INVESTIGATION OF FINITE ELEMENT SCHEMES FOR THE NAVIER STOKES EQUATIONS

by

AMRIK SINGH SANGHA

A thesis submitted for the degree of
Doctor of Philosophy
at the
University of Aston in Birmingham

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SUMMARY

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The main aim of this thesis has been to develop a reliable numerical method for the solution of the two-dimensional Navier-Stokes equations. The numerical method to be used was the finite element method. A literature survey revealed that a limitation common to all finite element methods available to date is that they only produce solutions for low Reynold's numbers. However, for aerodynamics applications, Reynold's numbers of the order of 10^6 are frequently encountered. At these levels conventional finite element methods break down completely. It was felt that this limitation could be overcome by the use of new types of shape functions.

The search for the new shape functions were carried out in three stages. Firstly a new method is presented for deriving shape functions for a wide class of second order ordinary differential equations with significant first order derivatives. The method is then extended to derive shape functions for a wide class of elliptic partial differential equations with similar properties. Several numerical examples are presented to illustrate the advantages of the new shape functions over the traditional polynomial shape functions.

The shape functions developed for partial differential equations are then used to construct a new finite element scheme for the Navier-Stokes equations. The scheme was implemented on a computer and the numerical results obtained indicated that the new scheme was more stable than the conventional schemes.

FINITE ELEMENTS FOR VISCOUS FLOWS

DEDICATION

To my Father, without whose earlier sacrifices this work would not have been possible.

ACKNOWLEDGEMENT

I would like to thank Malcolm Sylvester for his patient supervision of the work presented in this thesis and indeed for the many valuable suggestions he has made. My sincere thanks are also due to Professor N. Mullineux for his encouragement in the course of this research project. Last, but by no means least, I would like to thank Audrey Breakspear for her diligent typing of the final result.

CONTENTS

CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF THE PROBLEM

Over the last two decades or so finite element methods have made a deep impact in the field of structural mechanics. The finite element method has, in general, certain advantages over the traditional finite difference method. These are the ease with which irregular geometries non-uniform meshes and imposition of appropriate boundary conditions can be applied.

The use of finite element methods in the field of fluid mechanics is a relatively new innovation. It is anticipated that the method will again supersede the finite difference methods in the ways mentioned above.

In classical hydrodynamics, the fluid is assumed to be inviscid and incompressible, the so-called ideal fluid. An extensive mathematical theory has been developed for the ideal fluid and in addition finite element applications to potential flow problems may be found in (11, 29). Indeed it is fair to say that for potential flow problems the lead of the finite element method has already been established (46).

However, in real life problems fluids are in general viscous and compressible. The theory of ideal fluids fails to explain many phenomena of real fluids when the effects of viscosity and compressibility become important. In constructing a theory for real fluids it must be remembered that flows in many practical cases are turbulent in nature. Unfortunately most mathematical models of turbulent flow are empirical in nature and consequently analytic solutions to practical turbulent flow problems are not available. However, many important features of turbulent flows may be obtained from the study of laminar flows. Even then the complete set of equations describing laminar viscous flows are extremely complicated, non-linear and analytically intractable except in certain special cases.

In a vast number of engineering problems, substantial simplifying approximations can be made without the introduction of an intolerable error. Perhaps the most common approximation of viscous fluids is that the fluid is assumed to be incompressible. For example in many problems it may be assumed that steady state conditions prevail and the flow is two dimensional.

The equations under these assumptions are a system of non-linear elliptic partial differential equations. These are described in the next section.

1.2 THE NAVIER STOKES EQUATIONS

The two-dimensional, steady state flow of an incompressible viscous fluid is described by the solution of a system of coupled non-linear partial differential equations expressing mass conservation and the local transport and diffusion of momentum. In rectangular cartesian coordinates, the velocity distribution, $\underline{q}' = u'\underline{i} + v'\underline{j}$, satisfy the system

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + \frac{1}{\rho}\frac{\partial p'}{\partial x'} = \nu\nabla^2 u' \qquad (1.2.1)$$

$$u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + \frac{1}{\rho}\frac{\partial p'}{\partial y'} = \nu\nabla^2 v'$$

where ρ is the density and ν the kinematic viscosity. Non-dimensionalise equations (1.2.1) with respect to a characteristic length, L; and velocity, U_∞, and identify the Reynold's Number

$$Re = \frac{U_\infty L}{\nu}$$

Equations (1.2.1) in non-dimensional form become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{Re}\nabla^2 u \qquad (1.2.2)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{Re}\nabla^2 v$$

Numerous attempts have been made in the literature to solve equations (1.2.2) using the finite element method. These will be considered in detail in Chapter 3. A limitation common to all the methods available to date is that they only produce solutions for low values of the Reynold's number (of the order of a few hundred).

However, for aerodynamics applications, Reynold's numbers of the order of 10^6 are frequently encountered. At these levels conventional methods breakdown completely. The aim of this work has been to construct a finite element scheme which will produce a solution of the equations (1.2.2) for practical Reynold's numbers.

1.3 PLAN OF THIS THESIS

Chapter 2 presents that part of the basic theory of the finite element method which is most relevant to the work presented in this thesis. This chapter is included mainly for completeness. However, it does serve the purpose of setting up a notation and also some of the results are approached from a viewpoint which is different to the conventional viewpoint. It is suggested that the reader who is familiar with the mathematics of the finite element method does no more than skim through this chapter.

In Chapter 3 a survey is made of some of the most popular methods of solving equations (1.2.2) by finite element methods. The methods are carefully compared and their limitations noted.

The thesis then goes on to describe how some of the limitations can be overcome by the use of new types of shape functions.

In order to justify the use of the chosen shape functions the author begins in Chapter 4 with a discussion of ordinary differential equations, which exhibit similar characteristics to the Navier Stokes equations. In Chapter 5 the discussion is extended to a single elliptic partial differential equation which also exhibits characteristics similar to those of the Navier Stokes equations.

A method is presented for constructing shape functions for both ordinary and partial differential equations (of the type considered) which results in stable finite element schemes.

In Chapter 6 the theory developed is applied to the Navier Stokes equations. A new finite element scheme for the Navier Stokes equations is presented. This scheme is then used to make a numerical study of the Hiemenz flow problem for low values of the Reynold's number. At practical Reynold's numbers it is found that the iterative method for solving the non-linear finite element equations fails to converge. However, the results indicate that the new finite element scheme is probably stable for all Reynolds numbers.

In Chapter 7 stability of the new finite element scheme for practical Reynold's numbers is established. To achieve this the author by-passes the problem of convergence by studying a system of partial differential equations (the quasi Navier Stokes equations) which arise in each iteration of the solution procedure for the Navier Stokes equations. An analytic solution to the quasi Navier Stokes equations is obtained to validate the numerical results obtained from the new finite element scheme.

CHAPTER TWO

NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS USING FINITE ELEMENTS METHODS

2.1 Introduction

This Chapter gives a résumé of the theory required to apply finite element techniques to solve boundary value problems numerically. Although, many of the results quoted are well known, it is considered to be worthwhile including them for completeness. Also frequent reference will be made to some of the results in later chapters. Another significant function of this Chapter is to set up a notation and style of presentation to be used throughout this thesis.

The author strongly believes that the literature on the finite element method is full of obscurities. The main reason for this is the lack of a standard notation. It is hoped that the notation adopted in this thesis will be both illuminating and attractive.

2.2 Mathematical Statement of Boundary Value Problem

This Chapter is mainly concerned with boundary value problems where the dependent variable ϕ (a scalar, or more generally, a vector $\underline{\phi}$) in a domain R is to be determined by solving a field equation of the form

$$L\phi = f \quad \text{in } R, \tag{2.2.1}$$

Subject to boundary conditions of the form

$$B_i\phi = g_i \qquad i = 1,2,\ldots,P, \tag{2.2.2}$$

on the boundary ∂R of R. The operator L contains ϕ and its derivatives (up to some order p) with respect to the independent variables $x_1, x_2, \ldots x_n$, and will generally be a linear (elliptic) operator, i.e. ϕ and its derivatives appear linearly in it. The operators B_i contain ϕ and its derivatives (usually normal to the boundary) up to some order q, and in general f is a function of the independent variables. If ϕ is a vector $\underline{\phi}$, L and B_i will in general be matrices of differential operators, and f and g will be vectors.

A few definitions are in order.

The operator L is <u>self-adjoint</u> if for any two elements ϕ, ψ from its field of definition (domain R)

$$\int_R \phi L\psi dR = \int_R \psi L\phi dR \tag{2.2.3}$$

The operator L is <u>positive</u> if

$$\int_R \phi L\phi dR \geq 0 \tag{2.2.4}$$

and <u>positive</u> <u>definite</u> if in equations (2.2.4) the equality holds only for $\phi \equiv 0$. It may be shown that if L is positive definite then equation (2.2.1) cannot have more than one solution.

2.3 The Equivalent Variational Problem

The problem presented in the last section may some times be formulated variationally. This means that a functional $\chi(\phi)$ is to be minimised

$$\chi(\phi) = \min, \quad \phi \text{ in } R \qquad (2.3.1)$$

The functional is some integral of ϕ and its derivatives over R and/or ∂R. If the integrand of the functional is denoted by F, it is known from the Calculus of Variations that F satisfies the Euler-Lagrange equation. When the various derivatives of F are evaluated, this latter equation reduces to a differential equation in ϕ which, of course, is identical to the original domain equation (2.2.1).

The problem of finding a functional whose Euler-Lagrange equation is precisely the differential equation (2.2.1) is not an easy one. However, if the operator L is linear and positive definite then it may be shown that

$$\chi(\phi) = (L\phi,\phi) - 2(f,\phi) \qquad (2.3.2)$$

Some examples of variational principles for differential equations with Dirichlet boundary conditions are given below:

(1) Laplace's Equation

$$\chi(\phi) = \int_R \tfrac{1}{2}(\phi_x{}^2+\phi_y{}^2)\,dR$$

$$\phi_{xx} + \phi_{yy} = 0$$

$$\phi \text{ given on } \partial R.$$

(2) Poisson's Equation

$$\chi(\phi) = \int_R (\phi_x{}^2+\phi_y{}^2+2f\phi)\,dR$$

$$\phi_{xx} + \phi_{yy} = f$$

$$\phi \text{ given on } \partial R.$$

So far problems with Dirichlet boundary conditions have been considered. If now, for example, the functional

$$\chi(\phi) = \int_R F(x,y,\phi,\phi_x,\phi_y)\,dR \qquad (2.3.3)$$

is minimised, where ϕ is not given on ∂R, the necessary conditions are

$$\frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \phi_y}\right) = 0 \text{ in } R$$

$$\frac{\partial F}{\partial \phi_x}\cos\alpha + \frac{\partial F}{\partial \phi_y}\sin\alpha = 0, \text{ on } \partial R$$

where α is the angle which the normal to the surface makes with the horizontal. The condition on ∂R is the natural boundary condition of (2.3.3). If the boundary conditions are neither Dirichlet nor natural, $\chi(\phi)$ requires modification. Writing $\chi(\phi)$ as

$$\chi(\phi) = \int_R F(x,y,\phi,\phi_x,\phi_y)dR + \int_{\partial R} G(x,y,\phi,\phi_s,\phi_n)dS$$

The necessary conditions become Mitchell (41)

$$\frac{\partial F}{\partial \phi_x}\cos\alpha + \frac{\partial F}{\partial \phi_y}\sin\alpha + \frac{\partial G}{\partial \phi} - \frac{\partial}{\partial s}\left(\frac{\partial G}{\partial \phi_s}\right) = 0,$$

$$\frac{\partial G}{\partial u_n} = 0.$$

The function G is chosen so that the above equation coincides with the boundary conditions of the problem.

Examples of variational principles for differential equations with more general boundary conditions are

(i) Poissons Equation

$$\chi(\phi) = \int_R (\phi_x{}^2+\phi_y{}^2+2f\phi)dR - \int_{\partial R} (A\phi-2\beta)\phi dS$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} = f \text{ in } R$$

$$\frac{\partial\phi}{\partial n} + A\phi = \beta \text{ on } \partial R.$$

(ii) $$\chi(\phi) = \int_R \{a(\phi_x{}^2+\phi_y{}^2) + c\phi^2+2f\phi\}dR$$

$$- \int_{\partial R} a(A\phi^2-2\beta\phi)dS$$

$$(a\phi_x)_x + (a\phi_y)_y - c\phi = f \text{ in } R$$

$$\frac{\partial\phi}{\partial n} + A\phi = \beta \qquad \text{on } \partial R$$

Having found the variational equivalent of a problem the functional $\chi(\phi)$ may be minimised numerically. The finite element method, which is an extension of the classical Ritz method, attempts to minimise the functional χ in the following manner -

1. The region R is separated by imaginary lines into a number of subregions called 'finite elements' or simply elements. These elements may be of any shape although in this thesis only polygonal elements are used.
2. The elements are assumed to be interconnected at a discrete number of nodal points situated on their boundaries. The values of ϕ at these nodal points will be the basic unknown parameters of the problem.
3. A function (or functions) is chosen to define uniquely the value of ϕ within each element in terms of its nodal values.
4. The union of all such functions is taken to represent ϕ over the entire region R. Thus ϕ is represented

in terms of the unknown values of ϕ at the nodal points.

5. This representation of ϕ is substituted into the integral defining χ and the values of ϕ at the nodal points required to minimise χ are determined.

How these steps are carried out will be discussed in detail. Poisson's equation with Dirichlet boundary conditions will be used throughout this Chapter to illustrate the ideas introduced.

2.4 The Subdivision of the Region

The region R is subdivided into discrete subregions or finite elements with the boundaries of each element being plane or curvilinear faces, and with the adjacent boundaries of any pair of elements being coincident. The last condition means that where two faces of a pair of elements are in contact, the edges around these faces and the vertices of these faces are, respectively, coincident. Commonly used elements are of triangular, polygonal, or polyhedral form. At similar positions in each element, a number of points are identified as nodes or nodal points. These are generally at the vertices of the elements and at strategic positions of an edge or within the element. The subdivision of a two-dimensional region into triangular elements with nodes at the vertices is illustrated in figure (2.1) below.

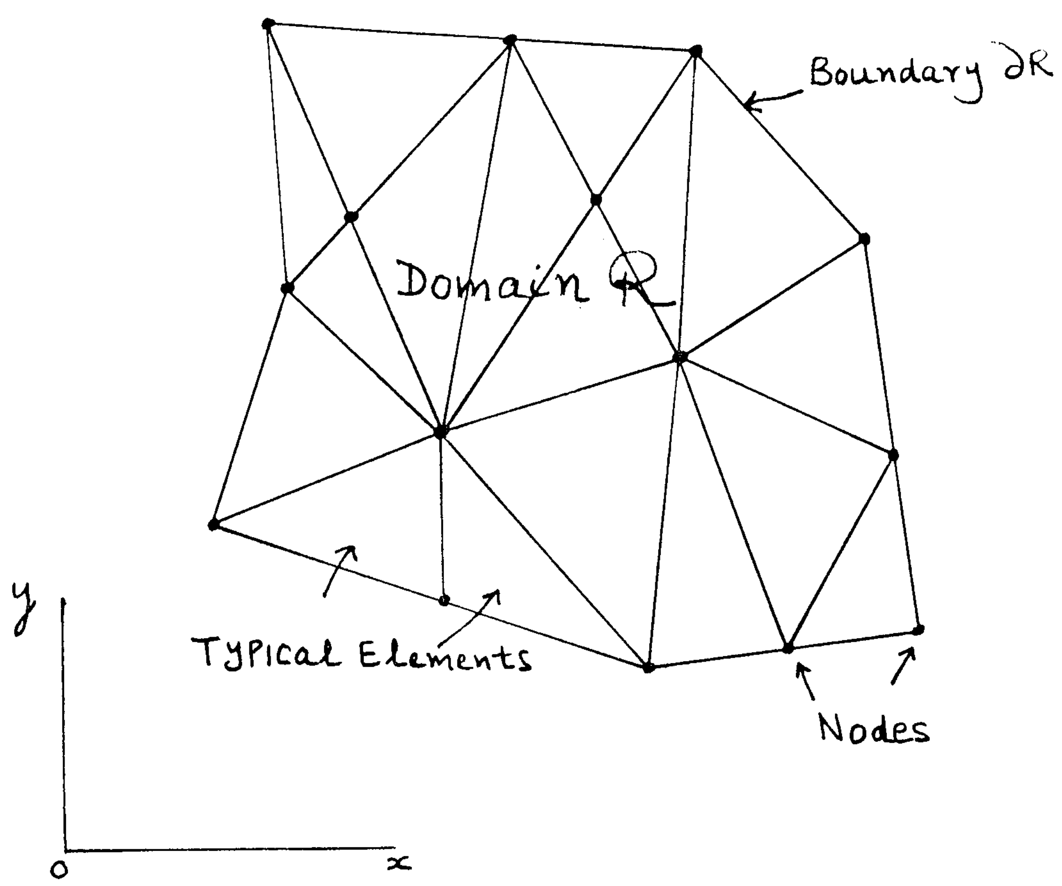


Figure (2.1)

Let the number of elements into which R is subdivided be ℓ, and the total number of nodes in and on the boundary of R be M. The number of nodes in a single element will be taken as m. The M total nodes are identified serially by the <u>system node numbers</u> 1,2,...,i,...,M. The ℓ elements are similarly identified by the <u>element numbers</u> 1,2,...,e,...ℓ. The value which the solution has at a node is the <u>nodal value</u> of ϕ, denoted by ϕ_i, where i is the <u>node number</u> of that node. When an element e is being considered on its own, the m nodal values of the element will be identified as $\phi_1,\phi_2,\ldots,\phi_m$, these being the node identifiers.

The notation described above is now clarified with reference to figure (2.2) below

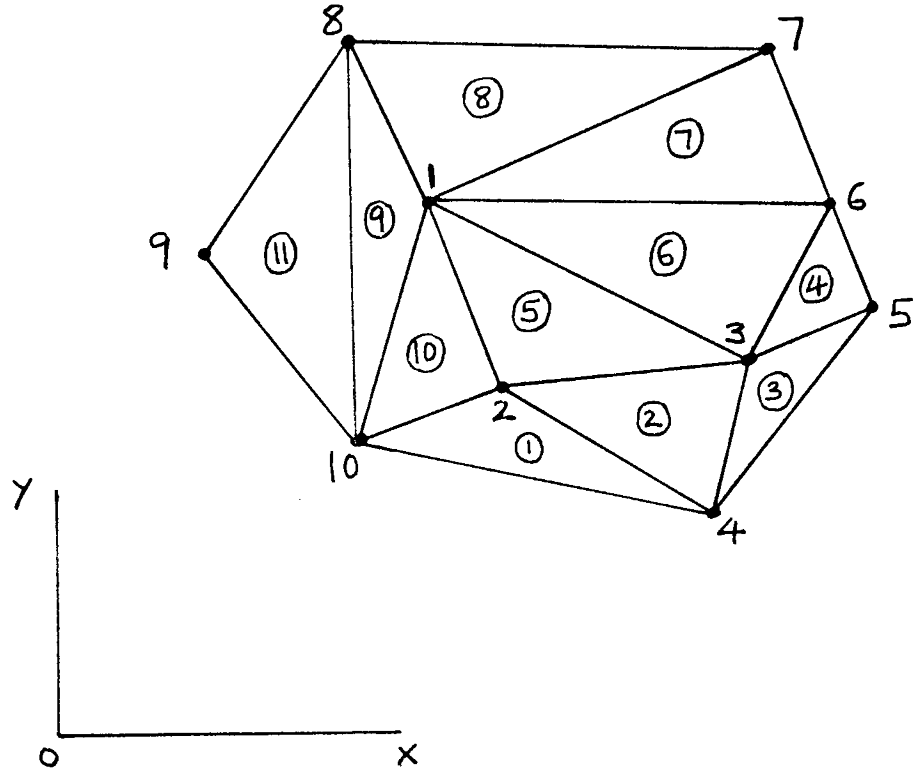


Figure (2.2)

Figure (2.2) depicts the subdivision of a two dimensional region into three node triangular elements. The circled numbers are element numbers whilst the uncircled numbers are node numbers. With the notation given above it is easily seen that for the grid in figure (2.2)

M = Total number of nodes = 10

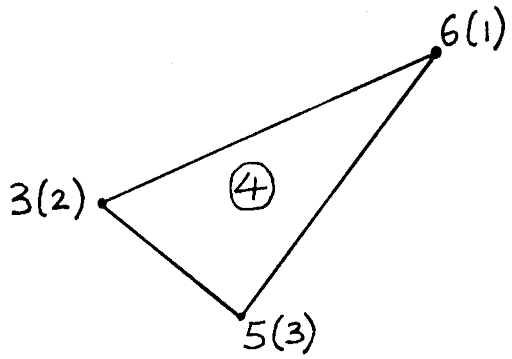
m = Number of nodes in an element = 3

ℓ = Total number of elements = 11

The node identifiers of an element should be clearly distinguished from the node numbers of that element. For example in element 4 there are 3 nodes, whose node numbers are, respectively, 3,5,6. The element node identifiers 1,2,3 would be allocated to the nodes of the element according to some convention (e.g. counter clockwise), and might correspond to the system node numbers as tabulated below

Element Node Identifiers	System Node Numbers
1	6
2	3
3	5

The situation for element 4 is also shown in the diagram below



where the numbers in brackets are node identifiers.

Having subdivided the region R as described above the aim is then to find the values of ϕ at the nodal points. This is accomplished by minimisation of the functional associated with the differential equation that is to be solved over the region R. However, before describing the minimisation procedure it is necessary to discuss a key idea in finite element work, namely that of "element shape functions".

2.5 The Element Shape Functions

The function ϕ is now defined piecewise over each elemental region. Within an element, e, it will be supposed that ϕ can be approximated by a linear combination of suitably chosen functions. The functional form of ϕ chosen is called the trial function. Thus over a typical element e a trial function is assumed to be

$$\phi(x,y) = \sum_{j=1}^{m} a_j f_j(x,y) \qquad (2.5.1)$$

where a_j's are constants and $f_j(x,y)$ $(j = 1,2,\ldots,m)$ are suitably chosen functions. If (2.5.1) is written down at each node of element e there results the following linear algebraic system of equations viz.

$$\phi(x_i,y_i) = \sum_{j=1}^{m} a_j f_j(x_i,y_i) \qquad (2.5.2)$$

$$i = 1,2,\ldots,m$$

Equation (2.5.2) may be written in matrix form as

$$\begin{bmatrix} f_1(x_1,y_1) & f_2(x_1,y_1) & \cdots & f_m(x_1,y_1) \\ f_1(x_2,y_2) & f_2(x_2,y_2) & \cdots & f_m(x_2,y_2) \\ \vdots & \vdots & & \vdots \\ f_1(x_m,y_m) & f_2(x_m,y_m) & \cdots & f_m(x_m,y_m) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_m \end{bmatrix} \quad (2.5.3)$$

Solving (2.5.3) for the constants a_j $(j=1,2,\ldots,m)$ and substituting back in (2.5.1) the trial function may be written in terms of the nodal values of ϕ as

$$\phi(x,y) = \sum_{j=1}^{m} N_j(x,y)\phi_j \quad (2.5.4)$$

The functions $N_j(x,y)$ $(j = 1,2,\ldots,m)$ are called the <u>shape functions</u>. From (2.5.4) it is easily seen that the shape functions satisfy the condition

$$N_j(x_K,y_K) = \delta_{jK} \quad (2.5.5)$$

The procedure described above for calculating the shape functions is rather tedious and not recommended. In the next section a formula is derived from which the shape functions may be calculated directly.

2.6 The Shape Function Formula

From (2.5.4) it is easily seen that

$$\phi(x,y) - N_1\phi_1 - N_2\phi_2 - \ldots - N_m\phi_m = 0$$

Using (2.5.1) to substitute for ϕ, ϕ_1, ϕ_2, ...,ϕ_m in the above equation gives

$$\sum_{j=1}^{m} a_j f_j(x,y) - N_1 \sum_{j=1}^{m} a_j f_j(x_1,y_1) - N_2 \sum_{j=1}^{m} a_j f_j(x_2,y_2)$$

$$- \ldots\ldots\ldots - N_m \sum_{j=1}^{m} a_j f_j(x_m,y_m) = 0.$$

Rearranging this equation gives

$$a_1\left[f_1(x,y) - \sum_{j=1}^{m} N_j f_1(x_j,y_j)\right] + a_2\left[f_2(x,y) - \sum_{j=1}^{m} N_j f_2(x_j,y_j)\right]$$

$$+ \ldots\ldots\ldots + a_m\left[f_m(x,y) - \sum_{j=1}^{m} N_j f_m(x_j,y_j)\right] = 0.$$

The last equation holds for all a_i $(i = 1,2,\ldots,m)$. Hence the coefficient of each of the a's must be zero. This gives rise to the following set of simultaneous algebraic equations.

$$\begin{bmatrix} f_1(x_1,y_1) & f_1(x_2,y_2) & \cdots & f_1(x_m,y_m) \\ f_2(x_1,y_1) & f_2(x_2,y_2) & \cdots & f_2(x_m,y_m) \\ \vdots & \vdots & & \vdots \\ f_m(x_1,y_1) & f_m(x_2,y_2) & \cdots & f_m(x_m,y_m) \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_m \end{bmatrix} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ \vdots \\ f_m(x,y) \end{bmatrix} \quad (2.6.1)$$

All the shape functions may be found directly from equation (2.6.1). This formula will be called the <u>shape function formula</u>. It is worth noting that every shape function $N_i(x,y)$ will be of the form

$$N_i(x,y) = \beta_{i1}f_1(x,y) + \beta_{i2}f_2(x,y) + \ldots + \beta_{im}f_m(x,y)$$

$$i = 1,2,\ldots,m.$$

Where β_{ij} is a function of the coordinates of the nodes of the element considered. Finally it must be mentioned that the formula is valid in any coordinate system although cartesian coordinates were used in the proof.

2.7 Conforming Two Dimensional Elements and their Shape Functions

The trial function is generally required to be continuous between adjacent elements. All elements for which this continuity requirement is satisfied are called <u>conforming elements</u>. The Shape Functions of the most commonly used two dimensional conforming elements will now be derived using equation (2.6.1).

(i) Four Node Rectangle

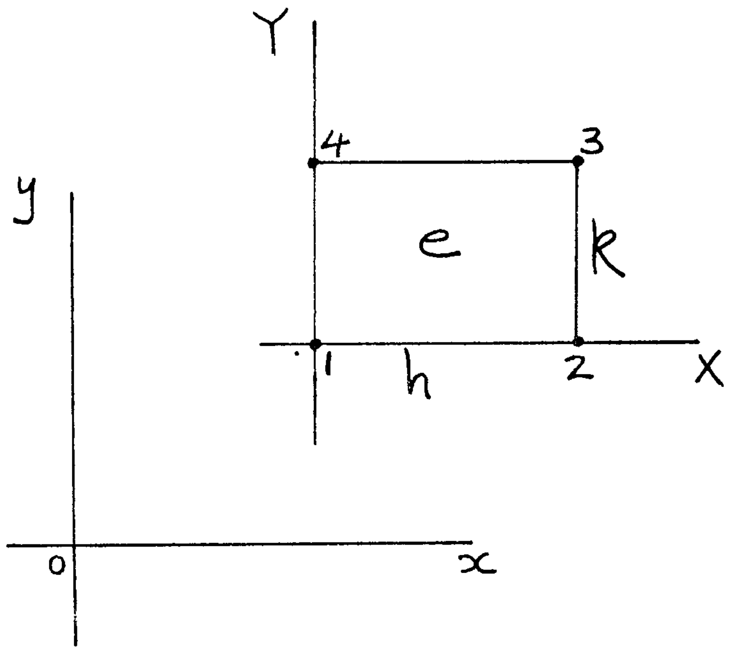


Figure (2.3)

In figure (2.3) e is a typical four node rectangular element of a finite element grid with node identifiers 1,2,3,4 as shown. To ensure that the element is conforming the trial function is chosen as

$$\phi = A + Bx + Cy + Dxy \qquad (2.7.1)$$

It is clear that this trial function is continuous between adjacent elements as its value at every point on a side of the element depends only on the nodal values on that side. The calculation of the shape functions may be simplified if the element is referred to the local (X,Y) coordinate system. The local and global coordinates are related by the transformation equations.

$$x = X + x_1 \tag{2.7.2}$$

$$y = Y + y_1 \tag{2.7.3}$$

Where (x_1, y_1) are the coordinates of node 1 in the (x,y) frame of reference. The trial function in the (X,Y) plane will be given by

$$\phi = A' + B'X + C'Y + D'XY \tag{2.7.4}$$

Using (2.6.1) the Shape functions are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & h & h & 0 \\ 0 & 0 & k & k \\ 0 & 0 & hk & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} 1 \\ X \\ Y \\ XY \end{bmatrix} \tag{2.7.5}$$

Solving (2.7.5) for the Shape functions gives

$$N_i(X,Y) = \beta_{i1} + \beta_{i2}X + \beta_{i3}Y + \beta_{i4}XY \tag{2.7.6}$$

$$i = 1,2,3,4$$

where

$$\beta_{11} = 1; \quad \beta_{12} = -\frac{1}{h}; \quad \beta_{13} = -\frac{1}{k}; \quad \beta_{14} = \frac{1}{hk}$$

$$\beta_{21} = 0; \quad \beta_{22} = \beta_{12}; \quad \beta_{23} = 0; \quad \beta_{24} = -\beta_{14}$$

$$\beta_{31} = 0; \quad \beta_{32} = 0; \quad \beta_{33} = 0; \quad \beta_{34} = \beta_{14}$$

$$\beta_{41} = 0; \quad \beta_{42} = 0; \quad \beta_{43} = \frac{1}{k}; \quad \beta_{44} = -\beta_{14}$$

The Shape functions $N_i(X,Y)$ may now be obtained in the global (x,y) coordinate system by using (2.7.2) and (2.7.3). However it will be seen later that this step is not necessary when the finite element method is applied to obtain numerical solutions to differential equations.

(ii) <u>The Three Node Triangle</u>

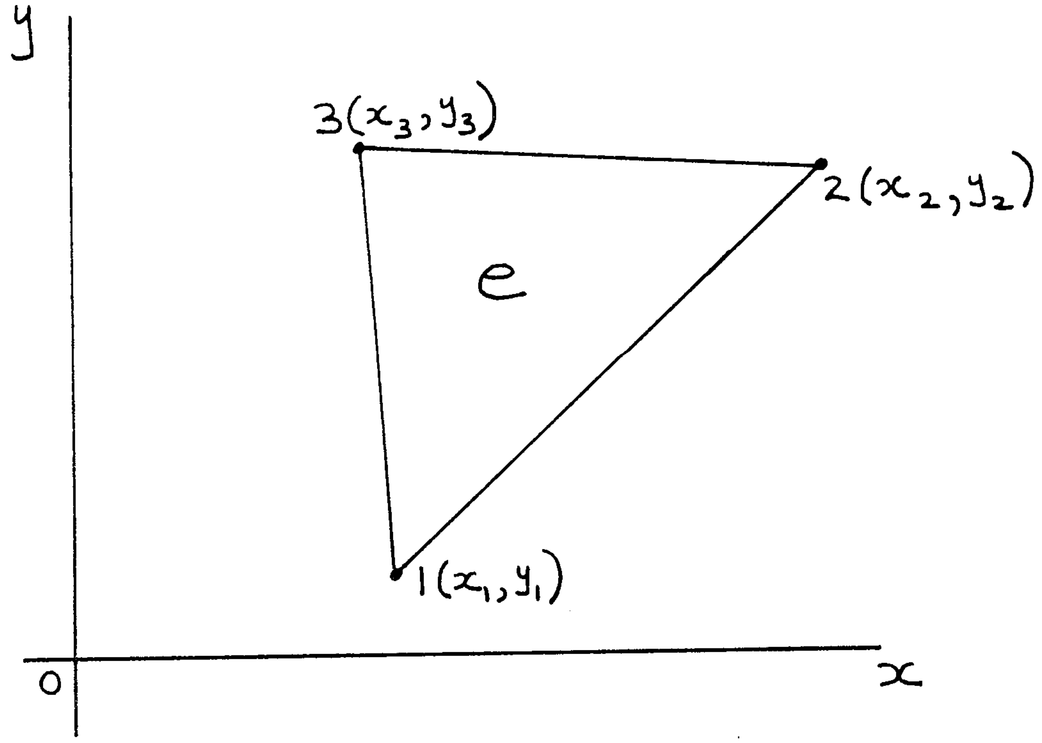


Figure (2.4)

A three node triangular element with node identifiers 1,2,3 is depicted in figure (2.4). The trial function which is continuous between adjacent elements is easily seen to be

$$\phi = A + Bx + Cy \qquad (2.7.7)$$

This is so because the value of ϕ at every point on a side of the element depends only on the nodal values of ϕ on that side. Using (2.6.1) the Shape functions are given by

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \qquad (2.7.8)$$

Solving (2.7.8) for the Shape functions using Cramer's rule gives

$$N_i(x,y) = \beta_{i1} + \beta_{i2}x + \beta_{i3}y \qquad (2.7.9)$$

$$i = 1,2,3,$$

where

$$\beta_{11} = \frac{(x_2y_3 - x_3y_2)}{2\Delta}; \quad \beta_{12} = \frac{(y_2-y_3)}{2\Delta}; \quad \beta_{13} = \frac{(x_3-x_2)}{2\Delta};$$

$$\beta_{21} = \frac{(x_3y_1 - x_1y_3)}{2\Delta}; \quad \beta_{22} = \frac{(y_3-y_1)}{2\Delta}; \quad \beta_{23} = \frac{(x_1-x_3)}{2\Delta};$$

$$\beta_{31} = \frac{(x_1y_2 - x_2y_1)}{2\Delta}; \quad \beta_{32} = \frac{(y_1-y_2)}{2\Delta}; \quad \beta_{33} = \frac{(x_2-x_1)}{2\Delta}$$

Δ being the area of triangle 1 2 3. The following determinantal formula for the area of the triangle has been used viz.

$$\Delta = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

(iii) <u>The Six Node Triangle</u>

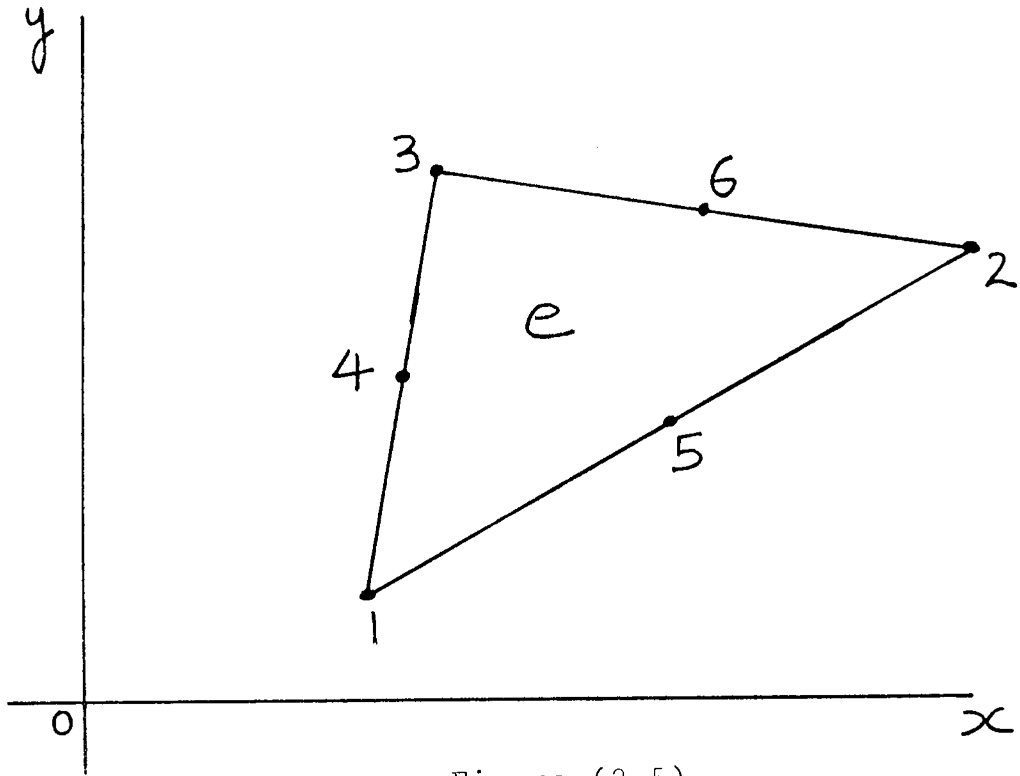


Figure (2.5)

A Six node triangular element with node identifiers 1,2,3,4,5,6 is depicted in figure (2.5). A full quadratic expansion containing six constants can be used as a trial function. Thus, for example, the function ϕ can be written as

$$\phi = A + Bx + Cy + Dx^2 + Exy + Fy^2 \qquad (2.7.10)$$

of course the usual problem of continuity of the function between adjacent elements has to be considered. It should be noticed that the variation of ϕ along any side of the element is now parabolic. Thus along any side such as 1 - 5 - 2 the function ϕ may be written as

$$\phi = a + bs + cs^2 \qquad (2.7.11)$$

The variation of ϕ being parabolic. The three values of ϕ at nodes 1, 5 and 2 uniquely determine the parabola, and continuity of the function at these nodes with those of the adjacent element automatically guarantees the continuity of the function throughout the interface 1 - 5 - 2.

Using (2.6.1) the Shape functions are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ x_1{}^2 & x_2{}^2 & x_3{}^2 & x_4{}^2 & x_5{}^2 & x_6{}^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 \\ y_1{}^2 & y_2{}^2 & y_3{}^2 & y_4{}^2 & y_5{}^2 & y_6{}^2 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix} \qquad (2.7.12)$$

Solving (2.7.12) the Shape functions may be written as

$$N_i(x,y) = \beta_{i1}+\beta_{i2}x+\beta_{i3}y+\beta_{i4}x^2+\beta_{i5}xy+\beta_{i6}y^2 \qquad (2.7.13)$$

$$i = 1,2,3,\ldots,6.$$

The coefficients β_{ij} $(i = 1,2,\ 6;\ j = 1,2,\ldots,6)$ are highly complicated functions of the coordinates (x_i, y_i) and will not be given here. In fact the use of cartesian coordinates for triangular elements is not recommended. A more suitable system of coordinates for triangular elements will be discussed in (2.14).

2.8 Minimisation of the Functional

The finite element technique of minimising a functional will now be described in detail for Poisson's equation.

It has already been stated that the solution of Poisson's equation with Dirichlet boundary conditions minimises the functional χ where

$$\chi = \iint_R \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + 2f\phi \right] dxdy \qquad (2.8.1)$$

If the value of χ associated with an element e is called χ^e then

$$\chi^e = \iint_e \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + 2f\phi \right] dxdy \qquad (2.8.2)$$

The superscript signifies that the integration is limited to the area of element e.

If the elements used are <u>conforming</u> then

$$\chi = \sum_e \chi^e \qquad (2.8.3)$$

where the summation is carried over all the elements.

The expression for ϕ over an element i.e.

$$\phi = \sum_j N_j(x,y)\phi_j \qquad (2.8.4)$$

is then substituted in (2.8.1). This gives χ to be a function of all the nodal values of ϕ and for an extremum it is required that

$$\frac{\partial \chi}{\partial \phi_i} = 0 \tag{2.8.5}$$

or equivalently from (2.8.3)

$$\sum_e \frac{\partial \chi^e}{\partial \phi_i} = 0 \tag{2.8.6}$$

$$i = 1,2,3,\ldots M$$

The solution of these simultaneous linear algebraic equations then gives the nodal values of ϕ. The equations (2.8.6) in matrix form are

$$[A]\ \{\phi\} = \{b\} \tag{2.8.7}$$

where [A] is an (M × M) square matrix. $\{\phi\}$ and {b} are (M × 1) column vectors. The details of constructing the elements of [A] and {b} are given in the next Section.

2.9 Construction of the Algebraic Equations

From (2.8.3)

$$\frac{\partial \chi}{\partial \phi_i} = \sum_e \frac{\partial \chi^e}{\partial \phi_i} \qquad (i = 1,2,\ldots,M) \tag{2.8.8}$$

Substituting for χ^e from (2.8.2) in above gives

$$\frac{\partial \chi}{\partial \phi_i} = \sum_e \iint_e \frac{\partial}{\partial \phi_i} \left[\phi_x{}^2 + \phi_y{}^2 + 2f\phi \right] dxdy$$

Hence

$$\frac{\partial \chi}{\partial \phi_i} = \sum_e \iint_e \left[2\phi_x \frac{\partial \phi_x}{\partial \phi_i} + 2\phi_y \frac{\partial \phi_y}{\partial \phi_i} + 2f \frac{\partial \phi}{\partial \phi_i} \right] dxdy$$

But over a typical element e

$$\phi = \sum_j N_j \phi_j$$

where j is a typical node number of element e.

Substituting this in the last equation gives

$$\frac{\partial \chi}{\partial \phi_i} = 2 \sum_e \iint_e \left[\left(\sum_j \frac{\partial N_j}{\partial x} \phi_j \right) \frac{\partial N_i}{\partial x} + \left(\sum_j \frac{\partial N_j}{\partial y} \phi_j \right) \frac{\partial N_i}{\partial y} + fN_i \right] dxdy$$

i.e.

$$\frac{\partial \chi}{\partial \phi_i} = 2 \sum_e \sum_j \left[\iint_e \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \right) dxdy \right] \phi_j$$

$$+ 2 \sum_e \left[\iint_e N_i f dxdy \right] \qquad (2.9.1)$$

Define

$$\left.\begin{aligned} \alpha_{ij}^{e} &= \iint_{e} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy \\ F_i^e &= \iint_{e} N_i f dxdy \end{aligned}\right\} \quad (2.9.2)$$

The superscript e signifies that the integration is confined to the element e. For an extremum it is required that

$$\frac{\partial \chi}{\partial \phi_i} = 0$$

so that (2.9.1) gives

$$\sum_{e} \sum_{j} \alpha_{ij}^{e} = - \sum_{e} F_i^e \qquad (2.9.3)$$

It will be noticed that only elements containing node i will contribute to

$$\frac{\partial \chi}{\partial \phi_i}$$

Consequently the summation in (2.9.3) need only be carried over such elements. If e_r is an element containing node i and j is a typical node number of e_r then (2.9.3) may be written

$$\sum_{e_r} \sum_{j \varepsilon e_r} \alpha_{ij}^{e_r} \phi_j = - \sum_{e_r} F_i^{e_r} \qquad (2.9.4)$$

Of course the superscript e_r means that the integration in (2.9.2) is confined to element e_r. For brevity the superscript e_r will be replaced by (r) so that (2.9.4) becomes

$$\sum_{e_r} \sum_{j \varepsilon e_r} \alpha_{ij}^{(r)} \phi_j = - \sum_{e_r} F_i^{(r)} \qquad (2.9.5)$$

Equation (2.9.5) is more conveniently written using set notation. Let

$N^{(r)}$ - be the set of nodes in e_r

E_i - be the set of elements e_r

Thus (2.9.5) may be written

$$\sum_{e_r \varepsilon E_i} \sum_{j \varepsilon N^{(r)}} \alpha_{ij}^{(r)} \phi_j = - \sum_{e_r} F_i^{(r)} \qquad (2.9.6)$$

Introducing the further two sets N_i and E_{ij} defined as

N_i - the set of nodes in E_i

E_{ij} - the set of elements containing node i and node j.

Equation (2.9.6) becomes

$$\sum_{j \varepsilon N_i} \phi_j \sum_{e_r \varepsilon E_{ij}} \alpha_{ij}^{(r)} = - \sum_{e_r \varepsilon E_i} F_i^{(r)} \qquad (2.9.7)$$

The step from (2.9.6) to (2.9.7) is easily understood if (2.9.6) is written in extenso.

Comparing (2.9.7) with (2.8.7) i.e.

$$[A]\{\phi\} = \{b\}$$

it is seen that

$$a_{ij} = \sum_{e_r \varepsilon E_{ij}} \alpha_{ij}^{(r)} \qquad (2.9.8)$$

$$b_i = - \sum_{e_r \varepsilon E_i} F_i^{(r)} \qquad (2.9.9)$$

where a_{ij} and b_i are typical elements of [A] and {b} respectively.

For illustration purposes consider the finite element grid shown below

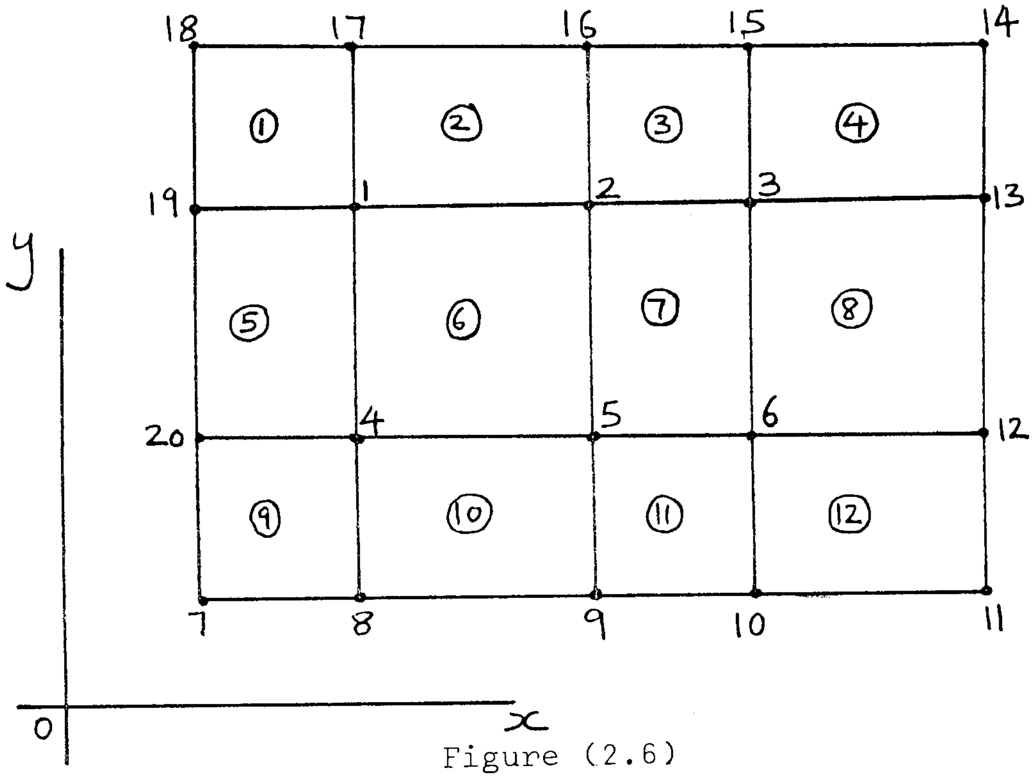


Figure (2.6)

It is required to solve Poisson's equation viz.

$$\nabla^2\phi = f \quad \text{in } R$$
$$\phi = g \quad \text{on } \partial R$$

where R is the rectangular region shown in figure (2.6). The function ϕ is prescribed on the boundary. Hence from the finite element grid it is obvious that the basic unknowns of the problem are $\phi_1, \phi_2, \ldots, \phi_6$. In order to find these nodal values of ϕ it is necessary to write (2.9.7) at all internal nodes. Suppose (2.9.7) is to be written down for node 1 then firstly the sets N_1 and E_{1j} have to be identified.

From figure (2.6) these sets are given by

$$N_1 = \{1, 4, 5, 2, 16, 17, 18, 19, 20\}$$

i	j	E_{ij}
1	1	{ 1 , 2 , 5 , 6 }
1	4	{ 5 , 6 }
1	5	{ 6 }
1	2	{ 2 , 6 }
1	16	{ 2 }
1	17	{ 1 , 2 }
1	18	{ 1 }
1	19	{ 1 , 5 }
1	20	{ 5 }
1	$j \notin N_1$	{ null set }

Equation (2.9.7) for node 1 gives

$$a_{11}\phi_1+a_{12}\phi_2+a_{14}\phi_4+a_{15}\phi_5+a_{1,16}\phi_{16}+a_{1,17}\phi_{17}+a_{1,18}\phi_{18}$$
$$+a_{1,19}\phi_{19}+a_{1,20}\phi_{20} = b_1 \qquad (2.9.10)$$

From (2.9.8) and (2.9.9) it is seen that

$$a_{11} = \alpha^1_{11} + \alpha^2_{11} + \alpha^5_{11} + \alpha^6_{11}$$

$$a_{12} = \alpha^2_{12} + \alpha^6_{12} \; ; \; a_{14} = \alpha^5_{14} + \alpha^6_{14}$$

$$a_{15} = \alpha^6_{15} \; ; \; a_{1,16} = \alpha^2_{1,16} \; ; \; a_{1,17} = \alpha^1_{1,17} + \alpha^2_{1,17}$$

$$a_{1,18} = \alpha^1_{1,18} \; ; \; a_{1,19} = \alpha^1_{1,19} + \alpha^5_{1,19} \; ; \; a_{1,20} = \alpha^5_{1,20}$$

$$a_{1j} = 0 \quad \text{for } j \notin N_1$$

$$b_1 = -(F_1^1 + F_1^2 + F_1^5 + F_1^6)$$

Equation (2.9.10) may also be written

$$a_{11}\phi_1 + a_{12}\phi_2 + a_{14}\phi_4 + a_{15}\phi_5 = R_1 \qquad (2.9.11)$$

Where

$$R_1 = b_1 - (a_{1,16}\phi_{16} + a_{1,17}\phi_{17} + a_{1,18}\phi_{18} + a_{1,19}\phi_{19} + a_{1,20}\phi_{20})$$

Notice that R_1 a known quantity since $\phi_{16}, \phi_{17}, \ldots, \phi_{20}$ are prescribed values. Similarly (2.9.7) may be written at nodes 2, 3, 4, 5 and 6. Hence six equations will be obtained for the six unknowns $\phi_1, \phi_2, \ldots, \phi_6$.

