DEVELOPMENTS AND EXTENSIONS OF ROTH'S METHOD OF DOUBLE FOURIER SERIES WITH APPLICATIONS TO THE SOLUTION OF ELECTRO-MAGNETIC FIELD PROBLEMS.

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Summary.

The first part of the thesis compares Roth's method with other methods, in particular the method of separation of variables and the finite cosine transform method, for solving certain elliptic partial differential equations arising in practice. In particular we consider the solution of steady state problems associated with insulated conductors in rectangular slots. Roth's method has two main disadvantages namely the slow rate of convergence of the double Fourier series and the restrictive form of the allowable boundary conditions. A combined Roth-separation of variables method is derived to remove the restrictions on the form of the boundary conditions and various Chebyshev approximations are used to try to improve the rate of convergence of the series. All the techniques are then applied to the Neumann problem arising from balanced rectangular windings in a transformer window.

Roth's method is then extended to deal with problems other than those resulting from static fields. First we consider a rectangular insulated conductor in a rectangular slot when the current is varying sinusoidally with time. An approximate method is also developed and compared with the exact method. The approximation is then used to consider the problem of an insulated conductor in a slot facing an air gap. We also consider the exact method applied to the determination of the eddy-current loss produced in an isolated rectangular conductor by a transverse magnetic field varying sinusoidally with time. The results obtained using Roth's method are critically compared with those obtained by other authors using different methods.

The final part of the thesis investigates further the application of Chebyshev methods to the solution of elliptic partial differential equations; an area where Chebyshev approximations have rarely been used. A Poisson equation

with a polynomial source term is treated first followed by a slot problem in cylindrical geometry.

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LIST OF SYMBOLS

$\underline{E}$	electric intensity	(volt/metre)
$\underline{B}$	magnetic flux density	(weber/square metre)
$\underline{D}$	electric displacement density	(coulomb/square metre)
$\underline{H}$	magnetic intensity	(ampere-turn/metre)
$\underline{J}$	current density	(ampere/square metre)
ρ	volume charge density	(coulomb/cubic metre)
ε_o	permittivity of free space	(8.854×10^{-12} farad/metre)
μ_o	permeability of free space	($4\pi \times 10^{-7}$ henry/metre)
μ_r	relative permeability	
ε_r	relative permittivity	
σ	conductivity	(siemens/metre)
V	scalar potential	(volt)
$\underline{A}$	vector potential	(weber/metre)
α	$\sqrt{\mu_r \ \mu_o \ \sigma \ \omega}$	
ω	angular frequency	(radians/second)
I	current	(ampere)
W	magnetic energy	(joules)
L	leakage inductance	(henry)
R	effective resistance	(ohm)
X	inductive reactance	(ohm)
*	denotes complex function	
~	denotes complex conjugate	
∇^2	Laplacian operator	

Introduction

Between the years 1927 and 1938, a French engineer, E.Roth, developed a mathematical method using double Fourier series for the solution of boundary value problems in electrical engineering. He applied the method to problems of heat conduction and to magnetic field problems in electrical machines and transformers although his methods are capable of much wider application.

Roth's first paper[1] considers the flow of heat in electrical machines and[2] deals with the thermal and magnetic fields of current carrying conductors, assumed infinitely long and of rectangular cross-section, in a long rectangular slot. A steady constant current flows in the axial direction. The problem is then two-dimensional and is illustrated by Fig.1. Roth assumes the centre line OO' to be a line of symmetry and the magnetic boundary conditions are those of infinite permeability along BA, AA', A'B' while BB' is a flux line.

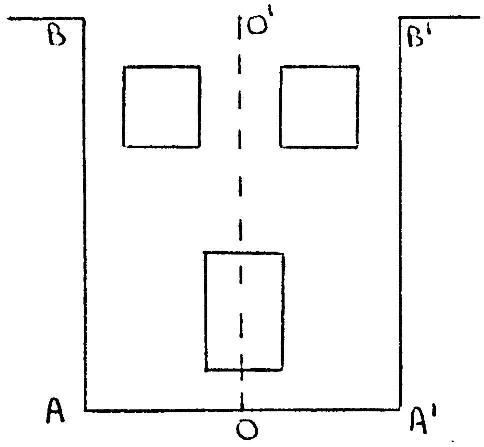

Fig. 1.

Roth next[3] deals with the leakage field in transformers considering the case of a rectangular transformer window with balanced rectangular windings. A steady constant axial current flows in the windings, and the permeability of the iron at the iron-air boundaries is again assumed to be infinite.

Roth's method involves the summation of a double Fourier series. The rate of convergence of such series is poor and although their numerical computation can be much improved if the recurrence properties of the trigonometric functions are used, Roth did not realise this. His next paper[4], written in collaboration with G.Kouskoff, seeks improved methods of summation of the double series and they achieve this by summation of the series over one variable.

Roth in[5] again considers the leakage field in transformers but with a more sophisticated configuration as shown in Figure 2. AC, A'C' and AA' are iron-air boundaries and the iron is assumed to be infinitely permeable. CC' lies along the centre line of the core and is, by symmetry, a flux line. The region BB'C'C represents the iron core and is assumed to have finite, constant permeability.

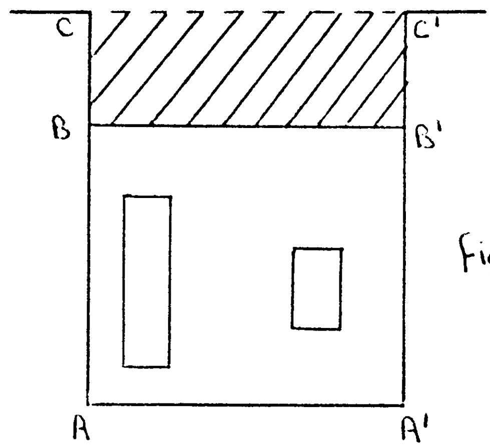

Fig. 2.

In[6] Roth generalised the configuration of[2] shown here in Figure 1. The permeability of the iron along AA' is still assumed to be infinite but along AB and A'B' the tangential magnetic field is finite and of arbitrary strength. Also BB' is no longer a flux line and the windings are not symmetrically distributed about OO'.

References[7] and [8] are included for completeness but are not considered in detail in the thesis. In[7] Roth applies his methods to a transformer problem with circular boundaries. The solution involves double series of Bessel and trigonometric functions and although Roth sums over the circular functions the labour involved is prohibitive. Reference[8] deals with the magnetic field of a system of rectangular parallel conductors.

In 1967, Hammond[9] wrote a critical account of

Roth's method listing its advantages and disadvantages and comparing it with the method of separation of variables. He considered the configuration of Figure 1 but with just one conductor in the slot. To obtain the solution by the method of separation of variables the slot must be divided into regions as shown in Figure 3. There is a separate solution valid in each region and continuity conditions must be met across all internal boundaries.

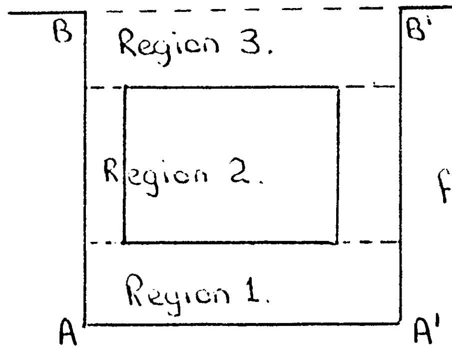

Fig. 3.

Roth's method on the other hand gives a single solution valid over the whole region of the slot. Hammond points out that to apply Roth's technique directly it is necessary that the slot boundaries are either flux lines or scalar equipotentials. This restriction does not apply to the method of separation of variables. Because Roth's solution is a double Fourier series, the rate of convergence is slow compared with the single series of the solution by separation of variables. Hammond concludes that Roth's method is not suitable for numerical computation. In the separation of variables solution, each individual term in the infinite series is a linearly independent solution of Laplace's equation and these individual terms can often be usefully identified with different parts of the boundary. In Roth's solution, only the complete expression satisfies the conditions of the problem.

Based on the claims made by Hammond in[9] both for and against Roth's method, the first part of this investigation consists of a detailed description of Roth's method together with a critical comparison of Roth's method with the method of separation of variables. The methods are compared from the following points of view:-

(i) ease of derivation of the mathematical solution valid over the whole region of the slot;

(ii) numerical computation of the solution;

(iii) rate of convergence of the series solution;

(iv) effect of increasing the number of conductors in the slot;

(v) form of allowable boundary conditions;

(vi) significance of individual terms in each form of solution;

(vii) form of allowable current density function;

(viii) the conductor cross-section.

For the solution by Roth's method, the relative merits of the Roth-Kouskoff techniques as described in[4] are discussed.

In[10], Mullineux and Reed point out the similarities between Roth's method and methods of solution using finite cosine transform techniques. In fact the two solutions are identical if the slot boundaries are flux lines or scalar equipotentials. However, unlike Roth's method, the transform method without modification can cope with more generalised forms of boundary condition. If the boundary conditions are of the Dirichlet type then the sine transform is used and if the boundary conditions are of the Neumann type the cosine transform is used. As with Roth's method, the Fourier transform technique results in a single solution valid over the whole region of the slot. Mullineux and Reed conclude that, since Roth's method appears to be a special case of the transform method able to cope with only a limited form of boundary condition, a better comparison would be between the transform method and the method of separation of variables.

To examine this conflict of opinion more fully we consider a configuration where the boundary conditions are of mixed type, the normal derivative being specified as known functions around three sides of the slot while the function itself is specified along the remaining side. To solve this problem using Roth's method, a combined Roth-separation of variables technique is derived and this is then compared with the finite Fourier transform approach. As a practical example of the methods we consider the problem of[5] described in Figure 2. An examination of Roth's solution as given in[5] is also included.

One of the chief disadvantages in the use of Roth's methods is the slow rate of convergence of the double Fourier series. In an effort to improve this we investigate the effect of using Chebyshev polynomials rather than circular functions in the solution. It is well known in the theory of numerical solution of ordinary differential equations that Chebyshev approximation speeds up the rate of convergence of the solution. Also, little work has been done to date on the application of Chebyshev polynomials to the solution of partial differential equations and so the treatment given here represents a significant advance in the current knowledge. Due to the difficulties associated with differentiation of Chebyshev polynomials, we consider first Chebyshev approximation in one direction only, taking each direction in turn. Then we allow Chebyshev variation in both directions simultaneously. The various Chebyshev approximations are compared with each other and with the double Fourier series method. The Fourier-Chebyshev method reduces the partial differential equation to a sequence of ordinary differential equations and two methods of solution are described. This is backed up by a theoretical investigation of the error due to the introduction of the Lanczos τ-terms.

Consideration is given to the possible methods of solution when the boundary conditions are either of the Dirichlet type or of the Neumann type, and the effect of slight perturbation of the boundary conditions is studied. In the double Chebyshev approximation, a new method for the evaluation of the Chebyshev coefficients is developed and this is capable of wider application.

Roth's method and the Chebyshev methods are then modified to solve the Neumann problem resulting from rectangular windings in a transformer window. Although Roth himself realised that, for a solution to exist at all when the surrounding iron is assumed infinitely permeable, the total net current in the window must be zero, later writers(11) and(12) do not emphasise this point. We consider here the simplest configuration possible i.e. a primary and a secondary winding balanced so that the total net current in the window is zero.

Prior to this investigation Roth's method has only been applied to problems resulting from static fields. The next part of the thesis endeavours to extend Roth's ideas to other classes of problem. In(9) Hammond suggested that one might try consideration of eddy-current phenomena. He states that a solution in Roth's form demands that the conductivity and permeability must be constant throughout the region under consideration so excluding most problems of practical interest. We consider first an insulated rectangular conductor in a rectangular slot when the current in the conductor is varying sinusoidally with time. A Roth solution is developed, valid throughout the whole region of the slot. The Fourier coefficients are not now calculable directly but are determined by a set of linear complex equations for which an iterative method of solution is developed. The effective resistance and inductive reactance are obtained by integration of the Poynting vector. With the conductor filling the slot, the results are compared with those given by Swann and Salmon in[13] for the fully open slot.

Due to the difficulty of having to solve a large set of equations for the Fourier coefficients, a simpler approximate model is developed. This involves a superposition of a separation of variables solution on a Roth solution. The results of this simplified model are then compared with those of the exact model for a practical range of insulation thicknesses. The approximate method is used to consider the problem of an insulated conductor in a slot facing an air gap. This problem is considered by Silvester in[14] who concludes that the insulation layer drastically affects the complex impedance. Due to the controversial nature of these conclusions we aim to use Roth's method to try to corroborate Silvester's findings.

In[15] Stoll obtained the eddy-current loss produced in a long conductor of rectangular cross-section by a transverse magnetic field which varies sinusoidally with time. The field is uniform and perpendicular to one side of the conductor. We use Roth's method to solve this problem and the purpose of doing this is twofold. Firstly it shows that Roth's method can be applied to problems other than those associated with insulated conductors in slots and secondly, the results obtained using Roth's method can be critically compared with those obtained by Stoll in[15]. Roth's exact method of solution must be used for this problem since the cross-sectional area of the non-conducting region is large.

The final part of the thesis reverts back to consideration of the use of Chebyshev polynomials in the solution of partial differential equations. It will be shown that Chebyshev approximations cannot be recommended for the determination of fields due to rectangular conductors in slots. The question remains as to whether there exist physical problems where Chebyshev methods would be superior to other methods, for example, the method of separation of variables or Roth's method. The first example to be considered is a Poisson equation with a polynomial source term and this is solved using Roth's method, the method of separation of variables and the double Chebyshev approximation. All three methods of solution are then compared. Then follows a more general example in cylindrical geometry, namely an infinitely long insulated conductor in an annular slot as described by Figure 4. It is assumed that there is a steady axial current flowing in the conductor. The Laplacian is expressed in cylindrical coordinates and the solution is obtained using a Fourier-Chebyshev approximation. To derive the solution by Roth's method or the method of separation of variables would involve the use of Bessel functions making these methods, if not impossible, then very cumbersome. Not only does this example serve to illustrate the power of Chebyshev methods in solving practical problems, it also provides a quantitative estimate of the effects of neglecting curvature in the solutions for the rectangular slot.

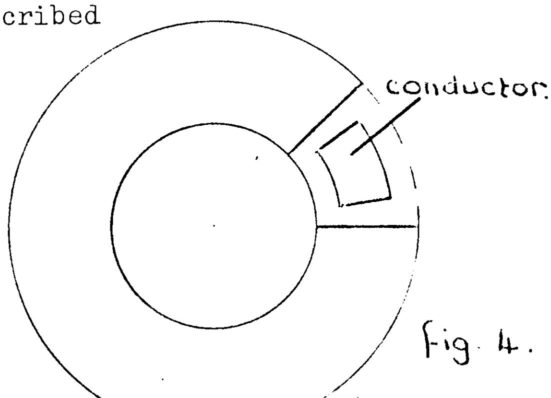

Fig. 4.

C H A P T E R 1.

THE ELECTRO-MAGNETIC FIELD EQUATIONS.

1.1) Introduction.

This chapter is based on Maxwell's equations and the mathematical simulation of the physical problems to be considered. Although many of the results quoted are well known, it is considered to be worthwhile including them both for completeness and in order to put the rest of the treatment on a firmly based foundation. Also we shall require them for frequent reference in later chapters.

1.2) The field equations.

At all interior points of bodies Maxwell's equations are satisfied, i.e.

$$\text{curl } \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \qquad 1.2(1)$$

$$\text{curl } \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J} \qquad 1.2(2)$$

$$\text{div } \underline{D} = \rho \qquad 1.2(3)$$

$$\text{div } \underline{B} = 0 \qquad 1.2(4)$$

1.3) Macroscopic properties of matter.

In free space,

$$\underline{D} = \varepsilon_o \underline{E} \qquad 1.3(1)$$

$$\underline{B} = \mu_o \underline{H} \qquad 1.3(2)$$

where ε_o and μ_o are constants.

For homogeneous isotropic bodies

$$\underline{D} = \varepsilon_r \varepsilon_o \underline{E} \qquad 1.3(3)$$

$$\underline{B} = \mu_r \mu_o \underline{H} \qquad 1.3(4)$$

where μ_r and ε_r are constant throughout the body.

1.4) Ohm's law.

For a conducting medium Ohm's law is satisfied i.e.

$$\underline{J} = \sigma\,\underline{E} \qquad 1.4(1)$$

where σ is constant.

1.5) The scalar and vector potentials.

Since div $\underline{B} = 0$ there is a vector $\underline{A}$ such that

$$\underline{B} = \text{curl}\,\underline{A} \qquad 1.5(1)$$

Hence, using equation 1.2(1)

$$\text{curl}\left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right) = \underline{0}$$

Thus there exists a scalar function V such that, to within a constant,

$$\underline{E} + \frac{\partial \underline{A}}{\partial t} = -\,\text{grad}\,V \qquad 1.5(2)$$

1.6) The potentials in an insulating medium.

Since the medium is an insulator the current density is everywhere zero.

a) Steady state equations.

In this case, all quantities are time invariant and equation 1.2(2) becomes

$$\text{curl}\,\underline{H} = \underline{0} \qquad 1.6(1)$$

Hence curl curl $\underline{A} = \underline{0}$

i.e. grad div $\underline{A} - \nabla^2\underline{A} = \underline{0}$

where $\nabla^2\underline{A} \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$.

Imposing the further condition that div $\underline{A} = 0$ this equation reduces to

$$\nabla^2\underline{A} = \underline{0} \qquad 1.6(2)$$

1.6) contd.

b) Sinusoidal variation with time.

$\frac{\partial}{\partial t}\underline{D}$ is neglected; the usual assumption for the range of frequencies to be considered. Hence equations 1.6(1),(2) are again satisfied.

Assuming sinusoidal variation with time t, we may write

$$\underline{A} = R\ell\left(e^{i\omega t}\ \underline{A}^*\right),\quad i = \sqrt{-1} \qquad 1.6(3)$$

where $\underline{A}^*$ is a complex vector function of position and $R\ell(z)$ denotes the real part of z. Equation 1.6(2) then reduces to

$$\nabla^2 \underline{A}^* = \underline{0} \qquad 1.6(4)$$

1.7) The potentials in a conducting medium.

a) Steady state equations.

All quantities are time invariant and Ohm's law is satisfied everywhere within the region. Combining equations 1.2(2), 1.3(4) and 1.5(1),

$$\begin{aligned} \mu_r\, \mu_o\, \underline{J} &= \text{curl}\ (\mu_r\, \mu_o\, \underline{H}) \\ &= \text{curl}\ \underline{B} \\ &= \text{curl curl}\ \underline{A} \\ &= \text{grad div}\ \underline{A} - \nabla^2 \underline{A}. \end{aligned}$$

$$\therefore\ \nabla^2 \underline{A} = -\mu_r\, \mu_o\, \underline{J} \quad \text{if div}\ \underline{A} = 0 \qquad 1.7(1)$$

b) Sinusoidal variation with time.

From equation 1.5(2),

$$\underline{E} = -\text{grad } V - \frac{\partial \underline{A}}{\partial t}$$

$$= \underline{E}_1 + \underline{E}_2$$

1.7) contd.

b) contd.

where $\underline{E_1} = -\text{ grad } V$ and

$$\underline{E_2} = -\frac{\partial \underline{A}}{\partial t}$$

From Ohm's law

$$\underline{J} = \sigma(\underline{E_1} + \underline{E_2})$$

$$= \underline{J_1} + \underline{J_2}$$

where $\underline{J_1} = \sigma \underline{E_1} = -\sigma \text{ grad } V$

and $\underline{J_2} = \sigma \underline{E_2} = -\sigma \frac{\partial \underline{A}}{\partial t}$

Equation 1.7(1) is still satisfied so that

$$\nabla^2 \underline{A} = -\mu_r \mu_o (\underline{J_1} + \underline{J_2})$$

Hence $\nabla^2 \underline{A} - \mu_r \mu_o \sigma \frac{\partial \underline{A}}{\partial t} = -\mu_r \mu_o \underline{J_1}$

Writing $\underline{A} = R\ell\left(e^{i\omega t} \underline{A}^*\right)$

and $\underline{J_1} = R\ell\left(e^{i\omega t} \underline{J_1}^*\right)$

this equation reduces to

$$\nabla^2 \underline{A}^* - \mu_r \mu_o \sigma \; i\omega \; \underline{A}^* = -\mu_r \mu_o \underline{J_1}^*$$

i.e. $\nabla^2 \underline{A}^* - i\,\alpha^2 \underline{A}^* = -\mu_r \mu_o \underline{J_1}^*$ 1.7(2)

1.8) <u>Boundary conditions</u>.

The boundary conditions across a surface dividing two media are

i) the normal component of $\underline{B}$ is continuous and

ii) the tangential components of $\underline{H}$ are continuous across a change in medium when the conductivity is bounded.

1.9) <u>A conductor in an open slot of an electrical machine</u>.

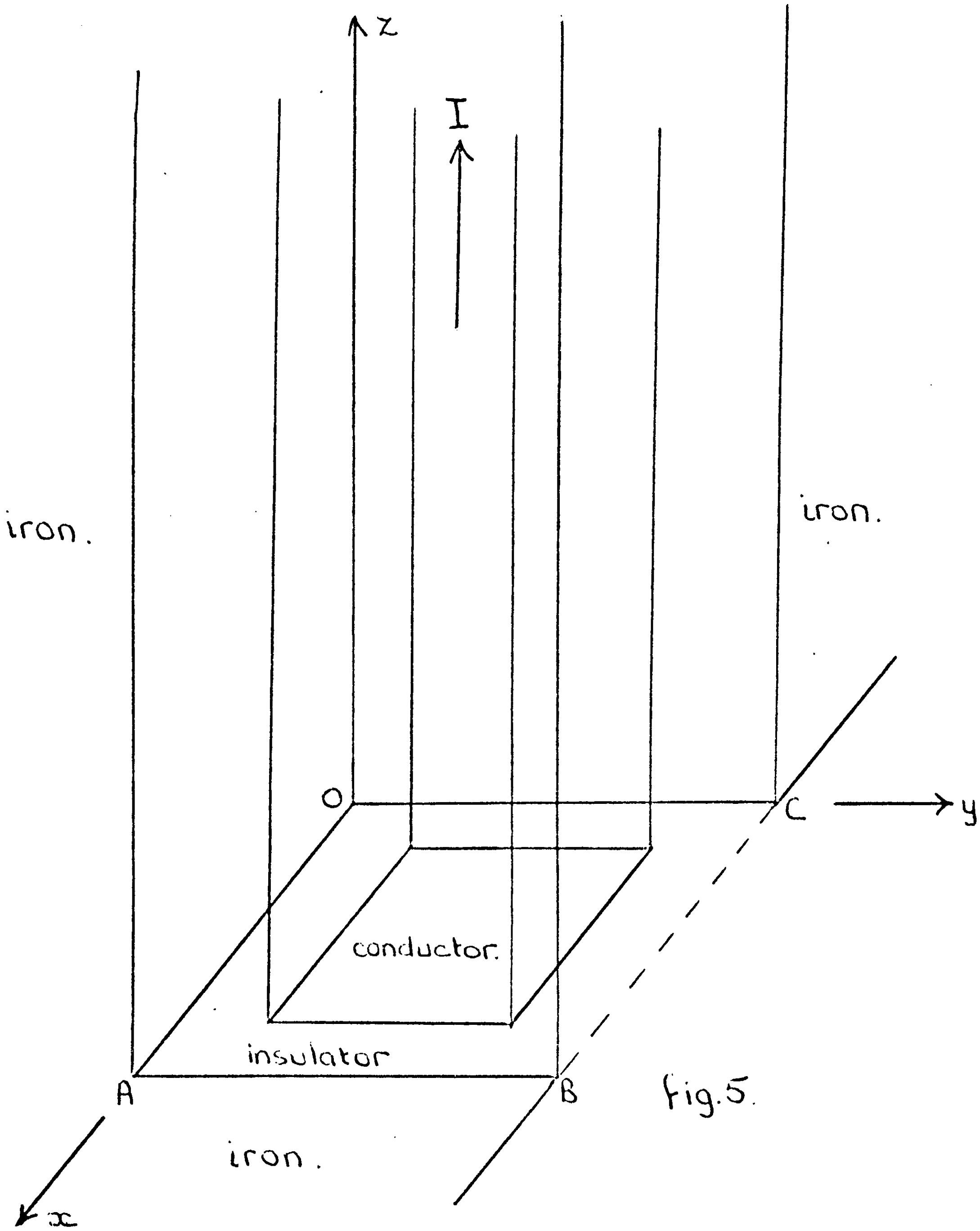

fig.5.

Both the conductor and the slot have rectangular cross-section and are infinite in length in the axial direction. A current I flows in the axial z-direction as shown in the diagram. The slot is surrounded on three sides by the iron core and the fourth side is open. With this configuration

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1.9) contd.

$$\underline{J} = (0,0,J_z)$$

and $$\underline{A} = (0,0,A_z) \quad \text{where } A_z \text{ is independent of } z.$$

a) <u>Constant steady current.</u>

When I is a constant steady current equations 1.6(2) and 1.7(1) apply and the equations to be solved are

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0 \text{ in the insulator} \qquad 1.9(1)$$

and $$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\mu_r\mu_0 J_z \text{ in the conductor} \qquad 1.9(2)$$

The components of $\underline{B}$ and $\underline{H}$ in the directions of the axes are given by

$$\underline{B} = \mu_r\mu_0\underline{H} = \left(\frac{\partial A_z}{\partial y}, \frac{-\partial A_z}{\partial x}, 0\right)$$

so that, if A_z is known as a function of x and y throughout the slot, then the magnetic intensity and flux density components can be calculated. The components of grad A_z are

$$\text{grad } A_z = \left(\frac{\partial A_z}{\partial x}, \frac{\partial A_z}{\partial y}, 0\right)$$

so that $\underline{B}.\text{grad } A_z = 0$

i.e. $\underline{B}$ and grad A_z are perpendicular vectors. This implies that flux lines are also lines of constant A_z and a plot of A_z is therefore a flux plot.

Consider now the cross-section of the slot OABC

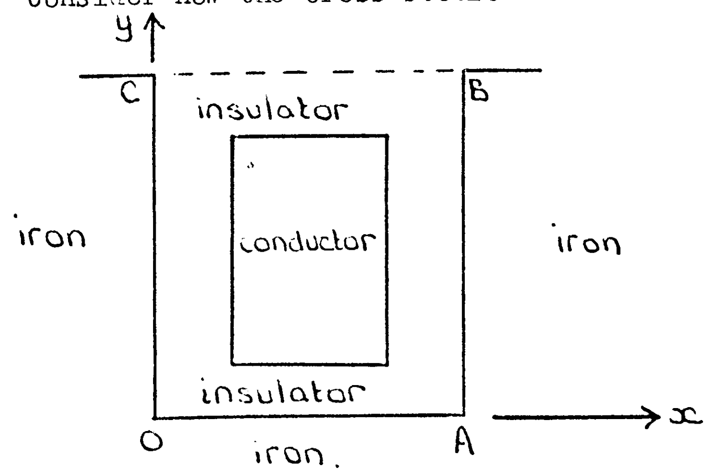


fig. 6.

1.9) contd.

a) contd.

In practice the permeability of the insulator is the same as that of the conductor so that across the insulator-conductor boundaries, the tangential and normal components of both $\underline{B}$ and $\underline{H}$ are continuous. Thus throughout the interior of the slot $\frac{\partial A_z}{\partial x}$ and $\frac{\partial A_z}{\partial y}$ are continuous. This implies that A_z must be continuous also. The iron is assumed to have infinite permeability so that at the iron-insulator boundaries the tangential components of $\underline{H}$ must be zero. This gives rise to the following boundary conditions:-

$$\frac{\partial A_z}{\partial x} = 0 \text{ on OC, AB} \qquad 1.9(3)$$

$$\frac{\partial A_z}{\partial y} = 0 \text{ on OA} \qquad 1.9(4)$$

It is assumed that the line BC is a flux line i.e. A_z = constant. Due to the form of equations 1.9(1) and (2), we may, without loss of generality, choose the value of this constant to be zero. Choice of a different value for this constant merely alters the level of potential. Thus we have a further boundary condition:-

$$A_z = 0 \text{ on BC} \qquad 1.9(5)$$

To obtain A_z and hence the field throughout the slot we have to solve equations 1.9(1) and (2) subject to the boundary conditions 1.9(3),(4) and (5).

b) <u>Current varying sinusoidally with time.</u>

Equations 1.6(4) and 1.7(2) now apply i.e.

in the insulator

1.9) contd.

b) contd.

$$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z^*}{\partial y^2} = 0 \qquad 1.9(6)$$

and in the conducting region

$$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z^*}{\partial y^2} - i\alpha^2 A_z^* = -\mu_r \mu_o J_{1z}^* \qquad 1.9(7)$$

where A_z^* and J_{1z}^* are given by

$$\underline{A}^* = (0,0,A_z^*)$$

$$\underline{J}_1^* = (0,0,J_{1z}^*)$$

Again $\frac{\partial A_z^*}{\partial x}$, $\frac{\partial A_z^*}{\partial y}$, and A_z^* are continuous throughout the slot.

By reasoning analogous to the steady current case the boundary conditions at the iron-insulator surfaces are:-

$$\frac{\partial A_z^*}{\partial x} = 0 \quad \text{on OC, AB} \qquad 1.9(8)$$

$$\frac{\partial A_z^*}{\partial y} = 0 \quad \text{on OA} \qquad 1.9(9)$$

Along BC, A_z^* is constant but because of the form of equations 1.9(6) and (7) we may not now assume this constant to be zero. Sufficient equations are now given for A_z^* to be determined throughout the slot.

1.10) <u>Rectangular windings in a transformer window.</u>

The configuration is exactly similar to that discussed in paragraph 1.9 except that the region OABC is now completely enclosed by the iron.

1.10) contd.

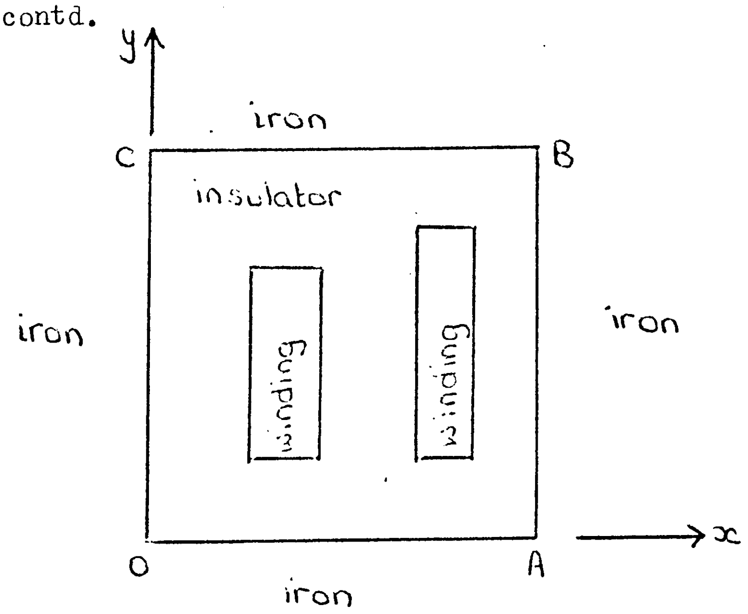


fig. 7.

Considering the case of constant steady current in the windings, equation 1.9(1) applies in the insulator and 1.9(2) in each winding. Boundary conditions 1.9(3) and (4) still apply but since BC is now an iron-insulator boundary the appropriate boundary condition is

$$\frac{\partial A_z}{\partial y} = 0 \text{ along BC} \qquad 1.10(1)$$

If I is the total net axial current flowing through the window OABC,

$$I = \oint_C \underline{H}.d\underline{s} \quad \text{where C is the path OABC.}$$

But $\underline{H} = \frac{1}{\mu_r \mu_o}\left(\frac{\partial A_z}{\partial y}, \; -\frac{\partial A_z}{\partial x}, \; 0\right)$ so that, taking account of the boundary conditions along C,

$$I = 0 \qquad 1.10(2)$$

Thus for a solution to exist with these boundary conditions, the total net current must be zero, which means that the simplest problem we can consider is that of two windings with axial currents I_1 and I_2 such that

1.10) contd.

$$I_1 + I_2 = 0 \qquad 1.10(3)$$

1.11) Magnetic energy and leakage inductance.

Consider the case of a constant steady axial current flowing in the conducting region. The magnetic energy W stored in a volume V is

$$W = \frac{1}{2} \iiint_V \underline{B} \cdot \underline{H} \, dV$$

$$= \frac{1}{2} \iiint_V \text{curl } \underline{A} \cdot \underline{H} \, dV$$

$$= \frac{1}{2} \iiint_V (\text{div}(\underline{A} \times \underline{H}) + \underline{A}.\text{curl } \underline{H}) dV$$

$$= \frac{1}{2} \iint_S (\underline{A} \times \underline{H}).d\underline{S} + \frac{1}{2} \iiint_V \underline{A}.\underline{J} \, dV$$

since curl $\underline{H} = \underline{J}$,

(S is the surface bounding the volume V).

Consider the volume V formed by taking unit axial length of the slot, cross-section OABC.

Now $\quad \underline{A} = (0,0,A_z)$

$$\underline{H} = \frac{1}{\mu_r \mu_o} \left(\frac{\partial A_z}{\partial y} , - \frac{\partial A_z}{\partial x} , 0 \right)$$

$$\therefore \underline{A} \times \underline{H} = \frac{1}{\mu_r \mu_o} \left(A_z \frac{\partial}{\partial x} A_z , A_z \frac{\partial}{\partial y} A_z , 0 \right)$$

Thus, taking account of the boundary conditions on the surface S, for the transformer window and for the open slot

$$\iint_S (\underline{A} \times \underline{H}).d\underline{S} = 0$$

If it is assumed that

$$\underline{J} = (0,0,J_z)$$

1.11) contd.

is constant within the volume V_c of the conducting region and zero elsewhere then

$$W = \frac{1}{2} J_z \iiint_{V_c} A_z \, dV$$

and so the magnetic energy per unit axial length of the conductor is

$$W = \frac{1}{2} J_z \iint_{S_c} A_z \, dx \, dy \qquad 1.11(1)$$

where S_c is the conductor cross-sectional area. The leakage inductance L (per unit axial length) is then given by

$$W = \frac{1}{2} L I^2 \qquad 1.11(2)$$

1.12) Complex impedance for current varying sinusoidally with time.

The complex Poynting vector is defined to be

$$\frac{1}{2} \underline{E}^* \times \tilde{\underline{H}}^*$$

where

$$\underline{E} = R\ell(\underline{E}^* e^{i\omega t})$$
$$\underline{H} = R\ell(\underline{H}^* e^{i\omega t})$$

and $\tilde{z}$ denotes the complex conjugate of z. $\underline{E}^*$, $\underline{H}^*$ are complex, vector functions of position only.

If S is a closed surface bounding a volume V and $d\underline{S}$ is in the direction of the outward normal to S,

$$\iint_S \frac{1}{2}(\underline{E}^* \times \tilde{\underline{H}}^*).d\underline{S} = \frac{1}{2}\iiint_V \mathrm{div}(\underline{E}^* \times \tilde{\underline{H}}^*)dV$$

$$= \frac{1}{2}\iiint_V (\tilde{\underline{H}}^*.\mathrm{curl}\, \underline{E}^* - \underline{E}^*.\mathrm{curl}\, \tilde{\underline{H}}^*)dV$$

1.12) contd.

From equation 1.2(1) $\text{curl}\,\underline{E}^* = -i\omega\underline{B}^*$ and from equation 1.2(2) $\text{curl}\,\underline{\tilde{H}}^* = \underline{\tilde{J}}^*$

$$\therefore \iint_S \frac{1}{2}(\underline{E}^* \times \underline{\tilde{H}}^*).\underline{dS} = -\frac{1}{2}i\omega\iiint_V \underline{\tilde{H}}^*.\underline{B}^* dV - \frac{1}{2}\iiint_V \underline{E}^*.\underline{\tilde{J}}^* dV$$

$$= -\frac{1}{2}i\omega\iiint_V \frac{\underline{\tilde{B}}^*.\underline{B}^*}{\mu_r\mu_o} dV - \frac{1}{2}\iiint_V \frac{\underline{J}^*.\underline{\tilde{J}}^*}{\sigma} dV$$

$\frac{1}{2}\iiint_V \frac{\underline{J}^*.\underline{\tilde{J}}^*}{\sigma} dV$ is real and equal to the average power in the volume V.

$$\text{Hence } \frac{1}{2}\iiint_V \frac{\underline{J}^*.\underline{\tilde{J}}^*}{\sigma} dV = \frac{1}{2} R\,\underline{I}^*.\underline{\tilde{I}}^*$$

where R is the effective resistance and current

$\underline{I} = R\ell(\underline{I}^* e^{i\omega t})$

$\frac{1}{2}\iiint_V \frac{\underline{\tilde{B}}^*.\underline{B}^*}{\mu_r\mu_o} dV$ is real and equal to the magnetic energy stored in V.

$$\therefore \frac{1}{2}\iiint_V \frac{\underline{\tilde{B}}^*.\underline{B}^*}{\mu_r\mu_o} dV = \frac{1}{2} L\,\underline{I}^*.\underline{\tilde{I}}^*$$

$$\therefore \iint_S \frac{1}{2}(\underline{E}^* \times \underline{\tilde{H}}^*).\underline{dS} = -\frac{1}{2}\underline{I}^*.\underline{\tilde{I}}^*(R + i\omega L) \qquad 1.12(1)$$

$$= -\frac{1}{2}\underline{I}^*.\underline{\tilde{I}}^*(R + iX) \qquad 1.12(2)$$

Thus, integrating the complex Poynting vector over a suitable surface enables us to find the effective resistance and reactance of unit length of the conductor. The evaluation of the integral is particularly simple due to the form of the boundary conditions.

1.13) Summary.

We have now set up the mathematical models of the physical systems we are to study. The subsequent chapters will be devoted to methods of solution of these models. Roth's method of double Fourier series will be described in detail and critically compared with other methods of solution. Where necessary, improvements and modifications will be made to his methods in order to make them applicable to a wide variety of practical problems. Roth himself and also later users of his methods dealt only with problems where the current in the conductor is constant and independent of time. We shall develop Roth's technique to solve problems where the applied field is varying sinusoidally with time.

CHAPTER 2.

DESCRIPTION OF ROTH'S METHOD AND COMPARISON WITH THE METHOD OF SEPARATION OF VARIABLES.

2.1) Introduction.

Roth's form of solution for N rectangular conductors in an infinitely long rectangular slot is described in detail. A constant steady axial current is flowing in each conductor. The identical problem is then solved using the method of separation of variables. The two methods for obtaining the vector potential over the whole cross-section of the slot are then critically compared.

2.2) Description of Roth's method.

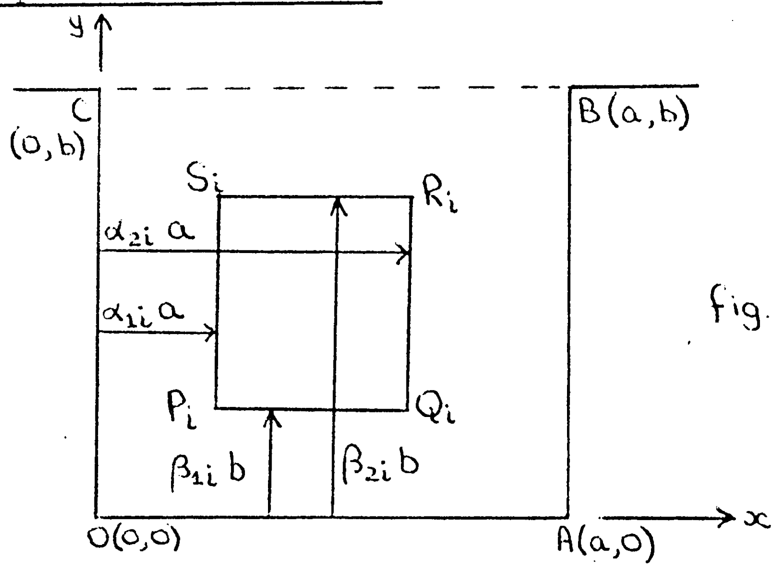


fig. 8.

Referring to section 1.9, OABC is the cross-section of the slot, with sides of length a and b metres as shown in Figure 8. O is taken as origin of coordinates. P_i Q_i R_i S_i is the cross-section of a typical conductor in the slot and the dimensions and position in the slot of this i'th conductor are as shown in the diagram. Let the current density associated with this i'th conductor be J_{zi}. Suppose there are N such non-overlapping conductors in the slot. Then from equations 1.9(1) and (2) we have to solve

2.2) contd.

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = \begin{cases} -\mu_r\mu_o J_{zi} & \text{in i'th conductor } (i = 1,2,3 \ldots N) \\ 0 & \text{in the insulator} \end{cases}$$

$$= -f(x,y) \quad \text{(say)}$$

subject to the boundary conditions(equations 1.9(3),(4) and (5)),

(i) $\frac{\partial A_z}{\partial x} = 0$ for $x = 0$, $0 \leqslant y \leqslant b$

(ii) $\frac{\partial A_z}{\partial x} = 0$ for $x = a$, $0 \leqslant y \leqslant b$

(iii) $\frac{\partial A_z}{\partial y} = 0$ for $y = 0$, $0 \leqslant x \leqslant a$

(iv) $A_z = 0$ for $y = b$, $0 \leqslant x \leqslant a$.

Assume an expression for A_z of the form

$$A_z = \sum_p \sum_q C_{pq} \cos px \cos qy.$$

Then A_z automatically satisfies boundary conditions (i) and (iii). In order to satisfy boundary condition (ii)

$$\sin pa = 0$$

$$\therefore \; p = 0, \frac{\pi}{a}, \frac{2\pi}{a} \; \ldots\ldots \; \frac{m\pi}{a}$$ where m is any positive integer, or zero.

Similarly using boundary condition (iv)

$$\cos qb = 0$$

$$\therefore \; q = \frac{\pi}{2b}, \frac{3\pi}{2b}, \frac{5\pi}{2b} \; \ldots\ldots \; (2k+1)\frac{\pi}{2b}$$ where k is any positive integer, or zero.

$$\therefore \; A_z = \sum_{m=o}^{\infty}{}' \sum_{k=0}^{\infty} C_{mk} \cos \frac{m\pi x}{a} \cos\left(k + \frac{1}{2}\right)\frac{\pi y}{b} \qquad 2.2(1)$$

where $\sum_{m=o}^{\infty}{}'$ denotes that a factor $\frac{1}{2}$ is to be included when $m = 0$.

2.2) contd.

The function f(x,y) defined over the whole slot may be expanded as a double Fourier series of the same form i.e.

$$f(x,y) = \frac{4}{\pi^2} \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty} \left\{ \sum_{i=1}^{N} \mu_r \mu_o J_{zi} \alpha_i(m) \beta_i(k) \right\} \cos\frac{m\pi x}{a} \cos\left(k+\frac{1}{2}\right)\frac{\pi y}{b} \qquad 2.2(2)$$

When J_{zi} is constant in each conductor, $\alpha_i(m)$ and $\beta_i(k)$ are given by

$$\alpha_i(m) = \frac{\sin(\alpha_{2i} m\pi) - \sin(\alpha_{1i} m\pi)}{m} \qquad (m \neq 0)$$

$$\alpha_i(0) = (\alpha_{2i} - \alpha_{1i})\pi$$

$$\beta_i(k) = \frac{\sin(\beta_{2i}(k+\frac{1}{2})\pi) - \sin(\beta_{1i}(k+\frac{1}{2})\pi)}{(k+\frac{1}{2})} \qquad 2.2(3)$$

If A_z given by 2.2(1) is to satisfy the differential equation

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -f(x,y),$$

where f(x,y) is given by 2.2(2), the coefficients C_{mk} are given by

$$C_{mk} = \frac{4}{\pi^2} \frac{\sum_{i=1}^{N} \mu_r \mu_o J_{zi} \; \alpha_i(m) \beta_i(k)}{\left\{ \frac{m^2 \pi^2}{a^2} + (k+\frac{1}{2})^2 \frac{\pi^2}{b^2} \right\}} \qquad \text{for } m = 0,1,2 \ldots \quad k = 0,1,2 \ldots \qquad 2.2(4)$$

obtained by equating coefficients of $\cos\frac{m\pi x}{a} \cos(k+\frac{1}{2})\frac{\pi y}{b}$.

Thus equations 2.2(1),(3) and (4) give the vector potential $\underline{A} = (0,0,A_z)$ at all points in the slot, cross-section OABC, with N rectangular conductors.

2.3) Convergence of the solution.

From equations 2.2(1) and (4)

$$|A_z| \leqslant \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} \left| C_{mk} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \right|$$

$$\leqslant \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} |C_{mk}|$$

and
$$|C_{mk}| \leqslant \frac{4}{\pi^2} \frac{\sum_{i=1}^{N} \mu_r \mu_o \left| J_{zi}\ \alpha_i(m)\beta_i(k)\right|}{\left\{\frac{m^2\pi^2}{a^2} + (k+\frac{1}{2})^2 \frac{\pi^2}{b^2}\right\}}$$

Now, from equations 2.2(3)

$$|\alpha_i(m)| \leqslant \frac{2}{m} \quad (m \neq 0)$$

$$|\alpha_i(0)| \leqslant (\alpha_{2i} - \alpha_{1i})\pi$$

$$|\beta_i(k)| \leqslant \frac{2}{(k+\frac{1}{2})}$$

Hence
$$|C_{ok}| \leqslant \frac{8\ \mu_o b^2}{(k+\frac{1}{2})^3 \pi^3} \sum_{i=1}^{N} \mu_r |J_{zi}| (\alpha_{2i}-\alpha_{1i})$$

and
$$|C_{mk}| \leqslant \frac{16\mu_o b^2 \sum_{i=1}^{N} \mu_r |J_{zi}|}{m(k+\frac{1}{2})\pi^4\left\{\frac{m^2b^2}{a^2} + (k+\frac{1}{2})^2\right\}}, \quad m \neq 0$$

$$\therefore |A_z| \leqslant \frac{4\mu_o b^2}{\pi^3} \sum_{i=1}^{N} \left\{\mu_r |J_{zi}| (\alpha_{2i}-\alpha_{1i})\right\} \sum_{k=o}^{\infty} \frac{1}{(k+\frac{1}{2})^3}$$

$$+ \frac{16\mu_o b^2}{\pi^4} \sum_{i=1}^{N} \left\{\mu_r |J_{zi}|\right\} \sum_{m=1}^{\infty} \sum_{k=o}^{\infty} \frac{1}{m(k+\frac{1}{2})\left\{\frac{m^2b^2}{a^2} + (k+\frac{1}{2})^2\right\}}$$

Now $\sum_{k=o}^{\infty} \frac{1}{(k+\frac{1}{2})^3}$ is convergent

$$\frac{m^2b^2}{a^2} + (k+\tfrac{1}{2})^2 \geqslant 2m\left(\frac{b}{a}\right)(k+\tfrac{1}{2})$$

2.3) contd.

$$\therefore \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m(k+\frac{1}{2})\left\{\frac{m^2b^2}{a^2} + (k+\frac{1}{2})^2\right\}} \leqslant \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2\left(\frac{b}{a}\right) m^2 (k+\frac{1}{2})^2}$$

i.e. $$\frac{1}{2\left(^{b}/_{a}\right)} \left(\sum_{m=1}^{\infty} \frac{1}{m^2}\right)\left(\sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{2})^2}\right)$$

which is convergent.

Hence the double Fourier series for $\mathbf{A}_z$ is absolutely convergent and so may be summed in any manner. Since the terms are of the same order of magnitude along the diagonals shown in the diagram of suffices below, the series is summed by diagonals, truncating at the M'th diagonal.

zero diagonal - 00 01 02 03 04 . . . . OM
first diagonal- 10 11 12 13 (1,M-1)
2nd diagonal - 20 21 22
3rd diagonal - 30 31
4th diagonal - 40
(M-2,2)
(M-1,1)
M'th diagonal - M,0

2.3) contd.

The truncated form of A_z is then

$$A_z = \sum_{m=0}^{M}{}' \left\{ \sum_{k=0}^{M-m} C_{mk} \cos(k+\tfrac{1}{2}) \frac{\pi y}{b} \right\} \cos\frac{m\pi x}{a} \qquad 2.3(1)$$

where the coefficients C_{mk} are given by 2.2(4). The numerical method for calculating this sum efficiently is described in Appendix 1.

2.4) <u>Single conductor in a slot.</u>

When considering just one conductor in the slot we may drop the i-suffix and writing

$$A_z = \frac{4\,\mu_r\mu_o\,J_z b^2}{\pi^4} F(x,y) \qquad 2.4(1)$$

then F(x,y) is non-dimensional and is given in truncated form by

$$F(x,y) = \sum_{m=0}^{M}{}' \left\{ \sum_{k=0}^{M-m} \frac{\alpha(m)\ \beta(k)}{\left\{\frac{b^2}{a^2}m^2 + (k+\frac{1}{2})^2\right\}} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \right\} \cos\frac{m\pi x}{a} \qquad 2.4(2)$$

where
$$\alpha(m) = \frac{\sin(\alpha_2 m\pi) - \sin(\alpha_1 m\pi)}{m} \qquad (m \neq 0)$$
$$\alpha(0) = (\alpha_2 - \alpha_1)\pi \qquad 2.4(3)$$
$$\beta(k) = \frac{\sin(\beta_2(k+\frac{1}{2})\pi) - \sin(\beta_1(k+\frac{1}{2})\pi)}{(k+\frac{1}{2})}$$

2.5) <u>Estimate of truncation error.</u>

Let S_{m+1} be the sum of all terms on the (m+1)th diagonal, neglected in the sum 2.4(2).

$$S_{m+1} = \sum_{r=0}^{m+1}{}' \frac{\alpha(r)\ \beta(m+1-r)}{\left\{r^2 \frac{b^2}{a^2} + \left(m+\frac{3}{2}-r\right)^2\right\}} \cos\frac{r\pi x}{a} \cos\left(m+\frac{3}{2}-r\right)\frac{\pi y}{b}$$

2.5) contd.

$$\therefore \left|S_{m+1}\right| \leqslant \sum_{r=0}^{m+1}{}' \frac{|\alpha(r)|\ |\beta(m+1-r)|}{\left\{r^2 \frac{b^2}{a^2} + \left(m+\frac{3}{2}-r\right)^2\right\}}$$

Using the results of Section 2.3,

$$\left|S_{m+1}\right| \leqslant \frac{(\alpha_2-\alpha_1)\pi}{\left(m+\frac{3}{2}\right)^3} + 4\sum_{r=1}^{m+1} \frac{1}{r\left(m+\frac{3}{2}-r\right)\left\{\frac{r^2b^2}{a^2} + \left(m+\frac{3}{2}-r\right)^2\right\}}$$

Considering the continuous function of r given by

$$f(r) = r\left(m+\frac{3}{2}-r\right)\left\{\frac{r^2b^2}{a^2} + \left(m+\frac{3}{2}-r\right)^2\right\}$$

then $f(r) = 0$ when $r = 0$ or $r = m + \frac{3}{2}$ and is positive for intermediate values of r. Hence the greatest term in the series $\sum_{r=1}^{m+1} \frac{1}{f(r)}$ occurs when $f(r)$ is least i.e. either when $r = 1$ or when $r = m+1$

$$\frac{1}{f(1)} = \frac{1}{\left(m+\frac{1}{2}\right)\left\{\frac{b^2}{a^2} + \left(m+\frac{1}{2}\right)^2\right\}}$$

$$\frac{1}{f(m+1)} = \frac{1}{(m+1)\left(\frac{1}{2}\right)\left\{\frac{b^2}{a^2}(m+1)^2 + \frac{1}{4}\right\}}$$

and both of these expressions are $O\left(\frac{1}{m^3}\right)$

$$\therefore \sum_{r=1}^{m+1} \frac{1}{f(r)} = O\left(\frac{1}{m^2}\right)$$

2.5) contd.

$$\therefore \quad |S_{m+1}| = O\left(\frac{1}{m^2}\right) \qquad 2.5(1)$$

Similarly $|S_{m+r}| = O\left(\frac{1}{(m+r-1)^2}\right)$ 2.5(2)

If the truncation error is E then

$$|E| \leqslant \sum_{r=1}^{\infty} |S_{m+r}| \qquad 2.5(3)$$

Consider $\sum_{r=o}^{\infty} \left(\frac{1}{m+r}\right)^2$.

Now $g(r) = \frac{1}{(m+r)^2}$ is a positive monotonic decreasing function of r as illustrated by Figure 9.

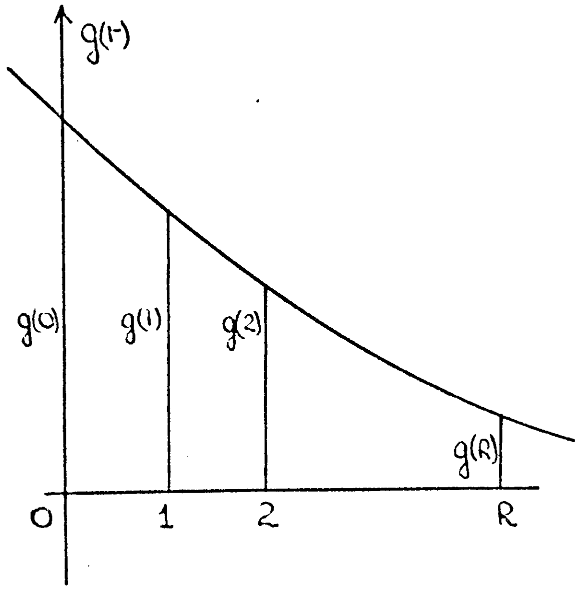


fig. 9.

