{\ell} + 0 + \tfrac{1}{2}m(2\sqrt{gd})^2\right) = \left(0 + mgd + \tfrac{1}{2}mV_1^2\right)$$

$$3gd = V_1^2 \text{ as before.}$$

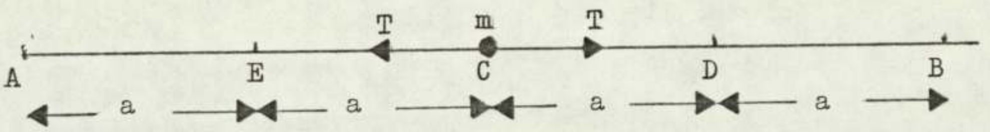
SWITCH ON.

EXAMPLE 5.

A light elastic string of natural length 2a, and modulus 8 mg, is stretched between two fixed points A and B, on a smooth horizontal plane, with AB = 4a. A particle, of mass m, is fastened to the midpoint of the string. Show, that if it is projected from the equilibrium position, towards B, with a velocity $2\sqrt{14ga}$, it will just reach B, in a time,

$$\frac{1}{4}\sqrt{\frac{a}{g}}\left\{\cos^{-1}\left(\frac{\sqrt{5}}{\sqrt{7}}\right) + \sqrt{2}\cos^{-1}\left(\frac{2}{3}\right)\right\}$$

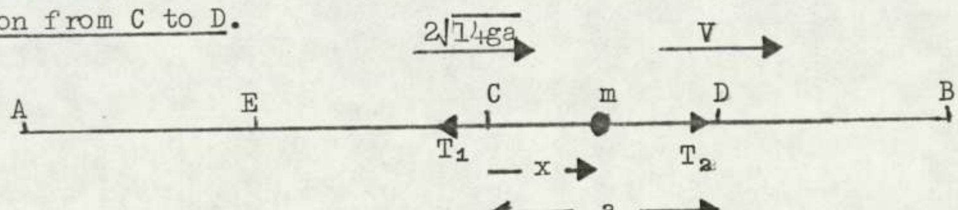
Equilibrium Position (Particle at C with AC = 2a = CB).



The mass is, effectively, dividing the string into two strings, each of natural length a, and modulus 8 mg, fastened at A and B. For the motion from C to D, there is tension in both strings, but from D to B, the right hand string is slack.
When you agree that the motion is in two parts,

SWITCH ON.

Motion from C to D.



Use 'F = ma'

$$m\frac{d^2x}{dt^2} = T_2 - T_1 = \frac{8mg}{a}(a-x) - \frac{8mg}{a}(a+x) = -\frac{16mgx}{a}$$

hence, $x = A\cos 4\sqrt{\frac{g}{a}}.t + B\sin 4\sqrt{\frac{g}{a}}.t$

(Continued on the next page.)

When $t = 0$, $x = 0$, hence $A = 0$.

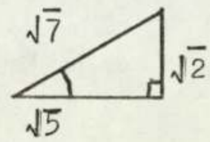
$$\frac{dx}{dt} = 4B\sqrt{\frac{g}{a}}\ \cos 4\sqrt{\frac{g}{a}}\ .t$$

when $t = 0$, $\frac{dx}{dt} = 2\sqrt{14ga}$, hence $B = a\sqrt{\frac{7}{2}}$

$$x = a\sqrt{\frac{7}{2}}\ \sin 4\sqrt{\frac{g}{a}}\ .t$$

Let the particle reach D at time $t = t_1$

then $a = a\sqrt{\frac{7}{2}}\ \sin 4\sqrt{\frac{g}{a}}\ .\ t_1$

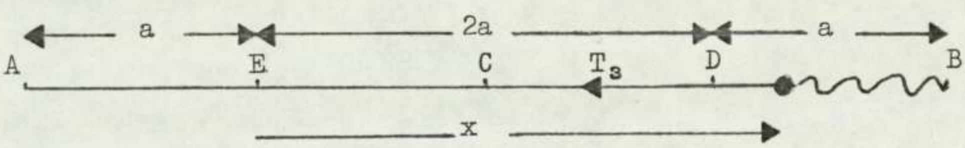
$t_1 = \frac{1}{4}\sqrt{\frac{a}{g}}\ \sin^{-1}\left(\sqrt{\frac{2}{7}}\right) = \frac{1}{4}\left(\sqrt{\frac{a}{g}}\ \cos^{-1}\left(\sqrt{\frac{5}{7}}\right)\right)$ []

At D, time t_1

$$V = 4a\sqrt{\frac{7g}{2a}}\ \cos 4\sqrt{\frac{g}{a}}.\ t_1 = 2\sqrt{10ga}$$

SWITCH ON.

Motion from D to B.



Use 'F = ma'

$$m\frac{d^2x}{dt^2} = -T_3 = -8\frac{mg}{a}x$$

$$x = C\cos 2\sqrt{\frac{2g}{a}}\ .t + D\sin 2\sqrt{\frac{2g}{a}}\ .t$$

For convenience, measure time from position D, with $t = 0$ when

$x = 2a$, then $C = 2a$.

and $x = 2a\cos 2\sqrt{\frac{2g}{a}}\ .t + D\sin 2\sqrt{\frac{2g}{a}}.\ t$

$$\frac{dx}{dt} = -4a\sqrt{\frac{2g}{a}}\ \sin 2\sqrt{\frac{2g}{a}}.t + 2D\sqrt{\frac{2g}{a}}\ \cos 2\sqrt{\frac{2g}{a}}.t$$

when $t = 0$ $\frac{dx}{dt} = 2\sqrt{10ga}$, hence $D = a\sqrt{5}$

$$x = 2a\cos 2\sqrt{\frac{2g}{a}}.t + a\sqrt{5}\sin 2\sqrt{\frac{2g}{a}}.\ t$$

SWITCH ON.

Rewrite x, as $R\cos(\theta-\alpha)$ with, $\theta = 2\sqrt{\frac{2g}{a}}\,.t$

i.e. $R\cos\theta\cos\alpha + R\sin\theta\sin\alpha = 2a\cos\theta + a\sqrt{5}\sin\theta = x$

Compare the coefficients of $\cos\theta$ and $\sin\theta$, giving

$$x = 3a\cos\left(2\sqrt{\frac{2g}{a}}\,.t - \alpha\right) \text{ with } \alpha = \cos^{-1}\left(\frac{2}{3}\right).$$

Let the particle reach B, at time $t = t_2$ with $x = 3a$,

$$\text{then } 3a = 3a\cos\left(2\sqrt{\frac{2g}{a}}\,.t_2 - \alpha\right)$$

$$t_2 = \frac{1}{2}\sqrt{\frac{a}{2g}}.\alpha = \frac{1}{2}\sqrt{\frac{a}{2g}}\cos^{-1}\left(\frac{2}{3}\right)$$

The total time to B, is $(t_1 + t_2)$ which gives the required answer.

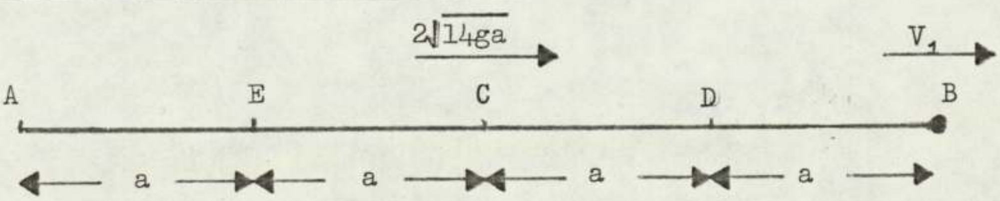
Let the velocity at B be V_1, when $t = t_2$.

$$V_1 = \left.\frac{dx}{dt}\right|_{t=t_2} = -6a\sqrt{\frac{2g}{a}}\sin\left(2\sqrt{\frac{2g}{a}}\,.t_2 - \alpha\right) = 0$$

i.e. The particle only just reaches B.

SWITCH ON.

Potential energies of the strings.



At the midpoint C, both strings are extended a distance a, and thus have equal P.E. (N.B. there is no gravitational P.E.)

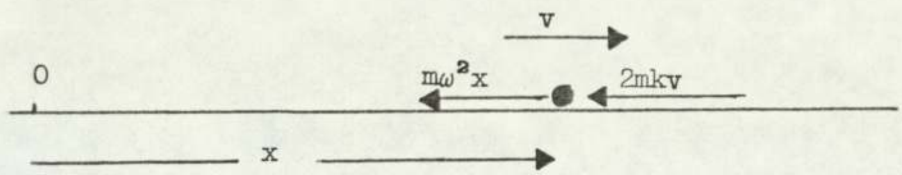
$$[P.E. + K.E.]_C = [P.E. + K.E.]_B$$

$$2\left(\frac{1}{2}\frac{8mga^2}{a}\right) + \frac{1}{2}m\left(2\sqrt{14ga}\right)^2 = \frac{1}{2}\frac{8mg}{a}(3a)^2 + \frac{1}{2}mV_1^2$$

$$0 = V_1 \quad \text{as before.}$$

SWITCH ON.

Damped Oscillations.



Resistance = 2mkv

Then $m\ddot{x} = -2mk\dot{x} - m\omega^2 x$

or $(D^2 + 2kD + \omega^2)\,x = 0$ where $D \equiv \frac{d}{dt}$.

The solution of this will depend upon the relative size of k and ω.

There are 3 cases.

Damping	Auxiliary equation
i) Heavy	Real roots
ii) Light	Complex roots
iii) Critical	Equal roots

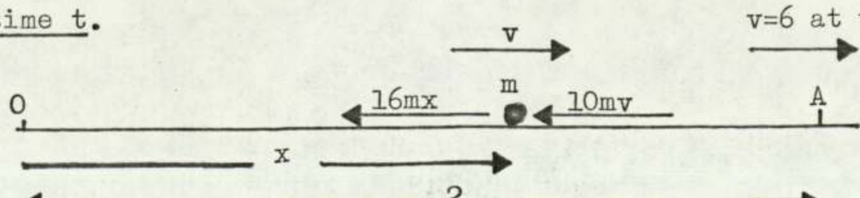
When you have studied this,

SWITCH ON.

EXAMPLE 6.

A particle of mass m, moving in a straight line, is subjected to a restoring force of 16m times the displacement, and a resistance of 10 m times the velocity. Obtain the differential equation of motion, and find the displacement and velocity at any time, if, initially, the displacement was 2m, and the velocity was 6m s^{-1}, directed away from the centre of the restoring force.

At time t.



When you have studied the directions of the forces, and velocities,

SWITCH ON.

Use 'F = ma' to give

$$m\ddot{x} = -\ 10\ mv - 16\ mx$$

$$\ddot{x} + 10\dot{x} + 16x = 0$$

Auxiliary equation is $m^2 + 10m + 16 = 0$

$$(m+2)(m+8) = 0$$

$$m = -2 \text{ or } -8$$

These are real and distinct roots, and, therefore, this represents a heavily damped oscillation.

$$x = A\ e^{-2t} + B\ e^{-8t}$$

when $t = 0$, $x = 2$ $\quad\therefore\ A + B = 2$

$$\frac{dx}{dt} = -\ 2A\ e^{-2t} - 8B\ e^{-8t}$$

when $t = 0$, $\frac{dx}{dt} = 6$ $\quad\therefore\ -A - 4B = 3$

Hence $B = -\frac{5}{3}$ and $A = \frac{11}{3}$

$$x = \frac{1}{3}(11e^{-2t} - 5e^{-8t}) \qquad = \frac{e^{-2t}}{3}\left[11 - \frac{5}{e^{6t}}\right]$$

This quantity is never negative, and theoretically x never = 0, but notice the very rapid speed of decay. The motion is, in fact, non-oscillatory, nearly all directed towards O, and $\dot{x}$ should always be negative after the particle turns towards O.

$$\frac{dx}{dt} = \frac{2e^{-2t}}{3}\left[20e^{-6t} - 11\right] = 0 \text{ when } t = \frac{1}{6}\ln\left(\frac{20}{11}\right)$$

SWITCH ON.

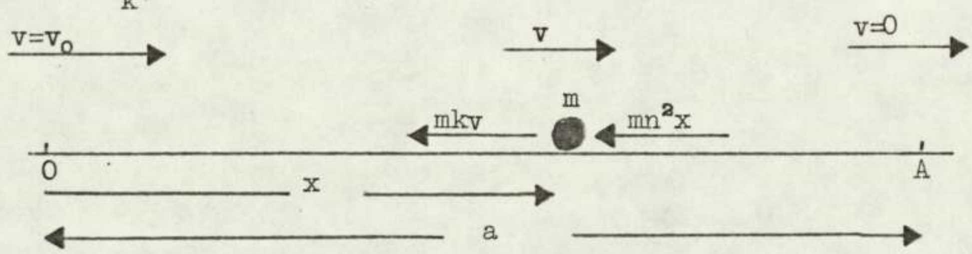
EXAMPLE 7.

A particle, of mass m, moves in a straight line, under a restoring force, mn^2x, and a resistance of mkv, with $4n^2 > k^2$, where x is the distance from a point O, on the line, and v is the velocity. Investigate the motion, if the particle is projected from O, with speed v_0, when t = 0. Show that,

i) the time it first comes to rest is independent of v_0,

ii) if x = a when it first come to rest, $v_0 = na\ \mathrm{EXP}\left[\frac{\tan^{-1}\lambda}{\lambda}\right]$,

when $\lambda = \frac{1}{k}\sqrt{4n^2-k^2}$.



When you have studied the forces and velocities,

SWITCH ON.

Use 'F = ma' from O to A.

$$m\ddot{x} = -\ mkv - mn^2x$$

$$(D^2 + kD + n^2)\ x = 0 \qquad \text{when } D \equiv \frac{d}{dt}$$

auxiliary equation has roots $-\frac{1}{2}k \pm \frac{i}{2}\sqrt{4n^2-k^2}$

These are complex, and the motion is lightly damped.

$$x = e^{-\frac{1}{2}kt}(A\ \cos\tfrac{1}{2}\sqrt{4n^2-k^2}.t + B\ \sin\tfrac{1}{2}\sqrt{4n^2-k^2}.t)$$

Initially t = 0, x = 0 hence A = 0.

i.e. $x = e^{-\frac{1}{2}kt}(B\sin\sigma t)$ where $\sigma = \frac{1}{2}\sqrt{4n^2-k^2}$.

(Continued on the next page).

$$\frac{dx}{dt} = -\tfrac{1}{2}ke^{-\frac{1}{2}kt}(B\ \sin\sigma t) + \sigma\ e^{-\frac{1}{2}kt}\ B\ \cos\sigma t$$

when $t = 0$ $\frac{dx}{dt} = v_0$, hence $B = \frac{v_0}{\sigma}$

$$x = \frac{v_0}{\sigma}\ e^{-\frac{1}{2}kt}\ \sin\ \sigma t$$

$$\frac{dx}{dt} = \frac{v_0}{\sigma}\left[\ e^{-\frac{1}{2}kt}\ (\sigma\ \cos\sigma t - \tfrac{1}{2}k\sin\sigma t)\right]$$

At rest, $\sigma\ \cos\sigma t - \frac{1}{2}k\ \sin\sigma t = 0$

for time t_1 to rest, take the smallest root of

$$\tan\ \sigma t_1 = \frac{2\sigma}{k} = \lambda(\text{given})$$

i.e. $t_1 = \frac{1}{\sigma}\ \tan^{-1}\lambda$, which is independent of v_0.

SWITCH ON.

Substitute $t = t_1$ and $x = a$ in the equation for x,

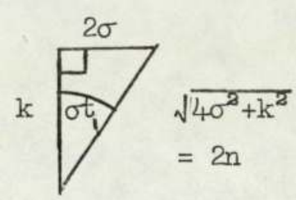
$$a = \frac{v_0}{\sigma}\ e^{-\frac{1}{2}kt_1}\ \sin\sigma t_1$$

$$a = \frac{v_0}{\sigma}\ \text{EXP}\left[-\ \frac{k}{2\sigma}\ \ \tan^{-1}\lambda\ \right]\ \sin\sigma t_1$$

By manipulation, $k^2 + 4\sigma^2 = 4n^2$

As $\tan\sigma t_1 = \frac{2\sigma}{k} = \lambda$

So $\sin\sigma t_1 = \frac{\sigma}{n}$.



Hence $a = \frac{v_0}{\sigma}\ \text{EXP}\left[-\ \frac{1}{\lambda}\ \tan^{-1}(\lambda)\ \right].\frac{\sigma}{n}$

$$\text{i.e. } v_0 = na\ \text{EXP}\left[\ \frac{\tan^{-1}\lambda}{\lambda}\right]$$

SWITCH ON.

EXAMPLE 8.

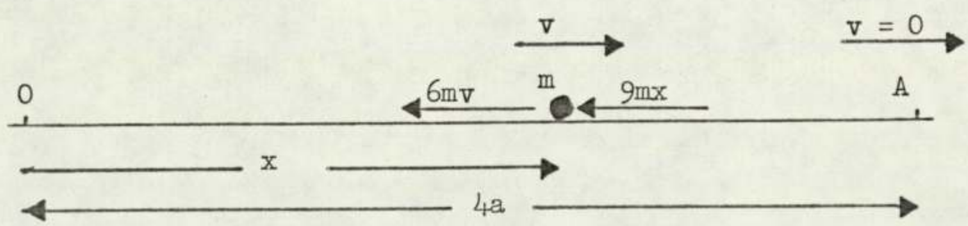
A particle, of mass m, moves in a straight line, under a restoring force, of 9mx newtons, and resistance, of 6mv N. If the displacement is 4a metres, when the particle starts from rest, find,

i) an expression for x, in terms of t,

ii) the velocity, after 2 seconds.

Identify the type of damping, before you look below.

At time t.



'F = ma' gives

$$m\ddot{x} = -6mv - 9mx$$

$$(D^2 + 6D + 9)x = 0 \qquad \text{where } D \equiv \frac{d}{dt}$$

$$(D+3)^2 x = 0$$

So the auxiliary equation has equal roots of -3.

This represents a critically damped motion.

$$x = (A+Bt)e^{-3t}$$

SWITCH ON.

---

i) when $t = 0$, $x = 4a$, $A = 4a$

$$x = (4a + Bt)\, e^{-3t}$$

$$\frac{dx}{dt} = e^{-3t}(B-12a-3Bt)$$

When $t = 0$, $\frac{dx}{dt} = 0$, $B = 12a$

i.e. $x = 4a(1+3t)e^{-3t}$

ii) $\frac{dx}{dt} = -36$ at e^{-3t}

when $t = 2$, $v = -72ae^{-6}$

This is a very small velocity, directed towards O.

SWITCH ON.

Forced Oscillations.

1. The differential equation is of the form

$$(D^2 + 2kD + \omega^2)x = f(t), \text{ where } D \equiv \frac{d}{dt}$$

and $f(t)$ is oscillatory.

2. The solution has two parts,

   i) transient, which comes from the complementary function,

   ii) steady state, which comes from the particular integral.

When you have studied these,

SWITCH ON.

EXAMPLE 9.

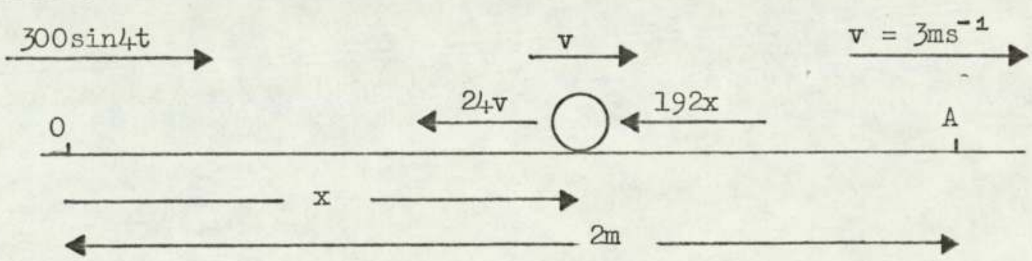
A mass of 3 kg moves in a straight line, and is acted upon by a restoring force of $192x$ newtons, a resistance of $24v$ N, and a periodic force $300 \sin 4t$ N. If the particle has an initial displacement of 2m, and a velocity of $3ms^{-1}$ directed away from the origin, find,

i) an expression for the transient motion,

ii) the steady state solution, and hence the amplitude and phase of the forced vibrations,

iii) the complete solution for x.

Time t.



When you have studied the diagram,

SWITCH ON.

---

i) Use 'F = ma'

$$3\ddot{x} = -24v - 192x + 300 \sin 4t$$

$$(D^2+8D+64)x = 100 \sin 4t \text{ with } D \equiv \frac{d}{dt}.$$

The C.F. is given by 185.

$$(D^2 + 8D + 64)x = 0$$

(Continued on the next page)

auxiliary equation has roots $-4 \pm i4\sqrt{3}$

C.F. is $e^{-4t}(A\cos 4\sqrt{3}.t + B\sin 4\sqrt{3}.t)$

SWITCH ON.

---

ii) The P.I. is

$$\frac{1}{D^2 + 8D + 64}.100\sin 4t \qquad \left[\text{Replace "}D^2\text{ by }-a^2\text{" i.e. by }-16\right]$$

$$= \frac{1}{-16+8D+64}.100\sin 4t = \frac{1}{D+6}\,\frac{25}{2}\sin 4t.$$

$$= \frac{25}{2}\,\frac{(D-6)}{(D^2-36)}.\sin 4t = \frac{25}{104}(6-D)\sin 4t$$

$$= \frac{25}{52}(3\sin 4t - 2\cos 4t)$$

$(3\sin 4t - 2\cos 4t)$ must be expressed in the form

$R\sin(4t-\alpha)$ to give the steady state solution.

$\frac{25\sqrt{13}}{52}\sin(4t-\alpha)$ with $\alpha = \tan^{-1}(\frac{2}{3})$

Amplitude $= \frac{25\sqrt{13}}{52}$, Phase $= \tan^{-1}(\frac{2}{3})$

The period is $\frac{\pi}{2}$, which is the same as that of the external force.

SWITCH ON.

---

iii) $x = e^{-4t}(A\cos 4\sqrt{3}.t + B\sin 4\sqrt{3}.t) + \frac{25}{52}(3\sin 4t - 2\cos 4t)$

When $t = 0$ $x = 2$ hence $A = \frac{77}{26}$

$x = e^{-4t}(\frac{77}{26}\cos 4\sqrt{3}.t + B\sin 4\sqrt{3}.t) + \frac{25}{52}(3\sin 4t - 2\cos 4t)$

$$\frac{dx}{dt} = e^{-4t}\left\{(4\sqrt{3}B - \frac{154}{13})\cos 4\sqrt{3}.t - (4B + \frac{154\sqrt{3}}{13})\sin 4\sqrt{3}.t\right\} + \frac{25}{52}(12\cos 4t + 8\sin 4t)$$

when $t = 0$ $\frac{dx}{dt} = 3$ hence $B = \frac{59\sqrt{3}}{78}$

Complete solution

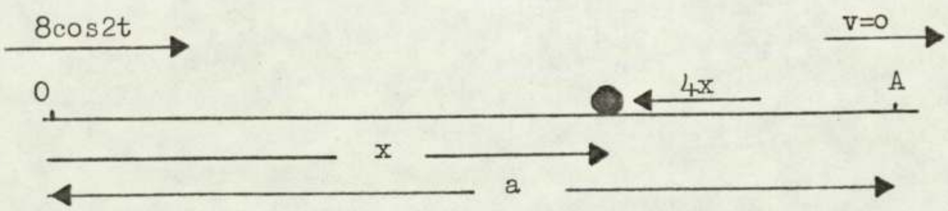
$$x = e^{-4t}(\frac{77}{26}\cos 4\sqrt{3}.t + \frac{59\sqrt{3}}{78}\sin 4\sqrt{3}.t) + \frac{25}{52}(3\sin 4t - 2\cos 4t)$$

SWITCH ON.