The coefficients $\alpha_{ij}^{(r)}$ and $F_i^{(r)}$ have to be evaluated to obtain a_{ij} and b_i. Now from (2.9.2)

$$\alpha_{ij}^{e} = \iint_e \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy$$

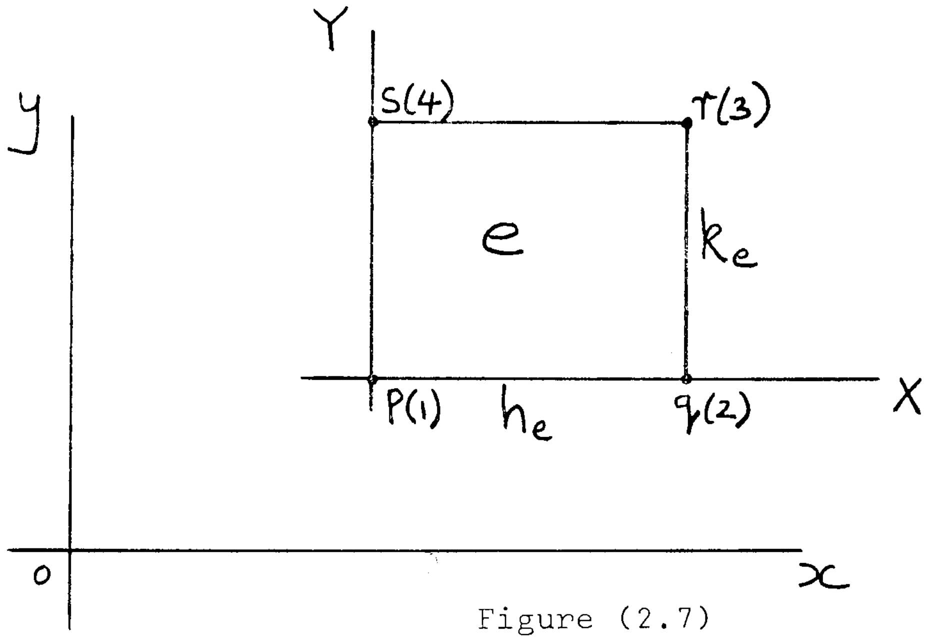


Figure (2.7)

The element e with node numbers p, q, r, s and corresponding node identifies 1, 2, 3, 4 is shown in figure (2.7). Transforming the double integral α^e_{ij} from global (x,y) to local (X,Y) coordinates gives

$$\alpha^e_{ij} = \int_0^{h_e} \int_0^{k_e} \left(\frac{\partial N_i}{\partial X} \frac{\partial N_j}{\partial X} + \frac{\partial N_i}{\partial Y} \frac{\partial N_j}{\partial Y} \right) dXdY$$

For the four node rectangle in figure (2.7) $N_i(X,Y)$ is given by

$$N_i(X,Y) = \beta^e_{i1} + \beta^e_{i2}X + \beta^e_{i3}Y + \beta^e_{i4}XY$$

$$i = 1,2,3,4$$

where β_{ij}^{e} are given in (2.7).

Thus

$$\alpha_{ij}^{e} = \int_{o}^{h_e}\int_{o}^{k_e}\Big[\left(\beta_{i2}^{e}+\beta_{i4}^{e}Y\right)\left(\beta_{j2}^{e}+\beta_{j4}^{e}Y\right)+\left(\beta_{i3}^{e}+\beta_{i4}^{e}X\right)\left(\beta_{j3}^{e}+\beta_{j4}^{e}X\right)\Big]dX\,dY$$

Let

$$P_{\xi,\eta}^{e} = \int_{o}^{h_e}\int_{o}^{k_e} X^{\xi}Y^{\eta}dX.dY = \frac{h_e^{\xi+1}k_e^{\eta+1}}{(\xi+1)\ (\eta+1)}$$

Then

$$\alpha_{ij}^{e} = (\beta_{i2}^{e}\ \beta_{j2}^{e}+\beta_{i3}^{e}\ \beta_{j3}^{e})P_{o,o}^{e} + (\beta_{i2}^{e}\ \beta_{j4}^{e}+\beta_{i4}^{e}\ \beta_{j2}^{e})P_{0,1}^{e}$$

$$+(\beta_{i3}^{e}\ \beta_{j4}^{e}+\beta_{i4}^{e}\ \beta_{j3}^{e})P_{1,0}^{e}+\beta_{i4}^{e}\ \beta_{j4}^{e}\ P_{0,2}^{e}+\beta_{i4}^{e}\ \beta_{j4}^{e}\ P_{2,0}^{e} \qquad (2.9.2)$$

$i = 1,\ 2,\ 3,\ 4$

$j = 1,\ 2,\ 3,\ 4$

Also from (2.9.2)

$$F_{i}^{e} = \iint_{e} N_{i}f\ dxdy$$

If f is approximated as a constant f_e over element e then

$$F_i^e = f_e \iint_e N_i(x,y)\,dxdy = f_e \iint_e N_i(X,Y)\,dXdY$$

i.e.

$$F_i^e = f_e \iint_e (\beta_{i1}+\beta_{i2}X+\beta_{i3}Y+\beta_{i4}XY)\,dXdY$$

$$F_i^e = f_e(\beta_{i1}P^e_{o,o}+\beta_{i2}P^e_{1,0}+\beta_{i3}P^e_{0,1}+\beta_{i4}P^e_{1,1}) \qquad (2.9.3)$$

$i = 1, 2, 3, 4$

2.10 Alternative Method of Inserting Dirichlet Boundary Conditions

For the illustrative example presented in the last section the finite element equations were written down only at the internal nodes, i.e. the nodes at which ϕ was to be determined. The resulting system of algebraic equations took the form

$$[A]\ \{\phi\} = \{b\}$$
$$(\overline{M} \times \overline{M})\ (\overline{M} \times 1) = (\overline{M} \times 1)$$

Where $\overline{M}$ is the number of internal nodes. In many cases (especially when dealing with simultaneous partial differential equations) it is found convenient to write down the finite element equations at all the nodes including boundary nodes. The resulting algebraic equations now take the form

$$[A]\{\phi\} = \{b\} \qquad (2.10.1)$$

$$(M \times M)(M \times 1) = (M \times 1)$$

Where of course as usual M is the total number of nodes. The Dirichlet boundary conditions are now applied directly to the matrix equation (2.10.1). For example if ϕ is specified at a node η on the boundary then the ηth equation in (2.10.1) is replaced by the equation viz.

$$\phi_\eta = \phi_\eta^*$$

where ϕ_η^* is the prescribed value of ϕ at node η. This gives the following rule for Dirichlet nodes.

Rule for Dirichlet Nodes

If η is a Dirichlet node (number) put zeros in the ηth row of the [A] matrix in equation (2.10.1) except for a 1 in the diagonal position, and put in the ηth row of the {b} vector the known value of ϕ for that node.

2.11 Continuity Requirements of the Trial Function

It was mentioned in (2.7) that the trial function is generally required to be continuous between adjacent elements.

This is usually the case for second order partial differential equations where the highest order derivative appearing in the associated functional is of order unity. Quite generally the functional χ is defined as

$$\chi = \int_R G(x,y,z,\phi,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z},\ldots,\frac{\partial^n\phi}{\partial x^n},\frac{\partial^n\phi}{\partial z^n})\, d(R)$$

where ϕ is the unknown function, n is the highest order derivative of ϕ and R is the region over which the solution is sought. If χ^e is the contribution of an element to the value of χ it is assumed that

$$\chi = \sum_e \chi^e \qquad (2.11.1)$$

This is generally only true if no 'infinite' values of G occur at element interfaces. The condition of continuity can therefore be restated by the requirement that <u>ϕ and all its derivatives up to order (n-1) be continuous and finite on the interface</u>.

<u>Example</u>

For the partial differential equation

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = f \text{ in } R$$

satisfying the boundary conditions that ϕ , $\frac{\partial\phi}{\partial n}$ are known on ∂R the associated functional χ is

$$\chi = \iint_R \{(\phi_{xx}^2 + \phi_{yy}^2) - 2f\phi\} dxdy \tag{2.11.2}$$

The highest order derivatives in (2.11.2) is of order two. This means that if (2.11.2) is to be minimized using finite element methods the trial function must be chosen so that ϕ and the first derivatives of ϕ are continuous between adjacent elements.

2.12 Notion of an Element Stiffness Matrix

It is quite apparent that the procedure described in (2.9) for constructing the algebraic equations is not suitable for computer implementation. However, if the equations are assembled by evaluating the contribution of an element to the global matrix then the process can be made automatic. In order to find the contribution of element e to $\frac{\partial \chi}{\partial \phi_i}$ it is necessary to compute $\frac{\partial \chi^e}{\partial \phi_i}$. Now

$$\chi^e = \iint_e \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + 2f\phi \right] dxdy$$

Proceeding as in (2.9) gives with the notation already introduced

$$\frac{\partial \chi^e}{\partial \phi_i} = \sum_j \alpha_{ij}^e \phi_j + F_i^e \tag{2.12.1}$$

Let e be as usual an m mode element with node identifiers 1,2,...,m. Writing (2.12.1) at each node of element e, i.e. for i = 1,2,...,m yields the equations

$$\begin{bmatrix} \frac{\partial \chi^e}{\partial \phi_1} \\ \frac{\partial \chi^e}{\partial \phi_2} \\ \vdots \\ \frac{\partial \chi^e}{\partial \phi_m} \end{bmatrix} = \begin{bmatrix} \alpha^e_{11} & \alpha^e_{12} & \cdots & \alpha^e_{1m} \\ \alpha^e_{21} & \alpha^e_{22} & \cdots & \alpha^e_{2m} \\ \vdots & \vdots & & \vdots \\ \alpha^e_{m1} & \alpha^e_{m2} & \cdots & \alpha^e_{mm} \end{bmatrix} \begin{bmatrix} \phi^e_1 \\ \phi^e_2 \\ \vdots \\ \phi^e_m \end{bmatrix} + \begin{bmatrix} F^e_1 \\ F^e_2 \\ \vdots \\ F^e_m \end{bmatrix} \qquad (2.12.2)$$

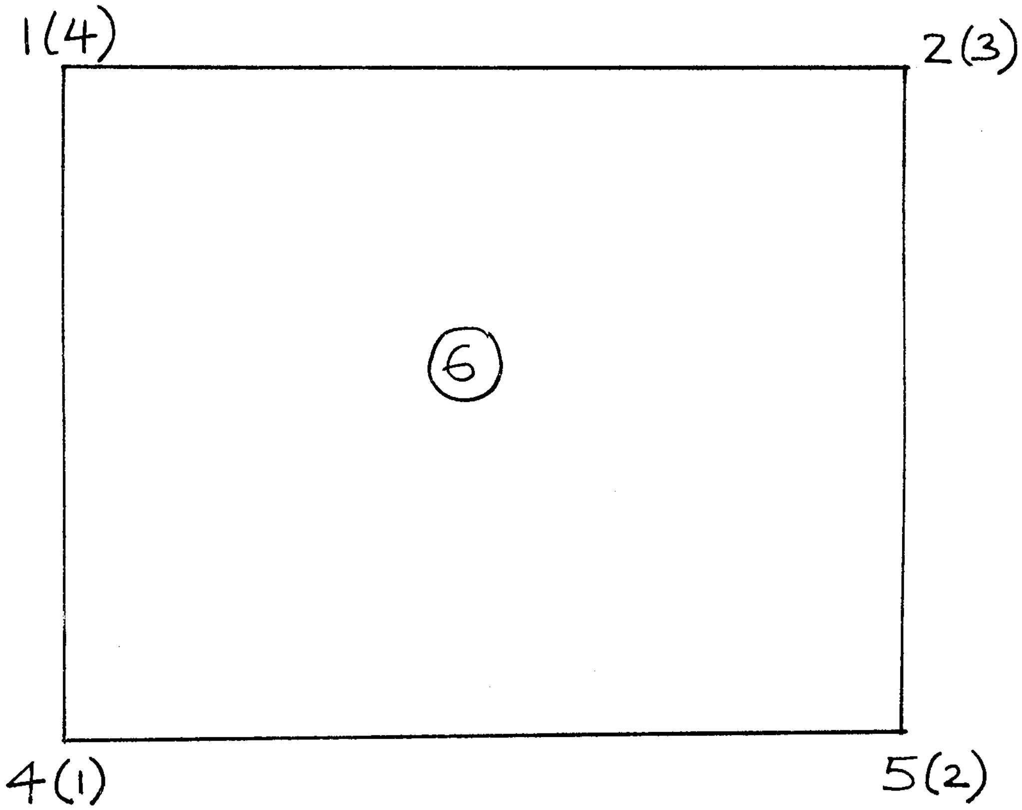
The (m × m) matrix in (2.12.2) is called the element stiffness matrix. The vector F in (2.12.3) will be called the element stiffness vector.

For example referring back to figure (2.6) equations (2.12.3) for element 6 would be

$$\begin{bmatrix} \frac{\partial \chi^6}{\partial \phi_1} \\ \frac{\partial \chi^6}{\partial \phi_2} \\ \frac{\partial \chi^6}{\partial \phi_3} \\ \frac{\partial \chi^6}{\partial \phi_4} \end{bmatrix} = \begin{bmatrix} \alpha^6_{11} & \alpha^6_{12} & \alpha^6_{13} & \alpha^6_{14} \\ \alpha^6_{21} & \alpha^6_{22} & \alpha^6_{23} & \alpha^6_{24} \\ \alpha^6_{31} & \alpha^6_{32} & \alpha^6_{33} & \alpha^6_{34} \\ \alpha^6_{41} & \alpha^6_{42} & \alpha^6_{43} & \alpha^6_{44} \end{bmatrix} \begin{bmatrix} \phi^6_1 \\ \phi^6_2 \\ \phi^6_3 \\ \phi^6_4 \end{bmatrix} + \begin{bmatrix} F^6_1 \\ F^6_2 \\ F^6_3 \\ F^6_4 \end{bmatrix}$$

(2.12.3)

Where the node identifiers are allocated to the node numbers of elements 6 as shown below



The element stiffness matrix and element stiffness vector in (2.12.3) may be evaluated using (2.19.12) and (2.19.13)

2.13 Construction of Algebraic Equations Using Element Stiffness Matrices

The final set of algebraic equations, viz.

$$\left\{\frac{\partial \chi}{\partial \phi_i}\right\} = [A]\ \{\phi\} + \{F\} = 0 \qquad (2.13.1)$$

are easier to construct using element stiffness matrices. In order to do this use is made of the equation

$$\frac{\partial \chi}{\partial \phi_i} = \sum_e \frac{\partial \chi^e}{\partial \phi_i} \qquad (2.13.2)$$

The procedure is summarised below: -

(i) $K = 0$

(ii) Initialise the elements of the global matrix [A] $(M \times M)$ and the vector {F} $(M \times 1)$ to zero.

(iii) $K = K+1$.

(iv) Take element number 'K' and compute its element stiffness matrix and element stiffness vector.

Let α^e_{ij} and F^e_i be typical elements of the ith row of the element stiffness matrix and element stiffness vector respectively. Where of course i and j are node identifiers.

(v) If I and J are node numbers corresponding to the node identifiers i and j respectively then perform the following operations -

$$\left.\begin{aligned} a_{IJ} &= a_{IJ} + \alpha_{ij}^{e} \\ F_I &= F_I + F_i^{e} \end{aligned}\right) \quad \begin{aligned} i &= 1,2,\ldots m \\ j &= 1,2,\ldots m \end{aligned}$$

where a_{IJ} and F_I are typical elements of [A] and {F} respectively.

(vi) If K is equal to ℓ (the total number of finite elements in the grid) then stop otherwise go to (iii).

It is not difficult to see that steps (i) - (vi) are in fact equivalent to writing down (2.13.2) for each node i. The equations corresponding to Dirichlet nodes are modified after assembly in the way described in (2.10).

This method of constructing the global matrix is most widely used and indeed is very suitable for computer implementation.

2.14 Use of Area Coordinates for Triangular Elements

A coordinate system will now be described which facilitates the use of triangular elements in finite element work.

Consider the point P inside a triangle (as shown in figure 2.8). Instead of specifying the position of P by rectangular cartesian coordinates (x,y), it may be specified relative to the triangle by the three areas A_1, A_2 and A_3, or, more conveniently, by the non-dimensional areas

$$L_1 = \frac{A_1}{A}; \quad L_2 = \frac{A_2}{A}; \quad L_3 = \frac{A_3}{A} \qquad (2.14.1)$$

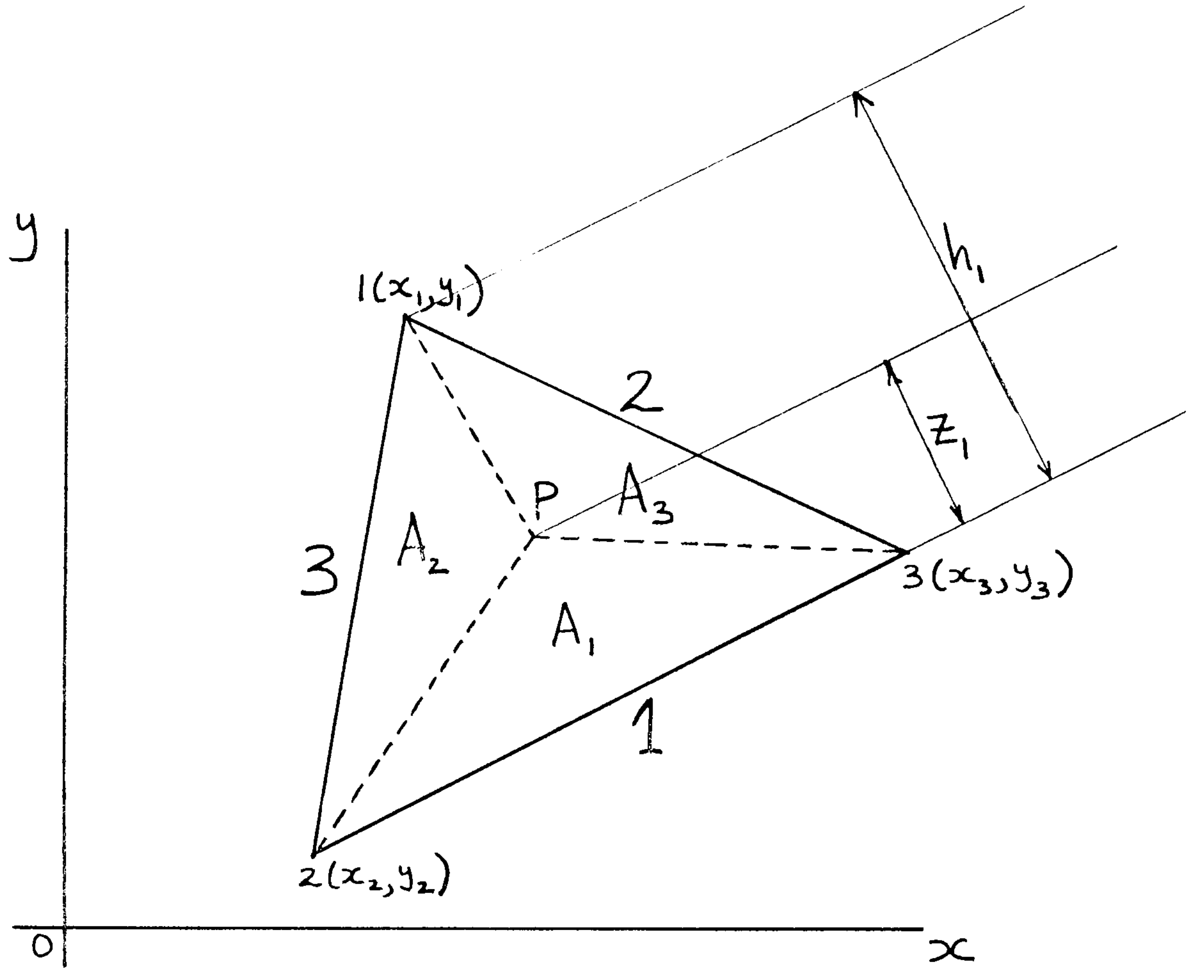


Figure (2.8)

It is clear from the foregoing relations that

$$L_1 + L_2 + L_3 = 1 \tag{2.14.2}$$

L_1, L_2 and L_3 will be used as coordinates of the point P. If desired the coordinate L_1 may also be considered to represent the non-dimensional distance

$$L_1 = \frac{z_1}{h_1} \tag{2.14.3}$$

of P from the element side 1.

Similarly the coordinates L_2 and L_3 may be interpreted as the distances from the other two sides. Thus, (L_1, L_2, L_3) constitutes a special type of cartesian coordinate system. Two coordinates suffice for specifying the pont P uniquely. This complies with the interdependence relation found in equation (2.14.2).

The area coordinates of the nodal points are -

$$\left.\begin{array}{llll} \text{Node 1} & L_1 = 1 & L_2 = L_3 = 0 \\ \text{Node 2} & L_2 = 1 & L_1 = L_3 = 0 \\ \text{Node 3} & L_3 = 1 & L_1 = L_2 = 0 \end{array}\right\} \qquad (2.14.4)$$

The equations for the sides of the triangle are -

$$\left.\begin{array}{ll} \text{Side 1} & L_1 = 0 \\ \text{Side 2} & L_2 = 0 \\ \text{Side 3} & L_3 = 0 \end{array}\right\} \qquad (2.14.5)$$

The area coordinates of the centre of gravity of triangle 123 are:

$$L_1 = L_2 = L_3 = \frac{1}{3} \qquad (2.14.6)$$

The relation between area coordinates and rectangular cartesian coordinates is established by writing down the linear relationship viz.

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \qquad (2.14.7)$$

Where the first two lines in (2.14.7) represent the conditions to be satisfied at the corners (compare with (2.14.4)) and the last line is merely a repetition of (2.14.2).

Area coordinates are expressed by rectangular cartesian coordinates by inversion of (2.14.7), yielding

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} y_{23} & x_{32} & x_2y_3-x_3y_2 \\ y_{31} & x_{13} & x_3y_1-x_1y_3 \\ y_{12} & x_{21} & x_1y_2-x_2y_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (2.14.8)$$

Where Δ is the area of triangle 123 and the notation $y_{23} = y_2-y_3$ etc. has been introduced.

The linear relationships of equation (2.14.7) shows that polynomials in (x,y) may be expressed by polynomials of the same degree in (L_1,L_2,L_3). The advantage of a polynomial in (L_1,L_2,L_3) is that these coordinates are invariant with respect to the shape and the orientation of the triangle, and that all integration formulas become remarkably simple. These advantages justify the introduction of this coordinate system.

It will now be shown that the shape functions of triangular elements can be written in terms of area coordinates.

(i) Three Node Triangle

The trial function usually taken for a three node triangle is a complete polynomial of degree unity.

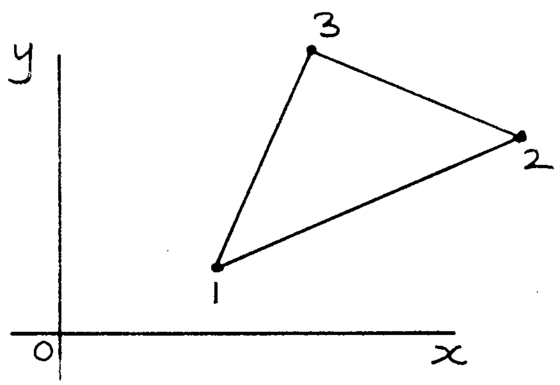
Now from (2.14.7) it is seen that

$$x = L_1(x_1 - x_3) + L_2(x_2 - x_3) + x_3 \quad (2.14.9)$$

$$y = L_1(y_1 - y_3) + L_2(y_2 - y_3) + y_3 \quad (2.14.10)$$

Thus any polynomial in (x,y) may be expressed by polynomials of the same degree in L_1 and L_2.

For the three node triangle shown below



The trial function in cartesian coordinates is

$$\phi(x,y) = A + Bx + Cy$$

By virtue of (2.14.9) and (2.14.10) the trial function in area coordinates may be taken as

$$\phi(L_1,L_2) = A' + B'L_1 + C'L_2$$

Using the "Shape Function Formula" the shape functions N_1, N_2, N_3 in terms of area coordinates are given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 \\ L_1 \\ L_2 \end{bmatrix} \qquad (2.14.11)$$

From (2.14.11) the shape functions are found to be

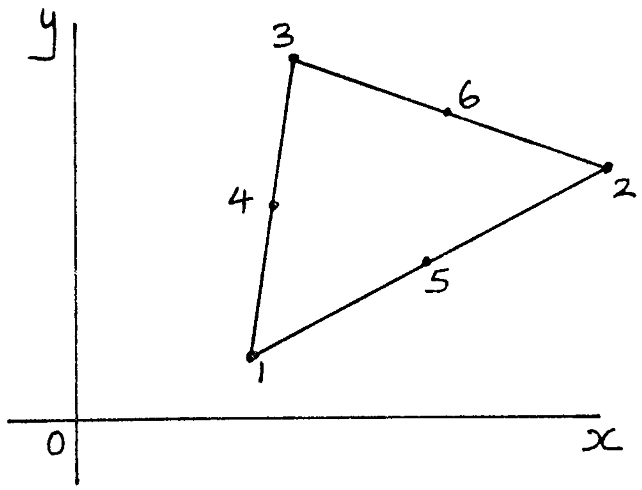
$$N_1 = L_1 ; \quad N_2 = L_2 ; \quad N_3 = 1 - L_1 - L_2 = L_3$$

Hence the shape functions for the three node triangle are simply the area coordinates.

(ii) <u>Six Node Triangle</u>

The trial function usually taken over a six node triangle is a complete polynomial of degree two.

Thus for a six node triangle shown below



the trial function in cartesian coordinates is

$$\phi(x,y) = A + Bx + Cy + Dxy + Ex^2 + Fy^2$$

By virtue of (2.14.9) and (2.14.10) the trial function in area coordinates may be taken as

$$\phi(L_1,L_2) = A' + B'L_1 + C'L_2 + D'L_1L_2 + E'L_1^2 + F'L_2^2$$

Using the "Shape Function Formula" the shape functions $N_i(L_1,L_2)$ $(i = 1,2,...,6)$ in terms of area coordinates are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{bmatrix} = \begin{bmatrix} 1 \\ L_1 \\ L_2 \\ L_1L_2 \\ L_1^2 \\ L_2^2 \end{bmatrix} \qquad (2.14.12)$$

Solving (2.14.12) for the shape function gives

$$N_1 = (2L_1-1)L_1; \qquad N_4 = 4L_1L_3$$
$$N_2 = (2L_2-1)L_2; \qquad N_5 = 4L_1L_2$$
$$N_3 = (2L_3-1)L_3; \qquad N_6 = 4L_2L_3$$

In a similar way shape functions for higher order triangular elements may be found in terms of the area coordinates (L_1,L_2,L_3)

2.15 Analytic Integration Using Area Coordinates

In finite element work various integrals have to be evaluated. These will be of the form

$$\int\int [G]dxdy \qquad (2.15.1)$$

in which [G] depends on N (shape function) and/or its derivatives with respect to global, i.e. x,y coordinates. If polynomial trial functions and polygonal elements are used then all the integrals will exhibit terms of the form

$$P_{\xi\eta} = \int\int_{e} x^{\xi} y^{\eta} dxdy \qquad (2.15.2)$$

ξ, η being positive integers

For example, in section (2.9) the solution of Poisson's equation using the four node rectangle involved the evaluation of $P_{\xi\eta}$. Of course, for rectangular elements $P_{\xi\eta}$ is easily evaluated analytically.

When using triangular elements it is found convenient to write [G] in terms of area coordinates. The only feature is presented by the fact that differentiation with respect to cartesian coordinates needs to be carried out. This is quite easy noting that

$$\frac{\partial}{\partial x} \equiv \frac{\partial L_1}{\partial x}\frac{\partial}{\partial L_1} + \frac{\partial L_2}{\partial x}\frac{\partial}{\partial L_2} + \frac{\partial L_3}{\partial x}\frac{\partial}{\partial L_3}$$

$$\frac{\partial}{\partial y} \equiv \frac{\partial L_1}{\partial y}\frac{\partial}{\partial L_1} + \frac{\partial L_2}{\partial y}\frac{\partial}{\partial L_2} + \frac{\partial L_3}{\partial y}\frac{\partial}{\partial L_3}$$

But from (2.14.8)

$$L_1 = \frac{1}{2\Delta}(y_{23}x + x_{32}y + (x_2y_3 - x_3y_2)) \text{ etc.}$$

This gives

$$\frac{\partial}{\partial x} \equiv \frac{1}{2\Delta}(y_{23}\frac{\partial}{\partial L_1} + y_{31}\frac{\partial}{\partial L_2} + y_{12}\frac{\partial}{\partial L_3})$$

$$\frac{\partial}{\partial y} \equiv \frac{1}{2\Delta}(x_{32}\frac{\partial}{\partial L_1} + x_{13}\frac{\partial}{\partial L_2} + x_{21}\frac{\partial}{\partial L_3})$$

[G] will now depend on N (shape function in terms of area coordinates) and/or its derivatives with respect to area i.e. L_1, L_2, L_3 coordinates. All the integrals will exhibit terms of the form

$$I = \iint_{e_{xy}} L_1^{\alpha} \; L_2^{\beta} \; L_3^{\gamma} \; dxdy$$

where α, β, γ are positive integers and e_{xy} is a typical triangular element in the (x;y) plane.

Fortunately an analytic expresseion for I is available and in fact

$$I = \iint_{e_{xy}} L_1^{\alpha} \quad L_2^{\beta} \quad L_3^{\gamma} \quad dxdy = \frac{\alpha! \ \beta! \ \gamma!}{(\alpha+\beta+\gamma+2)!} 2\Delta \qquad (2.15.3)$$

Δ = area of triangle, e_{xy}

Although the formula (2.15.3) is well known its proof it not. The following proof uses the beta and gamma functions in the deriviations.

The proof is divided into two parts -

(i) The first step is to show that any triangle may be mapped into a right angle triangle using area coordinates.

To this end consider a triangle in the (x,y) plane with vertices 1, 2, 3 as shown below

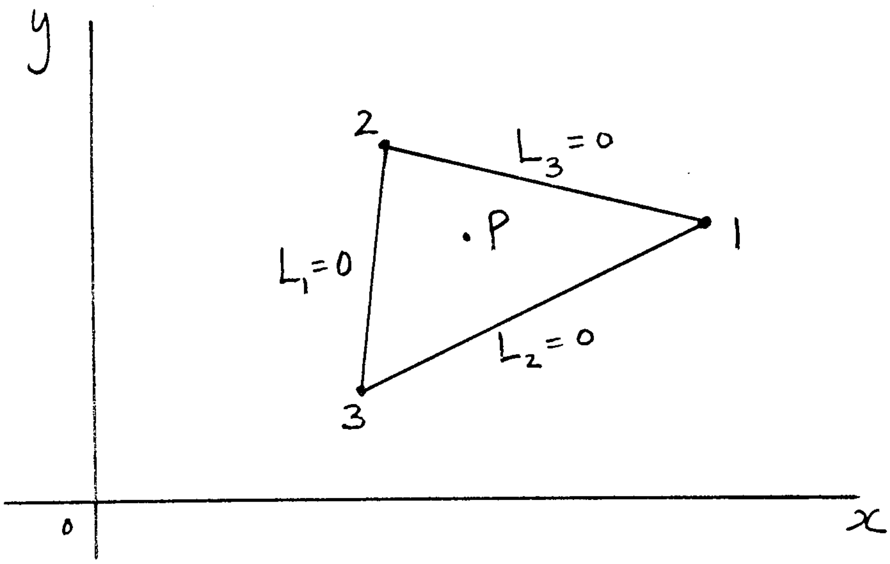


Figure (2.9)

The equation of side 12 is $L_3 = 0$ and similarly for sides 31 and 23. The coordinates L_1, L_2 and L_3 are not independent since

$$L_1 + L_2 + L_3 = 1 \qquad (2.15.4)$$

To avoid ambiguity L_1 and L_2 will be regarded as independent from now on. Whenever L_3 occurs it must be replaced by $(1-L_1-L_2)$ from (2.15.4). From (2.14.8)

$$L_1 = \frac{1}{2\Delta}[y_{23}x + x_{32}y + (x_2y_3-x_3y_2)] \qquad (2.15.5)$$

$$L_2 = \frac{1}{2\Delta}[y_{31}x + x_{13}y + (x_3y_1-x_1y_3)] \qquad (2.15.6)$$

The vertices of triangle 123 in figure (2.9) may easily be mapped into corresponding points in the (L_1,L_2) plane using (2.15.5) and (2.15.6). The result of doing this is given in tabular form below

Vertex	Cartesian coordinates (x,y)	Area coordinates (L_1,L_2)
1	(x_1,y_1)	(1, 0)
2	(x_2,y_2)	(0, 1)
3	(x_3,y_3)	(0, 0)

The sides of triangle 123 are easily mapped into the (L_1, L_2) plane. For example side 23 will map into $L_1 = 0$ since along 23 $L_1 = 0$ etc. The triangle in the (L_1, L_2) plane is shown below

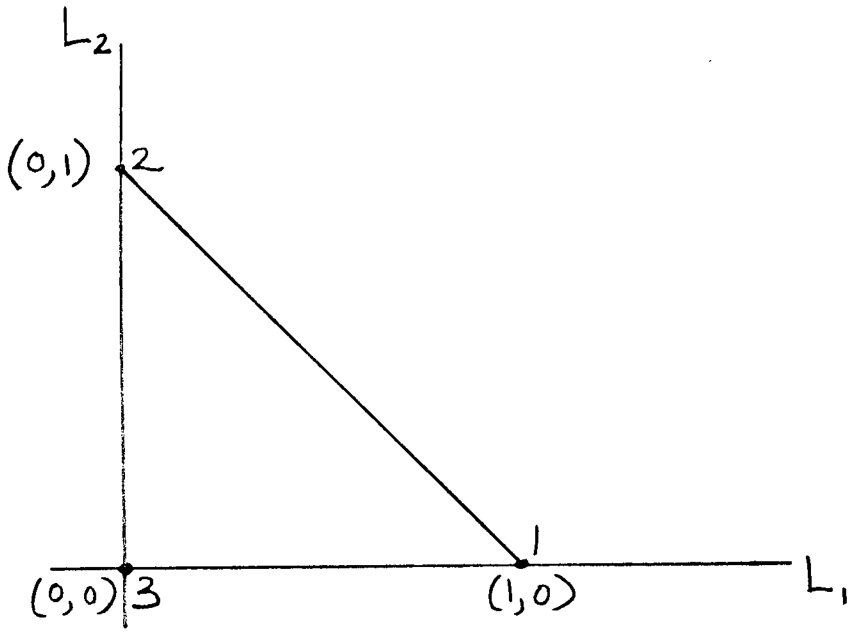


Figure (2.10)

This is a right angle triangle as required.

(ii) The second step is to prove the formula given in (2.15.3) viz.

$$I = \iint_{e_{xy}} L_1^{\alpha} \; L_2^{\beta} \; L_3^{\gamma} \; dxdy = \frac{\alpha!\beta!\gamma! \; 2\Delta}{(\alpha+\beta+\gamma+2)!}$$

writing $L_3 = 1-L_1-L_2$ from (2.15.4) in above gives

$$I = \iint_{e_{xy}} L_1^{\alpha} \; L_2^{\beta} \; (1-L_1-L_2)^{\gamma} dxdy$$

Changing variables from (x,y) to (L_1,L_2) in above integral gives

$$I = \iint_{e_{L_1L_2}} L_1^{\alpha} \; L_2^{\beta} \; (1-L_1-L_2)^{\gamma} |J| \, dL_1 dL_2 \qquad (2.15.7)$$

where

$e_{L_1L_2}$ - area of right angle triangle in (L_1,L_2) plane, shown in figure (2.10).

and J is the Jacobian of the transformation, i.e.

$$J = \frac{\partial(x,y)}{\partial(L_1,L_2)} = \begin{vmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial x}{\partial L_2} \\ \frac{\partial y}{\partial L_1} & \frac{\partial y}{\partial L_2} \end{vmatrix}$$

Now from (2.14.7)

$$x = L_1x_{13} + L_2x_{23} + x_3 \qquad (2.15.8)$$

$$y = L_1y_{13} + L_2y_{23} + y_3 \qquad (2.15.9)$$

where the notation $x_{13} = x_1 - x_3$ etc. has been used. Hence

$$J = \begin{vmatrix} x_{13} & x_{23} \\ y_{12} & y_{23} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 2\Delta$$

Thus equation (2.15.7) may be written

$$I = \iint_{e_{L_1 L_2}} L_1^{\alpha} \; L_2^{\beta} \; (1-L_1-L_2)^{\gamma} 2\Delta dL_1 dL_2$$

The limits of integration are obtained from figure (2.10) giving

$$I = 2\Delta \int_{0}^{1} L_1^{\alpha} \; dL_1 \int_{0}^{1-L_1} L_2^{\beta} \; (1-L_1-L_2)^{\gamma} dL_2$$

In the inner integral put

$$(1-L_1)y = L_2 \qquad \text{i.e.} \quad dL_2 = (1-L_1)dy$$

This gives

$$I = 2\Delta \int_{0}^{1} L_1^{\alpha} \; dL_1 \int_{0}^{1} (1-L_1)^{\beta} y^{\beta} (1-L_1)^{\gamma} (1-y)^{\gamma} (1-L_1) dy$$

or

$$I = 2\Delta \int_0^1 L_1^{\alpha} \quad (1-L_1)^{\beta+\gamma+1} dL_1 \int_0^1 y^{\beta}(1-y)^{\gamma} dy$$

Both integrals are now seen to be Beta functions since

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Thus

$$I = 2\Delta B(\alpha+1, \beta+\gamma+2) B(\beta+1, \gamma+1)$$

using

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

gives

$$I = 2\Delta \frac{\Gamma(\alpha+1)\Gamma(\beta+\gamma+2)}{\Gamma(\alpha+\beta+\gamma+3)} \quad \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)}$$

$$I = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} 2\Delta$$

Also using

$$\Gamma(N+1) = N!$$

$$I = \frac{\alpha! \ \beta! \ \gamma!}{(\alpha+\beta+\gamma+2)!} 2\Delta$$

as required.

2.16 Galerkin's Formulation

Not every differential equation has a solution which minimizes some functional. In such cases, the general principles of finite elements can be applied but some other criteria are required to write down the finite element equations to determine the ϕ_i. One possible approach, proposed originally by Galerkin for solving ordinary differential equations is to look for approximate solutions to the differential equations which are orthogonal to the shape functions. Thus for the equation viz:

$$L\phi = f \quad \text{in } R \qquad (2.16.1)$$

Orthogonallity is defined as a solution such that

$$\iint_R N_i(L\phi - f)\,dxdy = 0 \qquad (2.16.2)$$

for every N_i.

As only one shape function is defined for every node, then if (2.16.2) is written down at every node at which ϕ is to be determined, there will result precisely the same number of equations as unknowns. Solving these equations yields the required result. If the highest derivative appearing in the integrand of (2.16.2) is of order n then continuity of the trial function and all its derivatives up to order (n-1) are required on the interface. Green's theorem can often be used in (2.16.2) to reduce the order of the highest derivatives thus enabling the use of simpler elements. Finally it is mentioned that when a problem can be stated in a variational form, then Galerkin's approach leads to the same solution.

2.17 Galerkin's Approach Applied to Poisson's Equation

In order to highlight some aspects of Galerkin's formulation and also to introduce further notation consider Poisson's equation viz:

$$\nabla^2\phi = f \quad \text{in } R \tag{2.17.1}$$

Galerkin's criteria requires that

$$\iint_R N_i(\nabla^2\phi - f)\,dxdy = 0 \tag{2.17.2}$$

Using Green's theorem (to reduce the interelement continuity requirement) in the form

$$\iint_R N_i\nabla^2\phi\,dxdy = -\iint_R \text{grad}N_i . \text{grad}\,\phi\,dxdy$$

$$+ \oint_{\partial R} N_i \frac{\partial\phi}{\partial n}ds$$

Equation (2.17.2) becomes

$$\iint_R \operatorname{grad}N_i . \operatorname{grad}\phi \, dxdy + \iint_R N_i f dxdy = \oint_{\partial R} N_i \frac{\partial \phi}{\partial n} ds$$

Writing integrals over R and ∂R as sum of integrals over the elements gives

$$\sum_e \iint_e \left(\frac{\partial N_i}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial \phi}{\partial y} \right) dxdy + \sum_e \iint_e N_i \, dxdy$$

$$= \sum_e \oint_{\partial R_e} N_i \frac{\partial \phi}{\partial n} ds .$$

Where ∂R_e is the boundary of element e. Over element e put

$$\phi = \sum_j N_j \phi_j$$

so that the above equation becomes

$$\sum_e \sum_j \left[\iint_e \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy \right] \phi_j$$

$$+ \sum_e \iint_e N_i f dxdy = \sum_e \oint_{\partial R_e} N_i \frac{\partial \phi}{\partial n} ds \qquad (2.17.2)$$

Define the following quantities

$$\alpha_{ij}^{e} = \iint_{e} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy$$

$$F_i^e = \sum_{e} N_i f dxdy \quad ; \quad G_i^e = \oint_{\partial R_e} N_i \frac{\partial \phi}{\partial n} ds$$

(2.17.2) now becomes

$$\sum_{e} \sum_{j} \alpha_{ij}^{e} \phi_j = \sum_{e} \left(G_i^e - F_i^e \right) \qquad (2.17.3)$$

(2.17.3) is written at every node i at which ϕ is to be determined. Alternatively (2.17.3) may be written at all the nodes including boundary nodes. The equations corresponding to Dirichlet nodes can then be modified as explained in (2.10). It should also be noticed that the line integral G_i^e only contributes when i is a boundary node. Furthermore the integral is only to be taken along those sides of element e which actually coincide with the boundary.

The global matrix obtained by writing (2.17.3) at every node may also be constructed using the element stiffness matrix. Writing (2.17.3) at every node of an m node element with node identifiers 1,2,...,m gives

$$(E)_i^e = \sum_{j=1}^{m} \alpha_{ij}^e \quad \phi_j^e - (G_i^e - F_i^e) \tag{2.17.4}$$

$$i = 1,2,\ldots,m$$

where $(E)_i^e$ denotes the contribution of element e to equation I. Of course, I is the node number corresponding to the node identifier i. In matrix form (2.17.4) is

$$\begin{bmatrix} (E)_1^e \\ (E)_2^e \\ \vdots \\ (E)_m^e \end{bmatrix} = \begin{bmatrix} \alpha_{11}^e & \alpha_{12}^e & \cdots & \alpha_{1m}^e \\ \alpha_{21}^e & \alpha_{22}^e & \cdots & \alpha_{2m}^e \\ \vdots & \vdots & & \vdots \\ \alpha_{m1}^e & \alpha_{m2}^e & \cdots & \alpha_{mm}^e \end{bmatrix} \begin{bmatrix} \phi_1^e \\ \phi_2^e \\ \vdots \\ \phi_m^e \end{bmatrix} - \begin{bmatrix} (G_1^e - F_1^e) \\ (G_2^e - F_2^e) \\ \vdots \\ (G_m^e - F_m^e) \end{bmatrix} \tag{2.17.5}$$

The matrix on the right hand side of (2.17.5) having typical element α_{ij}^e is the element stiffness matrix for Poisson's equation. The global matrix is now easily constructed using the algorithm given in (2.13).

The next chapter continues with a critical survey of some of the most common finite element methods available to solve the Navier Stokes equations.

CHAPTER THREE

LITERATURE SURVEY

3.1 INTRODUCTION

The usual way of discretizing differential equations to obtain the finite element equations is to seek a variational equivalent of the original problem. This means that the Euler Lagrange equation(s) of the associated functional are precisely the differential equation(s) to be solved. Unfortunately, many systems do not possess a variational principle, and amongst these are the Navier Stokes equations. A crude proof of this fact was originally given by Millikan (16). A much more elegant proof using the frechet calculus may be found in (15, 16).

It is fair to say that most research workers have concentrated on seeking alternative methods of discretizing the Navier Stokes equations to obtain the finite element equations. A convenient alternative is the method of weighted residuals particularly the Galerkin Method. A discussion of several common formulations for the Navier Stokes equations (1.2.2) is presented below.

3.2 COMMON FORMULATIONS

It is convenient to restate equations (1.2.2) again. These were

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{Re} \nabla^2 u \qquad (3.2.1)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{Re} \nabla^2 v$$

It is worth noting that for transient viscous incompressible flow it is only necessary to augment the left hand side of the second and third equations in (3.2.1) by $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ respectively.

Stream Function Formulation

Equations (3.2.1) may be transformed into a single equation written on a stream function, ψ, defined to reduce the first of equations (3.2.1) to an identity

$$u = \frac{\partial \psi}{\partial y}$$
$$\qquad (3.2.2)$$
$$v = -\frac{\partial \psi}{\partial x}$$

Cross differentiating and subtracting the remaining equations (3.2.1) to eliminate the explicit appearance of pressure gives

$$\psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y = \frac{1}{Re} \nabla^4 \psi \qquad (3.2.3)$$

Probably the first attempt at solving equation (3.2.3) was made by Atkinson, Card, Irons (4). They considered the solution of (3.2.3) for creeping flows only. The Reynold's number of creeping flow problems is very small and consequently (3.2.3) reduces to the biharmonic equation. The solution of the biharmonic equation using a variational approach was given in (4). A formulation using Galerkin's approach may have application to higher Reynolds number flows. A clear disadvantage of the method is the requirement of higher order elements to ensure interelement continuity, while the advantage is that only one equation is required to describe the flow instead of three.

Olson (36) has presented a formulation for solving the whole of equation (3.2.3) using a pseudo variational principle. The Euler Lagrange equation of the functional

$$I = \iint \left[\frac{1}{2Re} \left(\nabla^2 \psi \right)^2 + \underline{\left(\psi_y \nabla^2 \psi \right)} \psi_x - \underline{\left(\psi_x \nabla^2 \psi \right)} \psi_y \right] dxdy$$

will yield the differential equation (3.2.3) provided the underlined bracketted terms are held constant. Since the functional I contains derivatives of ψ up to second order $C^{(1)}$ continuous elements are required. It was stated in (36) that this formulation has been tested on several examples and good results were obtained; but the author has seen no published results.

Stream Function and Vorticity Formulation

Equations (3.2.1) may be reduced to two equations involving the stream function and vorticity ω defined by -

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad (3.2.4)$$

Substituting for u and v from (3.2.2) into (3.2.4) gives the equation

$$\nabla^2\psi = -\omega \qquad (3.2.5)$$

The introduction of ψ and ω into the second and third of equations (3.2.1) and eliminating pressure gives

$$\frac{1}{Re}\nabla^2\omega = \psi_y\omega_x - \psi_x\omega_y \qquad (3.2.6)$$

The solution of (3.2.5) and (3.2.6) is not straightforward since generally the vorticity is not known on the boundary walls. A brief description of the solution procedure as presented by Taylor and Hood (17, 18, 19) is given below.

Now to solve equation (3.2.5) it is necessary to specify ψ or its normal derivatives on all boundaries. This implies that suitable velocity profiles have to be determined both on entrance to and exit from the region. Of course on the boundary walls the velocities are zero. The boundary values of ψ may be determined by fixing ψ at an arbitrary reference point on the boundary and using equations (3.2.3).

Also as the vorticity is not known anywhere an initial guess has to be made. Equation (3.2.5) can then be solved for the stream function. To solve (3.2.6) for the vorticity field it is necessary to know boundary values of vorticity. To this end let ψ and ω be given by the element approximation functions

$$\psi^e(x,y) = \sum_i N_i(x,y)\psi_i \qquad (3.2.7)$$

$$\omega^e(x,y) = \sum_i N_i(x,y)\omega_i \qquad (3.2.8)$$

Let ω_K be the vorticity at the boundary node K. Substituting for ψ from (3.2.7) in (3.2.5) gives

$$\nabla^2 (\sum_i N_i(x,y)\psi_i) = -\omega_K \qquad (3.2.9)$$

When two or more elements meet at a node K the average values obtained by (3.2.9) is used. Equation (3.2.6) may be solved by the finite element method for the vorticity field. In (19) it is suggested that it is more efficient to obtain the vorticity field from

$$\nabla^2\omega = 0 \qquad (3.2.10)$$

With boundary conditions provided by (3.2.9). The values of vorticity so obtained are substituted into (3.2.5) and the cycle repeated until convergence is obtained. Equation (3.2.6) may then be used in place of (3.2.10) to obtain the vorticity field. The advantage of this approach is that the symmetric matrices arising from (3.2.5) and (3.2.10) are identical and inversion is required only once at the beginning of the cycle. This procedure by Taylor and Hood (19) may be summarised as follows:

1. Determine suitable velocity profiles on inflow and exit
2. From a reference stream function obtain boundary conditions on ψ
3. Guess an initial ω distribution
4. Solve $\nabla^2\psi = -\omega$
5. Find the boundary ω values from $\omega_B = -\nabla^2\psi^e$
6. Solve $\nabla^2\omega = 0$

   Convergence

7. Solve $\nabla^2\psi = -\omega$ and find boundary values from $\omega_B = -\nabla^2\psi^e$

   Solve $\left(\frac{1}{Re}\nabla^2 - \psi_y\frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial y}\right)\omega = 0$

   Convergence

8. Solution obtained

The solution of (3.2.6) by finite elements requires further discussion. There are two possible ways of discretizing (3.2.6) either the term

$$(\psi_y\omega_x - \psi_x\omega_y)$$

is incorporated in the matrix for $\nabla^2\omega$ or it is put on the "right hand side" using the previously iterated values. It is mentioned in (19) that the latter approach proves to be unstable. The type of element required for this formulation is $C^{(0)}$ continuous and at least a third order polynomial, since second order derivatives are taken of the shape functions.