By the standard result,

$$\int_0^R g(r)dr + g(R) < \sum_{r=o}^{R} g(r) < \int_0^R g(r)dr + g(o)$$

i.e. $\frac{1}{m} - \frac{1}{m+R} + \frac{1}{(m+R)^2} < \sum_{r=o}^{R} g(r) < \frac{1}{m} - \frac{1}{m+R} + \frac{1}{m^2}$

2.5) contd.

$$\therefore \sum_{r=0}^{\infty} g(r) = O\left(\frac{1}{m}\right)$$

Combining this with equations 2.5(1), (2) and (3)

$$|E| = O\left(\frac{1}{m}\right) \qquad 2.5(4)$$

Thus the absolute error due to truncating the infinite series at the m'th diagonal should not be greater than $O\left(\frac{1}{m}\right)$. It should be emphasised here that this is an upper bound, not necessarily the least upper bound. In fact, in practice the truncation error is much less than $O\left(\frac{1}{m}\right)$ in magnitude as can be seen in the following table. The data used to obtain the values in the Table is as follows:-

$$\alpha_1 = 0.1, \ \alpha_2 = 0.8, \ \beta_1 = 0.1 \ \beta_2 = 0.7, \ \frac{b}{a} = 1.5$$

The values obtained by the method of separation of variables (solutions truncated at 30 terms) are given for comparison. It will be shown later that this latter solution has converged to within 0.01%.

2.5) contd.

Table 1. Values of F(x,y) at selected points in the slot truncating the double series at 10,20,30 diagonals.

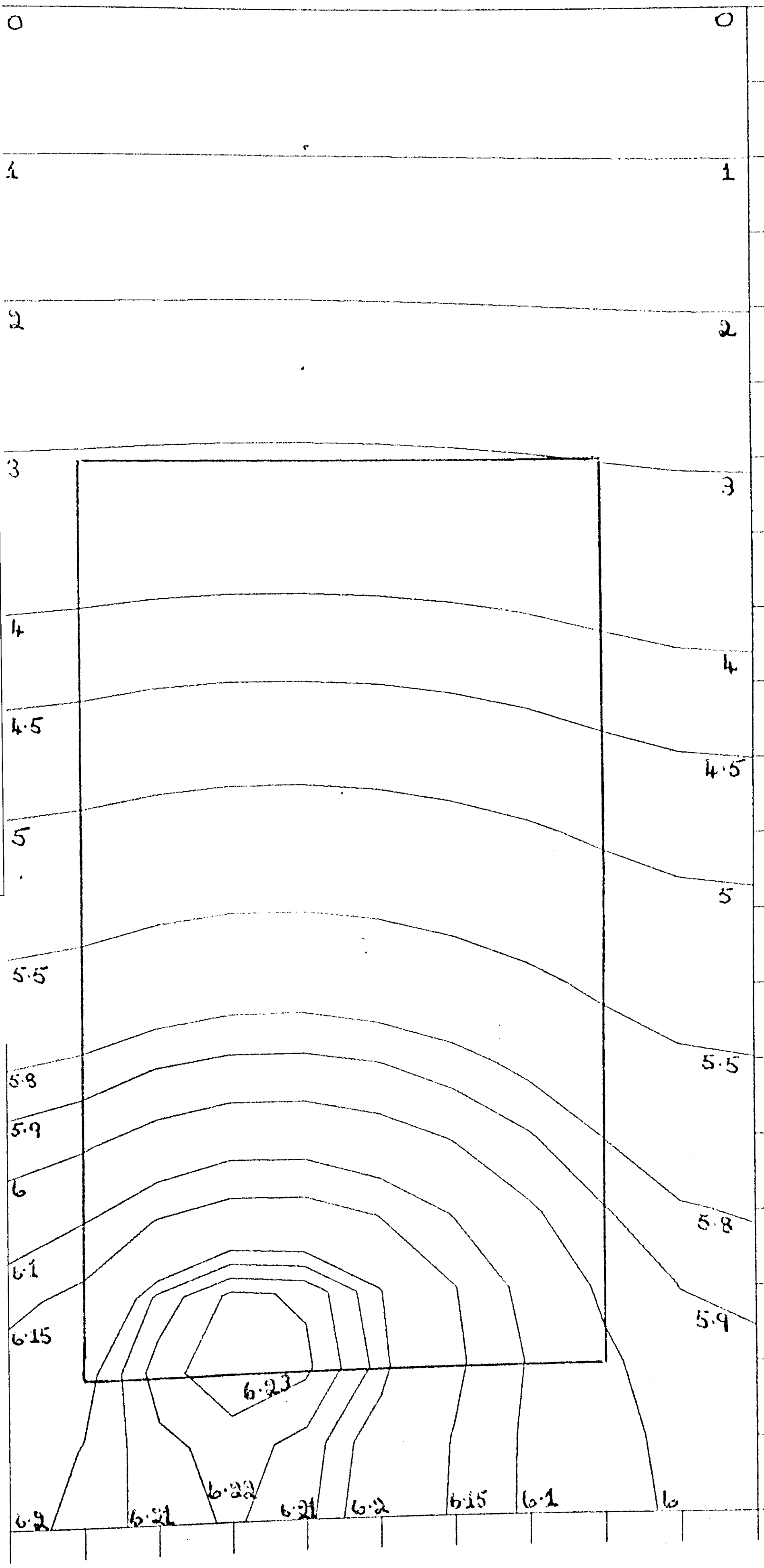
(x,y) \ m	10	20	30	Separation of variables
(0,0)	6.19295	6.19650	6.19603	6.19613
(0,b/2)	4.77017	4.77316	4.77299	4.77305
(a/2,0)	6.18270	6.18705	6.18624	6.18645
(a/2, b/2)	4.87935	4.88048	4.88068	4.88076
(a,0)	5.96918	5.96797	5.96795	5.96791
(a, b/2)	4.49420	4.49356	4.49346	4.49344

From the table it can be seen that F(x,y) has converged to within 0.1% when the infinite series has been summed up to and including the 10'th diagonal. As can be seen from the contour plot given in Figure 10, F(x,y) changes more markedly with y than with x. Because of this, in the evaluation of F(x,y) over the slot more mesh points were used in the y-direction and the mesh chosen was $\frac{x}{a} = 0(0.1)1, \frac{y}{b} = 0(0.05)1$. As explained in Section 1.9 this contour plot is also a flux plot. The times taken to evaluate F(x,y) over the given mesh on an I.C.L. 1903A computer (using the algorithm given in Appendix 1 to sum the Fourier series) are given in the following table.

Table 2. Computing time taken to evaluate F(x,y) over a mesh of (11 x 21) points (x,y)

M	Computing time in seconds	Number of terms in double Fourier series.
20	18	231
30	24	496

DATA FOR



Contours $F(x,y) = \text{constant}$. Fig. 10.

2.5) contd.

The times given include compilation time of the program and time for calculation of the coefficients C_{mk} in each case. It should also be noted that when M = 30 there are more than twice as many terms to be summed at each mesh point than for the case M = 20. Comparison of the corresponding computing times gives some indication of the efficiency of the numerical algorithm for summing a Fourier series described in Appendix 1.

2.6) Magnetic energy and leakage inductance.

From Section 1.11 the magnetic energy per unit axial length of the conductor is

$$W = \frac{1}{2} J_z \iint_{S_c} A_z \, dx\, dy$$

$$= \frac{2\mu_r \mu_o J_z^2 b^2}{\pi^4} {\sum_{m=o}^{M}}' \left\{ \sum_{k=o}^{M-m} \frac{\alpha(m)\beta(k)}{\left(\frac{b}{a}\right)^2 m^2 + (k+\frac{1}{2})^2} \int_{\beta_1 b}^{\beta_2 b} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} dy \right\} \int_{\alpha_1 a}^{\alpha_2 a} \cos\frac{m\pi x}{a}\, dx$$

Now $\displaystyle\int_{\beta_1 b}^{\beta_2 b} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} dy = \frac{b}{\pi}\beta(k)$

and $\displaystyle\int_{\alpha_1 a}^{\alpha_2 a} \cos\frac{m\pi x}{a}\, dx = \frac{a}{\pi}\alpha(m)$

$$\therefore W = \frac{2\mu_r \mu_o J_z^2 a b^3}{\pi^6} {\sum_{m=o}^{M}}' \left\{ \sum_{k=o}^{M-m} \frac{[\alpha(m)\ \beta(k)]^2}{\left(\frac{b}{a}\right)^2 m^2 + (k+\frac{1}{2})^2} \right\}$$

The total current I and J_z are related by

$$I = J_z(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)ab$$

2.6) contd.

$$\therefore W = \frac{2\mu_r \mu_0 I^2 (\frac{b}{a})}{\pi^6 (\alpha_2-\alpha_1)^2 (\beta_2-\beta_1)^2} \sum_{m=o}^{M}{}' \left\{ \sum_{k=o}^{M-m} \frac{[\alpha(m)\ \beta(k)]^2}{(\frac{b}{a})^2 m^2 + (k+\frac{1}{2})^2} \right\} \qquad 2.6(1)$$

The leakage inductance L per unit axial length is then, from equation 1.11(2)

$$L = \frac{4\mu_r \mu_0 (\frac{b}{a})}{\pi^6 (\alpha_2-\alpha_1)^2 (\beta_2-\beta_1)^2} \sum_{m=o}^{M}{}' \left\{ \sum_{k=o}^{M-m} \frac{[\alpha(m)\ \beta(k)]^2}{(\frac{b}{a})^2 m^2 + (k+\frac{1}{2})^2} \right\} \qquad 2.6(2)$$

Figure 11 shows the variation of

$$\sum_{m=o}^{M}{}' \left\{ \sum_{k=o}^{M-m} \frac{[\alpha(m)\ \beta(k)]^2}{\left(\frac{bm}{a}\right)^2 + (k+\frac{1}{2})^2} \right\}$$ with M when $\alpha_1 = 0.1$ $\alpha_2 = 0.8$,

$\beta_1 = 0.1$, $\beta_2 = 0.7$, $\frac{b}{a} = 1.5$. The graph shows that this double series converges rapidly. In addition it is an expression which is readily evaluated.

2.7) Advantages and disadvantages of Roth's method.

The chief advantage of Roth's method is that it gives a single solution for A_z which is valid over the whole region of the slot. This solution is in the form of a double Fourier series $\sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} C_{mk} \cos\frac{m\pi x}{a} \cos(k+\frac{1}{2})\frac{\pi y}{b}$ and each coefficient C_{mk} is a combination of circular functions. The C_{mk} are therefore easily calculated using a digital computer. Likewise, using the algorithm given in Appendix 1, the double series is readily evaluated at any point (x,y). The method of obtaining the solution is very straightforward and requires a minimum of mathematical manipulation.

29(a)

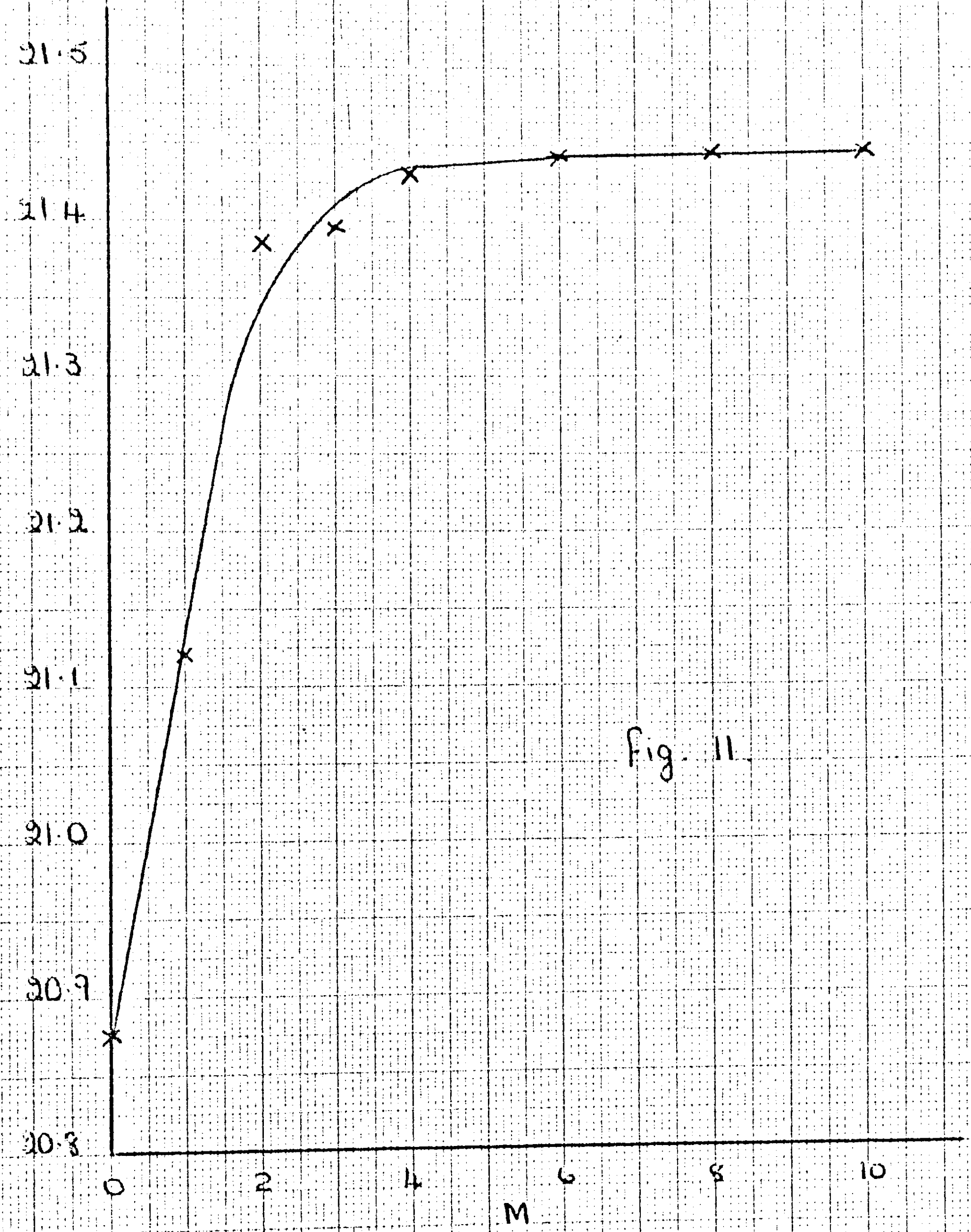


Fig. 11.

2.7) contd.

There are two main disadvantages of Roth's method, the first being the slow rate of convergence of the double Fourier series so that a large number of terms are required to give an accurate solution. For example, summing up to and including the 10th diagonal i.e. 66 terms gives a solution converged to within 0.1%. Obviously the rate of convergence would be worse than this in the calculation of $\underline{B} = \left(\frac{\partial}{\partial y} A_z, - \frac{\partial}{\partial x} A_z, 0 \right)$ and better in the calculation of magnetic energy (say) where the integral of A_z is required. However, with the availability of fast electronic computers, the slow rate of convergence is not considered to be a severe disadvantage when weighed against the ease of computation of the solution and the advantage of having a single expression valid over the whole region of the slot.

The other disadvantage is that to apply Roth's technique directly it is necessary that the slot boundaries are either flux lines or scalar equipotentials. This means that Roth's method cannot be applied, without modification, to regions bounded by material of finite permeability. However, we shall show in the next chapter how to superpose a Roth solution on a solution obtained by separation of variables to circumvent this difficulty.

A further advantage of Roth's method is that the solution is of the same simple form no matter how many conductors are in the slot. The coefficients C_{mk} now involve a sum of terms (equation 2.2(4)) and the solution then proceeds as before. We still have a single expression valid over the whole region of the slot.

2.7) contd.

In the Roth solution, only the complete expression satisfies the conditions of the problem. Each individual term in the double series does not satisfy Poisson's equation and can therefore have no physical significance.

Perhaps not applicable in most known practical cases at present, Roth's method can be applied when $J_z(x,y)\cos\frac{m\pi x}{a}\cos(k+\frac{1}{2})\frac{\pi y}{b}$ is any integrable function of x and y over the conductor area. It is, of course, possible that, in this case, the integration over the conductor area to give the coefficients C_{mk} might have to be performed numerically but this is no real disadvantage.

Also the method can still be used when the conductor cross-section is not rectangular. Pramanik in[16] considers the case of a conductor of triangular cross-section in a long rectangular slot. A conductor cross-section with any mathematically defined bounding curve can be dealt with by Roth's method using the techniques of double integration. Again it is possible that the integration might have to be performed numerically but this is not a severe disadvantage.

2.8) Roth-Kouskoff method for reduction of the double Fourier series to a single series.

From equation 2.4(2) the untruncated form of F(x,y) is

$$F(x,y) = \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} \frac{\alpha(m)\ \beta(k)}{\left\{\left(\frac{b}{a}\right)^2 m^2 + (k+\frac{1}{2})^2\right\}} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \qquad (2.8(1)$$

2.8) contd.

To illustrate the Roth-Kouskoff technique we shall reduce this double Fourier series to a single series. From reference [4] we extract the following results for

$0 < y < b$, $0 < \beta < 1$

$$\sum_{k=0}^{\infty} \frac{\sin(k+\frac{1}{2})\beta\pi \cos(k+\frac{1}{2})\frac{\pi y}{b}}{(k+\frac{1}{2})^3} = \frac{\pi^3}{2}\left(\beta - \frac{1}{2}\left(\frac{y}{b}\right)^2 - \frac{1}{2}\beta^2\right) \text{ if } \frac{y}{b} < \beta$$

$$= \frac{\pi^3}{2}(\beta)\left(1 - \frac{y}{b}\right) \text{ if } \frac{y}{b} > \beta \qquad 2.8(2)$$

$$\sum_{k=0}^{\infty} \frac{\sin(k+\frac{1}{2})\pi\beta \cos(k+\frac{1}{2})\frac{\pi y}{b}}{(k+\frac{1}{2})\left(\frac{b^2m^2}{a^2} + (k+\frac{1}{2})^2\right)} = \frac{\pi}{2\left(\frac{b^2m^2}{a^2}\right)}\left\{1 - \frac{\cosh\frac{m\pi y}{a}\cosh\frac{m\pi b}{a}(1-\beta)}{\cosh\frac{m\pi b}{a}}\right\} \text{ if } \frac{y}{b} < \beta$$

$$= \frac{\pi}{2\left(\frac{b^2m^2}{a^2}\right)}\left\{\frac{\sinh\frac{m\pi b}{a}\beta \sinh\frac{m\pi b}{a}\left(1 - \frac{y}{b}\right)}{\cosh\frac{m\pi b}{a}}\right\} \text{ if } \frac{y}{b} > \beta$$

Using these expressions in 2.8(1) we can reduce the double series for $F(x,y)$ to a single series. However it can be seen that we shall then have a different single series for each of the three regions $0 \leqslant \frac{y}{b} \leqslant \beta_1$, $\beta_1 \leqslant \frac{y}{b} \leqslant \beta_2$, $\beta_2 \leqslant \frac{y}{b} \leqslant 1$.

Thus by using this technique we have lost the simplicity of having a single expression for $F(x,y)$ valid over the whole region of the slot. In addition the resulting single series for each region is a more complicated mathematical expression which does not lend itself well to numerical computation. In fact, as will be seen later, by using the Roth-Kouskoff methods we have reduced the solution to the identical one which would be obtained by the method of separation of variables. We are to discuss this latter method in detail in the next paragraph together with its inherent disadvantages. Suffice it to say

2.8) contd.

at this stage that the Roth-Kouskoff method suffers from similar disadvantages.

2.9) Method of solution by separation of variables.

Consider first the case of a single rectangular conductor in the slot as illustrated in Figure 12. The differential equation to be solved in the region OABC together

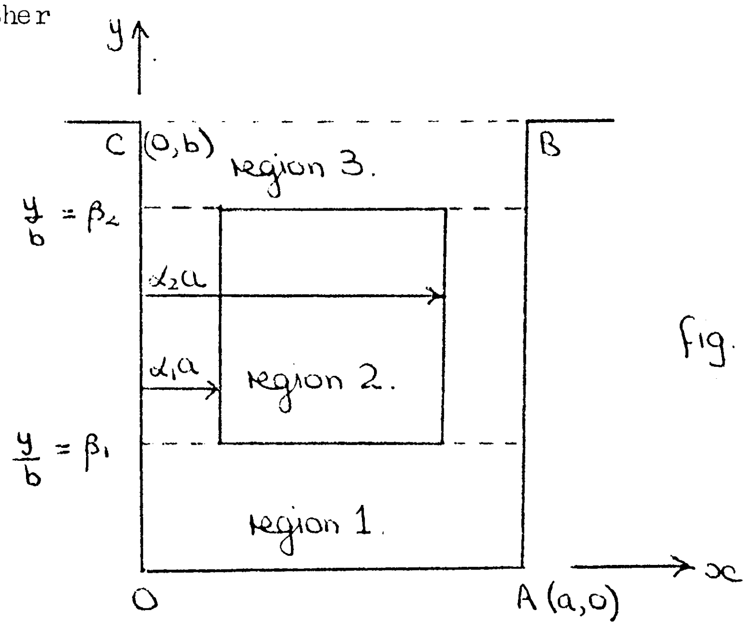


fig. 12.

with the associated boundary conditions are the same as given in Section 2.2, but to obtain a solution by separation of variables we have to divide the slot into three regions because the current distribution is limited to part of the slot. Choose these regions as follows:-

Region 1	$0 \leqslant \frac{y}{b} \leqslant \beta_1$,	$0 \leqslant x \leqslant a$,
Region 2	$\beta_1 \leqslant \frac{y}{b} \leqslant \beta_2$,	$0 \leqslant x \leqslant a$,
Region 3	$\beta_2 \leqslant \frac{y}{b} \leqslant 1$;	$0 \leqslant x \leqslant a$.

2.9) contd.

In regions 1 and 3, $$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0$$

In region 2 $$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = - \mu_r \mu_o J_z(x)$$

where
$$\begin{aligned} J_z(x) &= 0 \quad \text{if } 0 \leqslant \frac{x}{a} \leqslant \alpha_1 \\ &= J_z(\text{constant}) \text{ if } \alpha_1 < \frac{x}{a} < \alpha_2 \\ &= 0 \quad \text{if } \alpha_2 \leqslant \frac{x}{a} \leqslant 1 \end{aligned}$$

If A_{1z}, A_{2z} and A_{3z} represent A_z in the regions 1, 2, and 3 respectively then

$$A_{1z} = \sum_{m=0}^{\infty}{}' C_m \cos\frac{m\pi x}{a} \cosh\frac{m\pi y}{a}$$

and $$A_{3z} = \sum_{m=1}^{\infty} G_m \cos\frac{m\pi x}{a} \sinh\frac{m\pi b}{a} \left(1 - \frac{y}{b}\right) + \tfrac{1}{2} G_o\left(1 - \frac{y}{b}\right)$$

taking account of the boundary conditions at $x = 0$, $x = a$, $y = 0$ and $y = b$. The constants C_m and G_m will be evaluated by matching A_z and $\frac{\partial A_z}{\partial y}$ across the boundaries $\frac{y}{b} = \beta_1$ and $\frac{y}{b} = \beta_2$.

For region 2, the current density function $J_z(x)$ can be expanded as a half range Fourier cosine series given by

$$J_z(x) = \frac{2J_z}{\pi} \sum_{m=0}^{\infty}{}' \alpha(m) \cos\frac{m\pi x}{a}$$

where $\alpha(m)$ is given by equations 2.4(3).

Thus we have to solve in region 2

2.9) contd.

$$\frac{\partial^2 A_{2z}}{\partial x^2} + \frac{\partial^2 A_{2z}}{\partial y^2} = -\mu_r\mu_o \frac{2J_z}{\pi} \sum_{m=o}^{\infty}{}' \alpha(m) \cos\frac{m\pi x}{a}$$

The particular integral of this equation is of the form

$$\frac{2\mu_r\mu_o J_z a^2}{\pi^3} \sum_{m=1}^{\infty} \frac{\alpha(m)}{m^2} \cos\frac{m\pi x}{a} - \frac{\mu_r\mu_o J_z}{2\pi} \alpha(0)\, y^2$$

Thus in region 2

$$A_{2z} = \sum_{m=1}^{\infty} \left(D_m \cosh\frac{m\pi y}{a} + E_m \sinh\frac{m\pi y}{a}\right) \cos\frac{m\pi x}{a} + \frac{1}{2} D_o + \frac{1}{2} E_o y$$

$$- \frac{\mu_r\mu_o J_z}{\pi} \left\{\frac{\alpha(0)}{2} y^2 - \frac{2a^2}{\pi^2} \sum_{m=1}^{\infty} \frac{\alpha(m)}{m^2} \cos\frac{m\pi x}{a}\right\}$$

where D_m and E_m are constants to be found by equating A_z,

and $\frac{\partial A_z}{\partial y}$ across $\frac{y}{b} = \beta_1, \beta_2$.

As in equation 2.4(1) define functions F_1, F_2, F_3 valid in regions 1,2 and 3 respectively Then after much mathematical manipulation we obtain, from the continuity of A_z and $\frac{\partial A_z}{\partial y}$ at the boundaries $\frac{y}{b} = \beta_1, \beta_2$

$$F_1(x,y) = \frac{\pi^4}{8} (\alpha_2-\alpha_1)(\beta_2-\beta_1)(2-\beta_1-\beta_2)$$

$$+ \frac{\pi}{2\left(\frac{b}{a}\right)^2} \sum_{m=1}^{\infty} \frac{\alpha(m)}{m^2} \left\{\frac{\cosh\frac{m\pi b}{a}(1-\beta_1) - \cosh\frac{m\pi b}{a}(1-\beta_2)}{\cosh\frac{m\pi b}{a}}\right\} \cos\frac{m\pi x}{a} \cosh\frac{m\pi y}{a} \qquad 2.9(1)$$

$$F_2(x,y) = \frac{\pi^4}{8} (\alpha_2-\alpha_1)\left\{(\beta_2-\beta_1)(2-\beta_1-\beta_2) - \left(\frac{y}{b} - \beta_1\right)^2\right\}$$

$$+ \frac{\pi}{2\left(\frac{b}{a}\right)^2} \sum_{m=1}^{\infty} \frac{\alpha(m)}{m^2} \left\{1 - \frac{\sinh(\frac{m\pi b}{a}\beta_1)\sinh(\frac{m\pi b}{a}(1-\frac{y}{b}))}{\cosh\frac{m\pi b}{a}}\right.$$

$$\left. - \frac{\cosh\frac{m\pi b}{a}(1-\beta_2)\cosh\frac{m\pi y}{a}}{\cosh\frac{m\pi b}{a}}\right\} \cos\frac{m\pi x}{a} \qquad 2.9(2)$$

2.9) contd.

$$F_3(x,y) = \frac{\pi^4}{4}(\alpha_2-\alpha_1)(\beta_2-\beta_1)\left(1-\frac{y}{b}\right)$$

$$+ \frac{\pi}{2\left(\frac{b}{a}\right)^2}\sum_{m=1}^{\infty}\frac{\alpha(m)}{m^2}\left\{\frac{\sinh\frac{m\pi b}{a}\beta_2-\sinh\frac{m\pi b}{a}\beta_1}{\cosh\frac{m\pi b}{a}}\right\}\sinh\frac{m\pi b}{a}\left(1-\frac{y}{b}\right)\cos\frac{m\pi x}{a} \quad 2.9(3)$$

Using these results a flux plot for the whole area of the slot was obtained for the case $\alpha_1 = 0.1$, $\alpha_2 = 0.8$, $\beta_1 = 0.1$, $\beta_2 = 0.7$, $\frac{b}{a} = 1.5$ truncating each infinite series at first 30 terms and then 40 terms. The changes in the values of A_z between these two cases were within 0.01% so the solution was deemed to have converged when truncated at 30 terms.

2.10) Advantages and disadvantages of the method of solution by separation of variables applied to the problem of a conductor in a slot.

To apply the method of separation of variables for one rectangular conductor in a slot it is necessary to subdivide the slot into 3 regions with a different expression for A_z in each region. Thus we no longer have a single expression for A_z valid over the whole region of the slot. Each expression is a single Fourier series of the form

$$\sum_m C_m \cos\frac{m\pi x}{a} \left.\begin{matrix}\cosh\\ \sinh\end{matrix}\right\} \frac{m\pi y}{a}$$

and each coefficient C_m involves a combination of hyperbolic functions. Considerable mathematical manipulation is required in order to obtain the solutions over the whole slot since A_z and $\frac{\partial A_z}{\partial y}$ have to be matched across the internal boundaries between the regions. Having derived equations 2.9(1), (2) and (3), before a computer solution can be attempted it is necessary to scale the terms involving hyperbolic functions so that only exponential terms with negative exponent occur. Otherwise

2.10) contd.

terms like $\cosh\frac{m\pi x}{a}$ will cause numerical overload as m gets large. Again this involves considerable mathematical manipulation.

One of the advantages of the method is that it converges at least as fast as does $\sum_{m=1}^{\infty} \frac{1}{m^3}$ so that comparatively few terms are required for an accurate solution. The solution has converged to within 0.01% when 30 terms of each series are included.

The method of separation of variables does not suffer from the restriction that the slot boundaries should be either flux lines or scalar equipotentials. By superposition of solutions the method can be used with a boundary condition of the form

$$\text{(i)} \quad \frac{\partial A_z}{\partial x} = f(y) \text{ for } x = 0, \quad 0 \leqslant y \leqslant b$$

or

$$\text{(ii)} \quad A_z = f(y) \text{ for } x = 0, \quad 0 \leqslant y \leqslant b$$

for example, with similar conditions at the other slot boundaries.

Up to the present we have considered the case of a single conductor in the slot. As the number of conductors increases, so does the number of regions of division of the slot. For example suppose there are two conductors in the slot as shown in Figure 13.

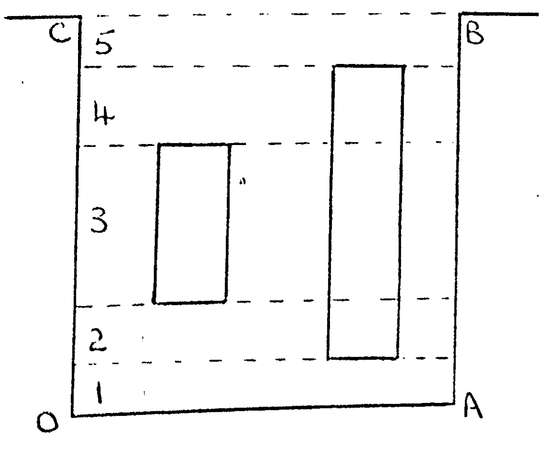


fig. 13.

2.10) contd.

The minimum number of regions required now is five. There will be a different solution valid in each region and continuity conditions must be met across all internal boundaries. This is a tedious and lengthy algebraic exercise and a severe limitation on the use of this method.

An advantage of the method of separation of variables is that each term of the series in the solution is a solution of Laplace's equation and can be sometimes usefully identified with different parts of the boundary so making the solution more meaningful.

Considering again the single conductor in the slot, to obtain the solution by the method of section 2.9 the current density function $J_z(x,y)$ in the conductor must be such that it can be expanded as a Fourier series in x. It is then necessary for the resulting series to be such that it is possible to evaluate the particular integral of the differential equation. A further constraint on the method is that the conductor must be of rectangular cross-section.

2.11) Conclusions.

Roth's method gives a single solution for A_z valid over the whole region of the slot. The derivation of the solution requires the minimum of effort and the resulting double Fourier series can readily be evaluated using a digital computer. The method of separation of variables requires the slot to be subdivided into regions with a separate solution valid in each region. Considerable mathematical effort is required to obtain the solutions since continuity conditions must be met across all internal boundaries. Even when the mathematical solution has been derived, further manipulation

2.11) contd.

is required to render the solutions suitable for digital computation. The rate of convergence of the Roth double Fourier series is slow by comparison with the single series of separation of variables. However with digital computing facilities available this is not considered to be too serious a disadvantage when weighed against the ease of computation of the solution and the advantage of having one expression for the whole slot. Summarising then, far less human effort is required to obtain a flux plot using Roth's method than is needed using the method of separation of variables.

Roth's method is particularly advantageous when there are several conductors in the slot. For a separation of variables solution, as the number of conductors increases so does the number of regions of subdivision of the slot. There is a different solution for each region and the algebra required to match the continuity conditions across the internal boundaries becomes prohibitive. Roth's solution on the other hand still consists of a single expression valid over the whole slot, the Fourier coefficients being slightly more complicated.

A disadvantage of Roth's method is that it requires the slot boundaries to be either flux lines or scalar equipotentials. The method of separation of variables can deal with more general forms of boundary condition making the latter more generally applicable in practice. In the next chapter we shall show how a Roth solution may be superposed on a separation of variables solution so combining the simplicity of Roth's solution for the discontinuities in current density in the slot with the generality of separation of variables for dealing with more generalised boundary conditions.

Only the complete Roth solution satisfies the

2.11) contd.

conditions of the problem; individual terms in the series cannot be interpreted meaningfully. In the separation of variables solution every term in the infinite series is a solution of Laplace's equation and can often be related to different parts of the slot boundary.

Using the Roth-Kouskoff methods, the double Fourier series of Roth may be summed over one variable to produce a single series. In doing this we lose the advantage of having a single series valid over the whole slot and in fact the solution becomes identical to that obtained by the method of separation of variables. So again we have the difficulties imposed by having different solutions in different regions of the slot and by having solutions which must be scaled before a numerical solution can be attempted. However by using Roth's method followed by the Roth-Kouskoff substitutions one avoids the problem of matching A_z and $\frac{\partial A_z}{\partial y}$ across the internal boundaries. On the whole, though, it is considered better to work with the double series directly and tolerate its slow rate of convergence.

Another advantage of Roth's method is that it can be used when the conductor cross-section is bounded by any mathematically defined closed curve. The method of separation of variables requires the conductor cross-section to be rectangular. Also with a general current density function in the conductor there might well be difficulties in finding the particular integral in the separation of variables solution. Roth's method can be applied with a general current density function but it is possible that in some

2,11) contd.

cases the coefficients C_{mk} in the double Fourier series might not be obtainable in closed form but would have to be evaluated using numerical integration.

CHAPTER 3.

GENERALISED BOUNDARY CONDITIONS.

3.1) Introduction.

In this chapter we shall show how a Roth solution may be superposed on a separation of variables solution so allowing Roth's method to be used when the slot boundaries are not flux lines or scalar equipotentials. Briefly the Roth solution is used to deal with the discontinuities in current density in the slot and the method of separation of variables is used to cope with the inhomogeneous boundary conditions. This Roth-separation of variables combination is then compared with the finite Fourier transform approach and we apply both methods of solution to the problem described in reference[5]. For completeness, Roth's treatment as given in[5] is also briefly discussed.

3.2) Description of the problem.

As described in Figure 8, there are N conductors in the slot and the differential equation to be solved throughout the region $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b$ is

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = - f(x,y) \qquad 3.2(1)$$

$$\text{where } f(x,y) = \frac{4}{\pi^2} \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty}{}' \left\{ \sum_{i=1}^{N} \mu_r \mu_o J_{zi} \alpha_i(m) \beta_i(k) \right\} \cos\frac{m\pi x}{a} \cos(2k+1)\frac{\pi y}{2b}$$

$$= \frac{4}{\pi^2} \sum_{m=0}^{\infty}{}' \sum_{r=0}^{\infty}{}' \left\{ \sum_{i=1}^{N} \mu_r \mu_o J_{zi} \alpha_i(m) \beta_i'(r) \right\} (1-(-1)^r) \cos\frac{m\pi x}{a} \cos\frac{r\pi y}{2b} \qquad 3.2(2)$$

where $2\beta_i'(2k+1) = \beta_i(k)$

and $\alpha_i(m), \beta_i(k)$ are given by equations 2.2(3)

[$\beta_i'(2k)\{1-(-1)^{2k}\} = 0$ but the term is included for completeness]

The boundary conditions are generalised as follows:-

3.2) contd.

$$\text{(i)} \quad \frac{\partial A_z}{\partial y} = f_1(x) \text{ when } y = 0,\ 0 \leqslant x \leqslant a$$

$$\text{(ii)} \quad \frac{\partial A_z}{\partial x} = f_2(y) \text{ when } x = 0,\ 0 \leqslant y \leqslant b$$

$$\text{(iii)} \quad \frac{\partial A_z}{\partial x} = f_3(y) \text{ when } x = a,\ 0 \leqslant y \leqslant b$$

$$\text{(iv)} \quad A_z = f_4(x) \text{ when } y = b,\ 0 \leqslant x \leqslant a$$

It is assumed that the functions $f_1(x)$, $f_2(y)$, $f_3(y)$ and $f_4(x)$ are known over the given intervals and that they satisfy the conditions necessary for expressibility as a Fourier series. Expand the functions $f_1(x)$ and $f_4(x)$ as half range Fourier cosine series over $[0,a]$ so that

$$f_1(x) = {\sum_{m=0}^{\infty}}' a_m \cos\frac{m\pi x}{a} \qquad 3.2(3)$$

and

$$f_4(x) = {\sum_{m=0}^{\infty}}' d_m \cos\frac{m\pi x}{a} \qquad 3.2(4)$$

The functions $f_2(y)$ and $f_3(y)$ are defined over the interval $[0,b]$. Assume that for $b < y \leqslant 2b$,

$$f_2(y) = -f_2(2b-y) \qquad 3.2(5)$$

and

$$f_3(y) = -f_3(2b-y) \qquad 3.2(6)$$

Then $f_2(y)$ and $f_3(y)$ can be expanded as half range Fourier cosine series over $[0,2b]$ in the form

$$f_2(y) = \sum_{k=0}^{\infty} b_k \cos(2k+1)\frac{\pi y}{2b}$$

$$= {\sum_{r=0}^{\infty}}' b_r'(1-(-1)^r)\cos\frac{r\pi y}{2b} \text{ where } 2b'_{2k+1} = b_k,\ (1-(-1)^{2k})b'_{2k} = 0 \qquad 3.2(7)$$

3.2) contd.

$$\text{and} \quad f_3(y) = \sum_{k=o}^{\infty} c_k \cos(2k+1)\frac{\pi y}{2b}$$

$$= \sum_{r=o}^{\infty}{}' \, C_r'(1-(-1)^r)\cos\frac{r\pi y}{2b} \text{ where } 2C'_{2k+1} = C_k, (1-(-1)^{2k})C'_{2k} = 0 \qquad 3.2(8)$$

Boundary conditions (i), (ii) and (iii) allow for an arbitrary, tangential magnetic field strength along the bottom and sides of the slot. The mouth of the slot is no longer a flux line (boundary condition (iv)).

3.3) The Roth-separation of variables solution.

$$\text{Write } A_z = A_1 + A_2 + A_3 + A_4 + A \qquad 3.3(1)$$

Briefly, A takes account of the conductors in the slot and A_1, A_2, A_3, A_4 take account of boundary conditions (i), (ii), (iii), (iv) respectively.

$$\text{Let } \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = - f(x,y)$$

so that A_1, A_2, A_3, A_4 all satisfy Laplace's equation

$$\text{i.e.} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad 3.3(2)$$

The boundary conditions to be satisfied by A, A_1, A_2, A_3, A_4 are given in the following table

3.3) contd.

Boundary	potentials A	A_1	A_2	A_3	A_4	A_z
(i) $y=0, 0\leqslant x\leqslant a$	$\frac{\partial A}{\partial y} = 0$	$\frac{\partial A_1}{\partial y} = f_1(x)$	$\frac{\partial A_2}{\partial y} = 0$	$\frac{\partial A_3}{\partial y} = 0$	$\frac{\partial A_4}{\partial y} = 0$	$\frac{\partial A_z}{\partial y} = f_1(x)$
(ii) $x=0, 0\leqslant y\leqslant b$	$\frac{\partial A}{\partial x} = 0$	$\frac{\partial A_1}{\partial x} = 0$	$\frac{\partial A_2}{\partial x} = f_2(y)$	$\frac{\partial A_3}{\partial x} = 0$	$\frac{\partial A_4}{\partial x} = 0$	$\frac{\partial A_z}{\partial x} = f_2(y)$
(iii) $x=a, 0\leqslant y\leqslant b$	$\frac{\partial A}{\partial x} = 0$	$\frac{\partial A_1}{\partial x} = 0$	$\frac{\partial A_2}{\partial x} = 0$	$\frac{\partial A_3}{\partial x} = f_3(y)$	$\frac{\partial A_4}{\partial x} = 0$	$\frac{\partial A_z}{\partial x} = f_3(y)$
(iv) $y=b, 0\leqslant x\leqslant a$	$A = 0$	$A_1 = 0$	$A_2 = 0$	$A_3 = o$	$A_4 = f_4(x)$	$A_z = f_4(x)$

TABLE 3.

As given in section 2.2

$$A = \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} C_{mk} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \cos\frac{m\pi x}{a} \qquad 3.3(3)$$

where the coefficients C_{mk} are given by equation 2.2(4).

Assume a separation of variables solution of equation 3.3(2) in the form $\phi = X(x)\, Y(y)$.

Then $\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda^2$ where λ is some constant.

Taking the positive sign the solutions are of the form

$$X = \begin{cases} \left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right\} \lambda x & \text{if } \lambda \neq 0 \\ \alpha + \beta x & \text{if } \lambda = 0 \end{cases}$$

$$Y = \begin{cases} \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} \lambda y & \text{if } \lambda \neq 0 \\ \gamma + \delta y & \text{if } \lambda = 0 \end{cases}$$

Taking the negative sigm, X is expressed in circular functions and Y in hyperbolic functions.

Taking account of boundary conditions (ii), (iii) and (iv) A_1 is of the form

3.3) contd.

$$A_1 = \sum_{m=1}^{\infty} P_m \cos\frac{m\pi x}{a} \sinh\frac{m\pi b}{a}\left(1 - \frac{y}{b}\right) + \tfrac{1}{2}P_o\left(1 - \frac{y}{b}\right) \qquad 3.3(4)$$

The coefficients P_m are found using boundary condition (i) and equation 3.2(3)

$$P_m\left(-\frac{m\pi}{a}\right) \cosh\frac{m\pi b}{a} = a_m (m \neq 0)$$

$$P_o\left(-\frac{1}{b}\right) = a_o \qquad 3.3(5)$$

From boundary conditions (i), (iii), (iv),

$$A_2 = \sum_{k=o}^{\infty} Q_k \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \cosh\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\left(1 - \frac{x}{a}\right)\right\} \qquad 3.3(6)$$

The coefficients Q_k are found from boundary condition (ii) and equation 3.2(7)

$$Q_k\left(-(k+\tfrac{1}{2})\frac{\pi}{b}\right)\sinh(k+\tfrac{1}{2})\frac{\pi a}{b} = b_k,\ k = 0,1,2 \ldots \qquad 3.3(7)$$

Proceeding in this way,

$$A_3 = \sum_{k=o}^{\infty} R_k \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \cosh(k+\tfrac{1}{2})\frac{\pi x}{b} \qquad 3.3(8)$$

where

$$R_k\left((k+\tfrac{1}{2})\frac{\pi}{b}\right) \sinh(k+\tfrac{1}{2})\frac{\pi a}{b} = C_k,\ k = 0,1,2 \ldots \qquad 3.3(9)$$

and $$A_4 = \sum_{m=1}^{\infty} S_m \cos\frac{m\pi x}{a} \cosh\frac{m\pi y}{a} + \tfrac{1}{2} S_o \qquad 3.3(10)$$

where $$S_m \cosh\frac{m\pi b}{a} = d_m,\quad m = 0,1,2 \ldots \qquad 3.3(11)$$

Thus we have obtained the solutions for A, A_1, A_2, A_3 and A_4 and hence for A_z from equation 3.3(1).

3.4) Discussion on the Roth-separation of variables method.

The method is a powerful one, extending Roth's technique to cope with general boundary conditions. Roth's method is used to obtain the vector potential solution due to the conductors in the slot, the method of separation of variables being inconvenient in this case for the reasons given in Chapter 2. We superpose on this solution potentials which arise due to the general form of the boundary conditions. These potentials are obtained using the method of separation of variables. Thus we are utilising Roth's methods when the use of separation of variables is inconvenient and so optimising the derivation of the complete solution for the whole slot.

The problem described in section 3.2 specifies the normal derivative of A_z round three sides of the slot and the variation of A_z along the fourth side, this being the natural extension of the insulated conductor problem described in Chapter 2. The Roth-separation of variables method described here can also be used when the boundary conditions specify either A_z around the slot boundary (Dirichlet form of boundary conditions) or the normal derivative of A_z around the slot boundaries (Neumann conditions). For the Dirichlet problem the Roth solution will be of the form

$$A = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} C_{mk} \sin\frac{m\pi x}{a} \sin\frac{k\pi y}{b} \quad (A = 0 \text{ around the slot})$$

and for the Neumann problem, of the form

$$A = \sum_{k=0}^{\infty}{}' \sum_{m=0}^{\infty}{}' C_{mk} \cos\frac{m\pi x}{a} \cos\frac{k\pi y}{b} \left(\frac{\partial A}{\partial n} = 0 \text{ around the slot}\right)$$

Obviously, therefore, the method has wide scope for application to a variety of problems. It is particularly useful when the

3.4) contd.

boundary conditions specify the function over part of the boundary and the normal derivative over the remainder. In this case Fourier transform methods are cumbersome as will be shown in the subsequent sections.

A further point in favour of the Roth-separation of variables method is that only the contribution to A_z from $\dot{A}$ is a double series. The remaining contributions are all single series with a consequently faster rate of convergence. Also, each term in each single series is itself a solution of Laplace's equation and can be identified with different parts of the boundary. This makes for a more meaningful physical interpretation of the solution.

On the other hand we have lost the advantage of having a single solution valid over the whole region of the slot and the contributions to A_z from A_1, A_2, A_3 and A_4 being series involving hyperbolic functions require further mathematical manipulation before a computer solution can be attempted as explained in Chapter 2 when considering the disadvantages of separation of variables solutions.

3.5) Fourier transform methods.

To solve the problem of section 3.2 by the transform method, consider the region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant 2b$. This will make for easier comparison with the Roth-separation of variables combination. The configuration is as shown in Figure 14.

Boundary condition (i) is as given in section 3.2. Boundary conditions (ii) and (iii) now become

3.5) contd.

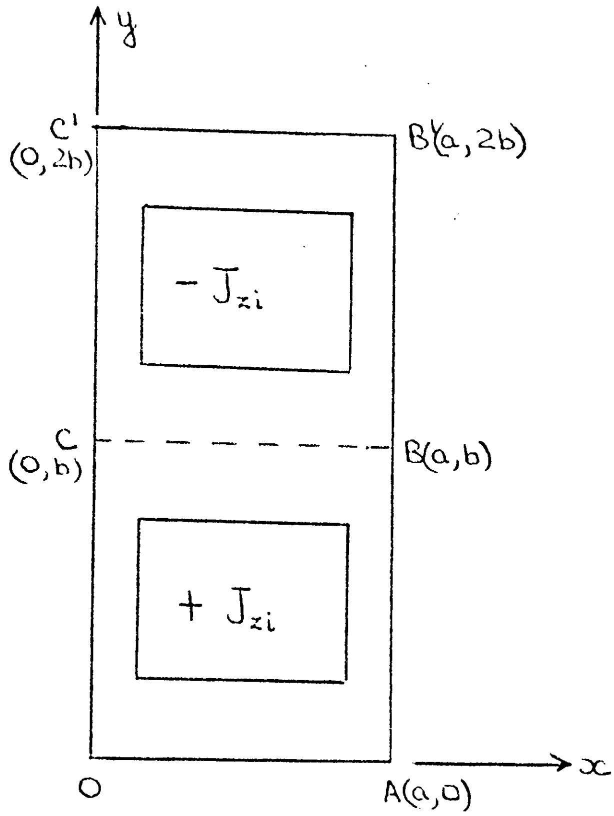


Figure 14.

(ii) $\dfrac{\partial A_z}{\partial x} = f_2(y)$ when $x = 0,\ 0 \leqslant y \leqslant 2b$

(iii) $\dfrac{\partial A_z}{\partial x} = f_3(y)$ when $x = a,\ 0 \leqslant y \leqslant 2b$

For $b < y \leqslant 2b$, the functions $f_2(y)$, $f_3(y)$ are given by equations 3.2(5) and (6) so that equations 3.2(7) and (8) still apply.

In order to use the finite cosine transform approach we must specify the normal derivative of A_z along B'C'. Assume therefore that the remaining boundary condition is

(iv) $\dfrac{\partial A_z}{\partial y} = f_5(x)$ when $y = 2b,\ 0 \leqslant x \leqslant a$.

$f_5(x)$ is an unknown function, which will be determined by making use of boundary condition (iv) of section 3.2 when the solution for A_z is known.

By taking the mirror images of the conductors as shown in Figure 14 the differential equation to be satisfied is

3.5) contd.

equation 3.2(1) where f(x,y) is given by equation 3.2(a).

Taking the finite cosine transform of equation 3.2(1) with respect to y, we get

$$\frac{d^2}{dx^2} A_z' - \left(\frac{r\pi}{2b}\right)^2 A_z' = f_1(x) - (-1)^r f_5(x) - \frac{4b}{\pi^2} \sum_{m=0}^{\infty}{}' \sum_{i=1}^{N} \mu_r \mu_o J_{zi} \alpha_i(m) \beta_i'(r) \{1-(-1)^r\} \cos\frac{m\pi x}{a} \qquad 3.5(1)$$

where $A_z' = \int_0^{2b} A_z \cos\frac{r\pi y}{2b}\, dy$

At $x = 0$, $\frac{dA_z'}{dx} = \int_0^{2b} f_2(y) \cos\frac{r\pi y}{2b}\, dy$

$= b_r'(1-(-1)^r)b$ using equation 3.2(7)

Similarly, at $x = a$, $\frac{dA_z'}{dx} = C_r'(1-(-1)^r)b$.

Taking the finite cosine transform of equation 3.5(1) with respect to x gives

$$A_z''(r,m)\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{r\pi}{2b}\right)^2\right\} = \frac{2ab}{\pi^2}(1-(-1)^r) \sum_{i=1}^{N} \mu_r \mu_o J_{zi} \alpha_i(m) \beta_i'(r) + \frac{a}{2}\left\{(-1)^r e_m - a_m\right\} + b(1-(-1)^r)\left\{(-1)^m c_r' - b_r'\right\} \qquad 3.5(2)$$

using equation 3.2(3) and writing

$$A_z''(r,m) = \int_0^a A_z' \cos\frac{m\pi x}{a}\, dx,$$

and $\int_0^a f_5(x) \cos\frac{m\pi x}{a}\, dx = \frac{a}{2} e_m$.

The coefficients e_m are unknown and to be determined using boundary condition (iv) of section 3.2. Note that $A_z''(0,0)$

3.5) contd.

is undefined and putting $r = m = 0$ in equation 3.5(2) gives

$$e_o = a_o \tag{3.5(3)}$$

so giving the unknown coefficient e_o.

The inverse transform is

$$A_z = \frac{4}{(2ab)} \sum_{m=o}^{\infty}{}' \sum_{r=o}^{\infty}{}' A_z''(r,m) \cos\frac{m\pi x}{a} \cos\frac{r\pi y}{2b} \tag{3.5(4)}$$

Considering separately odd and even values of r equation 3.5(2) becomes

$$A_z''(2k+1,m)\left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{(2k+1)\pi}{2b}\right)^2 \right\} = \frac{2ab}{\pi^2} \sum_{i=1}^{N} \mu_r \mu_o J_{zi} \alpha_i(m) \beta_i(k)$$

$$+ \frac{a}{2}(-e_m - a_m) + b((-1)^m C_k - b_k)$$

for $m = 0,1,2 \ldots, k = 0,1,2 \ldots$

$$A_z''(2k,m)\left\{ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi k}{b}\right)^2 \right\} = \frac{a}{2}(e_m - a_m) \quad \text{for } m = 0,1,2 \ldots$$

$$k = 0,1,2 \ldots \tag{3.5(5)}$$

The coefficients e_m, $m = 1,2, \ldots$ are obtained by applying boundary condition (iv) of section 3.2. From equations 3.5(4) and 3.2(4)

$$\frac{2}{ab} \sum_{r=o}^{\infty}{}' A_z''(r,m) \cos\frac{r\pi}{2} = d_m \quad \text{for } m = 0,1,2 \ldots$$

i.e.
$$\frac{2}{ab} \sum_{k=o}^{\infty}{}' A_z''(2k,m)(-1)^k = d_m \tag{3.5(6)}$$

Using equation 3.5(5) this becomes for $m \neq 0$

$$\frac{(e_m - a_m)}{b} \sum_{k=o}^{\infty}{}' \frac{(-1)^k}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{\pi k}{b}\right)^2} = d_m \tag{3.5(7)}$$

Making use of the known series

3.5) contd.

$$\sum_{k=0}^{\infty}{}' \frac{(-1)^k}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{k\pi}{b}\right)^2} = \frac{ab}{2m\pi \sinh\frac{m\pi b}{a}} \quad , m \neq 0$$

$$e_m = a_m + d_m \frac{2m\pi}{a} \sinh\frac{m\pi b}{a} \quad , m = 1,2,3 \ldots \qquad 3.5(8)$$

When m = 0 equation 3.5(6) becomes

$$A_z''(0,0) = ab\ d_0 \qquad 3.5(9)$$

Equations 3.5(3), (4),(5), (8) and (9) give the complete solution for A_z over the required region $0 \leq x \leq a$, $0 \leq y \leq b$. For the problem considered in Chapter 2 when the functions f_1, f_2, f_3 and f_4 are all identically zero, it can be seen that the solution obtained by the transform method is the identical one to that obtained by Roth's method.

3.6) Discussion on the Fourier transform method.

Fourier transform methods are most conveniently applied when the boundary conditions are either of the Dirichlet type when the sine transform is used or of the Neumann type when the cosine transform is used. For mixed boundary conditions of the type given in section 3.2, the method is not so suitable since considerable mathematical manipulation is required to obtain the solution.