EXAMPLE 10.

A particle, moves in a straight line, in a medium which exerts a negligible resistance. The restoring force is four times the displacement, and the external forcing agent is 8cos2t. Find the differential equation, and solve it completely, if, initially, the particle has zero speed, when the displacement is a.

Do not read this solution yet.



Use 'F = ma'

$$\frac{d^2x}{dt^2} = -4x + 8\cos 2t$$

$$\ddot{x} + 4x = 8\cos 2t$$

$$(D^2+4)x = 8\cos 2t \qquad \text{where } D \equiv \frac{d}{dt}$$

for C.F. $(D^2 + 4)x = 0$

i.e. $x_1 = A\cos 2t + B\sin 2t$

for P.I., use $\frac{1}{(D^2+4)} \cdot 8\cos 2t$.

i.e. Real part of $\frac{8}{(D-2i)(D+2i)} \cdot e^{2it}$

$$x_2 = 8R \cdot \frac{1}{(2i+2i)} \cdot te^{2it}$$

$$= 8R\left\{-\frac{it}{4}(\cos 2t + i.\sin 2t)\right\}$$

$$= \frac{8t}{4} \cdot \sin 2t = 2t.\sin 2t$$

general solution $= x_1+x_2 = A\cos 2t + B\sin 2t + 2t.\sin 2t$

(Continued on the next page.)

when $t = 0$, $x = a$ and $A = a$

$$x = a\cos 2t + (B+2t)\sin 2t$$

$$\frac{dx}{dt} = -2a\sin 2t + 2\sin 2t + 2(B+2t)\cos 2t$$

when $t = 0$, $\frac{dx}{dt} = 0$ hence $B = 0$

i.e. complete solution is $x = a\cos 2t + 2t\sin 2t$.

This motion has increasing amplitude of oscillations as the time increases. This is an example of resonance.

SWITCH ON.

SCRIPT FOR TAPE 7

OSCILLATIONS.

1. This is the tape to be used with Booklet 7 of 'A Revision Course in Particle Dynamics' by Kenneth Jackson, and is concerned with oscillations. These include examples about Simple Harmonic Motion, and both damped and forced oscillations.

You will remember that Simple Harmonic Motion of a particle moving in a straight line, in generated by a force which is always directed towards a fixed point on the line, and proportional to the distance of the particle from the point. Read the first section where this is revised for you. SWITCH OFF.

2. We shall start working through question one about Simple Harmonic Motion together. Read it carefully, and study the diagram. SWITCH OFF.

3. The diagram shows that rest position of the ring at A, where the string has minimum tension, and the pull on the ring is perpendicular to the wire. The displacement through a small distance x to B, produces a restoring force, which is the component of the tension T, directed along BA. Using the angle ALPHA, you can now write the equation of motion of the ring. Do this, and complete the question, checking it afterwards below. SWITCH OFF.

4. The second example is a more difficult one on stretching strings and I shall work through most of this to demonstrate the mehod. First of all, read the question, and study the equilibrium diagram overleaf. SWITCH OFF.

5. Notice that we are not told the unstretched length of the strings, or the extensions in the diagram, but we can use the equilibrium of the forces, and the over-all stretched length to give the relationships required. Write them down, and solve them to give these two unknown quantities. Check these afterwards below. SWITCH OFF.

6. In this situation we have symmetry on both sides of the vertical line through C, which is critical in this question. Now consider the vertical disturbance to be made from C. We will make it downwards for convenience. Draw a diagram of this for yourself, showing the displaced strings and their tensions. When this is complete check it with that in the booklet. SWITCH OFF.

7. With the help of this diagram we are able to write the equation of motion in terms of T_1 and theta. Then, bu using the cosine rule, we can find the length DN, for which we shall require a binomial expansion. The equation of motion is then expanded algebraically. Follow this closely in the next section of the booklet. SWITCH OFF.

8. This approximation must now be used in the equation of motion. Proceed with the algebraic reduction, neglecting x^2 and higher power terms. You should obtain the standard differential equation for simple harmonic

motion, and hence the period can be found. You will need some further algebraic manipulation to obtain the given answer. When you have finished, check this in the booklet. SWITCH OFF.

9. The next example is also concerned with a combination of strings. Read this carefully, and draw two diagrams, one for the equilibrium position, and another for the displaced particle. From the first you are able to confirm equilibrium, and from the second, write the outline equation of motion in terms of the tensions. Check your working below afterwards. SWITCH OFF.

10. Substitute for the tensions, neglect terms in x^2 and you will be able to complete the question. Remember that $\ddot{x}$ must have a restoring force proportional to x for S.H.M., and you will need to consider the sign at a later stage. Check your working below again. SWITCH OFF.

11. The fourth question concerns a time calculation. in a problem which is partly simple harmonic and needs to be worked in stages. Read question four carefully, and look at the diagram for the first part. SWITCH OFF.

12. As the displacement x is specified positive inthe upward direction, so the tension T_1 must be the positive force in the equation of motion. Using this, write the equation of motion and find the general solution of the differential equation. Check this afterwards, below. SWITCH OFF.

13. The constants A and B can now be found by

using the zero displacement, and given upwards velocity at the start of the motion. Carry out this evaluation, and then check in the next section. SWITCH OFF.

14. The string will become slack when the tension is zero, which occurs at the point A, with x = alpha. Use this condition in your solution to find the required time. When you have finished, check your working. SWITCH OFF.

15. If the particle has any kinetic energy remaining after the string is slack, it will rise freely in the air, so we need to calculate the velocity at A, to check this. Calculate this velocity and check it below. SWITCH OFF.

16. As you know the velocity V_1 at A, you are able to calculate the time of flight from A, back toA. For the time from A to D, remember that B is the centre of the complete S.H.M., and AD is the amplitude downwards. By using the appropriate fraction of the period, you can now finish the question. Check your answer overleaf, when you have done so. SWITCH off.

17. Notice that it is possible to find the velocity at A, without solving the differential equation. This can be done, using the conservation of energy. Follow this alternative method of working in the next section of the booklet. SWITCH OFF.

18. This energy method is a useful alternative when times are not required. Example 5 is also concerned with S.H.M. and is mainly for your own working. First read the question, and consider the diagram. SWITCH OFF.

19. To consider the motion from C to D, you will need another diagram showing the displacements and tensions. Write your equation of motion, and you should then be able to find the time to D, and the velocity at this point. When you have done this, check your working in the next section. SWITCH OFF.

20. For the remainder of the question, you will need another diagram for the motion from D to B. Remember that one string is now slack. Again, write the equation of motion, solve the differential equation, and find the constants. Do not go any further yet. Check your answer in the next section of the booklet. SWITCH OFF.

21. This equation is easier to deal with if it is expressed in the form R cos theta minus alpha. Do this, and find the values of R and alpha. You can now finish the question. Do so, and then check your working overleaf. SWITCH OFF.

22. Again an alternative procedure is provided by the conservation of energy, and it can be shown right at the beginning of the question, that the velocity at B is zero. Follow this in the next section. SWITCH OFF.

23. The next examples are about damped oscillations, which involves another term in the differential equation. Read the next section, which summarises this type of motion for you. SWITCH OFF.

24. Read example 6, which is concerned with damped oscillations, and we shall begin working through it together. Study the forces and velocities shown in the

diagram carefully. SWITCH OFF.

25. Notice the simplicity of the diagram with the particle, shown moving in the positive sense of x. This allows an easy formulation of the differential equation. The forces will, however, cause the particle to move towards O, for some of the time, and we expect the velocity v to be negative at such times. Write the equation of motion for this particle, identify the type of damping, and write out your solution, evaluating the constants from the initial conditions. Check this afterwards, overleaf. SWITCH OFF.

26. For the next example on damped oscillations, read example seven, and study the diagram closely. Notice how this neatly summarizes all of the available information. SWITCH OFF.

27. This example illustrates light damping. Confirm this, from the auxiliary equation of the differentail equation. Find the general solution, and use the initial conditions to determine the constants. Complete the first part of the question and check this afterwards below. SWITCH OFF.

28. For the second part of the question use $t = t_1$ where $x = a$ in the equation for x. By manipulation, obtain the quoted answer and when you have finished, check this below. SWITCH OFF.

29. Now read example 8, which should be tried mainly by your own efforts. When you have drawn the diagram written the equation of motion, identified the type of

damping, and found the general solution, check these in the booklet. SWITCH OFF.

30. You are now able to use the given conditions to answer part one of the question, and then find dx/dt to answer the second part. Check your solution below afterwards. SWITCH OFF.

31. A more complicated motion occurs when a particle is also subjected to an external force, and this can lead to forced oscillations. The essential points are again summarized for you in the next section of the booklet. SWITCH OFF.

32. Example 9 concerns forced oscillations, and we shall work through part of this question together. Read it and study the diagram below. SWITCH OFF.

33. Remember that with this type of question, that the differential equation has two parts, and we have to find the Complementary Function and Particular Integral separately. Also you cannot determine the constants in the C.F. until all the general solution has been found. You can now answer the first part of the question, which you should check afterwards. SWITCH OFF.

34. The steady state is found from the Particular Integral, as the transient solution decays exponentially. By rewriting the particular integral in its most compact form, you will be able to obtain the smplitude and phase of the steady state motion. When you have done this, check it below. SWITCH OFF.

35. The third part is a straightforward evaluation of the constants in the sum of the component solutions,

that is C.F. + P.I. The numbers will be a little awkward and it will help to write the P.I. in the first forms. Complete the question in this way, and then check your wolution. SWITCH OFF.

36. Solutions to problems in forced oscillations need a good mastery of the techniques for finding particular integrals. This is illustrated by the last example which concerns a forced motion without a resistance. Work this yourself, and check your solution afterwards. If you find you are unable to determine the P.I. in this, then follow the given solution immediately. SWITCH OFF.

37. This completes the work on oscillations. Would you rewind the tape before you remove it from the machine, please? Thank you!

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

BOOKLET 8 to be used with TAPE 8

"Restricted Motion"

Read the instructions on page 1 thoroughly.

(Omitted for convenience.)

Tangential and normal components of acceleration for a particle moving on a plane curve.

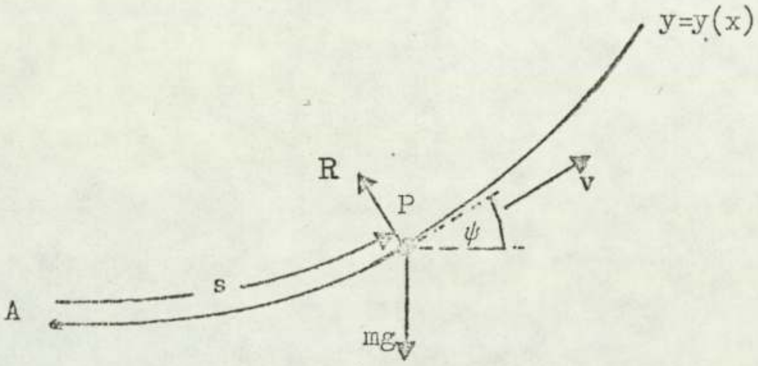
Tangential acceleration $= \ddot{s} = \dot{v} = v\,\dfrac{dv}{ds}$ (s increasing)

Normal acceleration $= \dfrac{v^2}{|\rho|}$ (along the inward normal)

with $\tan\psi = \dfrac{dy}{dx}$, with $-\dfrac{\pi}{2} < \psi \leqslant \dfrac{\pi}{2}$

$$\text{and } \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Consider a particle of mass m, moving on a given curve in a vertical plane.



Use 'F = ma'

along the tangent $\ddot{s} = -mg\sin\psi$

along the inward normal $\dfrac{mv^2}{|\rho|} = R - mg\cos\psi$

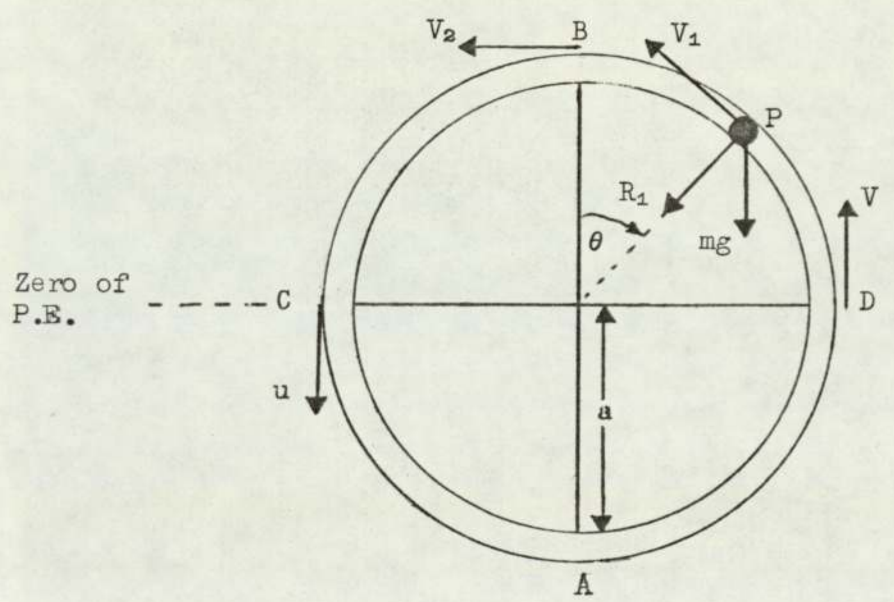
In the special case of circular motion, the acceleration along the inward normal reduces to $\dfrac{v^2}{a}$, where 'a' is the radius of the circle.

When you have studied this,

SWITCH ON.

EXAMPLE 1.

A particle is free to move, in a thin, smooth, vertical, circular tube, of radius 'a', of which A is the lowest point, and AB, and CD, the vertical, and horizontal, diameters respectively. The particle is projected downwards from C, with velocity u. If the pressure of the tube on the particle, changes from the outside to the inside, as the particle passes the midpoint of arc DB, find u, and determine the speed when the particle reaches B. If the part of the tube between B and C has no outer surface, at what point will the particle leave the surface?



When you have studied the diagram,

SWITCH ON

Kinetic energy at C = constant = Kinetic energy at D,

i.e. V = u

From D to B, 'F = ma' radially,

$$\frac{mV_1^2}{a} = R_1 + mg.\cos\theta$$

$$\text{i.e.} \quad R_1 = \frac{mV_1^2}{a} - mg.\cos\theta$$

(Continued on the next page)

By energy $[K.E. + P.E.]_P = [K.E. + P.E.]_D$

$$\tfrac{1}{2}mV_1^2 + mg.a.\cos\theta = \tfrac{1}{2}mu^2$$

$$V_1^2 = u^2 - 2ga.\cos\theta \qquad \ldots (1)$$

hence $R_1 = \dfrac{m(u^2-2ga.\cos\theta)}{a} - mg.\cos\theta = \dfrac{m(u^2-3ga.\cos\theta)}{a}$

Since $R_1 = 0$ when $\theta = 45°$

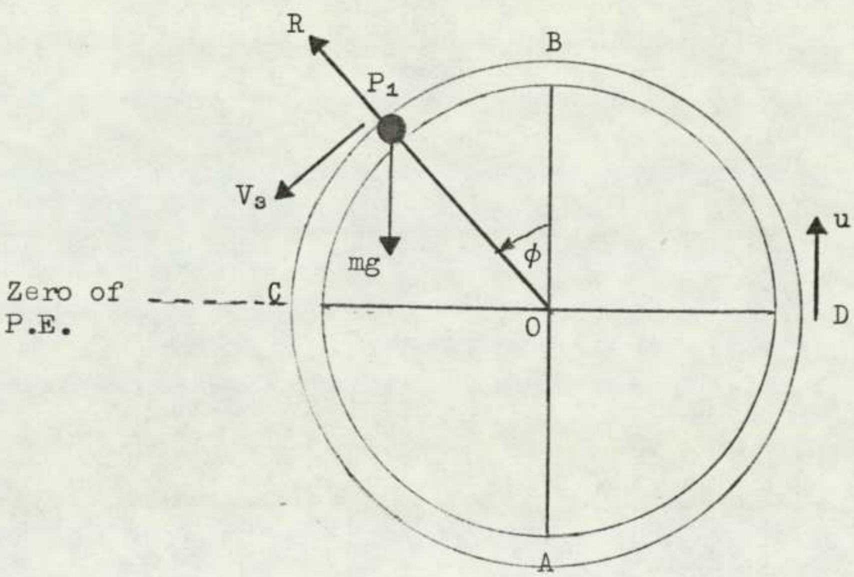
$$u^2 = 3ga.\cos45° \quad \text{i.e. } u = \left(\frac{3ga}{\sqrt{2}}\right)^{\frac{1}{2}}$$

Also, $\theta = 45°$ in (1) gives the critical velocity as $V_c = \dfrac{ga}{\sqrt{2}}$

SWITCH ON.

For velocity at B = V_2, put $\theta = 0°$ in (1)

and $V_2 = \sqrt{\dfrac{ga(3\sqrt{2}-4)}{2}}$



Particle moves to P_1 at angle ϕ to OB, with velocity V_3

By energy $[K.E. + P.E.]_{P_1} = [K.E. + P.E.]_D$

$$\tfrac{1}{2}mV_3^2 + mga.\cos\phi = \tfrac{1}{2}mga\frac{3}{\sqrt{2}}$$

'F = ma' radially $\dfrac{mV_3^2}{a} = mg.\cos\phi - R$

Put R = 0 and eliminate V_3^2, to give $\cos\phi = \dfrac{1}{\sqrt{2}}$

i.e. The particle will leave the surface at the midpoint of arc BC. This is only to be expected, as it shows the symmetry of this circular motion about the vertical diameter.

SWITCH ON.

200.

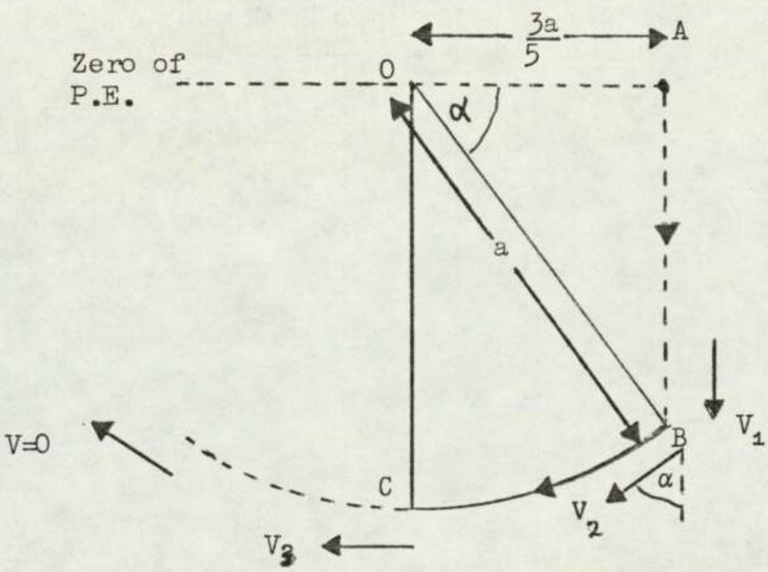
EXAMPLE 2.

A particle, attached to a fixed point O, by an inelastic string, of length 'a', is let fall, from a point in the horizontal through O, at a distance $\frac{3a}{5}$ from O. Find,

i) The impulse along the string immediately after the free fall ceases,

ii) the velocity of the particle when it is vertically below O,

iii) how far the particle rises vertically above this position.



When you agree that the motion should be examined in three parts,

SWITCH ON.

---

i) Distance fallen $= AB = \sqrt{a^2 - \left(\frac{3a}{5}\right)^2} = \frac{4}{5}a$

For free fall, use '$v^2 = u^2 + 2ax$'

$$V_1^2 = 2g.\frac{4a}{5}, \quad V_1 = 2\sqrt{\frac{2ga}{5}}$$

Momentum along OB, prior to the jerk $= mV_1\sin\alpha = \frac{8m}{5}\sqrt{\frac{2ga}{5}}$

Momentum along OB, after the jerk = 0

(Continued on the next page.)

i) contd.

Use 'Impulse = change in momentum'

impulsive tension $= 0 - \left(-\frac{8m}{5}\sqrt{\frac{2ga}{5}}\right) = \frac{8m}{5}\sqrt{\frac{2ga}{5}}$

Component V_2 is normal to the jerk in the string, and is not changed by the impulse.

ii) $V_2 = V_1 \cos\alpha = \frac{6}{5}\sqrt{\frac{2ga}{5}}$

$[K.E. + P.E.]_C = [K.E. + P.E.]_B$

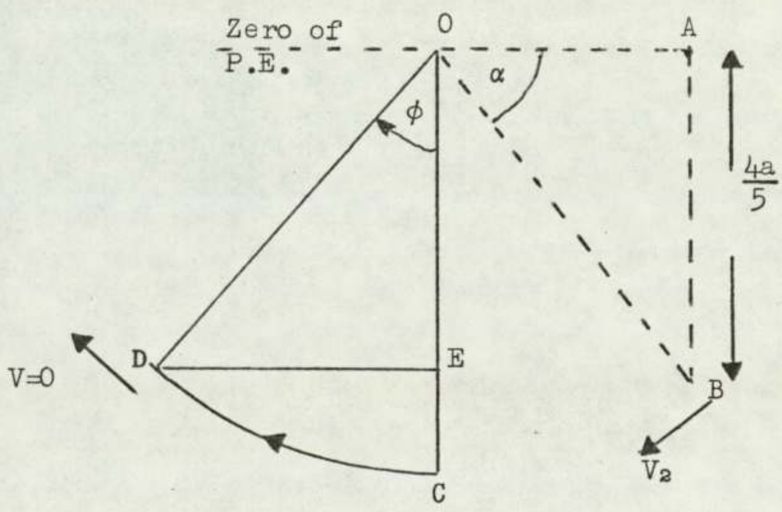
$\frac{1}{2}mV_3^2 - mga = \frac{1}{2}m\left(\frac{6}{5}\sqrt{\frac{2ga}{5}}\right)^2 - mg\frac{4a}{5}$

hence $V_3 = \frac{1}{5}\sqrt{\frac{122ga}{5}}$

SWITCH ON.

---

iii)



Particle rises to D where $D\hat{O}E = \phi$

$[K.E. + P.E.]_D = [K.E. + P.E.]_B$

$0 - mga\cos\phi = \frac{1}{2}m\left(\frac{6}{5}\sqrt{\frac{2ga}{5}}\right)^2 - \frac{4}{5}mga$

hence $\cos\phi = \frac{64}{125}$

Vertical distance $CE = a - a\cos\phi = \frac{61a}{125}$

SWITCH ON.