Several numerical results using the stream function and vorticity have been presented in the literature. Taylor and Hood (19) showed that for couette flow the stream function and vorticity formulation was very inefficient from the computational viewpoint. Baker (8), M. Ikenouchi and N. Kimura (22) used Galerkin's criteria and the stream function and vorticity formulation. In (8) numerical results were presented for flow in a duct at a Reynolds number of 200. In (22) the flow around a cylinder was studied and good agreement obtained for a Reynolds number of 40. Pin Tong (44) has used a variational principle and the stream function and vorticity formulation. Numerical results for flow around a cylinder upto a Reynolds number of 40 were obtained in this reference.

One major disadvantage of the stream function and vorticity formulation is that it is only applicable to two dimensional problems.

Velocity Pressure Formulation

It is possible to solve for the velocity and pressure field directly by descretizing equations (3.2.1) by Galerkin's method.

Let the element interpolation functions be defined by the equations:

$$u^e = \sum_j N_j u_j$$

$$v^e = \sum_j N_j v_j \qquad (3.2.11)$$

$$p^e = \sum_j N_j p_j$$

where for convenience the same shape functions are chosen for velocity and pressure. This is by no means necessary or even desirable. Galerkin's method applied to equations (3.2.1) lead to the following non-linear finite element equations

$$\sum_e \sum_j \left[A_{ij} u_j + B_{ij} V_j \right] = 0 \qquad (3.2.12)$$

$$\sum_e \sum_j \sum_K \left(C_{ijK} u_j u_K + D_{ijK} u_j v_K \right)$$

$$+ \frac{1}{Re} \sum_e \sum_j E_{ij} u_j + \sum_e \sum_j A_{ij} p_j = \sum_e F_i \qquad (3.2.13)$$

$$\sum_{e} \sum_{j} \sum_{K} \left(C_{ijK} u_K v_J + D_{ijK} v_j v_K \right)$$

$$+ \frac{1}{Re} \sum_{e} \sum_{j} E_{ij} V_j + \sum_{e} \sum_{j} B_{ij} p_j = \sum_{e} G_i \qquad (3.2.14)$$

Where e is a typical element, j and K are typical node numbers of element e. The various coefficients are defined below: -

$$A_{ij} = \iint_e N_i \frac{\partial N_j}{\partial x} \, dxdy \; ;$$

$$B_{ij} = \iint_e N_i \frac{\partial N_j}{\partial y} \, dxdy \; ;$$

$$C_{ijK} = \iint_e N_i \frac{\partial N_j}{\partial x} N_K dxdy \; ;$$

$$D_{ijK} = \iint_e N_i \frac{\partial N_j}{\partial y} N_K dxdy \; ;$$

$$E_{ij} = \iint_e \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy$$

$$F_i = \oint N_i \frac{\partial u}{\partial n} ds ;$$

$$G_i = \oint N_i \frac{\partial v}{\partial n} ds.$$

Notice that the integrations are confined to the element e. Green's theorem has been used on the Laplacian terms to reduce the order of interelement continuity. Consideration of the order of derivatives will show that the order of continuity required is $C^{(0)}$.

Taylor and Hood (18, 19) have applied this formulation to problems in the low and intermediate range of Reynolds numbers. They used the parabolic isoparametric element. Oden (32) has applied this formulation to a number of problems using the six node triangular element. It had been observed by research workers that the use of common trial functions for velocities and pressures lead to some inaccuracies in the pressure field. Kawahara, Yoshimura and Nakagawa (24) used perfect polynomial series to second order for velocities and linear polynomial series for pressure. This device gave good answers for all the variables. Hood and Taylor (20) conducted numerical experiments to confirm this and also presented tentative proposals to explain this phenomena.

3.3 LIMITATIONS OF CONVENTIONAL SHAPE FUNCTIONS

All the work to date has used polynomial shape functions. Using such shape functions solutions begin to breakdown for Reynolds numbers of the order of a few hundred. However, for aerodynamics applications, Reynolds numbers of the order of 10^6 are frequently encountered. At these levels conventional shape functions breakdown completely. Thus there is a very real need for a new approach to choosing the shape functions.

In order to develop the approach methodically the author starts in the next chapter by examining a class of ordinary differential equations which exhibit similar characteristics to the Navier Stokes equations.

CHAPTER FOUR

FINITE ELEMENT SCHEMES DERIVED FROM ORDINARY DIFFERENTIAL EQUATIONS

4.1

The aim of this chapter is to investigate the finite element solution of a wide class of ordinary differential equations. It will be shown that traditional polynomial trial functions used in conjunction with the finite element method lead to numerically unstable schemes. A new method of choosing the trial function to result in numerically stable schemes is presented.

The starting point is a second order constant coefficient ordinary differential equation. However, a great advantage of the method is that it may be readily adapted to apply to differential equations with variable coefficients including non-linear differential equations. In the next chapter the whole concept will be extended to cover partial differential equations.

To fix ideas consider the second order constant coefficient homogeneous differential equation

$$Ly + \mu y = 0 \qquad (4.1.1)$$

over [a,b], where L is a differential operator defined as

$$L \equiv \frac{d^2}{dx^2} - \lambda\frac{d}{dx} \qquad (4.1.2)$$

The non-homogeneous equation will be treated later although it is well-known that the stability of a difference equation is generally decided by its homogeneous solution (49).

For illustration purposes it is convenient to commence with the equation Ly = 0.

4.2 FINITE ELEMENT SOLUTION OF Ly = 0

Galerkin's criteria requires that

$$\int_a^b N_i Ly dx = 0 \qquad (4.2.1)$$

Integration by parts yields

$$\int_a^b \left[\frac{dN_i}{dx}\frac{dy}{dx} + \lambda N_i \frac{dy}{dx} \right] dx - \left[N_i y' \right]_a^b = 0$$

or

$$\sum_e \int_e \left(\frac{dN_i}{dx}\frac{dy}{dx} + \lambda N_i \frac{dy}{dx} \right) dx - \sum_e \left[N_i y' \right]_e = 0$$

Over an element write $y = \sum_j N_j y_j$ where j is a typical node of element e. The above equation reduces to

$$\sum_e \sum_j \left[\int_e \left(\frac{dN_i}{dx} \frac{dN_j}{dx} + \lambda N_i \frac{dN_j}{dx} \right) dx \right] y_j$$

$$- \sum_e \left[N_i y' \right]_e = 0 \qquad (4.2.2)$$

Define the following quantities

$$C^e_{ij} = \int_e N_i \frac{dN_j}{dx} dx \quad ; \qquad E^e_{ij} = \int_e \frac{dN_i}{dx} \frac{dN_j}{dx} dx$$

$$(4.2.3)$$

$$F^e_i = [N_i y']_e \qquad \text{and} \qquad \alpha^e_{ij} = \lambda C^e_{ij} + E^e_{ij}$$

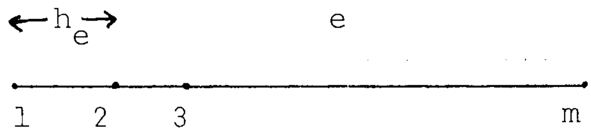
Equation (4.2.2) now takes the form

$$\sum_e \sum_j \left[\alpha^e_{ij} \right] y_j - \sum_e F^e_i = 0 \qquad (4.2.4)$$

This is a set of linear algebraic equations for the nodal values of y. For computer implementation of finite element algorithms it is generally desirable to construct the global matrix from the element stiffness matrix.

4.3 ELEMENT STIFFNESS MATRIX FOR Ly = 0

Consider an m node element e whose nodes are equally spaced and numbered as shown below



For this element (4.2.4) becomes

$$\sum_{j=1}^{m} \left[\alpha_{ij}^{e} \right] y_j^e - F_i^e = (E)_i^e \qquad (4.3.1)$$

$$i = 1, 2, \ldots, m$$

Where $(E)_i^e$ denotes the contribution to equation i from element e. The superscript signifies that the coefficients are evaluated solely for element e. In matrix form equation (4.3.1) looks like

$$\begin{bmatrix} (E)_1^e \\ (E)_2^e \\ \vdots \\ (E)_m^e \end{bmatrix} = \begin{bmatrix} \alpha_{11}^e & \alpha_{12}^e & \cdots & \alpha_{1m}^e \\ \alpha_{21}^e & \alpha_{22}^e & \cdots & \alpha_{2m}^e \\ \vdots & \vdots & & \vdots \\ \alpha_{m1}^e & \alpha_{m2}^e & & \alpha_{mm}^e \end{bmatrix} \begin{bmatrix} y_1^e \\ y_2^e \\ \vdots \\ y_m^e \end{bmatrix} - \begin{bmatrix} F_1^e \\ F_2^e \\ \vdots \\ F_m^e \end{bmatrix} \qquad (4.3.2)$$

The matrix on the right hand side of (4.3.2) having typical element α_{ij}^e is the element stiffness matrix for the equation $Ly = 0$. Each element is taken in turn and the entries of its associated element stiffness matrix are put into the appropriate locations of the global matrix. For example the element α_{ij}^e would be accumulated to the element a_{IJ} in the global matrix. Where I, J are the actual node numbers corresponding to the node identifiers i,j. The same procedure applies in constructing the right hand side of the final system of equations.

It will be noted from (4.2.3) that in order to evaluate α_{ij}^e it is necessary to compute the coefficients C_{ij} and E_{ij}. These coefficients are given below for linear (m = 2) and quadratic (m = 3) elements.

(i) Linear Elements (m = 2)

For linear elements the trial function is $y = A+Bx$. If the origin is taken to coincide with node 1 the shape functions are computed to be

$$N_1(x) = \left(\frac{h_e - x}{h_e}\right)$$

$$N_2(x) = \left(\frac{x}{h_e}\right)$$

The coefficients C_{ij}^e and E_{ij}^e as defined in (4.2.3) are easily found to be

$$C^e_{11} = -\tfrac{1}{2} \quad ; \quad C^e_{12} = -C^e_{11}$$

$$C^e_{21} = C_{11} \quad ; \quad C^e_{22} = -C^e_{11}$$

and

$$E^e_{11} = \frac{1}{h_e} \quad ; \quad E^e_{12} = -E^e_{11}$$

$$E^e_{21} = E_{12} \quad ; \quad E^e_{22} = E^e_{11}$$

(ii) Quadratic Elements (m = 3)

For quadratic elements the trial function is $y = A+Bx+Cx^2$. Again to simplify the algebra take the origin to coincide with node 1 of element. The shape functions are found to be

$$N_1(x) = \frac{(x-h_e)(x-2h_e)}{2h_e^2}$$

$$N_2(x) = \frac{x(2h_e - x)}{h_e^2}$$

$$N_3(x) = \frac{x(x - h_e)}{2h_e^2}$$

The coefficients C^e_{ij} and E^e_{ij} are

$$C^e_{11} = -\tfrac{1}{2} \quad ; \quad C^e_{12} = \frac{2}{3} \quad ; \quad C^e_{13} = -\frac{1}{6} \quad ;$$

$$C^e_{21} = -C^e_{12}; \quad C^e_{22} = 0 \quad ; \quad C^e_{23} = C^e_{12} \quad ;$$

$$C^e_{31} = -C^e_{13}; \quad C^e_{32} = -C^e_{12} \quad ; \quad C^e_{33} = -C^e_{11} \quad ;$$

and

$$E^e_{11} = \frac{7}{6h_e} \;; \quad E^e_{12} = \frac{-4}{3h_e} \;; \quad E^e_{13} = \frac{1}{6h_e} \;;$$

$$E^e_{21} = E^e_{12} \;; \quad E^e_{22} = \frac{8}{3h_e} \;; \quad E^e_{23} = E^e_{12} \;;$$

$$E^e_{31} = E^e_{13} \;; \quad E^e_{32} = E^e_{23} \;; \quad E^e_{33} = E^e_{11} \;.$$

4.4 DERIVATION OF NEW FINITE ELEMENT SCHEME FOR Ly = 0

The entries of the element stiffness matrix for the equation $Ly = 0$ both for linear and quadratic elements were derived in the last section. It is shown in appendix 1 that for linear elements a condition for stability requires that

$$h < \left|\frac{2}{\lambda}\right| \tag{4.4.1}$$

where h is the steplength. This is clearly prohibitive for large values of $|\lambda|$. In this section alternative shape functions are derived which obviates this severe restriction on h.

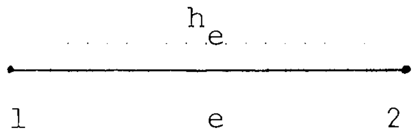
Now the general solution of the differential equation $Ly = 0$ is

$$y = A + Be^{\lambda x} \tag{4.4.2}$$

It is desired to construct a difference equation whose solution is identical to (4.4.2). To this end take as a trial function

$$y = A + Be^{\lambda x}$$

over a two node element as shown below



This element will be referred to as the "exponential element" for convenience. It is again convenient to take the origin to coincide with node 1 for this element. The shape functions are given by the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & e^{\lambda h_e} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\lambda x} \end{bmatrix} \qquad (4.4.3)$$

This gives

$$N_1(x) = \frac{e^{\lambda x} - e^{\lambda h_e}}{(1 - e^{\lambda h_e})}$$

$$N_2(x) = \frac{1 - e^{\lambda x}}{(1 - e^{\lambda h_e})}$$

The coefficients C^e_{ij} and E^e_{ij} are given below

$$C^e_{11} = -\tfrac{1}{2} \quad ; \quad C^e_{12} = -C^e_{11}$$

$$C^e_{21} = C_{11} \quad ; \quad C^e_{22} = -C^e_{11}$$

and

$$E^e_{11} = -\frac{\lambda(1+e^{\lambda h_e})}{2(1-e^{\lambda h_e})} \quad ; \qquad E^e_{12} = -E^e_{11}$$

$$E^e_{21} = E^e_{12} \qquad ; \qquad E^e_{22} = E^e_{11}$$

For large values of $|\lambda|$ some care has to be exercised in the numerical evaluation of the coefficient E^e_{11}. It must be arranged so that only negative exponential functions are computed for otherwise numerical overflow will occur. The three possible arrangements are shown below

$$E^e_{11} = -\frac{\lambda(e^{-\lambda h_e}+1)}{2(e^{-\lambda h_e}-1)} \qquad \text{for } \lambda > 0$$

$$E^e_{11} = \frac{1}{2h_e} \qquad \text{for } \lambda = 0$$

$$E^e_{11} = -\frac{\lambda(1+e^{\lambda h_e})}{2(1-e^{\lambda h_e})} \qquad \text{for } \lambda < 0$$

It is shown in appendix 1 that this choice of shape functions lead to a difference equation whose solution is identical to the differential equation.

4.5 NUMERICAL EXAMPLE

The differential equation considered is $y'' + Ry' = 0$ on the interval [0,1]. If the conditions $y(0) = 0$; $y(1) = 1$ are specified then the analytical solution is

$$y(x) = \frac{1}{(1-e^{R})} [1 - e^{Rx}]$$

It is interesting to note that as $R \to \infty$ $y(x) \to H(x)$, the unit step function.

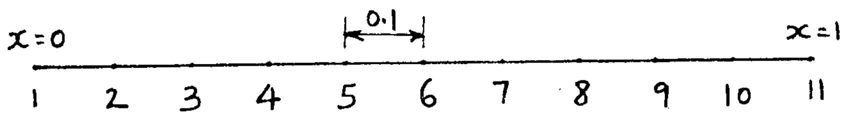
The differential equation $y'' + Ry' = 0$ was solved using the

(i) linear element

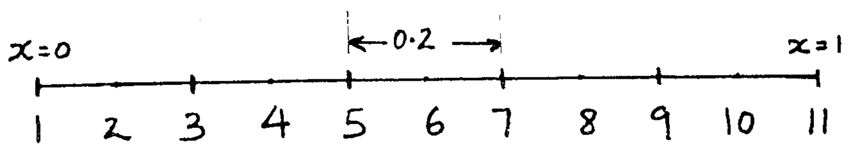
(ii) quadratic element

(iii) exponential element

for various values of R. The finite element grids are shown below



Linear and Exponential Elements



Quadratic Elements

In all cases the steplength h was taken to be 0.1. The results are presented in tabular form for R = 1, 10, 50, 1000 in figure (4.1). A graphical illustration of the results for these values of R may be found in figures (4.2) - figures (4.5).

For R = 1, it can be seen that there is not a great deal of discrepancy between the schemes. As R increases, the traditional linear and quadratic elements exhibit a Gibbs type oscillation. However, the "exponential element" continues to give the correct solution. For R > 1000, although not shown, the Gibbs oscillations associated with linear and quadratic elements increase dramatically but the "exponential element" continues to give the correct solution.

SOLUTIONS OF $Y'' + RY' = 0$ FOR VARIOUS VALUES OF R USING DIFFERENT ELEMENTS

R=1				
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	QUADRATIC ELEMENT
0.0	0.0	0.0	0.0	0.0
0.1	0.1505	0.1505	0.1506	0.1506
0.2	0.2868	0.2868	0.2868	0.2868
0.3	0.4100	0.4100	0.4101	0.4100
0.4	0.5215	0.5215	0.5216	0.5216
0.5	0.6225	0.6225	0.6226	0.6225
0.6	0.7138	0.7138	0.7139	0.7138
0.7	0.7964	0.7964	0.7965	0.7964
0.8	0.8711	0.8711	0.8712	0.8712
0.9	0.9388	0.9388	0.9388	0.9388
1.0	1.0	1.0	1.0	1.0

R= 10				
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	QUADRATIC ELEMENT
0.0	0.000	0.0	0.0	0.0
0.1	0.6321	0.6322	0.6667	0.6429
0.2	0.8647	0.8647	0.8889	0.8572
0.3	0.9503	0.9503	0.9630	0.9490
0.4	0.9817	0.9817	0.9877	0.9797
0.5	0.9933	0.9933	0.9959	0.9928
0.6	0.9976	0.9976	0.9986	0.9971
0.7	0.9991	0.9991	0.9996	0.9990
0.8	0.9997	0.9997	0.9999	0.9996
0.9	0.9999	0.9999	1.0000	0.9999
1.0	1.0	1.0	1.0	1.0

R=50				
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	QUADRATIC ELEMENT
0.0	0.0	0.0	0.0	0.0
0.1	0.9933	0.9933	1.4289	1.2240
0.2	1.0000	1.0000	0.8165	0.6994
0.3	1.0000	1.0000	1.0789	1.0695
0.4	1.0000	1.0000	0.9665	0.9109
0.5	1.0000	1.0000	1.0147	1.0228
0.6	1.0000	1.0000	0.9940	0.9748
0.7	1.0000	1.0000	1.0029	1.0087
0.8	1.0000	1.0000	0.9991	0.9942
0.9	1.0000	1.0000	1.0007	1.0044
1.0	1.0	1.0	1.0	1.0

R= 1000				
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	QUADRATIC ELEMENT
0.0	0.0	0.0	0.0	0.0
0.1	1.0000	1.0000	5.9469	5.7296
0.2	1.0000	1.0000	0.2332	0.2247
0.3	1.0000	1.0000	5.7228	5.6206
0.4	1.0000	1.0000	0.4485	0.4363
0.5	1.0000	1.0000	5.5160	5.5180
0.6	1.0000	1.0000	0.6472	0.6356
0.7	1.0000	1.0000	5.3251	5.4213
0.8	1.0000	1.0000	0.8307	0.8233
0.9	1.0000	1.0000	5.1488	5.3303
1.0	1.0	1.0	1.0	1.0

FIG 4.1

SOLUTION OF Y''+RY'=0 FOR R=1

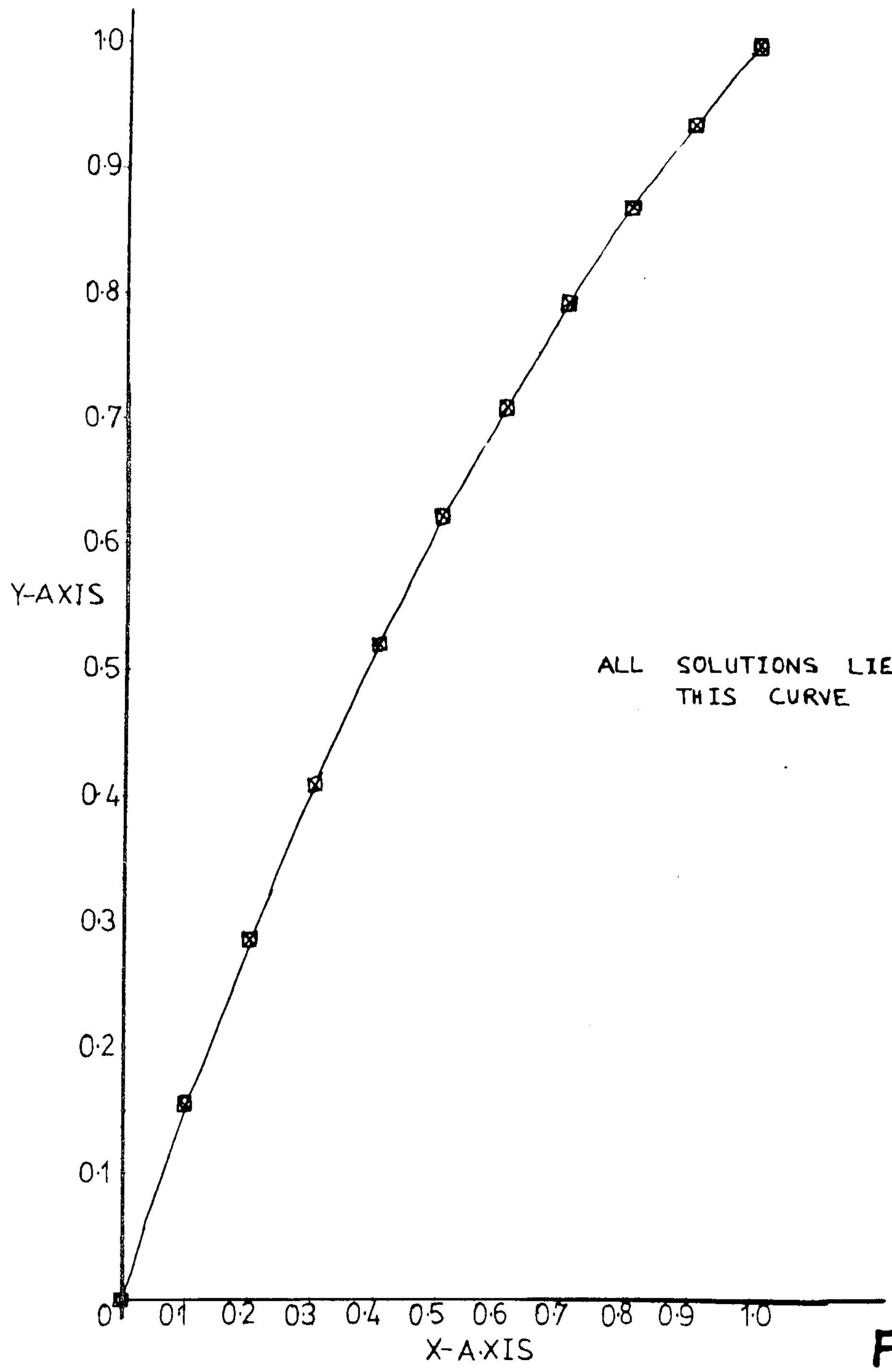


FIG 4·2

SOLUTION OF Y''+RY'=0 FOR R=10

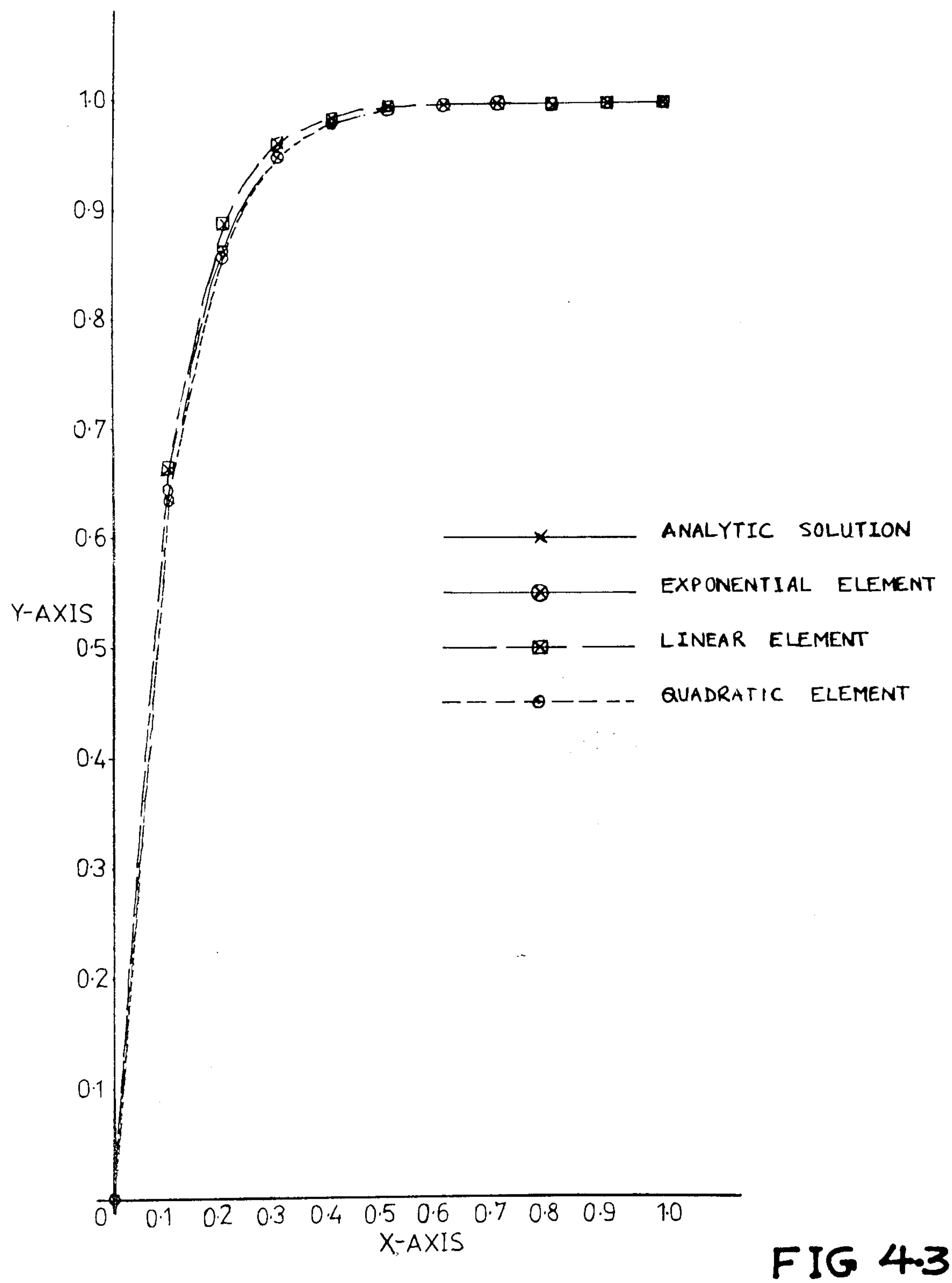


FIG 4·3

SOLUTION OF Y″+RY′=0 FOR R=50

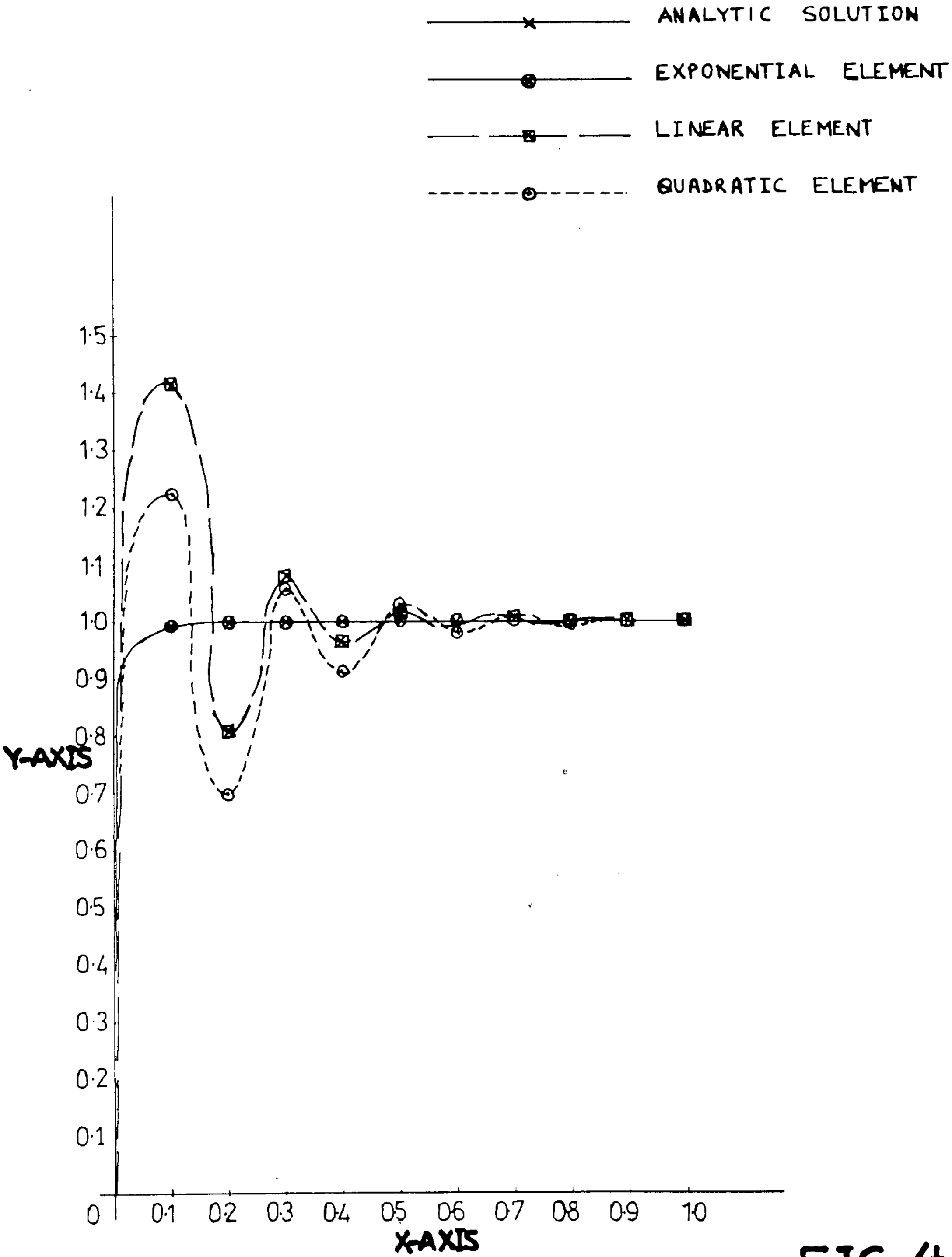


FIG 4·4

SOLUTION OF Y''+RY'=0 FOR R=1000

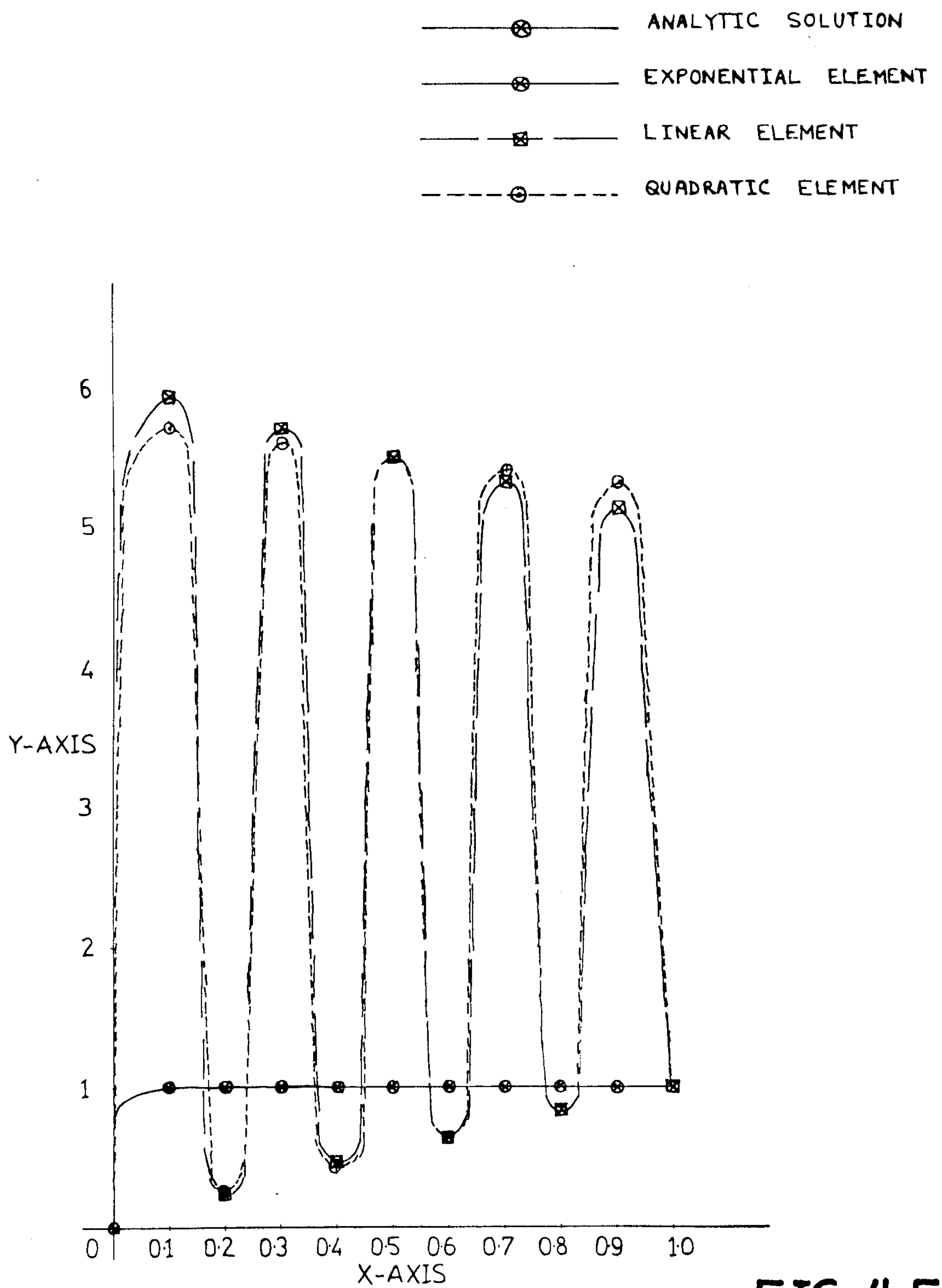


FIG 4·5

4.6 THE CASE OF λ A NON-CONSTANT

The finite element solution of $Ly = 0$ using traditional polynomial elements when λ is a function of x may involve numerical integration. This is easily seen since Galerkin's criteria leads to the following system of finite element equations (see (4.2.2.))

$$\sum_e \sum_j \left[\int_e \left(\frac{dN_i}{dx} \frac{dN_j}{dx} + \lambda(x) N_i \frac{dN_j}{dx} \right) dx \right] y_j$$

$$- \sum_e \left[N_i y' \right]_e = 0 \qquad (4.6.1)$$

For polynomial elements the shape functions and therefore the derivatives of the shape functions are polynomials. This means that the first term on the left hand side of (4.6.1) may be evaluated analytically. However, the second term, i.e.

$$\int_e \lambda(x) N_i \frac{dN_j}{dx} \, dx \qquad (4.6.2)$$

would in general require numerical integration. In practice Gaussian quadrature formulae are used.

For large values of $|\lambda(x)|$ the polynomial elements lead to numerically unstable schemes as inferred in appendix 1.

Now the general solution of $Ly = 0$ when λ is any function of x is not known. Thus the construction of a difference equation whose solution is identical to the solution of the differential equation $Ly = 0$ is not possible. However, in an interval of $x\,[x_i, x_{i+1}]$, sufficiently small, λ is approximately constant, $\bar{\lambda}$ say. In that interval the solution for y will be of the form

$$y(x) = A + Be^{\bar{\lambda}x} \qquad (4.6.3)$$

From (4.6.2) it can be seen that if $\lambda(x) = \bar{\lambda}$ over element e then

$$\int_e \lambda(x) N_i \frac{dN_j}{dx} dx \approx \bar{\lambda} \int_e N_i \frac{dN_j}{dx} dx \qquad (4.6.4)$$

Now using the generalised first mean value theorem, (4.6.2) also yields

$$\int_{x_i}^{x_{i+1}} \lambda(x) N_i \frac{dN_j}{dx}\, dx = \lambda(\xi) \int_{x_i}^{x_{i+1}} N_i \frac{dN_j}{dx} dx \qquad (4.6.5)$$

where

$$x_i < \xi < x_{i+1}$$

Thus comparing (4.6.4) and (4.6.5) it is seen that $\overline{\lambda}$ may be chosen so that

$$\overline{\lambda} = \lambda(\xi) \qquad (4.6.6)$$

The choice of ξ is somewhat arbitrary. The author has found it convenient to choose $\lambda(\xi)$ such that

$$\overline{\lambda} = \lambda(\xi) = \frac{\lambda(x_i) + \lambda(x_{i+1})}{2} \qquad (4.6.7)$$

This is clearly permissible from the intermediate value theorem assuming $\lambda(x)$ is a continuous function.

When λ is a function of both x and y it is first necessary to guess the solution. Let $\overline{y}_i^{(n)}$ be the value of y_i at the nth iteration. The function $\lambda(x,y)$ may be approximated as a constant over an element in the way described above. Thus for the (n+1)th iteration $\overline{\lambda}$ over an element e is given by

$$\overline{\lambda} = \lambda(\xi, y(\xi)) = \frac{\lambda(x_i, \overline{y}_i^{(n)}) + \lambda(x_{i+1}, \overline{y}_{i+1}^{(n)})}{2} \qquad (4.6.8)$$

Hence for the differential equation $Ly = 0$ it is only necessary to replace λ by $\overline{\lambda}$ in the coefficients given in (4.4).

This procedure means that the difference equation for a pivotal point does not have a solution identical to that of the differential equation. However, in a vanishingly small region about the pivotal point, for arbitrary h, the solution of the difference equation behaves exactly as the solution of the differential equation. This is opposed to the situation with polynomial elements when solutions of the homogeneous difference and differential equations only coincide for vanishingly small h. This suggests that the "exponential element" is inherently more stable than the polynomial elements.

4.7. NUMERICAL EXAMPLES

(1) This example illustrates the case of a linear differential equation with variable coefficients. The equation considered is

$$y'' + Rxy' = 0 \tag{4.7.1}$$

subject to $y(0) = 0$ and $y(1) = 1$.

The analytical solution is

$$y = \frac{\operatorname{erf}(\sqrt{R}x)}{\operatorname{erf}(\sqrt{R})} \tag{4.7.2}$$

where $\operatorname{erf}(\xi)$ is the error function defined as

$$\operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-t^2} dt$$

For large values of $|\xi|$ the asymptotic expansion of erf(ξ) is

$$\mathrm{erf}(\xi) \sim 1 + \frac{\xi e^{-\xi^2}}{\pi} \sum_{K=1}^{\infty} (-1)^K \frac{\Gamma(K-\frac{1}{2})}{\xi^{2K}}$$

This shows that erf(ξ) $\to$ 1 as $|\xi| \to \infty$. Hence from (4.7.2) it is clear that as $R \to \infty$;$y(x) \to H(x)$, the unit step function.

The differential equation $y'' + Rxy' = 0$ was solved using the

(i) linear element

(ii) exponential element

for various values of R. In all cases the steplength was chosen to be 0.1.

The results are presented in a table for R = 1, 25, 100, 1000 in figure (4.6). A graphical illustration of the results for these values of R may be found in figures (4.7) - figure (4.10). It is seen that as R increases the traditional linear element exhibits oscillations but the "exponential element" continues to give excellent agreement.

SOLUTION OF $Y''+RXY'=0$ FOR VARIOUS VALUES OF R USING DIFFERENT ELEMENTS

R=1				R=25			
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1166	0.1167	0.1167	0.1	0.3830	0.3829	0.3814
0.2	0.2323	0.2322	0.2322	0.2	0.6826	0.6826	0.6825
0.3	0.3454	0.3454	0.3454	0.3	0.8664	0.8664	0.8689
0.4	0.4553	0.4553	0.4552	0.4	0.9544	0.9545	0.9580
0.5	0.5611	0.5609	0.5608	0.5	0.9876	0.9876	0.9901
0.6	0.6613	0.6614	0.6613	0.6	0.9974	0.9973	0.9984
0.7	0.7559	0.7559	0.7559	0.7	1.0000	0.9995	0.9999
0.8	0.8441	0.8441	0.8441	0.8	1.0000	0.9999	1.0000
0.9	0.9256	0.9256	0.9255	0.9	1.0000	1.0000	1.0000
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

R=100				R=1000			
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.6826	0.6820	0.6791	0.1	1.0000	0.9977	1.1481
0.2	0.9544	0.9542	0.9701	0.2	1.0000	1.0000	0.9455
0.3	0.9974	0.9973	1.0024	0.3	1.0000	1.0000	1.0430
0.4	1.0000	0.9999	0.9995	0.4	1.0000	1.0000	0.9824
0.5	1.0000	1.0000	1.0002	0.5	1.0000	1.0000	1.0250
0.6	1.0000	1.0000	0.9999	0.6	1.0000	1.0000	0.9929
0.7	1.0000	1.0000	1.0000	0.7	1.0000	1.0000	1.0183
0.8	1.0000	1.0000	1.0000	0.8	1.0000	1.0000	0.9975
0.9	1.0000	1.0000	1.0000	0.9	1.0000	1.0000	1.0149
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