An advantage of the method is that it gives a single solution valid over the whole region of the slot in the form of a double Fourier series. Such a solution is readily evaluated numerically using the technique of Appendix 1 but the solution will require the summation of a large number of terms since the rate of convergence is poor.

The boundary conditions are included in the

3.6) contd.

Fourier coefficients and this makes the physical interpretation of the solution rather difficult and not so obvious as for the Roth-separation of variables solution. In addition each term of the series is not a solution of Laplace's equation and can have no physical significance. By summation over one variable of the parts of the double series arising from the boundary conditions, it is possible to obtain the identical solution to that obtained by the Roth-separation of variables combination but this is not recommended as it requires considerable mathematical effort. The Roth method is a much shorter route to the same destination.

3.7) Practical application.

As a practical application of the two methods described in this chapter consider the configuration shown in Figure 15.

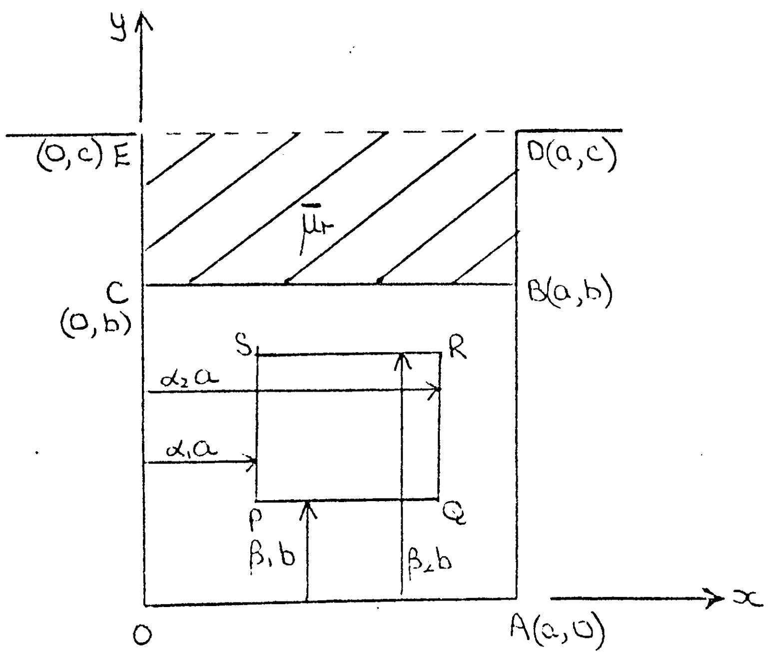


Figure 15.

PQRS is a single conductor in the region OABC. The boundaries OA, OE, AD are iron-air boundaries of a transformer and the iron is assumed to have infinite permeability. The boundary

3.7) contd.

ED represents the centre line of the core and is by symmetry a flux line. The rectangular region CBDE represents the core iron having a constant finite relative permeability $\bar{\mu}_r$. Let A_{1z}, A_{2z} be the z-components of vector potential in the regions OABC, CBDE respectively. Then the differential equation to be solved for A_{1z} is

$$\frac{\partial^2 A_{1z}}{\partial x^2} + \frac{\partial^2 A_{1z}}{\partial y^2} = \begin{cases} -\mu_r \mu_0 J_z & \text{in PQRS} \\ 0 & \text{elsewhere in OABC} \end{cases}$$

subject to the boundary conditions

(i) $\frac{\partial A_{1z}}{\partial y} = 0$ when $y = 0,\ 0 \leqslant x \leqslant a$

(ii) $\frac{\partial A_{1z}}{\partial x} = 0$ when $x = 0\ ,\ 0 \leqslant y \leqslant b$

(iii) $\frac{\partial A_{1z}}{\partial x} = 0$ when $x = a,\ \ 0 \leqslant y \leqslant b$

The differential equation to be satisfied by A_{2z} is

$\frac{\partial^2 A_{2z}}{\partial x^2} + \frac{\partial^2 A_{2z}}{\partial y^2} = 0$ in region CBDE subject to the boundary conditions

(i) $\frac{\partial A_{2z}}{\partial x} = 0$ when $x = 0,\ \ b \leqslant y \leqslant c$

(ii) $\frac{\partial A_{2z}}{\partial x} = 0$ when $x = a,\ \ b \leqslant y \leqslant c$

(iii) $A_{2z} = 0$ when $y = c,\ \ 0 \leqslant x \leqslant a$.

Across the boundary BC,

(i) $A_{1z} = A_{2z}$

(ii) $\frac{1}{\mu_r} \frac{\partial A_{1z}}{\partial y} = \frac{1}{\bar{\mu}_r} \frac{\partial A_{2z}}{\partial y}$

3.8) Solution for A_{2z}.

Using the method of separation of variables the solution for A_{2z} is given in the form

$$A_{2z} = \sum_{m=1}^{\infty} B_m \cos\frac{m\pi x}{a} \sinh\frac{m\pi}{a}(c-y) + \frac{B_0}{2}\left(1 - \frac{y}{c}\right) \qquad 3.8(1)$$

where the coefficients B_m are to be determined from the boundary conditions across CB.

3.9) Solution for A_{1z}.

Referring back to section 3.2, A_{1z} is to satisfy equation 3.2(1) where now N = 1 so that the i-suffix may be dropped. The functions f_1, f_2, f_3 are all identically zero and

$$f_4(x) = \sum_{m=1}^{\infty} B_m \cos\frac{m\pi x}{a} \sinh\frac{m\pi}{a}(c-b) + \frac{1}{2} B_0\left(1 - \frac{b}{c}\right) \qquad 3.9(1)$$

Hence from equation 3.2(4)

$$\begin{aligned} d_0 &= B_0\left(1 - \frac{b}{c}\right) \\ d_m &= B_m \sinh\frac{m\pi}{a}(c-b) \quad m = 1,2, \ldots \end{aligned} \qquad 3.9(2)$$

a) Using the Roth-separation of variables method.

As in section 2.4

$$A = \frac{4\mu_r \mu_0 J_z b^2}{\pi^4} \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty} C_{mk} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \cos\frac{m\pi x}{a} \qquad 3.9(3)$$

where $$C_{mk} = \frac{\alpha(m)\ \beta(k)}{\left(\frac{bm}{a}\right)^2 + \left(k+\frac{1}{2}\right)^2}$$

and $\alpha(m)$, $\beta(k)$ are given by equations 2.4(3). The contributions to A_{1z} from A_1, A_2, A_3 of section 3.3 are all identically zero and

3.9) contd.

a) contd.

$$A_4 = \sum_{m=1}^{\infty} S_m \cos\frac{m\pi x}{a} \cosh\frac{m\pi y}{a} + \tfrac{1}{2} S_o \qquad 3.9(4)$$

where $S_o = B_o\left(1 - \frac{b}{c}\right)$

$$S_m = \frac{B_m \sinh\frac{m\pi}{a}(c-b)}{\cosh\frac{m\pi b}{a}} \qquad m = 1,2,3 \ldots \qquad 3.9(5)$$

combining equations 3.3(11) and 3.9(2).

The solution is then

$$A_{1z} = A + A_4 \qquad 3.9(6)$$

The unknown coefficients B_m ($m = 0,1,2 \ldots$) are obtained from

$$\frac{1}{\mu_r}\left(\frac{\partial A_{1z}}{\partial y}\right)_{y=b} = \frac{1}{\mu_r}\left(\frac{\partial A_{2z}}{\partial y}\right)_{y=b}$$

Using equations 3.8(1), 3.9(3), (4), (5) and (6) and equating coefficients of $\cos\frac{m\pi x}{a}$ leads to

$$B_o = 4\,\bar{\mu}_r\,\mu o\,J_z\,\frac{bc}{\pi^3}\sum_{k=o}^{\infty}(-1)^k(k+\tfrac{1}{2})\,C_{ok} \qquad 3.9(7)$$

and $B_m\left(\frac{m\pi}{a}\right)\left\{\frac{1}{\bar{\mu}_r}\cosh\frac{m\pi}{a}(c-b) + \frac{1}{\mu_r}\sinh\frac{m\pi}{a}(c-b)\tanh\frac{m\pi b}{a}\right\}$

$$= \frac{4\mu o J_z b}{\pi^3}\sum_{k=o}^{\infty}(-1)^k(k+\tfrac{1}{2})C_{mk} \qquad \text{for } m = 1,2,3 \ldots \qquad 3.9(8)$$

Using known Fourier series, it is possible to sum these infinite series giving

$$B_o = 2\,\bar{\mu}_r\,\mu o\,J_z\,bc(\alpha_2-\alpha_1)(\beta_2-\beta_1) \qquad 3.9(9)$$

and $B_m\left(\frac{m\pi}{a}\right)\left\{\frac{1}{\bar{\mu}_r}\cosh\frac{m\pi}{a}(c-b) + \frac{1}{\mu_r}\sinh\frac{m\pi}{a}(c-b)\tanh\frac{m\pi b}{a}\right\}$

3.9) contd.

a) contd.

$$= 2\mu_0 J_z \frac{a\,\alpha(m)}{\pi^3 m}\left\{\frac{\sinh(m\pi\beta_2 \frac{b}{a}) - \sinh(m\pi\beta_1 \frac{b}{a})}{\cosh\frac{m\pi b}{a}}\right\} \qquad 3.9(10)$$

for m = 1,2,3 ...

We have now obtained the vector potential throughout the region OADE.

b) <u>Using the finite cosine transform method.</u>

Equations 3.5(3),(5), (8) and (9) become in this case

$$e_o = 0 \qquad 3.9(11)$$

$$A''_{1z}(2k+1,m)\left\{\left(\frac{m\pi}{a}\right)^2+\left(\frac{(2k+1)\pi}{2b}\right)^2\right\} = \frac{2ab}{\pi^2}\,\mu_r\mu_0 J_z\,\alpha(m)\beta(k)-\frac{a}{2}e_m \qquad 3.9(12)$$

$$A''_{1z}(2k,m)\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{k\pi}{b}\right)^2\right\} = \frac{a}{2}\,e_m \qquad 3.9(13)$$

$$e_m = \frac{2m\pi}{a}\sinh\left(\frac{m\pi b}{a}\right) B_m \sinh\frac{m\pi}{a}(c-b) \qquad 3.9(14)$$

(m = 1,2,3, ...) (using equation 3.9(2))

$$A''_{1z}(0,0) = ab\,B_o\left(1 - \frac{b}{c}\right) \quad \text{(using equation 3.9(2))} \qquad 3.9(15)$$

The solution is then from equation 3.5(4)

$$A_{1z} = \frac{2}{ab}\sum_{m=o}^{\infty}{}'\sum_{r=o}^{\infty}{}' A''_{1z}(r,m)\cos\frac{m\pi x}{a}\cos\frac{r\pi y}{2b} \qquad 3.9(16)$$

The coefficients B_m are obtained from the boundary condition

$$\frac{1}{\mu_r}\left(\frac{\partial A_{1z}}{\partial y}\right)_{y=b} = \frac{1}{\mu_r}\left(\frac{\partial A_{2z}}{\partial y}\right)_{y=b}$$

Equating coefficients of $\cos\frac{m\pi x}{a}$ and using equations 3.9(11),(12),(13),(14) and (15) the coefficients B_m (m = 0,1, ...) are found and are of course identical

3.9) contd.

b) contd.

to those obtained by the Roth method. Equation 3.9(16) then gives the potential A_{1z} over the whole region $0 \leq x \leq a,\ 0 \leq y \leq b$. If the contributions to this series which multiply the coefficients B_m are summed over the y variable the resulting solution for A_{1z} is the same as that obtained by the Roth-separation of variables combination.

3.10) Roth's solution as given in[5].

Roth in[5] derives a single solution for A_z in the form of a double Fourier series valid over the whole region $0 \leq x \leq a,\ 0 \leq y \leq c$. The method assumes an arbitrary current distribution on the surface $y = b$ to produce the same effect as the material of relative permeability $\bar{\mu}_r$. This surface current distribution is determined by considering the change in $\frac{\partial A_z}{\partial y}$ across $y = b$.

To satisfy the boundary conditions around the slot OADE we must have A_z of the form

$$A_z = {\sum_{m=o}^{\infty}}' \sum_{k=o}^{\infty} C_{mk} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{c} \qquad 3.10(1)$$

and A_z must satisfy the differential equation

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = \begin{cases} -\mu_r \mu_o J_z & \text{in the conducting region} \\ 0 & \text{elsewhere} \end{cases}$$

together with an unknown current strength $f(x)$ along $y = b$.

Hence the coefficients C_{mk} are given by

$$C_{mk}\left\{\frac{m^2\pi^2}{a^2} + \frac{(k+\frac{1}{2})^2\pi^2}{c^2}\right\}\frac{ac}{4} = \mu_r \mu_o J_z\ \alpha(m)\gamma(k)\frac{ac}{\pi^2} + \cos(k+\tfrac{1}{2})\frac{\pi b}{c}F(m) \qquad 3.10(2)$$

3.10) contd.

where $\gamma(k) = \dfrac{\sin(k+\frac{1}{2})\pi \frac{b}{c} \beta_2 - \sin(k+\frac{1}{2})\pi \frac{b}{c}\beta_1}{(k+\frac{1}{2})}$ 3.10(3)

and $F(m) = \displaystyle\int_0^a f(x) \cos\frac{m\pi x}{a}\, dx$ 3.10(4)

At the interface y = b it can be shown that

$$f(x) = 2\,\frac{(\bar{\mu} - 1)}{(\bar{\mu} + 1)} \left|\left(\frac{\partial A_z}{\partial y}\right)_{y=b}\right| \qquad \text{where } \bar{\mu} = \frac{\bar{\mu}_r}{\mu_r}$$

Combining this with equations 3.10(1) and (4),

$$F(m) = 2\,\frac{(\bar{\mu} - 1)}{(\bar{\mu} + 1)} \left| \sum_{k=o}^{\infty} C_{mk}(k+\tfrac{1}{2})\frac{\pi}{c}\sin(k+\tfrac{1}{2})\frac{\pi b'}{c} \right| \frac{a}{2} \qquad 3.10(5)$$

Substituting for C_{mk} from equation 3.10(2) gives F(m). Using series given in the Roth-Kouskoff paper[4], it is possible after much mathematical manipulation to obtain the coefficients F(m) in closed form. The coefficients C_{mk} are then known in closed form and the solution over the whole region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant c$ is given by equation 3.10(1).

3.11) <u>Conclusions.</u>

The Roth method as described in[5], although ingenious, is not recommended. Its main advantage is that it gives a single solution in the form of a double Fourier series valid over the whole region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant c$. However considerable mathematical effort is required if the coefficients F(m), m = 0,1,2 ..., are to be found in closed form. Although the double series lends itself well to numerical computation, the rate of convergence of the solution is slow and is particularly bad in the region of the discontinuity at y = b.

Both the transform method and the Roth-separation

3.11) contd.

of variables method consider the two regions of the slot separately, deriving a separation of variables single series in the region BCED. Thus, in this region, comparatively few terms are required to give a solution which has converged to a given accuracy. However the disadvantages associated with the numerical computation of a separation of variables solution (as described in Chapter 2) will, of course, be present. Consideration of the two regions separately leads to an easier interpretation of the physical conditions of the problem across the boundary $y = b$, but it does mean that we have lost the advantage of having a single solution valid over the whole region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant c$.

Considering now the derivation of the solution in the region OABD, the Roth-separation of variables method is straightforward and uses single series wherever possible. The double series arises as a result of the discontinuity in current density in the region. Associated with each part of the boundary where there is a generalised boundary condition, there is a single series solution obtained by the method of separation of variables. This again leads to a more direct physical interpretation of the solution. Summarising then, the Roth-separation of variables method combines the advantages of Roth's method for dealing with the discontinuities in current density with those of the method of separation of variables for dealing with the generalised boundary conditions. We have in this way optimised the derivation of the complete solution over the whole slot using Roth's method when the method of separation of variables is unsuitable for the reasons given in Chapter 2.

3.11) contd.

Although the transform method is very straightforward when the boundary conditions are either of the Dirichlet type when the finite sine transform is used or of the Neumann type when the finite cosine transform is used, it cannot be recommended when the boundary conditions are of mixed type as described in this chapter. In this case the method becomes unwieldy and its beauty is lost. The boundary conditions are included in the Fourier coefficients and this makes the physical interpretation of the solution rather difficult. An advantage of the method is that the solution consists of a single expression in the form of a double Fourier series which is readily summed numerically using the algorithms of Appendix 1 although a large number of terms are required to give a solution which has converged to within a given accuracy.

Based on the results of this chapter we must disagree with the assertions of Reed and Mullineux in[10]. For general boundary conditions the Roth-separation of variables method and the finite Fourier transform method are very different and for boundary conditions of mixed type, the Roth-separation of variables combination is to be preferred. When the boundary conditions are either of Dirichlet or of Neumann type, either of the two methods can be applied, and both are then straightforward. However we would still recommend the Roth-separation of variables combination since the physical interpretation of this combined solution is so much simpler. Also the solution consists of single series wherever possible. For either Dirichlet conditions of the form $A_z = 0$ or Neumann conditions $\frac{\partial A_z}{\partial n} = 0$ around the slot boundary, the solutions obtained by the two methods are identical.

CHAPTER 4

USE OF CHEBYSHEV POLYNOMIALS WITH ROTH'S METHODS.

4.1) Introduction.

One of the chief disadvantages in the use of Roth's methods is the slow rate of convergence of the double Fourier series. It is well known in the theory of numerical solutions of ordinary differential equations that the use of Chebyshev polynomials, rather than trigonometric functions, speeds up the rate of convergence of the solution. Also, very little work has been done to date on the application of Chebyshev polynomials to the solution of partial differential equations so the treatment given in this chapter sheds a great deal of light on the more general aspects of Chebyshev approximation. The configuration to be considered is that given in Figure 8 with a single conductor in the slot. Due to the difficulties associated with differentiation of Chebyshev polynomials we consider first Chebyshev variation in one direction only taking each direction in turn. Then we allow Chebyshev variation in both directions simultaneously.

4.2) Chebyshev variation in the y-direction only.

By considering the mirror image of the slot as shown in Figure 16 we may determine the solution over the whole region $0 \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$. In this case the boundary conditions in the y-direction are of the Dirichlet type i.e. $A_z = 0$ when $y = \pm b$, $0 \leqslant x \leqslant a$.

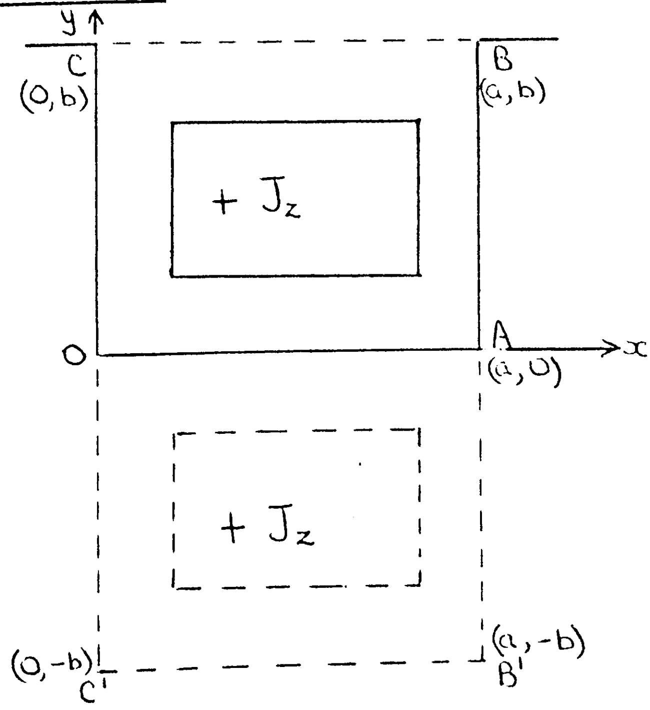


fig. 16.

4.2) contd.

In the x-direction the boundary conditions are

$$\frac{\partial A_z}{\partial x} = 0 \text{ when } x = 0,\ a;\ -b \leqslant y \leqslant b.$$

Hence we shall seek a solution in the form

$$A_z = \sum_{r=0}^{\infty}{}' \sum_{m=0}^{\infty}{}' C_{mr} \cos\frac{m\pi x}{a}\, T_{2r}\left(\frac{y}{b}\right) \qquad 4.2(1)$$

The boundary condition in the x-direction is then automatically satisfied. The current density profile over the whole region may be expressed in the form

$$f(x,y) = \mu_r\mu_0 \frac{4J_z}{\pi^2} \sum_{m=0}^{\infty}{}' \sum_{r=0}^{\infty}{}' \alpha(m)\,\delta(r) \cos\frac{m\pi x}{a}\, T_{2r}\left(\frac{y}{b}\right) \qquad 4.2(2)$$

using the orthogonal properties of the Chebyshev polynomials and circular functions. The coefficients $\alpha(m)$ are defined by equations 2.4(3) and the coefficients $\delta(r)$ are given by

$$\left.\begin{aligned} \delta(r) &= \frac{1}{r}\,(\sin 2r\xi_1 - \sin 2r\xi_2) \qquad r = 1,2 \ldots \\ \delta(0) &= 2(\xi_1 - \xi_2) \end{aligned}\right\} \qquad 4.2(3)$$

where $\xi_1 = \cos^{-1}\beta_1,\ \xi_2 = \cos^{-1}\beta_2$ 4.2(4)

Now $\dfrac{\partial^2 A_z}{\partial x^2} + \dfrac{\partial^2 A_z}{\partial y^2} = -f(x,y)$ where $f(x,y)$ is given by equation 4.2(2).

Write $F = A_z \Big/ \dfrac{4\mu_r\mu_0 J_z b^2}{\pi^4}$ so that

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -\frac{\pi^2}{b^2} \sum_{m=0}^{\infty}{}' \sum_{r=0}^{\infty}{}' \alpha(m)\,\delta(r) \cos\frac{m\pi x}{a}\, T_{2r}\left(\frac{y}{b}\right) \qquad 4.2(5)$$

Assume a solution for $F(x,y)$ in the form

4.2) contd.

$$F(x,y) = \sum_{m=o}^{\infty}{}' \phi_m\left(\frac{y}{b}\right) \cos\frac{m\pi x}{a} \qquad 4.2(6)$$

This automatically satisfies the boundary condition

$\frac{\partial F}{\partial x} = 0$ when $x = 0, a; -b \leqslant y \leqslant b$.

To satisfy the differential equation 4.2(5)

$$-\left(\frac{m\pi}{a}\right)^2 \phi_m\left(\frac{y}{b}\right) + \frac{d^2\phi_m}{dy^2} \doteq -\frac{\alpha(m)\pi^2}{b^2}\sum_{r=o}^{\infty}{}' \delta(r)T_{2r}\left(\frac{y}{b}\right) \qquad 4.2(7)$$

obtained by equating coefficients of $\cos\frac{m\pi x}{a}$, $m = 0,1,2 \ldots$ Equation 4.2(7) must be solved for each value of m. We have reduced the partial differential equation to a sequence of ordinary differential equations to determine the coefficients ϕ_m. From the boundary conditions at $y = \pm b$,

$$\phi_m = 0 \text{ when } y = \pm b \qquad 4.2(8)$$

4.3) Determination of the coefficients ϕ_m, $m = 0,1,2 \ldots$

a) Direct method.

The direct method solves equation 4.2(7) as it stands in the following way. Writing

$$Y = \frac{y}{b} \qquad 4.3(1)$$

assume that, dropping the m suffix,

$$\phi = \sum_{r=o}^{R}{}' a_r T_{2r}(Y) \qquad 4.3(2)$$

Now $\frac{d^2\phi}{dy^2} = \frac{1}{b^2}\frac{d^2\phi}{dY^2}$ and assume that

$$\frac{d^2\phi}{dY^2} = \sum_{r=o}^{R-1}{}' b_r T_{2r}(Y) \qquad 4.3(3)$$

4.3) contd.

where the coefficients b_r are to be calculated. Integrate this equation twice with respect to Y and then compare coefficients of $T_{2r}(Y)$, r = 1, ... R with equation 4.3(2) giving

$$a_r = \frac{1}{4(2r)}\left\{\frac{(b_{r-1}-b_r)}{2r-1} - \frac{(b_r-b_{r+1})}{2r+1}\right\} \quad r = 1,2, \ldots R$$

where $b_R = b_{R+1} \equiv 0$ 4.3(4)

Solving this set of equations for the coefficients b_r, r = 0,1, ... (R-1) gives

$$b_{R-1} = 4(2R)(2R-1)a_R$$

$$b_{R-2} = 4(1)(2R-2)(2R-3)a_{R-1}+4(2)(2R)(2R-2)a_R$$

$$b_{R-3} = 4(1)(2R-4)(2R-5)a_{R-2}+4(2)(2R-2)(2R-4)a_{R-1}+4(3)(2R)(2R-3)a_R$$

etc. 4.3(5)

Substituting the truncated form of ϕ given by equation 4.3(2) in the differential equation 4.2(7) with the right hand side truncated at r = R, there results a superfluous equation for the coefficients a_r. To make the equations consistent, rewrite equation 4.2(7) as

$$-\left(\frac{m\pi b}{a}\right)^2 \phi_m + \frac{d^2\phi_m}{dY^2} = -\pi^2\alpha(m)\sum_{r=0}^{R}{}' \delta(r)\ T_{2r}(Y) - \pi^2\tau_m\ T_{2R}(Y)$$

$$(m = 0,1,2 \ldots) \qquad 4.3(6)$$

Substituting for ϕ_m and $\frac{d^2\phi_m}{dY^2}$ and equating coefficients of $T_{2r}(Y)$, r = 0,1,2 ... R gives

$$-\left(\frac{m\pi b}{a}\right)^2 a_r + b_r = -\pi^2\ \alpha(m)\ \delta(r), \quad r = 0,1, \ldots (R-1) \qquad 4.3(7)$$

$$-\left(\frac{m\pi b}{a}\right)^2 a_R = -\pi^2\ \alpha(m)\ \delta(R) - \pi^2\tau_m \qquad 4.3(8)$$

4.3) contd.

together with the boundary equation

$$\sum_{r=0}^{R}{}' a_r = 0 \qquad 4.3(9)$$

This set of equations is solved successively for m = 0,1,2 ... M (truncating at M terms in the x-direction). For the method to be successful the coefficients a_r should decrease with increasing r so that each τ_m should be small giving a negligible error in the differential equation. The equations for the a_r assume a particularly simple form as can be seen by considering R = 4

$$\begin{bmatrix} \frac{1}{2} & 1 & 1 & 1 & 1 \\ \left(\frac{bm}{a}\right)^2 & -\frac{8}{\pi^3} & -\frac{64}{\pi^3} & -\frac{216}{\pi^3} & -\frac{512}{\pi^3} \\ 0 & \left(\frac{bm}{a}\right)^2 & -\frac{48}{\pi^3} & -\frac{192}{\pi^3} & -\frac{480}{\pi^3} \\ 0 & 0 & \left(\frac{bm}{a}\right)^2 & -\frac{120}{\pi^3} & -\frac{384}{\pi^3} \\ 0 & 0 & 0 & \left(\frac{bm}{a}\right)^2 & -\frac{224}{\pi^3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \delta(1) \\ \delta(2) \\ \delta(3) \\ \delta(4) \end{bmatrix} \alpha(m)$$

A Gaussian procedure was written to reduce the coefficient matrix, row by row, to upper triangular form. Note also that only the sub-diagonal elements need to be altered as m varies. Also from the form of the equations a_R will be small and hence τ_m is small so that the error due to the introduction of the τ-terms should be negligible. This will be considered in greater detail in section 4.4. Having obtained the coefficients a_r for each m, F is given by equation 4.2(6) and this double summation is calculated using the methods of Appendix 1.

4.3) contd.

b) Integrated method.

In the direct method, in order to evaluate the coefficients a_r, considering now the untruncated equation 4.2(7), we have to solve an infinite set of equations each with an infinite number of unknowns. Thus, on truncation, an infinite number of small terms are neglected in each equation. The integrated method leads to an infinite number of equations but with a finite number of terms in each so that each linear equation for the a_r arising from the differential equation will now be exact. We would expect therefore a more accurate solution using the integrated method. Integrating equation 4.2(7) twice with respect to Y gives

$$-\left(\frac{m\pi b}{a}\right)^2 \iint \phi_m \, dY \, dY + \phi_m = -\pi^2 \alpha(m) \sum_{r=1}^{\infty} Q(r) T_{2r}(Y) + K_1 Y + K_2 \qquad 4.3(10)$$

where $$Q(r) = \frac{1}{4}\left\{\frac{\delta(r-1)}{(2r-1)2r} - \frac{2\,\delta(r)}{(4r^2-1)} + \frac{\delta(r+1)}{(2r+1)2r}\right\} \qquad 4.3(11)$$

and K_1, K_2 are constants of integration.

If ϕ is given by equation 4.3(2) then

$$\iint \phi \, dY \, dY = \sum_{r=1}^{R+1} C_r T_{2r}(Y) + \text{constant}$$

where $$C_r = \frac{1}{4}\left\{\frac{a_{r-1}}{(2r-1)2r} - \frac{2a_r}{4r^2-1} + \frac{a_{r+1}}{(2r+1)2r}\right\} \quad r=1,2 \ldots R+1 \qquad 4.3(12)$$

and $a_{R+2} = a_{R+1} \equiv 0$.

To make the equations consistent on truncation write

$$-\left(\frac{m\pi b}{a}\right)^2 \iint \phi \, dY \, dY + \phi = -\pi^2 \alpha(m) \sum_{r=1}^{R+1} Q(r) T_{2r}(Y) + K_1 Y + K_2 - \pi^2 \tau_m T_{2R+2}(Y) \qquad 4.3(13$$

Comparing coefficients of $T_{2r}(Y)$ for $r = 1,2 \ldots (R+1)$

gives

4.3) contd.

$$-\left(\frac{m\pi b}{a}\right)^2 C_r + a_r = -\pi^2\,\alpha(m)Q(r) \quad r = 1,2, \ldots R \qquad 4.3(14)$$

$$-\left(\frac{m\pi b}{a}\right)^2 C_{R+1} = -\pi^2\,\alpha(m)\,Q(R+1) - \pi^2\tau_m \qquad 4.3(15)$$

which must be solved together with the boundary equation 4.3(9). Equations 4.3(14) may be written in the form

$$a_{r-1} + \lambda(r)\,a_r + \mu(r)\,a_{r+1} = K\,P(r), \quad r = 1,2 \ldots R \qquad 4.3(16)$$

where $a_{R+1} \equiv 0$,

$$\lambda(r) = -\left\{\frac{4r}{2r+1} + \frac{4(2r-1)(2r)}{\left(\frac{m\pi b}{a}\right)^2}\right\} \qquad 4.3(17)$$

$$\mu(r) = \frac{(2r-1)}{(2r+1)}$$

$$K = \frac{\pi^2\alpha(m)}{\left(\frac{m\pi b}{a}\right)^2}$$

$$P(r) = (2r-1)(2r)Q(r)$$

A recurrence algorithm to solve equation 4.3(9) together with a set of recurrence equations of the type given by 4.3(16) is given in Appendix 2. Having obtained the coefficients a_r the solution proceeds as for the direct method.

4.4) Error estimation.

There are two sources of error in the solution; the error due to truncation and the error due to the introduction of the τ-terms. Consider first the error E_τ due to the τ-terms. If $E_\tau = F - F_c$ where F_c is the calculated value of F and the truncation error is neglected, then the boundary conditions are

(i) $\frac{\partial E_\tau}{\partial x}$ when x = 0, a for $-b \leqslant y \leqslant b$

4.4) contd.

and (ii) $E_T = 0$ when $y = \pm b$ for $0 \leqslant x \leqslant a$.

Assume a solution in the form

$$E_T = \sum_{m=o}^{M}{}' \psi_m(Y) \cos\frac{m\pi x}{a} \qquad 4.4(1)$$

and then boundary condition (i) is automatically satisfied.

Using boundary condition (ii)

$$\psi_m(\pm 1) = 0 \text{ for } m = 0,1,2 \ldots M \qquad 4.4(2)$$

a) Direct method.

Using the direct method of determination of F_c, the equation to be satisfied by ψ_m is

$$-\left(\frac{m\pi b}{a}\right)^2 \psi_m + \frac{d^2\psi_m}{dY^2} = \pi^2\tau_m T_{2R}(Y),\ m = 0,1 \ldots M \qquad 4.4(3)$$

Integrating this equation twice with respect to Y gives

$$-\left(\frac{m\pi b}{a}\right)^2 \iint\psi_m(Y)dY\,dY + \psi_m(Y) = \pi^2\tau_m U_R(Y) + \alpha_m Y + \beta_m \qquad 4.4(4)$$

$$\text{where } U_R(Y) = \frac{1}{4}\left\{\frac{T_{2R+2}(Y)}{(2R+1)(2R+2)} - \frac{2T_{2R}(Y)}{4R^2-1} + \frac{T_{2R-2}(Y)}{(2R-1)(2R-2)}\right\} \qquad 4.4(5)$$

and α_m, β_m are constants of integration.

When m = 0, equation 4.4(4) gives

$$\psi o(Y) = \pi^2\tau o\, U_R(Y) + \alpha o Y + \beta o$$

Applying the boundary conditions 4.4(2) gives

$$\alpha o = 0, \quad \beta o = -\pi^2\tau o U_R(1)$$

$$\text{i.e.} \quad \beta o = -\frac{3\pi^2\tau o}{(4R^2-1)(4R^2-4)} \qquad 4.4(6)$$

Hence $\psi o(Y) = \pi^2\tau o(U_R(Y) - U_R(1))$ 4.4(7)

When $m \neq 0$, write $\psi_m(Y) = \pi^2\tau_m U_R(Y) + \psi_m^*(Y)$

4.4) contd.

a) contd.

Now $\iint U_R(Y)\, dY\, dY = 0\left(\frac{1}{R^4}\right)$ and is neglected by comparison with the other terms. Hence

$$-\left(\frac{m\pi b}{a}\right)^2 \iint \psi_m^*(Y)\, dY\, dY + \psi_m^*(Y) = \alpha_m Y + \beta_m$$

or $$-\left(\frac{m\pi b}{a}\right)^2 \psi_m^*(Y) + \frac{d^2\psi_m^*(Y)}{dY^2} = 0$$ which gives on integration

$$\psi_m^*(Y) = P_m \cosh\left(\frac{m\pi b}{a} Y\right) + Q_m \sinh\left(\frac{m\pi b}{a} Y\right) \qquad 4.4(8)$$

Applying the boundary condition,

$$Q_m = 0$$

$$P_m \cosh\left(\frac{m\pi b}{a}\right) = -\pi^2 \tau_m U_R(1)$$

so that

$$\psi_m(Y) = \pi^2 \tau_m \left\{ U_R(Y) - U_R(1) \frac{\cosh(\frac{m\pi b}{a} Y)}{\cosh(\frac{m\pi b}{a})} \right\} \qquad 4.4(9)$$

and $$E_\tau = \pi^2 \sum_{m=o}^{M}{}' \tau_m \left\{ U_R(Y) - U_R(1) \frac{\cosh(\frac{m\pi b}{a} Y)}{\cosh(\frac{m\pi b}{a})} \right\} \cos\frac{m\pi x}{a} \qquad 4.4(10)$$

Now $U_R(1) = 0\left(\frac{1}{R^4}\right)$ and the largest numerical value of $U_R(Y)$ occurs at $Y = 0$ and

$$U_R(0) = \frac{(-1)^{R+1}}{4(R^2-1)}$$

$$\therefore \quad |E_\tau| \leqslant \frac{\pi^2}{4(R^2-1)} \sum_{m=o}^{M}{}' |\tau_m| \qquad 4.4(11)$$

Using this result for $R = M = 10$ gives

$$|E_\tau| \leqslant 0.00586$$

4.4) contd.

b) Integrated method.

In this case the equation to be satisfied by ψ_m is

$$-\left(\frac{m\pi b}{a}\right)^2 \iint \psi_m(Y)\,dY\,dY + \psi_m(Y) = \pi^2 \tau_m\, T_{2R+2}(Y) + \alpha_m Y + \beta_m \qquad 4.4(12)$$

Solving in the same way as for the direct method,

$$E_\tau = \pi^2 \sum_{m=0}^{M}{}' \tau_m \left\{ T_{2R+2}(Y) - \frac{\cosh(\frac{m\pi b}{a}Y)}{\cosh(\frac{m\pi b}{a})} \right\} \cos\frac{m\pi x}{a} \qquad 4.4(13)$$

Note that in this case the neglected term is

$$\iint T_{2R+2}(Y)\,dY\,dY = O\left(\frac{1}{R^2}\right).$$

Now $\dfrac{\cosh(\frac{m\pi b}{a}Y)}{\cosh(\frac{m\pi b}{a})} \leqslant 1$ and $|T_{2R+2}(Y)| \leqslant 1$

$$|E_\tau| \leqslant 2\pi^2 \sum_{m=0}^{M}{}' |\tau_m| \qquad 4.4(14)$$

Using this result for R = M = 10 gives

$$|E_\tau| \leqslant 0.015$$

Comparing these results for the direct and integrated methods, it would appear that the direct method is significantly better. The following table gives the calculated values of F obtained by the direct and integrated methods, compared with the exact value (obtained by the method of separation of variables). The actual errors are also given and can be compared with the estimated values of E_τ.

4.4) contd.

TABLE 3. Values of F_c obtained by the direct and integrated methods.

point (x,y)	F (separation of variables)	direct method F_c	direct method $\lvert F-F_c \rvert$	integrated method F_c	integrated method $\lvert F-F_c \rvert$
$(0,0)$	6.19613	6.19822	0.00209	6.19896	0.00283
$(0,\frac{b}{2})$	4.77305	4.77176	0.00129	4.77378	0.00073
$(\frac{a}{2},0)$	6.18645	6.19379	0.00734	6.19159	0.00514
$(\frac{a}{2},\frac{b}{2})$	4.88076	4.88142	0.00066	4.88510	0.00434
$(a,0)$	5.96791	5.96853	0.00062	5.96969	0.00178
$(a,\frac{b}{2})$	4.49344	4.49428	0.00084	4.49433	0.00089
			$\lvert E_T \rvert \leqslant$ 0.~~000586~~ 00586		$\lvert E_T \rvert \leqslant 0.015$

$\alpha_1 = 0.1 \quad \alpha_2 = 0.8$

$\beta_1 = 0.1 \quad \beta_2 = 0.7$

$\frac{b}{a} = 1.5$

$R = M = 10$

From the table it can be seen that the estimates for E_T are reasonably good. Although the actual error is greater than the estimate at one or two isolated points, this could be accounted for by the truncation error. Both from the error estimation for the two methods and by inspection of the calculated values, contrary to expectations, the solution obtained by the integrated method was no more accurate (and at many points was worse) for a given R and M than that obtained by the direct method. In the latter method an infinite number of terms are neglected in each equation and consequently would be expected to be the poorer of the two. In the book by Fox and Parker(17), in discussing the equation $p(x)\frac{dy}{dx} + q(x)y = r(x)$ where p,q,r are polynomials in x, it is stated that the direct

4.4) contd.

method is better than the integrated method for cases when $p(x)$ is small compared with $q(x)$ over the range of x values considered. It quotes the case when $p(x) = x$, $q(x) = 2$ $(-1 \leqslant x \leqslant 1)$. We are solving the sequence of equations

$$-\left(\frac{m\pi b}{a}\right)^2 \phi_m(y) + \frac{d^2\phi_m}{dY^2} = -\alpha(m)\ \pi^2 \sum_{r=0}^{\infty}{}' \delta(r) T_{2r}(Y)$$

for $m = 0,1,2 \ldots$

Clearly as m increases, the coefficient of $\phi_m(Y)$ becomes dominant. Also Fox states that the direct method seems preferable for slowly convergent solutions and again my problem is of this type.

4.5) <u>Boundary conditions of the Neumann type.</u>

In choosing the Chebyshev variation for the y-direction, the boundary conditions for the sequence of ordinary differential equations are of the Dirichlet type. If we consider Chebyshev variation in the x-direction then we produce a similar sequence of ordinary differential equations but with boundary conditions of the Neumann type. Now we take the mirror image of the slot in the line OC and determine the solution over the region $-a \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. The current density profile is expressed in the form

$$f(x,y) = \frac{4\mu_r \mu_o J_z}{\pi^2} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty}{}' \beta(k)\eta(r)\cos(k+\tfrac{1}{2})\pi Y T_{2r}(X) \qquad 4.5(1)$$

4.5) contd.

where $Y = \frac{y}{b}$, $X = \frac{x}{a}$

$$\eta(r) = \frac{1}{r}(\sin 2r\zeta_1 - \sin 2r\zeta_2), \; r = 1,2 \ldots$$

$$\eta(0) = 2(\zeta_1 - \zeta_2) \qquad 4.5(2)$$

$$\zeta_1 = \cos^{-1} \alpha_1, \; \zeta_2 = \cos^{-1} \alpha_2 \qquad 4.5(3)$$

and the coefficients $\beta(k)$ are as given by equation 2.4(3).

We assume a solution for $F(x,y)$ in the form

$$F(x,y) = \sum_{k=0}^{\infty} \phi_k(X) \cos(k+\tfrac{1}{2})\pi Y \qquad 4.5(4)$$

and the differential equations for the coefficients ϕ_k are

$$-\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\right\}^2 \phi_k(X) + \frac{d^2\phi_k}{dX^2} = -\pi^2\left(\frac{a}{b}\right)^2 \beta(k) \sum_{r=0}^{\infty}{}' \eta(r) T_{2r}(X)$$

$$k = 0,1,2 \ldots \qquad 4.5(5)$$

The boundary conditions are

$$\frac{d\phi_k}{dX} = 0 \text{ when } X = \pm 1 \qquad 4.5(6)$$

Before detailed discussion of methods of solution of these equations, let us examine E_T for the direct method and the integrated method. Assume that E_T is given in the form

$$E_T = \sum_{k=0}^{N} \psi_k(X) \cos(k+\tfrac{1}{2})\pi Y \qquad 4.5(7)$$

with boundary condition

$$\frac{d\psi_k}{dX} = 0 \text{ when } X = \pm 1, \; k = 0,1,2 \ldots N \qquad 4.5(8).$$

4.5) contd.

a) Direct method.

The differential equation to be satisifed by ψ_k is

$$-\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\right\}^2 \psi_k(X) + \frac{d^2\psi_k}{dX^2} = \pi^2\tau_k \, T_{2R}(X) \qquad 4.5(9)$$

$$k = 0,1,2 \ldots$$

Using the methods of section 4.4

$$\psi_k(X) = \pi^2\tau_k \, U_R(X) + P_k\cosh\left\{(k+\tfrac{1}{2})\frac{\pi a}{b} X\right\} + Q_k\sinh\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}X\right\}$$

where P_k, Q_k are arbitrary constants to be determined from the boundary condition 4.5(8).

$$\text{Now } \frac{d}{dX} U_R(X) = \tfrac{1}{2}\left\{\frac{T_{2R+1}(X)}{2R+1} - \frac{T_{2R-1}(X)}{2R-1}\right\}.$$

Hence $Q_k = 0$

$$P_k \,(k+\tfrac{1}{2})\frac{\pi a}{b} \sinh\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\right\} = \frac{\pi^2\tau_k}{4R^2-1}$$

$$\text{Hence } \psi_k(X) = \pi^2\tau_k\left\{U_R(X) + \frac{1}{(4R^2-1)(k+\tfrac{1}{2})\frac{\pi a}{b}} \; \frac{\cosh\{(k+\tfrac{1}{2})\frac{\pi a}{b}X\}}{\sinh\{(k+\tfrac{1}{2})\frac{\pi a}{b}\}}\right\}$$

Thus the error due to the τ-terms is bounded and is

$O\left(\frac{1}{R^2}\sum_{k=0}^{N} |\tau_k|\right)$ for this method.

b) Integrated method.

The differential equation is now

$$-\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\right\}^2 \iint \psi_k(X)\,dX\,dX + \psi_k(X) = \pi^2\tau_k T_{2R+2}(X)$$

with approximate solution

$$\psi_k(X) = \pi^2\tau_k\left\{T_{2R+2}(X)\right\} + P_k\cosh\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}X\right\} + Q_k\sinh\left\{(k+\tfrac{1}{2})\frac{\pi a}{b} X\right\}$$

Putting $X = \cos\theta$,

4.5) contd.

b) contd.

$$\frac{d}{dX}\left\{T_{2R+2}(X)\right\} = \frac{d}{d\theta}\left\{\cos(2R+2)\theta\right\}\cdot\left(\frac{-1}{\sin\theta}\right)$$;)

$$= (2R+2)\,\frac{\sin(2R+2)\theta}{\sin\theta}$$

$$\therefore \text{ At } X = 1,\ \frac{d}{dX}\left\{T_{2R+2}(X)\right\} = \lim_{\theta \to 0}(2R+2)\frac{\sin(2R+2)\theta}{\sin\theta}$$

$$= (2R+2)^2$$

Similarly at $X = -1$, $\frac{d}{dX}\left\{T_{2R+2}(X)\right\} = -\,(2R+2)^2$

Hence applying the boundary condition 4.5(8)

$$Q_k = 0$$

$$P_k = \frac{-\pi^2\tau_k(2R+2)^2}{(k+\frac{1}{2})\frac{\pi a}{b}\,\sinh(k+\frac{1}{2})\frac{\pi a}{b}}$$

and $$\psi_k(X) = \pi^2\tau_k\left\{T_{2R+2}(X) - \frac{(2R+2)^2}{(k+\frac{1}{2})\frac{\pi a}{b}}\;\frac{\cosh\left\{(k+\frac{1}{2})\frac{\pi a}{b}X\right\}}{\sinh(k+\frac{1}{2})\frac{\pi a}{b}}\right\}$$

The magnitude of τ_k is $O\left(\frac{1}{kR}\right)$ and so, from the above equation, the magnitude of $\psi_k(X)$ increases with increasing R. If the integrated method is applied to the Neumann problem, the solution oscillates, a point which is not mentioned in the relevant texts. To conclude then, the integrated method is totally unsuitable for problems of the Neumann type and the direct method must be used. (6)
It is however possible to modify the direct method in order to circumvent the problem of having an infinite number of neglected terms in each equation. This is done in the following way.

Dropping the k-suffix assume that ace

4.5) contd.

$$\phi(X) = \sum_{r=0}^{R}{}' \, a_r \, T_{2r}(X) \qquad 4.5(10)$$

and
$$\frac{d^2\phi}{dX^2} = \sum_{r=0}^{R-1}{}' \, b_r \, T_{2r}(X) \qquad 4.5(11)$$

Then the coefficients a_r and b_r are related according to equation 4.3(4). The truncated form of equation 4.5(5) is

$$-\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\right\}^2 \phi_k(X) + \frac{d^2\phi_k}{dX^2} = -\pi^2\left(\frac{a}{b}\right)^2 \beta(k) \sum_{r=0}^{R}{}' \, \eta(r) \, T_{2r}(X)$$

$$-\pi^2 \tau_k \, T_{2R}(X) \qquad 4.5(12)$$

Using equations 4.5(10) and (11) and comparing coefficients of $T_{2r}(X)$, $r = 0,1,2 \ldots$ gives

$$-\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\right\}^2 a_r + b_r = -\pi^2\left(\frac{a}{b}\right)^2 \beta(k) \; \eta(r), \quad r = 0,1,2 \ldots (R-1) \qquad 4.5(13)$$

$$-\left\{(k+\tfrac{1}{2})\frac{\pi a}{b}\right\}^2 a_R = -\pi^2\left(\frac{a}{b}\right)^2 \beta(k) \; \eta(R) - \pi^2 \tau_k \qquad 4.5(14)$$

Integrating equation 4.5(11)

$$\frac{d\phi}{dX} = \tfrac{1}{2} b_0 T_1(X) + \tfrac{1}{2} \sum_{r=1}^{R-1} b_r \left\{ \frac{T_{2r+1}(X)}{2r+1} - \frac{T_{2r-1}(X)}{2r-1} \right\} + \text{constant}$$

Applying the boundary conditions 4.5(6)

$$\frac{1}{2} b_0 - \sum_{r=1}^{R-1} \frac{b_r}{(4r^2-1)} = 0 \qquad 4.5(15)$$

Equations 4.5(13) and (15) are now solved for the coefficients a_0, b_r, $r = 0,1 \ldots (R-1)$. If we define

$$c_r = -\frac{b_r}{(4r^2-1)} \qquad r = 0,1, \ldots (R-1) \qquad 4.5(16)$$

then equation 4.5(15) becomes

4.5) contd.

$$\sum_{r=0}^{R-1}{}' c_r = 0 \qquad 4.5(17)$$

and equations 4.5(13) (excluding the first which gives a_0) may be written in the form

$$C_{r-1} + \lambda(r)C_r + \mu(r)C_{r+1} = k\,P(r), \quad r = 1,2 \ldots (R-1) \qquad 4.5(18)$$

where $C_R \equiv 0$.

These equations 4.5(17) and (18) are then solved using the methods of Appendix 2 and the solution proceeds as before.

Alternatively the direct method is used exactly as described in section 4.3. There is found to be little difference in the rate of convergence of the solutions obtained by the direct method or the modified direct method.

4.6) Reduction of E_τ by perturbation of the boundary conditions.

In the application of Chebyshev polynomials to ordinary differential equations, it is sometimes possible to reduce the error in the solution by slight perturbation of the boundary conditions. With the exact boundary conditions, the solution is exact at the boundary but erroneous within the rectangle. Perturbation of the boundary conditions should result in a "smoothing" of the error over the whole region.

Consider now the estimation of E_τ using the direct method of solution as described in section 4.4(a). We have

$$\psi_0(Y) = \pi^2\tau_0\, U_R(Y) + \alpha_0 Y + \beta_0 \text{ and}$$

$$\psi_m(Y) = \pi^2\tau_m\, U_R(Y) + P_m \cosh\left(\frac{m\pi b}{a} Y\right) + Q_m \sinh\left(\frac{m\pi b}{a} Y\right)$$

$$m = 1,2, \ldots M,$$

where $\alpha_0, \beta_0, P_m, Q_m$ are constants of integration to be determined from the boundary conditions. Assume that the boundary conditions

4.6) contd.

as given are perturbed and now read

$$\left.\begin{array}{ll}\text{(i)} & \psi_m(1) = u_m \\ \text{(ii)} & \dfrac{d\psi_m}{dY} = v_m \text{ when } Y = 0\end{array}\right\} m = 0,1, \dots M \qquad 4.6(1)$$

where u_m, v_m are small. This gives

$$\alpha_0 = v_0$$

$$\beta_0 = u_0 - v_0 - \pi^2 \tau_0 U_R(1)$$

$$\left.\begin{array}{l} Q_m\left(\dfrac{m\pi b}{a}\right) = v_m \\ P_m \cosh\left(\dfrac{m\pi b}{a}\right) = u_m - \pi^2\tau_m U_R(1) - \dfrac{v_m \sinh(\frac{m\pi b}{a})}{\frac{m\pi b}{a}}\end{array}\right\} \begin{array}{l} m = 1,2 \dots M \\ 4.6(2)\end{array}$$

Hence $\psi_0(Y) = \pi^2\tau_0\{U_R(Y)-U_R(1)\} + v_0(Y-1) + u_0$

and $$\psi_m(Y) = \pi^2\tau_m U_R(Y) + \left\{u_m - \pi^2\tau_m U_R(1) - \frac{v_m \sinh(\frac{m\pi b}{a})}{\frac{m\pi b}{a}}\right\}\frac{\cosh(\frac{m\pi b}{a}Y)}{\cosh(\frac{m\pi b}{a})}$$

$$+ \frac{v_m}{m\pi b/a}\sinh\left(\frac{m\pi b}{a}Y\right), \quad m = 1,2 \dots M.$$

We may take $u_m = \pi^2\tau_m U_R(1)$, $m = 0,1, \dots M$ 4.6(3)

then $\psi_0(Y) = \pi^2\tau_0 U_R(Y) + v_0(Y-1)$

$$\psi_m(Y) = \pi^2\tau_m U_R(Y) - \frac{v_m}{\frac{m\pi b}{a}\cosh(\frac{m\pi b}{a})}\sinh\frac{m\pi b}{a}(1-Y), \quad m = 1,2 \dots M$$

Now $U_R(Y)$ attains its largest numerical value at $Y = 0$

i.e. $U_R(0) = \dfrac{(-1)^{R+1}}{4(R^2-1)}$.

Choose v_m, $m = 0,1,2 \dots M$ so that E_τ is zero at $Y = 0$

$$\therefore \quad v_0 = \frac{(-1)^{R+1}}{4(R^2-1)}\pi^2\tau_0 \qquad 4.6(4)$$

$$v_m = \frac{(-1)^{R+1}}{4(R^2-1)}\pi^2\tau_m\left\{\frac{m\pi b}{a}\coth\frac{m\pi b}{a}\right\} \quad m = 1,2 \dots M$$

4.6) contd.

Hence $\psi_o(Y) = \pi^2 \tau_o \left\{ U_R(Y) + \frac{(-1)^{R+1}}{4(R^2-1)} (Y-1) \right\}$

$$\psi_m(Y) = \pi^2 \tau_m \left\{ U_R(Y) - \frac{(-1)^{R+1}}{4(R^2-1)} \frac{\sinh\left\{\frac{m\pi b}{a}(1-Y)\right\}}{\sinh(\frac{m\pi b}{a})} \right\} \quad 4.6(5)$$

and $E_\tau = \sum_{m=o}^{M}{}' \psi_m(Y) \cos\frac{m\pi x}{a}$.