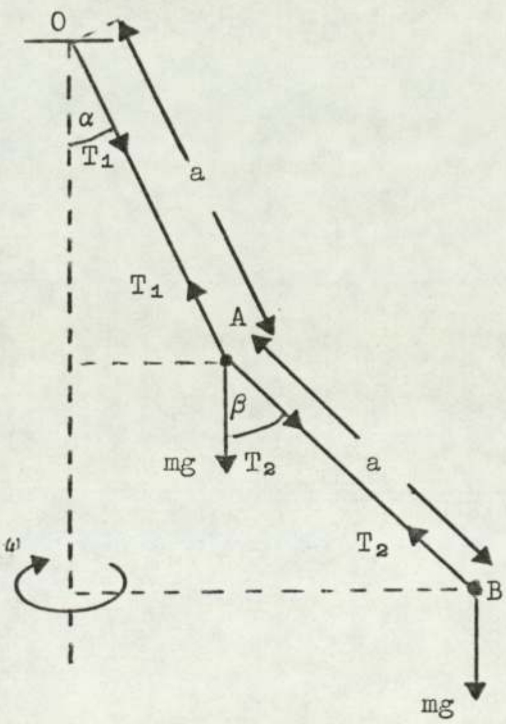
EXAMPLE 3.

A particle hangs by an inelastic string, of length 'a', from a fixed point, and a second particle of the same mass, hangs from the first, by an equal string. The whole moves with uniform angular velocity ω, about the vertical through the point of suspension, the strings making angles α and β with the vertical through the fixed point of suspension. Show that,

i) $\tan\alpha = \frac{a\omega^2}{2g}(2\sin\alpha + \sin\beta)$

ii) $\tan\beta = \frac{a\omega^2}{g}(\sin\alpha + \sin\beta)$

Hence show that if α and β are small, such a steady motion is only possible if $\beta/\alpha = \pm\sqrt{2}$, and when so, $a\omega^2$ has one of the values $(2 \pm \sqrt{2})g$.



When you understand this,

SWITCH ON.

For rotation, use 'F = ma' radially

at A, $m\omega^2 a \sin\alpha = T_1 \sin\alpha - T_2 \sin\beta$ ... (1)

at B, $m\omega^2 a (\sin\alpha + \sin\beta) = T_2 \sin\beta$ ... (2)

Vertical equilibrium,

for A, $T_1 \cos\alpha = mg + T_2 \cos\beta$ ... (3)

for B, $T_2 \cos\beta = mg$ ... (4)

SWITCH ON.

Use (3) and (4) $T_1 \cos\alpha = 2mg$

Rewrite (1) as $T_1 \sin\alpha = m\omega^2 a \sin\alpha + T_2 \sin\beta$

Use (2) to give $T_1 \sin\alpha = m\omega^2 a (2\sin\alpha + \sin\beta)$

Divide by $T_1 \cos\alpha$ to give the result

$$\tan\alpha = \frac{a\omega^2}{2g}(2\sin\alpha + \sin\beta) \quad \ldots (5)$$

Divide (2) by (4) to give

$$\tan\beta = \frac{a\omega^2}{g}(\sin\alpha + \sin\beta) \quad \ldots (6)$$

SWITCH ON.

If α, β are small angles,

$\tan\alpha = \sin\alpha = \alpha$ and $\tan\beta = \sin\beta = \beta$

Divide (6) by (5)

$\frac{\beta}{\alpha} = \frac{2(\alpha+\beta)}{(2\alpha+\beta)}$ which simplifies to $\frac{\beta}{\alpha} = \pm\sqrt{2}$.

This implies that β and α may have the same sense for the positive sign, as shown in the diagram, but β could be in the opposite sense for the negative sign, with B rotating in a smaller radius circle than A.

(Continued on the next page).

Rewrite (6) in the form

$$\frac{a\omega^2}{g} = \frac{\tan\beta}{\sin\alpha+\sin\beta} = \frac{\beta}{\alpha+\beta} = \frac{\frac{\beta}{\alpha}}{1+\frac{\beta}{\alpha}} = \frac{\pm\sqrt{2}}{1\pm\sqrt{2}}$$

which gives the second result after rationalisation.

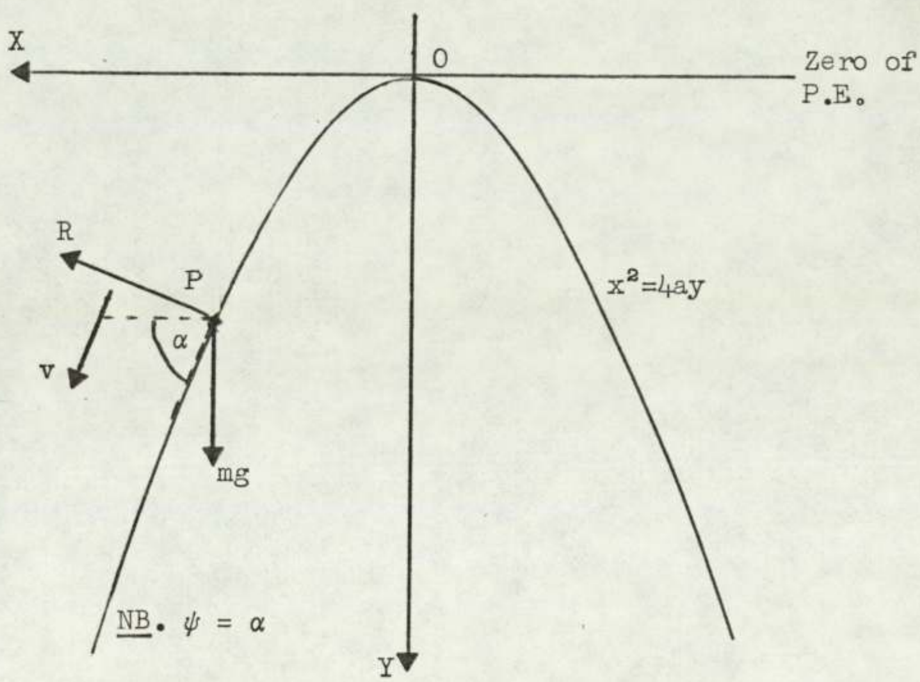
SWITCH ON.

EXAMPLE 4.

A bead is threaded on a smooth parabolic wire with axis vertical and vertex upwards. If the bead is slightly disturbed to the left from the vertex, find,

i) the velocity of the bead in any position,

ii) the reaction of the wire on the bead at a distance 2a vertically below the vertex.



When you understand this diagram,

SWITCH ON.

i) $[K.E. + P.E.]_P = [K.E. + P.E.]_O$

$$\tfrac{1}{2}mv^2 - mgy = 0$$

$$v^2 = 2gy$$

ii) Use 'F = ma' along the inward normal,

$$\frac{mv^2}{|\rho|} = mg\cos\alpha - R = mg\cos\psi - R$$

SWITCH ON.

$$x^2 = 4ay \text{ and } \frac{dy}{dx} = \frac{x}{2a}, \quad \frac{d^2y}{dx^2} = \frac{1}{2a}$$

$$\text{Hence } \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(4a^2+x^2)^{3/2}}{4a^2}$$

$$\tan\psi = \frac{dy}{dx} = \frac{x}{2a} \quad \text{so that } \cos\psi = \frac{2a}{\sqrt{x^2 + 4a^2}}$$

Substitute in $R = mg\cos\psi - \dfrac{mv^2}{|\rho|}$

to give $R = \dfrac{mg}{3\sqrt{3}}$ when $y = 2a$.

SWITCH ON.

EXAMPLE 5.

A particle moves on a smooth curve with equation $y = c\cos\left(\frac{x}{a}\right)$, which is fixed with its y axis directed vertically upwards. Initially the particle is at rest at $(0,c)$, and is slightly disturbed. Show that in the subsequent motion,

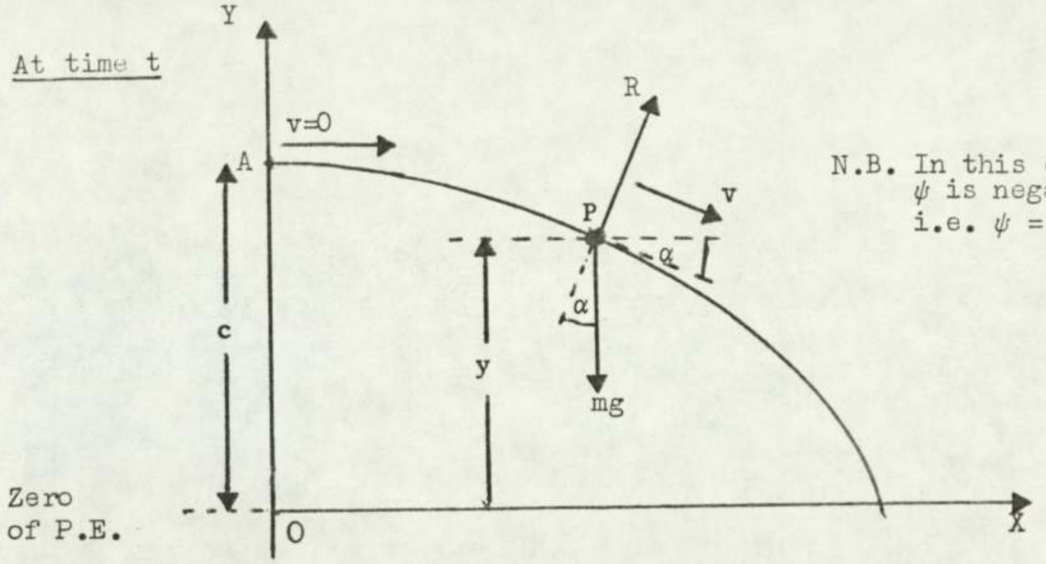
i) the velocity, v, is given by $v^2 = 4gc\sin^2\left(\frac{x}{2a}\right)$

ii) the reaction between the particle and the curve is

$$\frac{2mga(4a^2+c^2)}{(4a^2+3c^2)^{3/2}} \quad \text{when } y = \frac{c}{2}$$

(The diagram is on the next page.)

At time t



N.B. In this configuration ψ is negative i.e. $\psi = -\alpha$

When you have studied this,

SWITCH ON.

---

i) $[\text{K.E.} + \text{P.E.}]_P = [\text{K.E.} + \text{P.E.}]_A$

$$\tfrac{1}{2}mv^2 + mgy = 0 + mgc$$

$$\text{i.e. } v^2 = 2g(c-y)$$

N.B. In this configuration y is positive. Substitute for y to give $v^2 = 4gc\,\sin^2\!\left(\frac{x}{2a}\right)$

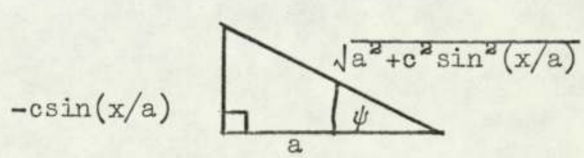
ii) Use 'F = ma' at P along the inward normal,

$$\frac{mv^2}{|\rho|} = mg\cos\alpha - R$$

$$\frac{dy}{dx} = -\frac{c}{a}\sin\left(\frac{x}{a}\right) \text{ and } \frac{d^2y}{dx^2} = -\frac{c}{a^2}\cos\left(\frac{x}{a}\right) = -\frac{y}{a^2}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = -\frac{\left[a^2+c^2\sin^2\left(\frac{x}{a}\right)\right]^{3/2}}{ay}$$

$$\tan\psi = \frac{dy}{dx} = \frac{-c\sin\left(\frac{x}{a}\right)}{a} \text{ i.e.}$$



$$\cos\alpha = \cos(-\psi) = \cos\psi = \frac{a}{\sqrt{a^2+c^2\sin^2(x/a)}}$$

(Continued on the next page.)

Substitute for $\cos\alpha$, $|\rho|$ and v^2 into

$$R = mg\cos\alpha - \frac{mv^2}{|\rho|}$$

to give $R = \dfrac{2mga(4a^2+c^2)}{(4a^2+3c^2)^{3/2}}$ when $y = \dfrac{c}{2}$

N.B. $\sin(x/a) = \dfrac{\sqrt{3}}{2}$ when $y = \dfrac{c}{2}$

SWITCH ON.

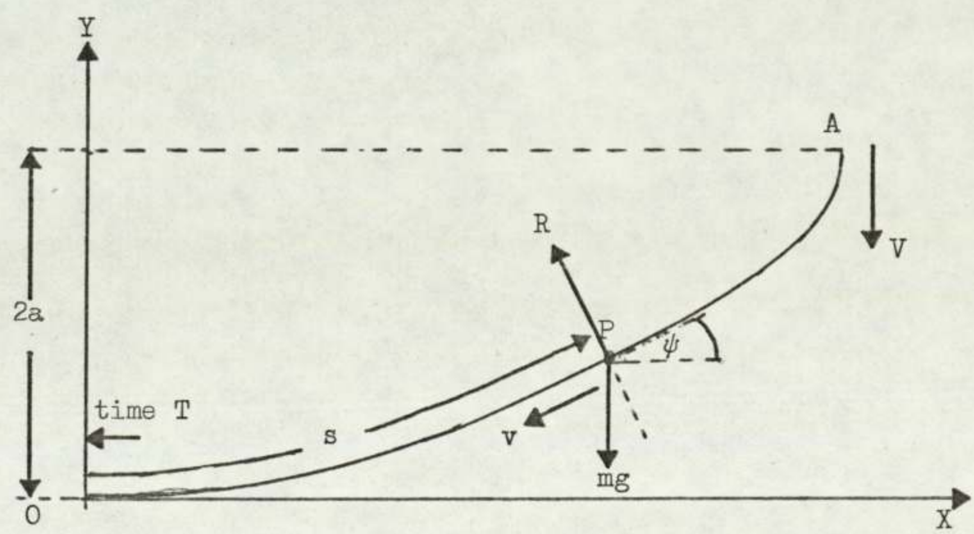
EXAMPLE 6.

A particle is projected, with velocity V, from the cusp of a smooth, inverted cycloid, down the arc. The parametric equations of the curve are,

$x = a(2\psi + \sin 2\psi)$; $y = a(1-\cos 2\psi)$

Show, that the time to reach the vertex is,

$$2\sqrt{\frac{a}{g}}\tan^{-1}\left(\frac{2\sqrt{ag}}{V}\right)$$



In this configuration, ψ is positive

$$x = a(2\psi + \sin 2\psi) \quad \ldots (1)$$

$$y = a(1 - \cos 2\psi) \quad \ldots (2)$$

The intrinsic equation of the cycloid is, $s = 4a\sin\psi$ ... (3)

When you understand this information,

SWITCH ON.

Use 'F = ma' tangentially,

$$m\frac{d^2s}{dt^2} = -\,mg\,\sin\psi = -\,\frac{mgs}{4a} \qquad \text{(from 3)}$$

$$v\frac{dv}{ds} = -\,\frac{gs}{4a}$$

Integrate, $\frac{v^2}{2} = -\,\frac{gs^2}{8a} + C$

At A, $\psi = \frac{\pi}{2}$ and $s = 4a$

i.e. $s = 4a$ when $v = -V$

hence $C = \frac{1}{2}V^2 + 2ag$

and $v^2 = V^2 + 4ag - \frac{gs^2}{4a}$

SWITCH ON.

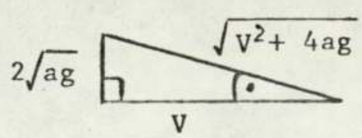
$$v = \frac{ds}{dt} = -\sqrt{V^2 + 4ag - \frac{gs^2}{4a}}$$

$$-\,dt = \frac{ds}{\sqrt{\frac{g}{4a}}\sqrt{\frac{(V^2+4ag)4a}{g} - s^2}}$$

$$-\int_0^T dt = \sqrt{\frac{4a}{g}}\int_{4a}^{0}\frac{ds}{\sqrt{\frac{4aV^2 + 16a^2g}{g} - s^2}}$$

$$-T = \sqrt{\frac{4a}{g}}\left[\sin^{-1}\left(\frac{s\sqrt{g}}{\sqrt{4aV^2+16a^2g}}\right)\right]_{4a}^{0} = -\sqrt{\frac{4a}{g}}\,\sin^{-1}\left(\frac{2\sqrt{ag}}{\sqrt{V^2+4ag}}\right)$$

$$T = 2\sqrt{\frac{a}{g}}\,\tan^{-1}\left(\frac{2\sqrt{ag}}{V}\right)$$



SWITCH ON.

$$\frac{d^2s}{dt^2} + \frac{gs}{4a} = 0$$

$$s = A\sin\tfrac{1}{2}\sqrt{\tfrac{g}{a}}.t + B\cos\tfrac{1}{2}\sqrt{\tfrac{g}{a}}.t$$

When $t = 0$, $s = 4a$, hence $B = 4a$

(Continued on the next page.)

$$\frac{ds}{dt} = \frac{A}{2}\sqrt{\frac{g}{a}}\cos\tfrac{1}{2}\sqrt{\frac{g}{a}}.t - 2a\sqrt{\frac{g}{a}}\sin\tfrac{1}{2}\sqrt{\frac{g}{a}}.t$$

When $t = 0$, $\dot{s} = -V$, and $A = -2V\sqrt{\frac{a}{g}}$

hence $s = -2V\sqrt{\frac{a}{g}}\sin\tfrac{1}{2}\sqrt{\frac{g}{a}}.t + 4a\cos\tfrac{1}{2}\sqrt{\frac{g}{a}}.t$

When $s = 0$, $t = T$

i.e. $0 = -2V\sqrt{\frac{a}{g}}\sin\tfrac{1}{2}\sqrt{\frac{g}{a}}.T + 4a\cos\tfrac{1}{2}\sqrt{\frac{g}{a}}.T$

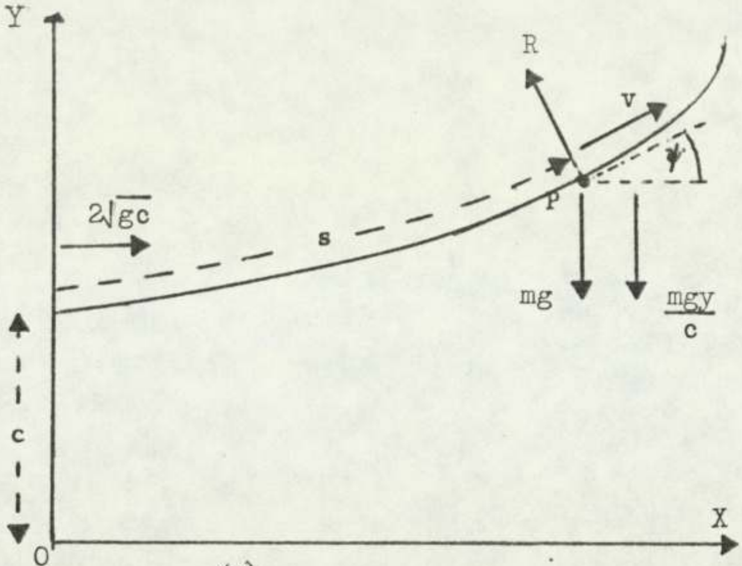
$$\text{hence } T = 2\sqrt{\frac{a}{g}}\tan^{-1}\left(\frac{2\sqrt{ag}}{V}\right)$$

SWITCH ON.

EXAMPLE 7.

A smooth tube is in the shape of a catenary, $y = c\cosh\left(\frac{x}{c}\right)$, and is fixed in the vertical plane, with its vertex downwards. A particle, of mass m, moves in the tube, under the action of gravity, and of an attractive force, towards the x axis, equal to mg/c times its distance from it. If the speed of the particle at the vertex is $2\sqrt{gc}$, show that it will come to instantaneous rest, at points distant $(2\sqrt{2}-1)c$ from the x axis. Find also, the reaction of the tube on the particle, at the vertex.

Do not look at the diagram overleaf, until you have tried to draw your own diagram, and write the equations of motion.



N.B. ψ is positive in this configuration

The catenary $y = c\cosh\left(\frac{x}{c}\right)$ has intrinsic equation

$s = c\tan\psi$

Use 'F = ma' at P,

tangentially $\quad m\ddot{s} = -mg\left(1 + \frac{y}{c}\right)\sin\psi \qquad \ldots (1)$

normally $\quad \frac{mv^2}{|\rho|} = R - mg\left(1 + \frac{y}{c}\right)\cos\psi \qquad \ldots (2)$

SWITCH ON.

---

from (1) $\quad v\frac{dv}{ds} = -g\left(1 + \frac{y}{c}\right)\frac{dy}{ds}$

$$vdv = -g\left(1 + \frac{y}{c}\right)dy$$

$$\frac{v^2}{2} = D - g\left(y + \frac{y^2}{2c}\right)$$

$D = \frac{4gc}{2} + g\left(c + \frac{c^2}{2c}\right) = \frac{7}{2}gc$

since $\left[\text{when } y = c, v = 2\sqrt{gc}\right]$

Hence $v^2 = 7gc - 2g\left(y + \frac{y^2}{2c}\right)$

Let $y = Y$ when $v = 0$, then

$$Y^2 + 2cY - 7c^2 = 0$$

$$Y = c(\pm 2\sqrt{2} - 1)$$

The negative sign is neglected, as y is always positive on this catenary.

So $Y = c(2\sqrt{2} - 1)$

SWITCH ON.

$$\left[\text{K.E.}\right]_{\text{end}} = \left[\text{K.E.}\right]_{\text{beginning}} + \left[\text{W.D. by forces}\right]$$

$$0 = \tfrac{m}{2}(4gc) - mg(Y-c) + \int_c^Y \left(\frac{-mg\,y}{c}\right) dy$$

After integration and simplification, this leads to the same quadratic equation for Y.

$$\text{i.e. } Y^2 + 2cY - 7c^2 = 0$$

SWITCH ON.

From (2) $\quad R = \dfrac{mv^2}{|\rho|} + mg\cos\psi\left(1 + \dfrac{y}{c}\right)$

$$y = c\cosh\left(\frac{x}{c}\right); \quad \frac{dy}{dx} = \sinh\left(\frac{x}{c}\right); \quad \frac{d^2y}{dx^2} = \frac{1}{c}\cosh\left(\frac{x}{c}\right)$$

$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{y^2}{c}$$

At the apex, $y = c$, $\psi = 0$, $v = 2\sqrt{gc}$

$$R = \frac{m(4gc)}{c} + mg\left(1 + \frac{c}{c}\right) = 6mg$$

SWITCH ON.

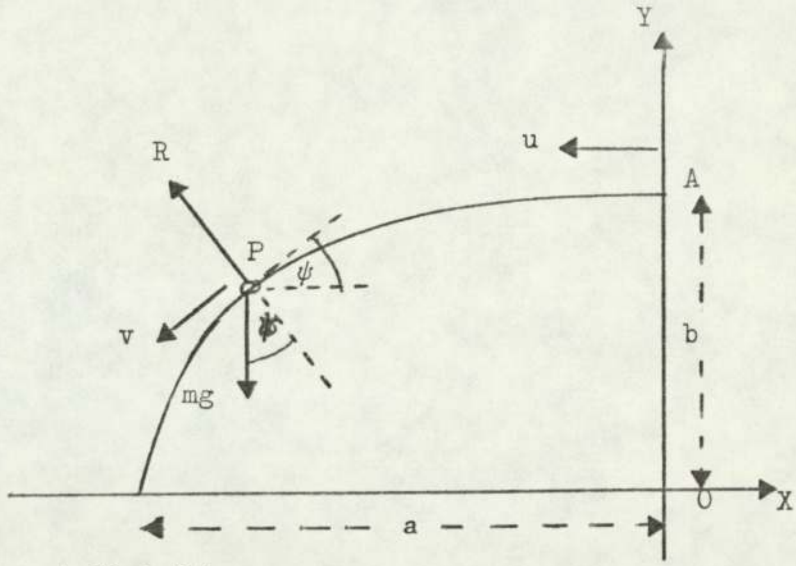
EXAMPLE 8.