FIG 4.6

SOLUTION OF Y″+RXY′=0 FOR R=1

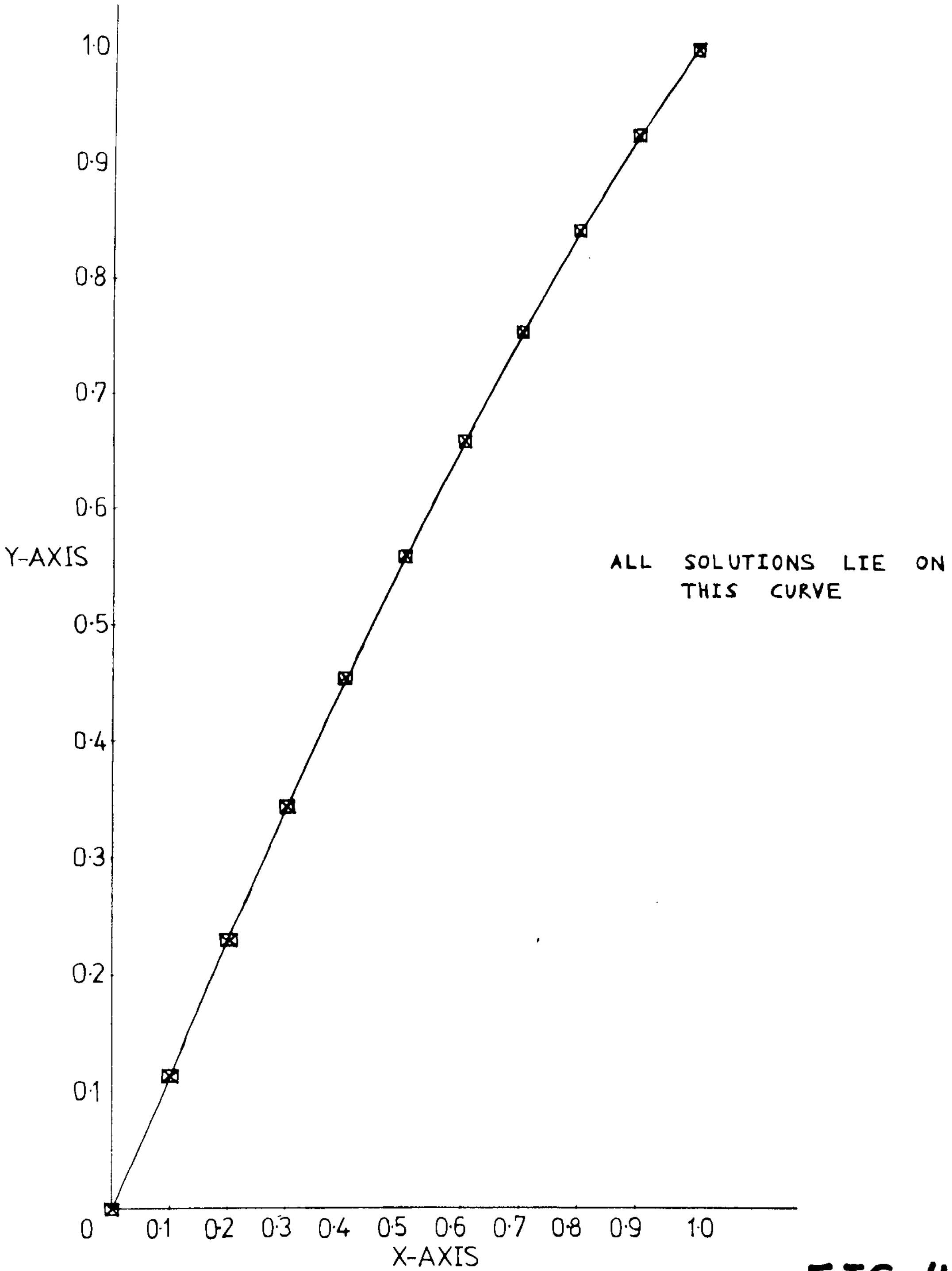


FIG 4·7

SOLUTION OF Y''+RXY'=0 FOR R=25

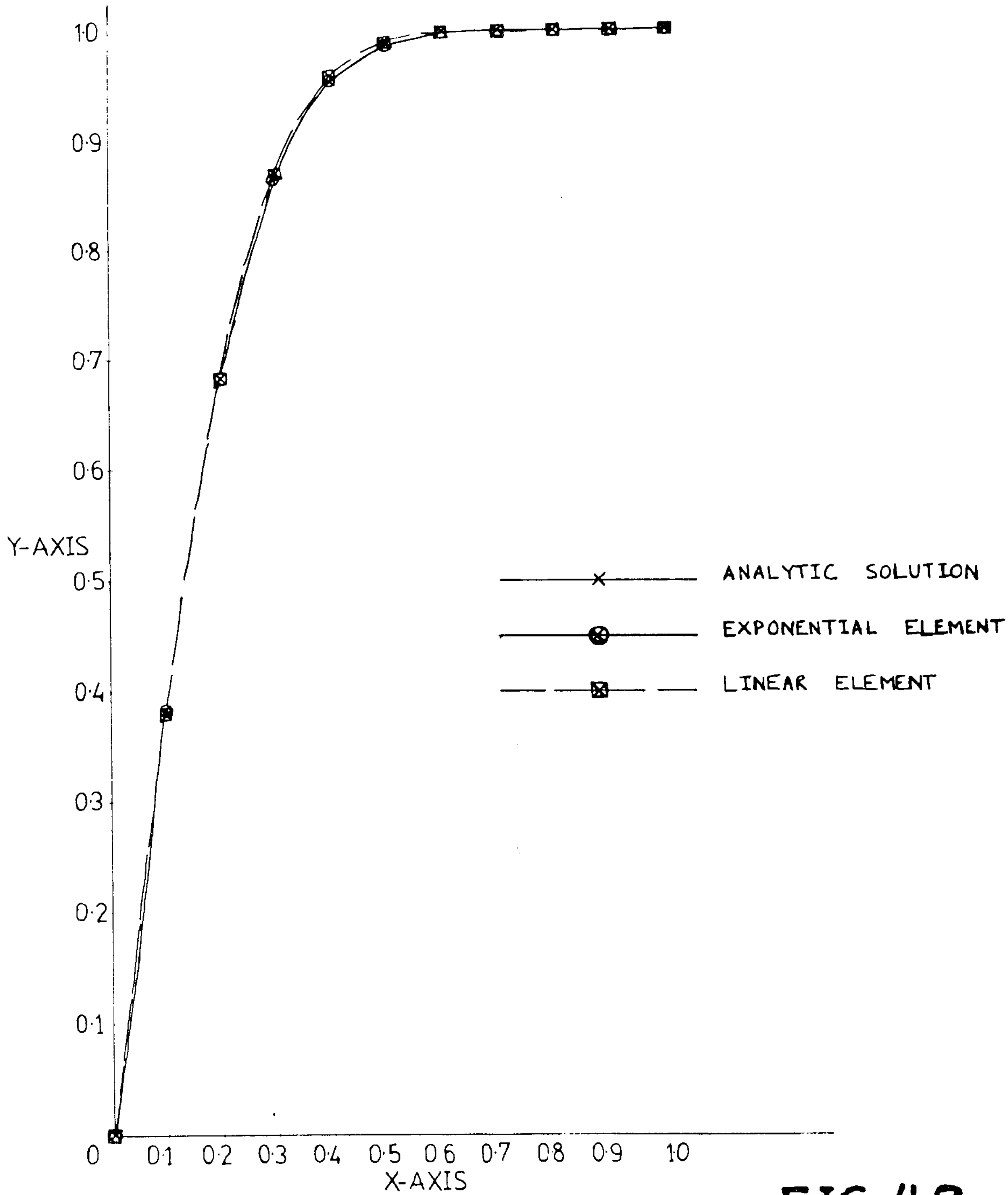


FIG 4·8

SOLUTION OF Y''+RXY'=0 FOR R=100

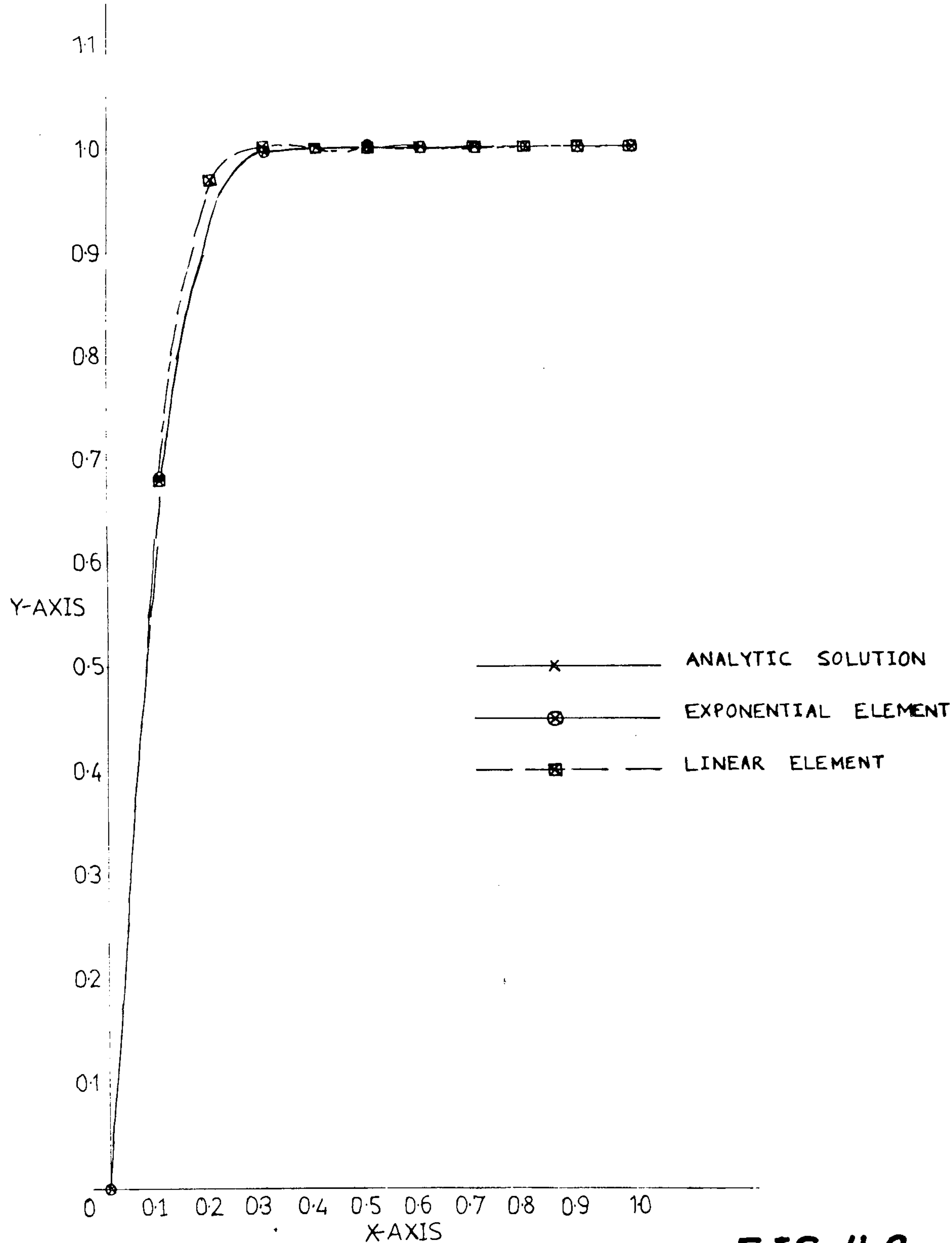


FIG 4·9

SOLUTION OF Y''+RXY'=0 FOR R=1000

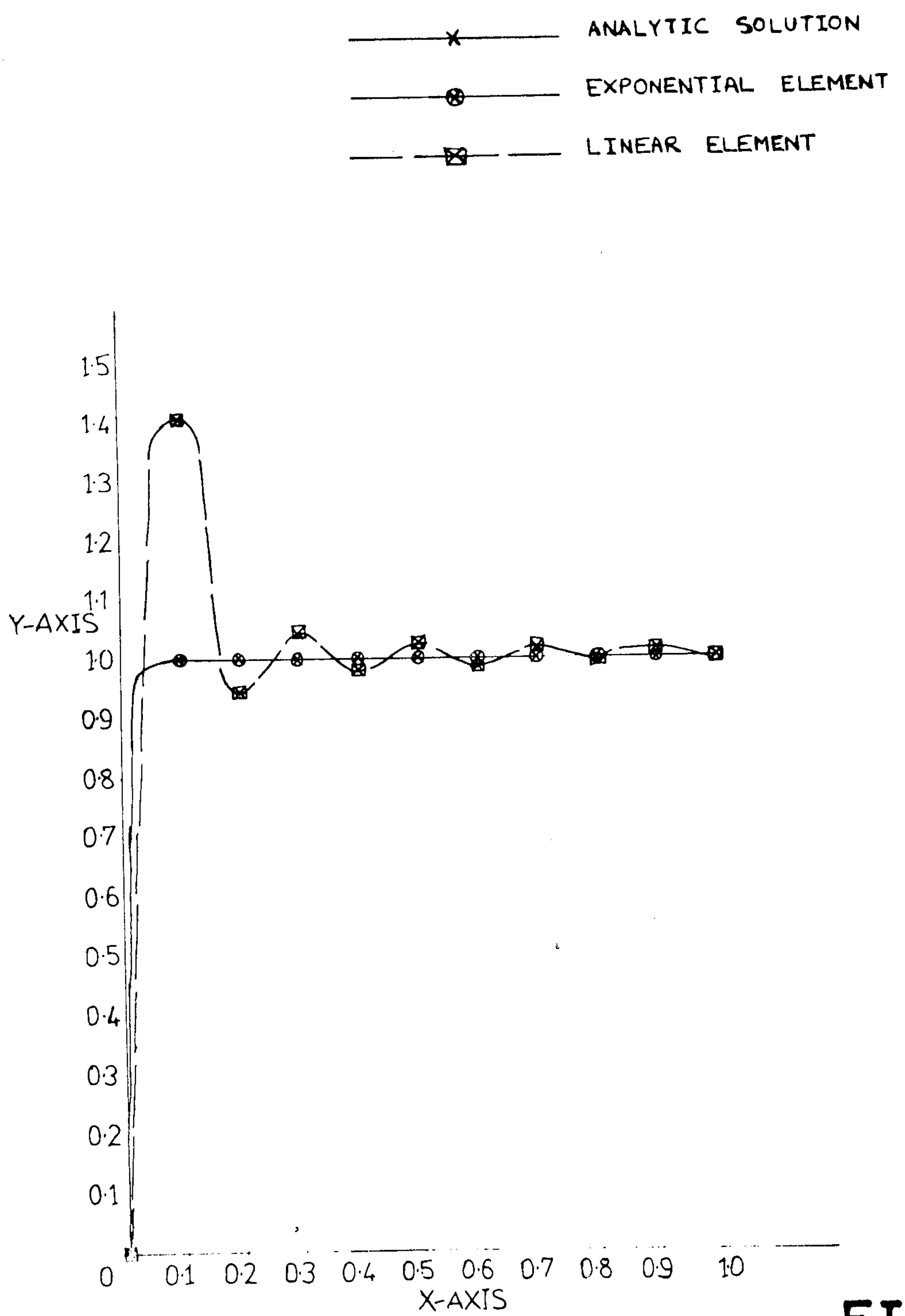


FIG 4·10

(2) This example presents the non-linear differential equation

$$y'' + Ryy' = 0 \tag{4.7.3}$$

With y(0) = 0 and y(1) = 1. The analytical solution is

$$y = \frac{\alpha(1-e^{-\alpha Rx})}{(1+e^{-\alpha Rx})} \tag{4.7.4}$$

where α is the root of the transcendental equation

$$(\alpha+1)e^{-R\alpha} + (1-\alpha) = 0 \tag{4.7.5}$$

This transcendental equation was solved by the Newton Raphson method to locate the root α for various values of R. It is worth noting that if $R > 1$ then $1 < \alpha < 2$ and as $R \to \infty$ then $\alpha \to 1$. This also implies that the solution $y(x) \to H(x)$ as $R \to \infty$.

Equation (4.7.3) was solved using the

(i) linear element

(ii) exponential element

for various values of R. In all cases the steplength was chosen to be 0.1.

The table of results for R = 1, 50, 100, 1000 is given in figure (4.11). The results are also illustrated graphically for the same values of R in figures (4.12) - figure (4.15). In all cases the iterative process was started from an initial guess of zero. Convergence was obtained within ten iterations in all cases. It is seen once again that the "exponential element" gives excellent agreement but for large R the linear elements breakdown.

SOLUTION OF $Y''+RYY'=0$ FOR VARIOUS VALUES OF R USING DIFFERENT ELEMENTS

R=1				R=50			
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.1189	0.1188	0.1188	0.1	0.9866	0.9528	1.0697
0.2	0.2363	0.2363	0.2362	0.2	0.9999	0.9996	0.9681
0.3	0.3511	0.3510	0.3509	0.3	1.0000	1.0000	1.0133
0.4	0.4618	0.4617	0.4616	0.4	1.0000	1.0000	0.9943
0.5	0.5676	0.5675	0.5674	0.5	1.0000	1.0000	1.0025
0.6	0.6676	0.6675	0.6674	0.6	1.0000	1.0000	0.9990
0.7	0.7611	0.7610	0.7609	0.7	1.0000	1.0000	1.0005
0.8	0.8478	0.8477	0.8477	0.8	1.0000	1.0000	0.9998
0.9	0.9274	0.9274	0.9273	0.9	1.0000	1.0000	1.0001
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

R=100				R=1000			
X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT	X	ANALYTIC SOLUTION	EXPONENTIAL ELEMENT	LINEAR ELEMENT
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.9999	0.9966	1.2379	0.1	1.0000	1.0000	3.2381
0.2	1.0000	1.0000	0.8150	0.2	1.0000	1.0000	0.4894
0.3	1.0000	1.0000	1.1150	0.3	1.0000	1.0000	3.2042
0.4	1.0000	1.0000	0.9270	0.4	1.0000	1.0000	0.6704
0.5	1.0000	1.0000	1.0565	0.5	1.0000	1.0000	3.1726
0.6	1.0000	1.0000	0.9721	0.6	1.0000	1.0000	0.8027
0.7	1.0000	1.0000	1.0293	0.7	1.0000	1.0000	3.1428
0.8	1.0000	1.0000	0.9915	0.8	1.0000	1.0000	0.9095
0.9	1.0000	1.0000	1.0170	0.9	1.0000	1.0000	3.1144
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

FIG 4.11

SOLUTION OF $Y''+RYY'=0$ FOR R=1

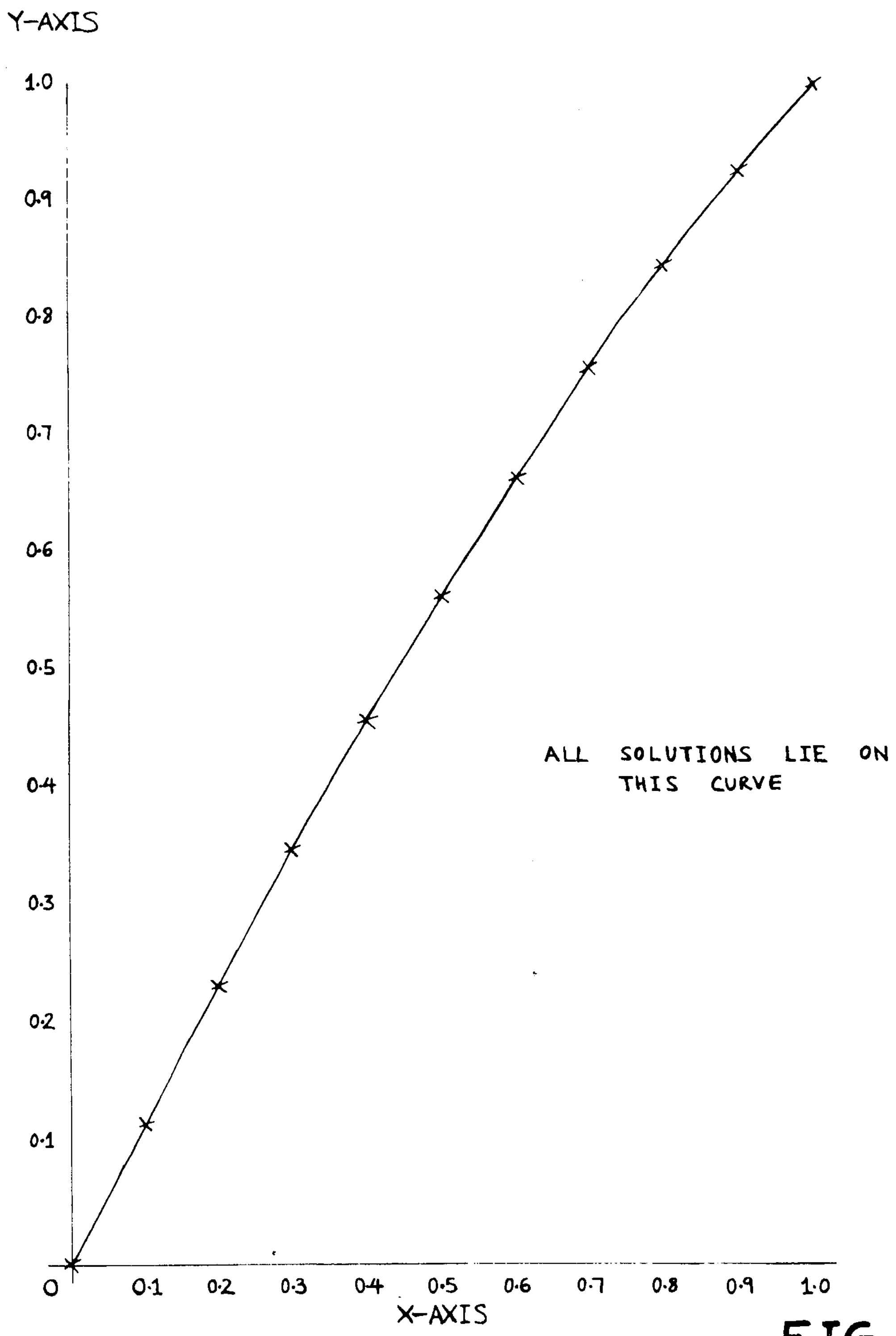


FIG 4.12

SOLUTION OF Y″+RYY′=0 FOR R=50

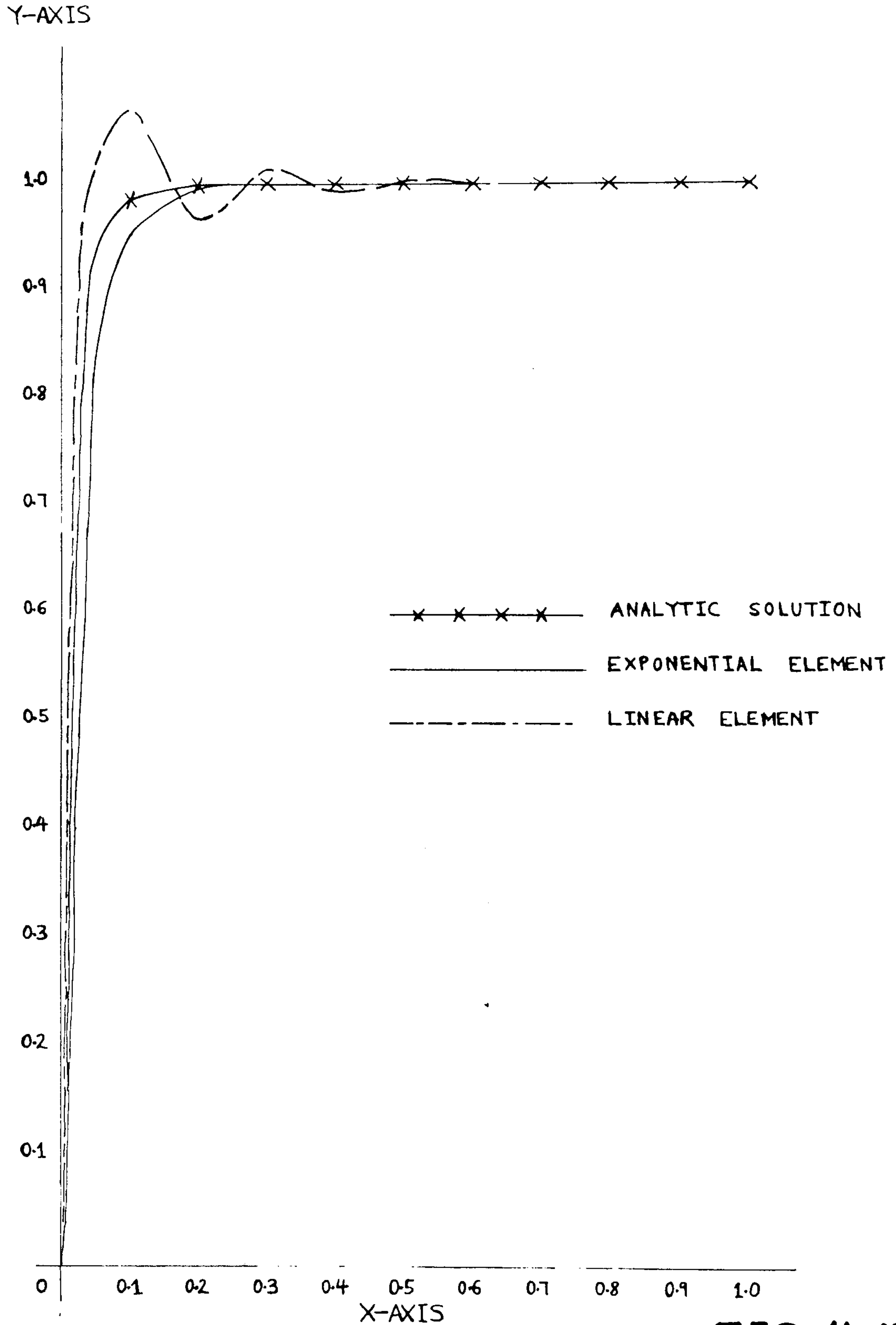


FIG 4.13

SOLUTION OF $Y'' + RYY' = 0$ FOR R=100

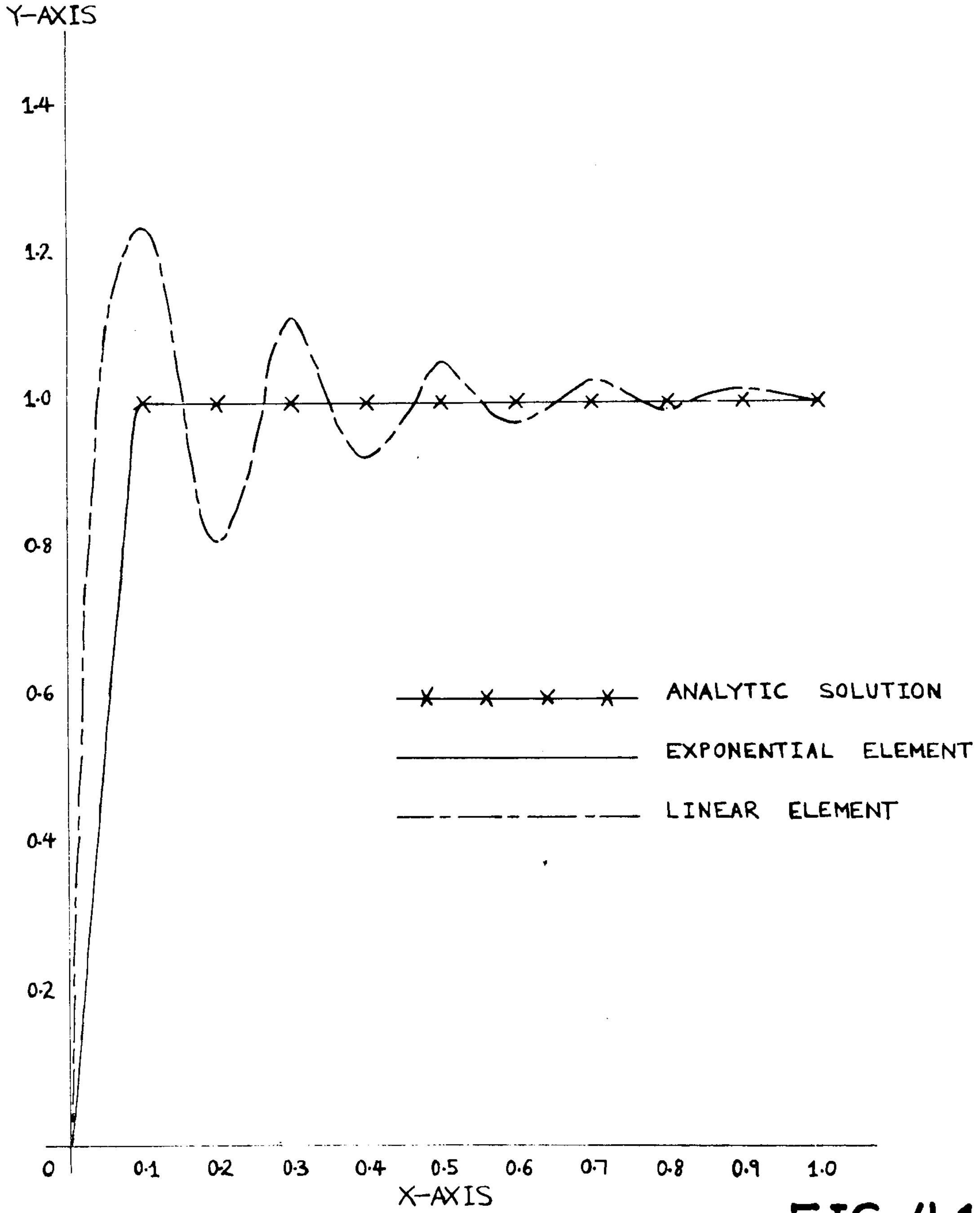


FIG 4·14

SOLUTION OF $Y'' + RYY' = 0$ FOR R = 1000

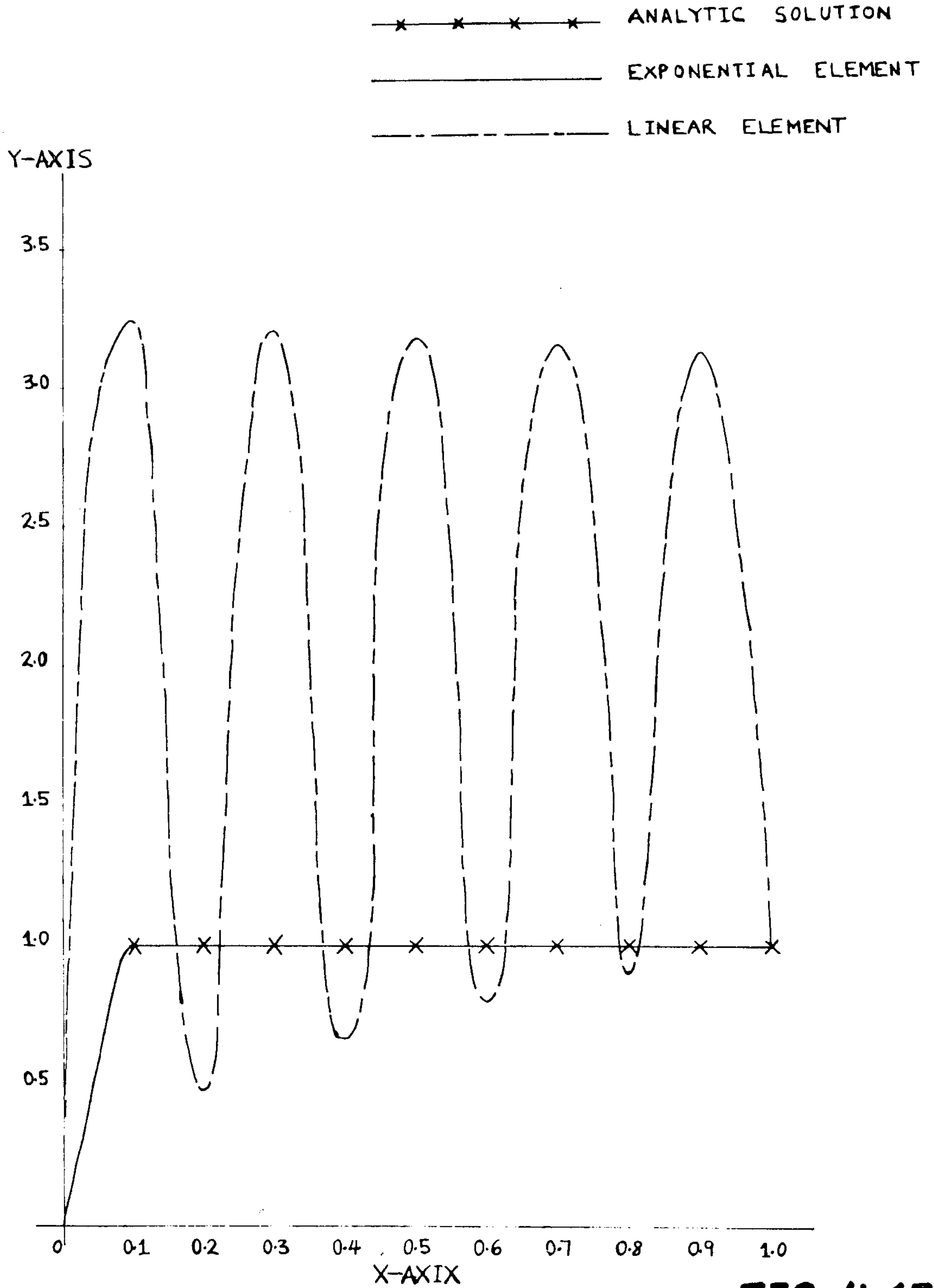


FIG 4.15

4.8 THE NON-HOMOGENEOUS EQUATION $Ly = f$

It has already been mentioned earlier that the stability of a difference equation is generally decided by its homogeneous equation [49]. This means that if a trial function of the form $y = A + Be^{\lambda x}$ is assumed over an element for the equation $Ly = f$ then stability for all values of $|\lambda|$ may be expected. Experience indicates that this is usually the case. However, in (4.10) it is shown how to construct a difference equation whose solution is identical to the solution of $Ly = f$ assuming λ and f to be constants. The method is also extended to the case when λ and f are functions of x and y. Both methods mentioned above are effective in overcoming the problem of stability associated with the non-homogeneous equation. However, the latter approach gives slightly better accuracy. But first, the finite element solution of the non-homogeneous equation must be formulated.

4.9 FINITE ELEMENT SOLUTION OF $Ly = f$

Galerkin's criteria requires that

$$\int_a^b N_i(Ly - f)dx = 0 \qquad (4.9.1)$$

Integration by parts yields

$$\int_a^b \left[\frac{dN_i}{dx}\frac{dy}{dx} + \lambda N_i \frac{dy}{dx}\right] dx + \int_a^b N_i f dx - \left[N_i y'\right]_a^b = 0$$

Assuming λ and f are constants for the moment this may be written as

$$\sum_e \int_e \left(\frac{dN_i}{dx}\frac{dy}{dx} + \lambda N_i \frac{dy}{dx}\right) dx + f \sum_e \int_e N_i dx$$

$$- \sum_e \left[N_i y'\right]_e = 0 \qquad (4.9.2)$$

Over an element write $y = \sum_j N_j y_j$ where j is a typical node of element e. The above equation reduces to

$$\sum_e \sum_j \left[\int_e \left(\frac{dN_i}{dx}\frac{dN_j}{dx} + \lambda N_i \frac{dN_j}{dx}\right) dx\right] y_j$$

$$+ \sum_e \int_e N_i f dx - \sum_e \left[N_i y'\right]_e = 0 \qquad (4.9.3)$$

Define the following quantities

$$C^e_{ij} = \int_e N_i \frac{dN_j}{dx} dx \quad ; \quad E^e_{ij} = \int_e \frac{dN_i}{dx} \frac{dN_j}{dx} dx$$

$$F^e_i = [N_i y']_e \quad \text{and} \quad \alpha^e_{ij} = \lambda C^e_{ij} + E^e_{ij}$$

$$G^e_i = f \int_e N_i dx$$

Equation (4.9.3) can now be written more compactly as

$$\sum_e \sum_j \left[\alpha^e_{ij}\right] y_j - \sum_e \left(F^e_i - G^e_i\right) = 0 \qquad (4.9.4)$$

With the usual notation the equations for an element for the non-homogeneous equation in matrix form are

$$\begin{bmatrix} (E)^e_1 \\ (E)^e_2 \\ \vdots \\ (E)^e_n \end{bmatrix} = \begin{bmatrix} \alpha^e_{11} & \alpha^e_{12} & \cdots & \alpha^e_{1m} \\ \alpha^e_{21} & \alpha^e_{22} & \cdots & \alpha^e_{2m} \\ \vdots & \vdots & & \vdots \\ \alpha^e_{m1} & \alpha^e_{m2} & \cdots & \alpha^e_{mm} \end{bmatrix} \begin{bmatrix} y^e_1 \\ y^e_2 \\ \vdots \\ y^e_m \end{bmatrix} - \begin{bmatrix} (F^e_1 - G^e_1) \\ (F^e_2 - G^e_2) \\ \vdots \\ (F^e_m - G^e_m) \end{bmatrix} \qquad (4.9.5)$$

The coefficients C^e_{ij}, E^e_{ij}, F^e_i and hence also α^e_{ij} are the same as those defined in (4.2.3). The values of these coefficients for the polynomial and "exponential" element have already been given earlier. The only new coefficients to be evaluated for the non-homogeneous equation are the G^e_i's. These are given below for the linear and "exponential" element

(i) Linear Element

$$G_1^e = \tfrac{1}{2}fh_e$$
$$G_2^e = \tfrac{1}{2}fh_e \qquad (4.9.6)$$

(ii) Exponential Element

$$G_1^e = -f\left[\frac{1}{\lambda} + \frac{h_e e^{\lambda h_e}}{(1-e^{\lambda h_e})}\right]$$
$$(4.9.7.)$$
$$G_2^e = f\left[\frac{1}{\lambda} + \frac{h_e}{(1-e^{\lambda h_e})}\right]$$

4.10 DERIVATION OF NEW FINITE ELEMENT SCHEME FOR Ly = f

If λ and f are constants then the general solution of the differential equation Ly = f is

$$y = A + Be^{\lambda x} - \frac{f}{\lambda}\left(x + \frac{1}{\lambda}\right) \qquad (4.10.1)$$

where, of course

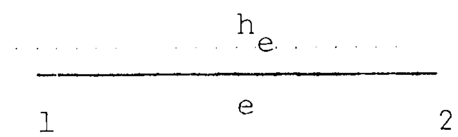
$A + Be^{\lambda x}$ - complementary function

$-\frac{f}{\lambda}\left(x + \frac{1}{\lambda}\right)$ - particular integral

It is desired to construct a difference equation whose solution is identical to (4.10.1). To this end take as a trial function

$$y = A + Be^{\lambda x} - \frac{f}{\lambda}\left(x + \frac{1}{\lambda}\right)$$

over a two node element as shown below



An element for which this trial function is assumed will be referred to as the "augmented exponential element". This name is suitable since the trial function for the homogeneous equation is augmented to deal with the non-homogeneous equation. To express the trial function in terms of the values of y at the nodes of element e proceed in the usual manner. Hence, writing down (4.10.1) at the two nodes of the element e gives

$$y_1 = A + Be^{\lambda x_1} - \frac{f}{\lambda}\left(x_1 + \frac{1}{\lambda}\right) \qquad (4.10.2)$$

$$y_2 = A + Be^{x_2} - \frac{f}{\lambda}\left(x_2 + \frac{1}{\lambda}\right) \qquad (4.10.3)$$

define

$$z = y + \frac{f}{\lambda}\left(x + \frac{1}{\lambda}\right) \qquad (4.10.4)$$

so that (4.10.2) and (4.10.3) may be written

$$z_1 = A + Be^{\lambda x_1} \tag{4.10.5}$$

$$z_2 = A + Be^{\lambda x_2} \tag{4.10.6}$$

This means that if $N_1(x)$ and $N_2(x)$ are the shape functions corresponding to a trial function of the form

$$z = A + Be^{\lambda x}$$

then

$$z = N_1 z_1 + N_2 z_2$$

using (4.10.4) this equation transforms into

$$y + \frac{f}{\lambda}\left(x + \frac{1}{\lambda}\right) = N_1\left[y_1 + \frac{f}{\lambda}(x_1 + \frac{1}{\lambda})\right] + N_2\left[y_2 + \frac{f}{\lambda}(x_2 + \frac{1}{\lambda})\right]$$

or

$$y = N_1 y_1 + N_2 y_2 + \chi(x) \tag{4.10.7}$$

where the function $\chi(x)$ is defined as

$$\chi(x) = -\frac{f}{\lambda}\left(x + \frac{1}{\lambda}\right) + \frac{f}{\lambda}\left(x_1 + \frac{1}{\lambda}\right)N_1 + \frac{f}{\lambda}\left(x_2 + \frac{1}{\lambda}\right)N_2 \tag{4.10.8}$$

Thus (4.10.7) gives the trial solution in terms of the nodal values of y. It is worth noting that if $f = 0$ then $\chi(x) \equiv 0$ in which case the trial function is identical to that used for the homogeneous equation.

4.11 FINITE ELEMENT SOLUTION OF Ly = f USING THE "AUGMENTED EXPONENTIAL ELEMENT"

It is now necessary to formulate the finite element solution of the differential equation Ly = f using the trial function given in (4.10.7). Galerkin's formulation leads to

$$\sum_e \int_j \left(\frac{dN_i}{dx}\frac{dy}{dx} + \lambda N_i \frac{dy}{dx}\right) dx + \sum_e \int_e N_i f dx$$

$$- \sum_e \Big[N_i y' \Big]_e = 0$$

Writing $y = \sum_j N_j y_j + \chi(x)$ over an element gives

$$\sum_e \sum_j \left[\int_e \left(\frac{dN_i}{dx}\frac{dN_j}{dx} + \lambda N_i \frac{dN_j}{dx}\right) dx \right] y_j$$

$$+ \sum_e \int_e \left(\frac{dN_i}{dx}\frac{d\chi}{dx} + \lambda N_i \frac{d\chi}{dx}\right) dx$$

$$+ \sum_e \int_e fN_i \, dx - \sum_e \Big[N_i y' \Big]_e = 0 \qquad (4.11.1)$$

Define the following quantities

$$C^e_{ij} = \int_e N_i \frac{dN_j}{dx} dx \;; \qquad E^e_{ij} = \int_e \frac{dN_i}{dx}\frac{dN_j}{dx} dx$$

$$F^e_i = [N_i y']_e \;; \qquad G^e_i = \int_e f N_i dx$$

$$S^e_i = \int_e \frac{dN_i}{dx}\frac{d\chi}{dx} \;; \qquad T^e_i = \int_e N_i \frac{d\chi}{dx} dx$$

$$\alpha^e_{ij} = \lambda C^e_{ij} + E^e_{ij} \;; \qquad \beta^e_i = \lambda T^e_i + S^e_i$$

Equation (4.11.1) may now be written

$$\sum_e \sum_j \left[\alpha^e_{ij}\right] y_j + \sum_e \left(\beta^e_i + G^e_i - F^e_i\right) = 0 \qquad (4.11.2)$$

4.12 CONSTRUCTION OF ELEMENT STIFFNESS MATRIX

The usual procedure leads to the following equations for a typical element e

$$\begin{bmatrix} (E)^e_1 \\ (E)^e_2 \end{bmatrix} = \begin{bmatrix} \alpha^e_{11} & \alpha^e_{12} \\ \alpha^e_{21} & \alpha^e_{22} \end{bmatrix} \begin{bmatrix} y^e_1 \\ y^e_2 \end{bmatrix} - \begin{bmatrix} (F^e_1 - G^e_1 - \beta^e_1) \\ (F^e_2 - G^e_2 - \beta^e_2) \end{bmatrix}$$

The coefficients C^e_{ij}, E^e_{ij}, F^e_{ij}, G^e_i and also α^e_{ij} are identical to those given for the "exponential element" in (4.4) and (4.9). The coefficients S^e_i and T^e_i are given below

$$S^e_1 = \frac{f}{\lambda} + \frac{f}{\lambda}(x_1+\frac{1}{\lambda})E^e_{11} + \frac{f}{\lambda}(x_2+\frac{1}{\lambda})E^e_{12}$$

$$S^e_2 = -\frac{f}{\lambda} + \frac{f}{\lambda}(x_1+\frac{1}{\lambda})E^e_{21} + \frac{f}{\lambda}(x_2+\frac{1}{\lambda})E^e_{22}$$

$$T^e_1 = \frac{f}{\lambda}\left[\frac{1}{\lambda}+\frac{he^{\lambda h}}{(1-e^{\lambda h})}\right] + \frac{f}{\lambda}(x_1+\frac{1}{\lambda})C^e_{11} + \frac{f}{\lambda}(x_2+\frac{1}{\lambda})C^e_{12}$$

$$T^e_2 = -\frac{f}{\lambda}\left[\frac{1}{\lambda}+\frac{h}{(1-e^{\lambda h})}\right] + \frac{f}{\lambda}(x_1+\frac{1}{\lambda})C^e_{21} + \frac{f}{\lambda}(x_2+\frac{1}{\lambda})C^e_{22}$$

4.13 THE CASE WHEN λ AND/OR f ARE NON-CONSTANTS

If λ and/or f are functions of x then the argument presented in (4.6) can clearly be repeated. Thus over an element $[x_i, x_{i+1}]$ it is only necessary to replace λ by $\overline{\lambda}$ and f by $\overline{f}$, where

$$\overline{\lambda} = \frac{\lambda(x_i) + \lambda(x_{i+1})}{2}$$

$$\overline{f} = \frac{f(x_i) + f(x_{i+1})}{2}$$

The procedure to be adopted when λ and f are functions of x and y is also similar to that described in (4.6).

4.14 NUMERICAL EXAMPLE

The equation considered is

$$\frac{d^2y}{dx^2} + \frac{R}{(1+x)} \frac{dy}{dx} = -R(1+x) \qquad (4.14.1)$$

subject to $y(0) = 0$; $y(1) = 1$.

The analytic solution is

$$y = \frac{R}{3(2+R)} \left[1-(1+x)^3 \right] + \frac{2(3+5R)}{3(2+R)[2^{1-R}-1]} \left[(1+x)^{1-R}-1 \right]$$

as $R \to \infty$

$$y \to \frac{1}{3} [11 - (1+x)^3]$$

It can be seen from (4.10.1) that if $|\lambda| >> |f|$ then the particular integral of $Ly = f$ is virtually zero. The parameter R is included on the right hand side of (4.14.1) to make the augmenting term $(-1)\frac{f}{\lambda}(x+\frac{1}{\lambda})$ significant.

Equation (4.14.1) was solved using

(i) Linear element

(ii) Exponential element

(iii) Augmented exponential element

for various values of R. The steplength was 0.1 in all cases. The results are depicted in a table in figure (4.16) for

R = 5,50, 100, 1000. The results are also illustrated graphically in figure (4.17) - figure (4.20) for the same values of R.

For R = 5, it can be seen that there is not a great deal of discrepancy between the schemes. As R increases the linear elements exhibit oscillations. However, the "exponential element" and the "augmented exponential element" give very good agreement even for very large R. The results indicate that the latter element gives slightly better accuracy than the former. Several other examples on the non-homogeneous equation were tried and they all confirmed the above observation.

SOLUTION OF $Y'' + R\left[\frac{1}{1+x}\right]Y' = R(1+x)^{-1}$ FOR VARIOUS VALUES OF R USING DIFFERENT ELEMENTS

R=5					R=50				
X	ANALYTIC SOLUTION	AUGMENTED EXPONENTIAL ELEMENT	EXPONENTIAL ELEMENT	LINEAR ELEMENT	X	ANALYTIC SOLUTION	AUGMENTED EXPONENTIAL ELEMENT	EXPONENTIAL ELEMENT	LINEAR ELEMENT
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.8228	0.8239	0.8239	0.8386	0.1	3.1071	3.1049	3.0866	4.5945
0.2	1.2994	1.3004	1.3004	1.3177	0.2	3.0098	3.0076	2.9918	2.4172
0.3	1.5635	1.5643	1.5643	1.5800	0.3	2.8599	2.8579	2.8447	3.0715
0.4	1.6888	1.6893	1.6893	1.7020	0.4	2.6846	2.6829	2.6721	2.6138
0.5	1.7171	1.7174	1.7173	1.7271	0.5	2.4824	2.4809	2.4723	2.5004
0.6	1.6733	1.6734	1.6733	1.6804	0.6	2.2513	2.2501	2.4 .35	2.2451
0.7	1.5722	1.5722	1.5722	1.5770	0.7	1.9894	1.9886	1.9838	1.9895
0.8	1.4230	1.4229	1.4229	1.4258	0.8	1.6949	1.6943	1.6912	1.6940
0.9	1.2312	1.2311	1.2311	1.2324	0.9	1.3657	1.3654	1.3639	1.3654
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

R=100					R=1000				
X	ANALYTIC SOLUTION	AUGMENTED EXPONENTIAL ELEMENT	EXPONENTIAL ELEMENT	LINEAR ELEMENT	X	ANALYTIC SOLUTION	AUGMENTED EXPONENTIAL ELEMENT	EXPONENTIAL ELEMENT	LINEAR ELEMENT
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	3.1791	3.1768	3.1192	5.5614	0.1	3.2186	3.2157	3.0849	−68.3056
0.2	3.0497	3.0476	2.9964	1.4387	0.2	3.0865	3.0840	2.9641	3.0743
0.3	2.8964	2.8946	2.8500	3.9627	0.3	2.9305	2.9282	2.8203	−71.1992
0.4	2.7176	2.7161	2.6781	2.0751	0.4	2.7485	2.7466	2.6514	4.6367
0.5	2.5114	2.5101	2.4787	2.9091	0.5	2.5386	2.5370	2.4555	−72.7975
0.6	2.2758	2.2748	2.2499	2.0708	0.6	2.2987	2.2975	2.2305	4.7260
0.7	2.0088	2.0080	1.9896	2.1374	0.7	2.0269	2.0260	1.9745	−73.2111
0.8	1.7085	1.7080	1.6958	1.6619	0.8	1.7212	1.7206	1.6854	3.4567
0.9	1.3729	1.3726	1.3666	1.4147	0.9	1.3796	1.3793	1.3612	−72.6172
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

FIG 4.16

SOLUTION OF $Y'' + \frac{R}{1+x} Y' = R(1+x)(-1)$ FOR R=5

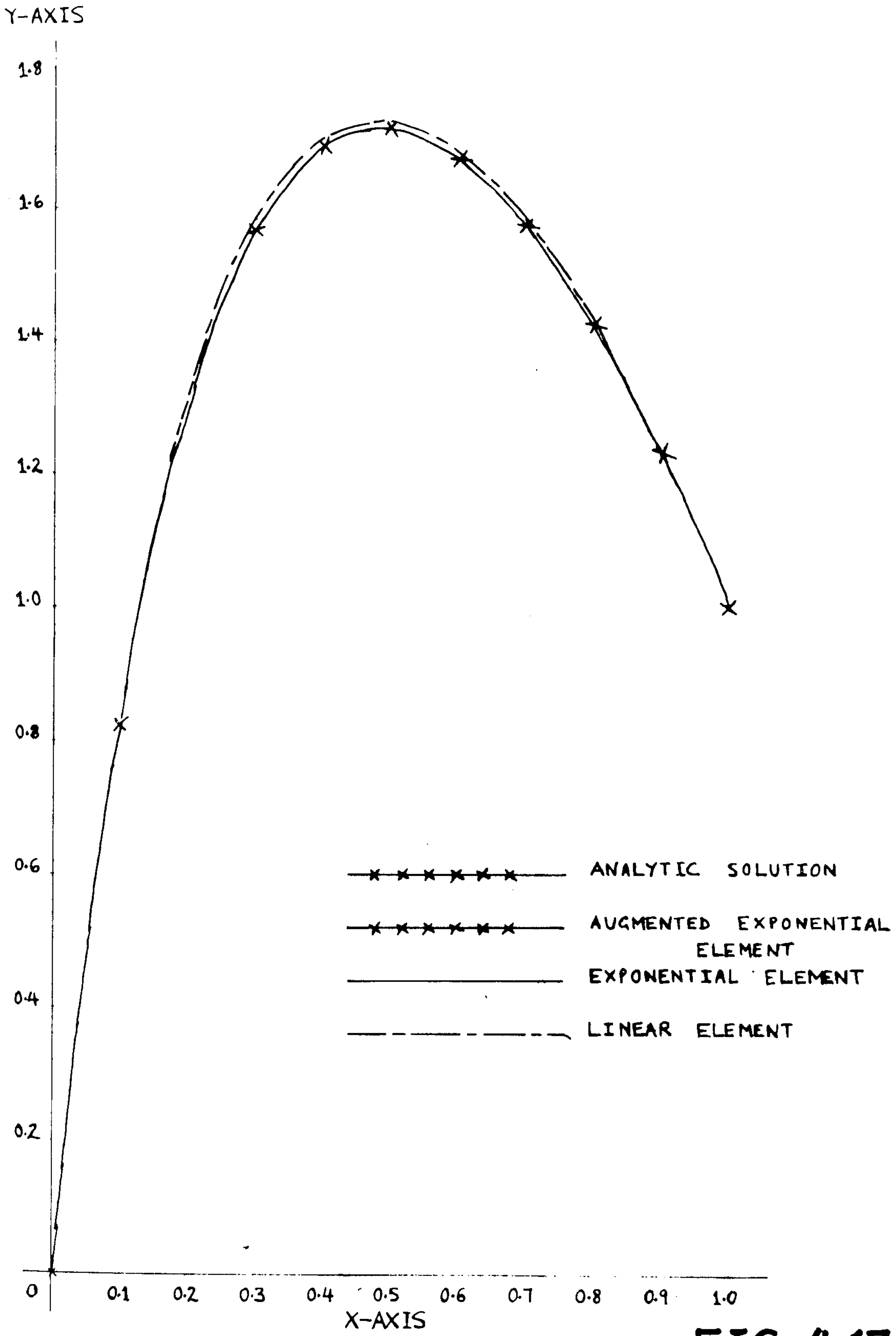


FIG 4·17

SOLUTION OF $Y'' + R\left[\frac{1}{1+x}\right]Y' = R(1+x)(1)$ FOR R=50

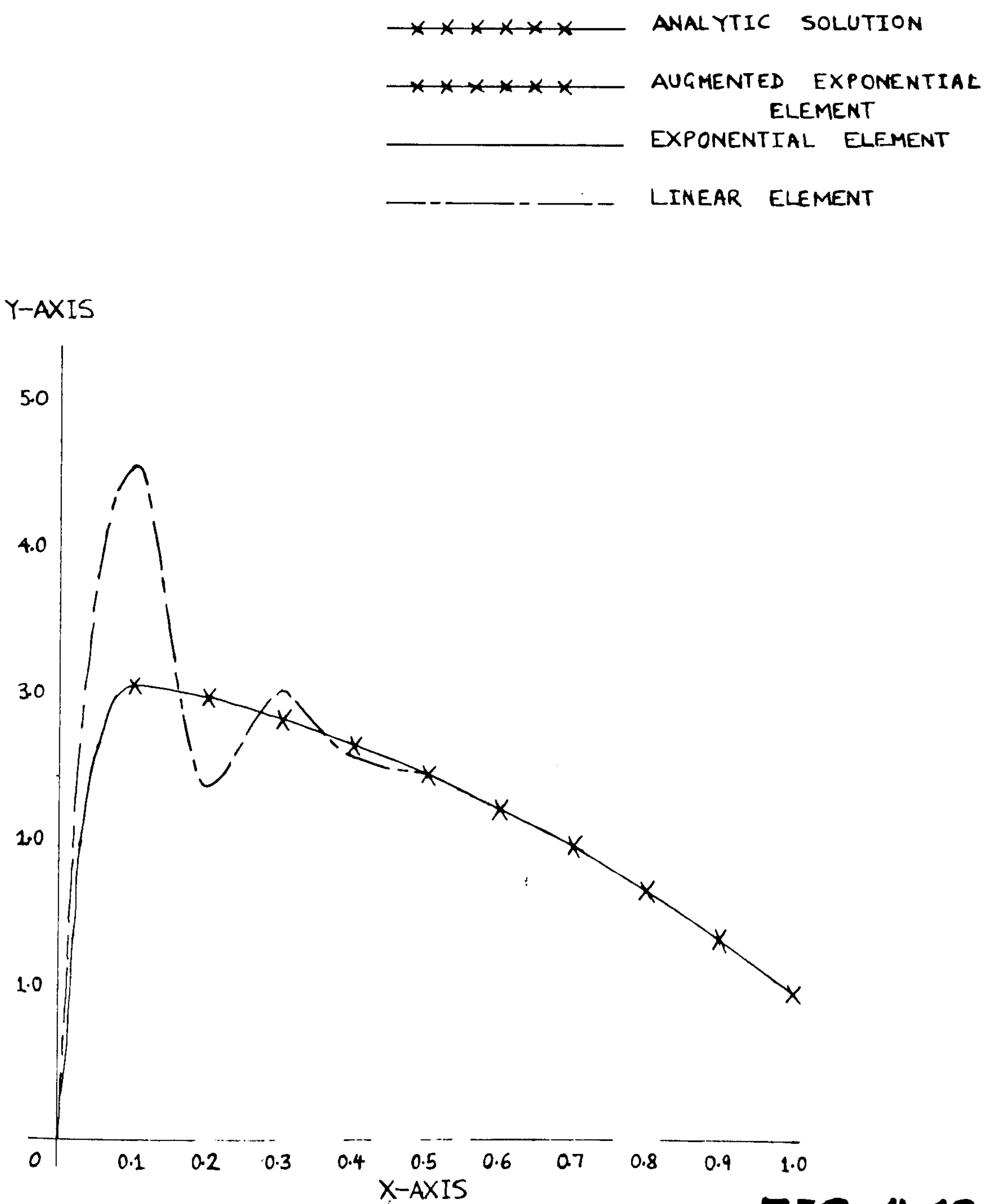


FIG 4.18

SOLUTION OF $Y'' + R\left[\frac{1}{1+x}\right]Y' = R\left((1+x)^{-1}\right)$ FOR R = 100