To apply these perturbations to the actual solution for F boundary conditions 4.2(8) are replaced by

(i) $\phi_m(1) = - u_m = - \pi^2 \tau_m U_R(1)$

(ii) $\frac{d\phi_m}{dY} = - v_m$ when $Y = 0$

$\Big\}$ $m = 0,1, \ldots M$

The amplitudes of these perturbations will be small since τ_m is small and R is large. It should be noticed that now $\mathbf{A}_z$ is no longer constant across the top of the slot but takes the form of a very low amplitude ripple. To implement boundary condition (i), equation 4.3(9) is replaced by

$$\sum_{r=o}^{R}{}' a_r = - \tau_m U_R(1)$$

which becomes on substituting for τ_m from equation 4.3(8)

$$\sum_{r=o}^{R-1}{}' a_r + a_R \left(1 + \left(\frac{m\pi b}{a}\right)^2 U_R(1)\right) = \pi^2 \alpha(m)\ \delta(r)\ U_R(1) \quad 4.6(6)$$

This is then solved with equations 4.3(7) to give the coefficients a_r as described in section 4.3(a). Boundary condition (ii) is not so readily implemented since the assumed evenness of the solution depends on the fact that $\frac{\partial F}{\partial y} = 0$ when $y = 0$. The simplest way to cater for this condition is to replace

4.6) contd.

ϕ_m by $\phi_m + \phi'_m$ where ϕ'_m is used to give boundary condition (ii).

Thus

$$-\left(\frac{m\pi b}{a}\right)^2 \phi'_m + \frac{d^2\phi'_m}{dY^2} = 0$$

where (i) $\phi'_m(Y) = 0$ when $Y = 1$

(ii) $\frac{d\phi'_m}{dY} = -v_m$ when $Y = 0$.

The solution of this is

$$\phi'_m(Y) = \frac{v_m}{\left(\frac{m\pi b}{a}\right)\cosh\left(\frac{m\pi b}{a}\right)} \sinh\frac{m\pi b}{a}(1-Y) \qquad m = 1,2 \ldots M$$

$$\phi'_o(Y) = v_o(1-Y).$$

Taking v_m, $m = 0,1,2 \ldots M$ as given by equations 4.6(4)

$$\phi'_m(Y) = \frac{\pi^2 \tau_m(-1)^{R+1}}{4(R^2-1)} \frac{\sinh\frac{m\pi b}{a}(1-Y)}{\sinh\frac{m\pi b}{a}} \qquad 4.6(7)$$

$$\phi'_o(Y) = \frac{\pi^2 \tau_o(-1)^{R+1}}{4(R^2-1)}(1-Y)$$

This solution is superposed on the solution for ϕ_m ($m = 0,1, \ldots M$), modified by the condition at $Y = 1$, and the solution over the whole slot then proceeds as before.

4.6) contd.

point(x,y)	F_c	$\|F-F_c\|$
(0, 0)	6.19884	0.00271
(0, $\frac{b}{2}$)	4.77074	0.00231
(0, b)	6×10^{-6}	6×10^{-6}
($\frac{a}{2}$, 0)	6.19096	0.00451
($\frac{a}{2}$, $\frac{b}{2}$)	4.88042	0.00034
($\frac{a}{2}$, b)	-2×10^{-5}	2×10^{-5}
(a, 0)	5.96819	0.00028
(a, $\frac{b}{2}$)	4.49335	0.00009
(a, b)	-3×10^{-6}	3×10^{-6}

TABLE 4.

Values of $|F-F_c|$ with boundary conditions perturbed

$R = M = 10$

$\alpha_1 = 0.1 \qquad \alpha_2 = 0.8$

$\beta_1 = 0.1 \qquad \beta_2 = 0.7$

$^b/_a = 1.5$

Table 4 shows the error in the calculated solution over a range of points in the slot. Comparing this with Table 3 of Section 4.4 it will be seen that perturbation of the boundary conditions has resulted in some "smoothing" of the error, the net effect producing a reduction in the error over most of the slot. Although comparatively little effort is required to perturb the boundary conditions in this way and so obtain some slight improvement in the overall error, it is felt to be hardly worthwhile since the same effect can be obtained more easily by increasing the number of terms taken in the double series. However, if there is not a great deal of computer storage available then this technique can be useful.

4.7) Comparison with Roth's method.

Both methods give a single solution valid over the whole region of the slot. Each solution is in the form of a double series and for each method the sum is readily evaluated numerically using the methods described in Appendix 1. Also, if the number of conductors in the slot is increased, 4,7) contd.

the difficulties in using either method are only marginally increased; the Fourier-Chebyshev method involving additional terms on the right hand sides of the equations for the coefficients a_{rm},

In Roth's method the Fourier coefficients are independent of each other and are obtained directly. The Fourier-Chebyshev method requires the solution of a sequence of ordinary differential equations to obtain the coefficients which are inter-dependent in this case. However the resulting linear equations for the coefficients are of a simple form whether the direct or integrated method is used and so the inter-dependence of the coefficients is felt to be not too serious a disadvantage.

Table 5 illustrates the rate of convergence of the solution with increasing R and M using the direct method of the Fourier-Chebyshev approximation.

	R	10	10	15	20
	M	10	15	15	15
point(x,y)	F (separation of variables)		F_c		
(0, 0)	6.19613	6.19822	6.19829	6.19610	6.19605
$(0, \frac{b}{2})$	4.77305	4.77176	4.77295	4.77283	4.77321
$(\frac{a}{2}, 0)$	6.18645	6.19379	6.19377	6.18454	6.18454
$(\frac{a}{2}, \frac{b}{2})$	4.88076	4.88142	4.88118	4.87950	4.87957
(a, 0)	5.96791	5.96853	5.96845	5.96796	5.96790
$(a, \frac{b}{2})$	4.49344	4.49428	4.49297	4.49302	4.49350

$\alpha_1 = 0.1$, $\alpha_2 = 0.8$, $\beta_2 = 0.7$, $\frac{b}{a} = 1.5$ TABLE 5.

$\beta_1 = 0.1$.

4.7) contd.

There would seem to be little virtue in taking unequal values of R and M. Comparing this table with Table 1 of section 2.5, it is seen that taking R = M = 15 gives a solution of comparable accuracy to that obtained by taking M = 20 using Roth's method. The number of terms summed in the two cases is 256 using the Fourier-Chebyshev method and 231 using Roth's method. Thus no saving is achieved by using Chebyshev polynomials. In addition, the computing time for a given accuracy is 50% greater using the Fourier-Chebyshev method rather than Roth's method for the mesh X = 0(0.1)1,Y = 0(0.05)1. This is a result of having to solve sets of linear equations for the coefficients in the double sum.

Let us now consider the case where we generalise the boundary conditions as described in section 3.2 but in one direction only (say the y-direction) i.e. the boundary conditions are now

(i) $\dfrac{\partial A_z}{\partial y} = f_1(x)$ when $y = 0;\ 0 \leqslant x \leqslant a$

(ii) $\dfrac{\partial A_z}{\partial x} = 0$ when $x = 0,a;\ 0 \leqslant y \leqslant b.$

(iii) $A_z = f_4(x)$ when $y = b;\ 0 \leqslant x \leqslant a.$

The functions $f_1(x)$ and $f_4(x)$ are expressible as Fourier series as given by equations 3.2(3) and (4). Then these boundary conditions can be directly incorporated into a Fourier-Chebyshev solution of the form given by equation 4.2(6) where now

$$\phi_m(^y/b) = \sum_{r=o}^{R}{}' a_{rm}\, T_r(^y/b) \quad .$$

The equations corresponding to 4.2(8) are

$\phi_m = d_m$ when $y = b$

4.7) contd.

and $\frac{d}{dy}\ \phi_m\left(\frac{y}{b}\right) = a_m$ when $y = 0$.

The solution then proceeds with only slight modifications using the direct Fourier-Chebyshev method. A similar modification can be made if the boundary conditions are generalised in the x-direction only. Using the Fourier-Chebyshev approximation, therefore, when the boundary conditions are generalised in one direction only, we can obtain a single solution over the whole area of the slot with only slight modification to the existing equations. To solve the same problem by Roth's method, we would need to combine with a separation of variables solution and so would not have a single solution valid over the whole region of the slot.

4.8) Double Chebyshev approximation.

Although the Fourier-Chebyshev approximation produced little significant improvement on Roth's method it was felt to be worthwhile to seek for a solution using a Chebyshev approximation in both directions simultaneously. This could well be an improvement on the Fourier-Chebyshev method and, in addition, very little work has been done to date on the solution of partial differential equations of the elliptic type using a double Chebyshev approximation.

We now consider the solution over the whole region $-a \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$, the configuration being as shown in Figure 17 where we are taking the mirror image of the slot in both coordinate axes.

4.8) contd.

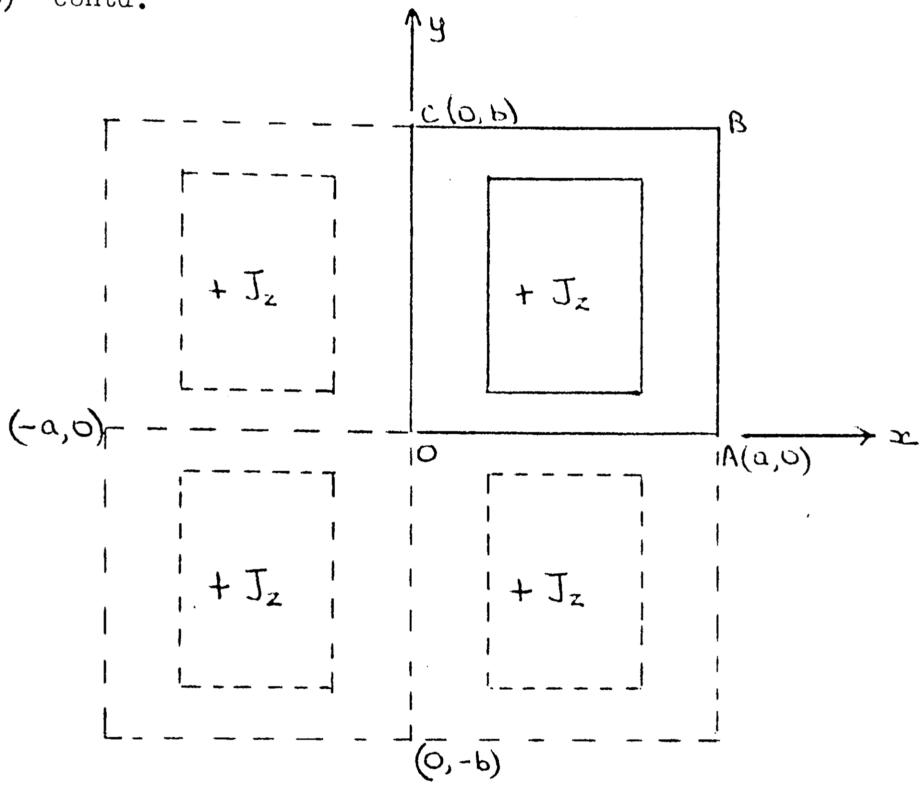


Figure 17.

The boundary conditions in the y-direction are of the Dirichlet type

i.e. (i) $A_z = 0$ for $y = \pm b$, $-a \leqslant x \leqslant a$

while in the x-direction we have Neumann boundary conditions

i.e (ii) $\frac{\partial A_z}{\partial x} = 0$ for $x = \pm a$, $-b \leqslant y \leqslant b$.

The current density profile over the whole region may be expressed as a double Chebyshev series

$$\text{i.e. } f(x,y) = \frac{4\mu_r\mu_o J_z}{\pi^3} \sum_{m=o}^{\infty}{}' \sum_{r=o}^{\infty}{}' \delta(r)\ \eta(m)\ T_{2r}(^y/b)\ T_{2m}\left(\frac{x}{a}\right) \qquad 4.8(1)$$

where $\delta(r)$, $\eta(m)$ are defined by equations 4.2(3) and 4.5(2) respectively. The differential equation to be solved for F(x,y) is therefore

4.8) contd.

$$\left(\frac{b}{a}\right)^2 \frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = -\pi^2 \sum_{m=0}^{\infty}{}' \sum_{r=0}^{\infty}{}' \delta(r)\ \eta(m)\ T_{2r}(Y)\ T_{2m}(X) \qquad 4.8(2)$$

We propose to solve this equation using the integrated method but noting the form of the boundary conditions, for the reasons given in section 4.5, we must integrate the equation only in the Y direction (where the boundary conditions are of the Dirichlet type). Integrating the equation twice with respect to Y gives

$$\left(\frac{b}{a}\right)^2 \iint \left(\frac{\partial^2 F}{\partial X^2}\right) dY\ dY + F = -\pi^2 \sum_{m=0}^{\infty}{}' \eta(m) T_{2m}(X) \left\{ \sum_{r=1}^{\infty} Q(r) T_{2r}(Y) + K_s Y + \frac{L_s}{2} \right\} \qquad 4.8(3)$$

where $Q(r)$ is defined by equation 4.3(11) and K_s, L_s are arbitrary constants of integration ($K_s = 0$ from symmetry).

Assume that

$$\frac{\partial^2 F}{\partial X^2} = \sum_{r=0}^{R}{}' \sum_{m=0}^{M}{}' a_{rm}\ T_{2m}(X)\ T_{2r}(Y) \qquad 4.8(4)$$

Then

$$F = \sum_{r=0}^{R}{}' T_{2r}(Y) \left\{ \sum_{m=0}^{M+1}{}' b_{rm}\ T_{2m}(X) \right\} \qquad 4.8(5)$$

$$\text{where } b_{rm} = \frac{1}{4}\left\{ \frac{a_{r,m-1}}{(2m-1)2m} - \frac{2a_{rm}}{4m^2-1} + \frac{a_{r,m+1}}{2m(2m+1)} \right\}, \qquad 4.8(6)$$

for $m = 1,2 \ldots M+1$

$$a_{r,M+2} = a_{r,M+1} \equiv 0$$

and $b_{r,o}$ is a constant of integration.

Also,

$$\iint \left(\frac{\partial^2 F}{\partial X^2}\right) dY\ dY = \sum_{m=0}^{M}{}' T_{2m}(X) \left\{ \sum_{r=1}^{R+1} C_{rm}\ T_{2r}(Y) \right\} \qquad 4.8(7)$$

$$\text{where } C_{rm} = \frac{1}{4}\left\{ \frac{a_{r-1,m}}{(2r-1)2r} - \frac{2a_{rm}}{4r^2-1} + \frac{a_{r+1,m}}{(2r+1)2r} \right\}, \qquad 4.8(8)$$

for $r = 1,2 \ldots (R+1)$

4.8) contd.

and $a_{R+2,m} = a_{R+1,m} \equiv 0$

(The constant of integration here is taken care of in L_s).

We truncate the right hand side of equation 4.8(3) and introduce τ terms so that the equation reads

$$\left(\frac{b}{a}\right)^2 \iint \left(\frac{\partial^2 F}{\partial X^2}\right) dY\, dY + F = -\pi^2 \sum_{m=o}^{M+1}{}' \eta(m) T_{2m}(X) \left\{ \frac{L_s}{2} + \sum_{r=1}^{R+1} Q(r) T_{2r}(Y) \right\}$$

$$+ \pi^2 \eta(M+1) Q(R+1) T_{2M+2}(X) T_{2R+2}(Y)$$

$$+ T_{2R+2}(Y) \sum_{m=o}^{M}{}' \tau_m T_{2m}(X) + T_{2M+2}(X) \sum_{r=1}^{R} \tau_r' T_{2r}(Y) \quad 4.8(9)$$

Substituting for F in this equation and equating coefficients of

$T_{2m}(X)\ T_{2r}(Y)$ for $\begin{cases} m = 0,1,2 \ \dots\ M+1 \\ r = 1,2 \ \dots\ R+1 \end{cases}$ gives

$$\left(\frac{b}{a}\right)^2 c_{rm} + b_{rm} = -\pi^2 \eta(m) Q(r) \quad \text{for } \begin{cases} m = 1,2 \ \dots\ M \\ r = 1,2 \ \dots\ R \end{cases} \quad 4.8(10)$$

$$\left(\frac{b}{a}\right)^2 \tfrac{1}{2} c_{r,o} + \tfrac{1}{2} b_{ro} = -\pi^2 \tfrac{1}{2} \eta(0) Q(r) \quad \text{for } r = 1,2 \ \dots\ R \quad 4.8(11)$$

$$b_{r,M+1} = -\pi^2 \eta(M+1) Q(r) + \tau_r' \quad \text{for } r = 1,2 \ \dots\ R \quad 4.8(12)$$

$$\left(\frac{b}{a}\right)^2 c_{R+1,m} = -\pi^2 \eta(m) Q(R+1) + \tau_m \quad \text{for } m = 0,1, \ \dots\ M \quad 4.8(13)$$

These are the total number of equations arising from the differential equation . Further equations for the coefficients a_{rm} arise from the boundary conditions.

From boundary condition (i)

$$\sum_{r=o}^{R}{}' b_{rm} = 0 \text{ for } m = 0,1, \dots (M+1) \quad 4.8(14)$$

Now $$\frac{\partial F}{\partial X} = \sum_{r=o}^{R}{}' T_{2r}(Y) \left\{ \sum_{m=1}^{M} \frac{a_{rm}}{2} \left\{ \frac{T_{2m+1}(X)}{2m+1} - \frac{T_{2m-1}(X)}{2m-1} \right\} + \tfrac{1}{2} a_{ro} T_1(X) \right\}$$

Hence boundary condition (ii) gives

4.8) contd.

$$\tfrac{1}{2}\, a_{ro} = \sum_{m=1}^{M} \frac{a_{rm}}{4m^2 - 1} \quad , \; r = 0,1, \ldots R \qquad 4.8(15)$$

Equations 4.8(12) and (13) are used to give the τ-terms and equations 4.8(11) and the first of (14) give the constants of integration $b_{r,o}$ $r = 0,1, \ldots R$. These equations will therefore be ignored at this stage. The remaining equations number $\{(R+1)(M+1) + 1\}$ while the number of unknowns a_{rm} is $(R+1)(M+1)$. Thus we would appear to have one surplus equation but in Appendix 3 it will be shown that one of the equations deduced from the boundary conditions (4.8(14) and (15)) is linearly dependent on the remainder. It will also be shown that this is still the case when the boundary conditions are of the Neumann or Dirichlet type all round the region or when they are mixed as in this problem.

4.9) Method of solution.

We have to solve equations 4.8(10), (14) and (15) (excluding one boundary equation) for the coefficients

$$a_{rm}, \begin{cases} r = 0,1, \ldots R \\ m = 0,1 \ldots M \end{cases}$$

Equation 4.8(10) may be written

$$a_{r-1,m} + \frac{(2r-1)2r}{(^b/a)^2}\left\{\frac{a_{r,m-1}}{(2m-1)(2m)} - 2a_{rm}\left(\frac{1}{4m^2-1} + \frac{(^b/a)^2}{4r^2-1}\right) + \frac{a_{r,m+1}}{(2m+1)2m}\right\}$$

$$+ \frac{(2r-1)}{(2r+1)}\, a_{r+1,m} = \frac{(2r-1)2r}{(^b/a)^2}\left(-4\pi^2\, \eta(m)Q(r)\right) \quad \begin{matrix} m = 1,2 \ldots M \\ r = 1,2 \ldots R \end{matrix}$$

where $a_{r,M+1} = a_{R+1,m} \equiv 0$

When $m = 1$, this equation involves a_{ro} but we may substitute for a_{ro} from equation 4.8(15) so that

4.9) contd.

$$a_{r-1,1} + \frac{(2r-1)2r}{(^b/a)^2}\left\{ \sum_{m=1}^{M} \frac{a_{rm}}{4m^2-1} - 2a_{r1}\left(\frac{1}{1.3} + \frac{(\frac{b}{a})^2}{4r^2-1}\right) + \frac{a_{r2}}{2.3}\right\} + \frac{(2r-1)}{(2r+1)} a_{r+1,1}$$

$$= \frac{(2r-1)2r}{(^b/a)^2}\left(-\,4\pi^2 \eta(1) Q(r)\right) \quad r = 1,2 \;\ldots\; R$$

Define vectors

$$\underline{\Phi}_r = \begin{bmatrix} a_{r1} \\ a_{r2} \\ a_{r3} \\ \vdots \\ a_{rM} \end{bmatrix} \quad \text{for } r = 0,1, \;\ldots\; R. \qquad 4.9(1)$$

Then the above equations may be written in matrix form as

$$\underline{\Phi}_{r-1} + \lambda(r)\, \underline{B}_r\, \underline{\Phi}_r + \mu(r)\, \underline{\Phi}_{r+1} = -\,4\pi^2\, \lambda(r)\, Q(r)\, \underline{T} \qquad 4.9(2)$$

for $r = 1,2, \;\ldots\ldots\; R$

where $\underline{\Phi}_{R+1} \equiv \underline{0}$

$$\lambda(r) = \frac{(2r-1)2r}{(^b/a)^2}$$

$$\mu(r) = \frac{(2r-1)}{(2r+1)}$$

$$\underline{T} = \begin{bmatrix} \eta(1) \\ \eta(2) \\ \vdots \\ \eta(M) \end{bmatrix}$$

4.9) contd.

and $$\underline{B}_r = \begin{bmatrix} K_r - \frac{2}{1.3} + \frac{1}{1.3}, & \frac{1}{2.3} + \frac{1}{3.5}, & \frac{1}{5.7}, & \frac{1}{7.9}, & \cdots\cdots, & \frac{1}{(2M-1)(2M+1)} \\ \frac{1}{3.4}, & K_r - \frac{2}{3.5}, & \frac{1}{4.5}, & 0, & \cdots\cdots, & 0 \\ 0, & \frac{1}{5.6}, & K_r - \frac{2}{5.7}, & \frac{1}{6.7}, & \cdots\cdots, & 0 \\ \vdots & 0, & \frac{1}{7.8}, & & & \\ \vdots & \vdots & 0, & & & \\ \vdots & \vdots & \vdots & & , & \frac{1}{(2M-1)(2M-2)} \\ 0, & 0, & 0, & \cdots\cdots \frac{1}{(2M-1)(2M)}, & & K_r - \frac{2}{4M^2-1} \end{bmatrix}$$

where $K_r = -2(^b/a)^2/(4r^2-1)$.

A further equation is obtained from the boundary equation 4.8(14). In terms of the coefficients a_{rm} this equation reads

$$\sum_{r=o}^{R}{}' \left\{ \frac{a_{r,m-1}}{(2m-1)2m} - \frac{2a_{rm}}{4m^2-1} + \frac{a_{r,m+1}}{2m(2m+1)} \right\} = 0, \quad m = 1,2 \ldots (M+1)$$

(the first of (14) is neglected at this stage since it involves the constants b_{ro}). Omitting the equation for m = 1 (one equation is linearly dependent on the others) these equations may be written in matrix form as

$$\underline{C}\ (\tfrac{1}{2}\underline{\Phi}_0 + \underline{\Phi}_1 + \ldots + \underline{\Phi}_R) = \underline{0}$$

4.9) contd.

where $\underline{C} = \begin{bmatrix} \frac{1}{3.4}, & \frac{-2}{5.3}, & \frac{1}{5.4}, & 0, & \cdots & & 0 \\ 0, & \frac{1}{5.6}, & \frac{-2}{7.5}, & \frac{1}{7.6}, & 0, & \cdots & 0 \\ \vdots & 0, & \frac{1}{7.8}, & \frac{-2}{9.7}, & \frac{1}{9.8}, & \cdots & 0 \\ \vdots & \vdots & 0 & & & & \vdots \\ \vdots & \vdots & \vdots & & & \frac{1}{(2M-1)2M}, & \frac{-2}{(2M+1)(2M-1)} \\ 0, & 0, & 0, & \cdots & & 0, & \frac{1}{(2M+1)(2M+2)} \end{bmatrix}$

Now $|\underline{C}| \neq 0$ so that $\underline{C}^{-1}$ exists.

Hence $$\sum_{r=0}^{R}{}' \underline{\Phi}_r = \underline{0} \qquad 4.9(3)$$

A recurrence algorithm to solve equations 4.9(2) and (3) is given in Appendix 2. Having obtained the vectors $\underline{\Phi}_r$, equations 4.8(15) give a_{ro} and 4.8(11) and the first of (14) give b_{ro}, r = 0,1,2 ... R. The solution for F is then calculated using equation 4.8(5), the double Chebyshev sum being evaluated using the methods of Appendix 1.

4.10) Error estimation.

Defining E_τ as in section 4.4, then E_τ satisfies the differential equation

$$\left(\frac{b}{a}\right)^2 \frac{\partial^2 E_\tau}{\partial X^2} + \frac{\partial^2 E_\tau}{\partial Y^2} = -\frac{\partial^2}{\partial Y^2}\left\{T_{2R+2}(Y)\sum_{m=0}^{M}{}' \tau_m T_{2m}(X) + T_{2M+2}(X)\sum_{r=1}^{R}{}' \tau_r T_{2r}(Y)\right\} \qquad 4.10(1)$$

where

(i) $E_\tau = 0$ for $Y = \pm 1$, $-1 \leqslant X \leqslant 1$

(ii) $\frac{\partial E_\tau}{\partial X} = 0$ for $X = \pm 1$, $-1 \leqslant Y \leqslant 1$

Considering the case R = M = 10, τ_m, m = 1,2 ... 10,

4.10) contd.

and τ_r', r = 2,3 ... 10 are negligible by comparison with τ_0, τ_1'. Hence equation 4.10(1) may be approximately written as

$$\left(\frac{b}{a}\right)^2 \frac{\partial^2 E_\tau}{\partial X^2} + \frac{\partial^2 E_\tau}{\partial Y^2} = - \frac{\partial^2}{\partial Y^2}\left\{\tfrac{1}{2}\,\tau_0\, T_{2R+2}(Y) + \tau_1'\, T_2(Y)T_{2M+2}(X)\right\} \quad 4.10(2)$$

Now $\frac{d^2}{dY^2}\{T_2(Y)\} = 4$ and assume that E_τ is given in the form

$E_\tau = \Theta(X) + \Phi(Y)$ where Θ, Φ are functions of X only, Y only, respectively. Hence equation 4.10(2) may be written

$$\left(\frac{b}{a}\right)^2 \frac{d^2\Theta}{dX^2} + 4\,\tau_1' T_{2M+2}(X) = -\frac{d^2\Phi}{dY^2} - \tfrac{1}{2}\,\tau_0 \frac{d^2}{dY^2}\, T_{2R+2}(Y)$$

$$= \lambda \text{ (say)} \quad 4.10(3)$$

where λ is some constant.

$$\text{Hence } \left(\frac{b}{a}\right)^2 \Theta = -\tau_1'\left\{\frac{T_{2M+4}(X)}{(2M+3)(2M+4)} - \frac{2T_{2M+2}(X)}{(2M+3)(2M+1)} + \frac{T_{2M}(X)}{(2M+1)2M}\right\}$$

$$+ \frac{\lambda X^2}{2} + P_\theta X + Q_\theta \quad 4.10(4)$$

(P_θ, Q_θ arbitrary constants)

$$\text{and } \Phi = -\tfrac{1}{2}\,\tau_0\, T_{2R+2}(Y) - \frac{\lambda Y^2}{2} + P_\phi Y + Q_\phi$$

(P_ϕ, Q_ϕ arbitrary constants)

Boundary condition (ii) gives

$$P_\theta = 0$$

$$\lambda = -\frac{4\tau_1'}{(2M+3)(2M+1)}$$

$$\text{Hence } E_\tau = -\frac{\tau_1'}{(b/a)^2}\left\{\frac{T_{2M+4}(X)}{(2M+3)(2M+4)} - \frac{2T_{2M+2}(X)}{(2M+3)(2M+1)} + \frac{T_{2M}(X)}{2M(2M+1)}\right\}$$

$$- \tfrac{1}{2}\,\tau_0\, T_{2R+2}(Y) + \frac{2\tau_1'}{(2M+3)(2M+1)}\left\{Y^2 - \frac{X^2}{(b/a)^2}\right\} + P_\phi Y + Q \quad 4.10(5)$$

$$\text{where } Q = \frac{Q_\theta}{(b/a)^2} + Q_\phi$$

4.10) contd.

Now the terms involving τ_1' are at most $0\left(\frac{|\tau_1'|}{M^2}\right)$ i.e. 5×10^{-4} while $|\tau_0| = 4 \times 10^{-3}$ (taking R = M = 10).

Hence $E_\tau \simeq \frac{1}{2}\tau_0(1 - T_{2R+2}(Y))$ 4.10(6)

applying boundary condition (i).

$$\therefore \quad |E_\tau| = 0(|\tau_0|) = 0(4 \times 10^{-3}) \qquad 4.10(7)$$

Table 6 shows the caloulated values of $|F-F_c|$ over a range of points in the slot for R = M = 10. From the the table it can be seen that the error estimate derived above gives a good indication of the absolute magnitude of the error.

TABLE 6.

(x,y)	F	F_c	$\|F-F_c\|$
(0,0)	6.19613	6.19888	0.00275
$(0,\frac{b}{2})$	4.77305	4.77542	0.00237
$(\frac{a}{2},0)$	6.18645	6.19144	0.00499
$(\frac{a}{2},\frac{b}{2})$	4.88076	4.88583	0.00507
(a,0)	5.96791	5.96948	0.00157
$(a,\frac{b}{2})$	4.49344	4.49471	0.00127

$\alpha_1 = 0.1$ $\alpha_2 = 0.8$, $\beta_1 = 0.1$, $\beta_2 = 0.7$

$^b/a = 1.5$, R = M = 10.

4.11) Comparison of the double Chebyshev method with the Fourier-Chebyshev and Roth's method.

Again we have a single solution valid over the whole region of the slot, the double sum being readily evaluated numerically. Negligible increase of difficulty follows if the number of conductors in the slot is increased. However, to derive and calculate the coefficients a_{rm} requires considerable mathematical and computational effort by comparison with the

4.11) contd.

other methods.

Table 7 shows the rate of convergence of the solution with increasing R and M

	R=M=	8	10	11
point	F	F_c	F_c	F_c
(0,0)	6.19613	6.20287	6.19888	6.19551
$(0,\frac{b}{2})$	4.77305	4.78131	4.77542	4.77290
$(\frac{a}{2},0)$	6.18645	6.19789	6.19144	6.18517
$(\frac{a}{2},\frac{b}{2})$	4.88076	4.87885	4.88583	4.88147
(a,0)	5.96791	5.97205	5.96948	5.96689
$(a,\frac{b}{2})$	4.49344	4.49603	4.49471	4.49236

TABLE 7

Comparing this with Table 1 of section 2.5, taking R = M = 11 gives comparable accuracy to that obtained by taking M = 20 in Roth's method. Thus some saving in the number of terms taken in the double sum has been achieved by using the double Chebyshev approximation. However the time taken to compute the solution using this method is four times that taken by Roth's method for the mesh of points $\frac{x}{a} = 0(0.1)1$, $\frac{y}{b} = 0(0.05)1$.

The double Chebyshev method does however have the advantage that it can cope directly with more generalised forms of boundary condition. For example suppose that we are considering the boundary conditions as given in section 3.2 where we will assume that the functions f_1, f_2, f_3 and f_4 are all polynomials. These polynomials can then be expressed as finite Chebyshev series

4.11) contd.

$$\text{e.g. } f_4(x) = \sum_{m=o}^{M}{}' d_m T_m\left(\frac{x}{a}\right) \qquad \text{(say)}.$$

$$\text{If } A_z = \sum_{r=o}^{R}{}' \sum_{m=o}^{M}{}' b_{rm} T_r(Y) T_m(X)$$

the boundary equations corresponding to this condition would read

$$\sum_{r=o}^{R}{}' b_{rm} = d_m \text{ for } m = 0,1, \ldots M$$

with appropriate modifications for the other boundary conditions. Before anything more can be said about this, it is necessary for further investigations to take place, as various questions now arise. For example, by how much would the equations for determination of the $\underline{\Phi}_r$ be affected and could the algorithm for their solution given in Appendix 2 still be utilised? Also, how would the linear dependence of the boundary equations be affected? However it would appear that the double Chebyshev method can be made to cope with more general boundary conditions and I propose to devote further research effort to this point.

4.12) <u>Conclusions.</u>

The use of Chebyshev polynomials rather than circular functions cannot be recommended for the type of problem under consideration where the current density function is discontinuous within the slot. However the experience gained in applying Chebyshev methods to an elliptic partial differential equation is felt to be very

4.12) contd.

worthwhile. Although not suitable for slot problems the question arises as to whether there exist problems where Chebyshev approximations would be the most suitable. One obvious possibility is the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

where $f(x,y)$ is a finite polynomial in x and y. Examples of this and other types of problem will be considered in Chapter 8. Also there are further avenues of investigation associated with the problems arising from more general forms of boundary condition as indicated in section 4.11. We have derived here a very powerful method of solution which will be shown in Chapter 8 to be particularly suitable for certain types of problem.

C H A P T E R 5

RECTANGULAR TRANSFORMER WINDOW.

5.1) Introduction.

This chapter aims to show how Roth's method and the Chebyshev methods can be adapted to solve the Neumann problem arising from rectangular windings in a transformer window.

The configuration to be considered is described in section 1.10. We shall solve the case of a transformer window with two balanced rectangular windings, i.e. with axial currents I_1, I_2 such that $I_1 + I_2 = 0$. Thus the geometrical configuration is as shown in Figure 18.

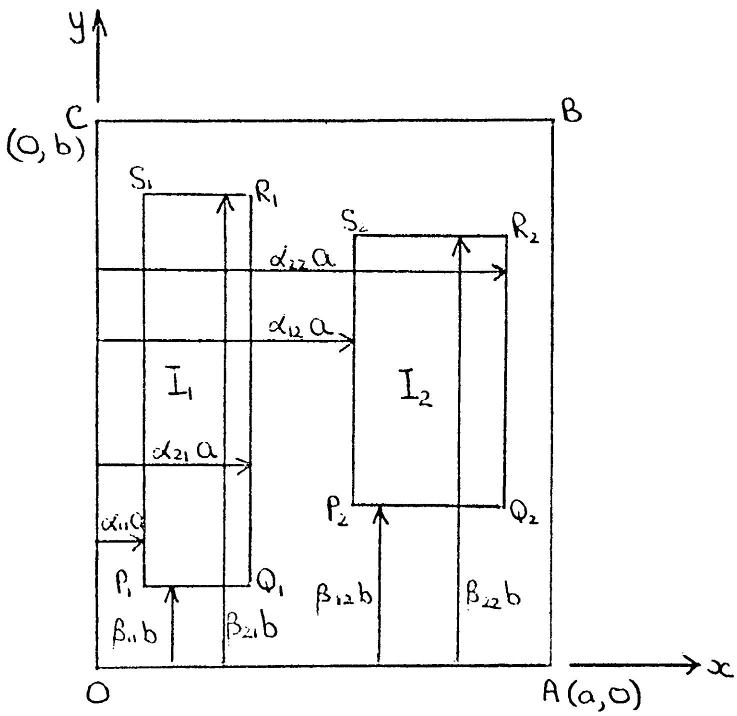


Figure 18.

J_{z1}, J_{z2} are the current densities in windings 1 and 2, respectively, OABC is the cross section of the window withthe windings $P_1Q_1R_1S_1$ and $P_2Q_2R_2S_2$. The differential equation to be satisfied throughout OABC is

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = \begin{cases} -\mu_r\mu_o J_{z1} & \text{in winding 1} \\ -\mu_r\mu_o J_{z2} & \text{in winding 2} \\ 0 & \text{elsewhere} \end{cases}$$

$$= -f(x,y) \quad \text{(say)} \qquad 5.1(1)$$

Referring to sections 1.9, 1.10, the boundary

5.1) contd.

conditions are

(i) $\dfrac{\partial A_z}{\partial x} = 0$ when $x = 0, a$; for $0 \leq y \leq b$

(ii) $\dfrac{\partial A_z}{\partial y} = 0$ when $y = 0, b$; for $0 \leq x \leq a$.

In this chapter we shall show how Roth's methods and the Chebyshev methods can be modified to solve this type of problem. No loss of generality occurs by considering only two windings; the methods all extend when there are several windings in the region OABC provided of course that the total net axial current within OABC is zero. Solution by the method of separation of variables is not considered as it is unsuitable for the reasons given in Chapter 2.

5.2) Roth's method.

To satisfy the boundary conditions the solution must be of the form

$$A_z = \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty}{}' C_{mk} \cos\frac{m\pi x}{a} \cos\frac{k\pi y}{b} \qquad 5.2(1)$$

as described in section 2.2. The function f(x,y) is expanded as a double Fourier series of the same form i.e.

$$f(x,y) = \frac{4}{\pi^2} \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty}{}' \left\{\mu_r\mu_o J_{z1}\alpha_1(m)\beta_1(k) + \mu_r\mu_o J_{z2}\alpha_2(m)\beta_2(k)\right\} \cos\frac{m\pi x}{a}\cos\frac{k\pi y}{b} \qquad 5.2(2)$$

where $\alpha_1(m)$, $\alpha_2(m)$ are given by equations 2.2(3) and

$$\left.\begin{aligned} \beta_i(k) &= \frac{\sin(\beta_{2i}k\pi) - \sin(\beta_{1i}k\pi)}{k}, \quad k \neq 0 \\ \beta_i(0) &= (\beta_{2i} - \beta_{1i})\pi \end{aligned}\right\} \qquad 5.2(3)$$

5.2) contd.

Substituting equations 5.2(1) and (2) into the differential equation and equating coefficients of $\cos\frac{m\pi x}{a} \quad \cos\frac{k\pi y}{b}$ gives

$$C_{mk}\left\{\frac{m^2}{a^2} + \frac{k^2}{b^2}\right\}\pi^2 = \frac{4}{\pi^2}\left\{J_{z1}\ \alpha_1(m)\beta_1(k) + J_{z2}\alpha_2(m)\beta_2(k)\right\}\mu_r\mu_o$$

$$m = 0,1,2\ ..\ ,\ k = 0,1,2\ ... \qquad 5.2(4)$$

Note that C_{oo} is undefined and equation 5.2(4) becomes in this case

$$\frac{4\mu_r\mu_o}{\pi^2}\left\{J_{z1}(\alpha_{21}-\alpha_{11})(\beta_{21}-\beta_{11}) + J_{z2}(\alpha_{22}-\alpha_{12})(\beta_{22}-\beta_{12})\right\} = 0 \qquad 5.2(5)$$

again expressing the fact that the total current within the region OABC must be zero. Without loss of generality we may take

$$C_{oo} = 0 \qquad 5.2(6)$$

A different value of C_{oo} merely gives a different level of potential within the region OABC. Note that with this value of C_{oo},

$$\int_{x=o}^{a}\int_{y=o}^{b} A_z\ dx\ dy = 0 \qquad 5.2(7)$$

Thus the solution is given by equation 5.2(1) where the coefficients C_{mk} are defined by equations 5.2(4) and (6) and the current densities J_{z1} and J_{z2} are related by equation 5.2(5). The double sum is evaluated by diagonals as described in section 2.3 using the numerical algorithm given in Appendix 1. Due to the symmetrical nature of the solution, values of A_z are calculated over the mesh $\frac{x}{a} = 0(0.05)1$, $\frac{y}{b} = 0(0.05)1$.

5.3) Fourier-Chebyshev approximation.

Due to the symmetry of the boundary conditions, it is felt to be immaterial in which direction we take the Chebyshev variation. Since the boundary conditions are of the Neumann type, the integrated method of solution as described in section 4.3 is unsuitable for the reasons given in section 4.5 and the direct method or modified direct method must be used as follows. Assuming Chebyshev variation in the y-direction and following the treatment given in section 4.2 we may expand the current density profile over the region $0 \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$, as

$$f(x,y) = \frac{4\mu_r\mu_o}{\pi^2} \sum_{m=o}^{\infty}{}' \sum_{r=o}^{\infty}{}' \left\{ \alpha_1(m)\,\delta_1(r)J_{z1} + \alpha_2(m)\,\delta_2(r)J_{z2} \right\} \cos\frac{m\pi x}{a}\, T_{2r}\left(\frac{y}{b}\right) \qquad 5.3(1)$$

where $\alpha_i(m)$, $i = 1,2, \ldots$ is given by equations 2.2(3)

$$\left.\begin{aligned} \text{and}\quad \delta_i(r) &= \frac{1}{r}(\sin 2r\xi_{1i} - \sin 2r\xi_{2i}) \quad r = 1,2 \ldots \\ \delta_i(0) &= 2(\xi_{1i} - \xi_{2i}) \end{aligned}\right\} \qquad 5.3(2)$$

$$\text{where } \xi_{1i} = \cos^{-1}\beta_{1i}\,,\quad \xi_{2i} = \cos^{-1}\beta_{2i} \qquad 5.3(3)$$

Defining $F = A_z \Big/ \dfrac{4\mu_r\mu_o J_{z1} b^2}{\pi^4}$,

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -\frac{\pi^2}{b^2} \sum_{m=o}^{\infty}{}' \sum_{r=o}^{\infty}{}' \left\{ \alpha_1(m)\,\delta_1(r) + J\ \alpha_2(m)\,\delta_2(r) \right\} \cos\frac{m\pi x}{a}\, T_{2r}\left(\frac{y}{b}\right) \Big\}$$

$$\begin{aligned} \text{where } J &= J_{z2}\big/J_{z1} \\ &= -\frac{(\alpha_{21}-\alpha_{11})(\beta_{21}-\beta_{11})}{(\alpha_{22}-\alpha_{12})(\beta_{22}-\beta_{12})} \end{aligned} \qquad 5.3(4)$$

since the total current within the slot must be zero.

Defining $F(x,y)$ in the form given by equation 4.2(6) we obtain

5.3) contd.

$$-\left(\frac{m\pi b}{a}\right)^2 \phi_m(Y) + \frac{d^2\phi_m}{dY^2} = -\pi^2 \sum_{r=0}^{\infty}{}' \left\{ \alpha_1(m)\,\delta_1(r) + J\ \alpha_2(m)\,\delta_2(r) \right\} T_{2r}(Y)$$

for $m = 0,1,2 \ldots$ 5.3(5)

where $Y = {}^{y}/b$.

From the boundary conditions at $y = \pm b$

$$\frac{d\phi_m}{dY} = 0 \text{ when } Y = \pm 1 \qquad 5.3(6)$$

Assuming that ϕ_m and $\frac{d^2\phi_m}{dY^2}$ are given by equations 4.3(2) and (3) respectively, the equations deducible from the differential equation are

$$-\left(\frac{m\pi b}{a}\right)^2 a_r + b_r = -\pi^2\left\{ \alpha_1(m)\ \delta_1(r) + J\ \alpha_2(m)\,\delta_2(r) \right\} \qquad 5.3(7)$$

$r = 0,1,2 \ldots (R-1)$

$$-\left(\frac{m\pi b}{a}\right)^2 a_R = -\pi^2\left\{ \alpha_1(m)\,\delta_1(R) + J\ \alpha_2(m)\ \delta_2(R) \right\} - \pi^2 \tau_m \qquad 5.3(8)$$

where the coefficients a_r, b_r are related by equations 4.3(4). The boundary condition 5.3(6) gives, as described in section 4.5,

$$\tfrac{1}{2}\, b_0 - \sum_{r=1}^{R-1} \frac{b_r}{(4r^2-1)} = 0 \qquad 5.3(9)$$

Equations 5.3(7) and (9) are solved for the coefficients a_o, b_r, $r = 0,1, \ldots (R-1)$, for $m = 1,2 \ldots M$.

Considering the case when $m = 0$, a_o is undefined and we would appear to have one surplus equation. If the coefficients b_r are given by equation 5.3(7) then

5.3) contd.

$$\tfrac{1}{2} b_0 - \sum_{r=1}^{R-1} \frac{b_r}{(4r^2-1)} = -\frac{\pi^2}{2}\left\{ \alpha_1(0)\ \delta_1(0) + J\ \alpha_2(0)\ \delta_2(0) \right\}$$

$$+ \pi^2 \sum_{r=1}^{R-1} \frac{1}{(4r^2-1)} \left\{ \alpha_1(0)\ \delta_1(r) + J\,\alpha_2(0)\,\delta_2(r) \right\}$$

Using the known Fourier series

$$x + \frac{\pi}{2}\cos x - \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{\sin 2nx}{n(4n^2-1)} \quad , \quad 0 \leqslant x \leqslant \pi,$$

$$-\frac{\delta(0)}{2} + \sum_{r=1}^{\infty} \frac{\delta(r)}{(4r^2-1)} = \frac{\pi}{2}(\beta_1-\beta_2)$$

Hence, to within the truncation error,

$$\tfrac{1}{2}b_0 - \sum_{r=1}^{R-1} \frac{b_r}{(4r^2-1)} = \frac{\pi^4}{2}\left\{ (\beta_{11}-\beta_{21})(\alpha_{21}-\alpha_{11}) + \frac{J_{z2}}{J_{z1}} (\beta_{12}-\beta_{22})(\alpha_{22}-\alpha_{12}) \right\}$$

$$= 0 \quad \text{if the total current is zero.}$$

Hence when $m = 0$, the coefficients b_r, $r = 0,1,\ (R-1)$, are given by equation 5.3(7) and the boundary equation 5.3(9) is automatically satisfied (to within the truncation error). In order to compare the solution with that obtained by Roth's method the constant a_0 $(m = 0)$ is chosen to satisfy the condition 5.2(7).

For $m = 1,2,3 \ldots M$ define

$$c_r = \frac{-\,b_r}{4r^2-1} \quad , \quad r = 0,1, \ldots (R-1)$$

and then equations 5.3(7) (excluding the first which gives a_0) may be written in the form

$$c_{r-1} + \lambda(r)\ c_r + \mu(r)\ c_{r+1} = k\,P(r), \quad r = 1,2 \ldots (R-1) \qquad 5.3(10)$$

where $c_R \equiv 0$

The boundary equation 5.3(9) becomes

5.3) contd.

$$\sum_{r=o}^{R-1}{}' c_r = 0 \qquad 5.3(11)$$

These equations are then solved using the methods of Appendix 2 and the solution proceeds as before. Alternatively we may use the direct method and solve directly for the coefficients a_r, for each m.

5.4) Double Chebyshev approximation.

Following the treatment of section 4.8,

$$f(x,y) = \frac{4\mu_r \mu_o J_{z1}}{\pi^2} \sum_{m=o}^{\infty}{}' \sum_{r=o}^{\infty}{}' \left\{ \delta_1(r)\ \eta_1(m) + J\ \delta_2(r)\eta_2(m) \right\} T_{2r}\left(\frac{y}{b}\right) T_{2m}\left(\frac{x}{a}\right) \qquad 5.4(1)$$

where $\delta_i(r)$ is given by equations 5.3(2) and

$$\left.\begin{aligned} \eta_i(m) &= \frac{1}{m}(\sin 2m\ \zeta_{1i} - \sin 2m\ \zeta_{2i}),\ m = 1,2 \ldots \\ \eta_i(0) &= 2(\zeta_{1i} - \zeta_{2i}) \end{aligned}\right\} \qquad 5.4(2)$$

$$\text{where } \zeta_{1i} = \cos^{-1} \alpha_{1i},\ \zeta_{2i} = \cos^{-1} \alpha_{2i} \qquad 5.4(3)$$

The differential equation to be solved for F(x,y) is therefore

$$\left(\frac{b}{a}\right)^2 \frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = -\pi^2 \sum_{m=o}^{\infty}{}' \sum_{r=o}^{\infty}{}' T_{2r}(Y)\ T_{2m}(X) \left\{ \delta_1(r)\eta_1(m) + J\ \delta_2(r)\eta_2(m) \right\} \qquad 5.4(4)$$

where $X = \frac{x}{a}$, $Y = {}^y/b$ and J is given by equation 5.3(4). The conditions at the boundary are

(i) $\frac{\partial F}{\partial X} = 0$ when $X = \pm 1$, for $-1 \leqslant Y \leqslant 1$

and (ii) $\frac{\partial F}{\partial Y} = 0$ when $Y = \pm 1$ for $-1 \leqslant X \leqslant 1$

Thus the boundary conditions are of the Neumann type in both directions and so no integration of the differential

5.4) contd.

equation is possible for the reasons given in section 4.5. We propose to use a version of the modified direct method as follows. Assume that

$$\frac{\partial^4 F}{\partial X^2 \partial Y^2} = \sum_{r=o}^{R}{}' \sum_{m=o}^{M}{}' a_{rm} T_{2m}(X) T_{2r}(Y) \qquad 5.4(5)$$

Briefly the method integrates this equation to obtain $\frac{\partial^2 F}{\partial X^2}$ and $\frac{\partial^2 F}{\partial Y^2}$ which are then substituted in equation 5.4(4). Equations for the coefficients a_{rm} are then obtained by comparing coefficients of $T_{2r}(Y)\ T_{2m}(X)$. Clearly this form of approximation could have been used for the slot problem of section 4.8. However to obtain F from equation 5.4(5) requires four integrations with an arbitrary function introduced at each stage. The approximation for $\frac{\partial^2 F}{\partial X^2}$ given in section 4.8 requires only two integrations to obtain the solution F and so the method of solution is less involved as will be seen from the treatment about to be given. As a general rule, we integrate the differential equation whenever the boundary conditions are of the Dirichlet type in a given direction. For a problem where the function is specified all round the boundary, integrate with respect to both X and Y and then approximate to F directly with a double Chebyshev series. Unfortunately with the Neumann problem defined here we have no choice but to use the approximation 5.4(5). Since we have the derivative specified round the boundary, some of the arbitrary functions can be dealt with during the integration process.

5.5) Method of solution.

Integrating equation 5.4(5)

$$\frac{\partial^2 F}{\partial Y^2} = \sum_{r=0}^{R}{}' \sum_{m=0}^{M+1}{}' b_{rm} T_{2m}(X) T_{2r}(Y)$$

where $$b_{rm} = \frac{1}{4}\left\{\frac{a_{r,m-1}}{(2m-1)2m} - \frac{2a_{rm}}{4m^2-1} + \frac{a_{r,m+1}}{2m(2m+1)}\right\} \qquad 5.5(1)$$

for $r = 0,1 \ldots R$, $m = 1,2, \ldots M+1$.

and $a_{r,M+2} = a_{r,M+1} \equiv 0$.

b_{ro}, $r = 0,1\ldots R$ are constants of integration.

Note that the other constants of integration are zero from the evenness of the solution.

Similarly $$\frac{\partial^2 F}{\partial X^2} = \sum_{m=0}^{M}{}' \sum_{r=0}^{R+1}{}' c_{rm} T_{2m}(X) T_{2r}(Y)$$

where $$c_{rm} = \frac{1}{4}\left\{\frac{a_{r-1,m}}{(2r-1)2r} - \frac{2a_{rm}}{4r^2-1} + \frac{a_{r+1,m}}{2r(2r+1)}\right\}$$

for $r = 1,2 \ldots (R+1)$, $m = 0,1 \ldots M$

and $a_{R+2,m} = a_{R+1,m} \equiv 0$.

c_{om}, $m = 0,1 \ldots M$ are constants of integration.

Truncating the right hand side of equation 5.4(4) and introducing τ terms gives

$$\left(\frac{b}{a}\right)^2 \frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = -\pi^2 \sum_{m=0}^{M+1}{}' \sum_{r=0}^{R+1}{}' T_{2r}(Y) T_{2m}(X)\left\{\delta_1(r)\eta_1(m) + J\,\delta_2(r)\eta_2(m)\right\}$$

$$+ \pi^2 T_{2M+2}(X)\, T_{2R+2}(Y)\left\{\delta_1(R+1)\eta_1(M+1) + J\,\delta_2(R+1)\eta_2(M+1)\right\}$$

$$+ T_{2R+2}(Y)\sum_{m=0}^{M}{}' \tau_m T_{2m}(X) + T_{2M+2}(X)\sum_{r=0}^{R}{}'\, \tau_r' T_{2r}(Y) \qquad 5.5(2)$$

Equating coefficients of $T_{2r}(Y)\, T_{2m}(X)$, $r = 0,1 \ldots (R+1)$, $m = 0,1 \ldots (M+1)$

5.5) contd.

gives

$$\left(\frac{b}{a}\right)^2 c_{rm} + b_{rm} = -\pi^2\left\{\delta_1(r)\ \eta_1(m) + J\ \delta_2(r)\ \eta_2(m)\right\}$$

for $r = 0,1 \ldots R;\ m = 0,1 \ldots M$ 5.5(3)

$$b_{r,M+1} = -\pi^2\left\{\delta_1(r)\ \eta_1(M+1) + J\ \delta_2(r)\ \eta_2(M+1)\right\} + \tau_r' \qquad 5.5(4)$$

for $r = 0,1, \ldots R$

$$\left(\frac{b}{a}\right)^2 c_{R+1,m} = -\pi^2\left\{\delta_1(R+1)\eta_1(m) + J\ \delta_2(R+1)\eta_2(m)\right\} + \tau_m \qquad 5.5(5)$$

for $m = 0,1 \ldots M$.

These are the total number of equations derivable from the differential equation.

We now deduce the equations derivable from the boundary conditions.

$$\frac{\partial F}{\partial Y} = \sum_{m=0}^{M+1}{}' T_{2m}(X)\left\{\tfrac{1}{2} b_{om} T_1(Y) + \sum_{r=1}^{R} \frac{b_{rm}}{2}\left\{\frac{T_{2r+1}(Y)}{2r+1} - \frac{T_{2r-1}(Y)}{2r-1}\right\}\right\}$$

(the arbitrary constants are zero from symmetry).