A smooth wire is bent into the form of an ellipse, and is fixed in a vertical plane, with the y axis vertical, ($x = a\cos\theta$, $y = b\sin\theta$). A bead is threaded on the wire, and projected from the highest point, with speed u. Find the reaction between the bead and the wire in the ensuing motion, and show that it vanishes, when the bead reaches the point given by, $(a^2-b^2)\sin^3\theta + 3b^2\sin\theta - 2b^2 = u^2b/g$ assuming that $a^2 > u^2b/g$.

(The diagram is overleaf)

At time t

N.B. ψ is positive in this configuration.



Zero of P.E.

When you have studied this,

SWITCH ON.

Use 'F = ma' along the inward normal,

$$\frac{mv^2}{|\rho|} = mg\cos\psi - R \qquad \ldots (1)$$

$$\Big[K.E. + P.E.\Big]_P = \Big[K.E. + P.E.\Big]_A$$

$$\tfrac{1}{2}mv^2 + mgb\sin\theta = \tfrac{1}{2}mu^2 + mgb$$

$$v^2 = u^2 + 2gb(1-\sin\theta)$$

SWITCH ON.

$$x = a\cos\theta \qquad \frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta \qquad \frac{d^2y}{dx^2} = \frac{b}{a}.\operatorname{cosec}^2\theta.\frac{d\theta}{dx} = -\frac{b}{a^2}.\frac{1}{\sin^3\theta}$$

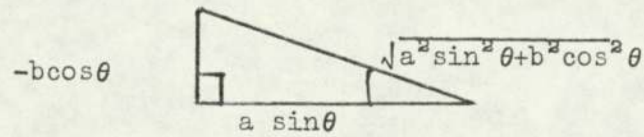
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = -\frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

$$\tan\psi = \frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

ψ is positive and hence $\tan\psi$ must be positive, but as $x = a\cos\theta$ is negative, so θ is obtuse and hence $\tan\psi$ does calculate to be positive.

(Continued on the next page).

$$\cos\psi = \frac{a\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$ which is positive for θ obtuse.



Substitute in (1) $R = mg\cos\psi - \dfrac{mv^2}{|\rho|}$

$$R = \frac{mga\sin\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}} - \frac{m[u^2+2gb(1-\sin\theta)]\,ab}{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}$$

which simplifies to,

$$R = \frac{mag[\sin^3\theta(a^2-b^2)+3b^2\sin\theta-2b^2]-mu^2ab}{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}$$

SWITCH ON.

Put $R = 0$ to give,

$$(a^2-b^2)\sin^3\theta + 3b^2\sin\theta - 2b^2 = \frac{u^2b}{g}$$

At the start of the motion $\theta = \frac{\pi}{2}$ and R will be positive if,

$$g\left[(a^2-b^2)+3b^2-2b^2\right] > u^2b$$

$$\text{i.e. } a^2 > \frac{u^2b}{g}$$

N.B. at the x axis $\theta = \pi$ and R is negative.

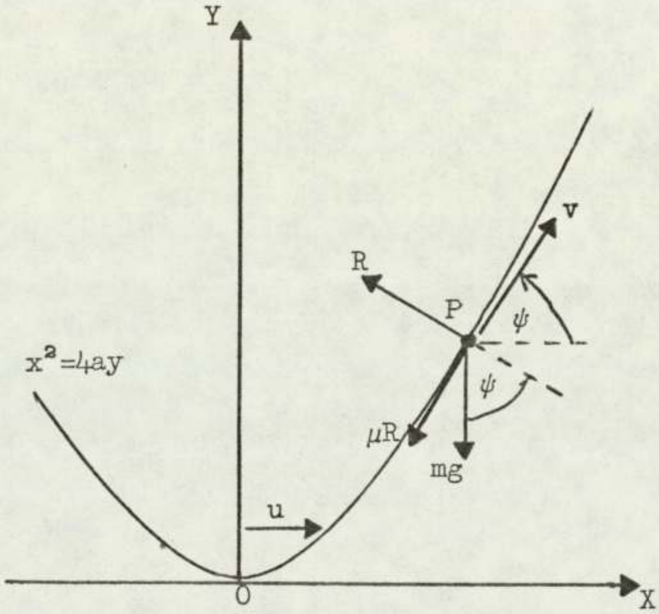
<u>SWITCH ON</u>.

EXAMPLE 9.

A rough, parabolic wire, $x^2 = 4ay$, is placed with its axis vertical, and its vertex downwards. A bead is projected along it, from the vertex, with velocity u. Show that the bead will first come to rest, where,

$$e^{\mu\theta} \sec\theta = \left(1 + \frac{u^2}{2ga}\right)^{\frac{1}{2}}$$

if μ is the coefficient of friction, and θ the angle between the tangent at the point, and the x axis.



N.B. ψ is positive acute in this configuration

When you have studied this,

SWITCH ON.

Use 'F = ma'

tangentially, $\quad m\ddot{s} = mv\dfrac{dv}{ds} = -mg.\sin\psi - \mu R$

inward normally, $\quad \dfrac{mv^2}{|\rho|} = R - mg\cos\psi$

eliminate R, $\quad v\dfrac{dv}{ds} + \dfrac{\mu v^2}{|\rho|} = -g(\sin\psi + \mu\cos\psi)$

(Continued on the next page).

Now $x^2 = 4ay$, and $\frac{dy}{dx} = \frac{x}{2a} = \tan\psi$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = 2a \sec^3\psi$$

So ρ is positive and $|\rho| = \rho = \frac{ds}{d\psi}$

Multiply through by ρ to give:

$$v\frac{dv}{d\psi} + \mu v^2 = -\ 2ag \sec^3\psi(\sin\psi + \mu\cos\psi)$$

SWITCH ON.

---

put $v^2 = p$ and $2v\frac{dv}{d\psi} = \frac{dp}{d\psi}$

$$\frac{dp}{d\psi} + 2\mu p = -\ 4ag \sec^2\psi(\tan\psi + \mu)$$

The I.F. $= e^{2\mu\psi}$

$$\frac{pe^{2\mu\psi}}{4ag} = -\int e^{2\mu\psi}(\sec^2\psi\tan\psi + \mu\sec^2\psi)d\psi$$

SWITCH ON.

Consider $\int e^{2\mu\psi}\sec^2\psi.\tan\psi\, d\psi + \mu\int e^{2\mu\psi}.\sec^2\psi\, d\psi$

$$= \int e^{2\mu\psi}\sec\psi(\sec\psi.\tan\psi)d\psi + \mu\int e^{2\mu\psi}.\sec^2\psi\, d\psi$$

$$= \int e^{2\mu\psi}\sec\psi\frac{d}{d\psi}(\sec\psi)d\psi + \mu\int e^{2\mu\psi}.\sec^2\psi\, d\psi.$$

$$= \left[\frac{e^{2\mu\psi}\sec^2\psi}{2} - \mu\int e^{2\mu\psi}\sec^2\psi\, d\psi\right] + \mu\int e^{2\mu\psi}\sec^2\psi\, d\psi + C$$

$$= \frac{e^{2\mu\psi}\sec^2\psi}{2} + C$$

SWITCH ON.

Hence $\frac{pe^{2\mu\psi}}{4ag} = -\frac{e^{2\mu\psi}\sec^2\psi}{2} + C_1$

when $\psi = 0$, $p = u^2$ and $C_1 = \frac{u^2}{4ag} + \frac{1}{2}$

In the general position

$$\frac{v^2 e^{2\mu\psi}}{4ag} = \frac{u^2}{4ag} + \frac{1}{2} - \frac{1}{2} e^{2\mu\psi} . \sec^2\psi$$

when $v = 0$ so $\psi = \theta$ and

$$\frac{u^2}{2ag} + 1 = e^{2\mu\theta} . \sec^2\theta$$

$$\text{i.e. } e^{\mu\theta} \sec\theta = \left(1 + \frac{u^2}{2ag}\right)^{\frac{1}{2}}$$

Take the positive root as the left hand side must be positive.

SWITCH ON.

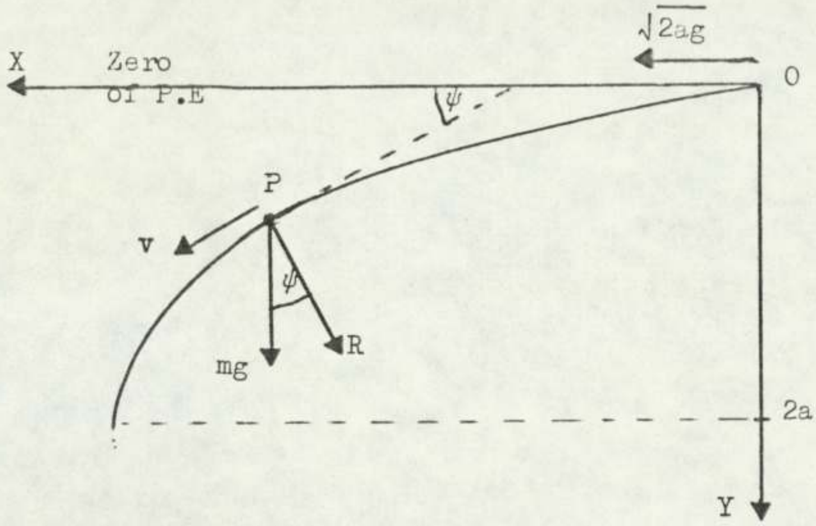
EXAMPLE 10.

A smooth wire in the shape of cycloidal arch (intrinsic equation, $s = 4a \sin\psi$), rests on its cusps with its axis of symmetry vertical. A particle is projected horizontally from the vertex, so that it is compelled to follow the arch internally. Show, that if the speed of projection is $\sqrt{2ag}$, the speed of the particle while still in contact with the wire, is

$$\sqrt{\left[2ag(1 + 2\sin^2\psi)\right]}$$

and the particle will fall from the wire when it has descended a vertical distance $\frac{1}{2}a$.

Do not read the solution overleaf until you have completed your own.



N.B. ψ is positive acute in the configuration and y is positive.

Energy equation

$$[K.E. + P.E.]_P = [K.E. + P.E.]_O$$

$$\tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m.2ag + 0$$

$$v^2 = 2g(a+y)$$

For the cycloid $y = a(1 - \cos 2\psi)$, hence

$$2g[a+a(1 - \cos 2\psi)] = v^2 = 2ag(1 + 2\sin^2\psi).$$

Use 'F = ma' at P. along the inward normal.

$$\frac{mv^2}{|\rho|} = mg\cos\psi + R$$

The particle will lose contact with the wire when R = 0

$$\text{i.e. } \frac{mv^2}{|\rho|} = mg\cos\psi$$

$\rho = \dfrac{ds}{d\psi} = 4a\cos\psi$ and hence $\cos\psi = \dfrac{\sqrt{3}}{2}$ and $y = \dfrac{a}{2}$

Notice how easy it is to find ρ from the intrinsic equation.

<u>SWITCH ON</u>.

SCRIPT FOR TAPE 8

RESTRICTED MOTION

1. This is the tape to be used with Booklet 8 of 'A Revision Course in Particle Dynamics', by Kenneth Jackson, and is concerned with 'Restricted Motion'. Early examples are about motion in a circle, and later ones about motion along general plane curves. In all of these, you will remember, a body is forced to travel along the curves by the interaction of forces, and their reaction to them. To consider these motions, it is convenient to use normal and tangential components of acceleration. These are revised for you in the first section of the booklet. Read this carefully. SWITCH OFF.

2. The reaction experienced by a particle will depend upon the physical nature of the problem. A bead on a wire, for example, can feel both inward and outward reactions, depending on its position, and always stay on the wire, but only a single direction of reaction is possible on an inner or outer surface. In such cases there is often a breakdown of the restricted motion. Now look at the first question and diagram carefully. We shall work through part of this, which deals with the special case of vertical motion on a circle. SWITCH OFF.

3. The conservation of energy, shows that the speed at D is the same as the starting speed. To describe

the motion between B and D, we introduce the angle theta measured from the fixed vertical OB. The reaction is zero when theta = 45 degrees, and hence we may calculate u, and the velocity of change over. Follow this working in the next section of the booklet. SWITCH OFF.

4. To determine the velocity at B, use the energy equation again. To finish the question, draw another diagram showing the motion between B and C, and the angle to the vertical OB. The particle will leave the open tube when the reaction is zero. Usethis condition with F = ma radially, and the energy equation to find where this happens. Check your method in the next section. SWITCH OFF.

5. The second question is also on circular motion, and we shall work through this in stages. Read it, and study the diagram. SWITCH OFF.

6. When the particle is at A, the string is slack, so it will fall freely from A to B when the string becomes taut, with an impulse along the inward radius. This impulse is the change of momentum in this direction. The particle will begin to move along the circular arc BC, with velocity V_2, which is the tangential component of the previous vertical velocity. Complete the working as far as V_3, and then check in the booklet. SWITCH OFF.

7. You will need another diagram showing the motion beyond C, and the energy equation will complete the question for you. Check this afterwards. SWITCH OFF.

8. The third example concerns a conical pendulum

with two strings. Read this carefully and study the diagram. SWITCH OFF.

9. As both A and B are rotating in horizontal circles about the vertical axis, we shall need equations for vertical equilibrium, as well as F = ma radially. Write out these equations, and then check them in the next section. SWITCH OFF.

10. By manipulation and division, you can introduce tan alpha and tan beta as required. Carry out these processes and then check your working. SWITCH OFF.

11. To consider the effects of making alpha and beta small introduce a simple radian approximation, and manipulate equations 5 and 6, to obtain the first result. Then use this result to complete the question. Check your working afterwards. SWITCH OFF.

12. Read question four which is about a bead on a parabolic wire, and we shall work through part of this together. This will involve the more general form of F = ma along the inward normal. Notice in the diagram, the direction of the axes and forces concerned. SWITCH OFF.

13. At P, the angle between the horizontal and the tangential velocity v, is labelled alpha. This angle is normally designated ψ, but difficulties will be avoided if this is only done when ψ is positive. This configuration is such that ψ is an acute positive angle and hence alph equals ψ. By using the energy equation you should be able to obtaine the general velocity. Write, alos, the equation of motiong along the inward normal, and

then check this afterwards. SWITCH OFF.

14. You can now find the reaction when $y = 2a$. Remember that $\tan \psi = dy/dx$, and p can be calculated from the formula involving the first and second derivations. Attempt this, and check your working afterwards. SWITCH OFF.

15. The fifth example concerns the motion of a particle along a different curve. Read this, and study the diagram carefully. SWITCH OFF.

16. This example shows the care that must be taken with the angle ψ, as in this configuation alpha = $-\psi$. Again, start by finding the velocity, using the energy equation, and then the normal equation of motion. Work with alpha, and later introduce the angle ψ. Try this question for yourself, checking it afterwards in the next section. SWITCH OFF.

17. You should now consider the next question, which is about motion along a cycloid. Read this, and study the diagram, so that we may work through part of this together. SWITCH OFF.

18. You will remember that the path of a point on a rolling circle, can be described in terms of the angle rolled through, and the radius. By noting the relationship between this angle and ψ, the parametric equations can be obtained. By elimination of x and y, we are able to produce the intrinsic equation between S and ψ. These are needed in this question. One particular advantage of the intrinsic equation is the ease with which the

radius of curvature can be found, although this is not required in this question. We start by finding the equation for tangential motion, and then integrate to find an expression for velocity. Follow this in the section below. SWITCH OFF.

19. This expression could, of course, be obtained by using the energy equation between A and P. Do this for yourself, using the x axis as the zero of potential energy. When you agree with this, switch on the tape again. SWITCH OFF.

20. To complete the question we need to find ds/dt and then integrate again from A to the origin. Choose the negative square root as S is decreasing while t increases. Do this and check your working afterwards. SWITCH OFF.

21. This is one way of solving this particular example, but notice in the equation of motion, that there is a restoring force, proportional to the displacement S. This means that the solution of the differential equation can be quoted immediately. Derive this alternative solution, and then check this in the next section of the booklet. SWITCH OFF.

22. The next question is about a particle moving in a tube in the form of a catenary. When you have read example 7, draw a diagram showing the forces and velocities. When you have completed this, write the normal and tangential equations of motion for the particle. Check these with those in the booklet, afterwards. SWITCH OFF.

23. The two downward foces will eventually bring

the particle to rest, and then start it moving towards the vertex again. Because the tube is smooth, the particle will oscillate continually about the vertex. We are again able to use two methods of finding s, energy or integration. Carry out the integration first using vdv/dS for $\ddot{S}$ and dy/ds for sinψ. The variables v and y can then be separated. The positions of instantaneous rest are found by the standard method of putting v = 0. Check these afterwards. SWITCH OFF.

24. You should now find v, using the energy equation Remember, you will need to find the work done by the variable force. Again, check this in the booklet afterwards. SWITCH OFF.

25. You will notice that in this second approach, we have applied the nergy equation directly from the start, to the position of rest, and avoided the step of putting v = 0. To find the reaction at the apex, use equation 2, with the appropriate values for v, p and ψ. Check this afterwards im the next section. SWITCH OFF.

26. The next question is about a bead on an elliptical wire, which will involve another radius of curvature. Read it, and study the diagram. SWITCH OFF.

27. Notice that by launching the bead to the left, we have a positive acute configuration for ψ. However, the x value is now negative and this must be borne in mind. Again, write the equation of motion for the particle along the inward normal, and use the energy equation to find v. When you have done this, check it below. SWITCH OFF.

28. You should now be able to find the reaction, but remember that parametric differentiation is needed to find dy/dx, and extra care should be taken with the second derivative. When you have finished this, checkit below. SWITCH OFF.

29. By equating R to zero, you are able to obtain the given equation. There is, however, the additional requirement that the reaction should be positive at the start of the motion. Insert the appropriate value for theta, and deduce the inequality. Check your method afterwards. SWITCH OFF.

30. The next question, example 9, introduces an extra complication with a frictional force acting along a parabolic wire. Read this, and look at the position of this force in the diagram. SWITCH OFF.

31. We can now write the normal and tangential equations of motion. Both of these contain R, which must be eliminated. As we wish to integrate with respect to psi, we must also manipulate ro. Follow this working in the next section. SWITCH OFF.

32. This differential equation reduces to a standard first order linear using the substitution $v^2 = p$. It can then be solved with an integrating factor. Determine this, and apply it to the equation. Check this afterwards. SWITCH OFF.

33. The remaining integral needs to be determined separately. The straightforward method of integrating this expression by parts, is to split the term involving the tangent into the secant and its differential coefficient.

Then carry out this integral. Afterwards, check this. SWITCH OFF.

34. This result needs to be considered in the general equation, and the constant of integration determined using initial velocity. To obtain the final result, find the angle given when v = zero in the expression. Check this in the next section afterwards. SWITCH OFF.

35. The last question is about a particle moving on the inside of a cycloidal wire, that is, it is only restricted on one side. Read it thoroughly and draw your diagram showing a position of the particle before it falls from the wire. Attempt this question in the standard way, that is, using energy to find the velocity, and making the reaction zero to find the point of departure. When you have finished, check your working overleaf. SWITCH OFF.

36. This completes the work on 'Restricted Motion'. Please rewind the tape before you remove it from the machine. Thank you!

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

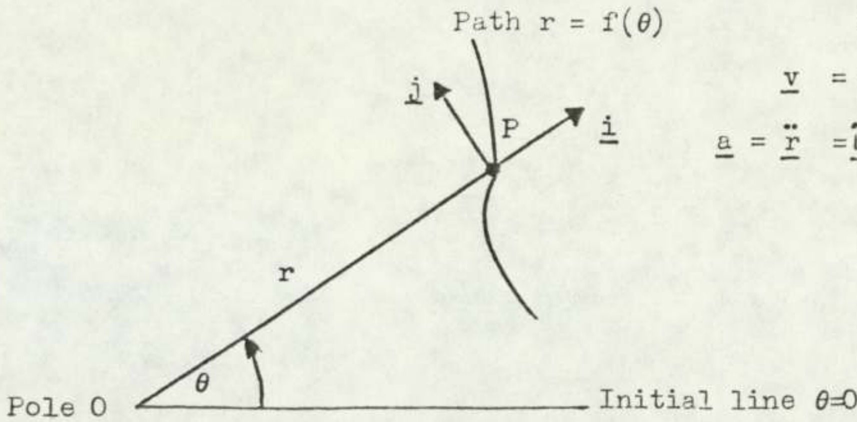
BOOKLET 9 to be used with TAPE 9

"Central Forces"

Read the instructions on page 1 thoroughly.

(Omitted for convenience.)

Polar Coordinates.



$$\underline{r} = r\underline{i}$$

$$\underline{v} = \dot{\underline{r}} = \dot{r}\underline{i} + r\dot{\theta}\underline{j}$$

$$\underline{a} = \ddot{\underline{r}} = \underline{i}(\ddot{r} - r\dot{\theta}^2) + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underline{j}$$

Hence the acceleration in the outward radial direction = $(\ddot{r} - r\dot{\theta}^2)$ and transverse acceleration = $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$ in the sense of θ increasing.

Under the action of a force directed towards the pole,

'F = ma' gives $\frac{m}{r}\cdot\frac{d}{dt}(r^2\dot{\theta}) = 0$ i.e. $r^2\dot{\theta} = C = r(r\dot{\theta})\Big/_{t=0}$

Note $|\underline{v}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \neq \dot{r}$

When you have studied this,

SWITCH ON.

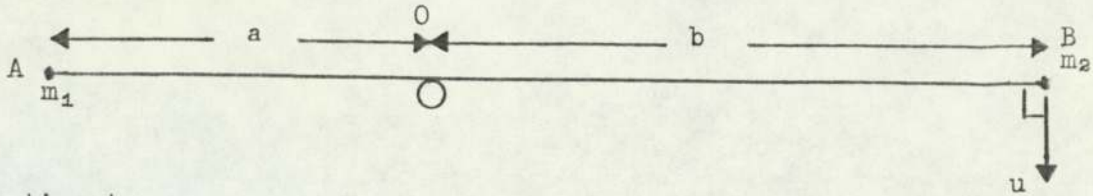
EXAMPLE 1

Masses m_1 and m_2 are attached to the ends of a light inextensible string, AOB, and rest on a smooth horizontal table. The string is in contact with a fixed smooth peg at O, and the portions of the string OA (= a) and OB (= b) are in a straight line. The mass m_2 is projected horizontally with a velocity u perpendicular to OB. If the string remains in contact with the peg and all motion takes place on the table, prove that the mass m_1 reaches the peg with velocity,

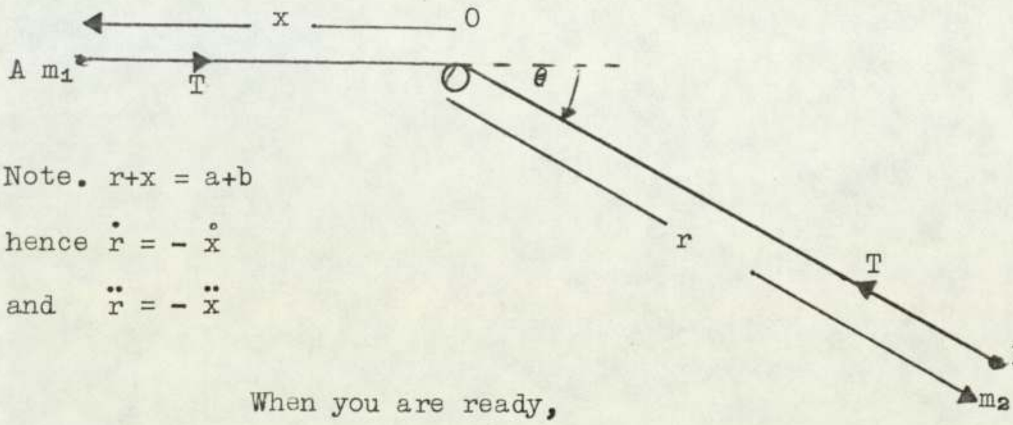
$$\frac{u}{(a+b)}\sqrt{\frac{m_2\,a(2b+a)}{(m_1+m_2)}}$$

The diagram is overleaf.