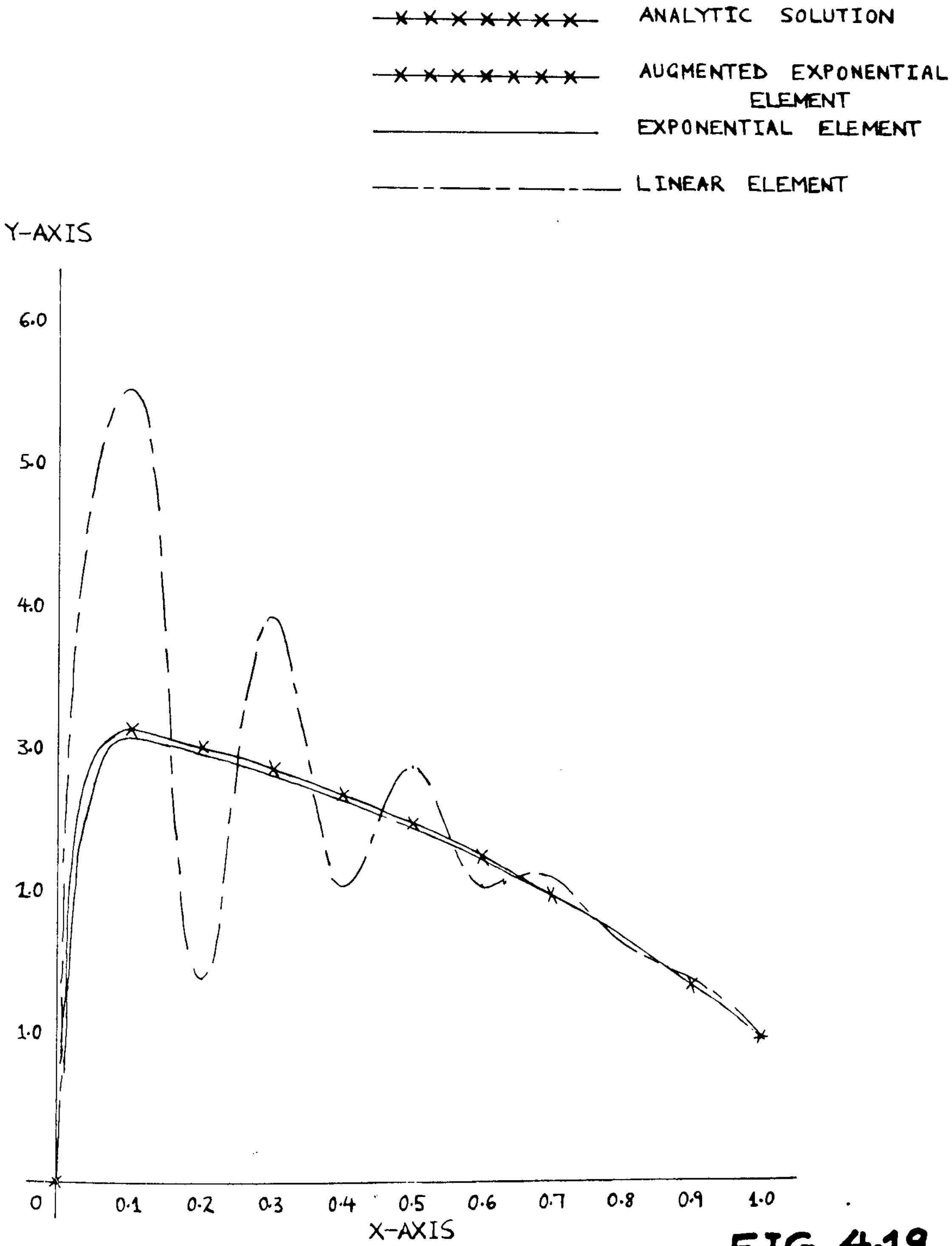


FIG 4.19

SOLUTION OF $Y''+R\left[\frac{1}{1+x}\right]Y'=R(1+x)(-1)$ FOR $R=10000$

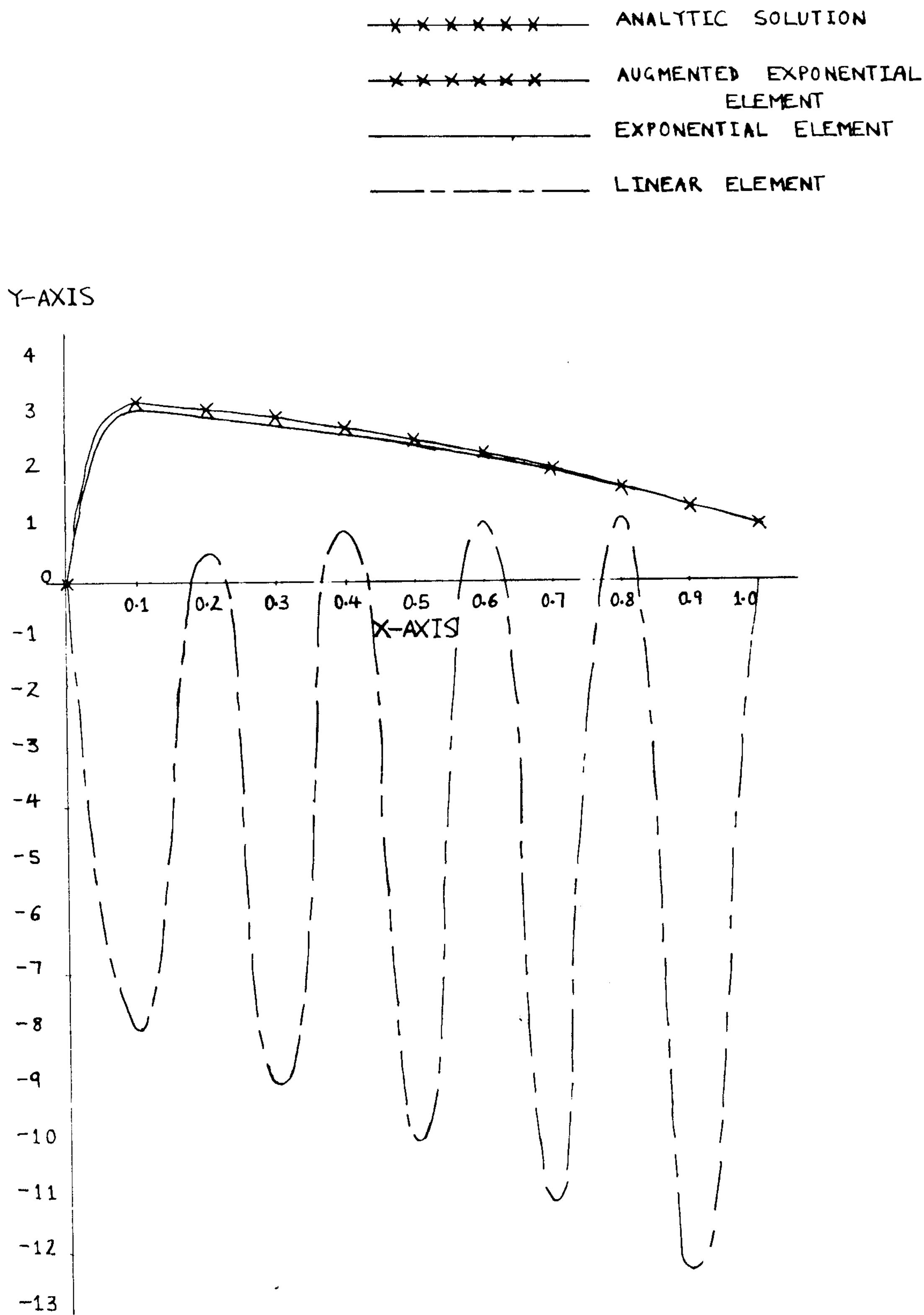


FIG 4·20

4.15 DISCUSSION OF THE EQUATION $Ly - \mu y = f$

To commence with consider the homogeneous differential equation $Ly - \mu y = 0$ and assume that λ and μ are constants. It is desired to construct a difference equation whose solution is identical to that of the differential equation. The general solution of the homogeneous equation is

$$y = Ae^{m_1 x} + Be^{m_2 x} \qquad (4.16.1)$$

Where m_1 and m_2 are roots of the auxillary equation viz.

$$m^2 - \lambda m - \mu = 0 \qquad (4.16.2)$$

Thus

$$m_1 = \frac{\lambda + \sqrt{\lambda^2 + 4\mu}}{2} \qquad (4.16.3)$$

$$m_2 = \frac{\lambda - \sqrt{\lambda^2 + 4\mu}}{2} \qquad (4.16.4)$$

The rest of the procedure is exactly the same as before. Thus a trial function of the form $y = Ae^{m_1 x} + Be^{m_2 x}$ is assumed over a two node element. If the origin is taken to coincide with node 1 of the element the shape functions are given by the equation

$$\begin{bmatrix} 1 & e^{m_1 h_e} \\ 1 & e^{m_2 h_e} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} e^{m_1 x} \\ e^{m_2 x} \end{bmatrix} \qquad (4.16.5)$$

The equation $Ly - \mu y = 0$ may be discretized by Galerkin's method as previously.

When λ and/or μ are functions of x then m_1 and m_2 are approximated as constants over an element in a similar way to that explained in (4.6). Thus over an element $[x_i, x_{i+1}]$

$$\overline{m}_1 = \frac{m(x_i) + m(x_{i+1})}{2} \qquad (4.16.6)$$

$$\overline{m}_2 = \frac{m(x_i) + m(x_{i+1})}{2} \qquad (4.16.7)$$

In the case when λ and/or μ are functions of both x and y then an initial guess has to be made at the solution and an iterative process set up. This has also been explained with reference to the equation $Ly = 0$ in (4.6).

There is one other difficulty that has to be resolved. It may happen that over some elements m_1 and m_2 turn out to be complex or even equal. In this case the appropriate trial function over that element must be used.

For example if $m = \alpha \pm j\beta$ over an element then the trial function for this element will be

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

The shape functions for this trial function will have to be computed and used in the evaluation of the element stiffness matrix for the element considered.

The non-homogeneous equation may be treated by augmenting the trial function for the homogeneous equation with a particular integral. As the argument is very similar to that for the differential equation Ly = f it is not proposed to go into details.

4.16 SUMMARY AND CONCLUSIONS

The dimensionless form of the one dimensional Navier Stokes equation is

$$\frac{d^2u}{dx^2} - Re\ u\frac{du}{dx} = Re\frac{dp}{dx} \qquad (4.16.1)$$

where Re is the Reynolds number, u is the x-component of velocity and p the pressure. If the pressure is a known function of x say p(x) (4.16.1) becomes

$$\frac{d^2u}{dx^2} - Re\ u\frac{du}{dx} = Re\,p'(x) \qquad (4.16.2)$$

The problem of stability associated with the Navier Stokes equations for large Reynolds numbers is well known.

The aim of this chapter has been to study the class of differential equations viz.

$$\frac{d^2y}{dx^2} - \lambda(x,y)\frac{dy}{dx} = f(x,y) \qquad (4.16.3)$$

- It will be noticed that this class of differential equations encompass the one dimensional Navier Stokes equation. A method has been presented for deriving shape functions for (4.16.3) to result in numerically stable schemes.

A number of numerical examples were also presented to illustrate the advantage of the new shape functions over the traditional polynomial shape functions.

The possible extension of the method to other differential operators were discussed in the last section.

In the next Chapter we examine a single elliptic partial differential equation which exhibits similar characteristics to those of the two-dimensional Navier Stokes equation.

CHAPTER FIVE

FINITE ELEMENT SCHEMES DERIVED FROM PARTIAL DIFFERENTIAL EQUATIONS

5.1

The canonical form of an elliptic partial differential equation may be taken as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \lambda\frac{\partial\phi}{\partial x} - \mu\frac{\partial\phi}{\partial y} + f\phi = g \qquad (5.1.1)$$

Define the differential operator L as

$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \lambda\frac{\partial}{\partial x} - \mu\frac{\partial}{\partial y}$$

so that (5.1.1) becomes

$$L\phi + f\phi = g \qquad (5.1.2)$$

The solution of a linear ordinary differential equation is a linear combination of a finite number of independent functions. It was this property which allowed the construction of a difference operator with an identical solution to the differential operator. However, the solution of (5.1.2) is generally a linear combination of an infinite number of independent functions. Thus it is not clear how the technique of the last chapter can be applied to (5.1.2) to obtain a difference operator whose solution is identical to (5.1.2).

One approach is to choose a finite number of particular solutions of (5.1.2) from the infinite number available. Suppose the functions $F_i(x,y)$ $(i = 1,2,\ldots,m)$ are particular solutions of (5.1.2) then the trial function

$$\phi = \sum_{i=1}^{m} a_i F_i(x,y) \qquad (5.1.3)$$

may be assumed over an m node two dimensional element. Galerkin's criteria can be used to discretize (5.1.2) in the usual way. There is one question which has to be answered. How are the functions $F_i(x,y)$ determined?

5.2 SEPARATION OF VARIABLES

It can be shown (12, 13, 14) that if a problem yields to the method of separation of variables and if the Galerkin method is applied in a certain way, then the two solutions are the same, provided the Galerkin method is carried through to completion. In fact the approximating functions in the Galerkin method must be the eigenfunctions found by the method of separation of variables. Such a result means simply that if the exact solution is contained in the trial function, then

Galerkin's method will find it, of course, in numerical calculations after obtaining an exact solution in the form of an infinite series, one calculates only a finite number of terms as a matter of practical necessity.

To commence with consider the differential equation $L\phi = 0$, i.e.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \lambda\frac{\partial\phi}{\partial x} - \mu\frac{\partial\phi}{\partial y} = 0 \qquad (5.2.1)$$

λ and μ will be assumed to be constants so that (5.2.1) can be tackled by the separation of variables. The case when λ and μ are general functions of x,y and ϕ will be treated subsequently.

Separating variables of (5.2.1) yields

$$\frac{X''}{X} - \lambda\frac{X'}{X} = -\frac{Y''}{Y} + \mu\frac{Y'}{Y} = k \qquad (5.2.2)$$

where X is a function of x only and Y is a function of y only, k is a constant. There are three separate cases to be considered.

(i) <u>$k < 0$</u>

Write $\qquad k = -\omega^2$

(5.2.2) gives

$$X'' - \lambda X' + \omega^2 X = 0 \qquad (5.2.3)$$

$$Y'' - \mu Y' - \omega^2 Y = 0 \qquad (5.2.4)$$

Define

$$p_1 = \frac{\lambda + \sqrt{\lambda^2 - 4\omega^2}}{2} \tag{5.2.5}$$

$$q_1 = \frac{\lambda - \sqrt{\lambda^2 - 4\omega^2}}{2} \tag{5.2.6}$$

$$r_1 = \frac{\mu + \sqrt{\mu^2 + 4\omega^2}}{2} \tag{5.2.7}$$

$$s_1 = \frac{\mu - \sqrt{\mu^2 + 4\omega^2}}{2} \tag{5.2.8}$$

X and Y are then given by

$$X = a_1 e^{p_1 x} + a_2 e^{q_1 x} \qquad \text{if} \quad \lambda^2 > 4\omega^2$$

$$Y = b_1 e^{r_1 y} + b_2 e^{s_1 y}$$

or

$$\phi = A_1 e^{(p_1 x + r_1 y)} + B_1 e^{(p_1 x + s_1 y)} + C_1 e^{(q_1 x + r_1 y)} + D_1 e^{(q_1 x + s_1 y)} \tag{5.2.9}$$

Thus the functions $e^{(p_1 x + r_1 y)}$, $e^{(p_1 x + s_1 y)}$, $e^{(q_1 x + r_1 y)}$, $e^{(q_1 x + s_1 y)}$ are particular solutions of (5.2.1). An infinite set of such solutions may be generated by giving ω different values.

(ii) k > 0

Write $k = \omega^2$

(5.2.2) gives

$$X'' - \lambda X' - \omega^2 X = 0 \qquad (5.2.10)$$

$$Y'' - \mu Y' + \omega^2 Y = 0 \qquad (5.2.11)$$

Define

$$p_2 = \frac{\lambda + \sqrt{\lambda^2 + 4\omega^2}}{2} \qquad (5.2.12)$$

$$q_2 = \frac{\lambda - \sqrt{\lambda^2 + 4\omega^2}}{2} \qquad (5.2.13)$$

$$r_2 = \frac{\lambda + \sqrt{\mu^2 - 4\omega^2}}{2} \qquad (5.2.14)$$

$$s_2 = \frac{\mu - \sqrt{\mu^2 - 4\omega^2}}{2} \qquad (5.2.15)$$

X and Y are then given by

$$X = a_3 e^{p_2 x} + a_4 e^{q_2 x}$$

$$Y = b_3 e^{r_2 y} + b_4 e^{s_2 y} \quad \text{if} \quad \mu^2 > 4\omega^2$$

Thus

$$\phi = A_2 e^{(p_2 x + r_2 y)} + B_2 e^{(p_2 x + s_2 y)} + C_2 e^{(q_2 x + r_2 y)}$$

$$+ D_2 e^{(q_2 x + s_2 y)}$$

(5.2.16)

Hence the functions $e^{(p_2x+r_2y)}$, $e^{(p_2x+s_2y)}$, $e^{(q_2x+r_2y)}$, $e^{(q_2x+s_2y)}$ are particular solutions of (5.2.1) and again an infinite number of such solutions may be generated by giving ω different values.

(iii) $\underline{k = 0}$

In this case it is easily seen that

$$\phi = A_3 + B_3e^{\lambda x} + C_3e^{\mu y} + D_3e^{\lambda x+\mu y} \qquad (5.2.17)$$

For practical purposes it is now necessary to choose a finite number of the particular solutions derived above. The question is which ones and how many?

From a computational point of view it is convenient to choose four particular solutions. This means that the element can be chosen to be a four node rectangle. As all the particular solutions are of the form $e^{\alpha x + \beta y}$ an added advantage of this choice is that all the integrations may be performed analytically. In selecting the four particular functions ease of implementation takes priority. In cases (i) and (ii) above λ and μ are required to satisfy.

$$\lambda^2 > 4\omega^2 \quad \text{in case (i)}$$
$$\mu^2 > 4\omega^2 \quad \text{in case (ii)}$$

If these inequalities for λ and μ are violated the solution for X in case (i) and Y in case (ii) will contain trigonometric terms. This means that the shape functions will be different for different ranges of values of λ and μ. For the case of non-constant λ and μ to be discussed later it would be necessary to choose appropriate shape functions for each element. Hence, this choice of eigenfunctions introduces a great computational complexity.

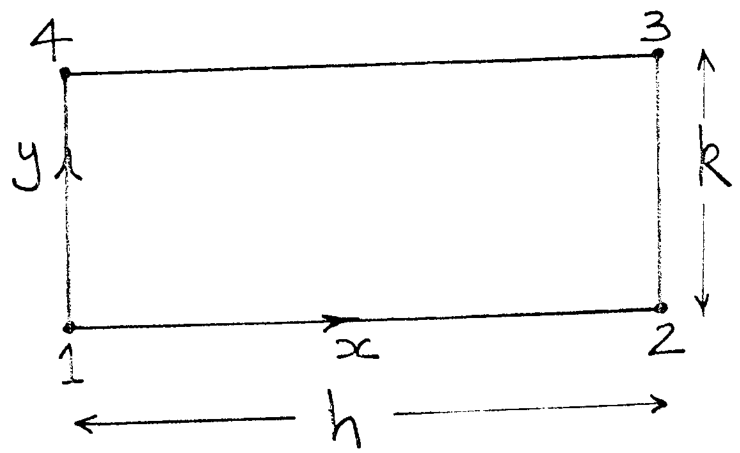
However, the eigenfunctions 1, $e^{\lambda x}$, $e^{\mu y}$, $e^{\lambda x+\mu y}$ derived in case (iii) above are suitable as there are no restrictions on λ and μ.

5.3 SHAPE FUNCTIONS FOR A TRIAL SOLUTION OF THE FORM $\phi = A + Be^{\lambda x} + Ce^{\mu y} + De^{\lambda x+\mu y}$

The functions 1, $e^{\lambda x}$, $e^{\mu y}$, $e^{\lambda x+\mu y}$ are solutions of the differential equation $L\phi = 0$ as established in the last section. A linear combination of these functions viz.

$$\phi = A + Be^{\lambda x} + Ce^{\mu y} + De^{\lambda x+\mu y} \tag{5.3.1}$$

is taken as a trial function over a four node rectangle as shown below



To simplify the algebra the axes are taken through node 1 as shown. Using the "shape function formula" the shape functions are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{\lambda h} & e^{\lambda h} & 1 \\ 1 & 1 & e^{\mu k} & e^{\mu k} \\ 1 & e^{\lambda h} & e^{\lambda h+\mu k} & e^{\mu k} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\lambda x} \\ e^{\mu y} \\ e^{\lambda x+\mu y} \end{bmatrix} \qquad (5.3.2)$$

Solving (5.3.2) for the shape functions yields

$$N_1 = \frac{(e^{\lambda x}-e^{\lambda h})(e^{\mu y}-e^{\mu k})}{(1-e^{\lambda h})\ (1-e^{\mu k})}$$

$$N_2 = \frac{(1-e^{\lambda x})(e^{\mu y}-e^{\mu k})}{(1-e^{\lambda h})(1-e^{\mu k})}$$

$$N_3 = \frac{(1-e^{\lambda x})(1-e^{\mu y})}{(1-e^{\lambda h})(1-e^{\mu k})}$$

$$N_4 = \frac{(e^{\lambda x}-e^{\lambda h})(1-e^{\mu y})}{(1-e^{\lambda h})(1-e^{\mu k})}$$

It is easily verified that these shape functions satisfy the usual conditions viz.

$$N_i(x_j, y_j) = \delta_{ij}$$

$$\sum_{i=1}^{4} N_i(x,y) = 1$$

The four node rectangular element with which the above shape functions are associated will be called the "exponential element".

5.4. FINITE ELEMENT SOLUTION OF $L\phi = 0$

Galerkin's criterion requires as usual

$$\iint_R N_i L\phi \, dxdy = 0 \tag{5.4.1}$$

where R is the two-dimensional region over which the solution of $L\phi = 0$ is required. Using Green's theorem in the form

$$\iint_R U\nabla^2 V dA = - \iint_R (\text{grad}U.\text{grad}V)dA + \oint_C U \frac{\partial V}{\partial n} ds$$

to reduce the interelement continuity requirement. Hence (5.4.1) may be written

$$-\iint_R (\text{grad}N_i \cdot \text{grad}\phi)\,dxdy + \oint_C N_i \frac{\partial\phi}{\partial n} ds - \iint_R \lambda N_i \frac{\partial\phi}{\partial x} dxdy$$

$$-\iint_R \mu N_i \frac{\partial\phi}{\partial y} dxdy = 0$$

or

$$\sum_e \iint_{r_e} \left\{ \left(\frac{\partial N_i}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial\phi}{\partial y} \right) + \left(\lambda N_i \frac{\partial\phi}{\partial x} + \mu N_i \frac{\partial\phi}{\partial y} \right) \right\} dxdy$$

$$-\sum_e \oint_{C_e} N_i \frac{\partial\phi}{\partial n} ds = 0 \qquad (5.4.2)$$

Where r_e is the region of a typical element and C_e its boundary. The summation is of course carried over all the elements.

Over a typical element write $\phi = \sum_j N_j\phi_j$. Thus (5.4.2) may be written

$$\sum_e \sum_j \left[\iint_{r_e} \left\{ \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) + \left(\lambda N_i \frac{\partial N_j}{\partial x} + \mu N_i \frac{\partial N_j}{\partial y} \right) \right\} dxdy \right] \phi_j$$

$$-\sum_e \left[\oint_{C_e} N_i \frac{\partial\phi}{\partial n} ds \right] = 0$$

(5.4.3)

Define the following quantities

$$C^e_{ij} = \iint_{r_e} N_i \frac{\partial N_j}{\partial x} dxdy; \qquad D^e_{ij} = \iint_{r_e} N_i \frac{\partial N_j}{\partial y} dxdy$$

$$E^e_{ij} = \iint_{r_e} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy$$

$$F^e_i = \oint_{C_e} N_i \frac{\partial \phi}{\partial n} ds.$$

and

$$\alpha^e_{ij} = \lambda C^e_{ij} + \mu D^e_{ij} + E^e_{ij}$$

(5.4.3) now becomes

$$\sum_e \sum_j \left[\alpha^e_{ij} \right] \phi_j - \sum_e F^e_i = 0 \qquad (5.4.4.)$$

5.5 CONSTRUCTION OF ELEMENT STIFFNESS MATRIX

The contribution of an element to the global matrix will now be evaluated. For an m node element e (5.4.4) may be written

$$\sum_{j=1}^{m} \left[\alpha_{ij}^{e} \right] \phi_{j}^{e} - F_{i}^{e} = (E)_{i}^{e} \qquad (5.5.1)$$

$i = 1, 2, \ldots, m.$

Where $(E)_i^e$ denotes the contribution to equation I from element e. I being the node number corresponding to the node identifier i.

In matrix form (5.5.1) is

$$\begin{bmatrix} (E)_1^e \\ (E)_2^e \\ \vdots \\ (E)_i^e \\ \vdots \\ (E)_m^e \end{bmatrix} = \begin{bmatrix} \alpha_{11}^e & \alpha_{12}^e & \cdots & \alpha_{1i}^e & \cdots & \alpha_{mm}^e \\ \alpha_{21}^e & \alpha_{22}^e & \cdots & \alpha_{2i}^e & \cdots & \alpha_{2m}^e \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{i1}^e & \alpha_{i2}^e & \cdots & \alpha_{ii}^e & \cdots & \alpha_{im}^e \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{m1}^e & \alpha_{m2}^e & \cdots & \alpha_{mi}^e & \cdots & \alpha_{mm}^e \end{bmatrix} \begin{bmatrix} \phi_1^e \\ \phi_2^e \\ \vdots \\ \phi_i^e \\ \vdots \\ \phi_m^e \end{bmatrix} - \begin{bmatrix} F_1^e \\ F_2^e \\ \vdots \\ F_i^e \\ \vdots \\ F_m^e \end{bmatrix} \qquad (5.5.2)$$

The superscript signifies that the coefficients are calculated solely for element e.

The coefficients C_{ij}^e, D_{ij}^e and E_{ij}^e are easily calculated analytically for the traditional four node rectangle. These coefficients may also be calculated analytically for the three node and six node triangle but it is necessary to use the known formula

$$\iint L_1^{\alpha} L_2^{\beta} L_3^{\gamma} dxdy = \frac{\alpha!\beta!\gamma!\ 2\Delta}{(\alpha+\beta+\gamma+\ 2)!} \qquad (5.5.3)$$

where L_i $(i = 1,2,3)$ are the area coordinates for node i. The integration is confined to a triangle having an area Δ.

The coefficients C^e_{ij}, D^e_{ij} and E^e_{ij} for traditional polynomial elements are given in appendix 2. It is interesting to note that other research workers have preferred to use numerical integration for triangular elements.

The coefficients for the "exponential element" can also be evaluated analytically. However, the expressions for these coefficients exhibit terms of the form $e^{\alpha\lambda h}$ and $e^{\beta\mu k}$, α β being constants. If it is desired to solve $L\phi = 0$ for large $|\lambda|$ and $|\mu|$ these coefficients must be arranged so that only negative exponential terms appear for otherwise numerical overflow will occur. All these coefficients arranged suitably are given in the next section.

5.6 ENTRIES OF ELEMENT SITFFNESS MATRIX FOR THE "EXPONENTIAL ELEMENT"

The coefficient α^e_{ij} in (5.5.2) for the "exponential element" are given below. First define the following quantities.

$$Q_1(\xi,\eta) = \begin{cases} \dfrac{(3e^{\xi\eta}-1)}{2\xi(1-e^{\xi\eta})} + \dfrac{\eta e^{2\xi\eta}}{(1-e^{\xi\eta})^2} & \xi < 0 \\ \dfrac{\eta}{3} & \xi = 0 \\ \dfrac{(3-e^{-\xi\eta})}{2\xi(e^{-\xi\eta}-1)} + \dfrac{\eta}{(e^{-\xi\eta}-1)^2} & \xi > 0 \end{cases}$$

$$Q_2(\xi,\eta) = \begin{cases} (-1)\left[\dfrac{\eta e^{\xi\eta}}{(1-e^{\xi\eta})} + \dfrac{(1+e^{\xi\eta})}{2\xi(1-e^{\xi\eta})}\right] & \xi < 0 \\ \dfrac{\eta}{6} & \xi = 0 \\ (-1)\left[\dfrac{\eta e^{-\xi\eta}}{(e^{-\xi\eta}-1)^2} + \dfrac{(e^{-\xi\eta}-1)}{2\xi(e^{-\xi\eta}-1)}\right] & \xi > 0 \end{cases}$$

$$Q_3(\xi,\eta) = \begin{cases} \dfrac{\eta}{(1-e^{\xi\eta})^2} + \dfrac{(3-e^{\xi\eta})}{2\xi(1-e^{\xi\eta})} & \xi > 0 \\ \dfrac{\eta}{3} & \xi = 0 \\ \dfrac{\eta e^{-2\xi\eta}}{(e^{-\xi\eta}-1)^2} + \dfrac{(3e^{-\xi\eta}-1)}{2\xi(e^{-\xi\eta}-1)} & \xi > 0 \end{cases}$$

$$Q_4(\xi,\eta) = \begin{cases} \dfrac{(-1)\xi(1+e^{\xi\eta})}{2(1-e^{\xi\eta})} & \xi < 0 \\ \dfrac{1}{2\eta} & \xi = 0 \\ \dfrac{(-1)\xi(e^{-\xi\eta}+1)}{(e^{-\xi\eta}-1)} & \xi > 0 \end{cases}$$

The coefficients C^e_{ij}, D^e_{ij}, and E^e_{ij} are given below

$$C^e_{11} = -\tfrac{1}{2}Q_1(\mu,k); \quad C^e_{12} = -C^e_{11}; \quad C^e_{13} = \tfrac{1}{2}Q_2(\mu,k); \quad C^e_{14} = -C^e_{13}$$

$$C^e_{21} = C^e_{11}; \quad C^e_{22} = -C_{11}; \quad C^e_{23} = C^e_{13}; \quad C^e_{24} = -C^e_{13}$$

$$C^e_{31} = -C^e_{13}; \quad C^e_{32} = C^e_{13}; \quad C^e_{33} = \tfrac{1}{2}Q_3(\mu,k); \quad C^e_{34} = C^e_{33}$$

$$C^e_{41} = -C^e_{13}; \quad C^e_{42} = C^e_{13}; \quad C^e_{43} = C^e_{33}; \quad C^e_{44} = -C^e_{33}$$

and

$$D^e_{11} = -\tfrac{1}{2}Q_1(\lambda,h); \quad D^e_{12} = \tfrac{1}{2}Q_2(\lambda,h); \quad D^e_{13} = -D^e_{12}; \quad D^e_{14} = -D^e_{11}$$

$$D^e_{21} = D^e_{12}; \quad D^e_{22} = -\tfrac{1}{2}Q_3(\lambda,h); \quad D^e_{23} = -D^e_{22}; \quad D^e_{24} = -D^e_{12}$$

$$D^e_{31} = D^e_{12}; \quad D^e_{32} = D^e_{22}; \quad D^e_{33} = -D^e_{22}; \quad D^e_{34} = -D^e_{12}$$

$$D^e_{41} = D^e_{11}; \quad D^e_{42} = D^e_{12}; \quad D^e_{43} = -D^e_{12}; \quad D^e_{44} = -D^e_{11}$$

and

$$E^e_{11} = Q_4(\lambda,h)Q_1(\mu,k) + Q_4(\mu,k)Q_1(\lambda,h)$$

$$E^e_{12} = (-1)Q_4(\lambda,h)Q_1(\mu,k) + Q_4(\mu,k)Q_2(\lambda,h)$$

$$E^e_{13} = (-1)Q_4(\lambda,h)Q_2(\mu,k) + (-1)Q_4(\mu,k)Q_2(\lambda,h)$$

$$E^e_{14} = Q_4(\lambda,h)Q_2(\mu,k) + (-1)Q_4(\mu,k)Q_1(\lambda,h)$$

$$E^e_{21} = E^e_{12}$$

$$E^e_{22} = Q_4(\lambda,h)Q_1(\mu,k) + Q_4(\mu,k)Q_3(\lambda,h)$$

$$E^e_{23} = Q_4(\lambda,h)Q_2(\mu,k) + (-1)Q_4(\mu,k)Q_3(\lambda,h)$$

$$E^e_{24} = (-1)Q_4(\lambda,h)Q_2(\mu,k) + (-1)Q_4(\mu,k)Q_2(\lambda,h)$$

$$E^e_{31} = E^e_{13}$$

$$E^e_{32} = E^e_{23}$$

$$E^e_{33} = Q_4(\lambda,h)Q_3(\mu,k) + Q_4(\mu,k)Q_3(\lambda,h)$$

$$E^e_{34} = (-1)Q_4(\lambda,h)Q_3(\mu,k)+Q_4(\mu,k)Q_2(\lambda,h)$$

$$E^e_{41} = E^e_{14}$$

$$E^e_{42} = E^e_{24}$$

$$E^e_{43} = E^e_{34}$$

$$E^e_{44} = Q_4(\lambda,h)Q_3(\mu,k)+Q_4(\mu,k)Q_1(\lambda,h)$$

α^e_{ij} may now be obtained from the relation

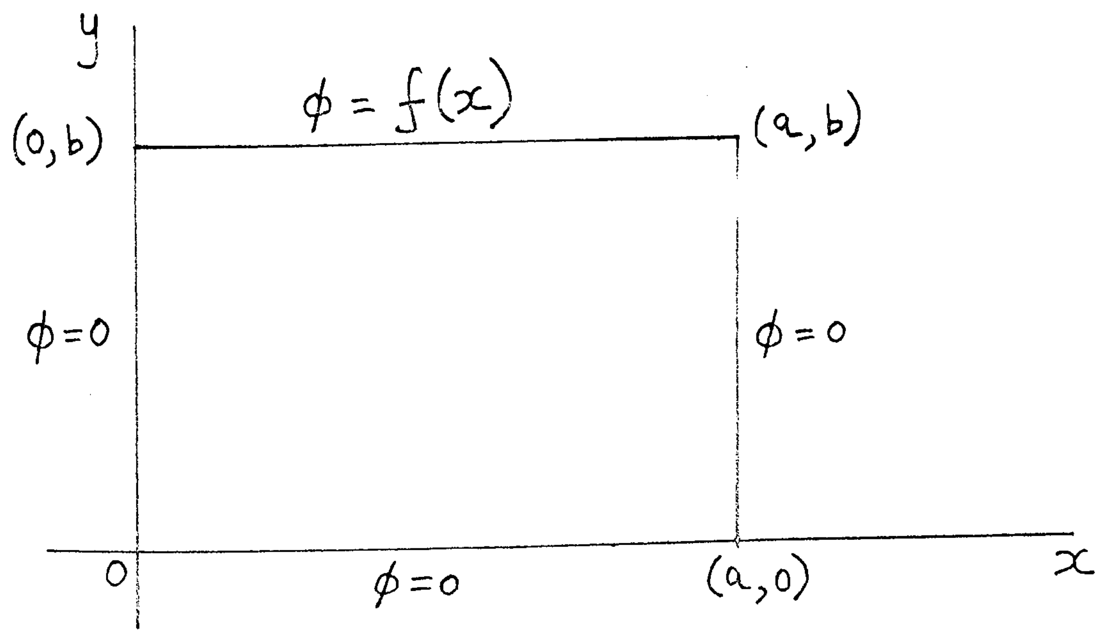
$$\alpha^e_{ij} = \lambda C^e_{ij} + \mu D^e_{ij} + E^e_{ij}$$

5.7 NUMERICAL EXAMPLE

The solution of $L\phi = 0$ i.e.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \lambda\frac{\partial\phi}{\partial x} - \mu\frac{\partial\phi}{\partial y} = 0 \qquad (5.7.1)$$

over a rectangle subject to the boundary conditions shown below is



$$\phi = \frac{2}{a} e^{\frac{1}{2}(\lambda x+\mu y)} e^{\frac{1}{2}\mu b} \sum_{n=1}^{\infty} I_n \left[\frac{\sinh(\frac{1}{2}\beta_n y)}{\sinh(\frac{1}{2}\beta_n b)} \right] \sin\left(\frac{n\pi x}{a}\right) \tag{5.7.2}$$

where

$$\beta_n = \sqrt{\lambda^2 + \mu^2 + \frac{4n^2\pi^2}{a^2}} \tag{5.7.3}$$

and

$$I_n = \int_0^a e^{-\frac{1}{2}\lambda x} f(x) \sin\left(\frac{n\pi x}{a}\right) dx \tag{5.7.4}$$

$f(x)$ was chosen to be $\sin(\frac{\pi x}{a})$ so that ϕ is continuous along the boundary and also with this choice I_n may be evaluated analytically. It is not difficult to show that

$$I_n = \frac{16\pi^2 n[1 + (-1)^n e^{-\frac{\lambda a}{2}}]}{\lambda^3 a^2 [1 + \frac{4(n-1)^2\pi^2}{\lambda^2 a^2}]\,[1 + \frac{4(n+1)^2\pi^2}{\lambda^2 a^2}]} \tag{5.7.5}$$

The analytic solution from (5.7.2) was evaluated for various values of λ and μ over the finite element grids shown in figure (5.1) - figure (5.6).

The numerical evaluation of the analytic solution is discussed in (5.13). It is worth noting that as $\lambda, \mu \to \infty$ then $\phi \to 0$. The differential equation (5.7.1) was solved using the

(i) Three node triangle

(ii) Traditional four node rectangle

(iii) Six node triangle

(iv) Exponential element

for various values of λ and μ.

In order to carry out convergence tests the finite element solutions were computed on a set of coarse and fine meshes see figure (5.1) - figure (5.6).

The results are illustrated graphically for λ, μ = 1, 10, 10^2, 10^3, 10^4 for the fine meshes in figure (5.7) - figure (5.11). The results for the coarse mesh are shown for λ, μ = 10, 10^3 in figure (5.12) and figure (5.13) respectively. It can be seen that refining the grid improves the accuracy for all the elements.

For $\lambda = \mu = 1$ there is not a great deal of discrepancy between the various elements. As λ and μ increase the traditional elements exhibit oscillations. However, the "exponential element" continues to give good agreement. Although not shown, but when $\lambda, \mu > 10^4$ the oscillations associated with the traditional elements increase dramatically. Finally it must be mentioned that although the graphs show the variation of the solution along the line $y = 0.3$, the same phenomenon was observed at other nodal points.