Hence from the boundary condition at $Y = 1$

$$\tfrac{1}{2} b_{om} = \sum_{r=1}^{R} \frac{b_{rm}}{(4r^2-1)} \quad \text{for } m = 0,1 \ \ldots\ (M+1) \qquad 5.5(6)$$

Similarly by considering $\frac{\partial F}{\partial X}$,

$$\tfrac{1}{2} c_{ro} = \sum_{m=1}^{M} \frac{c_{rm}}{(4m^2-1)} \quad \text{for } r = 0,1 \ \ldots\ (R+1) \qquad 5.5(7)$$

Equations 5.2(4) and (5) are used to give the τ-terms and will be neglected at this stage. The remaining equations number (R+1)(M+1) from the differential equation and (R+M+4) from the boundary conditions. The total number of unknowns is (R+1)(M+1) coefficients a_{rm} together with (R+M+2) constants of integration b_{ro}, c_{om}. Thus

5.5) contd.

we would appear to have two surplus equations. However in appendix 3 one equation of the boundary equations 5.2(6) and (7) will be shown to be linearly dependent on the remainder and, as in section 5.3, if the total current is zero within the slot one of the remaining equations is automatically satisfied. Let us consider separately the equations for the arbitrary constants b_{ro}, c_{om}. These are (i) from the boundary equations 5.5(6) and (7)

$$\tfrac{1}{2}\, b_{oo} = \sum_{r=1}^{R} \frac{b_{ro}}{4r^2 - 1} \qquad 5.5(8)$$

and $$\tfrac{1}{2}\, c_{oo} = \sum_{m=1}^{M} \frac{c_{om}}{4m^2 - 1} \qquad 5.5(9)$$

(ii) from the differential equation 5.5(3)

$$\left(\frac{b}{a}\right)^2 c_{ro} + b_{ro} = -\pi^2 \left\{ \delta_1(r)\ \eta_1(0) + J\ \delta_2(r).\eta_2(0) \right\} \qquad 5.5(10)$$

for r = 0,1,2 ... R

$$\left(\frac{b}{a}\right)^2 c_{om} + b_{om} = -\pi^2 \left\{ \delta_1(0)\ \eta_1(m) + J\ \delta_2(0)\ \eta_2(m) \right\} \qquad 5.5(11)$$

for m = 1,2,3 ... M

i.e. (R+M+3) equations in (R+M+2) unknowns.

From 5.5(8) and (9)

$$\tfrac{1}{2}\, b_{oo} + \tfrac{1}{2}\left(\frac{b}{a}\right)^2 c_{oo} = \sum_{r=1}^{R} \frac{b_{ro}}{4r^2-1} + \left(\frac{b}{a}\right)^2 \sum_{m=1}^{M} \frac{c_{om}}{(4m^2-1)}$$

$$= -\pi^2 \sum_{r=1}^{R} \left\{ \frac{\delta_1(r)\eta_1(0) + J\ \delta_2(r)\eta_2(0)}{4r^2-1} \right\} - \pi^2 \sum_{m=1}^{M} \frac{\{\delta_1(0)\eta_1(m) + J\,\delta_2(0)\eta_2(m)\}}{4m^2-1}$$

$$- \sum_{m=1}^{M} \frac{b_{om}}{4m^2-1} - \left(\frac{b}{a}\right)^2 \sum_{r=1}^{R} \frac{c_{ro}}{4r^2-1}$$ using 5.5(10) and (11).

5.5) contd.

But $b_{om} = 2 \sum_{r=1}^{R} \frac{b_{rm}}{(4r^2-1)}$ and $c_{ro} = 2 \sum_{m=1}^{M} \frac{c_{rm}}{(4m^2-1)}$

and b_{rm}, c_{rm} are given by equation 5.5(3).

Making use of the previous result

$$-\frac{\delta(0)}{2} + \sum_{r=1}^{\infty} \frac{\delta(r)}{4r^2 - 1} = \frac{\pi}{2}(\beta_1 - \beta_2)$$

and the exactly similar result obtained for $\eta(m)$ it follows that, if the net current is zero within the region OABC, the equation obtained by putting $r = 0$ in 5.5(10) is linearly dependent on the remaining equations (to within the truncation error). Thus the total number of equations is consistent if we ignore one of the boundary equations and the first of 5.5(10).

We now solve for the coefficients a_{rm}, $r = 0,1, \ldots R$; $m = 0,1 \ldots M$ using equation 5.5(3) with $r = 1,2, \ldots R$; $m = 1,2 \ldots M$, equation 5.5(6) with $m = 1,2 \ldots (M+1)$ and equation 5.5(7) for $r = 1,2 \ldots R$ (so neglecting the last boundary equation of 5.5(7)). Following the method of section 4.9 define vectors

$$\underline{\Phi}_r = \begin{bmatrix} a_{ro} \\ a_{r1} \\ \vdots \\ a_{rM} \end{bmatrix} \quad \text{for } r = 0,1 \ldots R \qquad 5.5(12)$$

The equations may then be written in matrix form as

$$\underline{C}\,\underline{\Phi}_{r-1} + \lambda(r)\,\underline{B}_r\,\underline{\Phi}_r + \mu(r)\,\underline{C}\,\underline{\Phi}_{r+1}$$
$$= -4\pi^2\,\lambda(r)\{\delta_1(r)\,\underline{T}_1 + J\,\delta_2(r)\,\underline{T}_2\} \qquad 5.5(13)$$

for $r = 1,2 \ldots R$

5.5) contd.

where $\underline{\Phi}_{R+1} \equiv \underline{0}$

$$\lambda(r) = \frac{(2r-1)2r}{(^b/a)^2}$$

$$\mu(r) = \frac{(2r-1)}{(2r+1)}$$

$$\underline{T}_i = \begin{bmatrix} 0 \\ \eta_i(1) \\ \eta_i(2) \\ \vdots \\ \eta_i(M) \end{bmatrix}, \quad i = 1,2.$$

$$\underline{C} = \begin{bmatrix} 1 & \frac{-2}{1.3} & \frac{-2}{3.5} & \cdots & \frac{-2}{(2M-1)(2M+1)} \\ & 1 & & & \\ & & 1 & & 0 \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$$

and

$$\underline{B}_r = \begin{bmatrix} k_r, & \frac{-2k_r}{1.3}, & \frac{-2k_r}{3.5}, & \frac{-2k_r}{5.7}, & \cdots & \frac{-2k_r}{(2M-1)(2M+1)} \\ \frac{1}{1.2}, & k_r - \frac{2}{1.3}, & \frac{1}{2.3}, & 0, & & 0 \\ 0, & \frac{1}{3.4}, & k_r - \frac{2}{3.5}, & \frac{1}{4.5}, & & \\ 0, & 0, & \frac{1}{5.6}, & k_r - \frac{2}{5.7}, & \frac{1}{6.7} & \\ \vdots & \vdots & \vdots & & & \\ 0, & 0, & 0, & & \frac{1}{(2M-1)2M}, & k_r - \frac{2}{4M^2-1} \end{bmatrix}$$

where $k_r = \dfrac{-2(^b/a)^2}{4r^2 - 1}$

Now $\underline{C}^{-1}$ exists and equations 5.5(13) can be written

5.5) contd.

$$\underline{\Phi}_{r-1} + \lambda(r)\underset{\sim}{C}^{-1}\underline{B}_r\underline{\Phi}_r + \mu(r)\underline{\Phi}_{r+1} = -4\pi^2\lambda(r)\left\{\delta_1(r)\underset{\sim}{C}^{-1}\underline{T}_1 + J\,\delta_2(r)\underset{\sim}{C}^{-1}\underline{T}_2\right\} \qquad 5.5(14)$$

for r = 1,2 ... R

A further equation for the unknown vectors $\underline{\Phi}_r$ is obtained from the boundary equations which may be written in the form

$$\underline{D}\left\{\sum_{r=o}^{R}{}' \frac{\underline{\Phi}_r}{(4r^2-1)}\right\} = \underline{0} \qquad 5.5(15)$$

where

$$\underline{D} = \begin{bmatrix} \frac{1}{1.2}, & \frac{-2}{1.3}, & \frac{1}{2.3}, & 0 & \cdots\cdots & 0 \\ 0, & \frac{1}{3.4}, & \frac{-2}{3.5}, & \frac{1}{4.5} & & 0 \\ \vdots & & & & & \\ 0, & 0, & \cdots\cdots & & & \frac{1}{(2M+1)(2M+2)} \end{bmatrix}$$

Thus $\underline{D}^{-1}$ exists and hence

$$\sum_{r=o}^{R}{}' \frac{\underline{\Phi}_r}{4r^2-1} = \underline{0} \qquad 5.5(16)$$

To obtain the solution by the method of Appendix 2 define

$\underline{\Psi}_r = \dfrac{\underline{\Phi}_r}{4r^2-1}$, r = 0,1 ... R and solve for the vectors $\underline{\Psi}_r$.

Hence the coefficients a_{rm} are found. The constants c_{om}, b_{ro} are found from equations 5.5(8). (9), (10), (11) and the τ terms from equations 5.5(4) and (5). The solution for F is then obtained by integration of the expressions for $\dfrac{\partial^2 F}{\partial X^2}$ and $\dfrac{\partial^2 F}{\partial Y^2}$, Comparison of the two results gives all the necessary constants of integration except for the constant term which is undefined. Its value is chosen to make

5.5) contd.

$$\int_0^1 \int_0^1 F \, dX \, dY = 0 .$$

5.6) Conclusions.

The purpose of this chapter was to show how Roth's method and the Chebyshev methods could be adapted to solve the Neumann problem due to a rectangular transformer window. Again the double Chebyshev method cannot be recommended. For this problem it becomes even more cumbersome to apply than for the slot problem.

For this problem, the rate of convergence for a given accuracy is significantly slower than for the slot problem whichever method of solution is used. This is illustrated in Table 8. One possible reason for this is that the solution is of the same order of magnitude as the coefficients.

Method	Number of terms in solution	Computing time	Values of F at: (0,0)	(0,b)	$(\frac{a}{2},0)$	$(\frac{a}{2},\frac{b}{2})$
Roth	231	-	4.41102×10^{-2}	7.48345×10^{-2}	-0.164901	-2.20341×10^{-2}
	496	34 secs	4.39840×10^{-2}	7.51031×10^{-2}	-0.164681	-2.18409×10^{-2}
	861	44 secs	4.40273×10^{-2}	7.48659×10^{-2}	-0.164786	-2.18835×10^{-2}
Fourier-Chebyshev	121	30 secs	4.39555×10^{-2}	7.49743×10^{-2}	-0.165106	-2.20763×10^{-2}
	256	27 secs	4.41492×10^{-2}	7.49364×10^{-2}	-0.165869	-2.19191×10^{-2}
	441	39 secs	4.40136×10^{-2}	7.49436×10^{-2}	-0.165159	-2.18760×10^{-2}
Double Chebyshev	121	99 secs	4.54808×10^{-2}	7.50805×10^{-2}	-0.165550	-2.33859×10^{-2}
	169	148 secs	4.56886×10^{-2}	7.49279×10^{-2}	-0.164267	-2.23141×10^{-2}
	196	184 secs	4.51835×10^{-2}	7.49919×10^{-2}	-0.165573	-2.28237×10^{-2}

TABLE 8.

Notes.

1) The data used to obtain this table is $\frac{b}{a} = \frac{\pi}{2}$,

$\alpha_{11} = 0.25$, $\alpha_{21} = 0.35$, $\beta_{11} = 0$, $\beta_{21} = 0.66667$,

$\alpha_{12} = 0.425$, $\alpha_{22} = 0.575$, $\beta_{12} = 0$, $\beta_{22} = 0.5$.

2) The computing times given are total occupation times

5.6) contd.

2) contd.

of the machine and include compilation, evaluation of the coeffcients and calculation of F over the grid X = 0(0.05)1, Y = 0(0.05)1. They are given in order to give a relative comparison between the three methods.

3) It was not possible to take more terms in the double Chebyshev method due to lack of sufficient computer storage.

Summarising then, although Chebyshev approximations are a possibility for this type of problem, they are not recommended. Roth's method is much simpler to apply and the derivation of the solution is comparatively trivial. It is possible, however, that the Chebyshev methods are wider in scope and this will be dealt with more fully in Chapter 8. By whatever method the solution is obtained, the rate of convergence for the transformer problem is much slower than that for the slot problem.

CHAPTER 6.

EDDY-CURRENT PROBLEMS.

6.1) Introduction.

In his papers, Roth considered only static fields. The purpose of this chapter is to investigate how Roth's methods can be applied to the problem of an insulated conductor in a rectangular slot when the current is varying sinusoidally with time. Most of the work to date on this subject has assumed that the effect of the insulation space is negligible. The Roth solution derived here gives a quantitative estimate of the effects due to the presence of the insulation.

The derivation of the exact Roth solution is somewhat complex and so a simpler approximate model is developed. The solution so obtained is then compared with the exact solution and the two methods are shown to give good agreement over a practical range of insulation thicknesses. The approximate method is then used to consider an insulated conductor in a slot facing an air gap, the problem considered by Silvester in[14].

6.2) Description of the problem.

The configuration is as given in Figure 8 of section 2.2 where we will now consider just one conductor PQRS in the slot. Referring back to section 1.9(b) we have to solve

$$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z^*}{\partial y^2} - i\,\alpha^2 A_z^* = -\,\mu_r \mu_0 J_{1z}^* \quad \text{in the conductor} \qquad 6.2(1)$$

and $$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z^*}{\partial y^2} = 0 \quad \text{in the insulator} \qquad 6.2(2)$$

subject to the boundary conditions

(i) $\dfrac{\partial A_z^*}{\partial x} = 0$ for $x = 0,\ 0 \leqslant y \leqslant b,$

(ii) $\dfrac{\partial A_z^*}{\partial x} = 0$ for $x = a,\ 0 \leqslant y \leqslant b,$

6.2) contd.

(iii) $\dfrac{\partial A_z^*}{\partial y} = 0$ for $y = 0$, $0 \leq x \leq a$

(iv) $A_z^* = \Omega$ for $y = b$, $0 \leq x \leq a$, where Ω is a complex constant.

Write $B_z^* = A_z^* - \Omega$ 6.3(3)

Then the boundary conditions on B_z^* become

(i) $\dfrac{\partial B_z^*}{\partial x} = 0$ for $x = 0, a$; $0 \leq y \leq b$

(ii) $\dfrac{\partial B_z^*}{\partial y} = 0$ for $y = 0$, $0 \leq x \leq a$

(iii) $B_z^* = 0$ for $y = b$, $0 \leq x \leq a$

which are of the form suitable for solution by Roth's method. The differential equations for B_z^* are

$$\frac{\partial^2 B_z^*}{\partial x^2} + \frac{\partial^2 B_z^*}{\partial y^2} - i\alpha^2 B_z^* = -\mu_r \mu_0 J_{1z}^* + i\alpha^2 \Omega \qquad 6.2(3)$$

in the conductor and

$$\frac{\partial^2 B_z^*}{\partial x^2} + \frac{\partial^2 B_z^*}{\partial y^2} = 0 \qquad 6.2(4)$$

in the insulator.

6.3) Method of solution.

As in section 2.2, in order to satisfy the boundary conditions B_z^* must be of the form

$$B_z^* = \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty} C_{mk} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \qquad 6.3(1)$$

where each C_{mk} is now a complex constant. If this is to satisfy the differential equation,

6.3) contd.

$$\sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} C_{mk}\left\{\frac{m^2\pi^2}{a^2} + (k+\tfrac{1}{2})^2 \frac{\pi^2}{b^2} + i\alpha^2\right\} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b}$$

$$= \begin{cases} (\mu_r\mu_o J^*_{1z} - i\alpha^2\Omega) & \text{in the conductor} \\ i\alpha^2 \sum_{p=o}^{\infty}{}' \sum_{q=o}^{\infty} C_{pq} \cos\frac{p\pi x}{a} \cos(q+\tfrac{1}{2})\frac{\pi y}{b} & \text{in the insulator} \end{cases}$$

Equations for the coefficients C_{mk} are obtained by multiplying this expression through by $\cos\frac{m\pi x}{a}$ $\cos(k+\frac{1}{2})\frac{\pi y}{b}$ and integrating over the whole area of the slot.

Writing $D_{mk} = C_{mk} \dfrac{\pi^4}{4b^2(\mu_r\mu_o J^*_{1z} - i\alpha^2\Omega)}$ 6.3(2)

the equations relating the coefficients D_{mk} are

$$PP(m,k)D_{mk} + i\left(\frac{\alpha b}{\pi}\right)^2 \sum_{p=o}^{\infty}{}' \sum_{q=o}^{\infty} AA(p,m)BB(q,k)D_{pq} = \alpha(m)\beta(k) \qquad 6.3(3)$$

$m = 0,1,2 \ldots, k = 0,1,2 \ldots$

where $\alpha(m)$, $\beta(k)$ are given by equations 2.4(3),

$$PP(m,k) = \left(\frac{b}{a}\right)^2 m^2 + (k+\tfrac{1}{2})^2, \qquad 6.3(4)$$

$$AA(p,m) = AA(m,p)$$

$$= \frac{1}{(p+m)\pi}\left\{\sin \alpha_2(p+m)\pi - \sin \alpha_1(p+m)\pi\right\}$$

$$+ \frac{1}{(p-m)\pi}\left\{\sin \alpha_2(p-m)\pi - \sin \alpha_1(p-m)\pi\right\} \quad (p \neq m)$$

$$AA(p,p) = \frac{1}{2p\pi}\left\{\sin \alpha_2 2p\pi - \sin \alpha_1 2p\pi\right\} + (\alpha_2 - \alpha_1) \quad (p \neq 0)$$

$$AA(0,0) = 2(\alpha_2 - \alpha_1) \qquad 6.3(5)$$

$$BB(q,k) = \frac{1}{(q+k+1)\pi}\left\{\sin \beta_2(q+k+1)\pi - \sin \beta_1(q+k+1)\pi\right\}$$

$$+ \frac{1}{(q-k)\pi}\left\{\sin \beta_2(q-k)\pi - \sin \beta_1(q-k)\pi\right\} \quad (q \neq k)$$

$$= BB(k,q)$$

6.3) contd.

$$BB(k,k) = \frac{1}{(2k+1)\pi}\left\{\sin \beta_2(2k+1)\pi - \sin \beta_1(2k+1)\pi\right\} + (\beta_2-\beta_1) \qquad 6.3(6)$$

Writing $D_{mk} = R\ell(D_{mk}) + iIm(D_{mk})$ and calculating the coefficients up to and including the M'th diagonal as described in section 2.3, equation 6.3(3) can be written

$$PP(m,k) \cdot R\ell(D_{mk}) - \left(\frac{\alpha b}{\pi}\right)^2 \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} AA(p,m)\ BB(q,k)\ Im(D_{pq})$$

$$= \alpha(m)\ \beta(k) \qquad 6.3(7)$$

$$PP(m,k)\ Im(D_{mk}) + \left(\frac{\alpha b}{\pi}\right)^2 \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} AA(p,m)\ BB(q,k)\ R\ell(D_{pq}) = 0 \qquad 6.3(8)$$

for $m = 0,1,2 \ldots M$, $k = 0,1, \ldots (M-m)$

An iterative procedure for solving a coupled set of equations of this type is given in Appendix 4. Having calculated the coefficients D_{mk}, A_z^* is given by

$$A_z^* = \Omega + \frac{4b^2}{\pi^4}(\mu_r \mu_0 J_{1z}^* - i\alpha^2\Omega) \sum_{m=o}^{M}{}' \sum_{k=o}^{(M-m)} D_{mk} \cos\frac{m\pi x}{a} \quad \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \qquad 6.3(9)$$

In order to have some measure of the degree of accuracy of the solution for a given M, the total axial current in the conductor was evaluated by two different methods. The answers should, of course, be identical but differ due to the truncation error. The magnitude of the difference is a measure of this error.

If I_z is the total current flowing in the conductor then $I_z = R\ell(I_z^* e^{i\omega t})$

and (i) $$I_z^* = \int_{x=\alpha_1 a}^{\alpha_2 a} \int_{y=\beta_1 b}^{\beta_2 b} J_z^*\, dx\, dy$$

6.3) contd.

$$= \int_{\alpha_1 a}^{\alpha_2 a} \int_{\beta_1 b}^{\beta_2 b} (J^*_{1z} + J^*_{2z}) dx\, dy$$

$$= \int_{\alpha_1 a}^{\alpha_2 a} \int_{\beta_1 b}^{\beta_2 b} (J^*_{1z} + (-\sigma i \omega) A^*_z) dx\, dy$$

using the results given in section 1.7. With A^*_z given by equation 6.3(9),

$$\mu_r \mu_o\, I^*_z = (\mu_r \mu_o J^*_{1z} - i\alpha^2 \Omega) ab \left\{ (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) - 4i \left(\frac{\alpha b}{\pi^3}\right)^2 {\sum_{m=o}^{M}}' \sum_{k=o}^{M-m} D_{mk} \alpha(m) \beta(k) \right\}$$

6.3(10)

Also (ii) $\mu_r \mu_o\, I^*_z = - \int_o^a \left(\frac{\partial A^*_z}{\partial y}\right)_{y=b} dx$

Using equation 6.3(9) this becomes

$$\mu_r \mu_o I^*_z = \frac{2ab}{\pi^3} (\mu_r \mu_o J^*_{1z} - i\alpha^2 \Omega) \sum_{k=o}^{M} D_{ok} (k+\tfrac{1}{2}) (-1)^k \qquad 6.3(11)$$

Examination of these two expressions for I^*_z shows that the series in equation 6.3(10) has a more rapid rate of convergence than that in equation 6.3(11).

M	$Re(kI^*_z)$ using 6.3(10)	$Re(kI^*_z)$ using 6.3(11)	$-Im(kI^*_z)$ using 6.3(10)	$-Im(kI^*_z)$ using 6.3(11)	
8	0.1553	0.1585	0.2174	0.2177	$\alpha b = 3$
12	0.1553	0.1543	0.2174	0.2173	
8	0.0504	0.0531	0.0901	0.0907	$\alpha b = 6$
12	0.0503	0.0496	0.0902	0.0898	

TABLE 9.

$$\left(k = \frac{\mu_r \mu_o}{(\mu_r \mu_o J^*_{1z} - i\alpha^2 \Omega) ab} \right)$$

The data used to obtain this table is

6.3) contd.

$\alpha_1 = 0.2$, $\alpha_2 = 1$, $\beta_1 = 0.1$, $\beta_2 = 0.9$, $\frac{b}{a} = 3$, corresponding to the conductor being symmetrically placed in a slot of total width 2a, as shown in Figure 19.

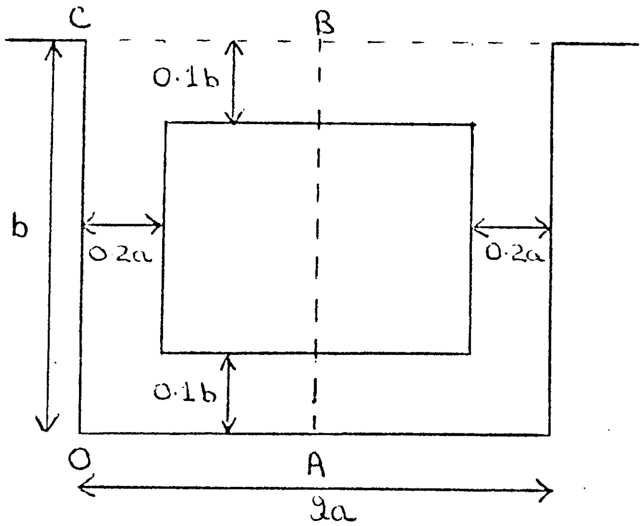


Figure 19.

Table 9 shows reasonable agreement between the values of I_z^* obtained by the two expressions 6.3(10) and (11) and this agreement is improved as more terms of the series are included. Also the series 6.3(10) has converged to within 0.2% for the range of frequencies to be considered and so the expression for I_z^* given by equation 6.3(10) will be used in the subsequent impedance calculations.

6.4) Evaluation of complex impedance.

Referring to section 1.12,

$$\iint_S \tfrac{1}{2}(\underline{E}^* \times \tilde{\underline{H}}^*).d\underline{S} = -\tfrac{1}{2}\, \underline{I}^*.\tilde{\underline{I}}^*(R+iX) \text{ where } X = \omega L, \quad 1.12(2)$$

and $d\underline{S}$ is in the direction of the outward normal to S, the closed surface bounding a given volume V.

Take as volume V, unit length of the slot, cross section OABC.

Now $\underline{E}^* = \underline{E}_1^* + \underline{E}_2^*$

6.4) contd.

$$= \frac{1}{\sigma} \underline{J}_1^* - i\omega \underline{A}^* \quad \text{using the results of section 1.7}$$

$$\therefore \underline{E}^* = (0,\ 0,\ \frac{1}{\sigma} J_{1z}^* - i\omega A_z^*) \qquad 6.4(1)$$

$$\underline{\tilde{H}}^* = \frac{1}{\mu_r \mu_o} \left(\frac{\partial}{\partial y} \tilde{A}_z^*,\ - \frac{\partial}{\partial x} \tilde{A}_z^*,\ 0 \right) \qquad 6.4(2)$$

$$\text{Hence } \underline{E}^* \times \underline{\tilde{H}}^* = \frac{1}{\mu_r \mu_o} \left\{ \frac{\partial \tilde{A}_z^*}{\partial x} \left(\frac{J_{1z}^*}{\sigma} - i\omega A_z^* \right),\ \frac{\partial \tilde{A}_z^*}{\partial y} \left(\frac{J_{1z}^*}{\sigma} - i\omega A_z^* \right), 0 \right\}$$

Taking account of the boundary conditions round the slot, the only contribution to the integral 1.12(2) arises from the side BC where $A_z^* = \Omega$.

$$\text{Hence } -I_z^* \tilde{I}_z^* (R+iX) = \frac{1}{\mu_r \mu_o} \left(\frac{J_{1z}^*}{\sigma} - i\omega\Omega \right) \int_o^a \left(\frac{\partial \tilde{A}_z^*}{\partial y} \right)_{y=b} dx$$

$$= - \tilde{I}_z^* \left(\frac{J_{1z}^*}{\sigma} - i\omega\Omega \right)$$

$$\text{Hence } R + iX = \frac{1}{I_z^*} \left(\frac{J_{1z}^*}{\sigma} - i\omega\Omega \right) \qquad 6.4(3)$$

If I_z^* is given by equation 6.3(10),

$$R+iX = \frac{1}{\sigma ab \left\{ (\alpha_2-\alpha_1)(\beta_2-\beta_1) - 4i \left(\frac{\alpha b}{\pi^3} \right)^2 \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} D_{mk}\ \alpha(m)\beta(k) \right\}} \qquad 6.4(4)$$

The resistance at zero frequency is

$$R_o = \frac{1}{\sigma ab(\alpha_2-\alpha_1)(\beta_2-\beta_1)} \qquad 6.4(5)$$

and thus the ratios $\frac{R}{R_o}$, $\frac{X}{R_o}$ can be calculated.

L_o, the inductance (at zero frequency) is evaluated from

6.4) contd.

$$L_o = \lim_{\omega \to o} \frac{X}{\omega}$$

$$= \frac{4b^2 \ \mu_r \mu_o \ \sigma}{\pi^6 \sigma ab(\alpha_2 - \alpha_1)^2(\beta_2 - \beta_1)^2} \lim_{\omega \to o} \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} R\ell(D_{mk})\,\alpha(m)\beta(k)$$

From equation 6.3(7), when $\omega = 0$,

$$R\ell(D_{mk}) = \frac{\alpha(m)\beta(k)}{PP(m,k)}$$

Hence $$L_o = \frac{4\mu_r \mu_o \ (\frac{b}{a})\ \sigma}{\pi^6(\alpha_2 - \alpha_1)^2(\beta_2 - \beta_1)^2 \sigma} \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} \frac{\alpha^2(m)\beta^2(k)}{PP(m,k)} \qquad 6.4(6)$$

and the ratio $\frac{L}{L_o}$ can be calculated for a range of frequencies. Equation 6.4(6) is of importance since it gives L_o, the inductance (at zero frequency), in closed form. Most techniques used for the solution of the type of problem considered here do not give an exact expression for L_o, the usual method being to evaluate it numerically for a sufficiently small value of ω.

Figures 20 and 21 show the variation of R/R_o , L/L_o with the frequency parameter αb for two insulation thicknesses and also for the case of the conductor completely filling the slot. The results obtained for this latter case agree with those obtained using the method described by Swann and Salmon in(13) for the fully open slot. From the graphs it can be seen that the effect of the insulation is to decrease the effective resistance and increase the effective inductance. When the insulation thickness is 5% of the slot dimensions, these effects are (when the frequency is given by $\alpha b = 6$

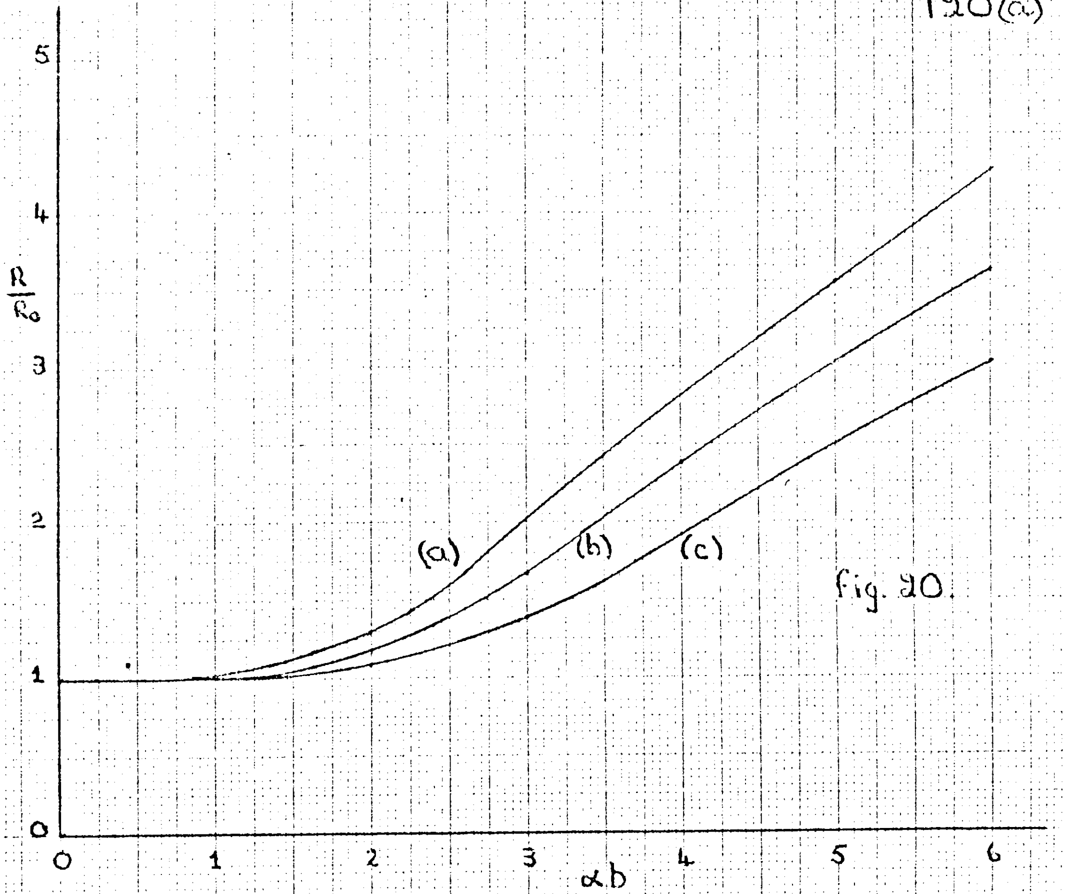


fig. 20.

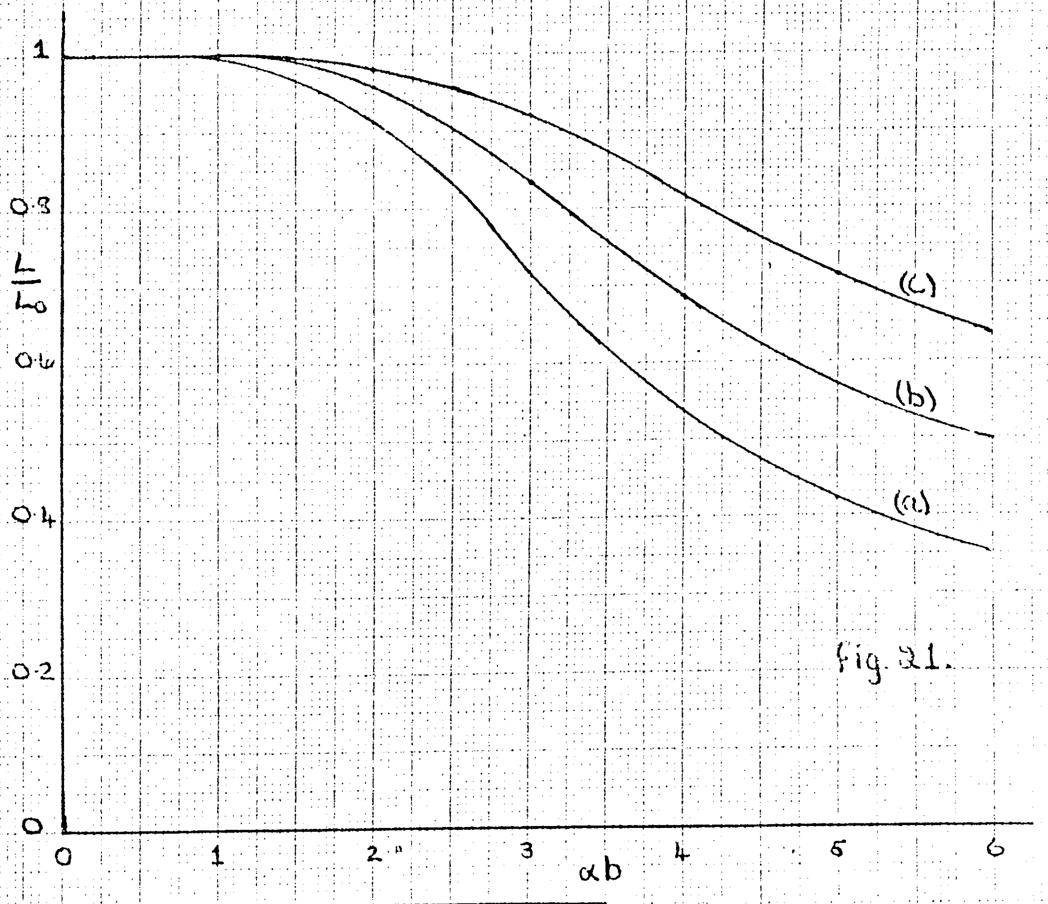


fig. 21.

DATA.

	α_1	α_2	β_1	β_2	b/a
(a)	0	1	0	1	3
(b)	0·1	1	0·05	0·95	3
(c)	0·2	1	0·1	0·9	3

6.4) contd.

(i) $\frac{R}{R_o}$ is decreased by 14% (approx)

(ii) $\frac{L}{L_o}$ is increased by 40% (approx).

If the insulation thickness is doubled, the graphs show that these changes are doubled.

6.5) Simpler approximate models.

Due to the difficulty of having to solve the coupled set of equations 6.3(7) and (8) by an iterative process, we now search for a simpler, more direct method. The exact solution shown in Figures 20 and 21 will be used to compare the approximate solutions.

a) First approximation.

This is based on the suggestions contained in Hammond's paper[9]. We assume that equation 6.2(4) is replaced by

$$\frac{\partial^2 B_z^*}{\partial x^2} + \frac{\partial^2 B_z^*}{\partial y^2} - i\alpha^2 B_z^* = 0 \qquad 6.5(1)$$

The boundary conditions are as before. Thus the differential equation is exact in the conductor but not in the insulator, where we have introduced an error term $i\alpha^2 B_z^*$, corresponding to an effective induced current in the insulator. Using Roth's method, the solution is of the form

$$B_z^* = \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty}{}' C_{mk} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \qquad 6.5(2)$$

6.5) contd.

a) contd.

$$\text{where } C_{mk} = \frac{4b^2}{\pi^4}(\mu_r\mu_o J^*_{1z} - i\alpha^2\Omega)\frac{\alpha(m)\beta(k)}{\left\{PP(m,k)+i\left(\frac{\alpha b}{\pi}\right)^2\right\}} \qquad 6.5(3)$$

$\alpha(m)$ and $\beta(k)$ are given by equations 2.4(3) and PP(m,k) is given by equation 6.3(4). Hence

$$A^*_z = \Omega + \frac{4b^2}{\pi^4}(\mu_r\mu_o J^*_{1z} - i\alpha^2\Omega)\sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} \frac{\alpha(m)\beta(k)\cos\frac{m\pi x}{a}\cos(k+\frac{1}{2})\frac{\pi y}{b}}{\left\{PP(m,k)+i\left(\frac{\alpha b}{\pi}\right)^2\right\}}$$

6.5(4)

which is comparable in form with equation 6.3(9). Using equation 6.3(10),

$$\mu_r\mu_o I^*_z = (\mu_r\mu_o J^*_{1z} - i\alpha^2\Omega)ab\left\{(\alpha_2-\alpha_1)(\beta_2-\beta_1) - 4i\left(\frac{\alpha b}{\pi^3}\right)^2 \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} \frac{\{\alpha(m)\ \beta(k)\}^2}{\left\{PP(m,k)+i\left(\frac{\alpha b}{\pi}\right)^2\right\}}\right\} \qquad 6.5(5)$$

and equation 6.4(4) becomes

$$R+iX = \frac{1}{\sigma ab\left\{(\alpha_2-\alpha_1)(\beta_2-\beta_1) - 4i\left(\frac{\alpha b}{\pi^3}\right)^2 \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} \frac{\{\alpha(m)\ \beta(k)\}^2}{\left\{PP(m,k)+i\left(\frac{\alpha b}{\pi}\right)^2\right\}}\right\}}$$

6.5(6)

This gives the same values for R_o and L_o as given in equations 6.4(5) and (6).

Although of a particularly simple and neat form, this model is totally inadequate since, as the frequency increases, the ratio R/R_o decreases even for very narrow insulation thicknesses.

6.5) contd.

b) Second approximation.

Write $A_z^* = A_1 + A_2$ 6.5(7)

where $\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} - i\alpha^2 A_1 = \begin{cases} -\mu_r\mu_o J_{1z}^* & \text{in the conductor} \\ 0 & \text{elsewhere} \end{cases}$ 6.5(8)

and $\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} - i\alpha^2 A_2 = 0$ for $\beta_1 b \leq y \leq \beta_2 b;\ 0 \leq x \leq a$ 6.5(9)

$\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} = 0$ for $0 \leq y < \beta_1 b,\ \beta_2 b < y \leq b;\ 0 \leq x \leq a$ 6.5(10)

The differential equation is exact in the conductor but not in the insulator. The boundary conditions are

(i) $\frac{\partial A_1}{\partial x} = 0;\ \frac{\partial A_2}{\partial x} = 0$ when $x = 0,\ 0 \leq y \leq b$

(ii) $\frac{\partial A_1}{\partial x} = 0;\ \frac{\partial A_2}{\partial x} = 0$ when $x = a,\ 0 \leq y \leq b$

(iii) $\frac{\partial A_1}{\partial y} = 0;\ \frac{\partial A_2}{\partial y} = 0$ when $y = 0,\ 0 \leq x \leq a$

(iv) $A_1 = 0;\ A_2 = \Omega$ when $y = b,\ 0 \leq x \leq a$.

A_1 is found using Roth's method which gives

$$A_1 = \frac{4b^2}{\pi^4}\mu_r\mu_o J_{1z}^* \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} \frac{\alpha(m)\beta(k)}{\left\{PP(m,k) + i\left(\frac{\alpha b}{\pi}\right)^2\right\}} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b}$$ 6.5(11)

A_2 is found using the method of separation of variables. In the region $0 \leq y < \beta_1 b$, taking account of the boundary conditions, A_2 is of the form

$$A_2 = \sum_{m=o}^{\infty}{}' a_m \cos\frac{m\pi x}{a} \cosh\frac{m\pi y}{a}\ ,\quad 0 \leq y \leq \beta_1 b,\ 0 \leq x \leq a$$ 6.5(12)

where a_m, m = 0,1,2 ... are constants to be determined.

Similarly in the region $\beta_2 b < y \leq b$,

6.5) contd.

b) contd.

$$A_2 = \sum_{m=1}^{\infty} b_m \sinh\frac{m\pi}{a}(b-y)\cos\frac{m\pi x}{a} + \tfrac{1}{2}b_0(b-y)+\Omega, \; \beta_2 b < y \leqslant b, \; 0 \leqslant x \leqslant a. \qquad 6.5(13)$$

where b_m, m = 0,1,2 ... are constants to be determined.

For the region $\beta_1 b \leqslant y \leqslant \beta_2 b$ write

$$A_2 = X(x)\,Y(y)$$

where X,Y are functions of x only, y only respectively.

Then $\frac{Y''}{Y} - i\alpha^2 = -\frac{X''}{X} = \lambda^2$ (say),

and the solutions are of the form

$$X = \begin{cases}\sin\lambda x & \text{if } \lambda \neq 0\\ \cos\lambda x\end{cases}$$

$$= \alpha + \beta x \quad \text{if } \lambda = 0$$

$$Y = \begin{cases}\sinh\sqrt{(\lambda^2+i\alpha^2)}\,y\\ \cosh\sqrt{(\lambda^2+i\alpha^2)}\,y\end{cases}$$

Taking account of the boundary conditions A_2 is of the form

$$A_2 = \sum_{m=0}^{\infty}{}' \cos\frac{m\pi x}{a}\left(c_m \cosh\lambda_1 y + d_m \sinh\lambda_1 y\right) \qquad 6.5(14)$$

$$\beta_1 b \leqslant y \leqslant \beta_2 b, \; 0 \leqslant x \leqslant a$$

where $\lambda_1^2 = \frac{m^2\pi^2}{a^2} + i\alpha^2.$

The constants a_m, b_m, c_m, d_m are evaluated by considering the continuity of A_2 and $\frac{\partial A_2}{\partial y}$ across the boundaries $y = \beta_1 b$ and $y = \beta_2 b$. This follows since the permeability is constant throughout the slot. The values obtained for the constants are

$$a_m = b_m = c_m = d_m = 0 \text{ for } m = 1,2,3 \ldots \qquad 6.5(15)$$

This means that the potential A_2 depends only on y.

The remaining constants are

6.5) contd.

b) contd.

$$a_0 = \frac{2\Omega}{\cosh q(\beta_2-\beta_1) + q(1-\beta_2)\sinh q(\beta_2-\beta_1)}$$

$$b_0 = -\alpha\sqrt{i}\,\sinh[q(\beta_2-\beta_1)]a_0 \qquad 6.5(16)$$

$$c_0 = a_0 \cosh q\beta_1$$

$$d_0 = -a_0 \sinh q\beta_1$$

where $q = \alpha b\sqrt{i}$.

Hence the vector potential A_2 is known and the potential A_z^* is given by equations 6.5(7) and (11). In the resulting expression for A_z^* we have two unknown quantities namely Ω and J_{1z}^*. Two relationships between these are obtained by deriving expressions for I_z^* using the results (i) and (ii) of section 6.3. The resulting equations are linear in the variables $\left(\frac{J_{1z}^* ab}{I_z^*}\right)$ and $\left(\frac{\Omega}{\mu_r\mu_0 I_z^*}\right)$ and so can readily be solved. Using equation 6.4(3)

$$R + iX = \frac{1}{\sigma ab}\left\{\left(\frac{J_{1z}^*\,ab}{I_z^*}\right) - \frac{i(\alpha b)^2}{b/a}\left(\frac{\Omega}{\mu_r\mu_0 I_z^*}\right)\right\}$$

and so the ratios R/R_0 and X/R_0 can be calculated. L_0 can be found using a limiting process and the expression so derived is identical to equation 6.4(6). Although this simplified model is inexact in the insulator we have adjusted the parameters Ω and J_{1z}^* to give the correct total current in the conductor.

Figures 22 and 23 show the variation of R/R_0 and L/L_0 with frequency over a practical range of insulation thicknesses using this simplified model. The curves obtained using the exact method are also given for comparison purposes. It can be seen that

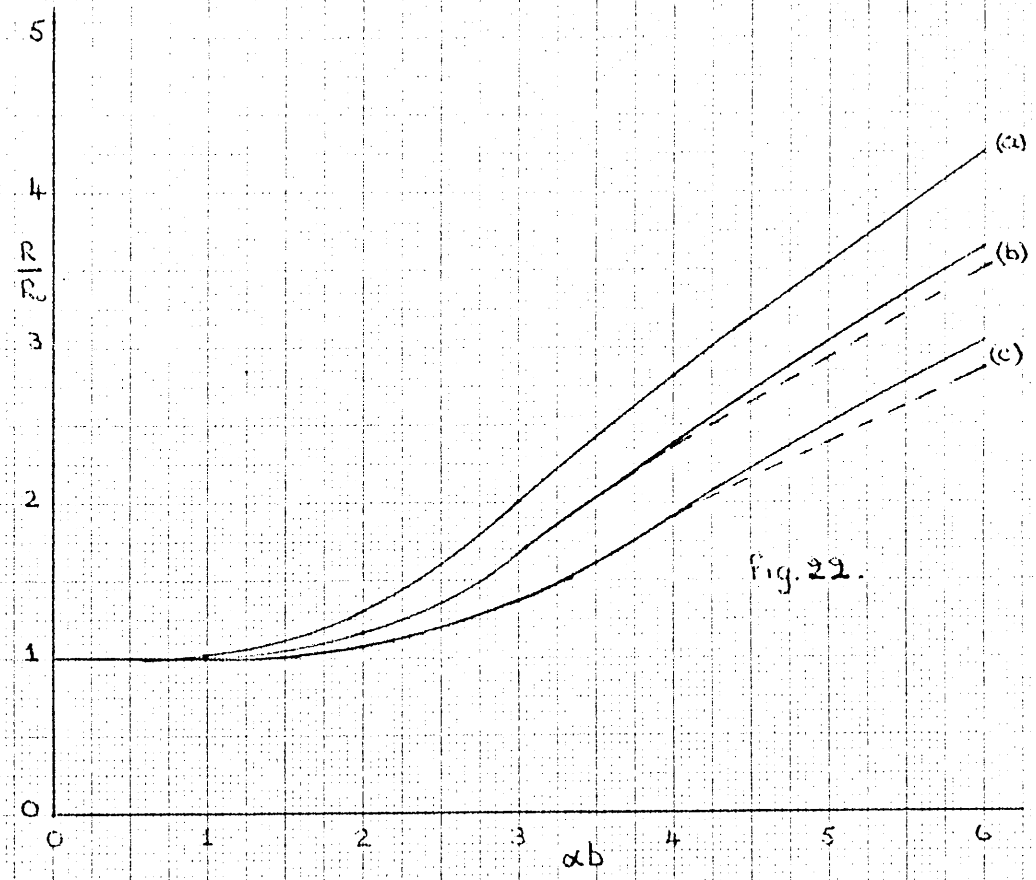


Fig. 22.

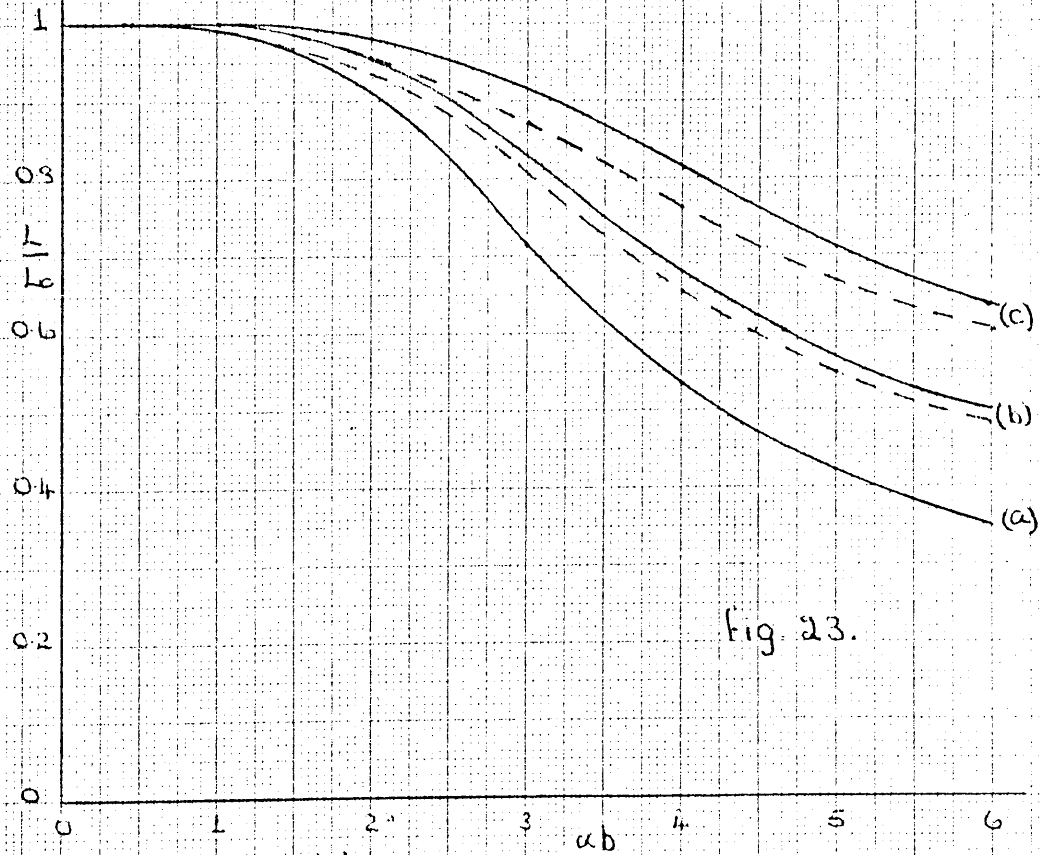


Fig. 23.

——— exact model.

– – – approximate model.

(Data as for figs. 20 and 21).

6.5) contd.

b) contd.

for an insulation thickness of the order of 10% of the slot dimensions, the agreement between the two models is as follows:-

for a frequency given by $\alpha b = 7$, the approximate model reduces both $^R/R_0$ and $^L/L_0$ by about 6%. As the insulation thickness decreases, this difference decreases linearly. If the conductor is assumed to fill the slot, the simplified model is, of course, exact and gives the identical results to those obtained by Swann and Salmon in[13]. As the frequency decreases the agreement between the two models improves in the case of $^R/R_0$, but stays fairly constant in the case of $^L/L_0$.

Because the simple model gives a good approximation to the exact solution, we now show how to use it to obtain a solution to the problem considered by Silvester in[14]. He obtains results for an insulated rectangular conductor in a slot facing an air gap, and these results are of a somewhat controversial nature since they indicate that the effects of the insulation are larger than would be expected. We aim to use the simplified model to try to refute or substantiate the findings of Silvester.

6.6) <u>Insulated conductor facing an air gap</u>.

The configuration is as shown in Figure 24.

6.6) contd.

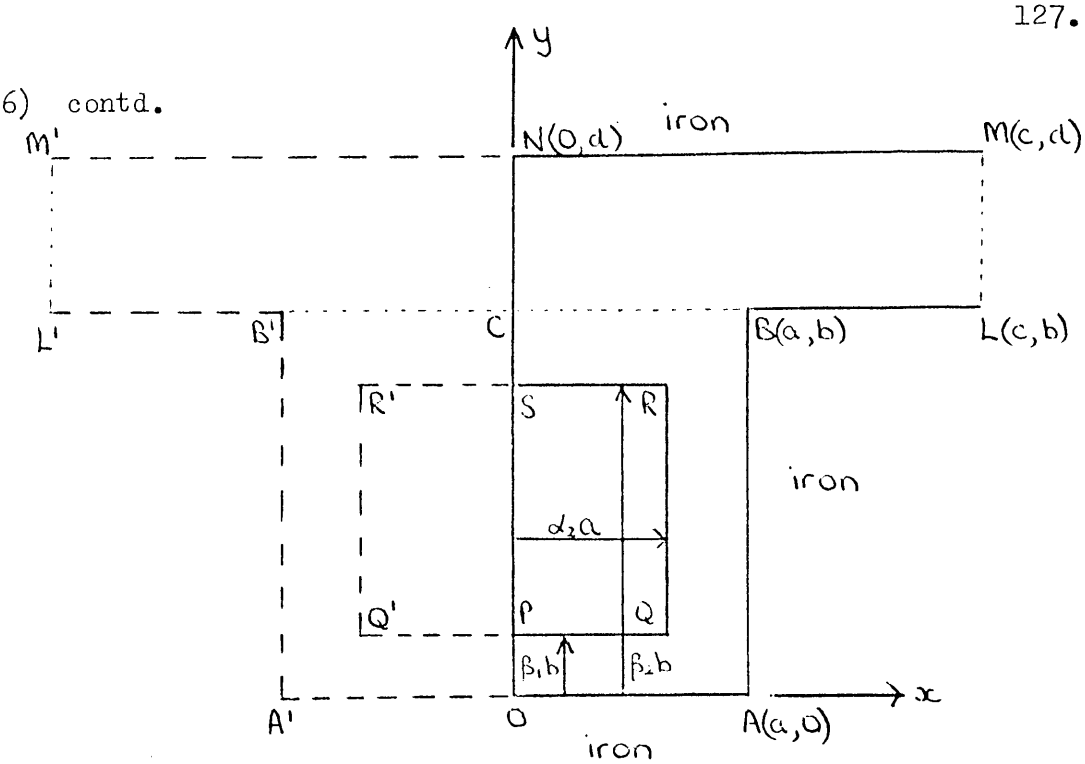


Figure 24.

It is assumed that the conductor is symmetrically placed in the slot A'ABB' so that we need only consider the solution in the region $x \geqslant 0$, $y \geqslant 0$. It is further assumed that the iron is of infinite permeability and that the line LM is a magnetic vector equipotential i.e. a flux line. As in section 6.2 we have to solve the differential equations

$$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z^*}{\partial y^2} - i\alpha^2 A_z^* = -\mu_r \mu_0 J_{1z}^* \text{ in the conductor PQRS}$$

and $$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0 \text{ elsewhere in the region OABLMN.}$$

The boundary conditions are

(i) $\frac{\partial A_z^*}{\partial x} = 0$ along ON, AB

(ii) $\frac{\partial A_z^*}{\partial y} = 0$ along OA, BL, MN

(iii) $A_z^* = \Omega$ along LM (Ω is a complex constant)

6.7) Method of solution.