Initial Position



At time t



Note. $r+x = a+b$

hence $\dot{r} = -\dot{x}$

and $\ddot{r} = -\ddot{x}$

When you are ready,

SWITCH ON.

Use 'F = ma' for m_1 at A

$$m_1\ddot{x} = -T \qquad \ldots (1)$$

for m_2 at B, radially, $m_2(\ddot{r} - r\dot{\theta}^2) = -T$ ... (2)

No transverse force gives $r^2\dot{\theta} = \text{constant} = r(r\dot{\theta})\Big|_{t=o} = bu$ ... (3)

When you agree with these,

SWITCH ON.

From (1) above, $m_1\ddot{r} = T$

Add to (2) $(m_1+m_2)\ddot{r} - m_2 r\dot{\theta}^2 = 0$

from (3), $r^2\dot{\theta}^2 = \dfrac{b^2u^2}{r^3}$,

Hence, $\ddot{r} = \dfrac{m_2\, b^2u^2}{r^3(m_1+m_2)}$

Multiply by $\dot{r}$ and integrate.

$$\frac{\dot{r}^2}{2} = C - \frac{1}{2r^2} \cdot \frac{m_2 b^2 u^2}{(m_1+m_2)}$$

when $\theta = 0$, $r = b$, $\dot{r} = 0$, $C = \dfrac{m_2 \cdot u^2}{2(m_1+m_2)}$

$$\dot{r}^2 = \frac{m_2 u^2}{(m_1+m_2)}\left(1 - \frac{b^2}{r^2}\right)$$

(Continued on the next page)

$$\dot{r} = \frac{u}{r}\sqrt{\frac{m_2(r^2-b^2)}{(m_1+m_2)}}$$

The positive root is required for m_1 to approach the peg.

When m_1 reaches the peg, $r = a + b$

giving, velocity of $m_1 = \dot{x} = -\dot{r} = \dfrac{-u}{(a+b)}\sqrt{\dfrac{m_2 a(2b+a)}{(m_1+m_2)}}$

Hence velocity m_1 is negative, as expected.

SWITCH ON.

$$[\text{K.E.} + \text{P.E.}]_t = [\text{K.E.} + \text{P.E.}]_0$$

In this case all the energy is kinetic.

$$\tfrac{1}{2}m_2(\dot{r}^2 + r^2\dot{\theta}^2) + \tfrac{1}{2}m_1(-\dot{x})^2 = \tfrac{1}{2}m_2u^2$$

$$\text{use} \qquad r^2\dot{\theta}^2 = \frac{b^2u^2}{r^2}$$

and hence $m_2\left(\dot{r}^2 + \dfrac{b^2u^2}{r^2}\right) + m_1\dot{r}^2 = m_2u^2$

and $\dot{r}^2 = \dfrac{m_2u^2(r^2-b^2)}{r^2(m_1+m_2)}$ as before.

SWITCH ON.

EXAMPLE 2.

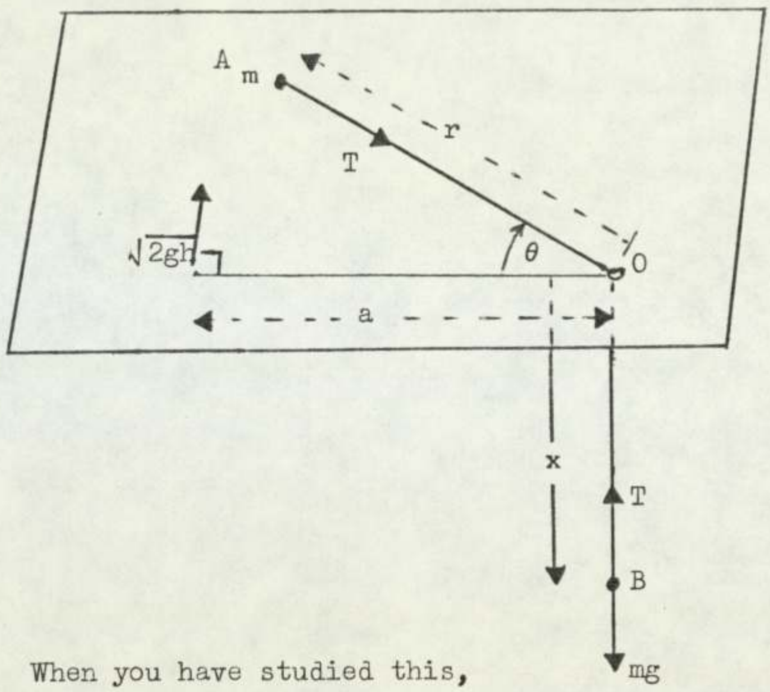
A particle, of mass, m, on a smooth table, is attached by a light string, passing through a small hole in the table, and carries an equal particle, hanging vertically. The former particle is projected along the table, at right angles to the string, with velocity $\sqrt{2gh}$, when at a distance 'a' from the hole. Find the tension in the string in the general position. If r is the distance from the hole, in the subsequent motion, show that,

i) $\dot{r}^2 = gh\left(1 - \dfrac{a^2}{r^2}\right) + g(a-r)$

ii) the lower particle will be pulled up to the hole, if the total length of the string is less than

$$\frac{h}{2} + \sqrt{ah + \frac{h^2}{4}}$$

(The diagram is overleaf).



When you have studied this,

SWITCH ON.

'F = ma' for A,

radially, $m(\ddot{r} - r\dot{\theta}^2) = - T$ ... (1)

transverse direction gives $r^2\dot{\theta} = \text{constant} = r(r\dot{\theta})\Big|_{t=0} = a\sqrt{2gh}$ (2)

for B, vertically, $m\ddot{x} = mg - T$ ... (3)

From the diagram, $x + r = \text{Constant} = C$ (say)

hence $\dot{x} = - \dot{r}$ . . . (4) and $\ddot{x} = - \ddot{r}$ ... (5)

When you agree with these,

SWITCH ON.

Rewrite (3) as $m(\ddot{r} + g) = T$ ... (6)

Eliminate $\ddot{r}$ with (1) to give

$$m(r\dot{\theta}^2 + g) = 2T$$

from (2) $$m\left(\frac{2a^2gh}{r^3} + g\right) = 2T$$

hence $$T = \tfrac{1}{2}mg\left(1 + \frac{2a^2h}{r^3}\right)$$

Add (1) and (6) to give,

$$m(2\ddot{r} - r\dot{\theta}^2 + g) = 0$$

From (2) $$2\ddot{r} = \frac{2a^2gh}{r^3} - g$$

(Continued on the next page).

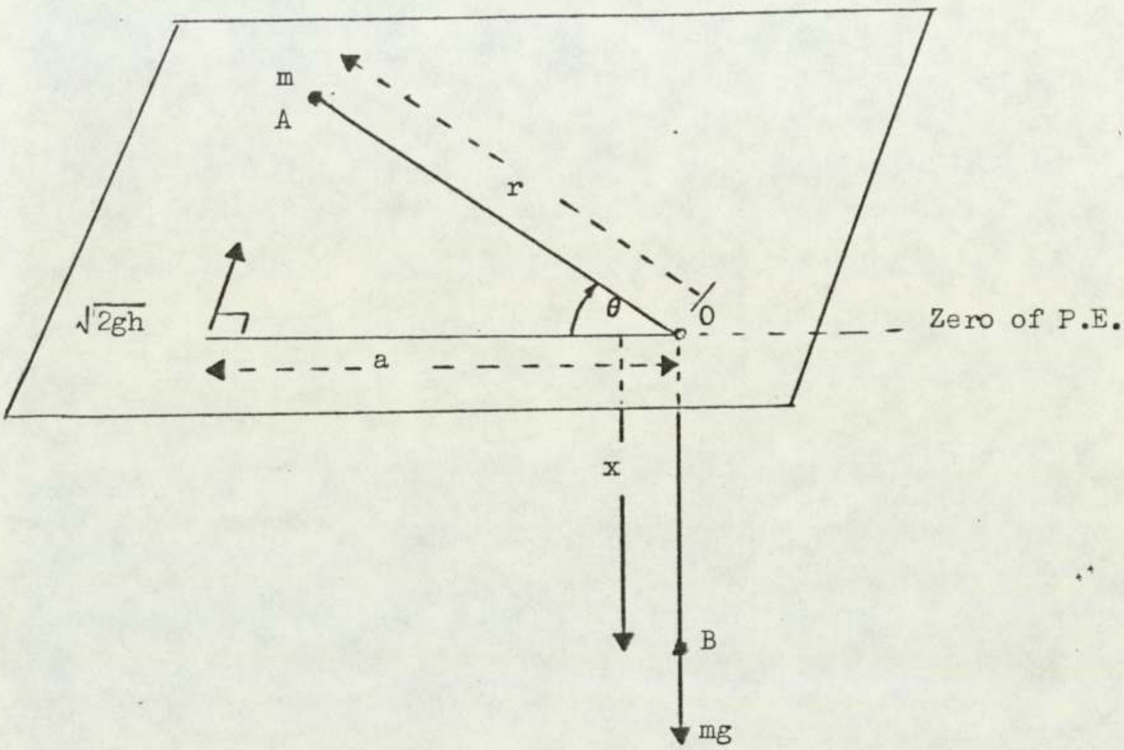
Multiply by $\dot{r}$ and integrate

$$\dot{r}^2 = D - \frac{a^2gh}{r^2} - gr$$

when $\theta = 0$, $\dot{r} = 0$ and $r = a$, hence $D = g(a+h)$

i.e. $$\dot{r}^2 = gh\left(1 - \frac{a^2}{r^2}\right) + g(a-r)$$

SWITCH ON.



let ℓ = original length of string below 0, then

$$x + r = \text{constant} = \ell + a$$

$$\text{or } \ell = r + x - a$$

$$[K.E. + P.E.]_t = [K.E. + P.E.]_0$$

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \tfrac{1}{2}m(-\dot{x})^2 - mgx = \tfrac{1}{2}m(2gh) - mg(r+x-a)$$

i.e. $$\dot{r}^2 + \frac{2a^2gh}{r^2} + \dot{r}^2 = 2gh - 2g(r-a)$$

$$\dot{r}^2 = gh\left(1 - \frac{a^2}{r^2}\right) + g(a-r)$$

SWITCH ON.

Write $$\dot{r}^2 = gh\left(\frac{r^2-a^2}{r^2}\right) - g(r-a)$$

$$= \frac{g}{r^2}(r-a)\left[h(r+a) - r^2\right]$$

(Continued on the next page)

$$\dot{r}^2 = \frac{g}{r^2}(r-a)\left[ah + \frac{h^2}{4} - \left(r - \frac{h}{2}\right)^2\right]$$

Notice that $\dot{r} = 0$ when $r = a$, which is the initial condition. For the hanging particle to rise in the subsequent motion, r must be greater than a

i.e. $(r-a) > 0$

Since $\dot{r}^2$ must be positive, so the square bracket must also be positive.

$$\text{i.e.}\quad ah + \frac{h^2}{4} > \left(r - \frac{h}{2}\right)^2$$

$$\text{Thus}\quad r < \frac{h}{2} + \sqrt{ah + \frac{h^2}{4}}$$

$$\text{and } r_{max} = \frac{h}{2} + \sqrt{ah + \frac{h^2}{4}}$$

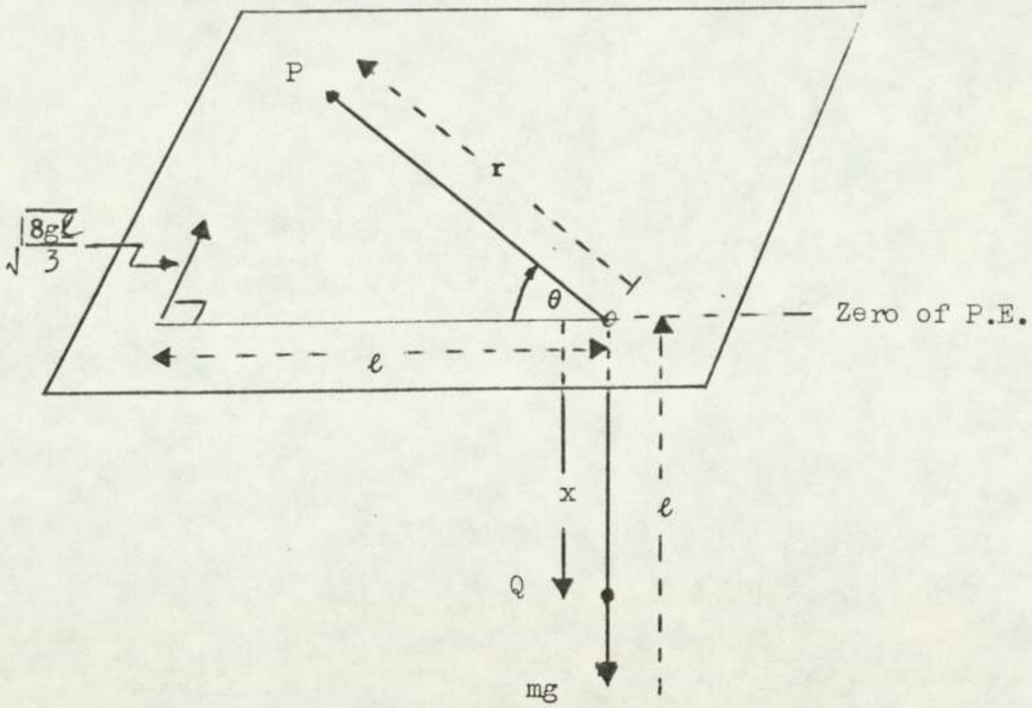
when $\dot{r}$ is again zero. Hence the hanging particle will be pulled to the hole if the total length of the string is less than r_{max}.

SWITCH ON.

EXAMPLE 3.

P and Q are two particles, each of mass m, connected by a light, inextensible string, of length 2ℓ, which passes through a small hole O, in a smooth horizontal table. P is free to slide on the table, Q hangs freely. Initially, OQ is of length ℓ, and P is projected at right angles to OP, with velocity $\sqrt{\frac{8g\ell}{3}}$. Find the maximum and minimum distances of P from O, in the ensuing motion. What do you conclude from one of these?

Do not read the solution overleaf until yours is complete.



From the diagram, $r+x = 2\ell$

hence, $\dot{r} = -\dot{x}$ . . . (1) and $\ddot{r} = -\ddot{x}$ ... (2)

Throughout the motion, $r^2\dot{\theta} = \text{constant} = r(r\dot{\theta})\Big|_{t=0} = \ell\sqrt{\frac{8g\ell}{3}}$ ... (3)

$$[\text{K.E.} + \text{P.E.}]_t = [\text{K.E.} + \text{P.E.}]_0$$

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \tfrac{1}{2}m(-\dot{x})^2 - mg(2\ell - r) = \tfrac{1}{2}m\left(\frac{8g\ell}{3}\right) - mg\ell$$

Hence, using (1) and (3), $\dot{r}^2 = \frac{7g\ell}{3} - \frac{4g\ell^3}{3r^2} - gr$

For max. and min. distances from 0, $\dot{r} = 0$

i.e. $3r^3 - 7r^2\ell + 4\ell^3 = 0$

$$(r-\ell)(3r^2 - 4r\ell - 4\ell^2) = 0$$

$$(r-\ell)(3r + 2\ell)(r-2\ell) = 0$$

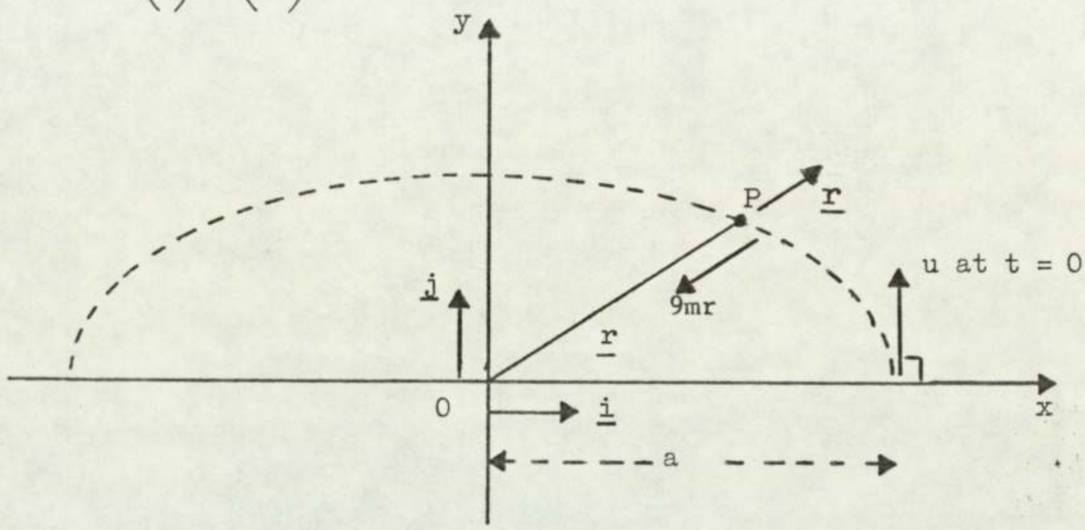
$r_{max} = 2\ell$ and $r_{min} = \ell$, as r cannot be negative.

When $r = 2\ell$, the particle Q has been lifted to the hole.

<u>SWITCH ON</u>.

EXAMPLE 4.

A particle P, of mass m, which moves in a plane, is attracted towards the origin O, of rectangular coordinates (x,y), by a force 9mr, where r = OP. Initially, the particle is projected from the point (a,0), with velocity u, parallel to the axis of y. Prove that the projectile describes the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{3y}{u}\right)^2 = 1$



When you have studied the vectors,

SWITCH ON.

$$'\underline{F} = m\underline{a}' \text{ gives}$$

$$m\ddot{\underline{r}} = -9mr\,\hat{\underline{r}} = -9m\underline{r}$$

$$\underline{i}\,\ddot{x} + \underline{j}\,\ddot{y} = -9(\underline{i}x + \underline{j}y)$$

Equate coefficients of $\underline{i}$ and $\underline{j}$ to give

$$\ddot{x} = -9x \quad \text{and} \quad \ddot{y} = -9y$$

Consider $\ddot{x} = -9x$, i.e. $\frac{d^2x}{dt^2} + 9x = 0$

$$x = A\cos 3t + B\sin 3t$$

when $t = 0$, $x = a$, hence $A = a$

$$\dot{x} = -3A\sin 3t + 3B\cos 3t$$

when $t = 0$, $\dot{x} = 0$, hence $B = 0$

i.e. $x = a\cos 3t$ ... (1)

SWITCH ON.

The differential equation for y,is $\ddot{y} + 9y = 0$

so $y = C \cos 3t + D \sin 3t$

when $t = 0$, $y = 0$, hence $C = 0$.

$$\dot{y} = 3D \cos 3t$$

when $t = 0$, $\dot{y} = u$, and $D = \frac{u}{3}$

$$\text{so} \quad y = \frac{u}{3} \sin 3t$$

$$\text{i.e.} \quad \frac{3y}{u} = \sin 3t$$

Rewrite (1) as $\frac{x}{a} = \cos 3t$

Square and add these equations to give

$$\left(\frac{x}{a}\right)^2 + \left(\frac{3y}{u}\right)^2 = 1 \qquad \text{which is the given orbit.}$$

SWITCH ON.

EXAMPLE 5.

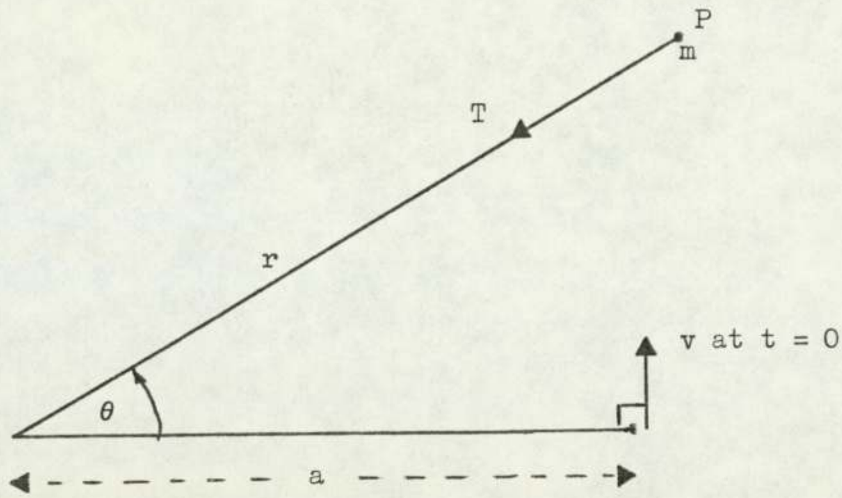
A particle,of mass m,is attached to an elastic string,of modulus λ, length 'a', the other end of which,is attached to a fixed point,on a smooth horizontal table. The particle lies on the table,and is projected with velocity v, at right angles to the string, which is initially just taut. Show that the greatest extension of the string in the subsequent motion is

$$\frac{amv^2}{\lambda}\left(\frac{a + b}{b^2}\right)$$

where b is the greatest length of the string.

Do not read the solution overleaf, until you have completed your own.

at time t



Throughout the motion, $r^2\dot{\theta}$ = constant = $r(r\dot{\theta})\Big|_{t=0}$ = av

P.E. in the string $= \dfrac{\lambda(r-a)^2}{2a}$

$$[\text{K.E.} + \text{P.E.}]_t = [\text{K.E.} + \text{P.E.}]_0$$

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\lambda(r-a)^2}{2a} = \tfrac{1}{2}mv^2$$

$$m\left(\dot{r}^2 + \frac{a^2v^2}{r^2}\right) + \frac{\lambda(r-a)^2}{a} = mv^2$$

$$\dot{r}^2 = v^2\left(1 - \frac{a^2}{r^2}\right) - \frac{\lambda(r-a)^2}{am}$$

$$= (r-a)\left[\frac{v^2}{r^2}(r+a) - \frac{\lambda}{am}(r-a)\right]$$

So $\dot{r} = 0$ when $r = a$, which is the start of the motion.