COARSE MESH USING FOUR NODE RECTANGLE

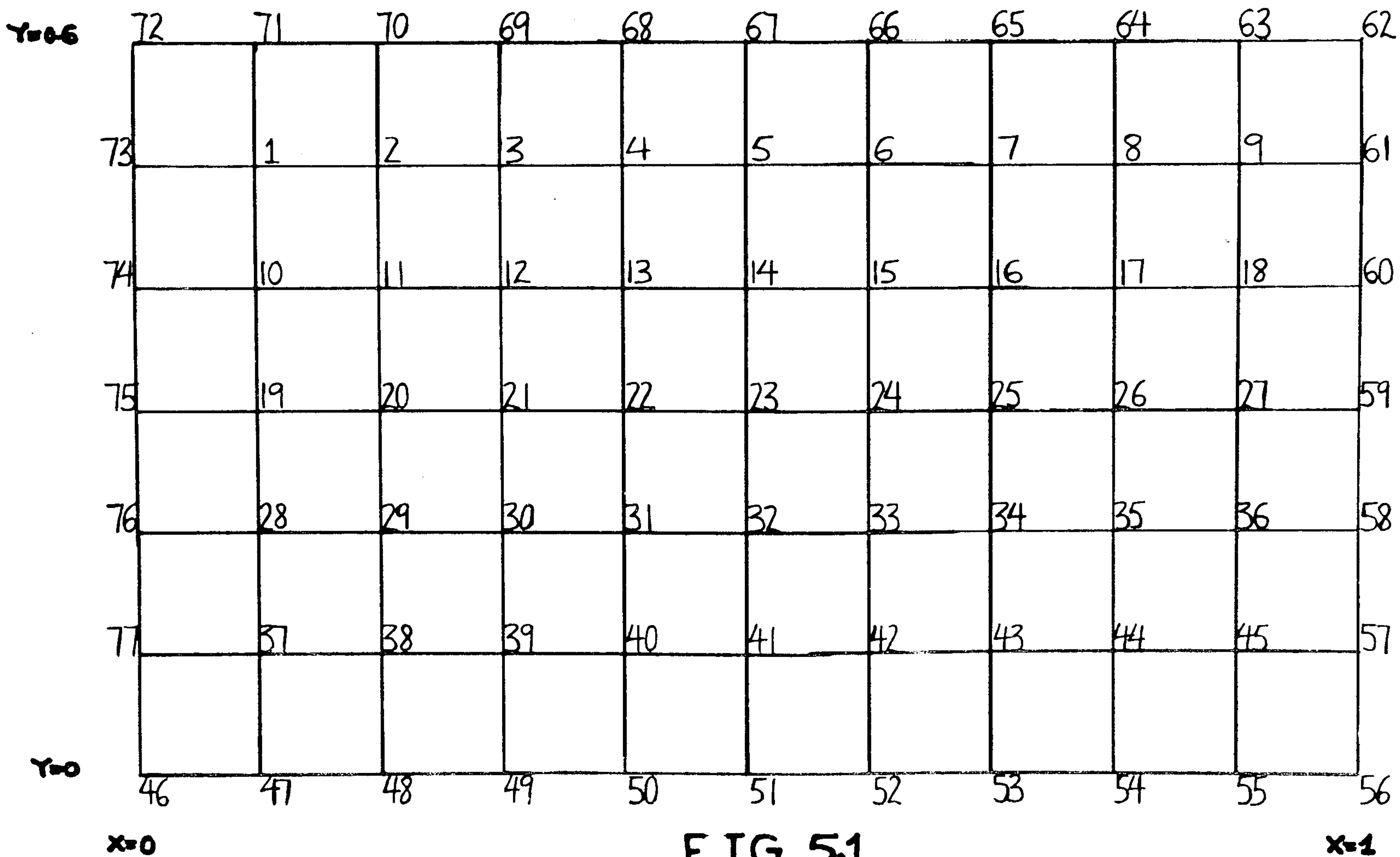


FIG 5.1

COARSE MESH USING THREE NODE TRIANGLE

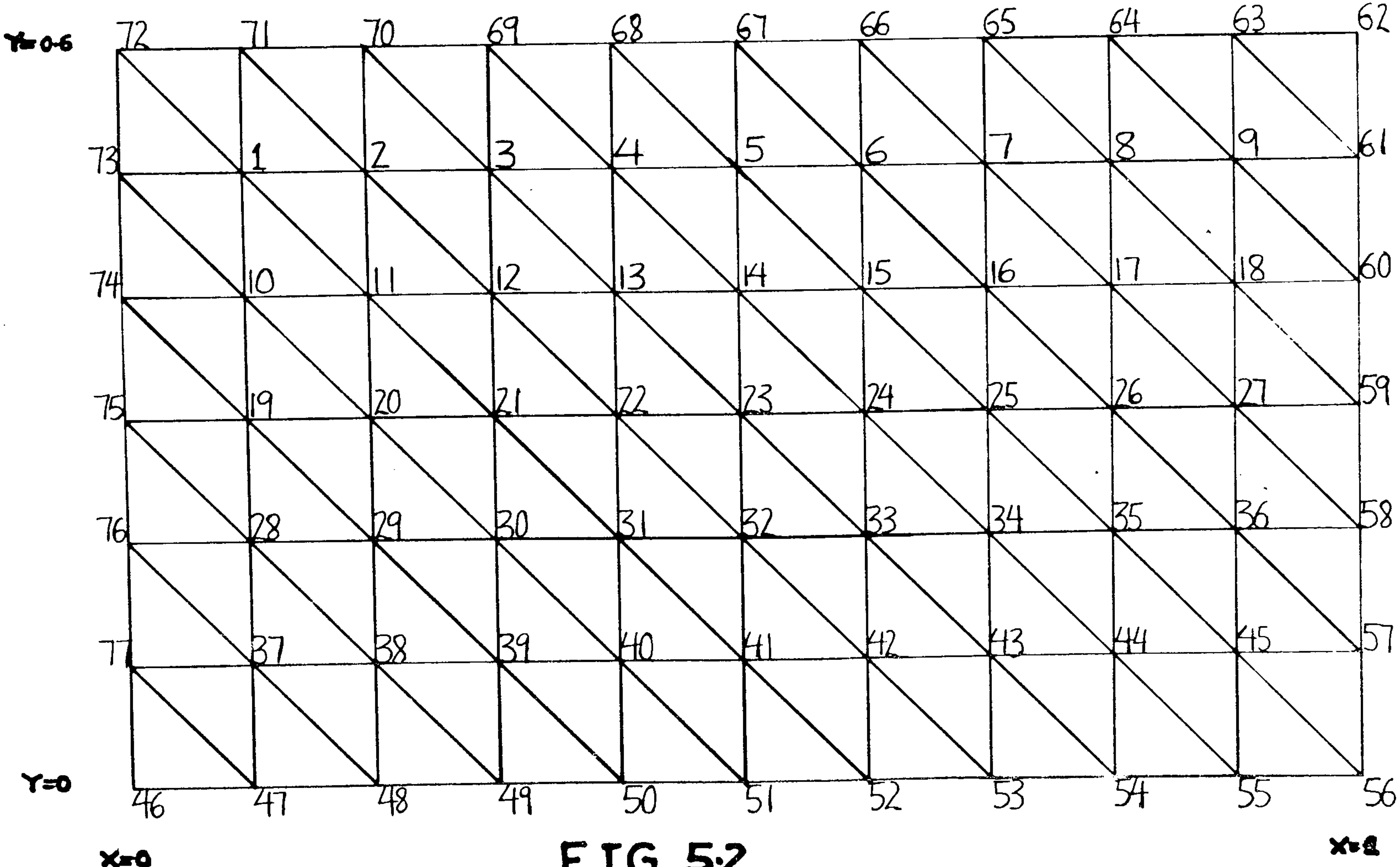


FIG 5·2

COARSE MESH USING SIX NODE TRIANGLE

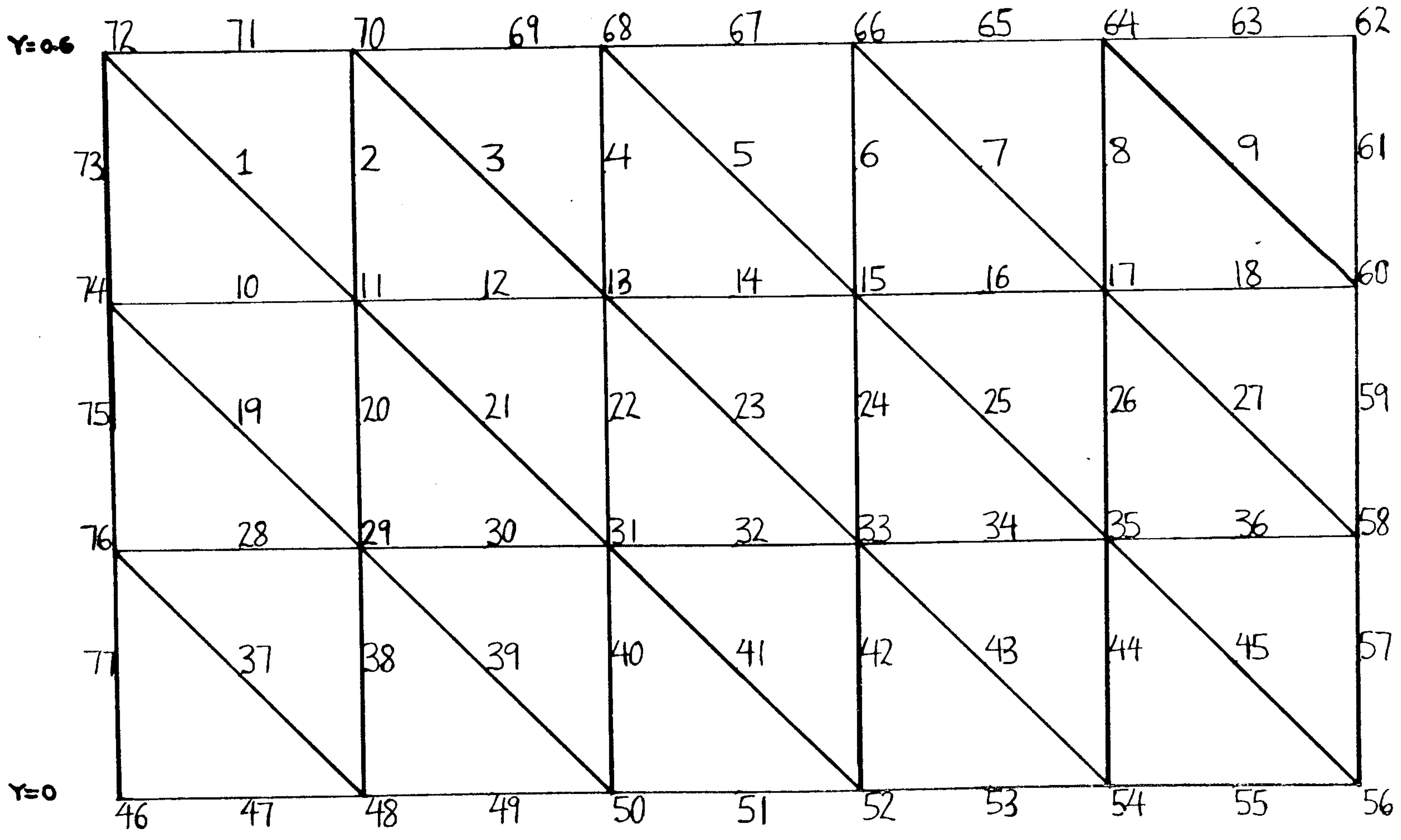


FIG 5.3

FINE MESH USING FOUR NODE RECTANGLE

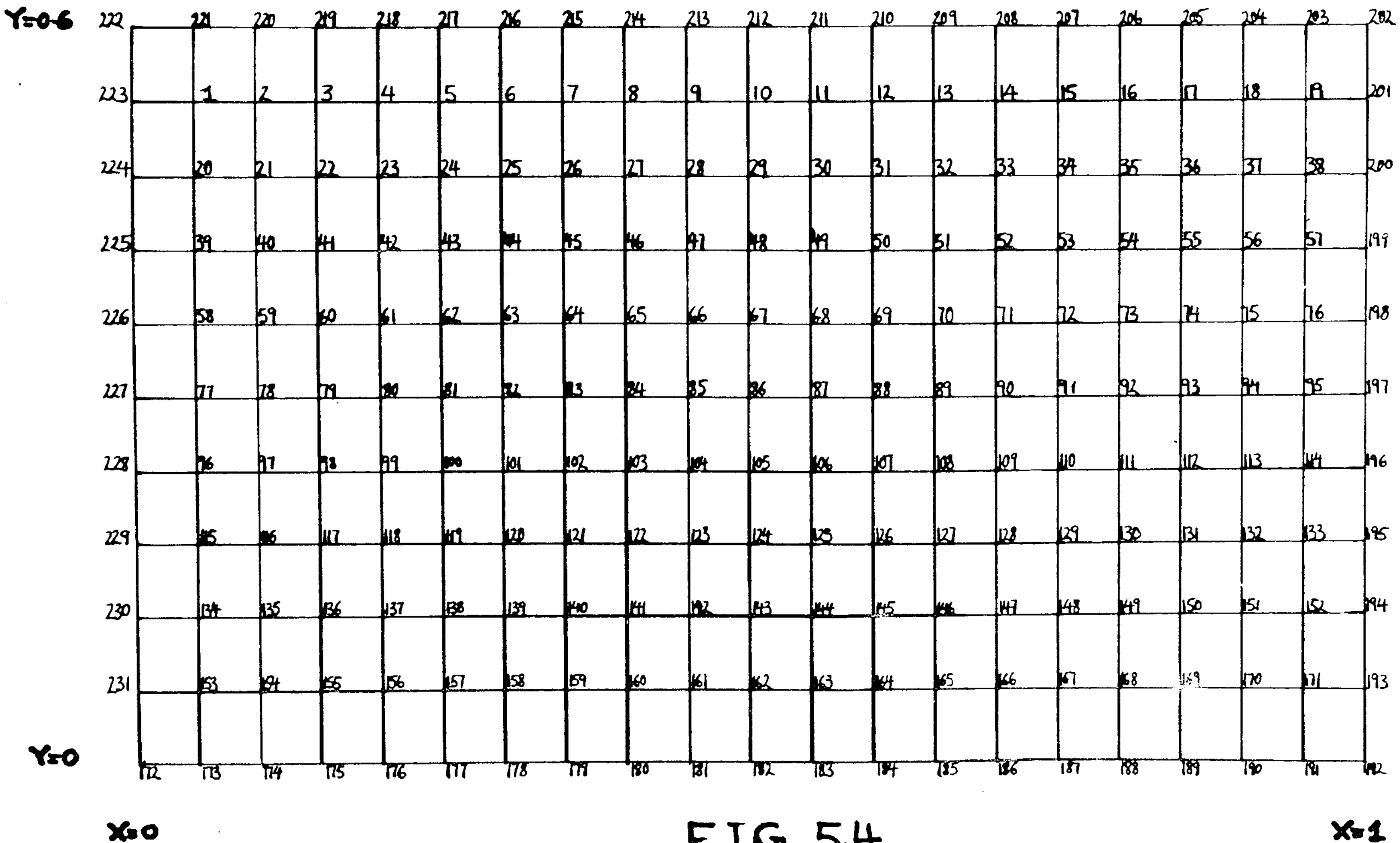


FIG 5.4

FINE MESH USING THREE NODE TRIANGLE

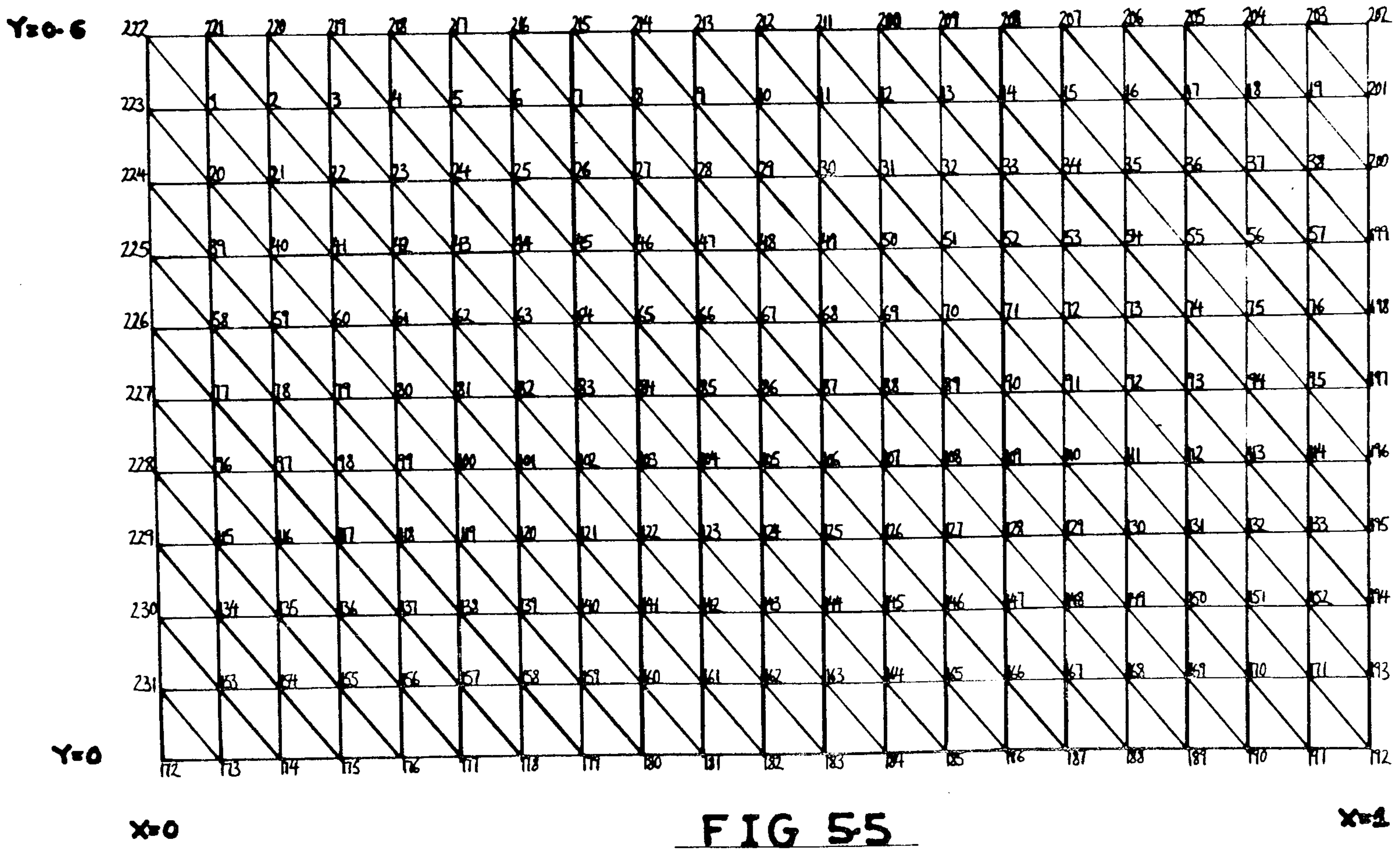


FIG 5.5

FINE MESH USING SIX NODE TRIANGLE

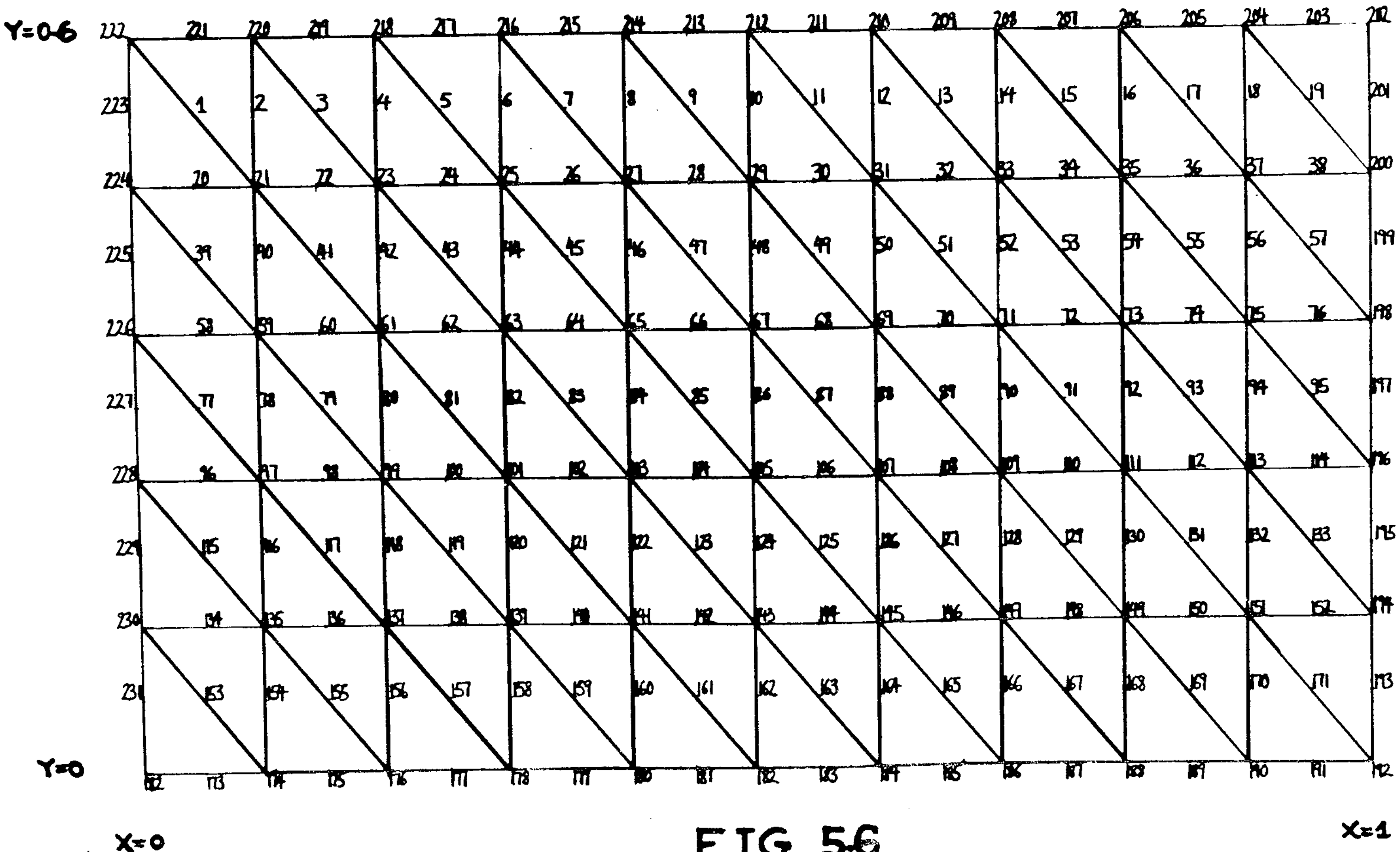
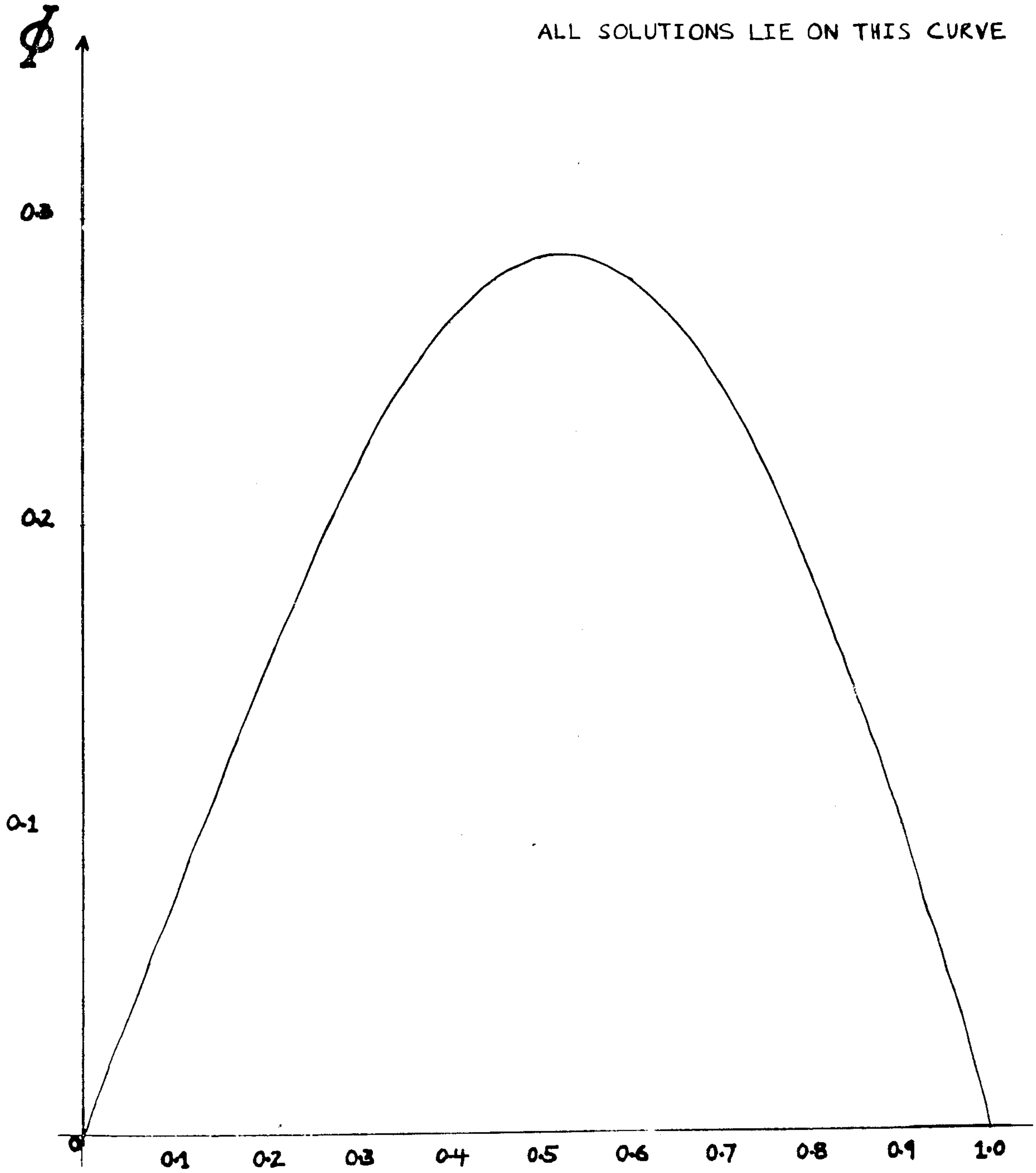


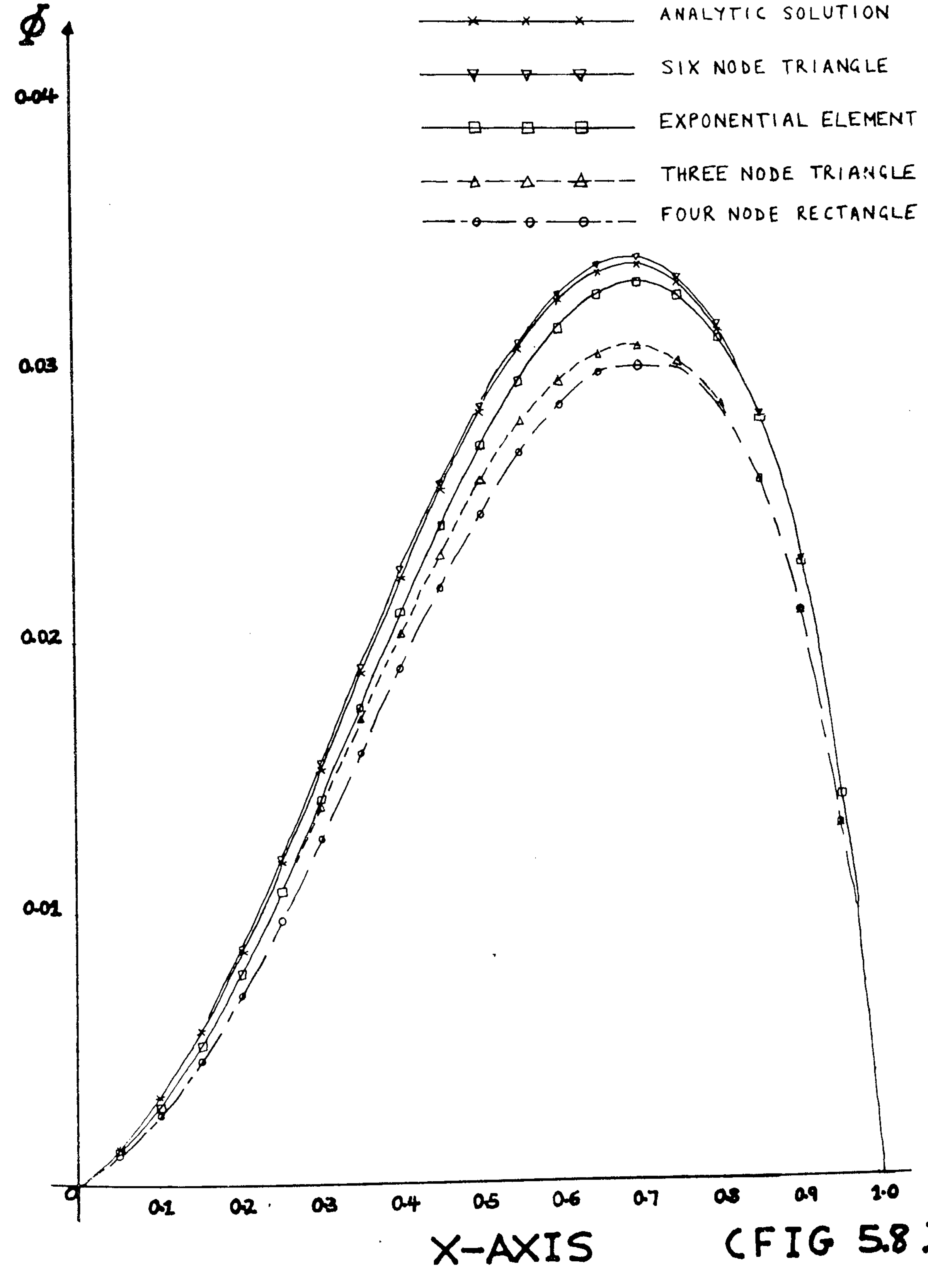
FIG 56

FINITE ELEMENT SOLUTION OF $L\phi=0$ ALONG $Y=0.3$ ON THE FINE MESH FOR $\lambda=\mu=1.0$



X-AXIS (FIG 5.7)

FINITE ELEMENT SOLUTION OF $L\phi=0$ ALONG $Y=0.3$ ON THE FINE MESH FOR $\lambda=\mu=10.0$



(FIG 5.8)

FINITE ELEMENT SOLUTION OF $L\phi=0$ ALONG $Y=0\cdot3$ ON THE FINE MESH FOR $\lambda=\mu=100\cdot0$

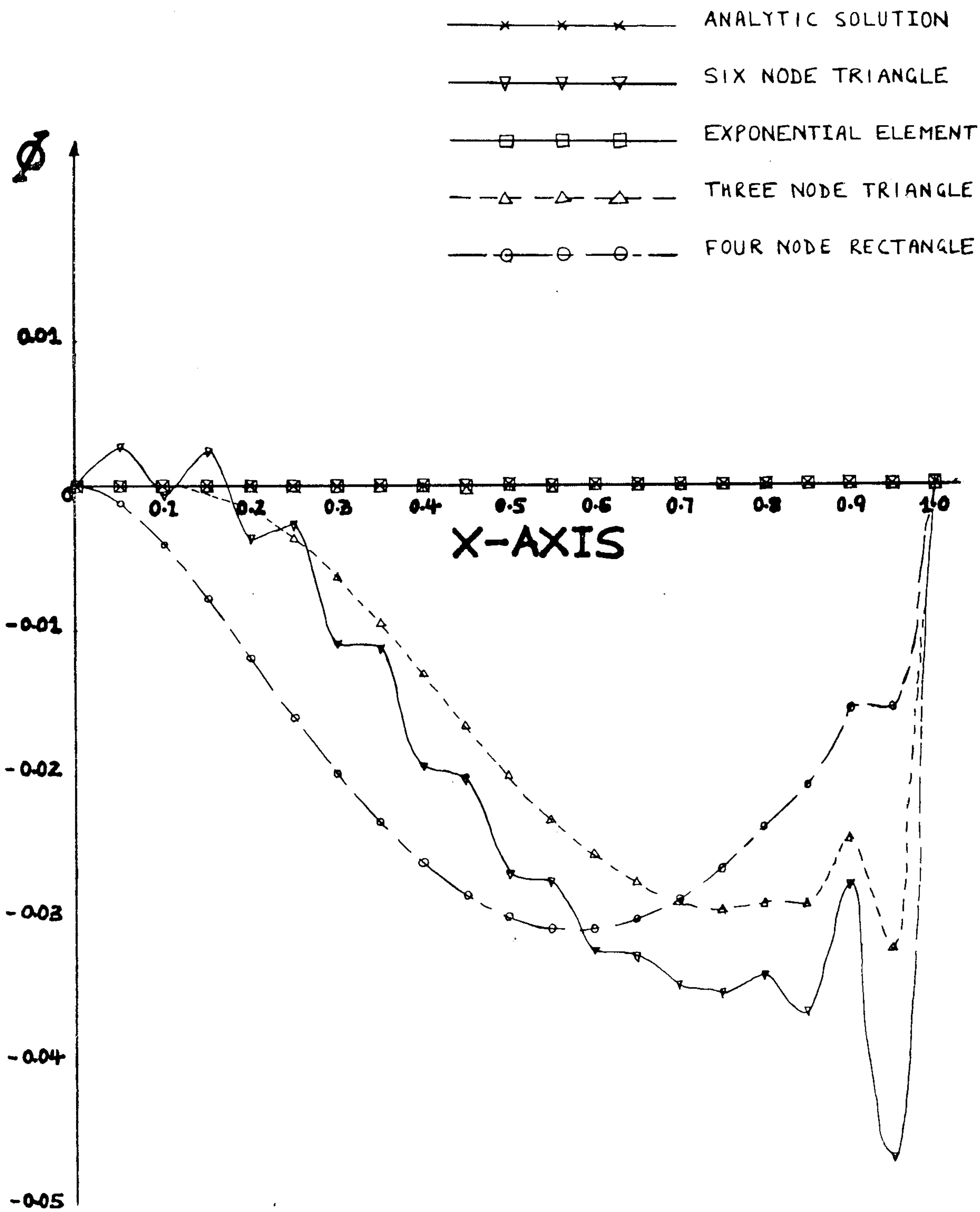


FIG 5.9

FINITE ELEMENT SOLUTION OF $L\phi = 0$ ALONG $Y = 0.3$ ON THE FINE MESH FOR $\lambda = \mu = 1000.0$

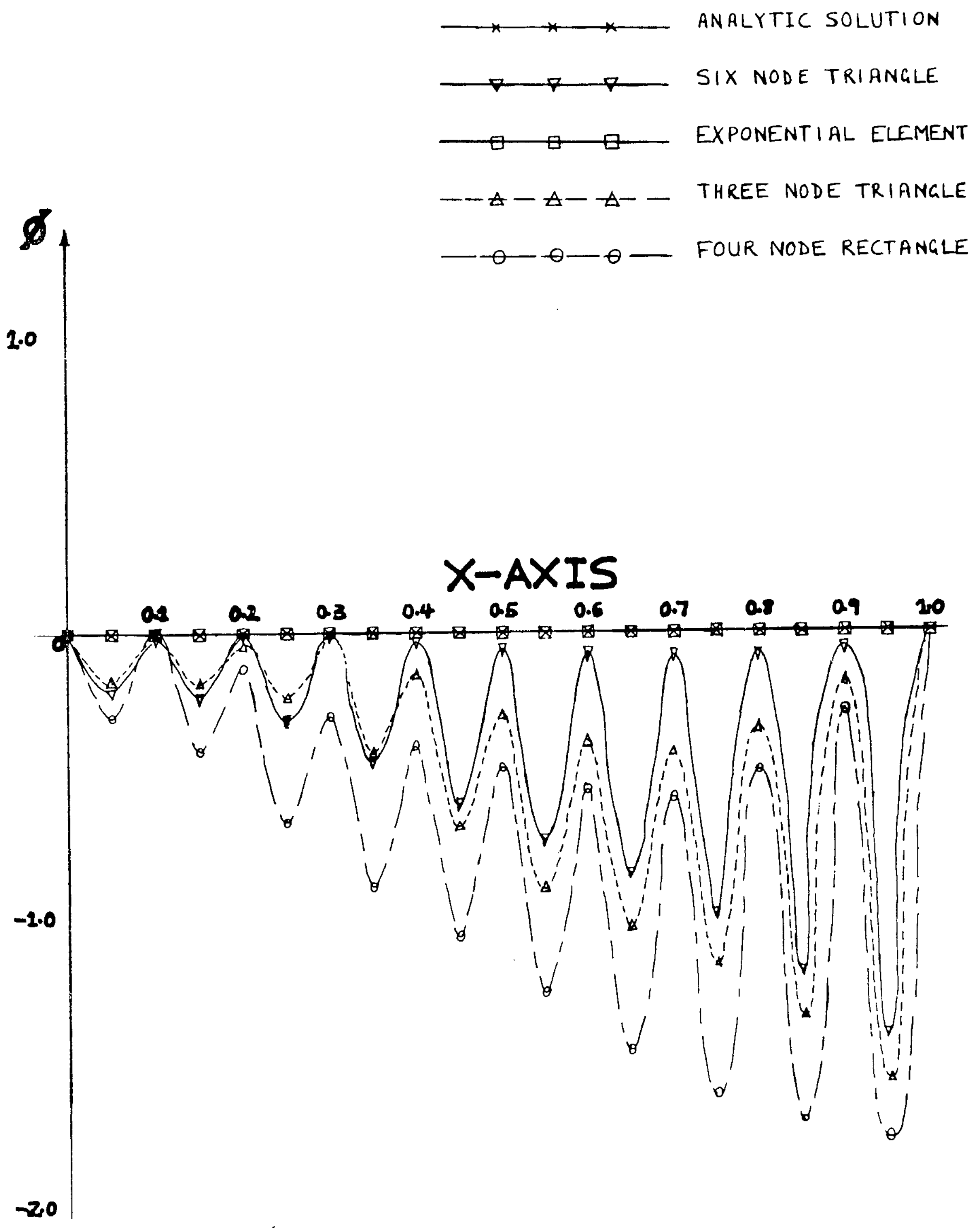


FIG 5.10

FINITE ELEMENT SOLUTION OF $L\phi=0$ ALONG $Y=0.3$ ON THE FINE MESH FOR $\lambda=\mu=10000.0$

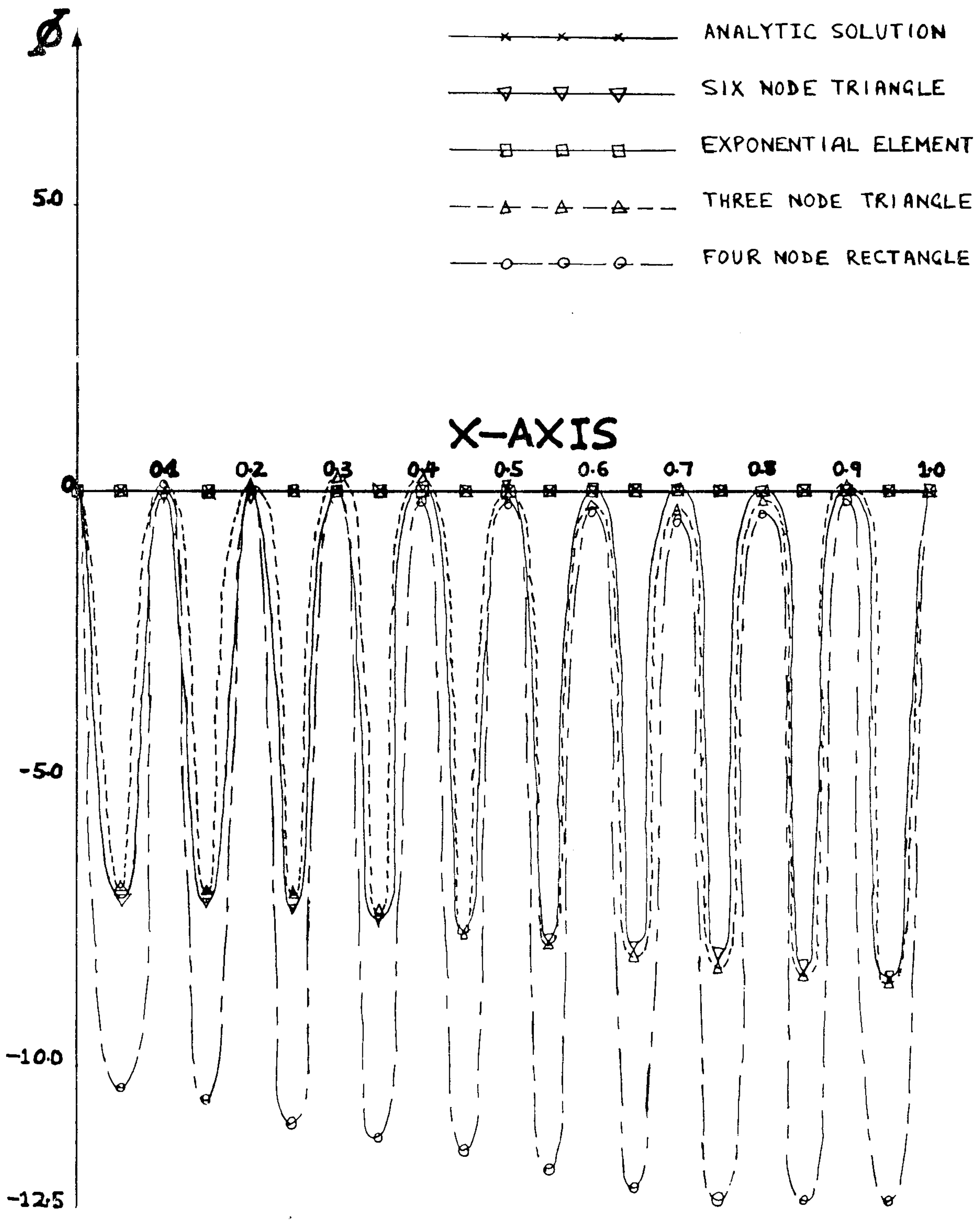
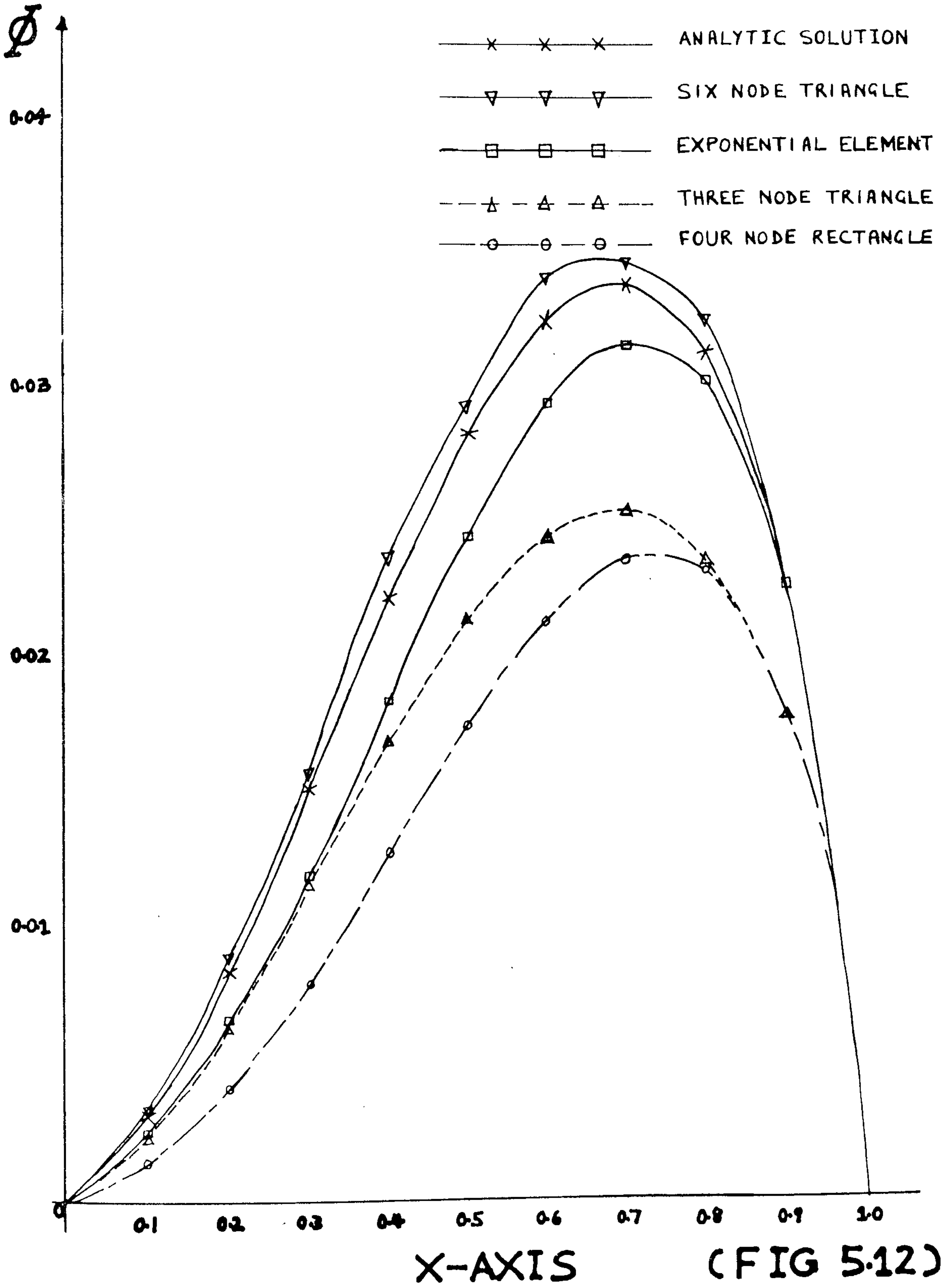


FIG 5.11

FINITE ELEMENT SOLUTION OF $L\phi=0$ ALONG Y=0.3 ON THE COARSE MESH FOR $\lambda=\mu=10.0$



(FIG 5.12)

FINITE ELEMENT SOLUTION OF $L\phi=0$ ALONG $Y=0.3$ ON THE COARSE MESH FOR $\lambda=\mu=1000.0$

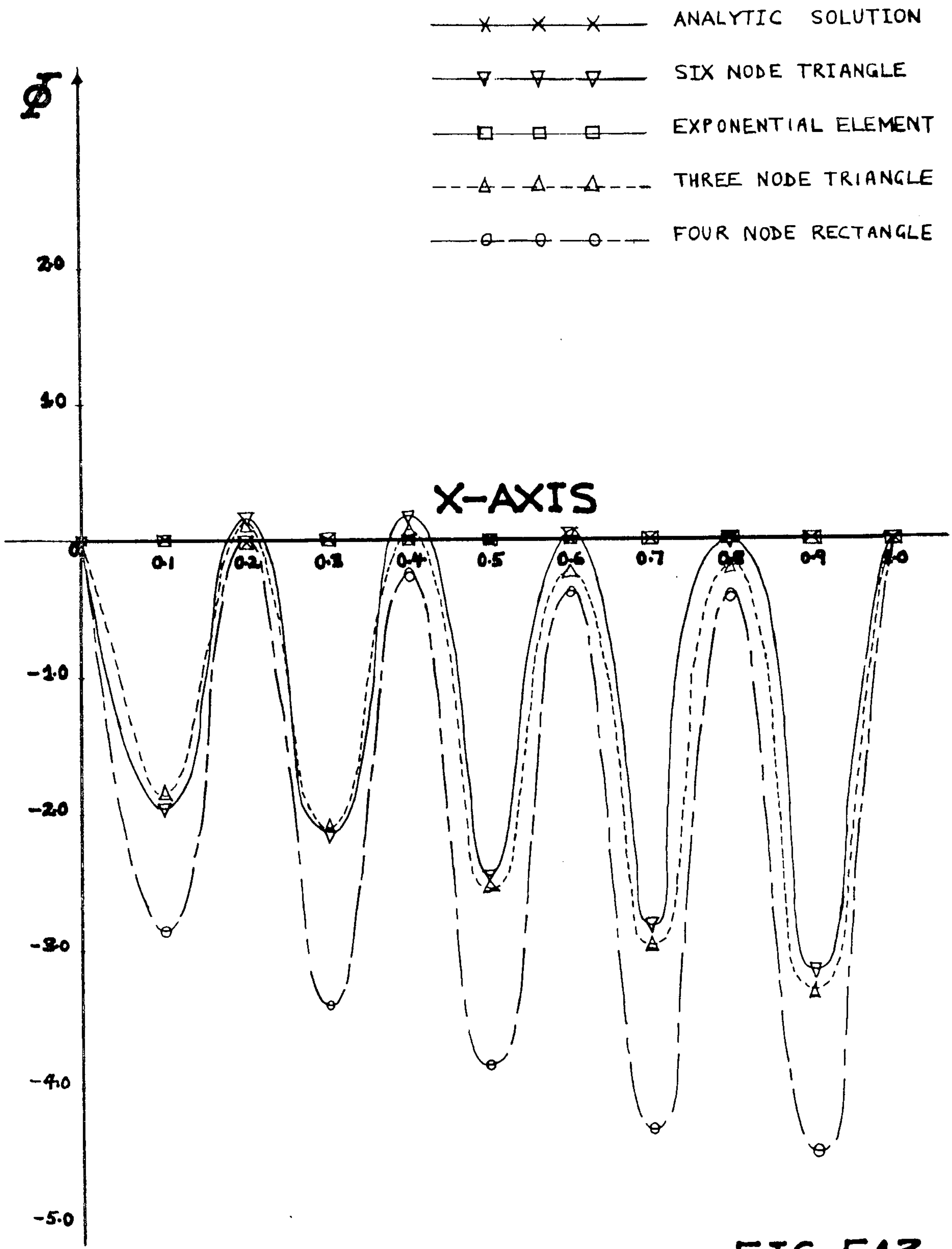


FIG 5.13

5.8 THE CASE OF λ AND/OR μ NON-CONSTANTS

When λ and μ are functions of x and y then the argument presented in (4.6) for ordinary differential equations can be applied here. Thus over a typical four node rectangular element with node identifiers 1234 $\lambda(x,y)$ and $\mu(x,y)$ can be approximated as constants $\overline{\lambda}$ and $\overline{\mu}$ respectively where

$$\overline{\lambda} = \frac{1}{4} \sum_{j=1}^{4} \lambda(x_j, y_j) \tag{5.8.1}$$

$$\overline{\mu} = \frac{1}{4} \sum_{j=1}^{4} \mu(x_j, y_j) \tag{5.8.2}$$

When λ and μ are functions of x and y and also ϕ it is first necessary to guess the solution. Let $\overline{\phi}_i^{(n)}$ be the value of ϕ_i at the nth iteration. Then for the (n+1)th iteration $\overline{\lambda}$ and $\overline{\mu}$ are given by

$$\overline{\lambda} = \frac{1}{4} \sum_{j=1}^{4} \lambda(x_j, y_j, \overline{\phi}_j^{(n)}) \tag{5.8.3}$$

$$\overline{\mu} = \frac{1}{4} \sum_{j=1}^{4} \mu(x_j, y_j, \overline{\phi}_j^{(n)}) \tag{5.8.4}$$

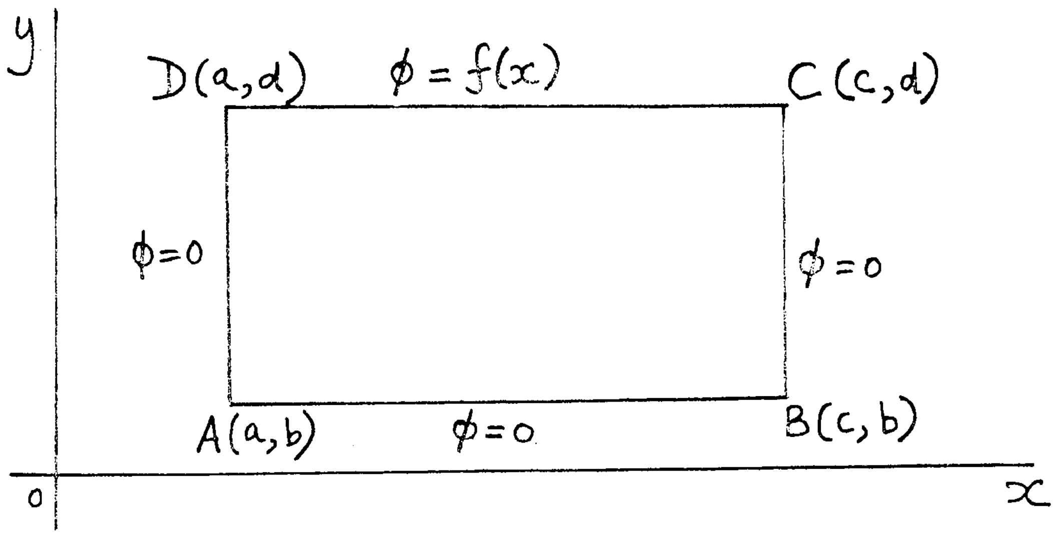
For the case of variable λ and μ the elements are actually non-conforming. However the above procedure yields good results even for a fairly coarse mesh as the next example shows.

5.9 NUMERICAL EXAMPLE

This example illustrates the case of non-constants λ and μ. The partial differential equation considered is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{2}{x}\frac{\partial\phi}{\partial x} + \frac{2}{y}\frac{\partial\phi}{\partial y} = 0 \qquad (5.9.1)$$

over a rectangle subject to the boundary conditions shown below



where

$$f(x) = \sin\frac{\pi(x-a)}{(c-a)}$$

It can be shown by separation of variables that the solution is

$$\phi = \frac{d(c+a)}{2xy}\,\frac{\sinh[\frac{\pi(y-b)}{(c-a)}]}{\sinh[\frac{\pi(d-b)}{(c-a)}]}\,\sin\left[\frac{\pi(x-a)}{(c-a)}\right] - \frac{16d(c-a)}{\pi^2 xy}$$

$$\sum_{r=1}^{\infty} \frac{r\sinh[\frac{2r\pi(y-b)}{(c-a)}]\,\sin[\frac{2r\pi(x-a)}{(c-a)}]}{(4r^2-1)^2\sinh[\frac{2r\pi(d-b)}{(c-a)}]} \qquad (5.9.2)$$

The numerical evaluation of this series is discussed in (5.13).

Comparing (5.9.1) with the standard equation $L\phi = 0$ it is seen that

$$|\lambda| = \frac{2}{|x|} \tag{5.9.3}$$

$$|\mu| = \frac{2}{|y|} \tag{5.9.4}$$

This means that $|\lambda|$ and $|\mu|$ will be large if $|x|$ and $|y|$ are small.

The differential equation (5.9.1) was solved using the

(i) Three node triangle

(ii) Traditional four node rectangle

(iii) Six node triangle

(iv) Exponential element

for various positions of the rectangle ABCD. It is seen from (5.9.3) and (5.9.4) that the position of the rectangle relative to the origin determines the magnitude of λ and μ. When the coordinates of the point A are large $|\lambda|$ and $|\mu|$ are accordingly small and vice versa.

The results are illustrated graphically for various positions of the rectangle ABCD in figure (5.14) - figure (5.18). The coarse mesh only was used to compute the finite element solutions. It is observed from the graphs that for small values of $|\lambda|$ and $|\mu|$ all solutions compare well with the analytic solution. As $|\lambda|$ and $|\mu|$ increase the traditional elements exhibit oscillations. However, the "exponential element" continues to give good agreement.