The solution in the region OABC is obtained using the simplified model described in section 6.5(b) except that boundary condition (iv) is replaced by

(iv) $A_1 = 0$; A_2 unspecified when $y = b$, $0 \leqslant x \leqslant a$.

A_1 is given by equation 6.5(11) (note that $\alpha_1 \equiv 0$ in this configuration).

The solution for A_2 follows as in section 6.5(b) except that in the region $\beta_2 b < y \leqslant b$, A_2 is in the form

$$A_2 = \sum_{m=1}^{\infty} \cos\frac{m\pi x}{a}\left\{b_m \cosh\frac{m\pi y}{a} + e_m \sinh\frac{m\pi y}{a}\right\} + \tfrac{1}{2}b_0 + \tfrac{1}{2}e_0 \frac{y}{b} \qquad 6.7(1)$$

Relationships between the constants a_m, b_m, c_m, d_m, e_m are obtained by considering the continuity of A_2 and $\frac{\partial A_2}{\partial y}$ across the boundaries $y = \beta_1 b$ and $y = \beta_2 b$. This gives for $m = 0,1,2,3 \ldots$, the coefficients b_m, c_m, d_m, e_m in terms of the coefficients a_m. Thus the potential in the region OABC is known in terms of the coefficients a_m, $m = 0,1, \ldots M$ (truncating at $m = M$).

The potential in the region CLMN is obtained using the method of separation of variables in the form

$$A_z^* = \Omega + \sum_{k=0}^{\infty} h_k \cos(k+\tfrac{1}{2})\frac{\pi x}{c} \cosh(k+\tfrac{1}{2})\frac{\pi}{c}(d-y) \qquad 6.7(2)$$

where h_k, $k = 0,1,2 \ldots$ are constants to be determined. Thus the potential throughout the region OABLMN is known in terms of the constants

a_m , $m = 0,1 \ldots M$

h_k , $k = 0,1 \ldots K$

truncating the series at $m=M$, $k=K$. These constants are evaluated by considering the continuity of A_z^* and

6.7) contd.

$\frac{\partial A_z^*}{\partial y}$ across the boundary BC using the method described in reference[18]. First consider the continuity of A_z^*.

For $0 \leq x \leq a$, $y = b$,

$$(A_1 + A_2)_{y=b} = \Omega + \sum_{k=0}^{K} h_k \cos(k+\tfrac{1}{2})\frac{\pi x}{c} \cosh(k+\tfrac{1}{2})\frac{\pi}{c}(d-b)$$

But $(A_1)_{y=b} = 0$ so that

$$(A_2)_{y=b} = \Omega + \sum_{k=0}^{K} h_k \cos(k+\tfrac{1}{2})\frac{\pi x}{c} \cosh(k+\tfrac{1}{2})\frac{\pi}{c}(d-b) \qquad 6.7(3)$$

$(A_2)_{y=b}$ is given by equation 6.7(1).

From equation 6.7(3),

$$\int_0^a (A_2)_{y=b} \cos\frac{m\pi x}{a} dx = \int_0^a \left\{\Omega + \sum_{k=0}^{K} h_k \cos(k+\tfrac{1}{2})\frac{\pi x}{c} \cosh(k+\tfrac{1}{2})\frac{\pi}{c}(d-b)\right\} \cos\frac{m\pi x}{a} dx$$

for $m = 0, 1, \ldots M$.

This produces the following (M+1) equations in the unknowns a_m, h_k, $m = 0,1 \ldots M$, $k = 0.1 \ldots K$.

For $m = 1, 2 \ldots M$,

$$\frac{a}{2}\left\{b_m \cosh\frac{m\pi b}{a} + e_m \sinh\frac{m\pi b}{a}\right\} = \sum_{k=0}^{K} h_k \cosh\left\{(k+\tfrac{1}{2})\frac{\pi}{c}(d-b)\right\} I(m,k) \qquad 6.7(4)$$

where $I(m,k) = \int_0^a \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi x}{c} dx$

$$= \frac{(-1)^m}{2\pi} c \sin\left\{(k+\tfrac{1}{2})\frac{\pi a}{c}\right\}\left\{\frac{1}{k+\frac{1}{2}+\frac{cm}{a}} + \frac{1}{k+\frac{1}{2}-\frac{cm}{a}}\right\}$$

$$\text{if } k+\tfrac{1}{2} \neq \frac{cm}{a} \qquad 6.7(5)$$

$$= \frac{a}{2} \text{ if } k+\tfrac{1}{2} = \frac{cm}{a}$$

When $m = 0$,

6.7) contd.

$$\frac{a}{2}(b_o + e_o) = a\Omega + \sum_{k=o}^{K} h_k \cosh\left\{(k+\tfrac{1}{2})\frac{\pi}{c}(d-b)\right\} I(0,k) \qquad 6.7(6)$$

Substituting for b_m, e_m in terms of a_m, $m = 0,1 \ldots M$ gives the required equations.

Now consider the continuity of $\frac{\partial A_z^*}{\partial y}$ along $y = b$.

$$\sum_{k=o}^{K} h_k\left\{-(k+\tfrac{1}{2})\frac{\pi}{c}\right\} \cos(k+\tfrac{1}{2})\frac{\pi x}{c} \sinh(k+\tfrac{1}{2})\frac{\pi}{c}(d-b)$$

$$= \begin{cases} \left(\frac{\partial A_1}{\partial y} + \frac{\partial A_2}{\partial y}\right)_{y=b} & \text{for } 0 \leqslant x \leqslant a \\ 0 & \text{for } a < x \leqslant c \end{cases}$$

Multiplying through this equation by $\cos(k+\frac{1}{2})\frac{\pi x}{c}$ and integrating between $x = 0$ and $x = c$ gives

$$\frac{c}{2} h_k\left\{-(k+\tfrac{1}{2})\frac{\pi}{c}\right\} \sinh(k+\tfrac{1}{2})\frac{\pi}{c}(d-b) = \int_o^a \left(\frac{\partial A_1}{\partial y} + \frac{\partial A_2}{\partial y}\right)_{y=b} \cos(k+\tfrac{1}{2})\frac{\pi x}{c}\, dx$$

for $k = 0,1 \ldots K$

which, on substituting for $\frac{\partial A_1}{\partial y}$, $\frac{\partial A_2}{\partial y}$, becomes

$$\frac{c}{2} h_k\left\{-(k+\tfrac{1}{2})\frac{\pi}{c}\right\} \sinh(k+\tfrac{1}{2})\frac{\pi}{c}(d-b) =$$

$$\frac{4b^2}{\pi^4}\mu_r\mu_o J_{1z}^* \sum_{m=o}^{M}{}' \sum_{p=o}^{M-m} \frac{\alpha(m)\beta(p)\{-(p+\frac{1}{2})\frac{\pi}{b}\}(-1)^p I(m,k)}{\left\{PP(m,p) + i\left(\frac{\alpha b}{\pi}\right)^2\right\}}$$

$$+ \sum_{m=1}^{M} \left(\frac{m\pi}{a}\right) I(m,k)\left\{b_m \sinh\frac{m\pi b}{a} + e_m \cosh\frac{m\pi b}{a}\right\} + \frac{1}{2}\frac{e_o}{b} I(0,k) \qquad 6.7(7)$$

for $k = 0,1, \ldots K$. All the coefficients a_m, h_k can now be calculated in terms of J_{1z}^* and Ω. As in section 6.5(b) we now obtain two further equations by considering the two expressions for I_z^* as given in section 6.3.

6.7) contd.

(i) $$I_z^* = \int_{x=0}^{\alpha_2 a} \int_{y=\beta_1 b}^{\beta_2 b} J_z^* \, dx \, dy$$

$$= \int_{0}^{\alpha_2 a} \int_{\beta_1 b}^{\beta_2 b} \left\{ J_{1z}^* - \sigma i \omega (A_1 + A_2) \right\} dx \, dy \qquad 6.7(8)$$

After some considerable reduction this produces an equation in J_{1z}^* and Ω.

Also (ii) $$I_z^* = -\frac{1}{\mu_r \mu_0} \int_0^a \left(\frac{\partial A_z^*}{\partial y} \right)_{y=b} dx$$

$$= \frac{1}{\mu_r \mu_0} \sum_{k=0}^{K} h_k \left\{ (k+\tfrac{1}{2}) \frac{\pi}{c} \right\} \sinh (k+\tfrac{1}{2}) \frac{\pi}{c} (d-b) \int_0^a \cos (k+\tfrac{1}{2}) \frac{\pi x}{c} \, dx$$

$$= \frac{1}{\mu_r \mu_0} \sum_{k=0}^{K} h_k \sinh \left\{ (k+\tfrac{1}{2}) \frac{\pi}{c} (d-b) \right\} \sin (k+\tfrac{1}{2}) \frac{\pi a}{c} \qquad 6.7(9)$$

Briefly, the method of solution of these equations is to eliminate the coefficients a_m using equations 6.7(4) and (6) (and the equations giving b_m, e_m in terms of a_m); to eliminate Ω using equations 6.7(6), (7) and (8); and then to solve the resulting equations for h_k, $k = 0, 1, \ldots K$ and J_{1z}^*.

It should be observed here that if the mathematical model were exactly true, the problem would now be mathematically over-defined. In fact, we are again adjusting the parameters Ω and J_{1z}^* to give us the correct total current flowing in the conductor and so compensating in some measure for the assumed form of differential equation in the insulator. For the simpler configuration of section 6.5 we showed that a good estimate of the internal impedance could be obtained in this way.

6.8) Evaluation of complex impedance.

Following the reasoning of section 6.4 and taking as the volume V, unit length of the slot, cross-section OABC,

$$-I_z^* \tilde{I}_z^*(R+iX) = \frac{1}{\mu_r\mu_o} \int_0^a \left(\frac{J_{1z}^*}{\sigma} - i\omega A_z^*\right)_{y=b} \left(\frac{\partial \tilde{A}_z^*}{\partial y}\right)_{y=b} dx \qquad 6.8(1)$$

The terms in the integrand may be calculated using equation 6.7(2) and then the determination of R and X follows. The resistance at zero frequency is

$$R_o = \frac{1}{\sigma ab \ \ \alpha_2(\beta_2 - \beta_1)} \qquad 6.8(2)$$

Thus the ratios $^R/R_o$, $^X/R_o$ can be calculated. L_o, the inductance at zero frequency, is taken to be the same as L evaluated at the frequency given by $\alpha b = 0.5$. Figure 21 shows that this is a reasonable approximation.

6.9) Comparison with Silvester's results.

The solution was obtained with the insulation width on three sides of the conductor, as (a) $\frac{1}{6}$ and (b) $\frac{1}{12}$ of the conductor width. We also considered the case of the conductor filling the slot. The height to width ratio of the conductor is 1.5 and the width of the air gap is half the conductor width in all cases. All these dimensions are as quoted in(14). From Silvester's paper it is not possible to deduce the values of the parameters β_2 and $^c/a$. From his diagrams it would appear that $\beta_2 = 1$ and we have chosen to take $\frac{c}{a} = 3$. The problem was solved truncating the infinite series at (i) M = K = 15

and (ii) M = K = 20.

On an I.C.L. 1903A computer, the computing times

6.9) contd.

taken to obtain the coefficients h_k, $k = 0,1 \ldots K$, the parameters Ω and J^*_{1z} and the ratios R'/R_o and X/R_o for a range of frequency given by $\alpha b = 0.5(1)\ 7.5$ are

(i) for $M = K = 15$, 20 minutes

(ii) for $M = K = 20$, 35 minutes.

Although these times seem long, Silvester himself quotes times of the order of minutes for each frequency considered.

For all insulation thicknesses considered, in going from $M = K = 15$ to $M = K = 20$, R/R_o changes by less than 0.2% and X/R_o changes by less than $1\frac{1}{2}$% at the frequency given by $\alpha b = 7.5$. For lower frequencies these differences are reduced. The solution was deemed to have converged to the required accuracy with $M = K = 20$.

Figures 25 and 26 show the variation of R/R_o and $(L_o - L)$ with frequency. The parameter p is the frequency parameter used by Silvester and is given by

$$p = \sqrt{\frac{\mu_r \mu_o \omega}{\pi R_o}} = (\alpha b) \sqrt{\frac{2\alpha_2(1-\beta_1)}{\pi(b/a)}}$$

For the conductor filling the slot, the results agree well with those of Silvester (Figures 7 and 8 in[14], which are enclosed in a pocket at the back of the thesis). However with some insulation present, the results do not exhibit the same features as those of Silvester and would appear to substantiate the generally held assumption that the presence of insulation does not materially alter the performance of the conductor. With insulation thickness $\frac{1}{6}$ of the conductor width the resistance ratio is reduced by 12% and $(L_o - L)$ is reduced by 20% approximately at the higher frequencies.

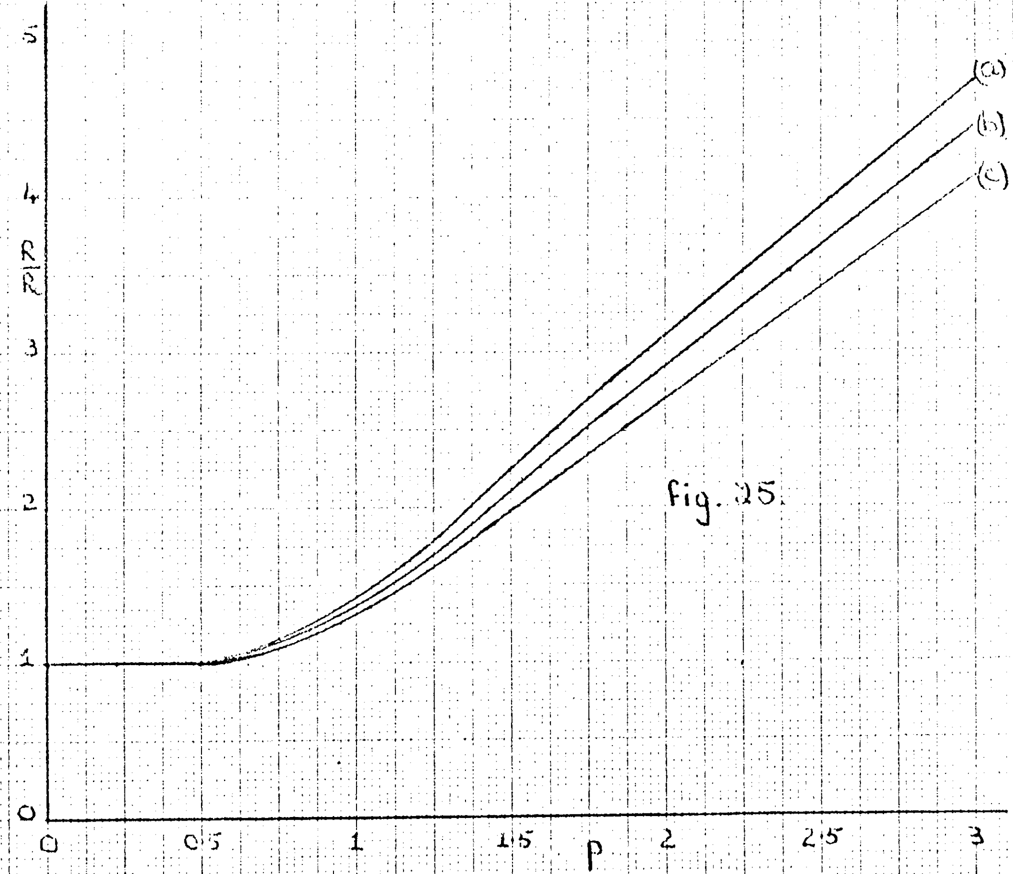


fig. 25.

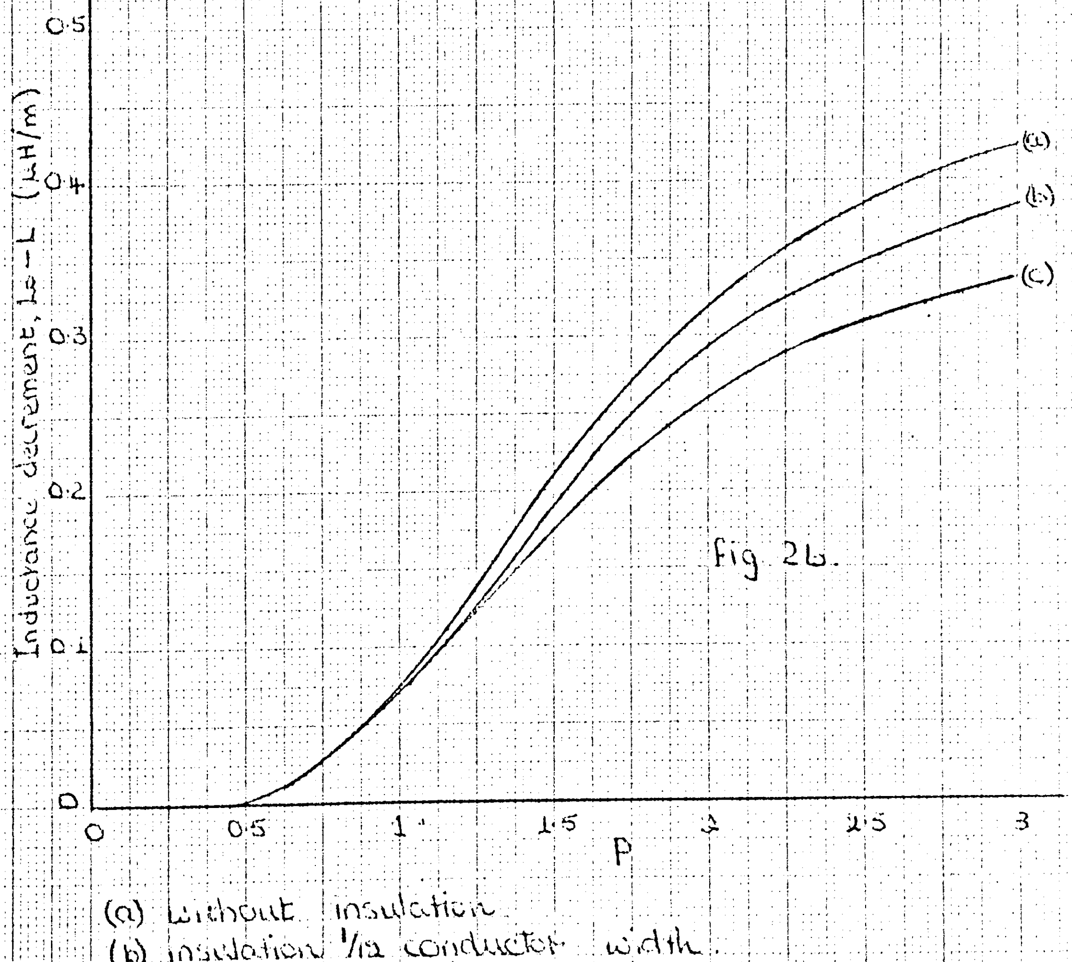


fig 26.

(a) without insulation
(b) insulation 1/12 conductor width.
(c) insulation 1/6 conductor width.

6.10) Conclusions.

We have shown how Roth's methods can be applied when the current in the conductor varies sinusoidally with time. The exact Roth method described here uses an iterative technique to solve the equations for the Fourier coefficients. This is a lengthy computational exercise and so a simpler approximate model has been developed and is shown to give good agreement with the exact solution. This approximate model is then used to solve the problem of an insulated conductor in a slot facing an air gap as given in(14).

The results of this chapter add weight to the arguments that the presence of insulation does not greatly affect the effective impedance of the conductor. This is contrary to the conclusions reached by Silvester in(14). The methods derived here can be used to give a quantitative estimate of the effects of the insulation at different frequencies and thus provide a valuable extension of the work of Roth in an area where his methods have not been applied before.

C H A P T E R 7.

RECTANGULAR CONDUCTOR IN A TRANSVERSE MAGNETIC FIELD.

7.1) Introduction.

In[15], Stoll obtained the eddy-current loss produced in a long conductor of rectangular cross-section by a transverse magnetic field which varies sinusoidally with time. The field is uniform and perpendicular to one side of the conductor. In this chapter we solve the same problem using Roth's method. The purpose of doing this is twofold. Firstly, it shows that Roth's method can be applied to problems other than those associated with insulated conductors in slots and secondly, the results obtained using Roth's method can be critically compared with those obtained by Stoll in[15]. Roth's exact method of solution as described in Chapter 6 must be used for this problem since the cross-sectional area of the non-conducting region is no longer small.

7.2) Description of the problem.

The configuration is as shown in Figure 27.

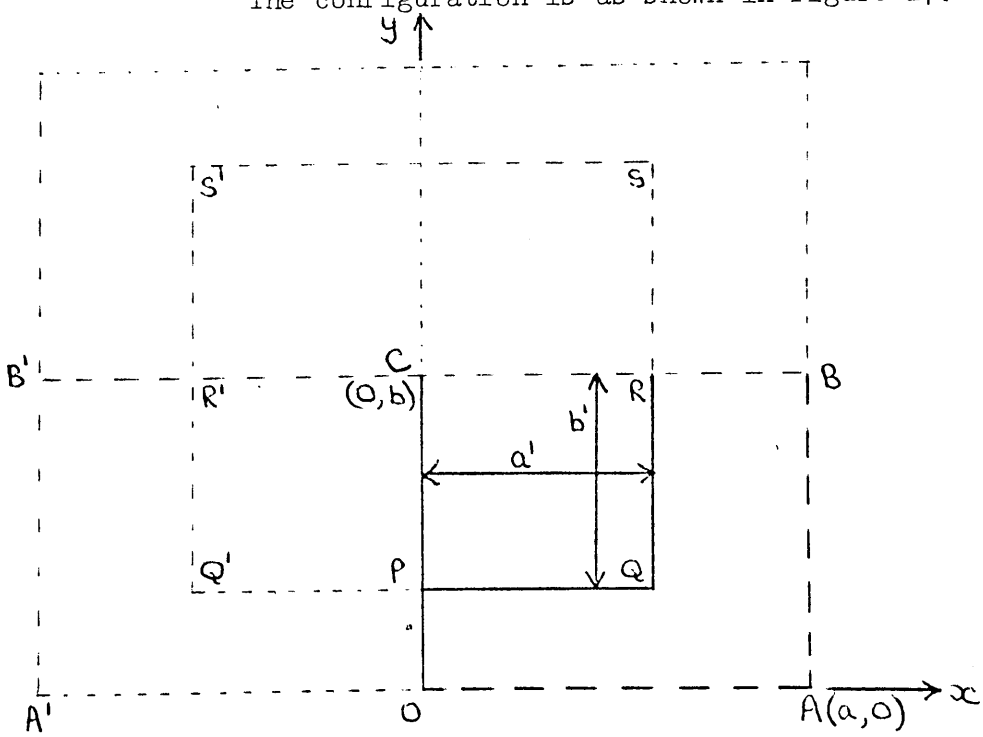


Figure 27.

7.2) contd.

QSS'Q' is a conductor with sides of lengths 2a' and 2b' as shown, situated in a uniform magnetic field

$$\underline{H} = R\ell\left(e^{i\omega t}\ \underline{H}^*\right) \qquad 7.2(1)$$

where $\underline{H}^* = (-H_x^*,\ 0,\ 0)$ 7.2(2)

the directions of the axes being as shown in the diagram. From symmetry we need only consider the solution in the region OABC and we also assume that there is no externally applied current in the conductor i.e. $J^*_{1z} \equiv 0$. The differential equations to be satisfied are then

$$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z^*}{\partial y^2} - i\alpha^2 A_z^* = 0 \text{ in the conducting region PQRC}$$

and $$\frac{\partial^2 A_z^*}{\partial x^2} + \frac{\partial^2 A_z^*}{\partial y^2} = 0 \text{ elsewhere in the region OABC}$$

(referring back to section 1.9).

The boundary conditions are as follows:-

(i) $\dfrac{\partial A_z^*}{\partial x} = 0$ when $x = 0,\ 0 \leqslant y \leqslant b$;

(ii) $A_z^* = 0$ when $y = b,\ 0 \leqslant x \leqslant a$,

(iii) $\dfrac{\partial A_z^*}{\partial x} = 0$ when $x = a,\ 0 \leqslant y \leqslant b$;

(iv) $\dfrac{\partial A_z^*}{\partial y} = -\mu_r\mu_o H_x^*$ when $y = 0,\ 0 \leqslant x \leqslant a$.

Boundary conditions (i) and (ii) follow from symmetry and conditions (iii) and (iv) follow from the result that

$$\underline{B}^* = \text{curl } \underline{A}^*$$

$$= \left(\frac{\partial}{\partial y} A_z^*,\ -\frac{\partial}{\partial x} A_z^*,\ 0\right)$$

$$= \mu_r\mu_o\left(-H_x^*,\ 0,\ 0\right).$$

It should be observed here that the boundary surface OAB should be sufficiently far from the conductor for the magnetic field

7.2) contd.

of the eddy currents to be negligible on OAB. This means that the area of the non-conducting region is unlikely to be small so that we must use Roth's exact method of solution.

7.3) Method of solution.

To render the problem suitable for solution by Roth's method write

$$A_z^* = B_z^* + \mu_r \mu_o (b-y) H_x^* \qquad 7.3(1)$$

The differential equations to be satisfied by B_z^* are then

$$\frac{\partial^2}{\partial x^2} B_z^* + \frac{\partial^2}{\partial y^2} B_z^* - i\alpha^2 B_z^* = i\alpha^2 \mu_r \mu_o (b-y) H_x^*$$

in the conducting region PQRC

and

$$\frac{\partial^2}{\partial x^2} B_z^* + \frac{\partial^2}{\partial y^2} B_z^* = 0 \text{ elsewhere in the region OABC.}$$

The boundary conditions on B_z^* are

(i) $\dfrac{\partial B_z^*}{\partial x} = 0$ when $x = 0,\ 0 \leqslant y \leqslant b$

(ii) $B_z^* = 0$ when $y = b,\ 0 \leqslant x \leqslant a$

(iii) $\dfrac{\partial B_z^*}{\partial x} = 0$ when $x = a,\ 0 \leqslant y \leqslant b$

(iv) $\dfrac{\partial B_z^*}{\partial y} = 0$ when $y = 0,\ 0 \leqslant x \leqslant a$.

Assume a solution for B_z^* in the form

$$B_z^* = \sum_{m=o}^{\infty}{}' \sum_{k=o}^{\infty} C_{mk} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \qquad 7.3(2)$$

which automatically satisfies the boundary conditions. To satisfy the differential equation

7.3) contd.

$$\sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty} C_{mk} \left\{ \frac{m^2\pi^2}{a^2} + (k+\tfrac{1}{2})^2 \frac{\pi^2}{b^2} + i\alpha^2 \right\} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b}$$

$$= \begin{cases} -\, i\alpha^2 \ \mu_r\mu_o(b-y)H_x^* & \text{in PQRC} \\ i\alpha^2 \displaystyle\sum_{p=0}^{\infty}{}' \sum_{q=0}^{\infty} C_{pq} \cos\frac{p\pi x}{a} \cos(q+\tfrac{1}{2})\frac{\pi y}{b} & \text{elsewhere in OABC} \end{cases}$$

Multiplying through this equation by $\cos\frac{m\pi x}{a} \cos(k+\frac{1}{2})\frac{\pi y}{b}$

and integrating over the whole area $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$

gives

$$PP(m,k)\ C_{mk} + i\left(\frac{\alpha b}{\pi}\right)^2 \sum_{p=0}^{\infty}{}' \sum_{q=0}^{\infty} C_{pq}\ AA(p,m)\ BB(q,k)$$

$$= 4i\left(\frac{\alpha b}{\pi}\right)^2 \left(\mu_r\mu_o\ H_x^*\ b\right) \alpha(m)\ \beta(k) \qquad 7.3(3)$$

$$m = 0,1,2\ldots,\ k = 0,1,2\ \ldots$$

where PP(m,k) is given by equation 6.3(4),

$$AA(p,m) = AA(m,p)$$

$$= \frac{1}{(p+m)\pi} \sin(p+m)\frac{a'}{a}\pi$$

$$+ \frac{1}{(p-m)\pi} \sin(p-m)\frac{a'}{a}\pi \qquad (p \neq m)$$

$$AA(p,p) = \frac{1}{2p\pi} \sin 2p\frac{a'}{a}\pi + \frac{a'}{a} \qquad (p \neq 0)$$

$$AA(0,0) = 2\frac{a'}{a} \qquad 7.3(4)$$

$$BB(q,k) = BB(k,q)$$

$$= \frac{-1}{(q+k+1)\pi} \sin(q+k+1)(1 - {b'}/{b})\pi$$

$$- \frac{1}{(q-k)\pi} \sin(q-k)(1 - {b'}/{b})\pi \qquad (q \neq k)$$

$$BB(q,q) = -\frac{1}{(2q+1)\pi} \sin(2q+1)(1 - {b'}/{b})\pi + {b'}/{b} \qquad 7.3(5)$$

7.3) contd.

$$\left.\begin{aligned}\alpha(m) &= \frac{1}{m\pi}\sin\frac{m\pi a'}{a} \qquad (m \neq 0)\\ \alpha(0) &= \frac{a'}{a}\end{aligned}\right\} \qquad 7.3(6)$$

$$\beta(k) = \frac{(-1)^k}{(k+\frac{1}{2})\pi}\left\{\frac{b'}{b}\cos(k+\tfrac{1}{2})\frac{\pi b'}{b} - \frac{1}{(k+\frac{1}{2})\pi}\sin(k+\tfrac{1}{2})\frac{\pi b'}{b}\right\} \qquad 7.3(7)$$

Truncating the series for B_z^* at m = M and summing by diagonals as before, equation 7.3(3) may be written (assuming that H_x^* is real)

$$PP(m,k)R\ell(C_{mk}) - \left(\frac{\alpha b}{\pi}\right)^2 \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} AA(p,m)BB(q,k)\ Im(C_{pq})$$

$$= 0 \qquad 7.3(8)$$

$$PP(m,k)Im(C_{mk}) + \left(\frac{\alpha b}{\pi}\right)^2 \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} AA(p,m)BB(q,k)R\ell(C_{pq})$$

$$= 4\left(\frac{\alpha b}{\pi}\right)^2 (\mu_r\mu_o H_x^* b)\ \alpha(m)\ \beta(k) \qquad 7.3(9)$$

for m = 0,1,2 .... M

k = 0,1 .... (M-m)

Again we have a coupled set of equations and the method of solution is described in Appendix 4. When the coefficients C_{mk} are known then A_z^* is given by equation 7.3(1) as

$$A_z^* = \mu_r\mu_o(b-y)H_x^* + \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} C_{mk}\cos\frac{m\pi x}{a}\cos(k+\tfrac{1}{2})\frac{\pi y}{b} \qquad 7.3(10)$$

7.4) <u>Eddy-current loss.</u>

Following the lines of[15], the eddy-current loss (E.C.L.) is given by

$$E.C.L. = \frac{1}{2\sigma}\int_{x=o}^{a'}\int_{y=b-b'}^{b} |J_{2z}|^2\ dx\ dy$$

7.4) contd.

$$= \frac{\omega^2\sigma}{2} \int_0^{a'} \int_{b-b'}^{b} |A_z^*|^2 \, dx \, dy \qquad 7.4(1)$$

since $J_{2z} = -\, i\omega\sigma A_z^*$

It should be observed here that the eddy-current loss could be calculated using the Poynting vector as in Chapter 6 but in order to compare the results with those obtained by Stoll, it was felt to be better to use the same expression as given in(15).

Writing $C_{mk} = \mu_r\mu_o b H_x^*, D_{mk}, \; m = 0,1 \ldots M; k=0,1\ldots(M-m)$ 7.4(2)

$$A_z^* = \mu_r\mu_o b H_x^* \left\{ 1 - \frac{y}{b} + \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} D_{mk} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \right\} \qquad 7.4(3)$$

$$\text{and } |A_z^*|^2 = \left\{ \mu_r\mu_o b H_x^* \right\}^2 f\left(\frac{x}{a}, \frac{y}{b}\right) \qquad 7.4(4)$$

$$\text{where } f\left(\frac{x}{a}, \frac{y}{b}\right) = \left\{ 1 - \frac{y}{b} + \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} R\ell(D_{mk}) \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \right\}^2$$

$$+ \left\{ \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} Im(D_{mk}) \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \right\}^2 \qquad 7.4(5)$$

$$\text{If E.C.L.} = \frac{(H_x^*)^2}{\sigma} F\left(\frac{a'}{\delta}, \frac{b'}{\delta}\right) \qquad 7.4(6)$$

$$\text{where the skin depth } \delta = \frac{\sqrt{2}}{\alpha} \qquad 7.4(7)$$

then

$$F\left(\frac{a'}{\delta}, \frac{b'}{\delta}\right) = \frac{(\alpha b)^4}{2} \left(\frac{a}{b}\right) \int_{\frac{x}{a}=0}^{\frac{a'}{a}} \int_{\frac{y}{b}=1-\frac{b'}{b}}^{1} f\left(\frac{x}{a}, \frac{y}{b}\right) d\left(\frac{x}{a}\right) d\left(\frac{y}{b}\right) \qquad 7.4(8)$$

Note that this gives the eddy-current loss over one quarter of the conductor area. The total eddy-current loss is given by $4F\left(\frac{a'}{\delta}, \frac{b'}{\delta}\right)$.

7.4) contd.

Now
$$\int_0^{\frac{a'}{a}} \int_{1-\frac{b'}{b}}^{1} \left\{ \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} Im(D_{mk}) \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \right\}^2 d\left(\frac{x}{a}\right) d\left(\frac{y}{b}\right)$$

$$= \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} \left\{ Im(D_{mk}) Im(D_{pq}) \int_0^{\frac{a'}{a}} \cos\frac{m\pi x}{a} \cos\frac{p\pi x}{a} d\left(\frac{x}{a}\right) \times \right.$$
$$\left. \left(\int_{1-\frac{b'}{b}}^{1} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \cos(q+\tfrac{1}{2})\frac{\pi y}{b} d\left(\frac{y}{b}\right) \right) \right\}$$

$$= \frac{1}{4} \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} Im(D_{mk}) Im(D_{pq}) AA(m,p) BB(q,k)$$

The remaining integrals are evaluated using similar techniques to give

$$F\left(\frac{a'}{\delta}, \frac{b'}{\delta}\right) = \frac{(\alpha b)^4}{2} \left(\frac{a}{b}\right) \left\{ \frac{1}{4} \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} AA(m,p) BB(q,k) \begin{bmatrix} Re(D_{mk}) Re(D_{pq}) \\ + Im(D_{mk}) Im(D_{pq}) \end{bmatrix} \right.$$
$$\left. + \frac{1}{3}\left(\frac{a'}{a}\right)\left(\frac{b'}{b}\right)^3 - 2 \sum_{m=o}^{M}{}' \sum_{k=o}^{M-m} Re(D_{mk}) \alpha(m) \beta(k) \right\} \quad 7.4(9)$$

where $\alpha(m)$, $\beta(k)$ are defined by equations 7.3(6) and (7).

It should be observed here that if $\frac{a'}{\delta}$, $\frac{b'}{\delta}$ are small (i.e. $\alpha a'$, $\alpha b'$ are small) then from equation 7.3(3) the coefficients C_{mk} are negligibly small and in this case

$$F\left(\frac{a'}{\delta}, \frac{b'}{\delta}\right) \doteqdot \frac{(\alpha b)^4}{2} \left(\frac{a}{b}\right) \frac{1}{3}\left(\frac{a'}{a}\right)\left(\frac{b'}{b}\right)^3$$
$$= \frac{1}{6} \alpha^4 (a')(b')^3$$
$$= \frac{1}{6} \cdot \frac{4}{\delta^4} a'(b')^3$$

Thus the total eddy current loss is given by

7.4) contd.

$$4\,F\left(\frac{a'}{\delta}\,,\,\frac{b'}{\delta}\right) = \frac{1}{6}\left(\frac{2a'}{\delta}\right)\left(\frac{2b'}{\delta}\right)^3$$

the well known result quoted in[15].

7.5) Results and Conclusions.

The solution was evaluated taking

$a = a' + K_1(2b')$ and

$b = b' + K_2(2b')$.

By varying K_1 and K_2 the south and east boundary surfaces can be moved independently away from the conductor until the magnetic field of the eddy currents is negligible on the outer boundary surface.

We consider first a square conductor given by the parameters $\frac{2a'}{\delta} = 2$, $\frac{2b'}{\delta} = 2$, and we assume initially that $K_1 = K_2$. The series solution for $F\left(\frac{a'}{\delta}, \frac{b'}{\delta}\right)$ changes by less than 0.5% as M is increased from 15 to 20. Figure 28 shows the variation of F with K_1, K_2. Referring to Figure 4 of[15] (Figures 4 and 6 of[15] are enclosed in a pocket at the back of the thesis), Stoll states that the clearance between the east boundary and the conductor should be greater than or equal to $1.5(2b')$ i.e. $K_1 \geq 1.5$ for the magnetic field of the eddy-currents to be negligible on the outer boundary. Figure 28 indicates that $K_1, K_2 \geq 2.5$ might be a better estimate in this case.

Keeping $\frac{2a'}{\delta} = 2$, $\frac{2b'}{\delta} = 2$ we now allow K_1, K_2 to vary independently. Figure 29 shows the variation of F in the two cases

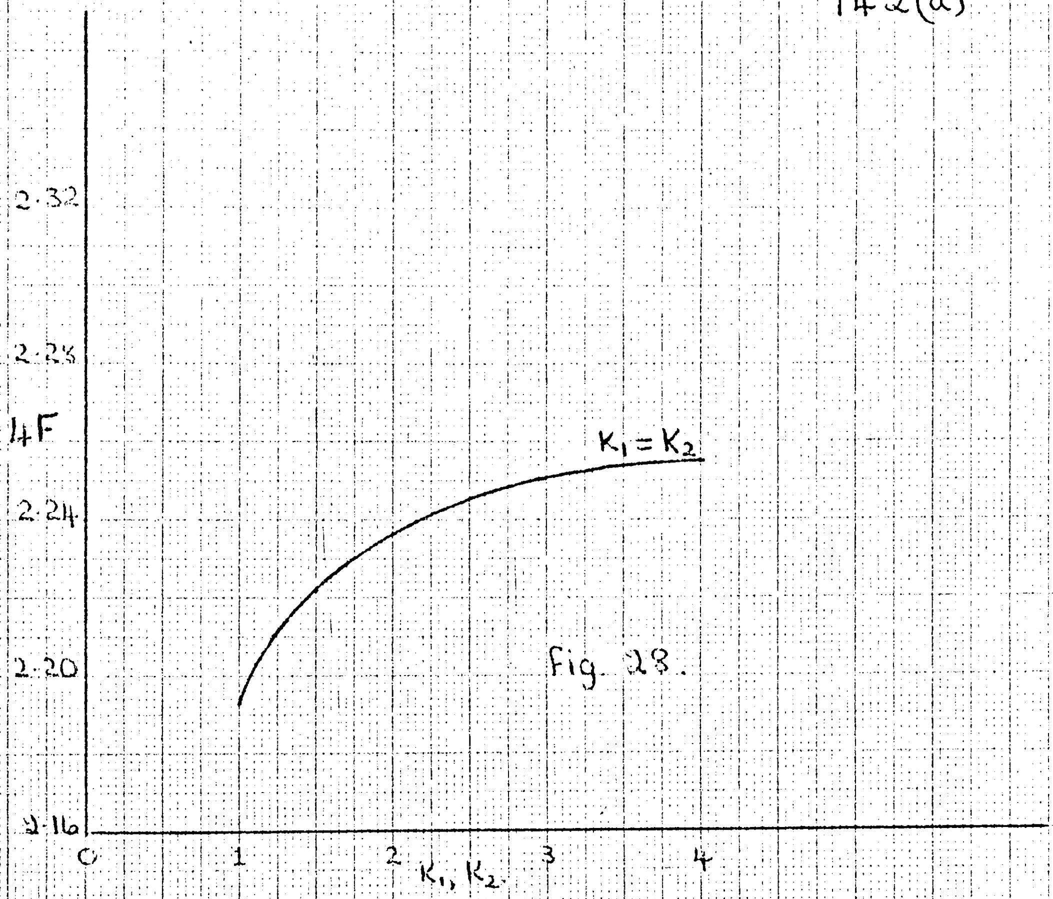


Fig. 28.

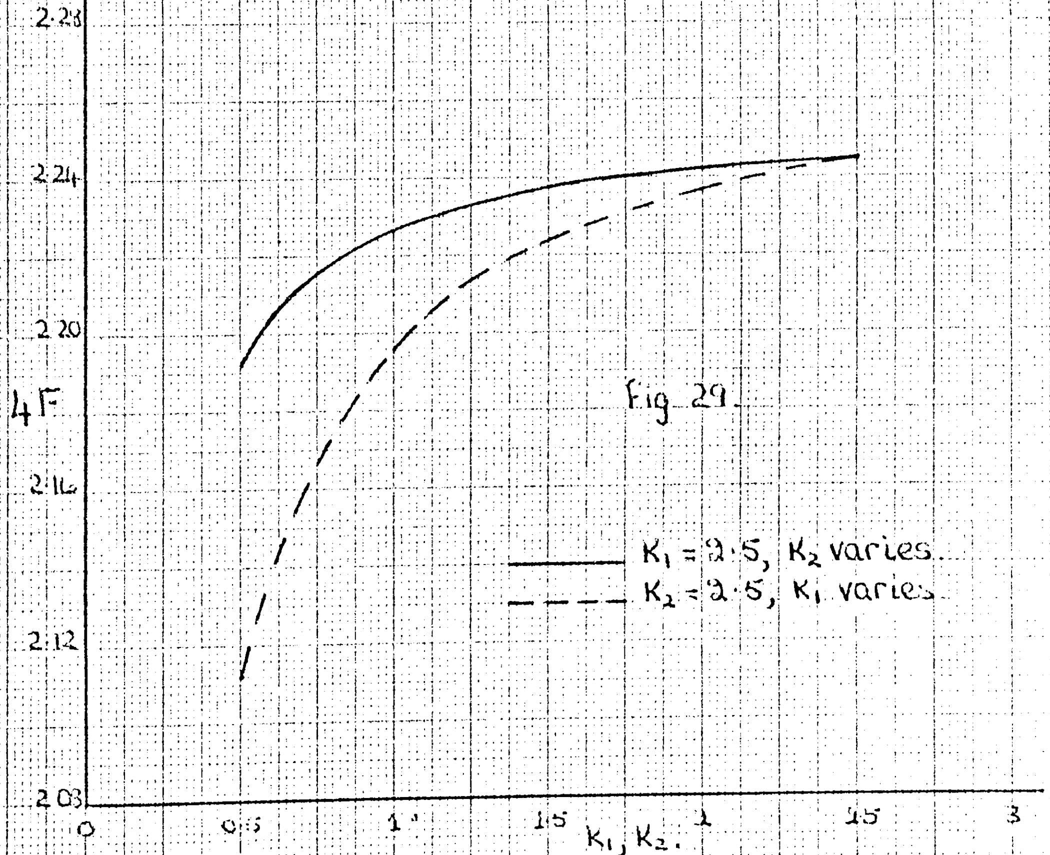


Fig. 29.

$$\frac{2a'}{\sigma} = 2, \quad \frac{2b'}{\sigma} = 2$$

7.5) contd.

(a) $K_1 = 2\frac{1}{2}$, K_2 varying

(b) $K_2 = 2\frac{1}{2}$, K_1 varying.

Again, all the solutions are changing by less than 0.5% as M is increased from 15 to 20. From the graph it can be seen that the position of the east boundary is the most critical so verifying the statement made by Stoll in[15]. Also from Figures 28 and 29, it would appear to be unnecessary to take $K_1 = K_2$. For agreement to within 0.5%, we may take $K_1 = 2\frac{1}{2}$, $K_2 = 1\frac{1}{2}$ or in general, a value of K_2 significantly less than that of K_1. The advantage of doing this will become apparent from the results of table 10 which illustrates the rate of convergence of the series solution in the two cases (a) $K_1 = K_2$ and (b) $K_2 < K_1$ and also the degree of agreement between the answers in the two cases.

	M \ $^{2a'}/\delta$	1	2	4	5	6	
(a)	15	1.233	2.263	4.110	4.989	5.853	$K_1 = K_2 = 4$
	20	1.227	2.255	4.095	4.970	5.831	
(b)	15	1.227	2.255	4.085	4.955	5.809	$K_1 = 4$, $K_2 = 2\frac{1}{2}$
	20	1.225	2.252	4.080	4.949	5.802	

TABLE 10

(All values quoted correspond to $^{2b'}/\delta = 2$)

From the table the following points should be noted

(i) the agreement between the two sets of figures (a) and (b) is within 0.5%

(ii) taking $K_1 = K_2 = 4$ the solution changes by less than 0.4% as M is increased from 15 to 20, but for case (b) this change is less than 0.2% indicating that the lower the value of K_2 the more rapid is the rate of convergence

7.5) contd.

(ii) contd.

of the solution. Ideally then we choose $K_2 < K_1$, with both values as low as possible but sufficiently large for the magnetic field of the eddy currents to be negligible on the outer boundary surface.

Figure 30 shows the variation of F with $2a'/\delta$ taking $2b'/\delta = 2$, $K_1 = 4$, $K_2 = 2\frac{1}{2}$. This is seen to give good agreement with the appropriate curve of Figure 6 of[15].

We next take a higher value of b' namely $2b'/\delta = 4$ and determine whether the conclusions rached up to now still apply. Taking $K_1 = K_2$, Table 11 illustrates both the rate of convergence of the solution and also the effect of moving the outer boundaries further from the conductor.

M \ $K_1 = K_2 =$	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
22	9.962	10.23	10.38				10.76
25		10.22	10.37	10.47	10.55	10.61	10.68
32				10.45	10.51	10.56	10.61

(Results given for $\frac{2a'}{\delta} = 3$, $\frac{2b'}{\delta} = 4$)

TABLE 11.

From the table it can be seen that

(i) the rate of convergence of the series for a given accuracy is poor and becomes worse the higher the values of K_1, K_2

(ii) for this higher value of b' it is necessary that the clearance between the east boundary and the conductor should be at least $3\frac{1}{2}(2b')$ if the effect of the magnetic field of the eddy currents on the

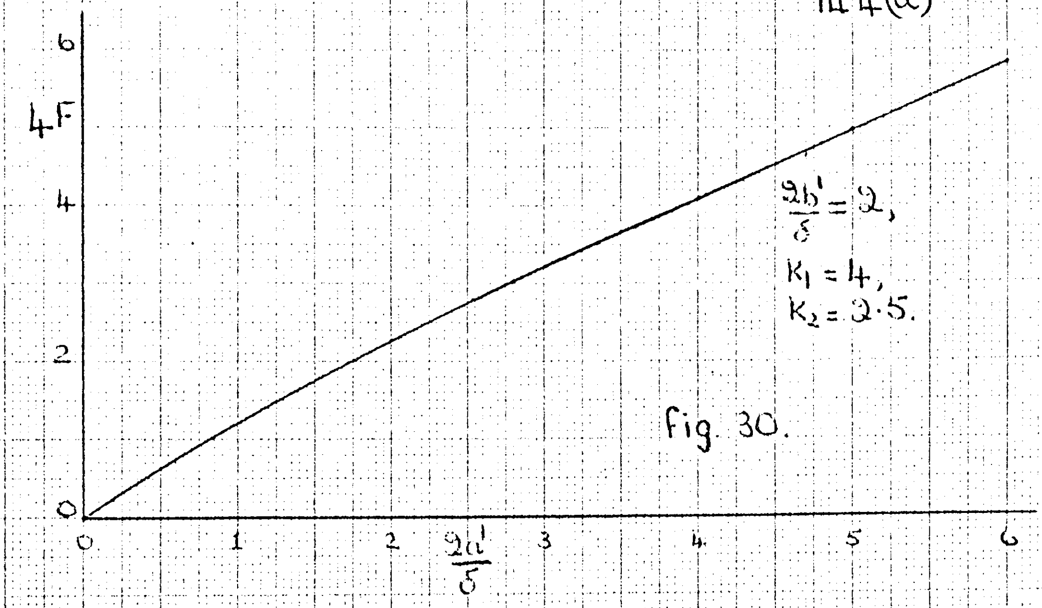

4F
6
4
2
0
0
1
2
3
4
5
6
2a'/δ
2b'/δ = 2,
K1 = 4,
K2 = 2·5.

Fig. 30.

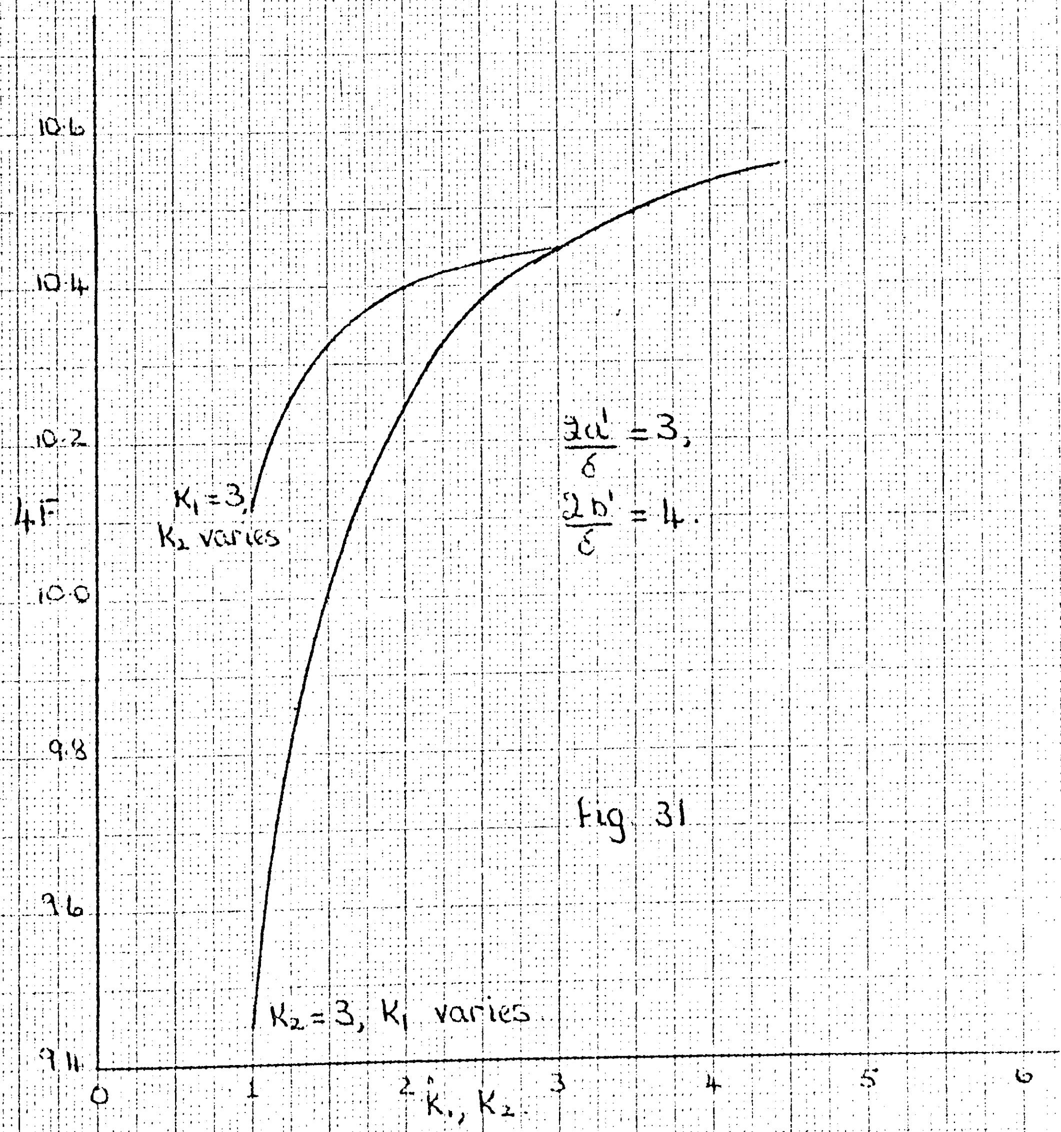

10·6
10·4
10·2
10·0
9·8
9·6
9·4
4F
K1 = 3,
K2 varies
K2 = 3, K1 varies
2a'/δ = 3,
2b'/δ = 4.
0
1
2
3
4
5
6
K1, K2

Fig. 31

7.5) contd.

(ii) contd.

outer boundary is to be less than or equal to 0.5%

(iii) taking $K_1 = K_2 = 1.5$ gives $4F\left(\frac{a'}{\delta}, \frac{b'}{\delta}\right) = 9.962$ which agrees well with the appropriate point on the curves of Figure 6 of[15]. However it is felt that a higher value of K_1, K_2 is required in this case with the corresponding $4F$ at 10.6 i.e. some 6% higher than the value given by Stoll.

It has been pointed out that the rate of convergence of the series solution is poor at the higher values of K_1, K_2. It is hoped that as in the earlier case i.e. $2b'/\delta = 2$ we may reduce K_2 so improving the rate of convergence but without materially affecting the absolute values of $F\left({}^{a'}/\delta, {}^{b'}/\delta\right)$. Figure 31 shows the variation of F in the two cases

(a) $K_1 = 3$, K_2 varying

(b) $K_2 = 3$, K_1 varying

with ${}^{2a'}/\delta = 3$, ${}^{2b'}/\delta = 4$. All the solutions are changing by less than 0.5% as M is increased from 25 to 30. Again the position of the east boundary is the most critical and we may take K_2 significantly less than K_1. Table 12 illustrates the rate of convergence of the series solution when $K_1 = 4\frac{1}{2}$, $K_2 = 3$.