$\dot{r}$ is also zero when $r = b$, where

$$\frac{v^2}{b^2}(b+a) - \frac{\lambda}{am}(b-a) = 0$$

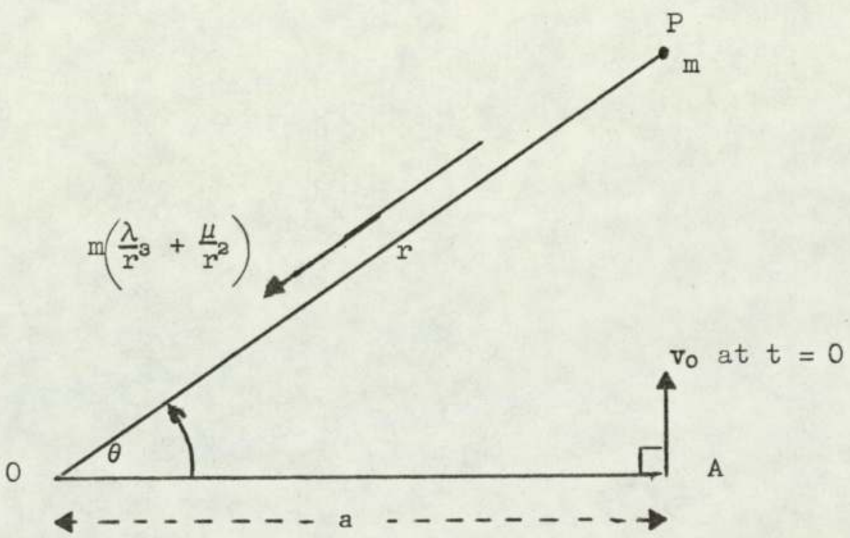
i.e. extension $= b - a = \dfrac{amv^2}{\lambda}\left(\dfrac{a+b}{b^2}\right)$

SWITCH ON.

EXAMPLE 6.

A particle, of mass m, is projected with velocity v_0, at a distance 'a' from the origin, and it moves under a central attractive force $m\left[\frac{\lambda}{r^3} + \frac{\mu}{r^2}\right]$. If the direction of projection is perpendicular to the radius vector, and $v_0^2 = \frac{\lambda}{a^2} + \frac{2\mu}{3a}$, show that the particle will again be moving at right angles to the radius vector, when $r = \frac{a}{2}$.

At time t



When you have studied this,

SWITCH ON.

Use 'F = ma',

radially, $m(\ddot{r} - r\dot{\theta}^2) = -m\left(\frac{\lambda}{r^3} + \frac{\mu}{r^2}\right)$

transverse direction, gives $r^2\dot{\theta} = \text{constant} = r(r\dot{\theta})\Big|_{t=0} = av_0$

When you have checked these,

SWITCH ON.

From these equations, $\ddot{r} = \dfrac{a^2v_0^2}{r^3} - \dfrac{\lambda}{r^3} - \dfrac{\mu}{r^2}$

Multiply by $\dot{r}$ and integrate,

$$\frac{\dot{r}^2}{2} = C - \frac{a^2v_0^2}{2r^2} + \frac{\lambda}{2r^2} + \frac{\mu}{r}$$

When $t = 0$, $\theta = 0$, $r = a$, and $\dot{r} = 0$

hence $C = \dfrac{v_0^2}{2} - \dfrac{\lambda}{2a^2} - \dfrac{\mu}{a} = \dfrac{\lambda}{2a^2} + \dfrac{\mu}{3a} - \dfrac{\lambda}{2a^2} - \dfrac{\mu}{a} = -\dfrac{2\mu}{3a}$

and $\dfrac{\dot{r}^2}{2} = \dfrac{\lambda}{2r^2} + \dfrac{\mu}{r} - \dfrac{2\mu}{3a} - \dfrac{a^2}{2r^2}\left(\dfrac{\lambda}{a^2} + \dfrac{2\mu}{3a}\right)$

i.e. $\dfrac{\dot{r}^2}{2} = \mu\left(\dfrac{1}{r} - \dfrac{2}{3a} - \dfrac{a}{3r^2}\right)$

When $\dot{r} = 0$ (i.e. at an apse) the particle moves at right angles to the radius vector.

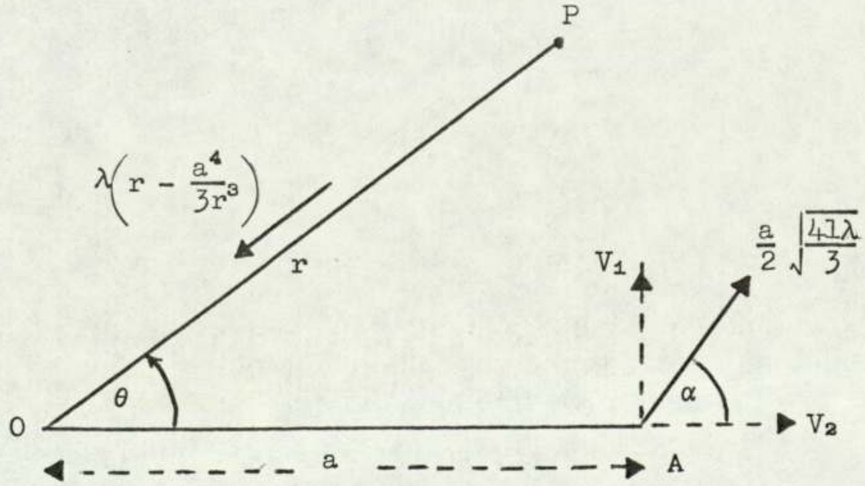
i.e. $$2r^2 - 3ar + a^2 = 0$$
$$(2r - a)(r - a) = 0$$

So $r = a$ at the start, and the other apse is at $r = \dfrac{a}{2}$.

SWITCH ON.

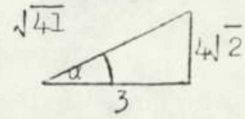
EXAMPLE 7.

A particle is projected from a point, at a distance 'a' from the centre of force, and moves under the action of the attractive force $\lambda\left(r - \dfrac{a^4}{3r^3}\right)$ per unit mass. The velocity of projection is $\dfrac{a}{2}\sqrt{\dfrac{41\lambda}{3}}$, at an angle $\tan^{-1}\left(\dfrac{4\sqrt{2}}{3}\right)$ with the outward radius vector. Show that the apsidal distances of the orbit are, 2a and $\dfrac{a\sqrt{3}}{2}$.



When you have studied this,
239.
SWITCH ON.

$$V_1 = \frac{a}{2}\sqrt{\frac{41\lambda}{3}}.\sin\alpha = \frac{a}{2}\sqrt{\frac{41\lambda}{3}}\cdot\frac{4\sqrt{2}}{\sqrt{41}} = 2a\sqrt{\frac{2\lambda}{3}}$$



$$V_2 = \frac{a}{2}\sqrt{\frac{41\lambda}{3}}.\cos\alpha = \frac{a}{2}\sqrt{\frac{41\lambda}{3}}\cdot\frac{3}{\sqrt{41}} = \frac{a\sqrt{3\lambda}}{2}$$

For the transverse motion $r^2\dot{\theta}$ = constant = $r(r\dot{\theta})\Big|_{t=o}$

$$= aV_1 = 2a^2\sqrt{\frac{2\lambda}{3}} \qquad \ldots \quad (1)$$

SWITCH ON.

'F = ma' radially,

$$m(\ddot{r} - r\dot{\theta}^2) = -m\lambda\left(r - \frac{a^4}{3r^3}\right)$$

From (1) $\quad \ddot{r} = \dfrac{3a^4\lambda}{r^3} - \lambda r$

Multiply by $\dot{r}$ and integrate,

$$\frac{\dot{r}^2}{2} = C - \frac{3a^4\lambda}{2r^2} - \frac{\lambda r^2}{2}$$

when $t = 0$, $\theta = 0$, $r = a$, and $\dot{r} = V_2 = \dfrac{a\sqrt{3\lambda}}{2}$

hence $C = \dfrac{19\lambda a^2}{8}$

$$\frac{\dot{r}^2}{2} = \frac{19\lambda a^2}{8} - \frac{3a^4\lambda}{2r^2} - \frac{\lambda r^2}{2}$$

At the apses, $\dot{r} = 0$

hence $\quad 4r^4 - 19a^2r^2 + 12a^4 = 0$

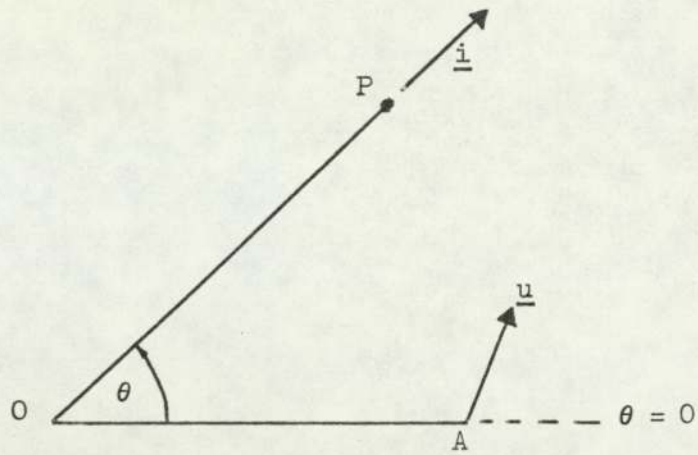
$$(4r^2 - 3a^2)(r^2 - 4a^2) = 0$$

$$r = \frac{a\sqrt{3}}{2} \text{ or } 2a$$

(Remember 'r' is always positive).

SWITCH ON.

Differential Equation of the orbit for a central attractive force.



$\underline{F} = -\,mF(r)\underline{i}$

$F(r) =$ the force function

The plane of the motion is defined by $\overline{OA}$ and $\underline{u}$.

The differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$$

where $u = \frac{1}{r}$, $F(u)$ = force function in terms of u.

and $h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=o}$

The solution of this differential equation contains two arbitrary constants, which must be evaluated from the given conditions.

When you are ready to proceed,

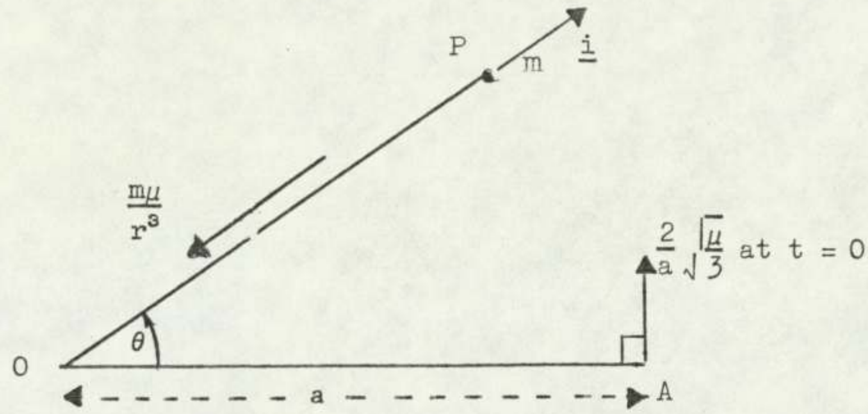
SWITCH ON.

EXAMPLE 8.

If a particle, P, moves in a plane under the action of an attractive force $\frac{\mu}{r^3}$ per unit mass towards the pole O, and the particle is projected from a point A, with velocity $\frac{2}{a}\sqrt{\frac{\mu}{3}}$, at right angles to OA, where OA = a, show that the particle describes the curve,

$$r = a\sec\frac{\theta}{2}$$

The diagram is overleaf.



When you are ready,

SWITCH ON.

$$F(r) = \frac{\mu}{r^3} \text{ and } F(u) = \mu u^3$$

$$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=o} = a.\frac{2}{a}\sqrt{\frac{\mu}{3}} = 2\sqrt{\frac{\mu}{3}}$$

D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^3 3}{4\mu u^2} = \frac{3u}{4}$$

i.e. $\frac{d^2u}{d\theta^2} + \frac{u}{4} = 0$

or $(D^2 + \frac{1}{4})u = 0$ where $D \equiv \frac{d}{d\theta}$

General solution is $u = A\sin\frac{\theta}{2} + B\cos\frac{\theta}{2}$

when $t = 0$, $\theta = 0$ and $u = \frac{1}{a}$, hence $B = \frac{1}{a}$

and $\frac{1}{r} = u = A\sin\frac{\theta}{2} + \frac{1}{a}\cos\frac{\theta}{2}$

SWITCH ON.

$$-\frac{1}{r^2}.\frac{dr}{dt} = \left(\frac{A}{2}\cos\frac{\theta}{2} - \frac{1}{2a}\sin\frac{\theta}{2}\right)\frac{d\theta}{dt}$$

$$-\dot{r} = \left(\frac{A}{2}\cos\frac{\theta}{2} - \frac{1}{2a}\sin\frac{\theta}{2}\right)r^2\dot{\theta}$$

when $t = 0, \theta = 0$, $\dot{r} = 0$, $r^2\dot{\theta} = h$, hence $A = 0$

$$\text{So } \frac{1}{r} = \frac{1}{a}\cos\frac{\theta}{2}, \text{ or } r = a\sec\frac{\theta}{2}$$

This curve is such that r increases steadily, as θ increases, and r eventually becomes infinite, when θ reaches 180°.

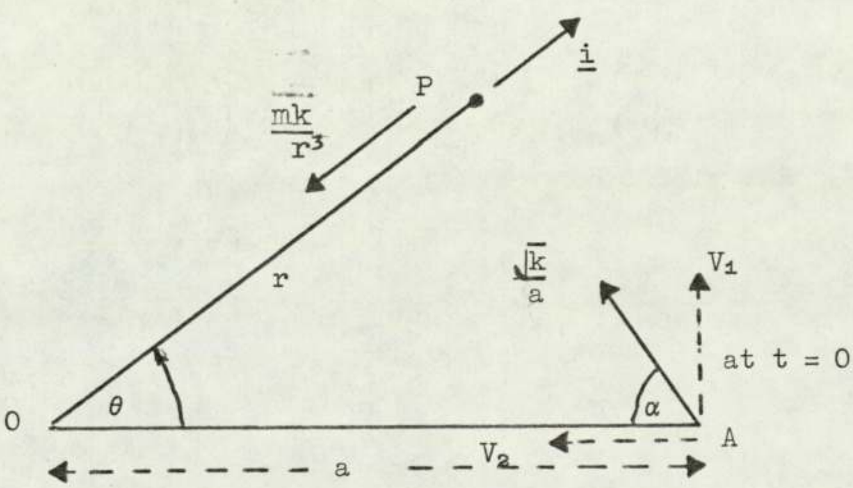
SWITCH ON.

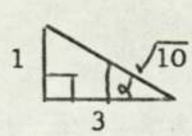
EXAMPLE 9.

A particle of mass, m, is attracted by a force $\frac{mk}{r^3}$, towards a fixed point O, where k is a constant. It is projected from a point A, which is distant 'a' from O, with velocity $\sqrt{\frac{k}{a}}$, at an angle $\tan^{-1}(\frac{1}{3})$ with $\overline{AO}$ (i.e. with the inward radial direction.) Show that the orbit is

$$r = a\,e^{-3\theta}.$$

Do not read this solution until you have completed your own.





$F(r) = \frac{k}{r^3}$, and $F(u) = ku^3$

$$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=0} = aV_1 = a\sqrt{\frac{k}{a}} \cdot \frac{1}{\sqrt{10}} = \sqrt{\frac{k}{10}}$$

D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$ which gives,

$$(D^2 - 9)u = 0 \text{ where } D \equiv \frac{d}{d\theta}$$

$$\text{or} \qquad (D-3)(D+3)u = 0$$

General solution is $\frac{1}{r} = u = Ae^{3\theta} + Be^{-3\theta}$

(Continued on the next page).

At $t = 0$, $\theta = 0$ and $u = \frac{1}{a}$

$$\text{i.e.} \quad \frac{1}{a} = A + B \qquad \ldots (1)$$

Differentiating with respect to t, gives

$$-\dot{r} = 3(r^2\dot{\theta})(Ae^{3\theta} - Be^{-3\theta})$$

when $t = 0$, $\theta = 0$, $r^2\dot{\theta} = h$

$$\text{and } \dot{r} = -V_2 = -\frac{\sqrt{k}}{a} \cdot \frac{3}{\sqrt{10}} = -\frac{3}{a}\sqrt{\frac{k}{10}}$$

$$\text{i.e.} \quad \frac{3}{a}\sqrt{\frac{k}{10}} = 3\sqrt{\frac{k}{10}}\,(A-B)$$

$$\text{hence } \frac{1}{a} = (A - B) \qquad \ldots (2)$$

From (1) and (2), $A = \frac{1}{a}$ and $B = 0$, so the orbit is,

$r = ae^{-3\theta}$, which is a spiral into the centre of force.

SWITCH ON.

APPENDIX 9 CONTINUED

SCRIPT FOR TAPE 9

CENTRAL FORCES

1. This is the tape to be used with Booklet 9 of 'A Revision Course in Particle Dynamics' by Kenneth Jackson. It is concerned with problems in two dimensions where the use of polar coordinates for velocity and acceleration components are appropriate. These are mainly connected with orbits, but also cover some examples on connected particles. You should read carefully the information revising this for you in the first section of the booklet. SWITCH OFF.

2. When a string is attached at some fixed point, or passes through a hole in a table top there is a natural pole for the motion, with a central force provided by the tension in the string. The first example is of this type, and we shall work through it together. Read the problem and study the two diagrams relevant to it. SWITCH OFF.

3. The peg O is the pole for the motion of B, while m_1 moves along a straight line AO. Notice that the variables x and r are measured from O, and remember the connections between velocity and acceleration components. As the string OB turns through angle θ, we can write the equations of motion at A and B. These are written for you below. Study them carefully. SWITCH OFF.

4. By eliminating T, θ and $\ddot{x}$ you can obtain an

expression for $\ddot{r}$, which can be integrated by using the integrating factor $\dot{r}$. Insert the initial conditions to obtain a general expression for the radial velocity of B. The question is completed by using the appropriate value for r when A reaches the peg. Check your work in the next section afterwards. SWITCH OFF.

5. This velocity may be also found quite neatly, using the energy equation, which is demonstrated for you in the next section. Notice how we need to use the two perpendicular components of velocity for the kinetic energy of m_2. SWITCH OFF.

6. The second question also concerns a string, but this time passing through a hole in a table. Read it, and study the diagram, and note that the hanging 'particle' introduces potential energy into this question. SWITCH OFF.

7. You should be able to write two equations of motion for A, another for the hanging particle B, and also the relationship between r and x using the information in this diagram. When you have completed these equations check them in the section below. SWITCH OFF.

8. You have here all the information required to solve the problem. Find the tension by eliminating $\ddot{r}$ between equations 1 and 3. The expression for $\dot{r}$ is again faound by using an integrating factor. Carry out these processes, and check them afterwards. SWITCH OFF.

9. As an alternative approach, you should also use the energy equation to deduce this expression for $\dot{r}$. Remember that there is potential energy for particle B

to be considered. When you have finished, check this in the next section of the booklet. SWITCH OFF.

10. The last answer concerning the length of string may be deduced from the expression for $\dot{r}^2$, which can only be positive or zero. By factorization and completion of the squarre, this expression can be rewritten as shown in the next section. The deduction on the length of string can then be completed. Follow this analysis carefully. SWITCH OFF.

11. The third example is similar to the previous one, and is for your own working. Again, use the energy equation. When you have finished, check your working on the next page. SWITCH OFF.

12. It is not always necessary to use polar coordinates when dealing with central forces, as is shown by example 4. Read the question and study the diagram carefully. SWITCH OFF.

13. When the central force is proportional to displacement, it is possible to work using cartesian coordinates. The question can be started by writing the equation of motion in vector form, as is shown in the next section. Follow this carefully. SWITCH OFF.

14. Solve the differential equation for y in the same way, and use the initial conditions to find the arbitrary constants. Eliminate the time t between the equations for x and y to obtain the ellipse. Check your working in the next section. SWITCH OFF.

15. Example five is for your own working and concerns an elastic string. You will find it convenient to use

polar coordinates and the energy equation again. Remember that the string will have a potential energy of its own. Check your working afterwards. SWITCH OFF.

16. The next example is about a more complex central force, and we shall work through this together. Read it, and study the diagram carefully. SWITCH OFF.

17. The action of this central attractive force is to cause the particle to move in the plane which is specified by the initial velocity and the radius vector. Thus the motion is again conveniently described using polar coordinates. Write the radial and transverse equations of motion in the usual way, and then check these below. SWITCH OFF.

18. Now find $\dot{r}$ by integration, using $\dot{r}$ as an integrating factor. You will find it helpful with this complicated force, to substitute for v nought in the constant of integration. Complete the problem, and remember that the particle moves at right angles to the radius vector when $\dot{v} = 0$. Afterwards, check your working overleaf. SWITCH OFF.

19. Example 7 is similar to the previous one. Read it, and note particularly in the diagram, the direction of the initial velocity. SWITCH OFF.

20. You will notice the added complication of having the particle projected at an angle alpha to the initial line OA. Resolve this velocity into its components V_1 and V_2 and find the constant of transverse motion. Check this working overleaf. SWITCH OFF.

21. Now find $\dot{r}$ by integration in the usual way. Remember, when you find the constant of integration, that the initial value of $\dot{r}$ is the velocity component V^2. Complete the question and then check your working below. SWITCH OFF.

22. In many problems connected with central attractive forces it is important to find the orbit. This can often be derived from a standard second order differential equation in u and θ, where $u = 1/r$. These ideas are revised for you overleaf. SWITCH OFF.

23. These equations are of fundamental importance in finding orbits and should be memorized. We shall illustrate the process in example 8, which we shall work through together. Read the example, and study the diagram. SWITCH OFF.

24. The advantage of this method, is that we can use our knowledge of differential equations to find a solution for u, and hence find the polar equation of the orbit. Follow the working for the first part, very carefully in the section below. SWITCH OFF.

25. As you see, we have evaluated one of the constants immediately from the initial conditions. The second may be obtained from the initial radial velocity, but we must differentiate with respect to the time variable, t, in order to do this. Again, follow the remainder of the working below. SWITCH OFF.

26. You should now try the last example by your own efforts. It is a little more difficult. Read it very

carefully, to ensure that you have the correct initial radial velocity, when you come to find the arbitrary constants. When you have finished, check your solution with that in the booklet. SWITCH OFF.

27. This completes the work on Central Forces. Please rewind the tape before you remove it from the machine. Thank you!

A REVISION COURSE IN PARTICLE DYNAMICS

by

Kenneth A.H.Jackson

BOOKLET 10 to be used with TAPE 10

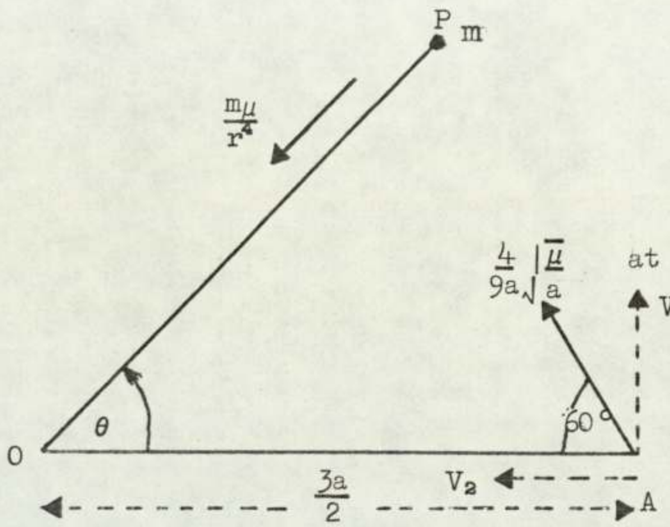
"Orbits"

Read the instructions on page 1 thoroughly.