FINITE ELEMENT SOLUTION OF

$$\nabla^2\phi + \frac{2}{x}\phi_x + \frac{2}{y}\phi_y = 0$$

[a=b=1.0 ; c=2.0 ; d=1.6]

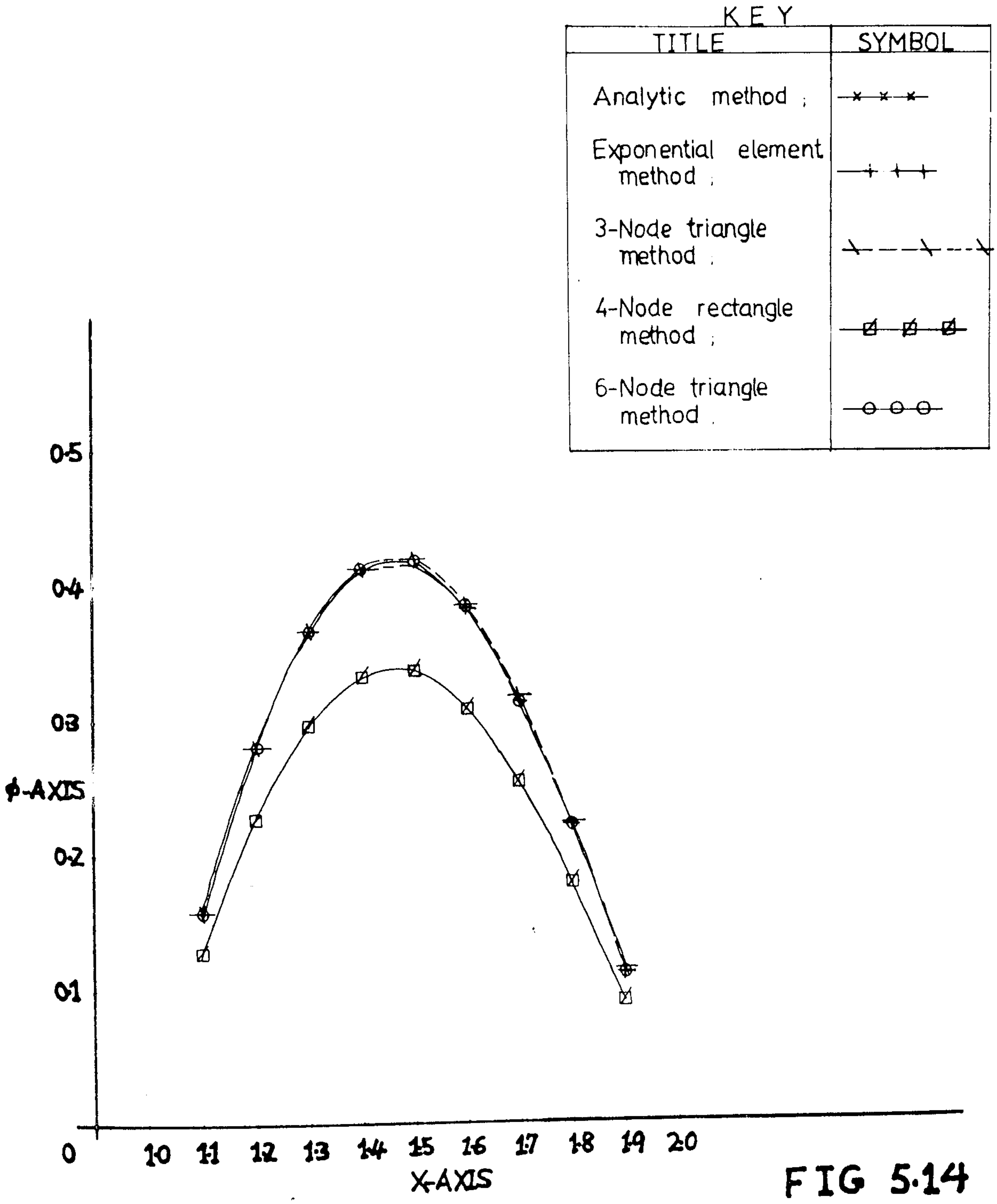


FIG 5.14

FINITE ELEMENT SOLUTION OF

$$\nabla^2\phi + \frac{2}{x}\phi_x + \frac{2}{y}\phi_y = 0$$

[a=b=0·2 ; c=1·2 ; d=0·8]

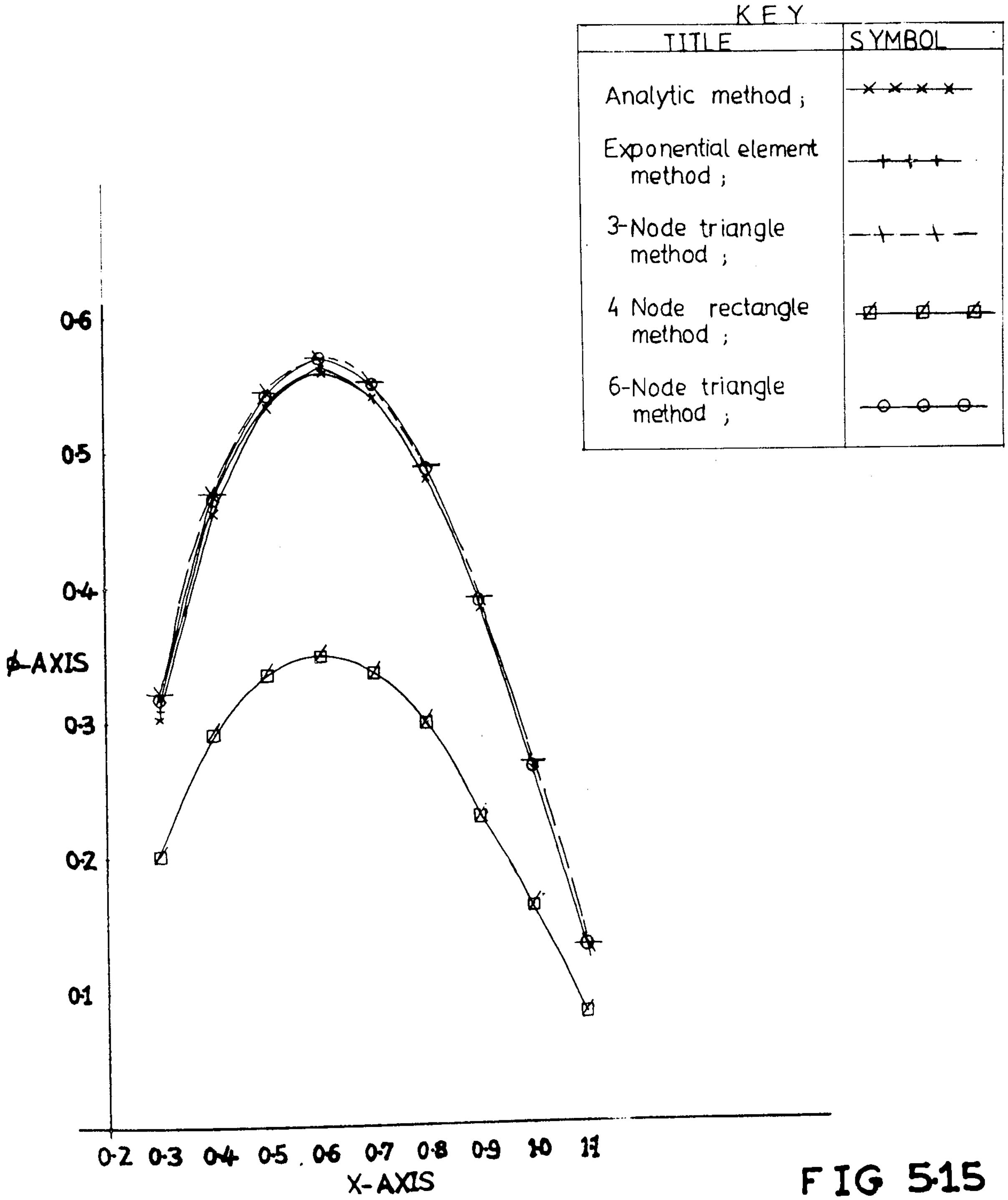


FIG 5.15

FINITE ELEMENT SOLUTION OF

$$\nabla^2\phi + \frac{2}{x}\phi_x + \frac{2}{y}\phi_y = 0$$

[a=b=0·02 ; c=1·02; d=0·62]

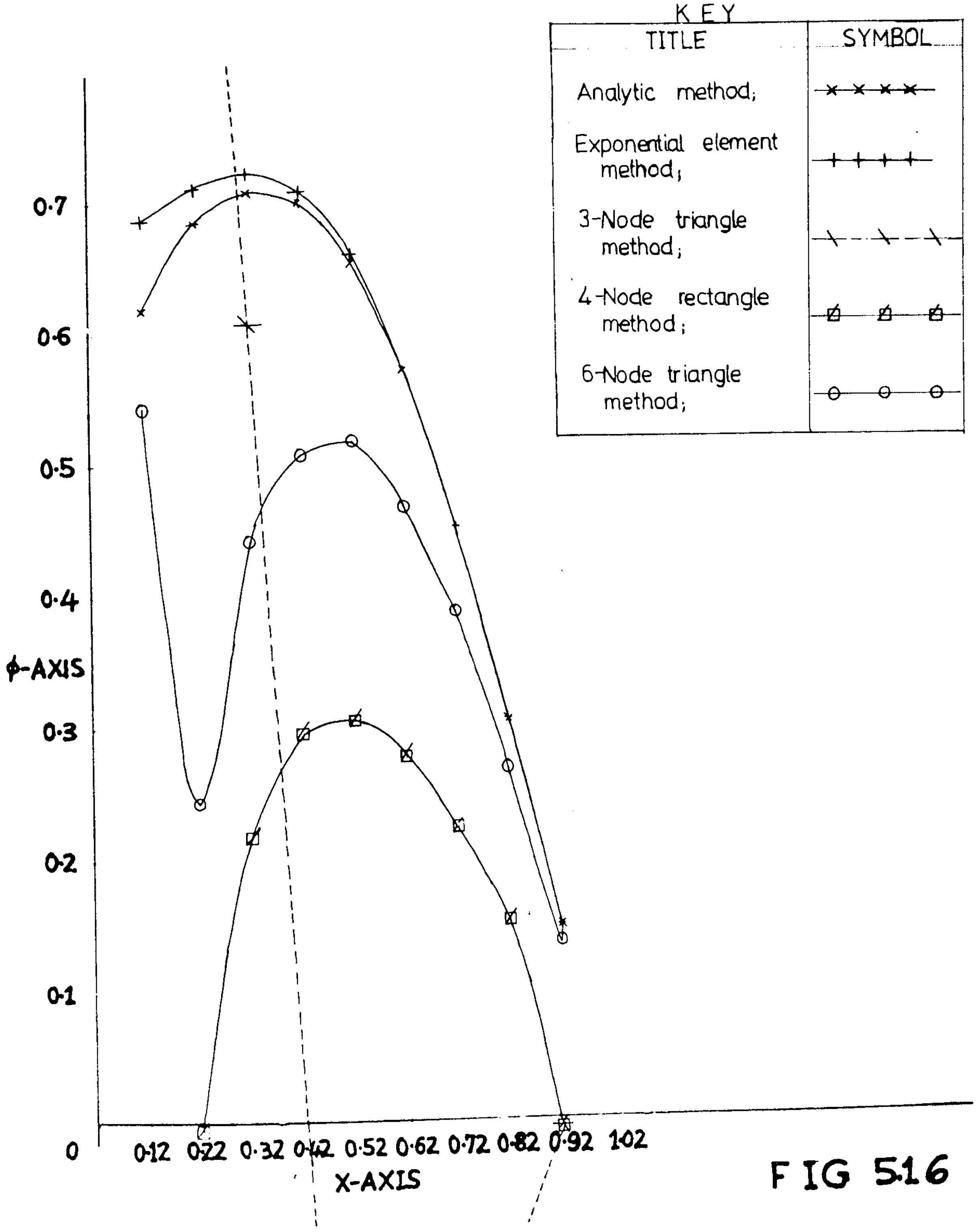
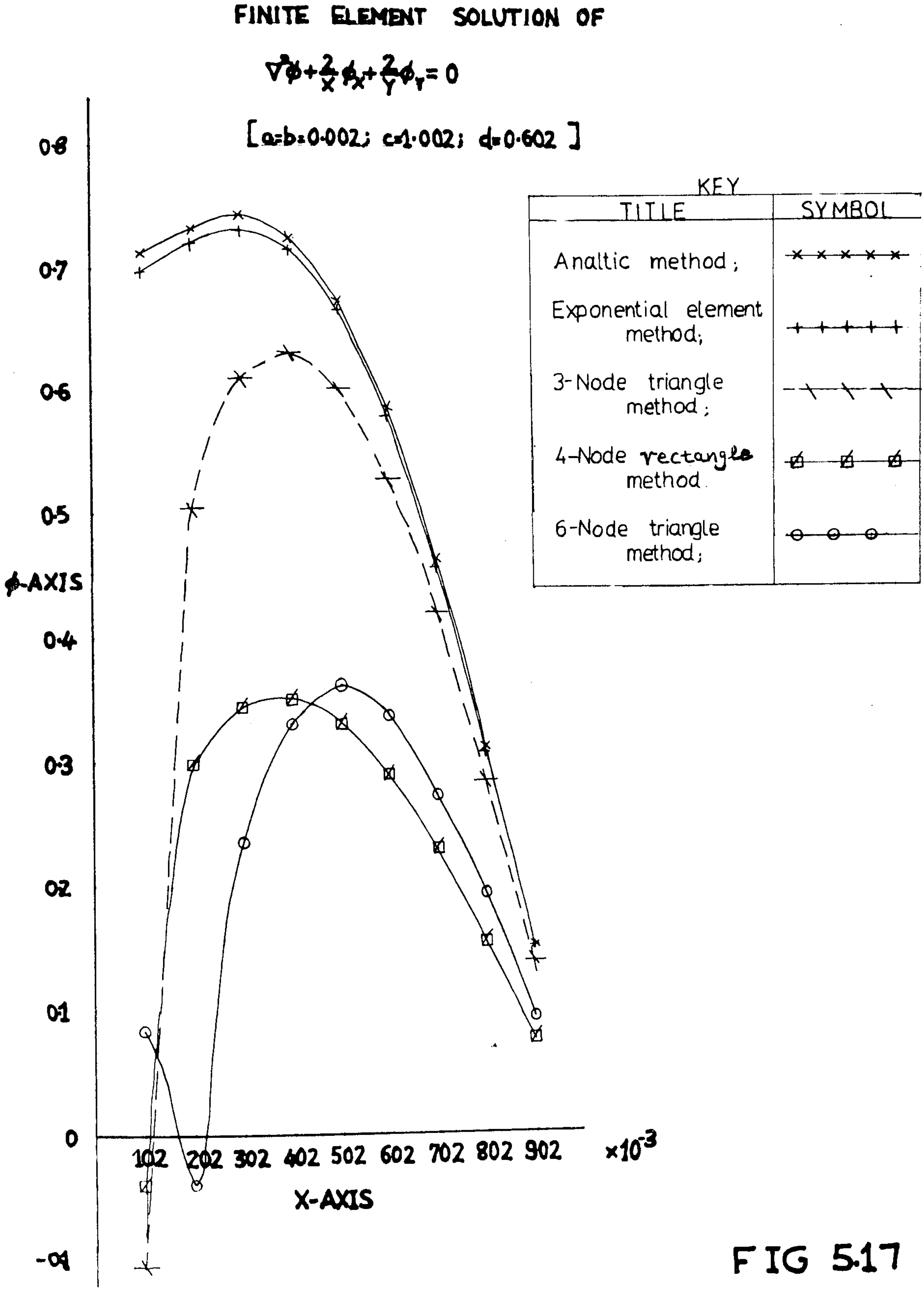


FIG 5.16



FINITE ELEMENT SOLUTION OF
$\nabla^2\phi + \frac{2}{X}\phi_X + \frac{2}{Y}\phi_Y = 0$
[a=b=0·002; c=1·002; d=0·602]
KEY
TITLE
SYMBOL
Analtic method;
Exponential element method;
3-Node triangle method;
4-Node rectangle method
6-Node triangle method;
0·8
0·7
0·6
0·5
φ-AXIS
0·4
0·3
0·2
0·1
0
-0·1
102
202
302
402
502
602
702
802
902
$\times 10^{-3}$
X-AXIS

FIG 5.17

FINITE ELEMENT SOLUTION OF

$$\nabla^2\phi + \frac{2}{X}\phi_X + \frac{2}{Y}\phi_Y = 0$$

[a=b=0·0002 ; c=1·0002;
d=0·6002]

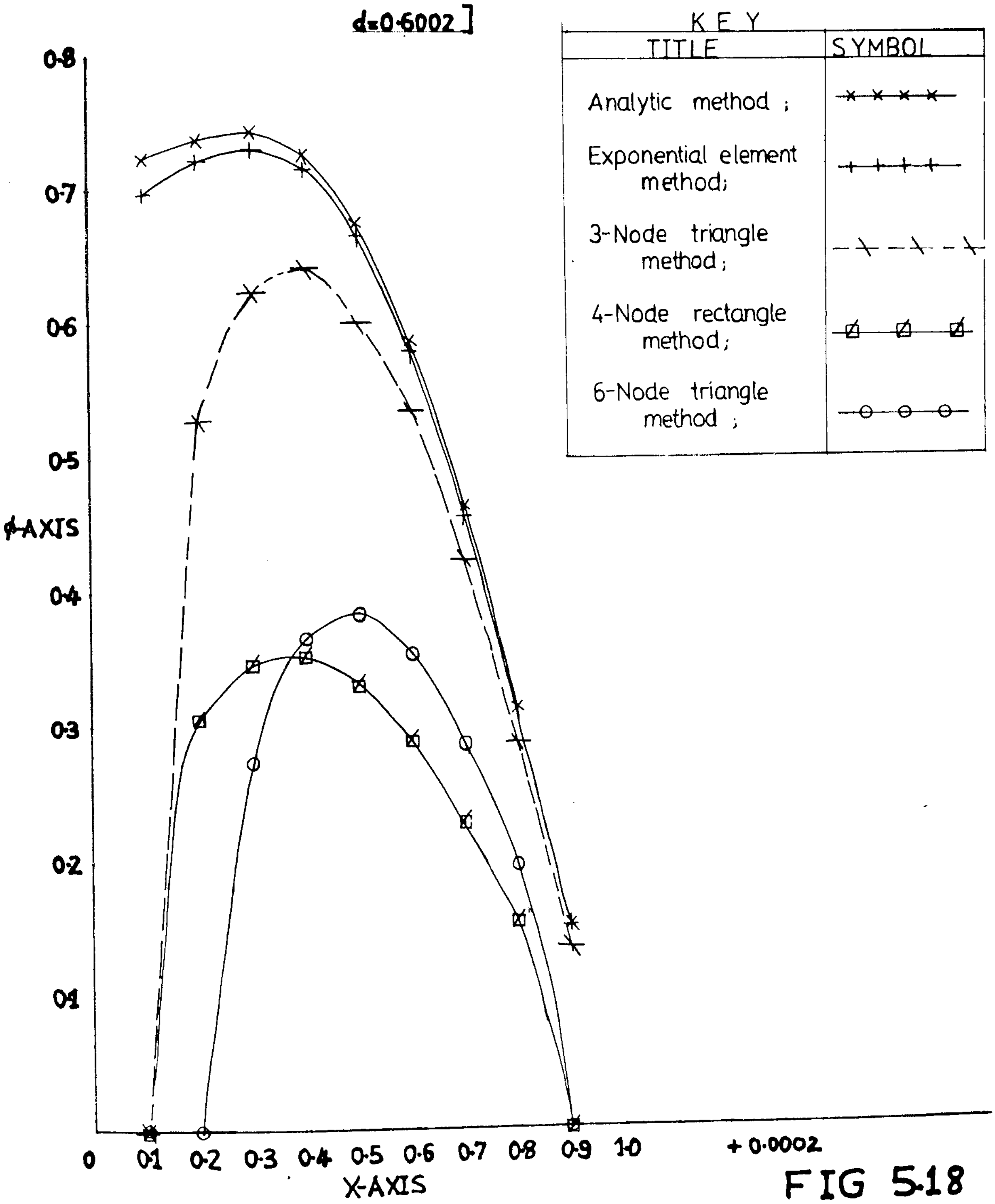


FIG 5.18

5.10 THE NON-HOMOGENEOUS EQUATION $L\phi = g$

It was mentioned in the last chapter that the stability of a difference equation is generally decided by its homogeneous solution (49). This means that if a trial function of the form $\phi = A+Be^{\lambda x}+Ce^{\mu y}+De^{\lambda x+\mu y}$ is assumed over an element for the equation $L\phi = g$ then stability for large values of $|\lambda|$ and $|\mu|$ may be expected. Experience indicates that this is usually the case.

As the equation $L\phi = g$ does not in general yield to the method of separation of variables it is difficult to see how the above procedure may be improved.

5.11 FINITE ELEMENT SOLUTION OF $L\phi = g$

Galerkin's criterion requires

$$\iint_R N_i(L\phi-g)dxdy = 0 \qquad (5.11.1)$$

The usual procedure leads to the following finite element equations

$$\sum_e \sum_j \left[\alpha_{ij}^e\right]\phi_j - \sum_e \left(F_i^e-G_i^e\right) = 0 \qquad (5.11.2)$$

The coefficients α_{ij}^e and F_i^e are precisely the same as in (5.4). The coefficient G_i^e is given by

$$G_i^e = \iint_{r_e} N_i g dx dy$$

The evaluation of G_i^e may require numerical integration depending on the functional form of g. The equations for an element with the usual notation for an m node element are

$$\begin{bmatrix} (E)_1^e \\ (E)_2^e \\ \vdots \\ (E)_i^e \\ \vdots \\ (E)_m^e \end{bmatrix} = \begin{bmatrix} \alpha_{11}^e & \alpha_{12}^e & \cdots & \alpha_{1m}^e \\ \alpha_{21}^e & \alpha_{22}^e & \cdots & \alpha_{2m}^e \\ \vdots & \vdots & & \vdots \\ \alpha_{i1}^e & \alpha_{i2}^e & \cdots & \alpha_{im}^e \\ \vdots & \vdots & & \vdots \\ \alpha_{m1}^e & \alpha_{m2}^e & \cdots & \alpha_{mm}^e \end{bmatrix} \begin{bmatrix} \phi_1^e \\ \phi_2^e \\ \vdots \\ \phi_i^e \\ \vdots \\ \phi_m^e \end{bmatrix} - \begin{bmatrix} (F_1^e - G_1^e) \\ (F_2^e - G_2^e) \\ \vdots \\ (F_i^e - G_i^e) \\ \vdots \\ (F_m^e - G_m^e) \end{bmatrix} \qquad (5.11.3)$$

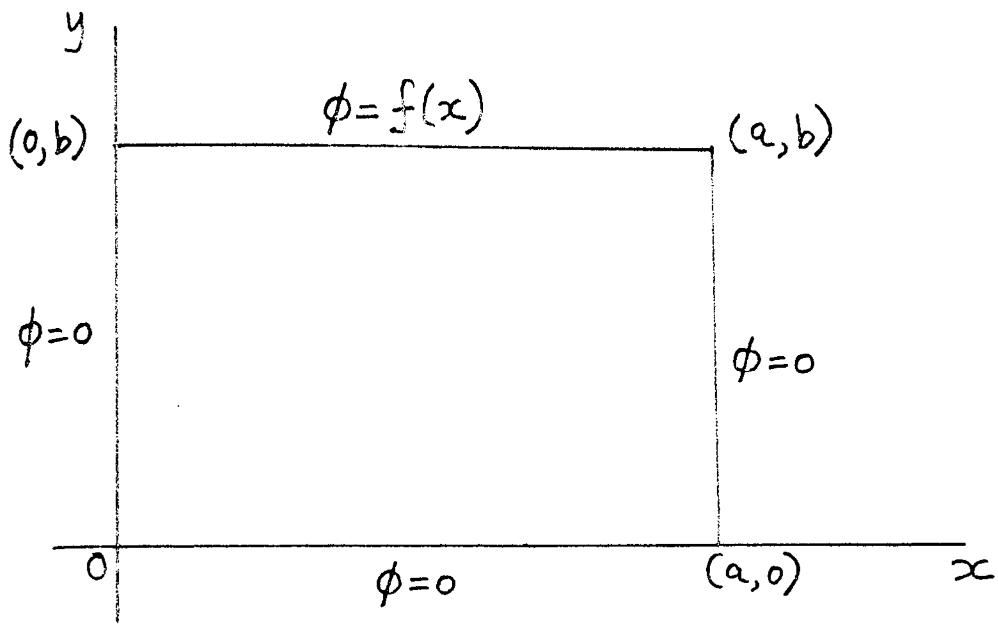
5.12 NUMERICAL EXAMPLE

The differential equation considered is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \lambda \frac{\partial\phi}{\partial x} - \mu \frac{\partial\phi}{\partial y} = k_o \qquad (5.12.1)$$

Where λ, μ and k_o are constants.

This equation was solved over a rectangle with the boundary conditions shown below



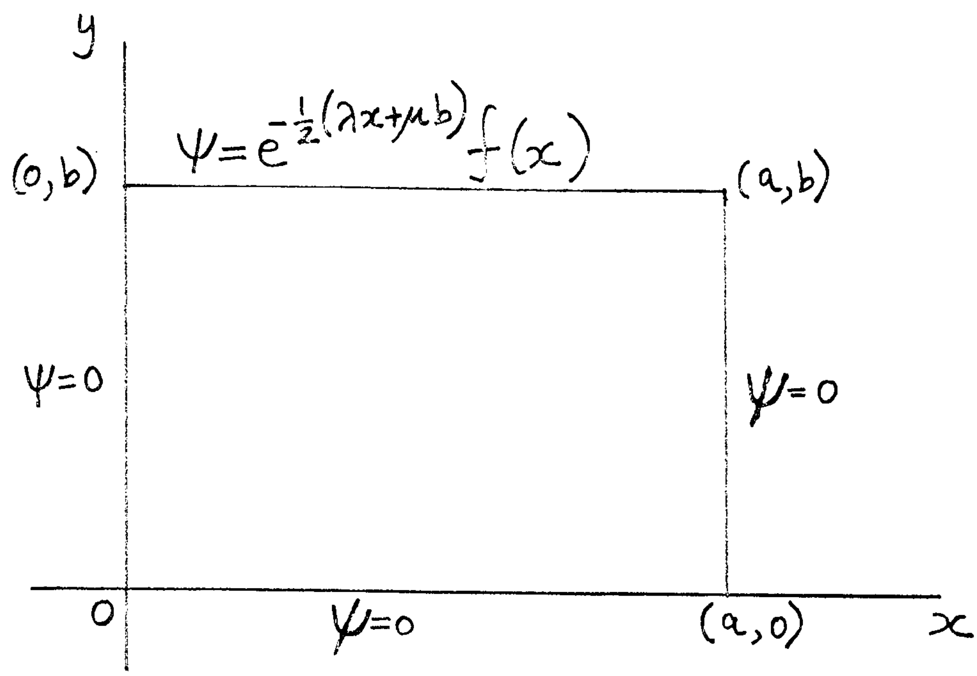
To solve (5.12.1) analytically the dependent variable was changed from ϕ to ψ by the following substitution

$$\phi = e^{\frac{1}{2}(\lambda x+\mu y)}\psi \qquad (5.12.2)$$

With the substitution (5.12.1) transforms into

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{1}{4}(\lambda^2+\mu^2)\psi = k_o e^{-\frac{1}{2}(\lambda x+\mu y)} \qquad (5.12.3)$$

and the boundary conditions into



Taking the finite sine transform of (5.12.3) w.r.t. x gives the following ordinary differential equation

$$\frac{d^2\bar{\psi}}{dy^2} - \frac{1}{4}\beta_n^2\bar{\psi} = k_o e^{-\frac{1}{2}\mu y} J_n \tag{5.12.4}$$

where

$$\beta_n^2 = \lambda^2 + \mu^2 + \frac{4n^2\pi^2}{a^2} \tag{5.12.5}$$

$$J_n = \int_0^a e^{-\frac{\lambda}{2}x} \sin\frac{n\pi x}{a}\, dx \tag{5.12.6}$$

The general solution of (5.12.4) is

$$\bar{\psi} = A\cosh(\tfrac{1}{2}\beta_n y) + B\sinh(\tfrac{1}{2}\beta_n y) + \alpha_n e^{-\frac{\mu}{2}y} \tag{5.12.7}$$

where

$$\alpha_n = \frac{4k_o J_n}{(\mu^2 - \beta_n^2)} \qquad (5.12.8)$$

The boundary conditions for (5.12.7) are that when

$$y = 0; \qquad \psi = 0 \qquad \therefore \qquad \overline{\psi} = 0$$
$$y = b; \qquad \psi = e^{-\frac{1}{2}(\lambda x + \mu b)} f(x) \; \therefore \; \overline{\psi} = e^{-\frac{1}{2}\mu b} I_n$$

where

$$I_n = \int_0^a e^{-\frac{\lambda}{2}x} f(x) \sin\left(\frac{n\pi x}{a}\right) dx \qquad (5.12.9)$$

The application of the above boundary conditions to (5.12.7) determines the arbitrary constants A and B. It is easily shown that this gives

$$\overline{\psi} = \alpha_n \left[\frac{e^{-\frac{1}{2}\mu y} - e^{-\frac{1}{2}\mu b}\sinh(\frac{1}{2}\beta_n y)}{\sinh(\frac{1}{2}\beta_n b)} + \frac{\sinh\frac{1}{2}\beta_n (y-b)}{\sinh(\frac{1}{2}\beta_n b)} \right]$$

$$+ \frac{e^{-\frac{1}{2}\mu b} I_n \sinh(\frac{1}{2}\beta_n y)}{\sinh(\frac{1}{2}\beta_n b)} \qquad (5.12.10)$$

Taking the inverse and using

$$\phi = \psi e^{\frac{1}{2}(\lambda x + \mu y)}$$

gives

$$\phi = \phi_1 + \phi_2 \qquad (5.12.11)$$

where

$$\phi_1 = \frac{2}{a} e^{\frac{1}{2}(\lambda x+\mu y)} \sum_{n=1}^{\infty} \alpha_n \left[e^{-\frac{1}{2}\mu y} - \frac{e^{-\frac{1}{2}\mu b}\sinh(\frac{1}{2}\beta_n y)}{\sinh(\frac{1}{2}\beta_n b)} + \frac{\sinh\frac{1}{2}\beta_n(y-b)}{\sinh(\frac{1}{2}\beta_n b)} \right] \sin\left(\frac{n\pi x}{a}\right)$$

$$(5.12.12)$$

and

$$\phi_2 = \frac{2}{a} e^{\frac{1}{2}(\lambda x+\mu y)} e^{-\frac{1}{2}\mu b} \sum_{n=1}^{\infty} I_n \left[\frac{\sinh(\frac{1}{2}\beta_n y)}{\sinh(\frac{1}{2}\beta_n b)} \right] \sin\left(\frac{n\pi x}{a}\right)$$

$$(5.12.13)$$

It will be noticed that ϕ_2 is merely the solution of $L\phi = 0$. Since $\alpha_n = 0$ when $k_o = 0$ this is to be expected. In this test problem $f(x)$ was chosen as

$$f(x) = \sin\left(\frac{\pi x}{a}\right) \tag{5.12.14}$$

With this choice of f(x) the quantities J_n, α_n and I_n as defined in equations (5.12.6), (5.12.8) and (5.12.9) respectively work out to be

$$J_n = \frac{4n\pi a[1-(-1)^n e^{-\frac{\lambda}{2}a}]}{[\lambda^2 a^2 + 4n^2\pi^2]} \tag{5.12.15}$$

$$\alpha_n = -\frac{16k_o\pi n[1-(-1)^n e^{-\frac{\lambda}{2}a}]}{\lambda^4 a[1+\frac{4n^2\pi^2}{\lambda^2 a^2}]^2} \tag{5.12.16}$$

$$I_n = \frac{16\pi^2 n[1+(-1)^n e^{-\frac{\lambda}{2}a}]}{\lambda^3 a^2[1+\frac{4(n-1)^2\pi^2}{\lambda^2 a^2}][1+\frac{4(n+1)^2\pi^2}{\lambda^2 a^2}]} \tag{5.12.17}$$

Now ϕ_1 and ϕ_2 are completely determined. The numerical computation of ϕ_1 and ϕ_2 will be discussed in(5.13).

The analytic solution $\phi(=\phi_1+\phi_2)$ was evaluated for various values of λ and μ over the finite element grids shown in figure (5.1) - figure (5.3). k_o was chosen to be unity.

The differential equation (5.12.1) was solved using the

(i) Three node triangle.

(ii) Four node rectangle.

(iii) Six node triangle.

(iv) Exponential element.

for λ, μ = 1,20.

It can be seen from figure (5.19) and figure (5.20) that there is not a great deal of discrepancy between the various elements for $\lambda = \mu = 1$. For $\lambda = \mu = 20$ the "exponential element" is more accurate than the traditional elements.

FINITE ELEMENT SOLUTION OF $L\phi = K_0$ ALONG $Y = 0.3$ ON THE COARSE MESH FOR $\lambda = \mu = K_0 = 1.0$

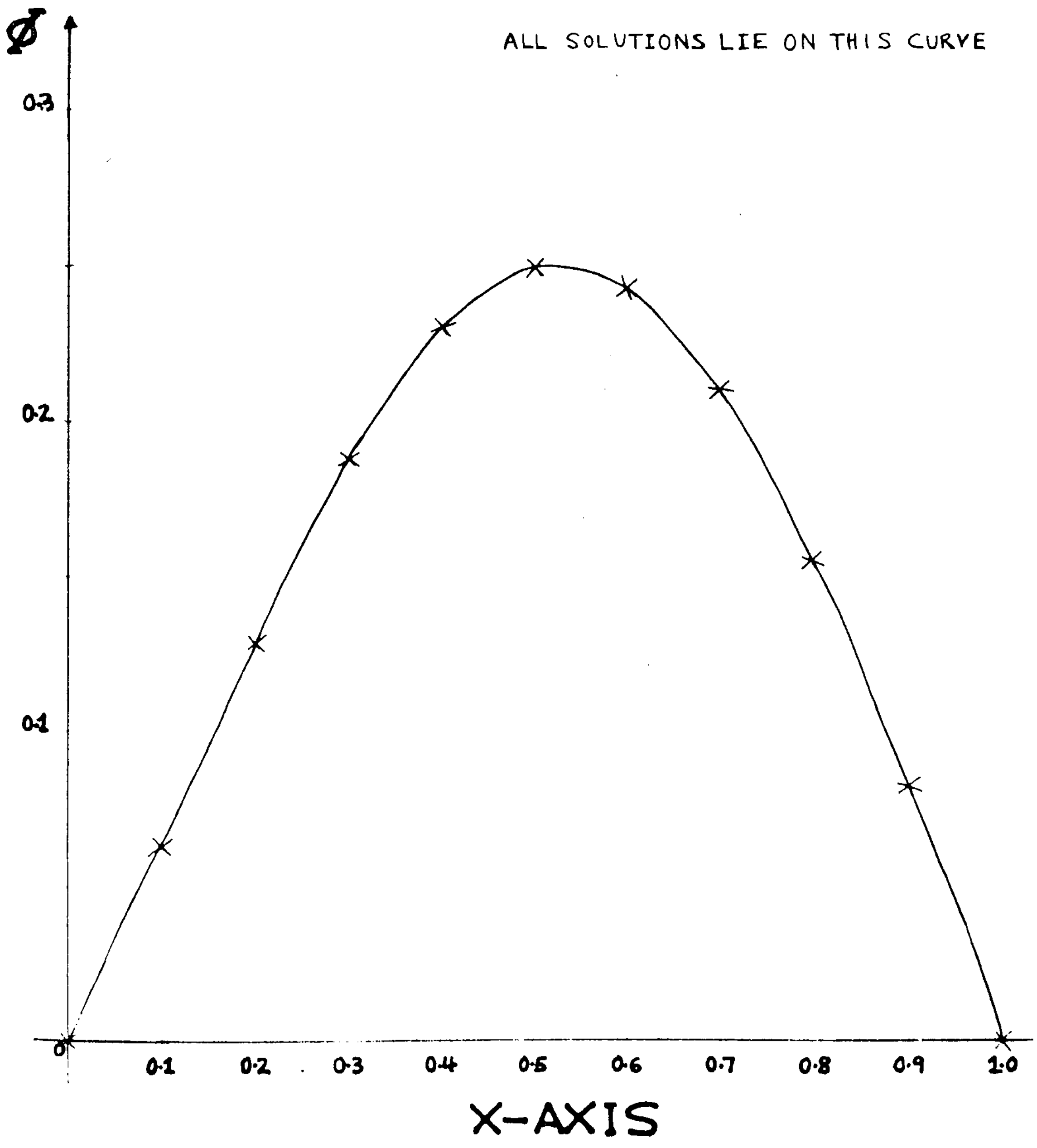


FIG 5.19

FINITE ELEMENT SOLUTION OF $L\phi = K_0$ ALONG $Y = 0.3$ ON THE COARSE MESH FOR $\lambda = \mu = 20.0$ AND $K_0 = 1.0$

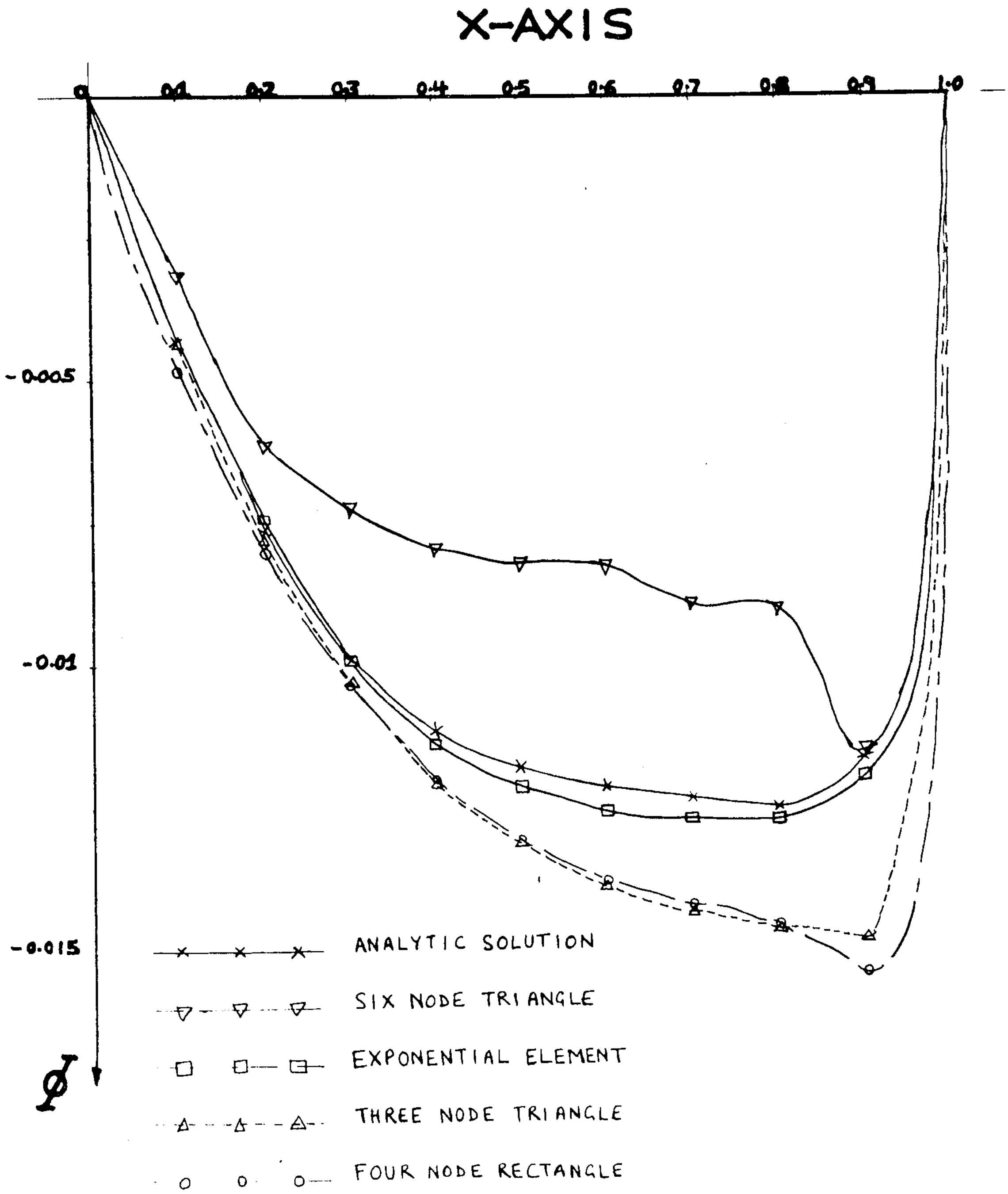


FIG 5.20

5.13 A NOTE ON THE NUMERICAL EVALUATION OF THE ANALYTIC SOLUTION

The analytic solutions to the test problems presented in this chapter are in the form of Infinite Series. These series contain hyperbolic functions and to prevent numerical overflow these must be expressed in terms of negative exponentials. For the test example presented in (5.7) the analytic solution in terms of negative exponentials is

$$\phi = 32^2\pi^2 \sum_{n=1}^{\infty} \frac{\sigma(n)}{\rho(n)}$$

where

$$\sigma(n) = n\left[1+(-1)^n e^{-\frac{\lambda}{2}a}\right] e^{\frac{1}{2}[\lambda x-(b-y)(\mu+\beta_n)-6\log\lambda a]}\left(1-e^{-\beta_n y}\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$\rho(n) = \left[1+\frac{4\pi^2(n-1)^2}{\lambda^2 a^2}\right]\left[1+\frac{4\pi^2(n+1)^2}{\lambda^2 a^2}\right]\left[1-e^{-\beta_n b}\right]$$

Similarly for the second test problem the analytic solution (5.9.2) may be written

$$\phi = \frac{d(c+a)e^{-\frac{\pi(d-y)}{(c-a)}}\left[1-e^{-\frac{2\pi(y-b)}{(c-a)}}\right]}{2xy\left[1-e^{\frac{-2\pi(d-b)}{(c-a)}}\right]}\sin\left[\frac{\pi(x-a)}{(c-a)}\right]$$

$$-\frac{16d(c-a)}{\pi^2 xy}\sum_{r=1}^{\infty}\frac{r e^{\frac{-2r\pi(d-y)}{(c-a)}}\left[1-e^{-\frac{4r\pi(y-b)}{(c-a)}}\right]}{(4r^2-1)^2\left[1-e^{-\frac{4r\pi(d-b)}{(c-a)}}\right]}\sin\left[\frac{2r\pi(x-a)}{(c-a)}\right]$$

The analytic solution (5.12.11) to the non-homogeneous equation can be written as

$$\phi = \phi_1 + \phi_2$$

where

$$\phi_1 = \frac{32k_o\pi}{a} \sum_{n=1}^{\infty} \frac{n\left[1-(-1)^n e^{-\frac{\lambda}{2}a}\right]}{\left[1+\frac{4\pi^2 n^2}{\lambda^2 a^2}\right]^2} \left[e^{\frac{1}{2}\{\lambda x-(\mu+\beta_n)(b-y)-8\log\lambda\}} \right.$$

$$\left(\frac{1-e^{-\beta_n y}}{1-e^{-\beta_n b}}\right) + e^{\frac{1}{2}\{\lambda x-y(\beta_n-\mu)-8\log\lambda\}}\left(\frac{1-e^{-\beta_n(b-y)}}{1-e^{-\beta_n b}}\right)$$

$$\left. - \underline{e^{\frac{1}{2}\{\lambda x-8\log\lambda\}}} \right] \sin\left(\frac{n\pi x}{a}\right)$$

$$\phi_2 = 32\pi^2 \sum_{n=1}^{\infty} \frac{\sigma(n)}{\rho(n)}$$

where

$$\sigma(n) = n\left[1+(-1)^n e^{-\frac{\lambda}{2}a}\right] e^{\frac{1}{2}[\lambda x-(b-y)(\mu+\beta_n)-6\log\lambda a]}\left(1-e^{-\beta_n y}\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$\rho(n) = \left[1 + \frac{4\pi^2(n-1)^2}{\lambda^2 a^2}\right]\left[1 + \frac{4\pi^2(n+1)^2}{\lambda^2 a^2}\right]\left[1-e^{-\beta_n b}\right]$$

It should be noticed that ϕ_I cannot be evaluated for very large λ as the underlined term becomes unbounded. The only other point warranting discussion is the number of terms to be taken in the series. In practice as each term is generated a partial sum s_n say can be formed and the process terminated when

$$\left|(s_{n+1} - s_n) / s_{n+m}\right|$$

is less than any prescribed level of accuracy.

5.14 SUMMARY AND CONCLUSIONS

The dimensionless form of the two dimensional Navier Stokes equation in the x-direction is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \lambda\frac{\partial u}{\partial x} - \mu\frac{\partial u}{\partial y} = \text{Re}\ \frac{\partial p}{\partial x} \qquad (5.14.1)$$

The usual notation has been employed of course

$$\lambda = \text{Re}\ u$$

$$\mu = \text{Re}\ v$$

If the pressure is known function of x and y such that

$$Re \frac{\partial p}{\partial x} = g(x,y)$$

then becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \lambda \frac{\partial u}{\partial x} - \mu \frac{\partial u}{\partial y} = g(x,y) \qquad (5.14.2)$$

For large Reynolds numbers $|\lambda|$ and $|\mu|$ are also large. The aim of this chapter has been to study differential equations of the type (5.14.2). A method has been presented for deriving shape functions for (5.14.2) to result in numerically stable schemes. A number of numerical examples were also presented to illustrate the advantage of the new shape functions over the traditional polynomial shape functions.

In the next Chapter we make use of the shape functions developed in this Chapter to construct a new finite element scheme for the Navier Stokes equations.

CHAPTER SIX

A NEW FINITE ELEMENT SCHEME FOR THE NAVIER STOKES EQUATIONS

6.1

Considerable effort and attention has been devoted to computations in viscous problems in the past. The partial differential equations to be solved are of course the non-linear Navier Stokes equations. When these are discretized by some numerical process a system of non-linear algebraic equations are obtained which are generally solved iteratively. Consequently the numerical techniques tend to require very large computer time, tend to lack accuracy and, in many cases, the iterative process for the non-linear algebraic equations fails to converge. Of course the most formidable problem of all is that numerical instability is encountered at practical Reynolds numbers.

A survey of many of the available finite element formulations for the Navier Stokes equations was given in Chapter 3. It was noted that polynomial elements are not suitable from the point of view of obtaining a solution for practical Reynolds numbers. However, in the beginning of this research project the author conducted numerous numerical studies of the Navier Stokes equations using traditional elements. Some notes on the derivation of finite element schemes using traditional polynomial elements may be found in Appendix 3.

In this Chapter a new finite element scheme is presented which uses exponential trial functions of the type developed in the last Chapter. Subsequent work will show that this new scheme is inherently more stable than schemes obtained using traditional polynomial trial functions.

6.2 FINITE ELEMENT FORMULATION USING THE NEW SCHEME

The Navier Stokes equations in non-dimensional form are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.2.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{Re} \nabla^2 u \qquad (6.2.2)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{Re} \nabla^2 v \qquad (6.2.3)$$

Defining λ and μ as

$$\lambda = Re * u \qquad (6.2.4)$$

$$\mu = Re * v \qquad (6.2.5)$$

equations (6.2.1) - (6.2.3) may be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.2.6)$$

$$\nabla^2 u - \lambda\frac{\partial u}{\partial x} - \mu\frac{\partial u}{\partial y} = Re\,\frac{\partial p}{\partial x} \qquad (6.2.7)$$

$$\nabla^2 v - \lambda\frac{\partial v}{\partial x} - \mu\frac{\partial v}{\partial y} = Re\,\frac{\partial p}{\partial y} \qquad (6.2.8)$$

In Chapter 5 it was shown that for differential equations of the type

$$\nabla^2 \phi - \lambda\frac{\partial \phi}{\partial x} - \mu\frac{\partial \phi}{\partial y} = 0$$

exponential trial functions were more suitable than the traditional polynomial trial functions. This was especially true for large values of $|\lambda|$ and $|\mu|$. It is clear from (6.2.4) and (6.2.5) that for practical Reynolds numbers $|\lambda|$ and $|\mu|$ will be large. The results obtained in Chapter 5 suggest that to solve the Navier Stokes equations for practical Reynolds numbers the following trial functions should be used for u and v i.e.

$$\left.\begin{aligned} u &= A + Be^{\lambda x} + Ce^{\mu y} + De^{\lambda x+\mu y} \\ v &= A + Be^{\lambda x} + Ce^{\mu y} + De^{\lambda x+\mu y} \end{aligned}\right) \qquad (6.2.9)$$

over a four node rectangle. The traditional trial function over a four node rectangle may be taken for pressure, i.e.

$$p = a + bx + cy + dxy \qquad (6.2.10)$$

If $N_j(x,y)$ are the shape functions corresponding to the exponential trial functions (see 5.3) and $M_j(x,y)$ are the shape functions corresponding to the trial function in (6.2.10) then over a typical element

$$u = \sum_j N_j u_j$$

$$v = \sum_j N_j v_j$$

$$p = \sum_j M_j p_j$$

We now apply Galerkin's criterion to discretize equations (6.2.6) - (6.2.8). For reasons which will emerge later the continuity equation is weighted with the shape function for pressure and the momentum equations are weighted with the shape function for velocities. This procedure yields the following finite element equations

$$\sum_e \sum_j \left[A^e_{ij} \, u_j + B^e_{ij} \, v_j \right] = 0 \qquad (6.2.11)$$

$$\sum_e \sum_j \left[\alpha^e_{ij} \right] u_j + Re \sum_e \sum_j \left[K^e_{ij} \right] p_j = \sum_e F^e_i \qquad (6.2.12)$$

$$\sum_e \sum_j \left[\alpha^e_{ij} \right] u_j + Re \sum_e \sum_j \left[L^e_{ij} \right] p_j = \sum_e G^e_i \qquad (6.2.13)$$

The various coefficients are defined below:

$$A^e_{ij} = \iint_e M_i \frac{\partial N_j}{\partial x} dxdy \qquad ; \qquad B^e_{ij} = \iint_e M_i \frac{\partial N_j}{\partial y} dxdy$$

$$C^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial x} dxdy \qquad ; \qquad D^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial y} dxdy$$

$$K^e_{ij} = \iint_e N_i \frac{\partial M_j}{\partial x} dxdy \qquad ; \qquad L^e_{ij} = \iint_e N_i \frac{\partial M_j}{\partial y} dxdy$$

$$E^e_{ij} = \iint_e \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy$$

$$\alpha^e_{ij} = \overline{\lambda} C^e_{ij} + \overline{\mu} D^e_{ij} + E^e_{ij}$$

$$F^e_i = \oint N_i \frac{\partial u}{\partial n} ds \qquad ; \qquad G^e_i = \oint N_i \frac{\partial v}{\partial n} ds$$

Where $\bar{\lambda}$ and $\bar{\mu}$ are the values of λ and μ respectively approximated as constants over element e in the way described in Chapter 5.

The coefficients defined above may all be evaluated analytically. However, the expressions for these coefficients contain terms of the form $e^{\gamma\lambda h}$ and $e^{\gamma\mu k}$ and these must be arranged so that only negative exponential are computed for otherwise numerical overflow will occur.