M \ $2a'/\delta$	$\frac{1}{2}$	1	2	3	4	5
20		7.186	9.260	10.676	11.921	13.167
25	4.627	7.059	9.214	10.586	11.859	13.094
30	4.594	7.009				

TABLE 12.

(All values quoted correspond to ${}^{2b'}/\delta = 4$.)

7.5) contd.

By comparing the entry for $2a'/\delta = 3$ in this table with the last column of Table 11 it will be seen that with the lower value of K_2 the rate of convergence is again improved while the value obtained for F is not significantly different.

Figure 32 shows the variation of $4F$ with $2a'/\delta$ taking $2b'/\delta = 4$, $K_1 = 4\frac{1}{2}$, $K_2 = 3$. The graph agrees well with the curve given in Figure 6 of[15] at the lower values of a'/δ but gives a value of F some 10% greater than that given by Stoll at $2a'/\delta = 5$. As has previously been pointed out this could be due to the outer boundary surface being too near the conductor in Stoll's calculations.

To complete the results, Figure 33 shows the variation of F when

(a) $K_1 = 3$, K_2 varies and

(b) $K_2 = 3$, K_1 varies

with $2a'/\delta = 4$, $2b'/\delta = 3$. In this case all the solutions are changing by less than 0.5% as M is increased from 20 to 25. The position of the east boundary is again the most critical and we may take K_2 less than K_1. Also the results show that the lower the value of K_2, the faster is the rate of convergence of the series solution. Thus we have shown in the three cases (i) $a' < b'$ (ii) $a' = b'$ and (iii) $a' > b'$ that the position of the east boundary is the most critical.

In this chapter we have described a further application of Roth's method and the results so obtained compare favourably with those obtained by Stoll in[15] using finite difference methods. For problems relating to insulated conductors in an applied field varying sinusoidally with time, we have shown that Roth's method is both powerful

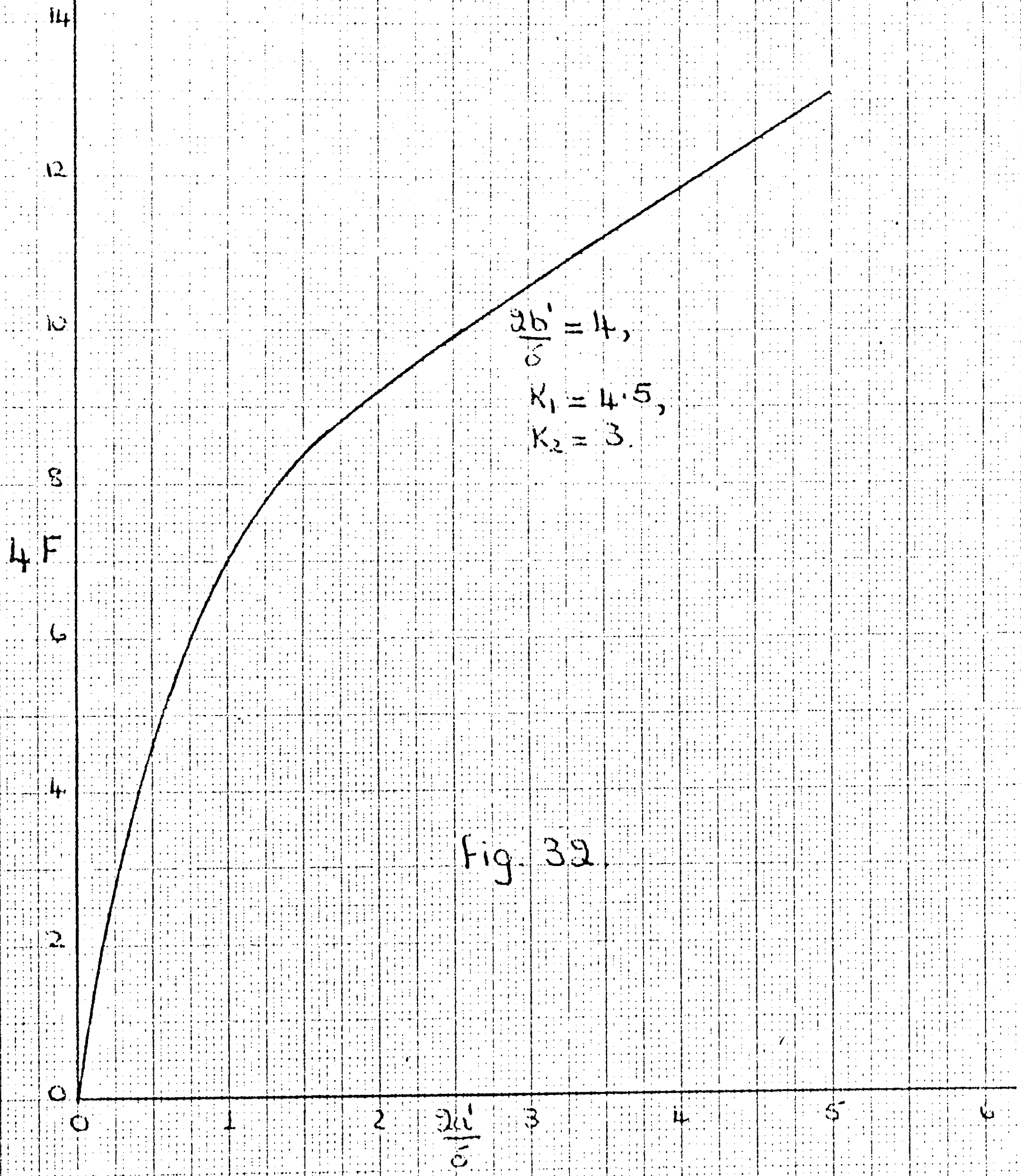

14
12
10
8
6
4
2
0
4F
$\frac{2b'}{\delta} = 4,$
$K_1 = 4\cdot5,$
$K_2 = 3.$
0
1
2
3
4
5
6
$\frac{2a'}{\delta}$

Fig. 32.

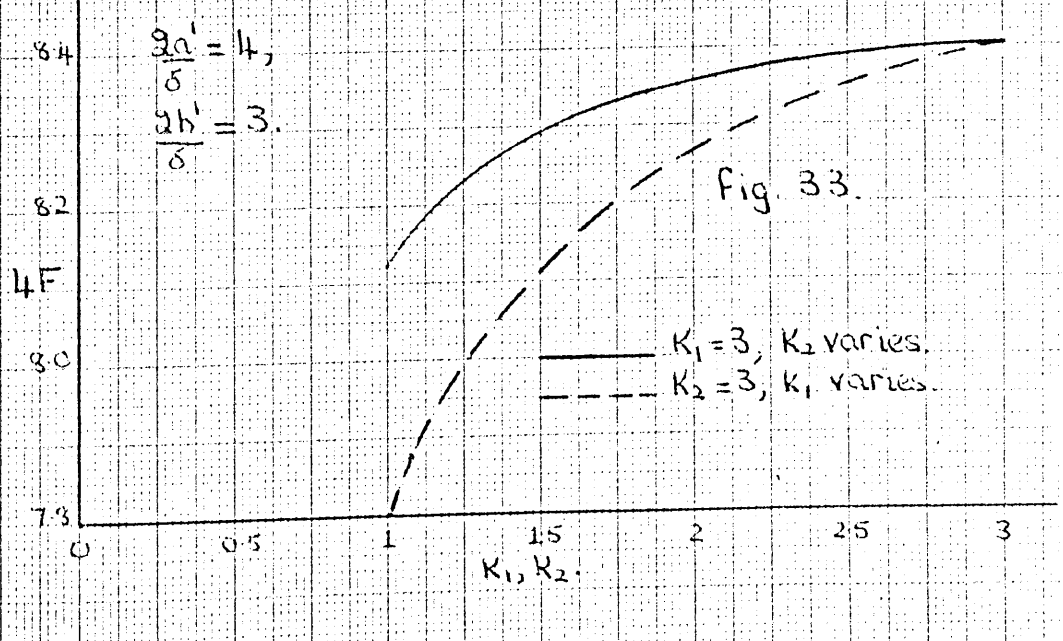

$\frac{2a'}{\delta} = 4,$
$\frac{2b'}{\delta} = 3.$
8·4
8·2
8·0
7·8
4F
$K_1 = 3$, K_2 varies.
$K_2 = 3$, K_1 varies.
0
0·5
1
1·5
2
2·5
3
K_1, K_2.

Fig. 33.

7.5) contd.

and straightforward in application. In addition, the form of solution for A_z^* so produced, being a double Fourier series, is both simple and also very convenient for the calculation of derived quantities e.g. complex impedance and magnetic field components. To the present, Roth's method has only been used for steady state fields and so the techniques described in Chapters 6 and 7 represent a major step forward in the study of possible further developments in the application of double Fourier series to the solution of partial differential equations.

CHAPTER 8.

FURTHER INVESTIGATION OF CHEBYSHEV METHODS APPLIED TO THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS.

8.1) Introduction.

Chapters 4 and 5 concluded that Chebyshev methods could not be recommended for the solution of differential equations of the form

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = f(x,y)$$

where $f(x,y)$ is a discontinuous function of x and y in the region under consideration. The resulting Chebyshev series approximation to $f(x,y)$ is an infinite series and there is, therefore, a truncation error in the solution for **A**.

The question remains as to whether there exist physical problems where Chebyshev methods of solution would be superior to other methods, for example, the method of separation of variables or Roth's method. We consider two special cases of the general elliptic differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial y^2} + 2H \frac{\partial^2 u}{\partial x \partial y} + 2G \frac{\partial u}{\partial x} + 2F \frac{\partial u}{\partial y} = - C \qquad 8.1(1)$$

$$(AB - H^2 > 0)$$

where A,B,C,F,G,H are finite polynomials in x and y.

The first example is given by

$$\begin{aligned} A &= B = 1, \\ F &= G = H = 0, \\ C &= 2 + \frac{x^2}{a^2} + \frac{y^2}{b^2} \end{aligned} \qquad 8.1(2)$$

(i.e. a Poisson equation with a polynomial source term) and we solve over the rectangular region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. The boundary conditions are assumed to be

$$\begin{aligned} &\text{(i)} \quad \frac{\partial u}{\partial x} = 0 \quad \text{when } x = 0,a \text{ for } 0 \leqslant y \leqslant b \\ &\text{(ii)} \quad \frac{\partial u}{\partial y} = 0 \quad \text{when } y = 0 \quad \text{for } 0 \leqslant x \leqslant a \\ &\text{(iii)} \quad u = 0 \quad \text{when } y = b \quad \text{for } 0 \leqslant x \leqslant a \end{aligned} \qquad 8.1(3)$$

With these boundary conditions, this configuration could,

8.1) contd.

of course, represent a conductor completely filling the rectangular slot with the steady state current density varying with position as given by the function C. The solution is obtained using the following three methods:-

(a) separation of variables

(b) Roth's method of double Fourier series

(c) double Chebyshev approximation

The solutions so obtained are then critically compared.

We next consider a more general example arising in cylindrical geometry, namely an infinitely long insulated conductor in an annular slot of cross-section $a \leq r \leq b$, $0 \leq \theta \leq \theta^*$ (with the usual notation). It is assumed that there is a steady axial current flowing in the conductor. The solution is obtained using a Fourier-Chebyshev approximation. To derive the solution by Roth's method or the method of separation of variables would involve the use of Bessel functions making these methods, if not impossible, then very cumbersome mathematically. This example not only serves to illustrate the power of the Chebyshev methods but also provides a quantitative estimate of the effects of neglecting curvature in our solutions for the rectangular slot.

8.2) Solution of equations 8.1(1), (2) and (3).

(a) Separation of variables method.

A particular integral of the equation is,

if $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$,

$$\frac{-1}{\{D^2+(D')^2\}} C(x,y) = \frac{1}{\{D^2+(D')^2\}}\left\{-2 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right\}$$

8.2) contd.

$$= \frac{1}{D^2(1 + (^{D'}/D)^2)} \left\{-1 - \frac{x^2}{a^2}\right\}$$

$$+ \frac{1}{(D')^2(1+(\frac{D}{D'})^2)} \left\{-1 - \frac{y^2}{b^2}\right\}$$

$$= -\frac{x^2}{2} - \frac{x^4}{12a^2} - \frac{y^2}{2} - \frac{y^4}{12b^2} \qquad 8.2(1)$$

Writing $u_0 = -\frac{x^2}{2} - \frac{x^4}{12a^2} - \frac{y^2}{2} - \frac{y^4}{12b^2}$, 8.2(2)

let $u = u^* + u_0$. 8.2(3)

Then the differential equation to be satisfied by u^* is

$$\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} = 0$$

subject to the boundary conditions

(i) $\frac{\partial u^*}{\partial x} = 0$ when $x = 0$, $0 \leqslant y \leqslant b$

(ii) $\frac{\partial u^*}{\partial x} = \frac{4a}{3}$ when $x = a$, $0 \leqslant y \leqslant b$

(iii) $\frac{\partial u^*}{\partial y} = 0$ when $y = 0$, $0 \leqslant x \leqslant a$

(iv) $u^* = \frac{x^2}{2} + \frac{x^4}{12a^2} + \frac{7b^2}{12}$ when $y = b$, $0 \leqslant x \leqslant a$

Now let $u^* = u_1 + u_2$

where $\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0$ and $\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$ and

(i) $\frac{\partial u_1}{\partial x} = 0$, $\frac{\partial u_2}{\partial x} = 0$ when $x = 0$, $0 \leqslant y \leqslant b$

(ii) $\frac{\partial u_1}{\partial x} = \frac{4a}{3}$, $\frac{\partial u_2}{\partial x} = 0$ when $x = a$, $0 \leqslant y \leqslant b$

(iii) $\frac{\partial u_1}{\partial y} = 0$, $\frac{\partial u_2}{\partial y} = 0$ when $y = 0$, $0 \leqslant x \leqslant a$

(iv) $u_1 = 0$, $u_2 = \frac{x^2}{2} + \frac{x^4}{12a^2} + \frac{7b^2}{12}$ when $y = b$, $0 \leqslant x \leqslant a$.

From boundary conditions (i), (iii) and (iv) the solution for u_1 is of the form

8.2) contd.

$$u_1 = \sum_{n=0}^{\infty} a_n \cos(n+\tfrac{1}{2})\frac{\pi y}{b} \cosh(n+\tfrac{1}{2})\frac{\pi x}{b} \qquad 8.2(4)$$

where the coefficients a_n, $n = 0,1,2 \ldots$ are to be determined from boundary condition (ii), i.e.

$$\frac{4a}{3} = \sum_{n=0}^{\infty} a_n(n+\tfrac{1}{2})\frac{\pi}{b} \sinh(n+\tfrac{1}{2})\frac{\pi a}{b} \cos(n+\tfrac{1}{2})\frac{\pi y}{b} \quad \text{for } 0 \leqslant y \leqslant b,$$

giving $\frac{4a}{3} \frac{b(-1)^n}{(n+\frac{1}{2})\pi} = \frac{b}{2} a_n(n+\tfrac{1}{2})\frac{\pi}{b} \sinh\left\{(n+\tfrac{1}{2})\frac{\pi a}{b}\right\}$

$$\text{i.e.} \quad a_n \sinh(n+\tfrac{1}{2})\frac{\pi a}{b} = \frac{8ab}{3} \frac{(-1)^n}{\{(n+\frac{1}{2})\pi\}^3} \qquad 8.2(5)$$

for $n = 0,1,2 \ldots$

Then u_1 is given by equations 8.2(4) and (5).

From boundary conditions (i), (ii) and (iii) the solution for u_2 is of the form

$$u_2 = \sum_{n=0}^{\infty}{}' \, b_n \cos\frac{n\pi x}{a} \cosh\frac{n\pi y}{a} \qquad 8.2(6)$$

and the coefficients b_n, $n = 0,1,2 \ldots$ are determined from boundary condition (iv).

$$\frac{x^2}{2} + \frac{x^4}{12a^2} + \frac{7b^2}{12} = \sum_{n=0}^{\infty}{}' \, b_n \cosh\frac{n\pi b}{a} \cos\frac{n\pi x}{a} \quad \text{for } 0 \leqslant x \leqslant a.$$

$$\text{Hence } \frac{a}{2} b_n \cosh\frac{n\pi b}{a} = \frac{1}{2} I(2,n) + \frac{1}{12a^2} I(4,n) + \frac{7b^2}{12} I(0,n) \qquad 8.2(7)$$

$$\text{where } I(m,n) = \int_0^a x^m \cos\frac{n\pi x}{a}\, dx \qquad 8.2(8)$$

$m = 0,1,2 \ldots$, $n = 0,1,2 \ldots$

It can be readily shown that

8.2) contd.

$$I(m,n) = \frac{a^{m+1}\, m(-1)^n}{n^2\pi^2} - \frac{a^2}{n^2\pi^2}\, m(m-1)I(m-2,n)$$

$$n \neq 0,\ m \geq 2 \qquad 8.2(9)$$

$$\left.\begin{aligned} \text{Now } I(0,n) &= \int_0^a \cos\frac{n\pi x}{a}\, dx = 0 \\ I(1,n) &= \frac{a^2}{n^2\pi^2}\left\{(-1)^n - 1\right\} \end{aligned}\right\} n \neq 0 \qquad 8.2(10)$$

Thus $I(m,n)$, $n \neq 0$, can be calculated for all $m = 0,1,2 \ldots$

$$\text{Now } I(m,0) = \int_0^a x^m\, dx$$

$$= \frac{a^{m+1}}{(m+1)} \text{ for } m = 0,1, \ldots \qquad 8.2(11)$$

$$\text{Hence } b_0 = \frac{7b^2}{6} + \frac{11a^2}{30} \qquad 8.2(12)$$

$$\text{and } b_n \cosh\frac{n\pi b}{a} = \frac{(-1)^n a^2}{n^2\pi^2}\left\{\frac{8}{3} - \frac{4}{n^2\pi^2}\right\} \qquad 8.2(13)$$

Thus u_2 is given by equations 8.2(6), (12) and (13) and so the complete solution for u is known. Clearly the method can be extended to the case where C(x,y) is any finite polynomial in x and y. This merely adds further complication to the evaluation of the coefficients a_n and b_n. Otherwise the solution proceeds as given here.

b) <u>Roth's method</u>.

We assume a solution of the form

$$u = \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty} C_{mk} \cos\frac{m\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b} \qquad 8.2(14)$$

so that the boundary conditions are automatically satisfied. To satisfy the differential equation,

8.2) contd.

$$\sum_{n=0}^{\infty}{}' \sum_{k=0}^{\infty} C_{nk} \left\{ \frac{n^2\pi^2}{a^2} + (k+\tfrac{1}{2})^2 \frac{\pi^2}{b^2} \right\} \cos\frac{n\pi x}{a} \cos(k+\tfrac{1}{2})\frac{\pi y}{b}$$

$$= 2 + \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{for } 0 \leq x \leq a, \quad 0 \leq y \leq b.$$

Hence

$$C_{nk} \left\{ \frac{n^2\pi^2}{a^2} + (k+\tfrac{1}{2})^2 \frac{\pi^2}{b^2} \right\} \frac{ab}{4} = 2I(0,n)\, J(0,k) + \frac{1}{a^2} I(2,n)\, J(0,k) + \frac{1}{b^2} I(0,n)\, J(2,k) \qquad 8.2(15)$$

where I(m,n) is defined by equation 8.2(8)

$$\text{and} \quad J(m,k) = \int_0^b y^m \cos(k+\tfrac{1}{2})\frac{\pi y}{b}\, dy \qquad 8.2(16)$$

$$m = 0,1,2 \ldots, \quad k = 0,1,2 \ldots$$

$$\text{Now } J(m,k) = \frac{b^{m+1}(-1)^k}{(k+\frac{1}{2})\pi} - \frac{m(m-1)b^2}{(k+\frac{1}{2})^2\pi^2} J(m-2,k) \qquad n \geq 2 \qquad 8.2(17)$$

$$\text{and } J(0,k) = \int_0^b \cos(k+\tfrac{1}{2})\frac{\pi y}{b}\, dy = \frac{b(-1)^k}{(k+\frac{1}{2})\pi}, \qquad 8.2(18)$$

$$J(1,k) = \frac{b^2}{(k+\frac{1}{2})\pi} \left\{ (-1)^k - \frac{1}{(k+\frac{1}{2})\pi} \right\} \qquad 8.2(19)$$

Thus J(m,k) is known for all m = 0,1,2 ..., k = 0,1,2 ...

Putting n = 0 gives

$$C_{ok} = \frac{4b^2(-1)^k}{(k+\frac{1}{2})^3\pi^3} \left\{ \frac{10}{3} - \frac{2}{(k+\frac{1}{2})^2\pi^2} \right\} \qquad 8.2(20)$$

and for n = 1,2,3 ...

$$C_{nk} = \frac{4b^2}{\pi^4} \; \frac{2(-1)^{n+k}}{n^2(k+\frac{1}{2})\pi\left\{ \frac{n^2b^2}{a^2} + (k+\frac{1}{2})^2 \right\}} \qquad 8.2(21)$$

The solution over the whole region $0 \leq x \leq a$, $0 \leq y \leq b$ is then given by equation 8.2(14). The sum is again calculated by diagonals as explained in Chapter 2.

8.2) contd.

(c) Double Chebyshev approximation.

Expressing C(x,y) in terms of Chebyshev polynomials we have to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\left\{ 3 + \frac{1}{2} T_2\left(\frac{x}{a}\right) + \frac{1}{2} T_2\left(\frac{y}{b}\right) \right\} \qquad 8.2(21)$$

Writing $F = u \left| \frac{4b^2}{\pi^4} \right.$, $X = \frac{x}{a}$, $Y = \frac{y}{b}$

this becomes

$$\left(\frac{b}{a}\right)^2 \frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = -\frac{\pi^4}{4}\left\{3 + \frac{1}{2} T_2(X) + \frac{1}{2} T_2(Y)\right\}$$

By comparison with equations 4.8(2) and (3) we may deduce the appropriate values of $\eta(m)$, $Q(r)$ and use the same method of solution as given in section 4.8.

8.3) Comparison between the three methods.

The solutions were evaluated numerically in all three cases truncating the infinite series as follows.

(a) For the method of separation of variables each infinite series was truncated at $n = N$.

(b) The solution by Roth's method was summed by diagonals and truncated at $m = M$.

(c) The double Chebyshev approximation is truncated at $r = R$, $m = M$.

Each solution was computed over a mesh given by $^x/a = 0(0.1)1$, $^y/b = 0(0.05)1$.

Considering first the ease of derivation of the mathematical solutions as given in section 8.2, Roth's method was again the most straightforward to apply. The method of separation of variables, although fairly mechanical also, is somewhat more lengthy. Although the double Chebyshev

8.3) contd.

approximation seems simple at this stage, it must be remembered that we have utilised all the previous techniques derived in Chapter 4. For all three methods, little additional difficulty is produced by increasing the degree of the polynomial C(x,y). Having derived the integral formulae for I(m,n) and J(m,k) it becomes trivial to deal with higher degree polynomials C(x,y) in both Roth's method and the method of separation of variables. It is, of course, inherent in the double Chebyshev approximation that a polynomial of any degree can be catered for. Considering now different forms of boundary condition, these can, of course, be readily incorporated into the method of separation of variables assuming that the boundary conditions are expressible as Fourier series. Some slight modification to Roth's method would be required in this case. It would be necessary to use a Roth-separation of variables combination as described in Chapter 3. Assuming that the boundary conditions are expressible either as polynomials or as Taylor series, the double Chebyshev approximation could still be used. With this method, once the Laplacian has been incorporated into the technique, there is not much additional difficulty in coping with fairly general boundary conditions. Two advantages of the method of separation of variables are firstly that its solution involves single series while the other methods each produce double series and secondly each term of its solution itself satisfies Laplace's equation and with care the terms can sometimes be identified with different parts of the boundary. This is not true of the solutions produced by the other two methods.

We now come to the numerical evaluation of the solution over the region $0 \leq x \leq a$, $0 \leq y \leq b$. Roth's method

8.3) contd.

is again the simplest. The coefficients C_{mk} are evaluated directly and the solution itself is simply and efficiently computed using the methods of Appendix 1. The double Chebyshev method has been described at length in Chapter 4. Suffice it to say here that although a fairly involved technique is required to evaluate the coefficients, once obtained the solution then proceeds as simply as for Roth's method. The method of separation of variables does not lend itself well to numerical computation due to the presence of the hyperbolic functions which must be expressed in exponential form and scaled so as to avoid numerical overload. To sum the Fourier series, the methods of Appendix 1 are again used.

Table 13 illustrates the rate of convergence of the solution for each method.

Point	Separation of variables		Roth's method		Double Chebyshev	
	N = 10	N = 30	M = 10	M = 30	R=M=3	R=M= 5
(0,0)	30.027	30.027	30.030	30.028	30.027	30.027
(0, $^b/2$)	22.832	22.832	22.832	22.832	22.832	22.832
(0,0.95b)	3.0627	3.0616	3.0542	3.0619	3.0622	3.0616
($^a/2$,0)	30.414	30.414	30.416	30.414	30.414	30.414
($^a/2$, $^b/2$)	23.185	23.185	23.185	23.185	23.185	23.185
($^a/2$,0.95b)	3.1361	3.1369	3.1287	3.1372	3.1378	3.1369
(a,0)	30.931	30.916	30.915	30.914	30.915	30.913
(a, $^b/2$)	23.651	23.649	23.650	23.649	23.648	23.649
(a,0.95b)	3.1762	3.2601	3.2434	3.2557	3.2516	3.2551
Computing times (secs)	34	72	16	22	33	34

TABLE 13.

(In calculating the values for this table $^b/a = 1.5$ and the computing times quoted include compilation time, time to evaluate the coefficients in each series and time to compute the solution over the grid $^x/a = 0(0.1)1$, $^y/b = 0(0.05)1$. They are given

8.3) contd.

merely to provide a qualitative comparison between the methods).

From the table it can be seen that for all three methods the rate of convergence is poorest near the top of the slot i.e. near $y = b$ becoming worse as the point (a,b) is approached. The double Chebyshev method gives by far the fastest rate of convergence. For corresponding solutions of a given accuracy, the number of terms summed in each method is

(a) separation of variables - 62 $(N = 30)$

(b) Roth's method -496 $(M = 30)$

(c) double Chebyshev method - 36 $(R = M = 5)$.

It is an advantage to have as few terms as possible when calculating quantities derived from u and the table clearly illustrates the power of the Chebyshev method in dealing with a Poisson equation with a source term $C(x,y)$ which is a polynomial.

The computing times given again verify the speed of computation of the Roth solution, and also show the difficulty of evaluating numerically the separation of variables solution. In comparing the times quoted for the double Chebyshev method it should be remembered that the methods of Chapter 4 and the resulting computer programme were produced for a more general case and so could be made somewhat more efficient for problems of the type considered here.

8.4) The annular slot.

The configuration is as shown in Figures 34 and 35.

8.4) contd.

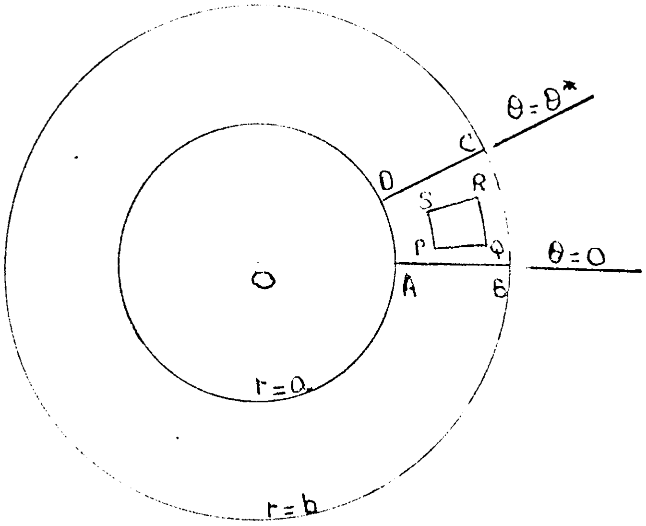


Figure 34.

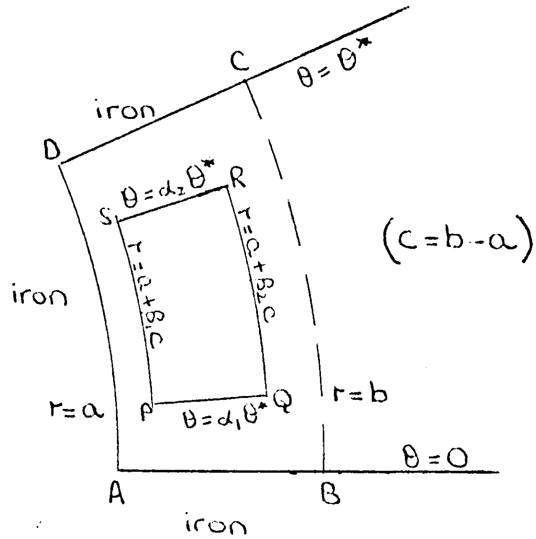


Figure 35.

ABCD defined by $a \leqslant r \leqslant b$, $0 \leqslant \theta \leqslant \theta^*$ represents the cross-section of an infinitely long annular slot. PQRS, $a+\beta_1 c \leqslant r \leqslant a+\beta_2 c$, $\alpha_1\theta^* \leqslant \theta \leqslant \alpha_2\theta^*$, is the cross-section of an infinitely long, insulated conductor in the slot and we consider the case of a steady-state, axial current flowing in the conductor. It is assumed that the slot is surrounded on three sides (CD, DA and AB) by iron of infinite permeability and that the fourth side BC is a flux line.

In cylindrical coordinates (r,θ,z), defining A_z as $\underline{A} = (0,0,A_z)$, where A_z depends on r and θ, the differential equation to be satisfied by A_z is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} = \begin{cases} -\mu_r\mu_o J_z & \text{in PQRS} \\ 0 & \text{elsewhere in ABCD} \end{cases} \qquad 8.4(1)$$

where $\underline{J} = (0,0,J_z)$.

To apply the Chebyshev methods we require a transformation which maps the region ABCD i.e. $a \leqslant r \leqslant b$, $0 \leqslant \theta \leqslant \theta^*$ on to the region $-1 \leqslant R \leqslant 1$, $0 \leqslant \Theta \leqslant 1$. Such a transformation is

$$R = \frac{2}{c}(r-a) - 1 \quad ; \quad \Theta = \frac{\theta}{\theta^*}, \quad (c = b - a) \qquad 8.4(2)$$

The inverse transformation is

8.4) contd.

$$r = (R+1)\frac{c}{2} + a \quad ; \quad \theta = \theta^{*}\Theta \qquad 8.4(3)$$

Under this transformation equation 8.4(1) becomes

$$\frac{4}{c^2}\frac{\partial^2 A_z}{\partial R^2} + \frac{1}{\{(R+1)\frac{c}{2} + a\}}\cdot\left(\frac{2}{c}\right)\frac{\partial A_z}{\partial R} + \frac{1}{\{(R+1)\frac{c}{2} + a\}^2 (\theta^*)^2}\frac{\partial^2 A_z}{\partial \Theta^2}$$

$$= \begin{cases} -\mu_r \mu_0 J_z & \text{for } \alpha_1 \leqslant \Theta \leqslant \alpha_2 \ , \ 2\beta_1 - 1 \leqslant R \leqslant 2\beta_2 - 1 \\ 0 & \text{elsewhere in } -1 \leqslant R \leqslant 1, \ 0 \leqslant \Theta \leqslant 1 \end{cases}$$

$$= -f(R,\Theta) \text{ (say)} \qquad 8.4(4)$$

To solve this equation we shall use the Fourier-Chebyshev method as described in Chapter 4, using Fourier approximation in the Θ-direction and Chebyshev variation in the R-direction.

The boundary conditions are as follows:-

(i) $\frac{\partial A_z}{\partial r} = 0$ when $r = a$ for $0 \leqslant \theta \leqslant \theta^*$

(ii) $\frac{\partial A_z}{\partial \theta} = 0$ when $\theta = 0, \theta^*$ for $a \leqslant r \leqslant b$

(iii) $A_z = 0$ when $r = b$ for $0 \leqslant \theta \leqslant \theta^*$

(In fact A_z is constant along BC but there is no loss of generality in taking the value of this constant as zero. Any other constant value merely alters the potential level).

In the (R,Θ) plane these conditions become

(i) $\frac{\partial A_z}{\partial R} = 0$ when $R = -1$ for $0 \leqslant \Theta \leqslant 1$

(ii) $\frac{\partial A_z}{\partial \Theta} = 0$ when $\Theta = 0,1$ for $-1 \leqslant R \leqslant 1$

(iii) $A_z = 0$ when $R = 1$ for $0 \leqslant \Theta \leqslant 1$.

Taking account of the form of these boundary conditions, the current density function $f(R,\Theta)$ is expanded as

8.4) contd.

$$f(R,\theta) = \frac{4\mu_r\mu_o J_z}{\pi^3} \sum_{m=o}^{\infty}{}' \sum_{n=o}^{\infty}{}' \alpha(m)\beta(n)\cos m\pi\theta\, T_n(R) \qquad 8.4(5)$$

where

$$\alpha(0) = \pi(\alpha_2 - \alpha_1)$$

$$\alpha(m) = \frac{\sin\alpha_2 m\pi - \sin\alpha_1 m\pi}{m}, \quad m = 1,2,3 \ldots$$

$$\beta(0) = \xi_1 - \xi_2 \qquad 8.4(6)$$

$$\beta(n) = \frac{\sin n\xi_1 - \sin n\xi_2}{n}, \quad n = 1,2,3 \ldots$$

$$\xi_1 = 2\cos^{-1}\sqrt{\beta_1} = \cos^{-1}(2\beta_1 - 1)$$

$$\xi_2 = 2\cos^{-1}\sqrt{\beta_2} = \cos^{-1}(2\beta_2 - 1)$$

Writing $F = A_z \Big/ \dfrac{4\mu_r\mu_o J_z c^2}{\pi^4}$, 8.4(7)

we have to solve

$$4\frac{\partial^2 F}{\partial R^2} + \frac{4}{(R+1+\frac{2a}{c})}\frac{\partial F}{\partial R} + \frac{4}{(\theta^*)^2(R+1+\frac{2a}{c})^2}\frac{\partial^2 F}{\partial\theta^2} = -\pi^2 \sum_{m=o}^{\infty}{}' \sum_{n=o}^{\infty}{}' \alpha(m)\beta(n)\, T_n(R)\cos m\pi\theta$$

subject to the boundary conditions

(i) $\dfrac{\partial F}{\partial R} = 0$ when $R = -1$ for $0 \leqslant \theta \leqslant 1$

(ii) $\dfrac{\partial F}{\partial \theta} = 0$ when $\theta = 0,1$ for $-1 \leqslant R \leqslant 1$

(iii) $F = 0$ when $R = 1$ for $0 \leqslant \theta \leqslant 1$.

8.5) Method of solution.

Assume a solution in the form

$$F = \sum_{m=o}^{M}{}' \phi_m(\mathring{R}) \cos m\pi\theta \qquad 8.5(1)$$

(truncating at $m = M$).

With this form of solution boundary condition (ii) is

8.5) contd.

automatically satisfied. Substituting in the differential equation and equating coefficients of cos $m\pi\Theta$ gives, for m = 0,1,2 ... M,

$$4\frac{d^2\phi_m}{dR^2} + \frac{4}{(R+1+\frac{2a}{c})}\frac{d\phi_m}{dR} - 4\frac{m^2\pi^2}{(\theta^*)^2}\frac{1}{(R+1+\frac{2a}{c})^2}\phi_m$$

$$= -\pi^2 \sum_{n=o}^{\infty}{}' \alpha(m)\beta(n)\, T_n(R)$$

i.e. $$\left(R+1+\frac{2a}{c}\right)^2 \frac{d^2\phi_m}{dR^2} + \left(R+1+\frac{2a}{c}\right)\frac{d\phi_m}{dR} - \frac{m^2\pi^2}{(\theta^*)^2}\phi_m$$

$$= -\frac{\pi^2\alpha(m)}{4}\sum_{n=o}^{\infty}{}'\beta(n)\, T_n(R)\left(R+1+\frac{2a}{c}\right)^2$$

for m = 0,1,2 ... M 8.5(2)

To satisfy boundary conditions (i) and (iii),

(i) $\frac{d\phi_m}{dR} = 0$ when $R = -1$

(ii) $\phi_m = 0$ when $R = +1$

for m = 0,1,2 ... M

Due to the derivative form of boundary condition at R = -1, the modified direct method as described in section 4.5 is used here. Assume that, dropping the m-suffix,

$$\frac{d^2\phi}{dR^2} = \sum_{n=o}^{N}{}' a_n T_n(R) \qquad 8.5(3)$$

Note that, from the form of the differential equation, we can no longer assume the solution to be even.

Writing $$\frac{d\phi}{dR} = \sum_{n=o}^{N+1}{}' b_n T_n(R) \qquad 8.5(4)$$

8.5) contd.

then $b_n = \frac{1}{2n}\left(a_{n-1} - a_{n+1}\right)$ for $n = 1,2, \ldots (N+1)$ 8.5(5)

where $a_{N+1} = a_{N+2} = 0$

(b_0 is an arbitrary constant of integration).

$$\text{If } \phi = \sum_{n=0}^{N+2}{}' c_n T_n(R) \qquad 8.5(6)$$

then $c_n = \frac{1}{2n}\left(b_{n-1} - b_{n+1}\right)$ for $n = 1,2 \ldots (N+2)$ 8.5(7)

$$= \frac{1}{4}\left\{\frac{a_{n-2}}{n(n-1)} - \frac{2a_n}{(n^2-1)} + \frac{a_{n+2}}{n(n+1)}\right\} \qquad 8.5(8)$$

for $n = 2,3 \ldots (N+2)$

where $a_{N+1} = a_{N+2} = a_{N+3} = a_{N+4} = 0$, $b_{N+2} = b_{N+3} = 0$.

These expressions for ϕ, $\frac{d\phi}{dR}$, $\frac{d^2\phi}{dR^2}$ are substituted in equation 8.5(2).

$$\left.\begin{aligned} &\text{Also, } \left(R+1+\frac{2a}{c}\right)^2 = \frac{1}{2}T_2(R) + 2\left(1+\frac{2a}{c}\right)T_1(R) + \left(1+\frac{2a}{c}\right)^2 + \frac{1}{2} \\ &\text{and } \left(R+1+\frac{2a}{c}\right) = T_1(R) + \left(1+\frac{2a}{c}\right) \end{aligned}\right\} \qquad 8.5(9)$$

In addition we make use of the well known result,

$$T_m(R)\,T_n(R) = \frac{1}{2}\left\{T_{m+n}(R) + T_{|m-n|}(R)\right\} \qquad 8.5(10)$$

so that

$$T_2(R)\sum_{n=0}^{N}{}' a_n T_n(R) = \frac{1}{2}a_1T_1(R) + \frac{1}{2}\sum_{n=2}^{N+2} a_{n-2}T_n(R) + \frac{1}{2}\sum_{n=0}^{N-2} a_{n+2}T_n(R) \qquad 8.5(11)$$

$$T_1(R)\sum_{n=0}^{N}{}' a_n T_n(R) = \frac{1}{2}\sum_{n=1}^{N+1} a_{n-1}T_n(R) + \frac{1}{2}\sum_{n=0}^{N-1} a_{n+1}T_n(R) \qquad 8.5(12)$$

8.5) contd.

Substituting in the differential equation and equating coefficients of $T_n(R)$, for $n = 0,1, \dots (N+2)$ gives

$$\frac{1}{4}\left\{a_{n-2} + a_{n+2}\right\} + \left(1 + \frac{2a}{c}\right)\left\{a_{n-1} + a_{n+1}\right\} + a_n\left\{\left(1 + \frac{2a}{c}\right)^2 + \frac{1}{2}\right\}$$

$$+ \frac{1}{2}\left\{b_{n-1} + b_{n+1}\right\} + \left(1 + \frac{2a}{c}\right) b_n - \frac{m^2 \pi^2}{(\theta^*)^2} c_n = - \pi^2 \alpha(m) Q(n) \qquad 8.5(13)$$

for $n = 2,3 \dots (N+2)$ $\left(\text{with } a_{N+1} = a_{N+2} = a_{N+3} = a_{N+4} = 0 \quad b_{N+2} = b_{N+3} = 0\right)$

where $$Q(n) = \frac{1}{4}\left\{ \frac{1}{4}(\beta(n-2)+\beta(n+2)) + \left(1 + \frac{2a}{c}\right)\left(\beta(n-1) + \beta(n+1)\right)\right.$$

$$\left. + \beta(n)\left[\left(1 + \frac{2a}{c}\right)^2 + \frac{1}{2}\right]\right\} \qquad 8.5(14)$$

for $n = 2,3, \dots (N+2)$

The coefficients b_n and c_n are given in terms of a_n by equations 8.5(5) and (8).

Considering $n = 0,1$ gives, respectively

$$\frac{1}{4} a_2 + \left(1 + \frac{2a}{c}\right) a_1 + \left\{\left(1 + \frac{2a}{c}\right)^2 + \frac{1}{2}\right\}\frac{1}{2} a_0 + \frac{1}{2} b_1 + \left(1 + \frac{2a}{c}\right)\frac{1}{2} b_0$$

$$- \frac{m^2 \pi^2}{(\theta^*)^2} \frac{1}{2} c_0 = -\pi^2 \alpha(m) Q(0) \qquad 8.5(15)$$

$$\frac{1}{4}(a_1+a_3) + \left(1 + \frac{2a}{c}\right)(a_0 + a_2) + a_1\left\{\left(1 + \frac{2a}{c}\right)^2 + \frac{1}{2}\right\} + \frac{1}{2}(b_0 + b_2)$$

$$+ \left(1 + \frac{2a}{c}\right) b_1 - \frac{m^2 \pi^2}{(\theta^*)^2} c_1 = -\pi^2 \alpha(m) Q(1) \qquad 8.5(16)$$

where $$Q(0) = \frac{1}{4}\left\{ \frac{1}{4} \beta(2) + \left(1 + \frac{2a}{c}\right)\beta(1) + \frac{1}{2}\beta(0)\left\{\left(1 + \frac{2a}{c}\right)^2 + \frac{1}{2}\right\}\right\} \qquad 8.5(17)$$

$$Q(1) = \frac{1}{4}\left\{\frac{1}{4}\left(\beta(1)+\beta(3)\right) + \left(1 + \frac{2a}{c}\right)\left(\beta(0)+\beta(2)\right) + \beta(1)\left[\left(1 + \frac{2a}{c}\right)^2 + \frac{1}{2}\right]\right\} \qquad 8.5(18)$$

In addition we have the equations derivable from the

8.5) contd.

boundary conditions i.e.

$$\text{(i)} \quad \sum_{n=o}^{N+1}{}' \; b_n(-1)^n = 0 \text{ and} \tag{8.5(19)}$$

$$\text{(ii)} \quad \sum_{n=o}^{N+2}{}' \; c_n = 0 \tag{8.5(20)}$$

Thus we have (N+5) equations in total and the total number of unknowns is (N+3) i.e. $a_o, a_1 \ldots a_N, b_o, c_o$. Hence we have two surplus equations and the truncated form of the right hand side of equation 8.5(2) must be modified to read

$$-\pi^2 \alpha(m) \sum_{n=o}^{N+2} Q(n)\, T_n(R) + \sigma_m T_{N+1}(R) + \tau_m T_{N+2}(R) \tag{8.5(21)}$$

The equations corresponding to (N+1), (N+2) of the set 8.5(13) then read, respectively,

$$\frac{1}{4} a_{N-1} + \left(1 + \frac{2a}{c}\right) a_N + \frac{1}{2} b_N + \left(1 + \frac{2a}{c}\right) b_{N+1} - \frac{m^2 \pi^2}{(\theta^*)^2} c_{N+1}$$

$$= -\pi^2 \alpha(m)\, Q(N+1) + \sigma_m \tag{8.5(22)}$$

$$\frac{1}{4} a_N + \frac{1}{2} b_{N+1} - \frac{m^2 \pi^2}{(\theta^*)^2} c_{N+2} = -\pi^2 \alpha(m)\, Q(N+2) + \tau_m \tag{8.5(23)}$$

These two equations are used to give σ_m, τ_m and are ignored at this stage. Equations 8.5(19) and (20) are used to give the constants of integration b_o and c_o. The remaining equations can then be written in the form

$\underline{A}\underline{x} = \underline{b}$ where $\underline{x} = \begin{bmatrix} a_o \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$ and the (N+1) × (N+1)

8.5) contd.

matrix A is a 5-band matrix with non-zero elements in the first two rows. It is possible that an algorithm for the inversion of such a matrix could be developed along the lines given in Appendix 2 but at this stage the equations were solved using a Gaussian elimination procedure. Having obtained the coefficients a_n, for each m, the solution over the whole region $-1 \leqslant R \leqslant 1$, $0 \leqslant \theta \leqslant 1$ is given by equations 8.5(1), (6), (8), (19) and (20). The double series is evaluated using the methods of Appendix 1.

8.6) Error estimation.

Again there are two sources of error in the solution; the truncation error and the error due to the introduction of the terms σ_m, τ_m for m = 0,1,2 ... M. E_τ is the error due to the τ-terms and the truncation error is neglected at this stage. The previous analysis of Chapter 4 has shown that E_τ gives a good estimate of the actual error.

$$E_\tau = F - F_c \qquad 8.6(1)$$

where F_c is the calculated value of F.

The differential equation to be satisfied by E_τ is

$$4\frac{\partial^2 E_\tau}{\partial R^2} + \frac{4}{(R+1+\frac{2a}{c})}\frac{\partial E_\tau}{\partial R} + \frac{4}{(\theta^*)^2(R+1+\frac{2a}{c})^2}\frac{\partial^2 E_\tau}{\partial \theta^2}$$

$$= \sum_{m=0}^{M}{}' \frac{-4\{\sigma_m T_{N+1}(R) + \tau_m T_{N+2}(R)\} \cos m\pi\theta}{(R+1+\frac{2a}{c})^2} \qquad 8.6(2)$$

subject to the boundary conditions

(i) $\frac{\partial E_\tau}{\partial R} = 0$ when $R = -1$ for $0 \leqslant \theta \leqslant 1$

8.6) contd.

(ii) $\frac{\partial E_T}{\partial \Theta} = 0$ when $\Theta = 0,1$ for $-1 \leq R \leq 1$

(iii) $E_T = 0$ when $R = 1$ for $0 \leq \Theta \leq 1$.

Assume a solution in the form

$$E_T = \sum_{m=o}^{M}{}' \psi_m(R) \cos m\pi\Theta \qquad 8.6(3)$$

so that boundary condition (ii) is automatically satisfied. Substituting in equation 8.6(2), the equation to be satisfied by $\psi_m(R)$ is

$$\left(R + 1 + \frac{2a}{c}\right)^2 \frac{d^2\psi_m}{dR^2} + \left(R + 1 + \frac{2a}{c}\right)\frac{d\psi_m}{dR} - \frac{m^2\pi^2}{(\theta^*)^2} \psi_m$$

$$= - \left(\sigma_m T_{N+1}(R) + \tau_m T_{N+2}(R)\right) \qquad 8.6(4)$$

(for $m = 0,1 \ldots M$)

subject to the boundary conditions

(i) $\frac{d\psi_m}{dR} = 0$ when $R = -1$

(ii) $\psi_m = 0$ when $R = 1$.

Referring back to Figure 34, for practical configurations, the width of the annulus c is small by comparison with a. Also $0 \leq R + 1 \leq 2$.

Hence, as a first approximation (R+1) can be neglected by comparison with $\left(\frac{2a}{c}\right)$, and equation 8.6(4) can be written

$$\frac{4a^2}{c^2} \frac{d^2\psi_m}{dR^2} + \frac{2a}{c} \frac{d\psi_m}{dR} - \frac{m^2\pi^2}{(\theta^*)^2} \psi_m = - \left(\sigma_m T_{N+1}(R) + \tau_m T_{N+2}(R)\right) \qquad 8.6(5)$$

for $m = 0,1, \ldots M$.

Considering first the case $m = 0$, the complementary function is

8.6) contd.

$$A_o + B_o \exp\left(-\frac{cR}{2a}\right)$$

where A_o, B_o are arbitrary constants. The particular integral is given by

$$\frac{1}{\frac{2a}{c} D\left(\frac{2a}{c} D + 1\right)} \left\{ -\sigma_o T_{N+1}(R) - \tau_o T_{N+2}(R) \right\} \text{ where } D \equiv \frac{d}{dR}$$

Now $\frac{1}{D} T_n(R) = \frac{1}{2}\left\{\frac{T_{n+1}(R)}{(n+1)} - \frac{T_{n-1}(R)}{(n-1)}\right\}$ for $n \geq 2$, so that,

expanding in powers of $^1/D$, a first approximation to the particular integral is given by

$$\frac{c^2}{4a^2} \frac{1}{D^2}\left\{-\sigma_o T_{N+1}(R) - \tau_o T_{N+2}(R)\right\}$$

i.e.
$$\frac{c^2}{16a^2}\left\{-\frac{\sigma_o T_{N-1}(R)}{N(N-1)} - \frac{\tau_o T_N(R)}{N(N+1)} + \frac{2\sigma_o T_{N+1}(R)}{N(N+2)} + \frac{2\tau_o T_{N+2}(R)}{(N+1)(N+3)} - \frac{\sigma_o T_{N+3}(R)}{(N+2)(N+3)} - \frac{\tau_o T_{N+4}(R)}{(N+3)(N+4)}\right\} \quad 8.6(6)$$

Applying the boundary conditions gives

$$B_o = \left(\frac{c}{2a}\right) \exp\left(\frac{-c}{2a}\right)(-1)^N\left\{\frac{\sigma_o}{N(N+2)} - \frac{\tau_o}{(N+1)(N+3)}\right\} \quad 8.6(7)$$

$$A_o = -B_o \exp\left(\frac{-c}{2a}\right) + \frac{3c^2}{4a^2 N(N+3)}\left\{\frac{\sigma_o}{(N-1)(N+2)} + \frac{\tau_o}{(N+1)(N+4)}\right\} \quad 8.6(8)$$

The maximum absolute error is likely to occur when $R = -1$ (note that $E_\tau = 0$ when $R = 1$) and, considering only terms $O\left(\frac{1}{N^2}\right)$,

$$\psi_o(-1) = \left(\frac{c}{2a}\right)(-1)^N\left\{\frac{\sigma_o}{N(N+2)} - \frac{\tau_o}{(N+1)(N+3)}\right\}\left\{1 - \exp\left(\frac{-c}{a}\right)\right\} \quad 8.6(9)$$

For m = 1,2 ... M a first approximation to the particular

8.6) contd.

integral is again given by the expression 8.6(6) with σ_0, τ_0 replaced by σ_m, τ_m. The complementary function is

$$A_m \exp(\gamma_1 R) + B_m \exp(\gamma_2 R)$$

where γ_1, γ_2 are the roots of the equation

$$\frac{4a^2}{c^2}\gamma^2 + \frac{2a}{c}\gamma - \frac{m^2\pi^2}{(\theta^*)^2} = 0$$

i.e. $$\gamma_1, \gamma_2 = \frac{c}{4a}\left\{-1 \pm \sqrt{1 + \frac{4m^2\pi^2}{(\theta^*)^2}}\right\}$$

For $m \geqslant 1$, $\frac{2m\pi}{\theta^*} >> 1$ so that

$$\gamma_1, \gamma_2 \doteqdot \pm \frac{m\pi c}{2a\theta^*}$$

Applying the boundary conditions gives, considering only terms $O\left(\frac{1}{N^2}\right)$,

$$\gamma_1 A_m e^{\gamma_1} \cosh 2\gamma_1 = (-1)^{N+1} \frac{c^2}{8a^2}\left\{\frac{\sigma_m}{N(N+2)} - \frac{\tau_m}{(N+1)(N+3)}\right\}$$

$$B_m = - A_m e^{2\gamma_1} \qquad 8.6(10)$$

where $\gamma_1 = - \gamma_2 = \frac{m\pi c}{2a\theta^*}$.

Hence to $O\left(\frac{1}{N^2}\right)$,

$$\psi_m(-1) = \frac{\tanh 2\gamma_1}{\gamma_1}(-1)^N \frac{c^2}{4a^2}\left\{\frac{\sigma_m}{N(N+2)} - \frac{\tau_m}{(N+1)(N+3)}\right\} \qquad 8.6(11)$$

for $m = 1,2 \ldots M$

An estimate of E_τ at $R = -1$ (i.e. along $r = a$) is then given by equations 8.6(3), (9) and (11). On examination of the results for various values of $\left(\frac{c}{a}\right)$, θ^* and N it can be shown that $\psi_m(-1)$ for $m = 1,2 \ldots M$ is negligible by comparison with $\psi_0(-1)$ so that, approximately,

8.6) contd.

$$(E_\tau)_{R=-1} = \frac{1}{2}\psi_0(-1)$$

$$= \left(\frac{c}{4a}\right)(-1)^N\left\{\frac{\sigma_0}{N(N+2)} - \frac{\tau_0}{(N+1)(N+3)}\right\}\left(1 - \exp\left(\frac{-c}{a}\right)\right) \quad 8.6(12)$$

(Note that this approximate error estimate is independent of θ and θ^*.)

Table 14 gives the values of $(E_\tau)_{R=-1}$ obtained from equation 8.6(12) for the cases N = 10,15 (with $\frac{c}{a} = 0.3$).

N	σ_0	τ_0	$(E_\tau)_{R=-1}$
10	8.33138	-49.8128	0.008
15	16.625	-30.6447	-0.003

TABLE 14.