(Omitted for convenience.)

EXAMPLE 1.

A particle, of mass m, moves under the action of the central attractive force $\frac{m\mu}{r^4}$, which is always directed towards the fixed point O. Find the orbit, if the particle was projected from the point A, with velocity $\frac{4}{9a}\sqrt{\frac{\mu}{a}}$, at an angle of 60° with AO, where AO = 3a/2.



When you have studied this diagram,

SWITCH ON.

$$F(r) = \frac{m\mu}{r^4} \quad \text{and} \quad F(u) = \mu u^4$$

$$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=o} = \frac{3a}{2} V_1 = \frac{3a}{2}\cdot \frac{4}{9a}\sqrt{\frac{\mu}{a}} \cdot \frac{\sqrt{3}}{2} = \sqrt{\frac{\mu}{3a}}$$

D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$, which becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^4 \cdot 3a}{\mu u^2} = 3au^2$$

$$\text{i.e. } \frac{d^2u}{d\theta^2} = 3au^2 - u$$

SWITCH ON.

(i) put $p = \frac{du}{d\theta}$

then $\frac{d^2u}{d\theta^2} = \frac{dp}{d\theta} = \frac{dp}{du} \cdot \frac{du}{d\theta} = \frac{dp}{du}.p$

Hence $p\frac{dp}{du} = 3au^2 - u$ ... (1)

(ii) It is necessary to find p when t = 0 in order to determine the constant of integration, and to do this, p must be re-written as follows,

$$p = \frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt} \cdot \frac{dt}{d\theta}$$

$$\text{i.e. } p = -\frac{1}{r^2\dot{\theta}}\dot{r} = -\frac{\dot{r}}{h}$$

SWITCH ON.

$$p\,dp = (3au^2 - u)du$$

$$\frac{p^2}{2} = au^3 - \frac{u^2}{2} + C$$

when $t = 0$, $u = \frac{2}{3a}$, $\dot{r} = -V_2 = -\frac{2}{9a}\sqrt{\frac{\mu}{a}}$

$$\text{hence } p = \left(-\sqrt{\frac{3a}{\mu}}\right)\left(-\frac{2}{9a}\sqrt{\frac{\mu}{a}}\right) = \frac{2}{3a\sqrt{3}}$$

thus $\frac{1}{2}\left(\frac{2}{3a\sqrt{3}}\right)^2 = a\left(\frac{2}{3a}\right)^3 - \frac{1}{2}\left(\frac{2}{3a}\right)^2 + C$ giving $C = 0$

and $p^2 = 2au^3 - u^2 = u^2(2au - 1)$

SWITCH ON.

$$p = \pm\, u\sqrt{2au - 1}$$

Now $p = -\frac{\dot{r}}{h}$ so that $\dot{r} = -h\left(\pm\, u\sqrt{2au - 1}\right)$

Initially $\dot{r}$ is negative, and this will be so if the positive root is chosen.

$$\text{i.e. } \frac{du}{d\theta} = p = u\sqrt{2au - 1}$$

$$\frac{du}{u\sqrt{2au - 1}} = d\theta \qquad \text{... (2)}$$

continued overleaf-

put $2au - 1 = z^2$ with $z = \sqrt{2au - 1}$

Then $u = \frac{z^2+1}{2a}$ and $du = \frac{z}{a}\,dz$

SWITCH ON.

$$d\theta = \frac{2\,dz}{(z^2 + 1)}$$

$$\theta = 2\tan^{-1}z + K$$

When $t = 0, \theta = 0$, $u = \frac{2}{3a}$, making $z = \frac{1}{\sqrt{3}}$

so $K = -\frac{\pi}{3}$.

Hence $1 + \tan^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right) = \frac{2a}{r} = \sec^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)$

and $r = 2a\cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)$

Notice that $r = 0$ when $\frac{\theta}{2} + \frac{\pi}{6} = \frac{\pi}{2}$ i.e. $\theta = \frac{2\pi}{3}$

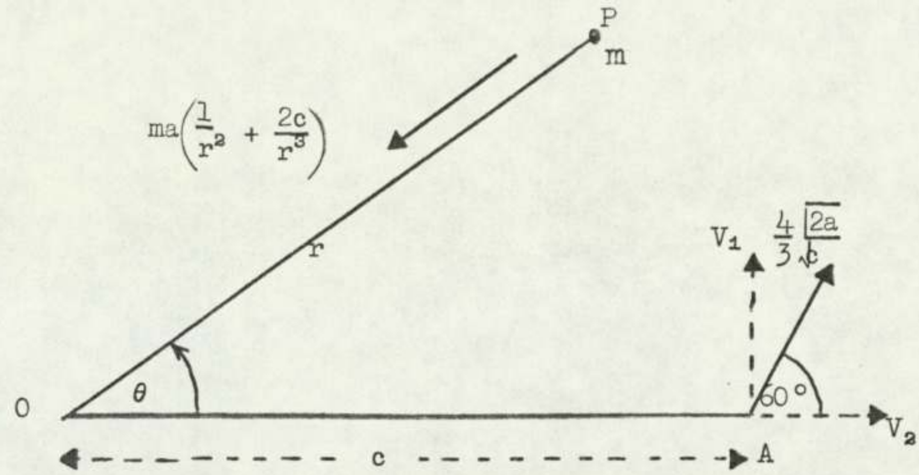
and the particle has been attracted right into the centre of force.

SWITCH ON.

EXAMPLE 2.

A particle of mass m, is attracted towards a fixed point O, with a force $ma\left(\frac{1}{r^2} + \frac{2c}{r^3}\right)$, where r is the distance of the particle from O. It is projected from a point A, at a distance c from O, with a velocity $\frac{4}{3}\left(\frac{2a}{c}\right)^{\frac{1}{2}}$ at an angle of 60° with OA. Find the orbit, and the distance of the particle from O when $\theta = 120°$.

Do not turn over until you have drawn a diagram and simplified the differential equation of the orbit.



$F(r) = a\left(\frac{1}{r^2} + \frac{2c}{r^3}\right)$ and $F(u) = a(u^2 + 2cu^3)$

$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=0} = cV_1 = 2\sqrt{\frac{2ac}{3}}$

D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$, which becomes,

$$\frac{d^2u}{d\theta^2} + \frac{u}{4} = \frac{3}{8c}$$

SWITCH ON.

$$\frac{1}{r} = u = A\cos\frac{\theta}{2} + B\sin\frac{\theta}{2} + \frac{3}{2c}$$

when $\theta = 0$, $r = c$, hence $A = -\frac{1}{2c}$

i.e. $u = \frac{1}{r} = \frac{-1}{2c}\cos\frac{\theta}{2} + B\sin\frac{\theta}{2} + \frac{3}{2c}$

Diff w.r.t. time,

$$-\frac{1}{r^2}\dot{r} = \left(\frac{1}{4c}\sin\frac{\theta}{2} + \frac{B}{2}\cos\frac{\theta}{2}\right)\dot{\theta}$$

when $t = 0, \theta = 0$, $\dot{r} = V_2 = \frac{2}{3}\sqrt{\frac{2a}{c}}$

$$r^2\dot{\theta} = h = 2\sqrt{\frac{2ac}{3}}$$

i.e. $-\frac{2}{3}\sqrt{\frac{2a}{c}} = \frac{B}{2} \cdot 2\sqrt{\frac{2ac}{3}}$, hence $B = \frac{-2}{c\sqrt{3}}$

(Continued on the next page)

So the orbit is $\frac{2c}{r} = 3 - \cos\frac{\theta}{2} - \frac{4}{\sqrt{3}}\sin\frac{\theta}{2}$

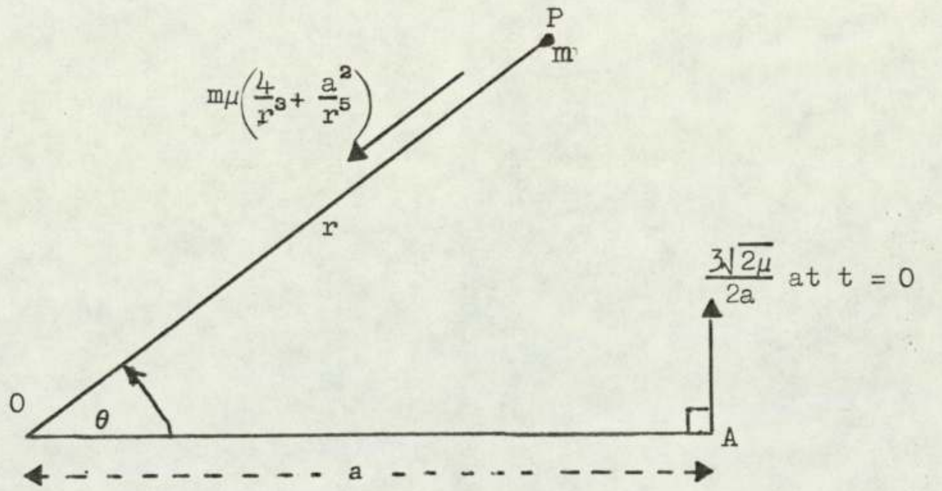
Put $\theta = 120°$ to give $r = 4c$

SWITCH ON.

EXAMPLE 3.

A particle, of mass m, which moves under an attractive force $m\mu\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$ is projected from an apse at $r = a$ with speed $\frac{3\sqrt{2\mu}}{2a}$. Find the orbit.

Do not read the solution until you have completed your own.



$F(r) = m\mu\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$ and $F(u) = m\mu u^3(a^2u^2 + 4)$

$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=o} = \frac{a3\sqrt{2\mu}}{2a} = 3\frac{\sqrt{\mu}}{\sqrt{2}}$

D.E. of the orbit $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$ reduces to

$$\frac{d^2u}{d\theta^2} = \frac{2a^2u^3 - u}{9}$$

(Continued on the next page).

Put $p = \frac{du}{d\theta}$, then $\frac{d^2u}{d\theta^2} = p\frac{dp}{du}$

Hence $\frac{p^2}{2} = \frac{a^2u^4}{18} - \frac{u^2}{18} + C$

We also have $p = -\frac{\dot{r}}{h}$

when $t = 0$, $\theta = 0$, $u = \frac{1}{a}$, $\dot{r} = 0$.

So $p = 0$, and hence $C = 0$

and $$p^2 = \frac{u^2}{9}(a^2u^2 - 1)$$

or $$\frac{du}{d\theta} = p = \frac{u}{3}\sqrt{a^2u^2 - 1}$$

[N.B. The initial conditions give no indication whether the positive or negative root should be taken as $\dot{r} = 0$. For convenience take the positive root.]

Now $u = \frac{1}{r}$, and $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

Substitute to give $$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{3r}\sqrt{\frac{a^2}{r^2} - 1}$$

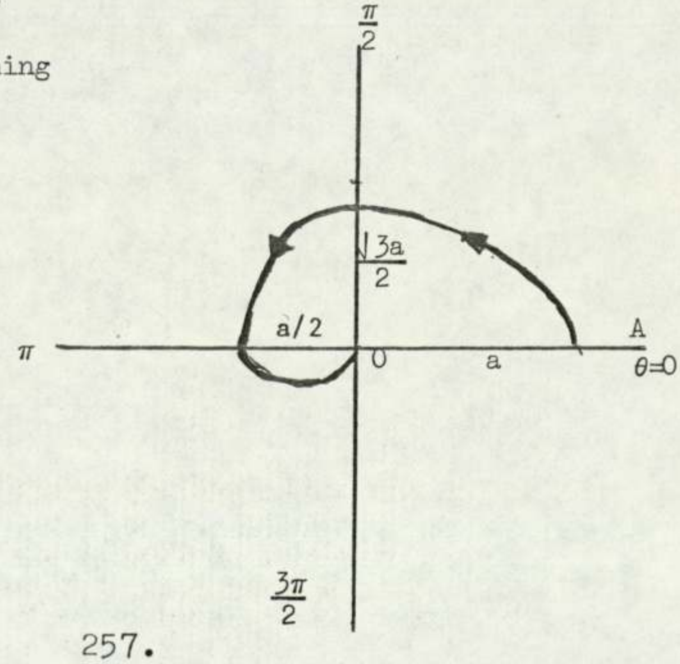
hence $$\frac{dr}{\sqrt{a^2 - r^2}} = -\frac{d\theta}{3}$$

$$\sin^{-1}\left(\frac{r}{a}\right) = K - \frac{\theta}{3}$$

where $\theta = 0$, $r = a$, hence $K = \frac{\pi}{2}$

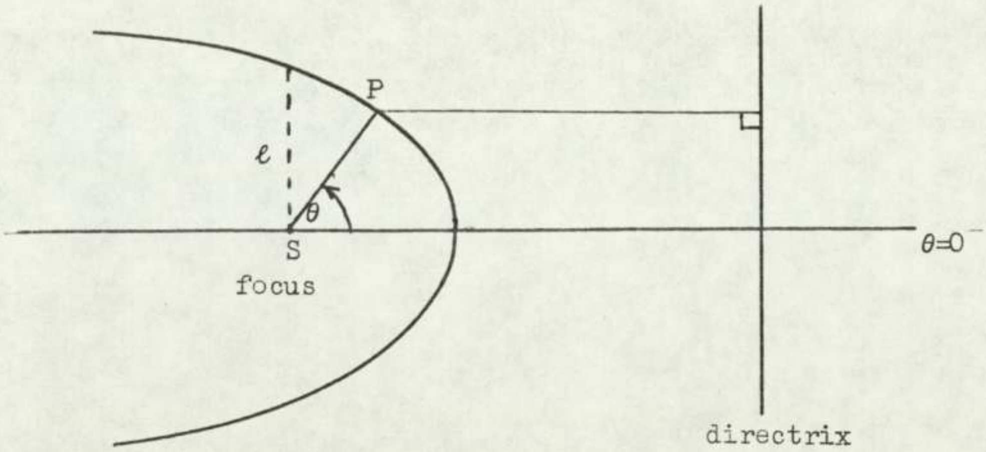
i.e. $r = a\sin\left(\frac{\pi}{2} - \frac{\theta}{3}\right) = a\cos\frac{\theta}{3}$

This particle spirals into the centre of force, reaching O when $\theta = \frac{3\pi}{2}$



SWITCH ON.

Conics in polar coordinates



The equations are of the form $\frac{\ell}{r} = 1 + e\cos\theta$

ℓ is the semi latus rectum

eccentricity $e < 1$ for an ellipse

" $e = 1$ for a parabola

" $e > 1$ for a hyperbola

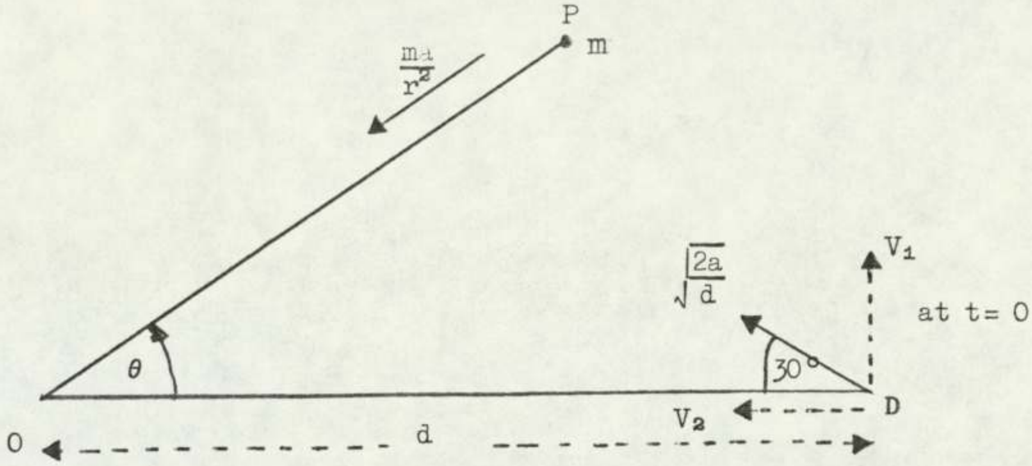
When you have revised this work,

SWITCH ON.

EXAMPLE 4.

A particle of mass m moves under the influence of a force $\frac{ma}{r^2}$, which is always directed towards a fixed point O, with OP = r, and 'a' a constant. The particle is projected from a point D, where OD = d, with a speed $\sqrt{\frac{2a}{d}}$, at an angle of 30° to DO. Show that the orbit is $r = \frac{d}{4}\sec^2\left(\frac{\theta}{2} + \frac{2\pi}{3}\right)$ where the pole is at O, and the initial line is OD. Also, sketch this orbit.

Do not look at the next section until you have drawn the diagram, and produced the general solution.



$F(r) = \frac{ma}{r^2}$ and $F(u) = au^2$

$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=o} = dV_1 = \sqrt{\frac{ad}{2}}$

D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$, which becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{2}{d}$$

Hence $\frac{1}{r} = u = A\cos\theta + B\sin\theta + \frac{2}{d}$

SWITCH ON.

---

when $t = 0$, $\theta = 0$, and $r = d$, so that $A = -\frac{1}{d}$

diff. w.r.t. time

$$-\frac{1}{r^2}\dot{r} = (-A\sin\theta + B\cos\theta)\dot{\theta}$$

when $t = 0$, $\theta = 0$ and $\dot{r} = -V_2 = -\sqrt{\frac{3a}{2d}}$

Also $r^2\dot{\theta} = h = \sqrt{\frac{ad}{2}}$ hence $B = \frac{\sqrt{3}}{d}$

Hence the orbit is

$$\frac{1}{r} = \frac{2}{d} - \frac{1}{d}\cos\theta + \frac{\sqrt{3}}{d}\sin\theta$$

First, convert the term $\frac{2}{d}$ into one, by multiplying through by $\frac{d}{2}$

(Continued on the next page)

$$\text{i.e. } \frac{d}{2} \cdot \frac{1}{r} = 1 - \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$

Now combine the circular functions together.

$$\text{i.e. } \frac{d}{2} \cdot \frac{1}{r} = 1 + \cos\theta\cos\frac{4\pi}{3} - \sin\theta\sin\frac{4\pi}{3}$$

Hence $\frac{d}{2} \cdot \frac{1}{r} = 1 + \cos\left(\theta + \frac{4\pi}{3}\right)$

Put $\theta + \frac{4\pi}{3} = \phi$ then

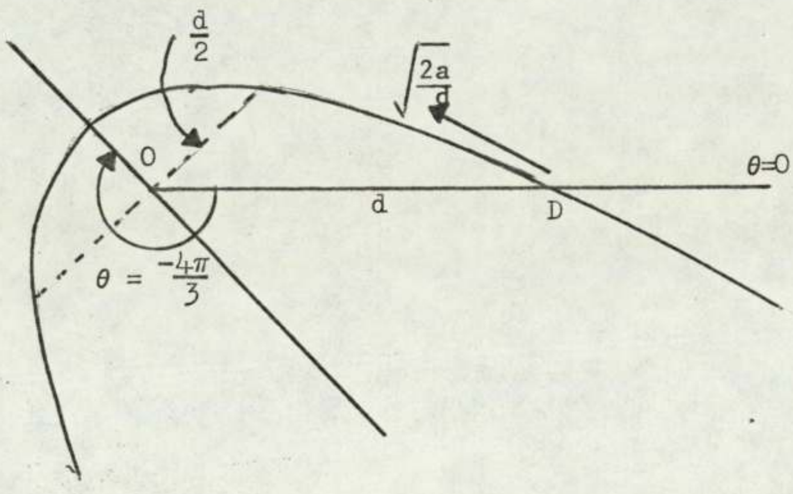
$\frac{d}{2}\cdot\frac{1}{r} = 1 + \cos\phi$, which agrees completely with the standard equation, in terms of ϕ in place of θ.

SWITCH ON.

Semi latus rectum $= \ell = \frac{d}{2}$

Eccentricity $= e = 1$, hence a parabola.

Axis is given by $\phi = 0 = \theta + \frac{4\pi}{3}$, i.e. $\theta = -\frac{4\pi}{3}$ indicating a clockwise rotation of $\frac{4\pi}{3}$



$$\frac{d}{2} \cdot \frac{1}{r} = 1 + \cos\left(\theta + \frac{4\pi}{3}\right) = 2\cos^2\left(\frac{\theta}{2} + \frac{2\pi}{3}\right)$$

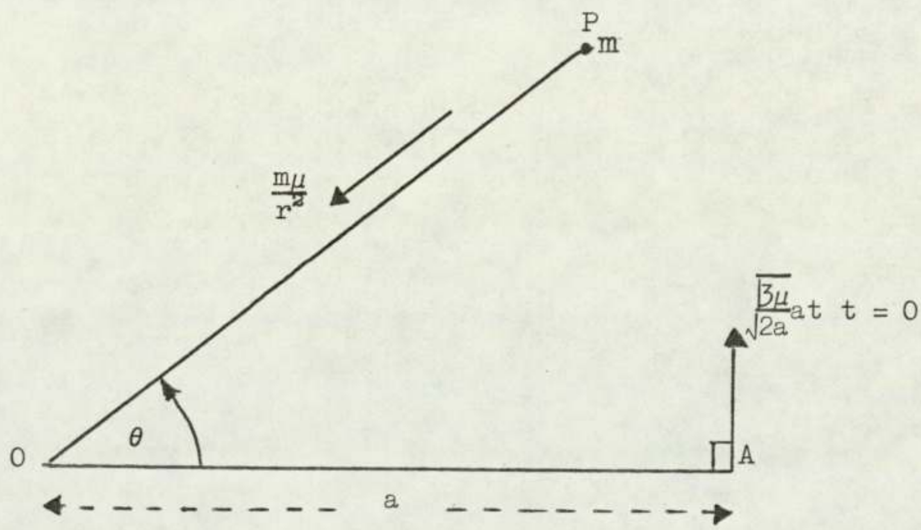
$$\text{i.e. } r = \frac{d}{4}\sec^2\left(\frac{\theta}{2} + \frac{2\pi}{3}\right)$$

SWITCH ON.

EXAMPLE 5.

A body of mass m, moves under a central attractive force $\frac{m\mu}{r^2}$, and is projected from a point distant 'a' from the centre of the force, at right angles to the radius vector, with velocity $\sqrt{\frac{3\mu}{2a}}$. Show that the orbit is an ellipse. Sketch the orbit, and find the period.

Do not read the next section until you have found the equation of the orbit.



$$F(r) = \frac{m\mu}{r^2} \quad \text{and} \quad F(u) = m\mu u^2$$

$$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=o} = a\sqrt{\frac{3\mu}{2a}} = \sqrt{\frac{3\mu a}{2}}$$

D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$, which becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{2}{3a}$$

$$\text{i.e.} \quad (D^2+1)u = \frac{2}{3a} \qquad \text{where } D \equiv \frac{d}{d\theta}$$

$$\frac{1}{r} = u = A\cos\theta + B\sin\theta + \frac{2}{3a}.$$

(Continued on the next page).