Table 15 illustrates the rate of convergence of the solution over the whole slot.

point (R,θ) \ M=N=	10	15	20
(-1,0)	5.7123	5.7209	5.7174
$(-1,\frac{1}{2})$	5.6992	5.7093	5.7052
(-1,1)	5.4502	5.4580	5.4548
(0,0)	4.3194	4.3246	4.3234
$(0,\frac{1}{2})$	4.4703	4.4699	4.4697
(0,1)	3.9892	3.9916	3.9905
Computing time (secs)	41	70	127

TABLE 15.

(The data used to obtain the table is:- $\frac{c}{a} = 0.3$, $\theta^* = 0.2$, $\alpha_1 = 0.1$, $\alpha_2 = 0.8$, $\beta_1 = 0.1$, $\beta_2 = 0.7$).

By comparison of tables 14 and 15, it can be seen that the error estimate given by equation 8.6(12) is good. In addition table 15 shows the dependence of the error on $(-1)^N$,

8.6) contd.

again a property predicted by equation 8.6(12). Further examination of Table 15 illustrates the previous assumption that the maximum absolute error is likely to occur when $R = -1$.

The computing times given in Table 15 include compilation time of the programme, time taken to evaluate the coefficients a_{mn} and time for computation of the solution over the mesh of points $R = -1(0.1)1$, $\Theta = 0(0.1)1$. These times indicate the efficiency of the Chebyshev method in solving problems of this type.

8.7) Magnetic energy and leakage inductance.

Following the reasoning of section 1.11, the magnetic energy per unit axial length of this conductor is

$$W = \frac{1}{2} J_z \int_{r=a+\beta_1 c}^{a+\beta_2 c} dr \int_{\theta=\alpha_1 \theta^*}^{\alpha_2 \theta^*} A_z \, r \, d\theta \qquad 8.7(1)$$

$$= \frac{1}{2} J_z \cdot \frac{4\mu_r \mu_o J_z c^2}{\pi^4} \int_{R=2\beta_1-1}^{2\beta_2-1} \frac{c}{2} \, dR \int_{\Theta=\alpha_1}^{\alpha_2} \left\{ \sum_{m=o}^{M}{}' \phi_m(R) \cos m\pi\Theta \right\} \left\{ (R+1)\frac{c}{2}+a \right\} \theta^* d\Theta$$

using equations 8.4(3) and (7), and 8.5(1).

Performing the integration with respect to Θ gives

$$W = \frac{\mu_r \mu_o J_z^2 c^3 \theta^*}{\pi^5} \int_{R=2\beta_1-1}^{2\beta_2-1} \sum_{m=o}^{M}{}' \alpha(m) \phi_m(R) \left\{ \frac{c}{2} T_1(R) + a + \frac{c}{2} \right\} dR \qquad 8.7(2)$$

Using equations 8.5(6) and (12), this expression can easily be calculated.

The total current I and J_z are related by

8.7) contd.

$$I = J_z \frac{1}{2} \left\{ (\beta_1+\beta_2)c+2a \right\} \left(\beta_2-\beta_1\right) \left(\alpha_2-\alpha_1\right) c\theta^* \qquad 8.7(3)$$

and $W = \frac{1}{2} L I^2$ where L is the leakage inductance per unit axial length. Hence L can be calculated. Equation 2.6(2) gives the leakage inductance for a rectangular conductor in a rectangular slot. For comparison purposes and to distinguish between the two quantities, we now denote by L_{rect} the leakage inductance given by equation 2.6(2).

In comparing the annular slot with the rectangle we take

$$\alpha_1 = 0.1, \quad \alpha_2 = 0.8, \ \beta_1 = 0.1, \ \beta_2 = 0.7$$

for both configurations. We choose the depth b of the rectangular slot to be equal to the width c of the annulus. The width of the rectangular slot is assumed to be $^2/_3$ b. Considering then the annular slot, for a given ratio ($^c/a$), the parameter θ^* is chosen so that the two conductor areas (i.e. the rectangular conductor and the annular conductor) are the same. In this case, if J_z is the same for both conductors, then the total current I will be the same for both. Table 16 gives the ratio $^L/L_{rect}$ for the two cases $^c/a$ = 0.3, 0.15 for the data given above.

$^c/a$	$^L/L_{rect}$
0.3	0.906
0.15	0.948

TABLE 16.

Thus, assuming the slot to be rectangular gives a value for the leakage inductance which is greater than the actual value, the order of magnitude of the difference being given by the table.

8.7) contd.

Another quantity often used in practice for comparison purposes is the total flux per unit length of the conductor. With the configurations given here and in Chapter 2, the total flux per unit length is just the maximum value of A_z in the slot. From equations 2.4(1) and 8.4(7), since we have assumed b = c and the current densities to be the same in each case, the maximum value of F for both the rectangular slot and the annular slot measures this total flux.

	Max.F	Ratio $\left(F/F_{rect}\right)$	Max.F occurs at
$\frac{c}{a} = 0.3$	5.7433	0.921	R= -0.8 θ = 0.3
$\frac{c}{a} = 0.15$	5.9669	0.957	R= -0.8 θ = 0.3
rectangle	6.2384	1	$\frac{y}{b} = 0.1$ $\frac{x}{a} = 0.3$

TABLE 17.

Table 17 gives the ratio of the total flux for the annular conductor to the total flux for the rectangular conductor. Again if the conductor is assumed to be rectangular, a value for the total flux is obtained which is in excess of the true value. The table also shows the point in the slot at which F is greatest. This is seen to be, in all three cases, on the boundary of the conductor parallel to and furthest from the top of the slot BC. Also, in all cases, this maximum point is at a distance 30% of the slot width from the side of the slot.

8.8) Summary.

From the previous work of Chapters 4 and 5, it seemed likely that Chebyshev methods of solution would be very suitable for differential equations of the type given by equation 8.1(1). In this chapter we have considered two special cases, the first being

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = - C(x,y)$$

where C(x,y) is a finite polynomial in x and y. The advantage of using the double Chebyshev approximation for equations of this type is that the rate of convergence of the double series solution is very fast and comparatively few terms are required for a solution which has converged to a given accuracy. Further research is indicated in this area. In particular further investigation is required into the additional difficulties produced by allowing more general forms of boundary condition.

The second example chosen is even more general than the first, solving a Poisson equation in cylindrical coordinates. To solve this problem by Roth's method or the method of separation of variables is difficult involving as it does the use of Bessel functions. We solve the problem using a Fourier-Chebyshev approximation and the method is shown to be quite straightforward in application. The double series has a fairly rapid rate of convergence and the computing time to obtain the solution over the whole slot is not excessive. In addition to demonstrating the usefulness of Chebyshev approximations to this type of practical problem, the solution is used to derive an estimate of the leakage inductance. The method shows that by assuming the slot to

8.8) contd.

be rectangular we over-estimate the leakage inductance

(i) by 10% approximately in slots where the ratio $^c/a$ is 0.3 and (ii) by 5% approximately in slots where $^c/a = 0.15$. These effects are the same if, instead of considering the leakage inductance, we evaluate the total flux per unit length of the conductor.

By these two particular examples, we have shown that Chebyshev methods are particularly suitable for the solution of elliptic partial differential equations of the type given by equation 8.1(1).

C O N C L U S I O N S.

The results and conclusions relevant to each chapter have been given in the appropriate sections of the text. Thus, what follows will be a summary of the main conclusions which have been deduced from this research.

The first objective of the research programme was to compare Roth's method with other methods, in particular the method of separation of variables and the finite cosine transform method, for solving certain elliptic partial differential equations arising in practice. In particular we consider the solution of steady state problems associated with insulated conductors in rectangular slots.

Considering first the mathematical derivation of the solution, Roth's method is to be preferred. It gives a single solution valid over the whole region of the slot and the method is both simple and straightforward to apply no matter how many conductors are in the slot. The method of separation of variables requires the slot to be divided into regions with a separate solution valid in each region. Continuity conditions must be met across all internal boundaries and if there are several conductors in the slot then the amount of algebraic effort required to do this becomes prohibitive. Further, Roth's method can be used when the conductor cross-sections are bounded by any mathematically defined closed curves but the method of separation of variables requires the conductors to be rectangular.

To obtain the numerical solution over the whole slot area, again Roth's method is preferable. Although other writers,[9] and [10], have stated that Roth's method cannot be recommended for numerical work due to the slow rate of convergence of the double Fourier series, as a result of this research we are bound to disagree with them. If use is made of the algorithms givenin Appendix 1, particularly with digital computing facilities available, then, although a comparatively large number of terms must be summed to produce a solution which has converged to a given accuracy, the method of summation is both efficient and simple. On the other hand, the form of the separation of variables solution, containing as it does complex combinations of hyperbolic functions, does not lend itself well to numerical computation. Having said this we cannot recommend the Roth-Kouskoff technique for summation of the double series over one variable. The resulting solution is identical to that obtained using the method of separation of variables so that we have lost the advantage of a single solution valid over the whole slot and again there is the cumbersome form of numerical solution.

To apply Roth's method directly it is necessary that the slot boundaries are either flux lines or scalar equipotentials. It should be observed here that, in this case, Roth's method and the finite cosine transform method are very similar. The solution obtained in each case is identical and the methods differ only in the way the Fourier coefficients are obtained. However, unlike Roth's method, the transform method, without modification, can cope with more generalised forms of boundary condition. It is suitable when the boundary conditions are either of the Dirichlet type when the sine transform is used or of the Neumann type when the cosine transform is used. When the generalised boundary conditions are of mixed type, the transform method is not recommended.Considerable mathematical manipulation is required to obtain the solution in this case and the beauty of the method is lost. Also, it is inherent in the transform method that the boundary conditions are included in the Fourier coefficients and this makes the physical interpretation of the solution difficult.

With more general boundary conditions, Roth's method is combined with a separation of variables solution. Roth's method is used for that part of the solution resulting from the conductors in the slot and a separation of variables solution is derived for each part of the boundary. The method is simple and straightforward to apply combining the advantages of both methods wherever possible. Also, since each part of the boundary can be identified with a particular part of the solution, the physical interpretation of the Roth-separation of variables combination becomes trivial. Summarising then, for general boundary conditions of mixed type, the Roth-separation of variables combination is preferable to the transform method, although with this combined solution we no longer have a single solution valid over the whole region of the slot.

In order to try to improve the rate of convergence of the double Fourier series, various Chebyshev approximations were considered. Considering first the Fourier-Chebyshev approach, the method involves the solution of a sequence of ordinary differential equations for which two methods are described, the direct method and the integrated method. For boundary conditions of the Dirichlet type there is little to choose between the two methods but for boundary conditions of the Neumann type, the integrated method is totally unsuitable and the direct method or its modified version must be used. It is possible to produce a "smoothing" of the error in the solution by slight perturbation of the boundary conditions. In general this is hardly worthwhile since the same effect can be achieved more easily by increasing the number of terms taken in the double series.

The double Chebyshev method represents a significant contribution to the existing knowledge concerning the application of Chebyshev approximations to the solution of elliptic partial differential equations. The method of solution given here including the algorithm of Appendix 2 is both elegant and efficient and is capable of application to a wider class of problems as is indicated in Chapter 8. Unfortunately Chebyshev methods cannot be recommended for the solution of problems concerning insulated rectangular conductors in slots. The current density is a discontinuous function in the region under consideration and its resulting Chebyshev approximation is an infinite series. There is, therefore, a truncation error present in the solution.

Roth's method, the Fourier-Chebyshev method and the double Chebyshev approximation can all be used to solve the Neumann problem arising from balanced rectangular windings in a transformer window. It should be observed here that if the surrounding iron is assumed to be infinitely permeable then, for the solution to exist at all, the total current within the window must be zero.

The second major objective of the research programme was to determine whether Roth's methods could be applied to classes of problems other than those resulting from static fields. The first problem considered is that of an insulated conductor in a rectangular slot when the current is varying sinusoidally with time. An extension of Roth's method is developed to give the exact solution for the vector potential within the slot and this new technique together with its associated numerical algorithm given in Appendix 4 is capable of application to a wide variety of practical problems. An alternative approximate technique is also derived and is shown to give good agreement with the exact method over a practical range of insulation thicknesses. This approximation is then used to consider the problem of an insulated conductor in a slot facing an air gap, the problem considered by Silvester in[14]. The results given by this research add weight to the arguments that the presence of insulation does not greatly affect the effective impedance of the conductor. This is contrary to the findings of Silvester whose results show the effects of the insulation to be considerable.

The second application of the extension of Roth's method considers the eddy-current loss produced in a rectangular conductor by a transverse magnetic field varying sinusoidally with time. Again the method is shown to be both powerful and simple to use. The results compare favourably with those obtained by Stoll in[15] but they indicate that the position of the outer boundaries of the region under consideration is a critical factor in the interpretation of the results.

Further examples of the use of Chebyshev methods applied to the solution of more general elliptic partial differential equations illustrate the power and scope of these methods. For a Poisson equation with a source term which is a polynomial, the rate of convergence of the double series solution obtained using the double Chebyshev method is significantly faster than that when Roth's method or the method of separation of variables is used. The problem in cylindrical geometry is an excellent example of Chebyshev approximation. To solve this problem by Roth's method or the method of separation of variables would be extremely difficult involving the use of Bessel functions. The Fourier-Chebyshev method results in a solution of simple form which is valid over the whole region of the annular slot and from which any derived quantities e.g. the magnetic field components or the leakage inductance can be easily calculated. Also the solution is again readily evaluated numerically using the algorithms given in Appendix 1. The calculations given here show that by assuming the slot to be rectangular we over-estimate the true value of the leakage inductance.

Further investigation is required into the application of Chebyshev methods to the solution of elliptic partial differential equations. In particular consideration should be given to more general forms of boundary condition. If the boundary conditions are expressible as polynomials or Taylor series then these can be readily incorporated into the technique but further research is needed to determine exactly how the method of solution must be modified, in particular what amendments must be made to the algorithms given in Appendix 2. Also how would the linear dependence of the boundary equations as described in Appendix 3 be affected?

A further application of the extension of Roth's method would be to derive the vector potential in a hollow water-cooled conductor in a machine slot when the current in the conductor is varying sinusoidally with time. This problem has been considered by Stoll in[19] using finite difference methods.

Another area of future research might concern the application of the Fourier-Chebyshev method to the annular slot problem described in Chapter 8 when the current in the conductor is varying sinusoidally with time.

APPENDIX 1.

NUMERICAL SUMMATION OF FOURIER AND CHEBYSHEV SERIES.

1(a) $\sum_{n=0}^{N}{}' \; a_n \cos\frac{n\pi x}{\ell}$ $\left(\sum_{n=0}^{N}{}' \right.$ denotes that a factor $\frac{1}{2}$ is to be taken when $\left. n = 0 \right)$

Now $\cos(n+1)\frac{\pi x}{\ell} + \cos(n-1)\frac{\pi x}{\ell} = 2\cos\frac{n\pi x}{\ell} \cos\frac{\pi x}{\ell}$

i.e. $\cos(n+1)\frac{\pi x}{\ell} - 2\cos\frac{\pi x}{\ell} \cos\frac{n\pi x}{\ell} + \cos(n-1)\frac{\pi x}{\ell} = 0$ A1(1)

Define coefficients b_n, $n = 0,1, \ldots (N+1)$ as follows:-

$$b_{N+1} = 0$$

$$b_N = a_N$$

$$b_n = a_n + 2\cos\frac{\pi x}{\ell} b_{n+1} - b_{n+2} \text{ for } n = (N-1),(N-2)\ldots 1,0$$

A1(2)

Then $\sum_{n=0}^{N}{}' \; a_n \cos\frac{n\pi x}{\ell} = b_N \cos\frac{N\pi x}{\ell} + \sum_{n=0}^{N-1}{}' \left\{ b_n - 2\cos\frac{\pi x}{\ell} b_{n+1} + b_{n+2} \right\} \cos\frac{n\pi x}{\ell}$

$= \frac{1}{2}(b_0 - b_2)$ using equation A1(1)

Thus, the series is summed by evaluating just one cosine term namely $2\cos\frac{\pi x}{\ell}$. The numerical calculation of the coefficients b_n, $n = 0,1 \ldots (N+1)$ then follows trivially.

1(b) $\sum_{n=0}^{N} a_n \cos(n+\frac{1}{2})\frac{\pi x}{\ell}$

In this case

$$\cos(n+1+\tfrac{1}{2})\frac{\pi x}{\ell} + \cos(n-1+\tfrac{1}{2})\frac{\pi x}{\ell} = 2\cos(n+\tfrac{1}{2})\frac{\pi x}{\ell} \cos\frac{\pi x}{\ell}$$

or $\cos(n+1+\frac{1}{2})\frac{\pi x}{\ell} - 2\cos\frac{\pi x}{\ell} \cos(n+\frac{1}{2})\frac{\pi x}{\ell} + \cos(n-1+\frac{1}{2})\frac{\pi x}{\ell} = 0$ A1(3)

If the coefficients b_n, $n = 0,1 \ldots (N+1)$ are defined as in equations A1(2), then, using the identity A1(3),

$$\sum_{n=o}^{N} a_n \cdot \cos\left(n+\tfrac{1}{2}\right)\frac{\pi x}{\ell} = (b_o - b_1)\ \cos\frac{\pi x}{2\ell}$$

1(c) $\sum_{n=o}^{N}{}' \ a_n\ T_n(x)$

The recurrence relation for the Chebyshev polynomials $T_n(x)$ is

$$T_{n+1}(x) - 2x\, T_n(x) + T_{n-1}(x) = 0 \qquad \text{A1(4)}$$

Hence, in this case, define coefficients b_n, $n = 0,1, \ldots (N+1)$ given by

$$b_{N+1} = 0$$

$$b_N = a_N$$

$$b_n = a_n + 2x\, b_{n+1} - b_{n+2} \text{ for } n = (N-1),(N-2) \ldots 1,0 \qquad \text{A1(5)}$$

Then $\sum_{n=o}^{N}{}' \ a_n\ T_n(x) = b_N\ T_N(x) + \sum_{n=o}^{N-1}{}' \left\{b_n - 2x\, b_{n+1} + b_{n+2}\right\} T_n(x)$

$$= \frac{1}{2}\,(b_o - b_2) \quad \text{using equation A1(4).}$$

1(d) $\sum_{n=o}^{N}{}' \ a_n\ T_{2n}(x)$

The corresponding recurrence relation for the Chebyshev polynomials of even order is

$$T_{2n+2}(x) - 2(2x^2-1)\ T_{2n}(x) + T_{2n-2}(x) = 0 \qquad \text{A1(6)}$$

The coefficients b_n, $n = 0,1, \ldots (N+1)$ are then defined by

$$b_{N+1} = 0$$

$$b_N = a_N$$

$$b_n = a_n + 2(2x^2-1)b_{n+1} - b_{n+2},\ n = (N-1),(N-2) \ldots 1,0 \qquad \text{A1(7)}$$

$$\sum_{n=0}^{N}{}' a_n T_{2n}(x) = b_N T_{2N}(x) + \sum_{n=0}^{N-1}{}' \left\{ b_n - 2(2x^2-1)b_{n+1} + b_{n+2} \right\} T_{2n}(x)$$

$$= \frac{1}{2}(b_0 - b_2) \text{ from the identity A1(6).}$$

Notes.

(a) Similar algorithms can be derived for Fourier sine series and sums of Chebyshev polynomials of odd order but these are not given here since they are not required.

(b) To test the speed of the computing algorithms given here, the following Fourier series was used.

$$\sum_{n=0}^{N}{}' a_n \cos n\pi x + \sum_{n=1}^{N} A_n \sin n\pi x = \begin{cases} 1 & \text{for } \alpha \leqslant x \leqslant 1 \\ 0 & \text{for } -1 \leqslant x < \alpha \end{cases}$$

giving $a_0 = 1-\alpha$

$$a_n = -\frac{1}{n\pi} \sin n\pi\alpha \quad \text{for } n = 1,2 \ldots N$$

$$A_n = -\frac{1}{n\pi}\left\{(-1)^n - \cos n\pi\alpha\right\} \quad \text{for } n = 1,2 \ldots N$$

The series was evaluated for $x = -1(0.1)1$ using

(i) algorithm 1(a) and the corresponding algorithm for the sine series

(ii) direct summation of the series.

An internal computer procedure was used to determine the actual computing time taken to sum the series in each case. Using the algorithms the time taken was one quarter of that used when the direct method of summation was employed. The savings when evaluating a double Fourier series over a rectangular region are even greater. It similarly follows that the summation of Chebyshev series is much more efficient numerically when the algorithms given here are used.

(c) It should be observed that when the algorithms given here are used in a computer, it is possible to overwrite the

(c) contd.

coefficients b_n as soon as each coefficient is no longer needed. This is an advantage if computer storage is in short supply.

A P P E N D I X 2

Solution of the equations:-

$$\tfrac{1}{2}\underline{\phi}_0 + \underline{\phi}_1 + \underline{\phi}_2 \;\ldots\; + \underline{\phi}_R = \underline{0}$$

$$\underline{\phi}_{r-1} + \lambda(r)\,\underline{B}_r\,\underline{\phi}_r + \mu(r)\,\underline{\phi}_{r+1} = -4\pi^2\lambda(r)\,Q(r)\,\underline{T}$$

$$\text{for } r = 1,2 \ldots R$$

$$\underline{\phi}_{R+1} = \underline{0}$$

$\underline{\phi}_r$, $r = 0,1 \ldots R$, is an $(M \times 1)$ unknown vector,

$\underline{B}_r$, $r = 0,1 \ldots R$, is an $(M \times M)$ known matrix

$\underline{T}$ is an $(M \times 1)$ known vector and

$\lambda(r)$, $\mu(r)$, $Q(r)$ are any known scalar quantities.

The technique evaluates the vectors $\underline{\phi}_r$ using the method of Gaussian elimination. Having obtained the vector $\underline{\phi}_R$, it was not considered advisable to calculate $\underline{\phi}_{R-1}$, $\underline{\phi}_{R-2}$ ... $\underline{\phi}_0$ by backward substitution in the equations as they stand since this could well propagate errors due to the large terms in the matrix $\lambda(r)\underline{B}_r$. Accordingly the elimination was done in the following way.

Consider the case given by R = 4. This special case illustrates the derivation of the general algorithm

	$\underline{\phi}_0$	$\underline{\phi}_1$	$\underline{\phi}_2$	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r1)	$\underline{I}$	$2\underline{I}$	$2\underline{I}$	$2\underline{I}$	$2\underline{I}$	$\underline{0}$
r2)	$\underline{I}$	$\lambda(1)\underline{B}_1$	$\mu(1)\underline{I}$	$\underline{0}$	$\underline{0}$	$-4\pi^2\lambda(1)Q(1)\underline{T}$
r3)	$\underline{0}$	$\underline{I}$	$\lambda(2)\underline{B}_2$	$\mu(2)\underline{I}$	$\underline{0}$	$-4\pi^2\lambda(2)Q(2)\underline{T}$
r4)	$\underline{0}$	$\underline{0}$	$\underline{I}$	$\lambda(3)\underline{B}_3$	$\mu(3)\underline{I}$	$-4\pi^2\lambda(3)Q(3)\underline{T}$
r5)	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{I}$	$\lambda(4)\underline{B}_4$	$-4\pi^2\lambda(4)Q(4)\underline{T}$

($\underline{I}$ is the $(M \times M)$ unit matrix and $\underline{0}$ is the $(M \times M)$ zero matrix)

(new row 2 = old row 2 - row 1)

	$\underline{\phi}_0$	$\underline{\phi}_1$	$\underline{\phi}_2$	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r1)	$\underline{I}$	$2\underline{I}$	$2\underline{I}$	$2\underline{I}$	$2\underline{I}$	$\underline{0}$
r2)	$\underline{0}$	$\lambda(1)\underline{B}_1-2\underline{I}$	$(\mu(1)-2)\underline{I}$	$-2\underline{I}$	$-2\underline{I}$	$-4\pi^2\lambda(1)Q(1)\underline{T}$
r3)	$\underline{0}$	$\underline{I}$	$\lambda(2)\underline{B}_2$	$\mu(2)\underline{I}$	$\underline{0}$	$-4\pi^2\lambda(2)Q(2)\underline{T}$
r4)	$\underline{0}$	$\underline{0}$	$\underline{I}$	$\lambda(3)\underline{B}_3$	$\mu(3)\underline{I}$	$-4\pi^2\lambda(3)Q(3)\underline{T}$
r5)	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{I}$	$\lambda(4)\underline{B}_4$	$-4\pi^2\lambda(4)Q(4)\underline{T}$

The first equation will be used to obtain $\underline{\phi}_0$ and so will now be discarded. Premultiply row 2 by

$$\underline{X}_1 = (\lambda(1)\underline{B}_1 - 2\underline{I})^{-1}$$

	$\underline{\phi}_1$	$\underline{\phi}_2$	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r2)	$\underline{I}$	$(\mu(1)-2)\underline{X}_1$	$-2\underline{X}_1$	$-2\underline{X}_1$	$-4\pi^2\lambda(1)Q(1)\underline{X}_1\underline{T}$
r3)	$\underline{I}$	$\lambda(2)\underline{B}_2$	$\mu(2)\underline{I}$	$\underline{0}$	$-4\pi^2\lambda(2)Q(2)\underline{T}$
r4)	$\underline{0}$	$\underline{I}$	$\lambda(3)\underline{B}_3$	$\mu(3)\underline{I}$	$-4\pi^2\lambda(3)Q(3)\underline{T}$
r5)	$\underline{0}$	$\underline{0}$	$\underline{I}$	$\lambda(4)\underline{B}_4$	$-4\pi^2\lambda(4)Q(4)\underline{T}$

(new row 3 = old row 3 - row 2)

Write $\underline{Y}_2 = \lambda(2)Q(2)\underline{I} - \lambda(1)Q(1)\,\underline{X}_1$

	$\underline{\phi}_1$	$\underline{\phi}_2$	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r2)	$\underline{I}$	$(\mu(1)-2)\underline{X}_1$	$-2\underline{X}_1$	$-2\underline{X}_1$	$-4\pi^2\lambda(1)Q(1)\underline{X}_1\underline{T}$
r3)	$\underline{0}$	$\lambda(2)\underline{B}_2-(\mu(1)-2)\underline{X}_1$	$\mu(2)\underline{I}+2\underline{X}_1$	$2\underline{X}_1$	$-4\pi^2\ \underline{Y}_2\ \underline{T}$
r4)	$\underline{0}$	$\underline{I}$	$\lambda(3)\underline{B}_3$	$\mu(3)\underline{I}$	$-4\pi^2\lambda(3)Q(3)\underline{T}$
r5)	$\underline{0}$	$\underline{0}$	$\underline{I}$	$\lambda(4)\underline{B}_4$	$-4\pi^2\lambda(4)Q(4)\underline{T}$

The first equation of this set will be used to obtain $\underline{\phi}_1$ and will now be discarded. Premultiply row 3 by $\underline{X}_2 = (\lambda(2)\underline{B}_2-\mu(1)\underline{X}_1+2\underline{X}_1)^{-1}$

	$\underline{\phi}_2$	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r3)	$\underline{I}$	$\mu(2)\underline{X}_2+2\underline{X}_2\underline{X}_1$	$2\underline{X}_2\underline{X}_1$	$-4\pi^2\ \underline{X}_2\underline{Y}_2\underline{T}$
r4)	$\underline{I}$	$\lambda(3)\underline{B}_3$	$\mu(3)\underline{I}$	$-4\pi^2\lambda(3)Q(3)\underline{T}$
r5)	$\underline{0}$	$\underline{I}$	$\lambda(4)\underline{B}_4$	$-4\pi^2\lambda(4)Q(4)\underline{T}$

Now new row 4 = old row 4 - row 3 and

$$\underline{Y}_3 = \lambda(3)Q(3)\underline{I} - \underline{X}_2\underline{Y}_2$$

	$\underline{\phi}_2$	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r3)	$\underline{I}$	$\mu(2)\underline{X}_2+2\underline{X}_2\underline{X}_1$	$2\underline{X}_2\underline{X}_1$	$-4\pi^2\ \underline{X}_2\underline{Y}_2\underline{T}$
r4)	$\underline{0}$	$\lambda(3)\underline{B}_3-\mu(2)\underline{X}_2-2\underline{X}_2\underline{X}_1$	$\mu(3)\underline{I}-2\underline{X}_2\underline{X}_1$	$-4\pi^2\underline{Y}_3\underline{T}$
r5)	$\underline{0}$	$\underline{I}$	$\lambda(4)\underline{B}_4$	$-4\pi^2\lambda(4)Q(4)\underline{T}$

Again the first equation is discarded (it is used to solve for $\underline{\phi}_2$), and row 4 is premultipled by

$$\underline{X}_3 = (\lambda(3)\underline{B}_3 - \mu(2)\underline{X}_2 - 2\underline{X}_2\underline{X}_1)^{-1}$$

	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r4)	$\underline{I}$	$\mu(3)\underline{X}_3 - 2\underline{X}_3\underline{X}_2\underline{X}_1$	$-4\pi^2\underline{X}_3\underline{Y}_3\underline{T}$
r5)	$\underline{I}$	$\lambda(4)\underline{B}_4$	$-4\pi^3\lambda(4)Q(4)\underline{T}$

Subtracting and writing

$$\underline{Y}_4 = \lambda(4)Q(4)\underline{I} - \underline{X}_3\,\underline{Y}_3$$

	$\underline{\phi}_3$	$\underline{\phi}_4$	R.H.S.
r4)	$\underline{I}$	$\mu(3)\underline{X}_3 - 2\underline{X}_3\underline{X}_2\underline{X}_1$	$-4\pi^2\underline{X}_3\underline{Y}_3\underline{T}$
r5)	$\underline{0}$	$\lambda(4)\underline{B}_4 - \mu(3)\underline{X}_3 + 2\underline{X}_3\underline{X}_2\underline{X}_1$	$-4\pi^2\underline{Y}_4\,\underline{T}$

Hence $\underline{\phi}_4 = -\,4\pi^3\,\underline{X}_4\,\underline{Y}_4\,\underline{T}$

where $\underline{X}_4 = (\lambda(4)\underline{B}_4 - \mu(3)\underline{X}_3 + 2\underline{X}_3\underline{X}_2\underline{X}_1)^{-1}$

This indicates the following general procedure for calculating $\underline{\phi}_R$.

(i) Evaluate $\underline{X}_1 = (\lambda(1)\underline{B}_1 - 2\underline{I})^{-1}$,

$$\underline{Y}_1 = \lambda(1)Q(1)\underline{I}, \qquad \text{A2(1)}$$

$$\underline{Z}_1 = \underline{I}$$

(ii) Define recurrence relations as follows:-

$$\underline{Y}_n = \lambda(n)Q(n)\underline{I} - \underline{X}_{n-1}\,\underline{Y}_{n-1},$$

$$\underline{Z}_n = -\,\underline{X}_{n-1}\,\underline{Z}_{n-1} \qquad \text{A2(2)}$$

$$\underline{X}_n = (\lambda(n)\underline{B}_n - \mu(n-1)\underline{X}_{n-1} - 2\underline{Z}_n)^{-1}$$

for $n = 2,3,4 \ldots R$.

$$\text{Then } \underline{\phi}_R = -\,4\pi^3\underline{X}_R\,\underline{Y}_R\,\underline{T} \qquad \text{A2(3)}$$

Unfortunately, in this method all the $\underline{X}_n$, $\underline{Y}_n$, $\underline{Z}_n$ for $n = 1,2 \ldots R$ must be stored since they are needed again for the back substitution. Consider again the case $R = 4$.

$$\underline{\phi}_3 = -4\pi^2\underline{X}_3\underline{Y}_3\underline{T} - \mu(3)\underline{X}_3\underline{\phi}_4 - 2\underline{Z}_4\,\underline{\phi}_4$$

$$\underline{\phi}_2 = -4\pi^2\underline{X}_2\underline{Y}_2\underline{T} - \mu(2)\underline{X}_2\underline{\phi}_3 - 2\underline{Z}_3(\underline{\phi}_3+\underline{\phi}_4)$$

$$\underline{\phi}_1 = -4\pi^3\underline{X}_1\underline{Y}_1\underline{T} - \mu(1)\underline{X}_1\underline{\phi}_2 - 2\underline{Z}_2\sum_{r=2}^{4}\underline{\phi}_r$$

$$\underline{\phi}_0 = -2\sum_{r=1}^{4}\underline{\phi}_r = -2\underline{Z}_1\sum_{r=1}^{4}\underline{\phi}_r$$

Hence, in the general case,

$$\underline{\phi}_n = -4\pi^2 \underline{X}_n \, \underline{Y}_n \, \underline{T} - \mu(n) \underline{X}_n \, \underline{\phi}_{n+1} - 2\underline{Z}_{n+1} \sum_{r=n+1}^{R} \underline{\phi}_r \qquad \text{A2(4)}$$

for n = (R-1), (R-2) ... 1

$$\underline{\phi}_0 = -2 \sum_{r=1}^{R} \underline{\phi}_r \qquad \text{A2(5)}$$

Equations A2(3), (4) and (5) together with the recurrence relations A2(1) and (2) give the complete solution for the vectors $\underline{\phi}_r$, r = 0,1,2 ... R.

The method can easily be modified to solve the set of linear equations

$$a_{r-1} + \lambda(r) a_r + \mu(r) a_{r+1} = K\,P(r)$$

for r = 1,2, ... R

$$a_{R+1} = 0$$

$$\tfrac{1}{2}a_0 + a_1 + a_2 + \dots + a_R = 0$$

for the unknowns a_r, r = 0,1 ... R. $\lambda(r)$, $\mu(r)$, $P(r)$ and K are known coefficients. In this case

$$x_1 = \frac{1}{(\lambda(1)-2)} \qquad \text{A2(6)}$$

$$y_1 = P(1)$$

$$z_1 = 1$$

and the recurrence relations are

$$y_n = P(n) - x_{n-1} y_{n-1}$$

$$z_n = -x_{n-1} z_{n-1} \qquad \text{A2(7)}$$

$$x_n = \frac{1}{(\lambda(n) - \mu(n-1) x_{n-1} - 2z_n)}$$

for n = 2,3 ... R.

The solution is then given by

$$a_R = K x_R y_R$$

and $$a_n = K x_n y_n - \mu(n) x_n a_{n+1} - 2 z_{n+1} \sum_{r=n+1}^{R} a_r \qquad A2(8)$$

for $n = (R-1),(R-2) \ldots 2,1$ A2(9)

$$a_0 = -2 \sum_{r=1}^{R} a_r \qquad A2(10)$$

In this case also, the coefficient $\lambda(r)$ is large and the algorithm, as given here, has been shown to give a more accurate solution than that obtained by using the original equations as they stand for the back substitution to determine a_{R-1}, a_{R-2} .... a_0.

APPENDIX 3.

LINEAR DEPENDENCE OF THE BOUNDARY EQUATIONS IN THE DOUBLE CHEBYSHEV APPROXIMATION.

We assume that the solution $F(X,Y)$ of the differential equation

$$\left(\frac{b}{a}\right)^2 \frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = f(X,Y)$$

over the region $-1 \leqslant X \leqslant 1$, $-1 \leqslant Y \leqslant 1$ is given in the form

$$F = \sum_{r=0}^{R}{}' \sum_{m=0}^{M}{}' a_{rm} T_{2r}(Y) T_{2m}(X) \qquad A3(1)$$

Thus F is an even function of both X and Y.

(a) Dirichlet boundary conditions.

The boundary conditions in this case are

(i) $F = 0$ when $X = \pm 1$, $-1 \leqslant Y \leqslant 1$

(ii) $F = 0$ when $Y = \pm 1$, $-1 \leqslant X \leqslant 1$.

Boundary condition (i) gives

$$\sum_{r=0}^{R}{}' T_{2r}(Y) \sum_{m=0}^{M}{}' a_{rm} = 0.$$

This must be true for all values of Y so that

$$\sum_{m=0}^{M}{}' a_{rm} = 0 \quad \text{for} \quad r = 0,1, \ldots R \qquad A3(2)$$

Similarly boundary condition (ii) gives

$$\sum_{r=0}^{R}{}' a_{rm} = 0 \quad \text{for } m = 0,1, \ldots M \qquad A3(3)$$

Equations A3(2) and (3) may be written in matrix form as follows:-

$$\underline{A}\ \underline{x} = \underline{0}.$$

$\underline{x}$ is the $\{(R+1)(M+1) \times 1\}$ vector of unknowns given by

(a) contd.

$$\underline{x} = \begin{bmatrix} a_{00} \\ a_{01} \\ \vdots \\ a_{cM} \\ a_{10} \\ \vdots \\ a_{1M} \\ \vdots \\ a_{R0} \\ \vdots \\ a_{RM} \end{bmatrix}$$

and $\underline{A}$ is the $[(R+M+2) \times \{(R+1)(M+1)\}]$ partitioned matrix

$$\left[\begin{array}{cccccc} \underline{r} & \underline{o} & \underline{o} & \cdots\cdots & \underline{o} \\ \underline{o} & \underline{r} & \underline{o} & \cdots\cdots & \underline{o} \\ \underline{o} & \underline{o} & \underline{r} & \cdots\cdots & \underline{o} \\ \vdots & & & & \\ \underline{o} & \underline{o} & \underline{o} & \cdots\cdots & \underline{r} \\ \tfrac{1}{2}\underline{I} & \underline{I} & \underline{I} & \cdots\cdots & \underline{I} \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} \text{boundary condition (i) } \{(R+1) \text{ rows in total}\} \\ \text{boundary condition (ii) } \{(M+1) \text{ rows}\} \end{array}$$

where $\underline{I}$ is the $(M + 1) \times (M + 1)$ unit matrix,

$\underline{r}$ is the $1 \times (M + 1)$ row vector with elements $(\tfrac{1}{2} \;\; 1 \;\; 1 \;\; \ldots. \;\; 1)$

$\underline{o}$ is the $1 \times (M + 1)$ zero vector

Now the rank of the matrix $\underline{A}$ is unaltered by elementary row and column operations on the sub-matrices.

Hence rank $(\underline{A})$ =

(a) contd.

$$\text{rank}\begin{bmatrix} \tfrac{1}{2}\underline{I} & \underline{I} & \underline{I} & \cdots & \underline{I} \\ \underline{r} & \underline{o} & \underline{o} & \cdots & \underline{o} \\ \underline{o} & \underline{r} & \underline{o} & \cdots & \underline{o} \\ \underline{o} & \underline{o} & \underline{r} & \cdots & \underline{o} \\ \vdots & & & & \\ \underline{o} & \underline{o} & \underline{o} & \cdots & \underline{r} \end{bmatrix} =$$

$$\text{rank}\begin{bmatrix} \tfrac{1}{2}\underline{I} & \underline{I} & \underline{I} & \cdots & \underline{I} \\ \underline{o} & -2\underline{r} & -2\underline{r} & \cdots & -2\underline{r} \\ \underline{o} & \underline{r} & \underline{o} & \cdots & \underline{o} \\ \underline{o} & \underline{o} & \underline{r} & \cdots & \underline{o} \\ \vdots & & & & \\ \underline{o} & \underline{o} & \underline{o} & \cdots & \underline{r} \end{bmatrix}$$

(new second row = old second row $-2\underline{r}$ (row 1))

=

$$\text{rank}\begin{bmatrix} \tfrac{1}{2}\underline{I} & \underline{I} & \underline{I} & \cdots & \underline{I} \\ \underline{o} & -2\underline{r} & -2\underline{r} & \cdots & -2\underline{r} \\ \underline{o} & \underline{o} & -\underline{r} & \cdots & -\underline{r} \\ \underline{o} & \underline{o} & \underline{r} & \cdots & \underline{o} \\ \vdots & & & & \\ \underline{o} & \underline{o} & \underline{o} & \cdots & \underline{r} \end{bmatrix}$$

(new row 3 = old row 3 + $\tfrac{1}{2}$ row 2)

=

$$\text{rank}\begin{bmatrix} \tfrac{1}{2}\underline{I} & \underline{I} & \underline{I} & \cdots & & \underline{I} \\ \underline{o} & -2\underline{r} & -2\underline{r} & \cdots & & -2\underline{r} \\ \underline{o} & \underline{o} & -\underline{r} & \cdots & & -\underline{r} \\ \underline{o} & \underline{o} & \underline{o} & -\underline{r} & \cdots & -\underline{r} \\ \underline{o} & \underline{o} & \underline{o} & \underline{r} & \cdots & \underline{o} \\ \vdots & \vdots & \vdots & & & \\ \underline{o} & \underline{o} & \underline{o} & & & \underline{r} \end{bmatrix}$$

(new row 4 = old row 4 + row 3)

Proceeding in this way, adding successive rows, we finally obtain

(a) contd.

$$\text{rank}\ (\underline{A}) = \text{rank} \begin{bmatrix} \tfrac{1}{2}\underline{I} & \underline{I} & \underline{I} & \cdots & \cdots & \underline{I} \\ \underline{0} & -2\underline{r} & -2\underline{r} & & & -2\underline{r} \\ \underline{0} & \underline{0} & -\underline{r} & -\underline{r} & \cdots & -\underline{r} \\ \underline{0} & \underline{0} & \underline{0} & -\underline{r} & \cdots & -\underline{r} \\ \vdots & & & & & \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \cdots & -\underline{r} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} \end{bmatrix} \begin{matrix} (M+1)\ \text{rows} \\ \\ (R+1)\ \text{rows} \\ \text{in total} \end{matrix}$$

$$= \underline{R + M + 1}$$

Hence, one of the equations A3(2) and (3) is linearly dependent on the remainder and so one of the equations must be discarded in solving for the coefficients a_{rm}.

(b) Mixed boundary conditions.

Suppose that boundary condition (i) of case (a) is now $\frac{\partial F}{\partial X} = 0$ when $X = \pm 1$, $-1 \leq Y \leq 1$,

boundary condition (ii) remaining as for case (a).

Assume that $\partial F/\partial X$ is given by

$$\frac{\partial F}{\partial X} = \sideset{}{'}\sum_{r=0}^{R} T_{2r}(Y) \sum_{m=1}^{M} b_{rm}\, T_{2m-1}(X) \qquad \text{A3(4)}$$

so that the coefficients a_{rm}, b_{rm} are related by the equations

$$a_{rm} = \frac{1}{2.2m} \left\{ b_{rm} - b_{r,m+1} \right\} \qquad \text{A3(5)}$$

for $r = 0, 1, \ldots\ R$

$m = 1, 2\ \ldots\ M$

where $b_{r,M+1} \equiv 0.$

Now boundary condition (i) gives

$$\sum_{m=1}^{M} b_{rm} = 0 \text{ for } r = 0, 1, \ldots\ R \qquad \text{A3(6)}$$

Solving equations A3(5) for the coefficients b_{rm} gives

(b) contd.

$$b_{rM} = 2.(2M)\, a_{rM}$$

$$b_{r,M-1} = 2.(2M-2)a_{r,M-1} + 2(2M)a_{rM}$$

$$b_{r,M-2} = 2.(2M-4)a_{r,M-2} + 2(2M-2)a_{r,M-1} + 2(2M)a_{rM}$$

etc.

Hence, by addition, equation A3(6) becomes

$$\sum_{m=1}^{M} m^2 a_{rm} = 0 \quad \text{for } r = 0,1, \ldots R \qquad \text{A3(7)}$$

This equation now replaces equation A3(2) of case (a). Thus the matrix $\underline{A}$ is identical in form with that defined previously except that the vector $\underline{r}$ is replaced by the $[1 \times (M+1)]$ row vector $\underline{s}$ given by

$$\underline{s} = (0 \quad 1^2 \quad 2^2 \quad \ldots . M^2).$$

The argument then proceeds exactly as for case (a) (with $\underline{s}$ replacing $\underline{r}$) and again there is one equation linearly dependent on the rest and thus one of the boundary equations must be discarded. Obviously, from symmetry considerations, if the boundary conditions are of the form

(i) $F = 0$ when $X = \pm 1, \quad -1 \leqslant Y \leqslant 1$

(ii) $\dfrac{\partial F}{\partial Y} = 0$ when $Y = \pm 1, -1 \leqslant X \leqslant 1$

we again have one redundant boundary equation.

(c) Neumann boundary conditions.

The boundary conditions in this case are

(i) $\dfrac{\partial F}{\partial X} = 0$ when $X = \pm 1, \quad -1 \leqslant Y \leqslant 1$

(ii) $\dfrac{\partial F}{\partial Y} = 0$ when $Y = \pm 1, \quad -1 \leqslant X \leqslant 1$

As for case (b) boundary condition (i) gives rise to the set of equations A3(7). Similarly boundary condition (ii) leads to the equations

(c) contd.

$$\sum_{r=1}^{R} r^2 \; a_{rm} = 0 \text{ for } m = 0,1,2 \ldots M \qquad \text{A3(8)}$$

so that in this case the matrix $\underline{A}$ is given by

$$\underline{A} = \left[\begin{array}{ccccc}
\underline{s} & \underline{o} & \underline{o} & \cdots\cdots & \underline{o} \\
\underline{o} & \underline{s} & \underline{o} & & \underline{o} \\
\underline{o} & \underline{o} & \underline{s} & & \underline{o} \\
\vdots & & & & \\
\underline{o} & \underline{o} & \underline{o} & \cdots\cdots & \underline{s} \\
\underline{0} & 1^2\underline{I} & 2^2\underline{I} & & R^2\underline{I}
\end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} (R+1) \text{ rows in total} \\ (M+1) \text{ rows} \end{array}$$

where $\underline{s}$ is as defined for case (b)

$\underline{o}$ is the $1 \times (M+1)$ zero vector

$\underline{0}$ is the $(M+1) \times (M+1)$ zero matrix

$$\text{rank}(\underline{A}) = \text{rank} \left[\begin{array}{ccccc}
\underline{s} & \underline{o} & \underline{o} & \cdots\cdots & \underline{o} \\
\underline{0} & 1^2\underline{I} & 2^2\underline{I} & \cdots\cdots & R^2\underline{I} \\
\underline{o} & \underline{s} & \underline{o} & \cdots\cdots & \underline{o} \\
\underline{o} & \underline{o} & \underline{s} & \cdots\cdots & \underline{o} \\
\vdots & & & & \\
\underline{o} & \underline{o} & \underline{o} & & \underline{s}
\end{array}\right]$$

$$= \text{rank} \left[\begin{array}{cccccc}
\underline{s} & \underline{o} & \underline{o} & \cdots & \cdots & \underline{o} \\
\underline{0} & 1^2\underline{I} & 2^2\underline{I} & \cdots & \cdots & R^2\underline{I} \\
\underline{o} & \underline{o} & -2^2\underline{s} & -3^2\underline{s} & & -R^2\underline{s} \\
\underline{o} & \underline{o} & \underline{s} & \underline{o} & & \underline{o} \\
\vdots & & & & & \\
\underline{o} & \underline{o} & \underline{o} & \underline{o} & \cdots\cdots & \underline{s}
\end{array}\right] \quad \begin{array}{l} \text{(new row 3} \\ = \text{old row 3} \\ - \underline{s} \text{ (row 2))} \end{array}$$

(c) contd.

$$= \text{rank} \begin{bmatrix} \underline{s} & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & 1^2\underline{I} & 2^2\underline{I} & 3^2\underline{I} & & R^2\underline{I} \\ \underline{0} & \underline{0} & -2^2\underline{s} & -3^2\underline{s} & & -R^2\underline{s} \\ \underline{0} & \underline{0} & \underline{0} & -3^2\underline{s} & & -R^2\underline{s} \\ \vdots & & & & & \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & & \underline{s} \end{bmatrix} \quad \begin{matrix}\text{(new row 4 =} \\ 2^2 \text{ old row 4} \\ \text{+ row 3)}\end{matrix}$$

Proceeding in this way,

$$\text{rank}\ (\underline{A}) = \text{rank} \begin{bmatrix} \underline{s} & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & 1^2\underline{I} & 2^2\underline{I} & & R^2\underline{I} \\ \underline{0} & \underline{0} & -2^2\underline{s} & & -R^2\underline{s} \\ \vdots & & & & \\ \underline{0} & \underline{0} & \underline{0} & \cdots & -R^2\underline{s} \\ \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} \end{bmatrix} \begin{matrix} \} \ (M+1) \text{ rows} \\ \\ \} \ R \text{ rows in total} \end{matrix}$$

$$= R + M + 1$$

Thus, there is again one equation which is linearly dependent on the remainder and so one equation must be discarded.

In all three cases (i.e. Dirichlet, Neumann and mixed boundary conditions) there is one boundary equation linearly dependent on the remainder when F is assumed to be an even function of both X and Y.

APPENDIX 4.

ITERATIVE SOLUTION OF A COUPLED SET OF EQUATIONS.

We have to solve a coupled set of equations of the form

$$PP(m,k)\ X_{mk} - \lambda \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} AA(p,m)\ BB(q,k)\ Y_{pq}$$

$$= A(m)\ B(k) \qquad A4(1)$$

$$PP(m,k)\ Y_{mk} + \lambda \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} AA(p,m)\ BB(q,k)\ X_{pq} = 0 \qquad A4(2)$$

for $m = 0,1, \ldots\ M$

$k = 0,1, \ldots\ (M-m)$

X_{mk}, Y_{mk} are the unknowns to be determined and PP(m,k), AA(m,k), BB(m,k), A(m), B(k) and λ are known coefficients. From equation A4(2),

$$Y_{pq} = \frac{-\lambda}{PP(p,q)} \sum_{i=o}^{M}{}' \sum_{j=o}^{M-i} AA(i,p)\ BB(j,q)\ X_{ij} \qquad A4(3)$$

for $p = 0,1 \ldots\ M$

$q = 0,1 \ldots\ (M-p)$,

Substituting for Y_{pq} in equation A4(1) gives

$$PP(m,k)X_{mk}+\lambda^2 \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} \frac{AA(p,m)BB(q,k)}{PP(p,q)} \sum_{i=o}^{M}{}' \sum_{j=o}^{M-i} AA(i,p)BB(j,q)X_{ij}$$

$$= A(m)\ B(k) \qquad A4(4)$$

for $m = 0,1, \ldots\ M$

$k = 0,1\ \ldots\ (M-m)$.

The coefficient of X_{mk} in equation A4(4) is

$$PP(m,k) + \lambda^2 \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} \frac{\{AA(p,m)BB(q,k)\}^2}{PP(p,q)} \quad \text{for } m \neq 0$$

since $AA(p,m) = AA(m,p)$

$BB(q,k) = BB(k,q)$

If m = 0 the coefficient of X_{ok} is

$$PP(0,k) + \frac{\lambda^2}{2} \sum_{p=o}^{M}{}' \sum_{q=o}^{M-p} \frac{\{AA(p,o)BB(q,k)\}^2}{PP(p,q)}$$

Define arrays as follows:-

$$G_1(p,k) = \sum_{q=o}^{M-p} \frac{[BB(q,k)]^2}{PP(p,q)}, \quad p = 0,1 \dots M, \quad k = 0,1 \dots M.$$

$$G_2(m,k) = \sum_{p=o}^{M}{}' [AA(p,m)]^2 \, G_1(p,k), \quad m = 0,1, \dots M, \quad k = 0,1 \dots M-m,$$

$$F_1(i,q) = \sum_{j=o}^{M-i} BB(j,q)X_{ij}, \quad i = 0,1 \dots M, \quad q = 0,1 \dots M$$

$$F_2(p,q) = \sum_{i=o}^{M}{}' AA(i,p) \, F_1(i,q) \quad p = 0,1 \dots M, \quad q = 0,1 \dots (M-p)$$

$$F_3(p,k) = \sum_{q=o}^{M-p} \frac{BB(q,k)}{PP(p,q)} F_2(p,q), \quad p = 0,1 \dots M, \quad k = 0,1 \dots M$$

$$F_4(m,k) = \sum_{p=o}^{M}{}' AA(p,m) \, F_3(p,k), \quad m = 0,1 \dots M, \quad k = 0,1 \dots M-m$$

An iterative process for the unknowns X_{mk} is then given by

$$X_{mk}^{(n)} \left\{ PP(m,k) + \lambda^2 G_2(m,k) \right\} = A(m)\,B(k) - \lambda^2 F_4^{(n-1)}(m,k) + \lambda^2 G_2(m,k) X_{mk}^{(n-1)}$$

for $m \neq 0$ A4(5)

and

$$X^{(n)}_{ok}\left\{PP(0,k)+\frac{\lambda^2}{2}G_2(0,k)\right\} = A(0)B(k) - \lambda^2 F_4^{(n-1)}(0,k) + \frac{\lambda^2}{2}G_2(0,k)\,X^{(n-1)}_{ok} \qquad A4(6)$$

for $m = 0,1,2 \ldots M$

$k = 0,1,2 \ldots (M-m)$

($X^{(n)}_{mk}$ denotes the value of X_{mk} after the n'th iteration).

To improve the rate of convergence of the iterative scheme an accelerating factor α was used given by

$$X'^{(n)}_{mk} = \alpha X^{(n)}_{mk} + (1-\alpha)X'^{(n-1)}_{mk} \qquad (0 < \alpha \leq 1) \qquad A4(7)$$

and the values of $X'^{(n-1)}_{mk}$ were used in the iterative scheme given by equations A4(5) and (6). Having obtained the values of the coefficients X_{mk}, then the values of Y_{mk} are obtained from equation A4(3).

Notes.

1. It is not desirable to insert each new value of the unknown X_{mk} as it is found since this involves modification of all the arrays F_1, F_2, F_3 and F_4 at each stage of the calculation. The method used calculates a complete set of iterates $X'^{(n)}_{mk}$ for $m = 0,1 \ldots M$, $k = 0, 1 \ldots (M-m)$ and then recalculates the arrays $F_1^{(n)}$, $F_2^{(n)}$, $F_3^{(n)}$ and $F_4^{(n)}$ ready for the next complete iteration.
2. The accelerating factor α was determined by trial and error. For the simple conductor in the slot as described in Chapter 6 the best value of α was found to be unity. The problem of the rectangular conductor in a transverse magnetic field required a value of α of 0.5. The iterative scheme for this latter problem

2. contd.

had a much slower rate of convergence than that for the slot problem. This was presumably due to the relatively much larger area of the insulating region.

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