When $t = 0$, $\theta = 0$, and $r = a$, hence $A = \dfrac{1}{3a}$

Diff. w.r.t. time $-\dfrac{1}{r^2}\dot{r} = (-A\sin\theta + B\cos\theta)\dot{\theta}$

When $t = 0, \theta = 0$ and $\dot{r} = 0$, hence $B = 0$.

The equation of the orbit is $\dfrac{3a}{2} \cdot \dfrac{1}{r} = 1 + \tfrac{1}{2}\cos\theta$

Hence $\ell = \dfrac{3a}{2}$ and $e = \tfrac{1}{2} < 1$, i.e. an ellipse.

SWITCH ON.

If α, β are the semi major/minor axes respectively, then

$$\beta^2 = \alpha^2(1-e^2) \qquad \text{and} \qquad \ell = \frac{\beta^2}{\alpha}$$

So $\beta^2 = \dfrac{3\alpha^2}{4} = \dfrac{3a}{2}.\alpha$ and $\alpha = 2a$

Also $\beta^2 = \left(\dfrac{3a}{2}\right).2a$ and $\beta = a\sqrt{3}$

The period is calculated from the constant of the transverse motion.

$$h = r^2\dot{\theta} = r^2\frac{d\theta}{dt}$$

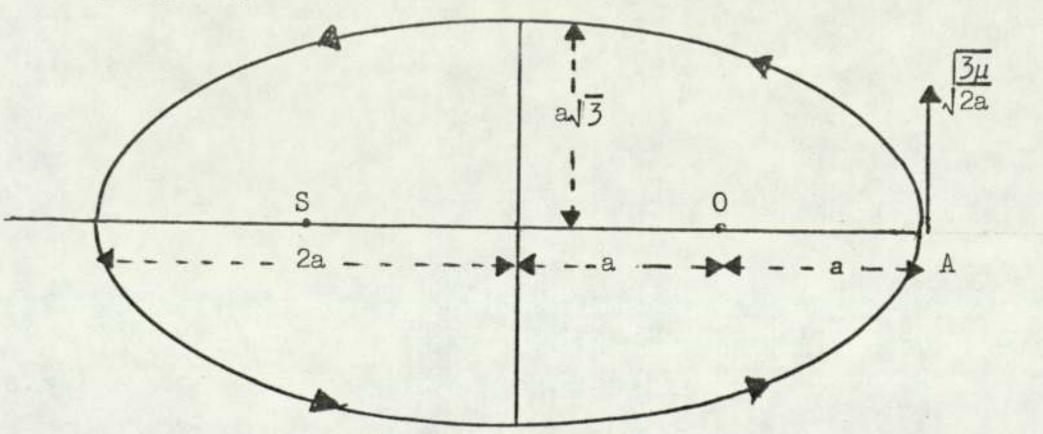
$$\text{So} \quad \tfrac{1}{2}h\,dt = \tfrac{1}{2}r^2\,d\theta$$

$$\text{and} \quad \frac{h}{2}\int_0^T dt = \int_0^{2\pi} \tfrac{1}{2}r^2\,d\theta$$

i.e. $\dfrac{hT}{2}$ = area of the ellipse = $\pi\,\alpha\beta$

So $T = 2\pi\sqrt{\dfrac{2}{3\mu a}}\,(2a)(a\sqrt{3}) = 4\sqrt{2}\pi\dfrac{a^{3/2}}{\mu^{1/2}}$

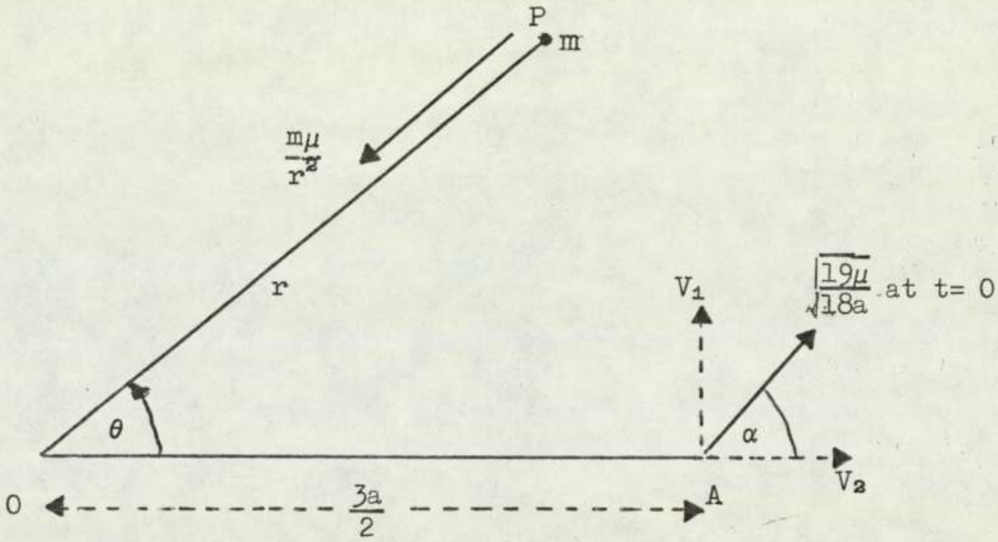
The phase angle in this orbit was zero, hence the major axis coincides with the initial line i.e. $\theta = 0$.



EXAMPLE 6.

A particle, of mass, m, is attracted by a force $\frac{m\mu}{r^2}$ towards a fixed point O. If it was projected from a fixed point A, distant $\frac{3a}{2}$ from O, at an angle $\tan^{-1}\left(\frac{4}{\sqrt{3}}\right)$ to $\overline{OA}$, with speed $\sqrt{19\mu/18a}$ show that it describes an ellipse. Find the eccentricity and semi latus rectum of this. Sketch the orbit, and find its period.

Do not look at the solution until you have completed the question.



$F(r) = \frac{m\mu}{r^2}$ and $F(u) = \mu u^2$

$$h = r^2\dot{\theta} = r(r\dot{\theta})\Big|_{t=0} = \frac{3a}{2} V_1 = \frac{3a}{2} \sqrt{\frac{19\mu}{18a}} \cdot \frac{4}{\sqrt{19}} = \sqrt{2\mu a}$$

D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{F(u)}{h^2u^2}$, which becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{2a}$$

$$\frac{1}{r} = u = A\cos\theta + B\sin\theta + \frac{1}{2a}$$

When $t = 0$, $\theta = 0$ and $r = \frac{3a}{2}$ hence $A = \frac{1}{6a}$

$$\tan\alpha = \frac{4}{\sqrt{3}}$$

(Continued on the next page).

Diff. w.r.t. time, $\quad -\frac{1}{r^2}\dot{r} = (-A\sin\theta + B\cos\theta)\dot{\theta}$

when $t = 0, \theta = 0, \dot{r} = V_2 = \frac{\sqrt{\mu}}{\sqrt{6a}}$ and $r^2\dot{\theta} = \sqrt{2\mu a}$

Hence $B = -\frac{1}{a2\sqrt{3}}$

$$\frac{1}{r} = \frac{1}{2a} + \frac{1}{6a}\cos\theta - \frac{1}{a2\sqrt{3}}\sin\theta$$

$$\frac{2a}{r} = 1 + \frac{2}{3}\cos\left(\theta + \frac{\pi}{3}\right)$$

gives $\ell = 2a$ and $e = \frac{2}{3} < 1$ i.e. an ellipse.

If α and β are the semi axes, then

$$\beta^2 = \left(1 - \frac{4}{9}\right)\alpha^2 \quad \text{and} \quad \beta^2 = 2a.\alpha$$

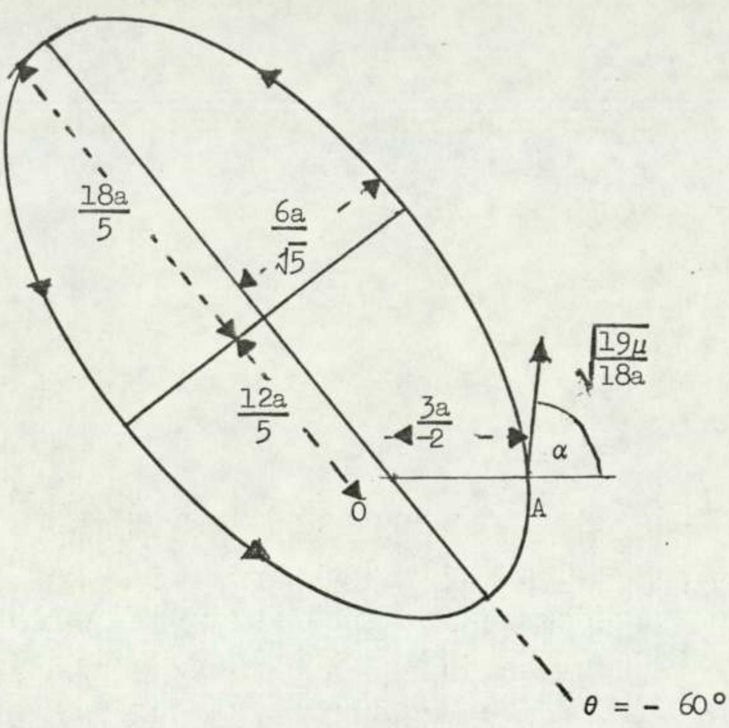
hence $\quad \alpha = \frac{18a}{5} \;, \; \beta = \frac{6a}{\sqrt{5}}$

For inclination of major axis, put $\left(\theta + \frac{\pi}{3}\right) = 0$

hence $\theta = -\frac{\pi}{3}$ gives a clockwise rotation of 60° about the focus.

Period, $T.\frac{\sqrt{2\mu a}}{2} = \pi.\frac{18a}{5}.\frac{6a}{\sqrt{5}}$ hence $T = \frac{108\pi}{5}\frac{\sqrt{2}}{\sqrt{5}}\frac{a^{3/2}}{\mu^{1/2}}$

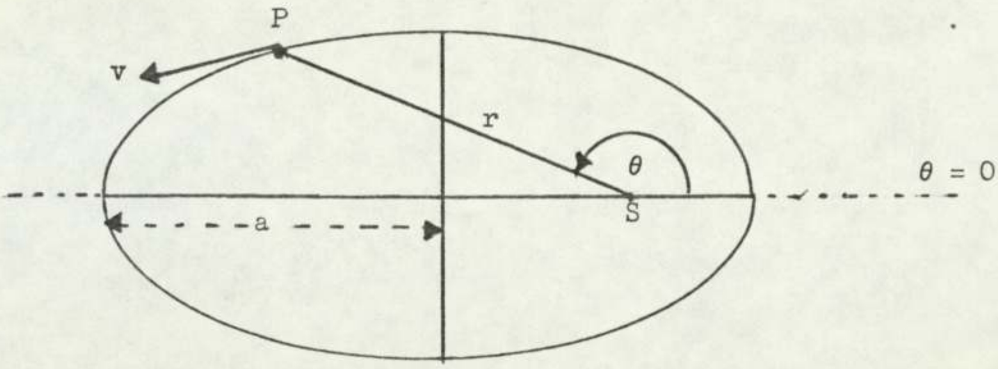
<u>Sketch of orbit.</u>



<u>SWITCH ON.</u>

Velocity in elliptic orbit.

Consider a particle, of mass m, moving under an attractive force $\frac{m\mu}{r^2}$ in an elliptic orbit with major axis 2a.



The velocity at any point P, can be calculated directly, from the radial distance r from the formula

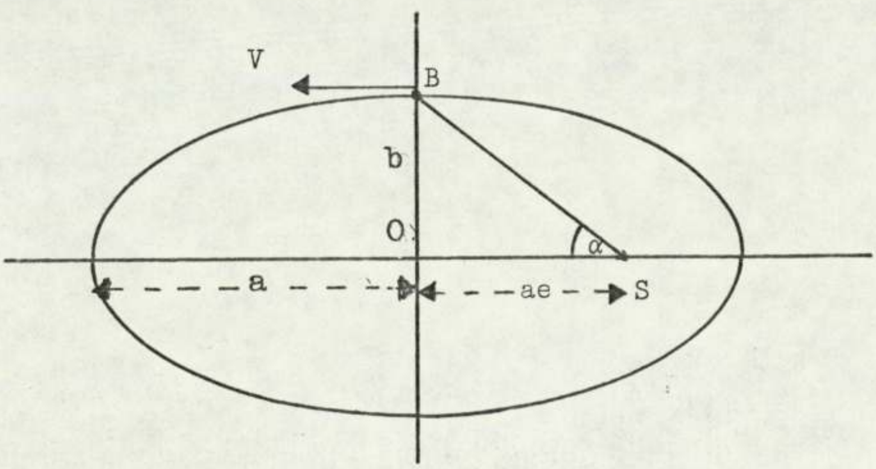
$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

When you have revised this,

SWITCH ON.

EXAMPLE 7.

A particle is moving in an elliptic orbit of eccentricity $\frac{1}{2}$, and semi major axis a, under a central force $\frac{m\mu}{r^2}$. If at one end of the minor axis, its velocity is suddenly doubled, find the eccentricity of the new orbit.



When you have studied this, SWITCH ON.

Since SB = a, so $V^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right)$ i.e. $V = \sqrt{\frac{\mu}{a}}$.

The new start velocity is $2V = 2\sqrt{\frac{\mu}{a}}$

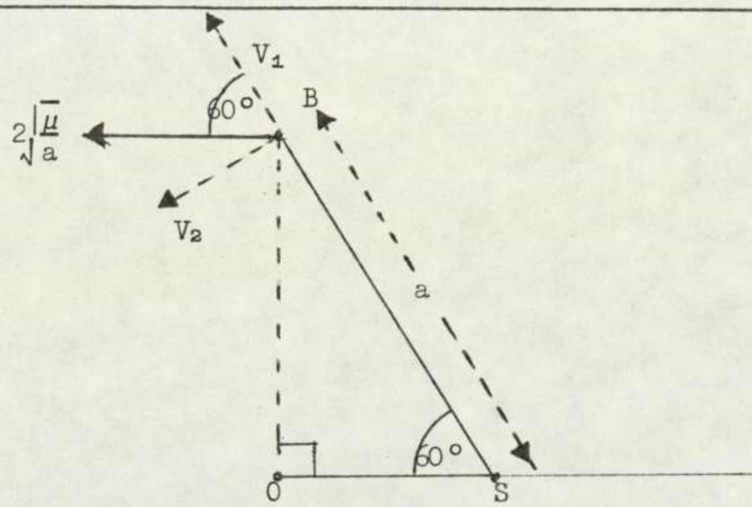
$\cos\alpha = \frac{OS}{SB} = \frac{ae}{a} = e = \frac{1}{2}$

i.e. $\alpha = 60°$ and $\sin\alpha = \frac{\sqrt{3}}{2}$

h for the new orbit $= r(r\dot{\theta})\Big|_{t=o} = a\ 2V\sin\alpha$

i.e. $h = a2\sqrt{\frac{\mu}{a}}\cdot\frac{\sqrt{3}}{2} = \sqrt{3\mu a}$

SWITCH ON.



D.E. of the orbit is $\frac{d^2u}{d\theta^2} + u = \frac{1}{3a}$

$$\frac{1}{r} = u = A\cos\theta + B\sin\theta + \frac{1}{3a}$$

when $t = 0, \theta = 0$, $r = a$, hence $A = \frac{2}{3a}$

Diff. w.r.t. time, $-\frac{1}{r^2}\dot{r} = (-A\sin\theta + B\cos\theta)\dot{\theta}$

When $t = 0, \theta = 0$, $\dot{r} = V_1 = \sqrt{\frac{\mu}{a}}$

$$r^2\dot{\theta} = h = \sqrt{3\mu a}$$

hence $$B = -\frac{1}{a\sqrt{3}}$$

So $\frac{1}{r} = \frac{1}{3a} + \frac{2}{3a}\cos\theta - \frac{1}{a\sqrt{3}}\sin\theta$

(continued on the next page).

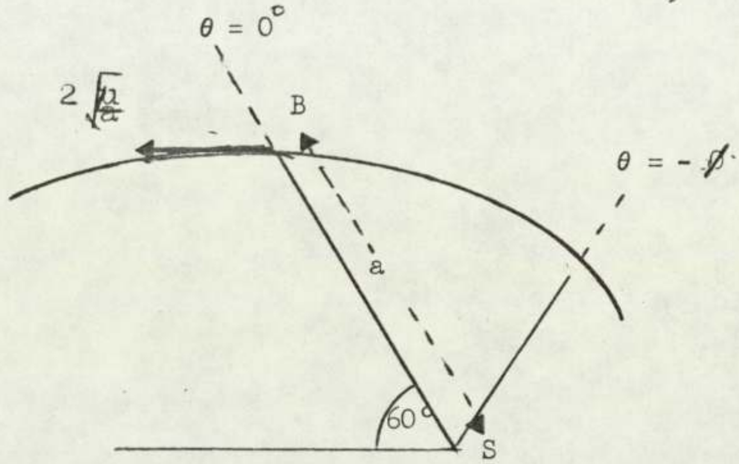
$$\frac{3a}{r} = 1 + 2\cos\theta - \sqrt{3}\,\sin\theta$$

i.e. $$\frac{3a}{r} = 1 + \sqrt{7}\,\cos(\theta+\phi), \text{ where } \phi = \tan^{-1}\frac{\sqrt{3}}{2}$$

hence $\ell = 3a$, and $e = \sqrt{7} > 1$.

So the new orbit is a hyperbola with the axis given by

$\theta = -\,\phi$.



SWITCH ON.

SCRIPT FOR TAPE 10

ORBITS

1. This tape is to be used with Booklet 10 of 'A Revision Course in Particle Dynamics', by Kenneth Jackson. It is concerned with more difficult problems on central forces, and also considers in particular, the inverse square law. All of this work is a natural progression of Booklet 9 and uses the standard differential equation for inverse polars i.e. $u = 1/r$ and θ. Read example 1, study the diagram, and we shall then work through it together. SWITCH OFF.

2. Notice in this question, the direction of the initial velocity, producing an inwards component. The first step in all these problems will be to write the differential equations of the orbit, determine the force function, and the constant h, and to substitute these into the equation. Then the equation must be simplified as far as possible, without attempting any integration. Do this, for the first question, and then check your working below. SWITCH OFF.

3. You will notice, that we cannot integrate this on sight, and to make progress, it is convenient to make a substitution, which must be developed in two ways. Follow this working in the next section. SWITCH OFF.

4. The result of this substitution is to give

equation (1) in p and u which is variable, separable and easy to integrate. The second expression gives p in terms of the radial velocity, which is known when t = 0. Now carry out this integration and evaluate the constant. When you have done this, look at the next section, to check your working. SWITCH OFF.

5. We can now take a square root, but we must be careful with the sign. The variables need to be separated again and prepared for the final integration. To carry this out however, we need to remove the square root by an appropriate substitution. Follow these operations in the next section. SWITCH OFF.

6. By completing the substitution on the left side of equation (2), you will be able to integrate, and then evaluate the constant. Replace u as 1/r, and, with a little manipulation finish the question. Check your working afterwards. SWITCH OFF.

7. The second question has a more complicated force function, yet leads to a simple D.E. We shall work through this in stages. When you have read the question, draw a diagram and write down the D.E. of the orbit. When you have simplified this, check it. SWITCH OFF.

8. This is a standard differential equation with constant coefficients and can be solved in the normal way, using the particular integral and complementary function. Write out the general solution and find the two arbitrary constants. This will give you the orbit which can be

used to calculate the required distance. Check your answer when you have finished. SWITCH OFF.

9. You have now seen two different methods for integrating the differential equation of the orbit. The next example, number three, uses the first method. I.e. substitution for $du/d\theta$ as p. Read the question, draw the diagram and work through this one by your own efforts. You will find the second integration easy in terms of r and θ, rather than u and θ. Check your complete solution afterwards. SWITCH OFF.

10. The remainder of this booklet is concerned with the inverse square law of attractive force, which is very important as it is the Law of Gravitation. With this law of force, the orbits are conics, and the polar equations of the curves are summarised for you in the next section. SWITCH OFF.

11. Example 4 is a typical inverse square law question, and you can start the working by yourself. Read the question, and when you have drawn a diagram, write the differential equation of the orbit, and find its general solution. Check your working in the next section. SWITCH OFF.

12. You can now find the two arbitrary constants in the normal way, using the initial conditions. Then try to express your answer in the standard form for a conic in polar coordinates, by manipulation. Check this result in the next section. SWITCH OFF.

13. You should now be able to state the length of

the semi latus rectum, to give the eccentricity and decide which type of conic it is. Also, find the direction of the axis and sketch the curve. Finally, manipulate your equation to give the stated answer. Check your working in the next section of the booklet. SWITCH OFF.

14 Example 5 also depends on the inverse square law and this time the orbit is an ellipse. Read the question carefully, draw a diagram, find the orbit equation, and determine l and e. When you get this far, check your working. SWITCH OFF.

15. From l and e, you can now calculate the semi-axes of the ellipse. It is convenient to call then alpha and beta, as the symbol 'a' has already been used in this question. The orbit can be sketched, and the period found. Follow this in the next section of the booklet. SWITCH OFF.

16. The next example is of exactly the same type as question 5, and you should work this completely yourself. Check your result in the next section afterwards. SWITCH OFF.

17. The last example concerns the velocity at any point in the orbit of an ellipse. Before you consider this, read the next section of the booklet which revises this result for you. SWITCH OFF.

18. You should now read example seven, and study the diagram drawn for this. SWITCH OFF.

19. The new velocity at B will enable us to calculate the value of h for the new orbit, and thus solve its differential equation. Notice, however, that the tangential

velocity V, will have components along SB, and at right angles to SB and hence we require the angle alpha. We are told the eccentricity of the ellipse, and you should be able to find the trig. functions of this angle. Do thi, and also calculate the new value of h, and then check your results. SWITCH OFF.

<u>20</u>. You should now draw a new diagram, showing the new velocity components. Then you can write the D.E. of the orbit, and obtain the general solution. Determine the constants in this, and then write your orbit in standard form. You will then be able to obtain the eccentricity and the type of orbit. When this is complete, check your working. SWITCH OFF.

<u>21</u>. This completes the work on 'Orbits'. Please rewind the tape before you remove it from the machine. Thank you!

Bibliography

...hor	Title	Date	Publisher
...an	Report on the future of broadcasting	1977	Annan HMSO
...ber. C.	Applied Mathematics for Engineers and Scientists	1960	E.U.P.
...eton	Textbook of Dynamics	1978	E. Horwood
...ke.M.L.	Higher Education in the Ancient World	1971	R.K.P.
... R.S.	Solving Problems in Dynamics	1968	Oliver & Boyd
...art. M.B.	Principles of Dynamics	1976	Routledge & K. Paul
...er. J.C.	Introduction to applied mathematics 2nd revised edition	1974	O.U.P.
...ph & Leahy	Programmed Physics Part IV	1967	J. Wiley
...en. D.F.	A Course in Applied Maths Vol.1	1967	E.U.P.
Mack & Manchester	Journal of Further and Higher Education		
...er & Shell	Report on The Education Act 1870		H.M.S.O.
...cott. J.	Mechanics of Particles and hinged bodies	1966	Longman
...ey. A.S.	Dynamics Part 1	1960	Longman
...n. Joan	Education and Society in Tudor England	1966	C.U.P.
...t. H.	Advanced Dynamics	1951	MacMillan
...field. A.M.	Journal of Education Research		
...ing ...mphrey	Intermediate Mechanics/Dynamics	1976	Longman