6.3 ENTRIES OF ELEMENT STIFFNESS MATRIX FOR THE NEW SCHEME

First define the following quantities:

$$Q_1(\xi,\eta) = \begin{cases} \dfrac{(3e^{\xi\eta}-1)}{2\xi(1-e^{\xi\eta})} + \dfrac{\eta e^{2\xi\eta}}{(1-e^{\xi\eta})^2} & \xi < 0 \\[2ex] \dfrac{\eta}{3} & \xi = 0 \\[2ex] \dfrac{(3-e^{-\xi\eta})}{2\xi(e^{-\xi\eta}-1)} + \dfrac{\eta}{(e^{-\xi\eta}-1)^2} & \xi > 0 \end{cases}$$

$$Q_2(\xi,\eta) = \begin{cases} (-1)\left[\dfrac{\eta e^{\xi\eta}}{(1-e^{\xi\eta})^2} + \dfrac{(1+e^{\xi\eta})}{2\xi(1-e^{\xi\eta})}\right] & \xi < 0 \\[2ex] \dfrac{\eta}{6} & \xi = 0 \\[2ex] (-1)\left[\dfrac{\eta e^{-\xi\eta}}{(e^{-\xi\eta}-1)^2} + \dfrac{(e^{-\xi\eta}-1)}{2\xi(e^{-\xi\eta}-1)}\right] & \xi > 0 \end{cases}$$

$$Q_3(\xi,\eta) = \begin{cases} \dfrac{\eta}{(1-e^{\xi\eta})^2} + \dfrac{(3-e^{\xi\eta})}{2\xi(1-e^{\xi\eta})} & \xi > 0 \\[2ex] \dfrac{\eta}{3} & \xi = 0 \\[2ex] \dfrac{e^{-2\xi\eta}}{(e^{-\xi\eta}-1)^2} + \dfrac{(3e^{-\xi\eta}-1)}{2\xi(e^{-\xi\eta}-1)} & \xi > 0 \end{cases}$$

$$Q_4(\xi,\eta) = \begin{cases} \dfrac{(-1)\xi(1+e^{\xi\eta})}{2(1-e^{\xi\eta})} & \xi < 0 \\[2ex] \dfrac{1}{2\eta} & \xi = 0 \\[2ex] \dfrac{(-1)\xi(e^{-\xi\eta}+1)}{(e^{-\xi\eta}-1)} & \xi > 0 \end{cases}$$

$$S_1(\xi,\eta) = \begin{cases} (-1)\left[\dfrac{1}{(1-e^{\xi\eta})} + \dfrac{1}{\xi\eta}\right] & \xi < 0 \\[2ex] -\frac{1}{2} & \xi = 0 \\[2ex] (-1)\left[\dfrac{e^{-\xi\eta}}{(e^{-\xi\eta}-1)} + \dfrac{1}{\xi\eta}\right] & \xi > 0 \end{cases}$$

$$S_2(\xi,\eta) = \begin{cases} \dfrac{e\xi\eta}{(1-e^{\xi\eta})} + \dfrac{1}{\xi\eta} & \xi < 0 \\[2ex] -\frac{1}{2} & \xi = 0 \\[2ex] \dfrac{1}{(e^{-\xi\eta}-1)} + \dfrac{1}{\xi\eta} & \xi > 0 \end{cases}$$

$$S_3(\xi,\eta) = \begin{cases} (-1)\left[\dfrac{1}{\xi(1-e^{\xi\eta})} + \dfrac{\eta e^{\xi\eta}}{2(1-e^{\xi\eta})} + \dfrac{1}{\xi^2\eta}\right] & \xi < 0 \\ -\dfrac{\eta}{6} & \xi = 0 \\ (-1)\left[\dfrac{e^{-\xi\eta}}{\xi(e^{-\xi\eta}-1)} + \dfrac{\eta}{2(e^{-\xi\eta}-1)} + \dfrac{1}{\xi^2\eta}\right] & \xi > 0 \end{cases}$$

$$S_4(\xi,\eta) = \begin{cases} \dfrac{e^{\xi\eta}}{\xi(1-e^{\xi\eta})} - \dfrac{\eta e^{\xi\eta}}{2(1-e^{\xi\eta})} + \dfrac{1}{\xi^2\eta} & \xi < 0 \\ -\dfrac{\eta}{6} & \xi = 0 \\ \dfrac{1}{\xi(e^{-\xi\eta}-1)} - \dfrac{\eta}{2(e^{-\xi\eta}-1)} + \dfrac{1}{\xi^2\eta} & \xi > 0 \end{cases}$$

$$S_5(\xi,\eta) = \begin{cases} \dfrac{1}{\xi(1-e^{\xi\eta})} + \dfrac{\eta}{2(1-e^{\xi\eta})} + \dfrac{1}{\xi^2\eta} & \xi < 0 \\ -\dfrac{\eta}{3} & \xi = 0 \\ \dfrac{e^{-\xi\eta}}{\xi(e^{-\xi\eta}-1)} + \dfrac{\eta e^{-\xi\eta}}{2(e^{-\xi\eta}-1)} + \dfrac{1}{\xi^2\eta} & \xi > 0 \end{cases}$$

$$S_6(\xi,\eta) = \begin{cases} \dfrac{\eta}{2(1-e^{\xi\eta})} - \dfrac{e^{\xi\eta}}{\xi(1-e^{\xi\eta})} - \dfrac{1}{\xi^2\eta} & \xi < 0 \\ -\dfrac{\eta}{3} & \xi = 0 \\ \dfrac{\eta e^{-\xi\eta}}{2(e^{-\xi\eta}-1)} - \dfrac{1}{\xi(e^{-\xi\eta}-1)} - \dfrac{1}{\xi^2\eta} & \xi > 0 \end{cases}$$

Let the length and height of the rectangular element be h and k respectively. Then also define

$$q_j = Q_j(\bar{\lambda}, h) \qquad j = 1,2,3,4$$
$$q'_j = Q_j(\bar{\mu}, k) \qquad j = 1,2,3,4$$
$$s_j = S_j(\bar{\lambda}, h) \qquad j = 1,2,3,4$$
$$s'_j = S_j(\bar{\mu}, k) \qquad j = 1,2,3,4$$

All the coefficients are given below in terms of q_j, q'_j, s_j and s'_j.

$$A^e_{11} = s_1 s'_3 \;;\quad A^e_{12} = -A^e_{11} \;;\quad A^e_{13} = (-1) s_1 s'_5 \;;\quad A^e_{14} = -A^e_{13}$$
$$A^e_{21} = s_2 s'_3 \;;\quad A^e_{22} = -A^e_{21} \;;\quad A^e_{23} = (-1) s_2 s'_5 \;;\quad A^e_{24} = -A^e_{23}$$
$$A^e_{31} = s_2 s'_4 \;;\quad A^e_{32} = -A^e_{31} \;;\quad A^e_{33} = (-1) s_2 s'_6 \;;\quad A^e_{34} = -A^e_{33}$$
$$A^e_{41} = s_1 s'_4 \;;\quad A^e_{42} = -A^e_{41} \;;\quad A^e_{43} = (-1) s_1 s'_6 \;;\quad A^e_{44} = -A^e_{43}$$

and

$$B^e_{11} = s'_1 s_3 \;;\quad B^e_{12} = s'_1 s_5 \;;\quad B^e_{13} = -B^e_{12} \;;\quad B^e_{14} = -B^e_{11}$$
$$B^e_{21} = s'_1 s_4 \;;\quad B^e_{22} = s'_1 s_6 \;;\quad B^e_{23} = -B^e_{22} \;;\quad B^e_{24} = -B^e_{21}$$
$$B^e_{31} = s'_2 s_4 \;;\quad B^e_{32} = s'_2 s_6 \;;\quad B^e_{33} = -B^e_{32} \;;\quad B^e_{34} = -B^e_{31}$$
$$B^e_{41} = s'_2 s_3 \;;\quad B^e_{42} = s'_2 s_5 \;;\quad B^e_{43} = -B^e_{42} \;;\quad B^e_{44} = -B^e_{41}$$

and

$$C^e_{11} = -\tfrac{1}{2} q'_1 \;;\quad C^e_{12} = -C^e_{11} \;;\quad C^e_{13} = \tfrac{1}{2} q'_2 \;;\quad C^e_{14} = -C^e_{13}$$
$$C^e_{21} = C^e_{11} \;;\quad C^e_{22} = -C^e_{11} \;;\quad C^e_{23} = C^e_{13} \;;\quad C^e_{24} = -C^e_{13}$$
$$C^e_{31} = C^e_{13} \;;\quad C^e_{32} = C^e_{13} \;;\quad C^e_{33} = \tfrac{1}{2} q'_3 \;;\quad C^e_{34} = C^e_{33}$$
$$C^e_{41} = C^e_{13} \;;\quad C^e_{42} = C^e_{13} \;;\quad C^e_{43} = C^e_{33} \;;\quad C^e_{44} = -C^e_{33}$$

and

$$D^e_{11} = -\tfrac{1}{2}q_1 \;;\quad D^e_{12} = \tfrac{1}{2}q_2 \;;\quad D^e_{13} = -D^e_{12} \;;\quad D^e_{14} = -D^e_{11}$$

$$D^e_{21} = D^e_{12} \;;\quad D^e_{22} = -\tfrac{1}{2}q_3 \;;\quad D^e_{23} = -D^e_{22} \;;\quad D^e_{24} = -D^e_{12}$$

$$D^e_{31} = D^e_{12} \;;\quad D^e_{32} = D^e_{22} \;;\quad D^e_{33} = -D^e_{22} \;;\quad D^e_{34} = -D^e_{12}$$

$$D^e_{41} = D^e_{11} \;;\quad D^e_{42} = D^e_{12} \;;\quad D^e_{43} = -D^e_{12} \;;\quad D^e_{44} = -D^e_{11}$$

and

$$E^e_{11} = q_4 q_1' + q_4' q_1 \;;\quad E^e_{12} = (-1)q_4' q_1 + q_4' q_2$$

$$E^e_{13} = (-1)q_4 q_2' + (-1)q_4' q_2 \;;\quad E^e_{14} = q_4 q_2' + (-1)q_4' q_1$$

$$E^e_{21} = E^e_{12} \;;\quad E^e_{22} = q_4 q_1' + q_4' q_3$$

$$E^e_{23} = q_4 q_2' + (-1)q_4' q_3 \;;\quad E^e_{24} = (-1)q_4 q_2' + (-1)q_4' q_2$$

$$E^e_{31} = E^e_{13} \;;\quad E^e_{32} = E^e_{23} \;;\quad E^e_{33} = q_4 q_3' + q_4' q_3$$

$$E^e_{34} = (-1)q_4 q_3' + q_4' q_2 \;;\quad E^e_{41} = E^e_{14} \;;\quad E^e_{43} = E^e_{34}$$

$$E^e_{44} = q_4 q_3' + q_4' q_1$$

and

$$K^e_{11} = s_2 s_3' \;;\quad K^e_{12} = -K^e_{11} \;;\quad K^e_{13} = -s_2 s_4' \;;\quad K^e_{14} = -K^e_{13}$$

$$K^e_{21} = s_1 s_3' \;;\quad K^e_{22} = -K^e_{21} \;;\quad K^e_{23} = -s_1 s_4' \;;\quad K^e_{24} = -K^e_{23}$$

$$K^e_{31} = s_1 s_5' \;;\quad K^e_{32} = -K^e_{31} \;;\quad K^e_{33} = -s_1 s_6' \;;\quad K^e_{34} = -K^e_{33}$$

$$K^e_{41} = s_2 s_5' \;;\quad K^e_{42} = -K^e_{41} \;;\quad K^e_{43} = -s_2 s_6' \;;\quad K^e_{44} = -K^e_{43}$$

and

$$L^e_{11} = s_2' s_3 \;;\quad L^e_{12} = s_2' s_4 \;;\quad L^e_{13} = -L^e_{12} \;;\quad L^e_{14} = -L^e_{11}$$

$$L^e_{21} = s_2' s_5 \;;\quad L^e_{22} = s_2' s_6 \;;\quad L^e_{23} = -L^e_{22} \;;\quad L^e_{24} = -L^e_{21}$$

$$L^e_{31} = s_1' s_5 \;;\quad L^e_{32} = s_1' s_6 \;;\quad L^e_{33} = -L^e_{32} \;;\quad L^e_{34} = -L^e_{31}$$

$$L^e_{41} = s_1' s_3 \;;\quad L^e_{42} = s_1' s_4 \;;\quad L^e_{43} = -L^e_{42} \;;\quad L^e_{44} = -L^e_{41}$$

α^e_{ij} may be computed from the relation

$$\alpha^e_{ij} = \overline{\lambda} C^e_{ij} + \overline{\mu} D^e_{ij} + E^e_{ij}$$

6.4 IMPOSING AND HANDLING BOUNDARY CONDITIONS

The most natural boundary conditions consist of the velocities prescribed on solid boundaries and on the sides of any obstacle. The specification of boundary conditions on the inlet and exit planes of a contained flow problem is a more difficult task.

Boundary conditions of velocity, pressure and/or normal derivatives of velocity are easy enough to incorporate into the solution procedure. As usual the finite element equations are written at each and every node and the boundary conditions are applied directly to the global system of algebraic equations. If for example a velocity component at node K on the boundary is known the corresponding component of the kth momentum equation is deleted and replaced by a 1 on the diagonal and the prescribed value on the right hand side vector. If the pressure is known at the kth node then the continuity equation at the kth node is deleted and replaced as above. This choice is somewhat arbitrary and certainly it makes no difference if another choice of equations to be deleted is made provided that the choice is consistent with the matrix assembly.

6.5 NUMERICAL EXAMPLE

Stagnation in Plane Flow (Hiemenz flow).

Consider fluid arriving from the y-axis and impinging on a flat wall placed at y = 0. The fluid divides into two streams on the wall and leaves in both directions see Figure (6.1) below

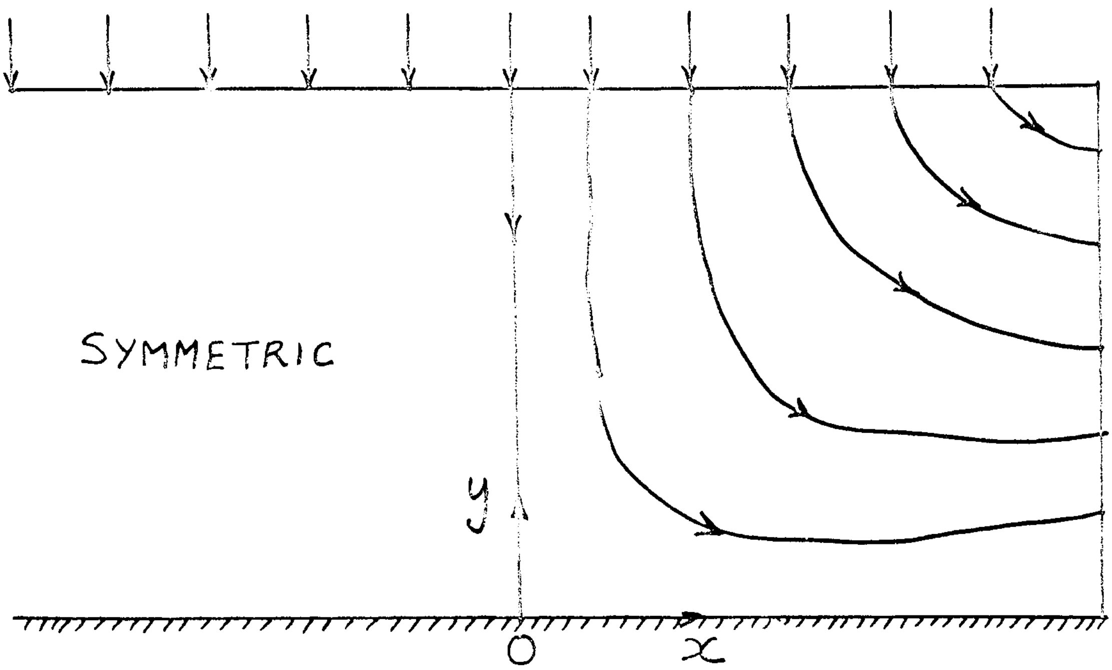


FIGURE 6.1

The origin 0 is the stagnation point. The velocity and pressure distribution for this kind of flow are given by

$$u = ax\phi'(\eta) \tag{6.5.1}$$

$$v = -\sqrt{\frac{a}{Re}}\,\phi(\eta) \tag{6.5.2}$$

$$p = p_o - \tfrac{1}{2}a^2x^2 - \tfrac{1}{2}\left(\frac{a}{Re}\right)[\phi^2(\eta)+2\phi'(\eta)] \tag{6.5.3}$$

Where a is a constant at our disposal p_o is the stagnation pressure and as usual Re is the Reynolds number. η is given by

$$\eta = \sqrt{a\ Re}\quad y \tag{6.5.4}$$

and the function $\phi(\eta)$ satisfies the non-linear ordinary differential equation

$$\phi''' + \phi\phi'' - \phi'^2 + 1 = 0 \tag{6.5.5}$$

The solution of (6.5.5) may be found in Schlichting (39) and is reproduced in tabular form in Figure (6.3). Owing to the symmetry of the problem, flow in the first quadrant was considered. The boundary conditions used to obtain a solution are shown below

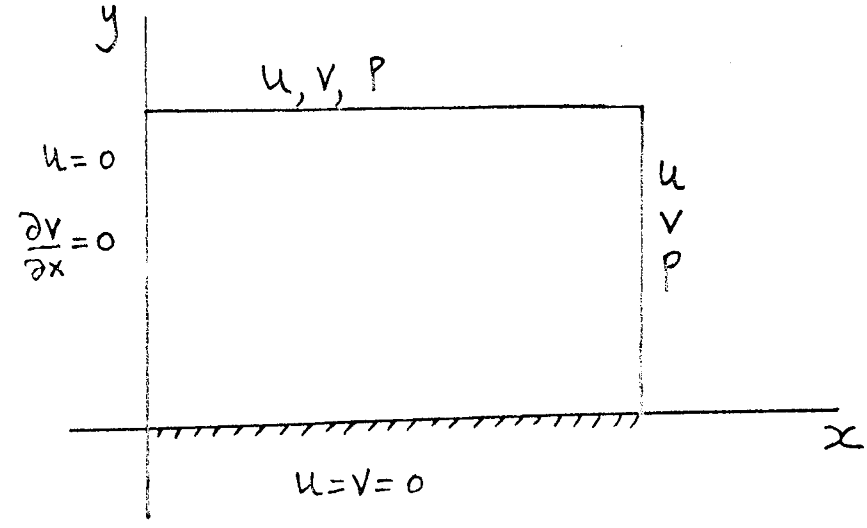


FIGURE 6.2

$\eta = \sqrt{aRe}\ y$	ϕ	ϕ'	ϕ''
0	0	0	1.2326
0.2	0.0233	0.2266	1.0345
0.4	0.0881	0.4145	0.8463
0.6	0.1867	0.5663	0.6752
0.8	0.3124	0.6859	0.5251
1.0	0.4592	0.7779	0.3980
1.2	0.6220	0.8467	0.2938
1.4	0.7967	0.8968	0.2110
1.6	0.9798	0.9323	0.1474
1.8	1.1689	0.9568	0.1000
2.0	1.3620	0.9732	0.0658
2.2	1.5578	0.9839	0.0420
2.4	1.7553	0.9905	0.0260
2.6	1.9538	0.9946	0.0156
2.8	2.1530	0.9970	0.0090
3.0	2.3526	0.9984	0.0051
3.2	2.5523	0.9992	0.0028
3.4	2.7522	0.9996	0.0014
3.6	2.9521	0.9998	0.0007
3.8	3.1521	0.9999	0.0004
4.0	3.3521	1.0000	0.0002
4.2	3.5521	1.0000	0.0001
4.4	3.7521	1.0000	0.0000
4.6	3.9521	1.0000	0.0000

FIGURE 6.3

The above table shows that the velocity components u and v vary rapidly near the wall y = 0 for large values of Re. Consequently the finite element mesh should be more refined in this region. The new scheme was used to obtain a numerical solution on the finite element mesh shown in figure (6.4) for various values of the Reynolds number.

FINITE ELEMENT MESH FOR THE NAVIER STOKES EQUATIONS USING THE NEW SCHEME

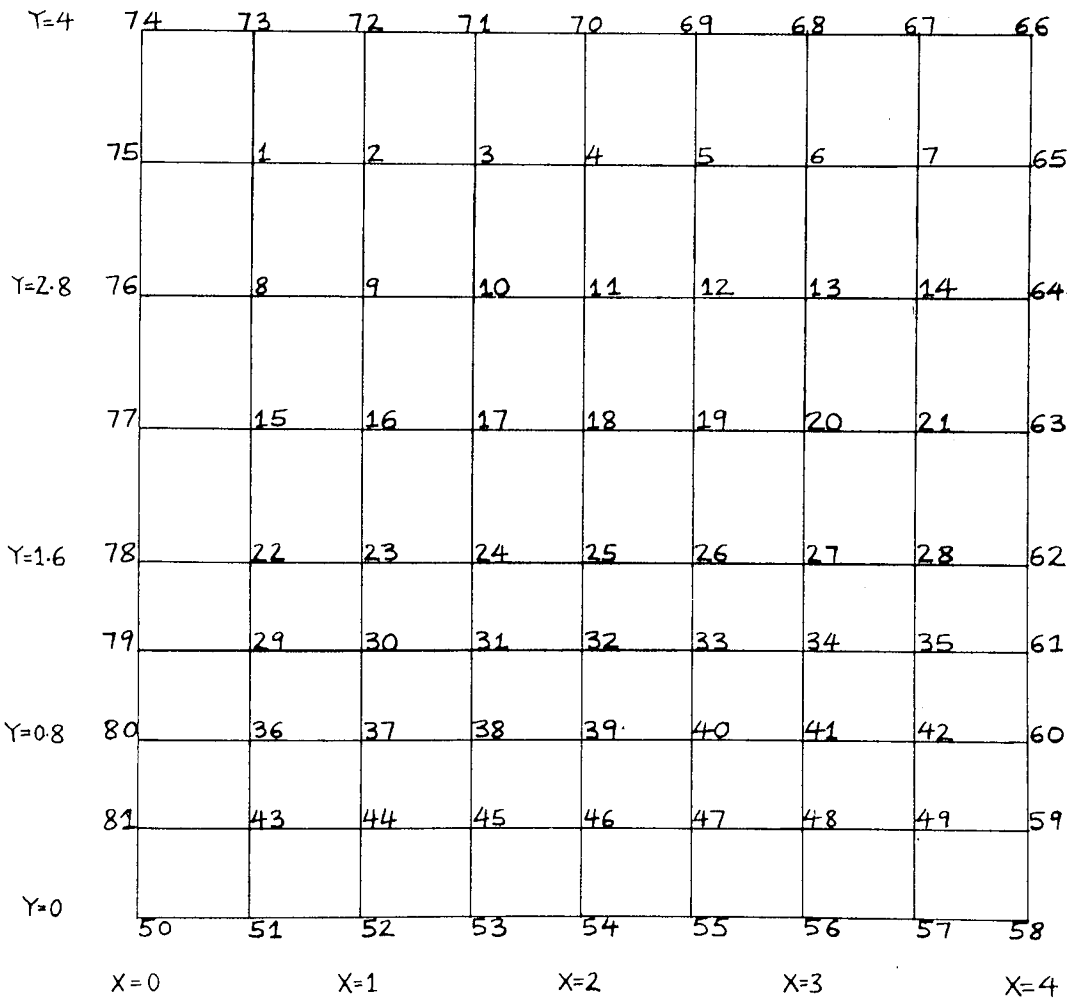


FIG 6·4

6.6 DISCUSSION OF RESULTS

The results are illustrated graphically in Figure (6.5) - Figure (6.7) for Re = 1. The results show that the velocity distribution obtained using the new finite element scheme compares well with the analytic solution.

However, the results for pressure are subject to inaccuracies. Refining the grid does improve the accuracy of all the variables but the pressure field is always less accurate than the velocity field. This anomaly also arises when traditional polynomial finite element schemes are used. The use of "mixed interpolation" overcomes this problem and will be discussed in the next section.

For large Reynolds numbers ($\simeq 10^4$) the iterative process for the non-linear finite element equations failed to converge. But the results indicated that the new scheme was probably stable. In the next Chapter we shall show that this scheme is inherently more stable than the traditional polynomial trial functions.

DISTRIBUTION OF u ALONG Y=1·6 FOR Re=1 USING THE NEW SCHEME

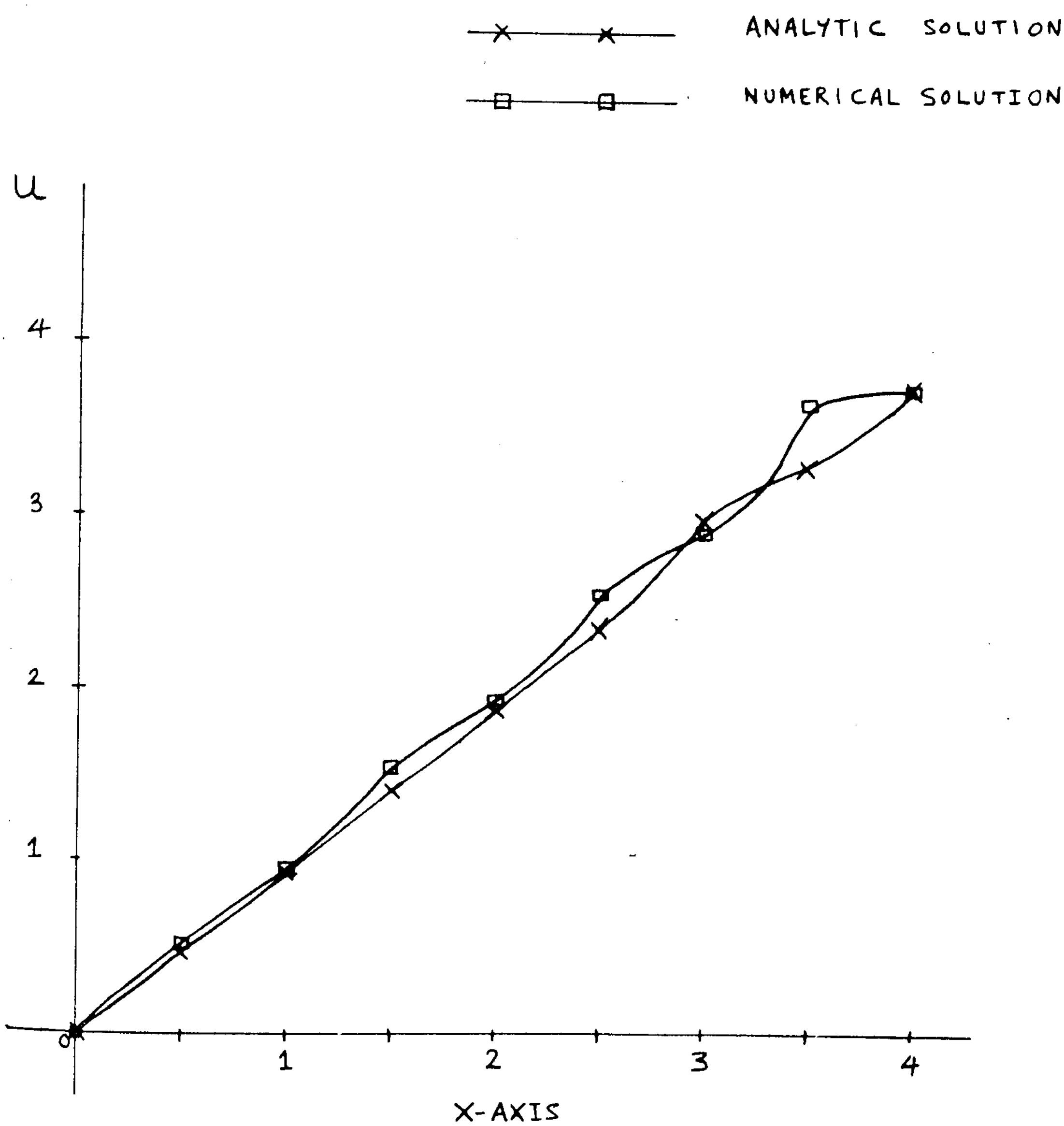


FIG 6·5

DISTRIBUTION OF V ALONG Y=1·6 FOR Re=1 USING THE NEW SCHEME

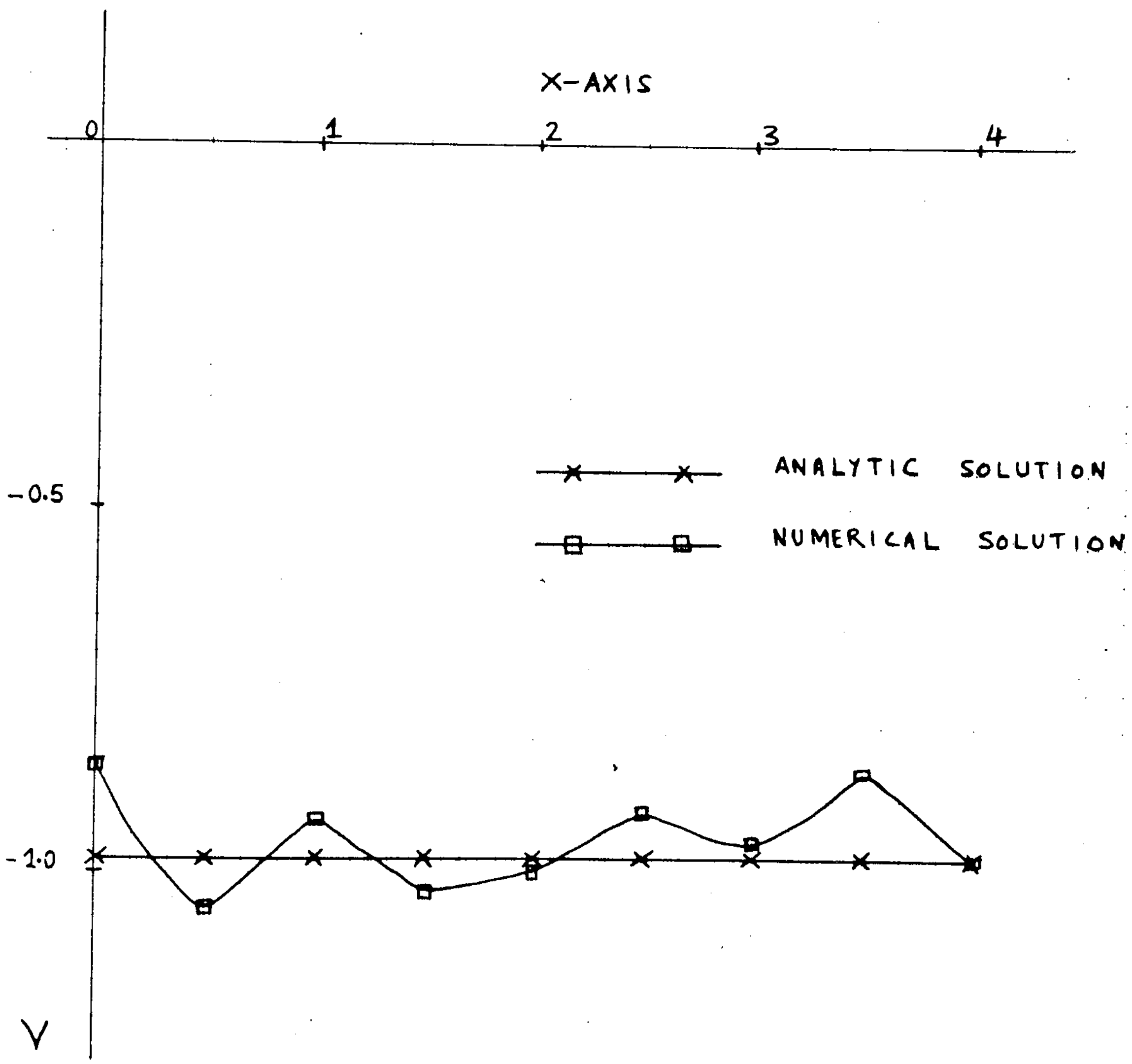


FIG 6·6

DISTRIBUTION OF P ALONG Y=1·6 FOR Re=1 USING THE NEW SCHEME

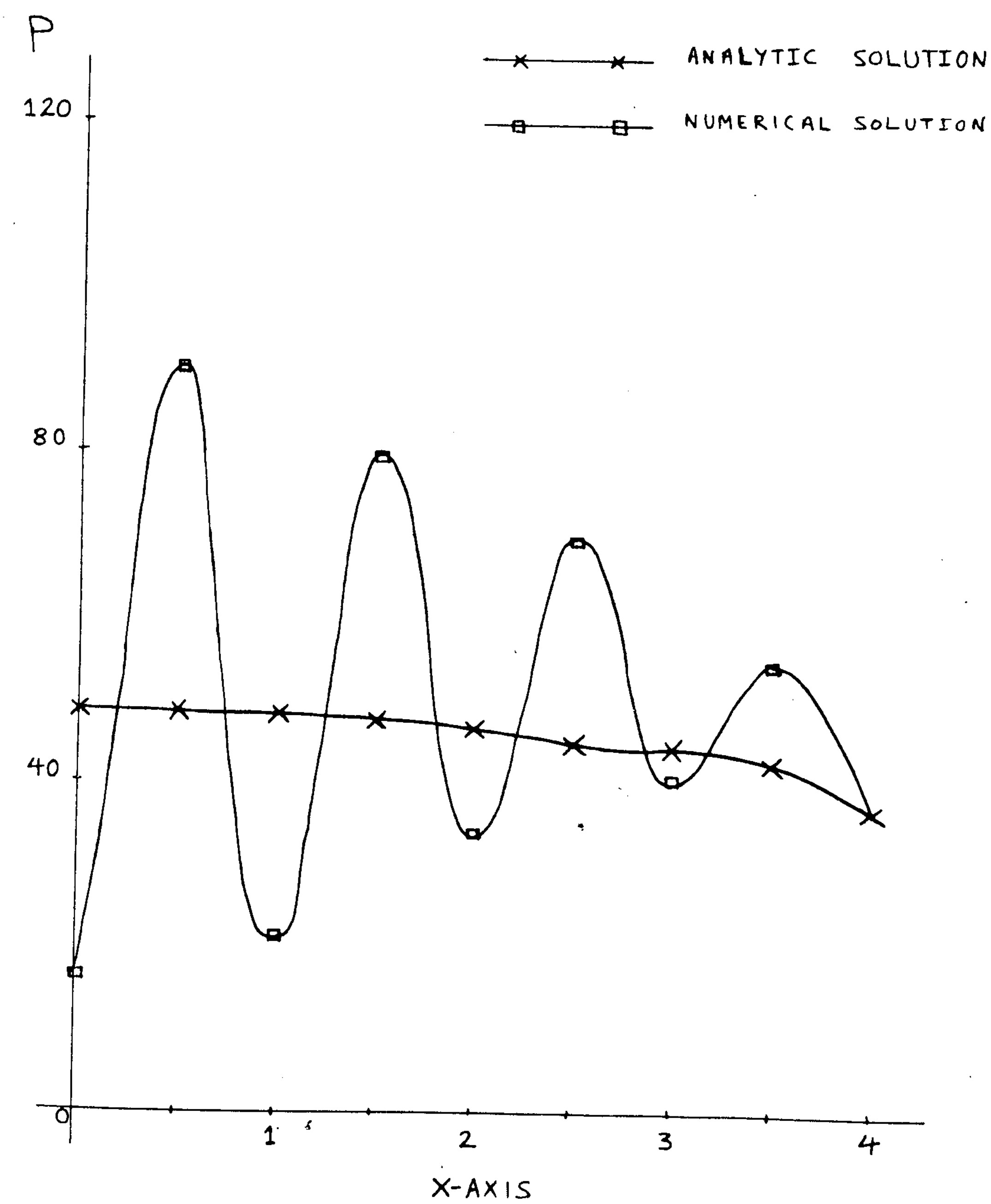


FIG 6·7

6.7 DISCUSSION OF MIXED INTERPOLATION

So the new finite element scheme presented yields an acceptable velocity field but the associated pressures are, in particular instances, subject to inaccuracies. Actually the same phenomenon was observed by other research workers long ago when they applied Galerkin's method combined with polynomial trial functions to solve the Navier Stokes equations. The reason for this anomaly was not initially known, and it was assumed that the effect might be due to incorrectly posed boundary conditions or to too coarse a mesh. Further investigations by researchers showed that neither of these possibilities could account for the discrepency. It was finally suggested that the same polynomial trial function for both velocity and pressure fields was incorrect under certain circumstances.

The next obvious step was to use different trial functions for the variables so as to produce accurate results simultaneously for both velocity and pressure fields. It was eventually discovered (although not clear from the literature by whom) that if a quadratic trial function is used for velocities and a linear trial function is used for pressure, and in addition if the continuity equation is weighted with the shape function for pressure and the momentum equations with the shape functions for velocities, then accurate results could be obtained for all the variables simultaneously. A very useful element which has been used for this purpose is the six node triangle. Pressure is assumed to be linear over the element and is therefore only interpolated at the corner nodes. Velocities are assumed to be quadratic and are interpolated at all the nodes of the element. Several research workers have used this so called mixed interpolation. Although it works but as far as the author is aware a rigorous justification of it is not available in the literature.

The phenomenon described above was observed when the numerical example (Hiemenz flow) presented in (6.5) was solved using traditional polynomial trial functions for low Reynolds numbers. Of course, for large Reynolds numbers the traditional schemes are unstable. The finite element grids are shown in Figure (6.8) - Figure (6.10). The pressure distribution along y =const for Re = 1 and Re = 25 is depicted in Figure (6.11) and Figure (6.12) respectively. It is seen that the pressure distribution obtained using mixed interpolation agrees well with the analytic solution. However, the pressure distribution obtained using common interpolation for all the variables is subject to inaccuracies. The velocity distribution in all cases compared well with the analytic solution.

FINITE ELEMENT MESH FOR THE NAVIER STOKES EQUATIONS USING THE TRADITIONAL THREE NODE TRIANGLE

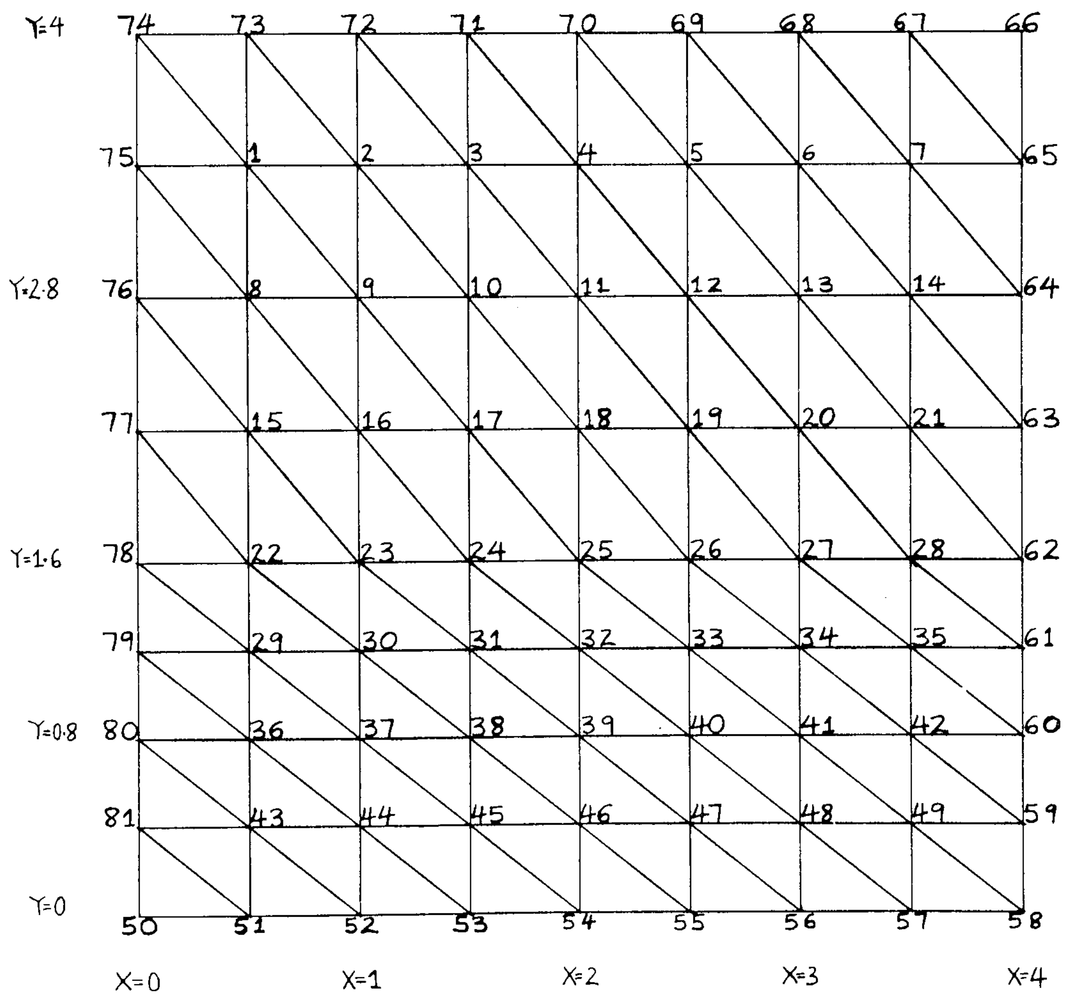


FIG 6·8

FINITE ELEMENT MESH FOR THE NAVIER STOKES EQUATIONS USING THE TRADITIONAL SIX NODE TRIANGLE

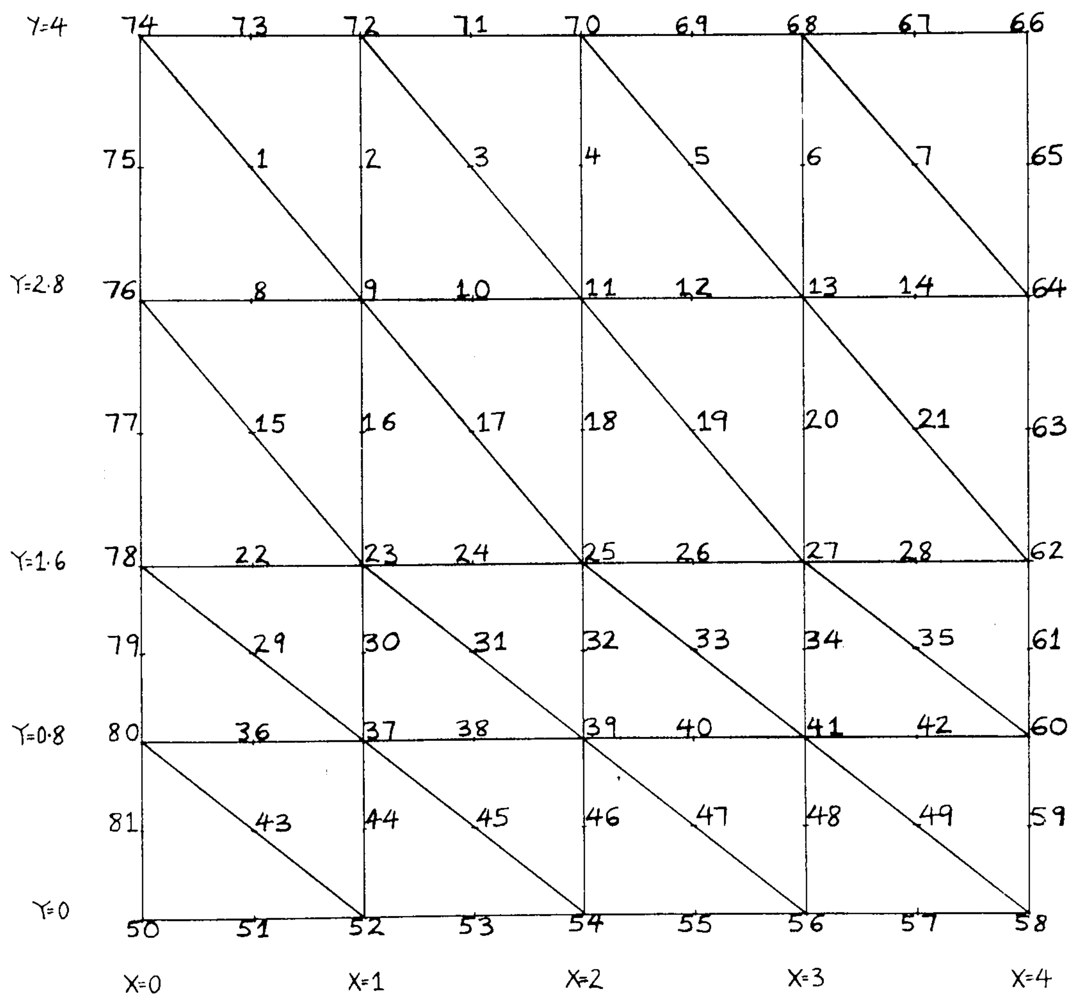


FIG 6.9

FINITE ELEMENT MESH FOR THE NAVIER STOKES EQUATIONS USING THE TRADITIONAL SIX NODE MIXED ELEMENT

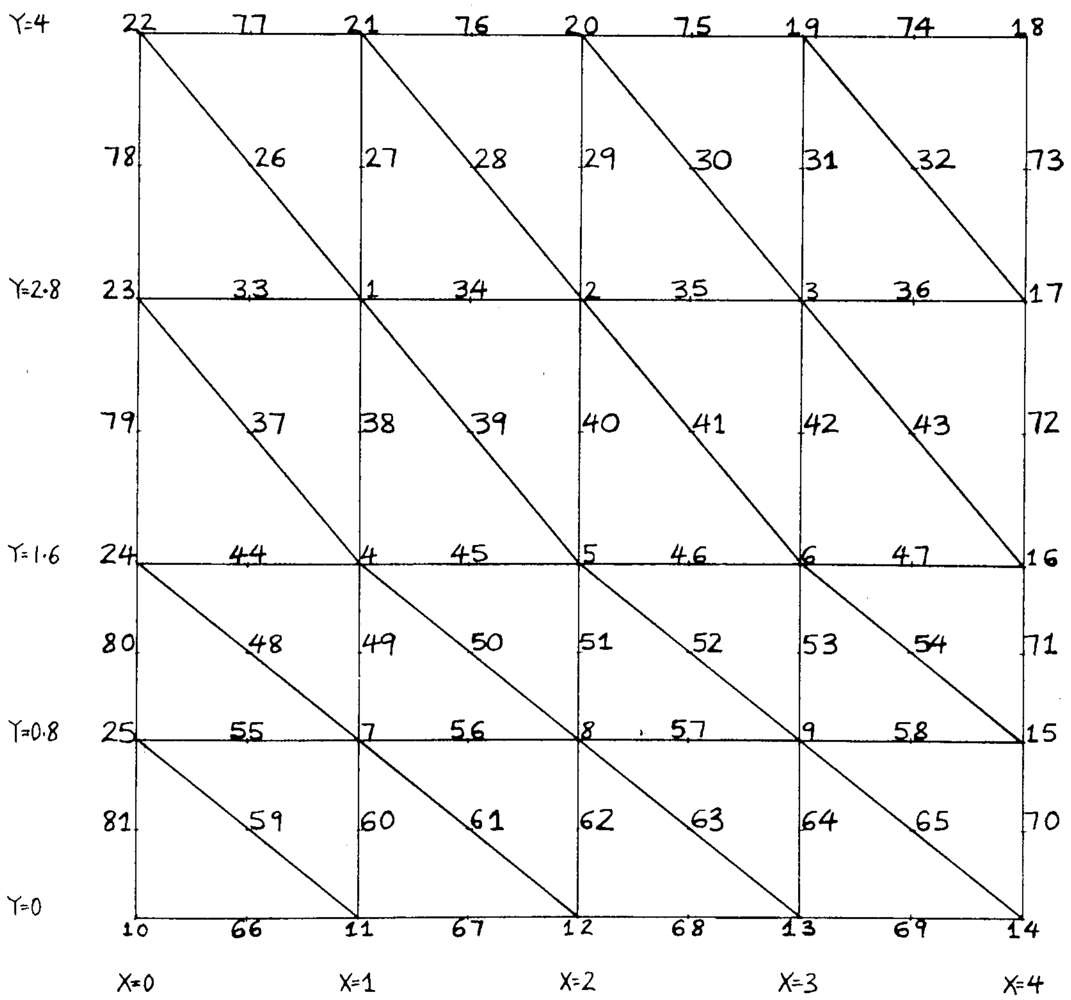
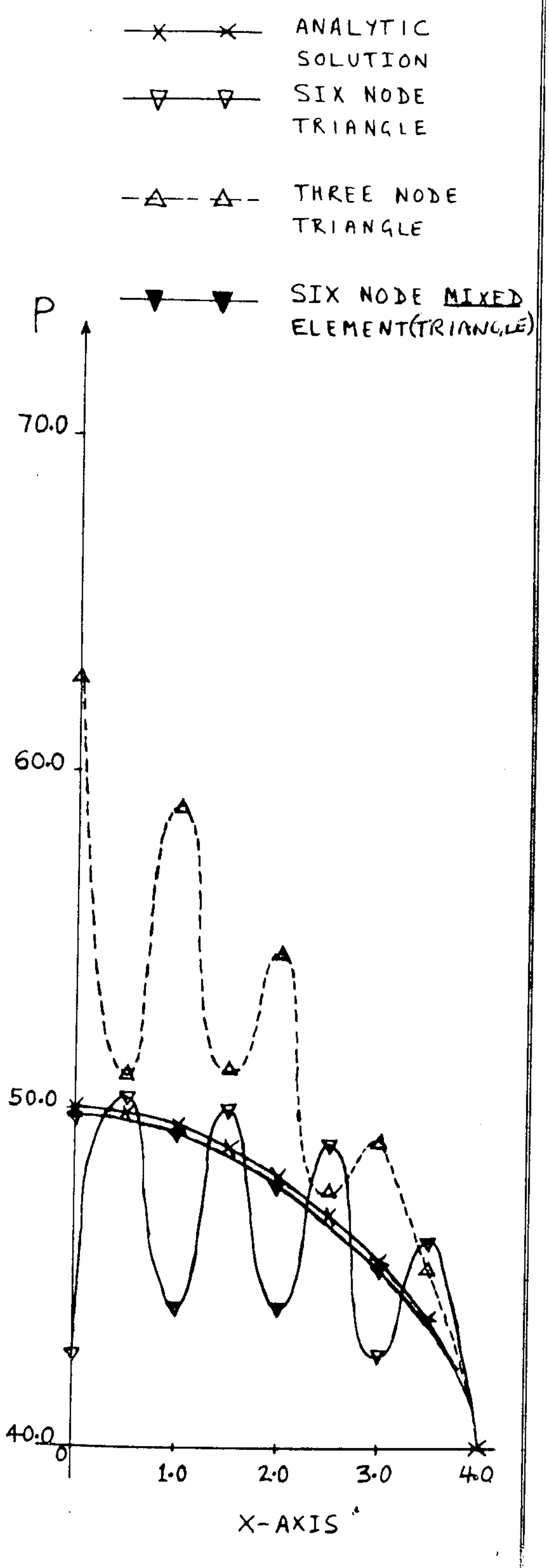


FIG 6·10

PRESSURE DISTRIBUTION ALONG Y=0 FOR Re=1 USING THE TRADITIONAL SCHEMES



PRESSURE DISTRIBUTION ALONG Y=1.6 FOR Re=1 USING THE TRADITIONAL SCHEMES

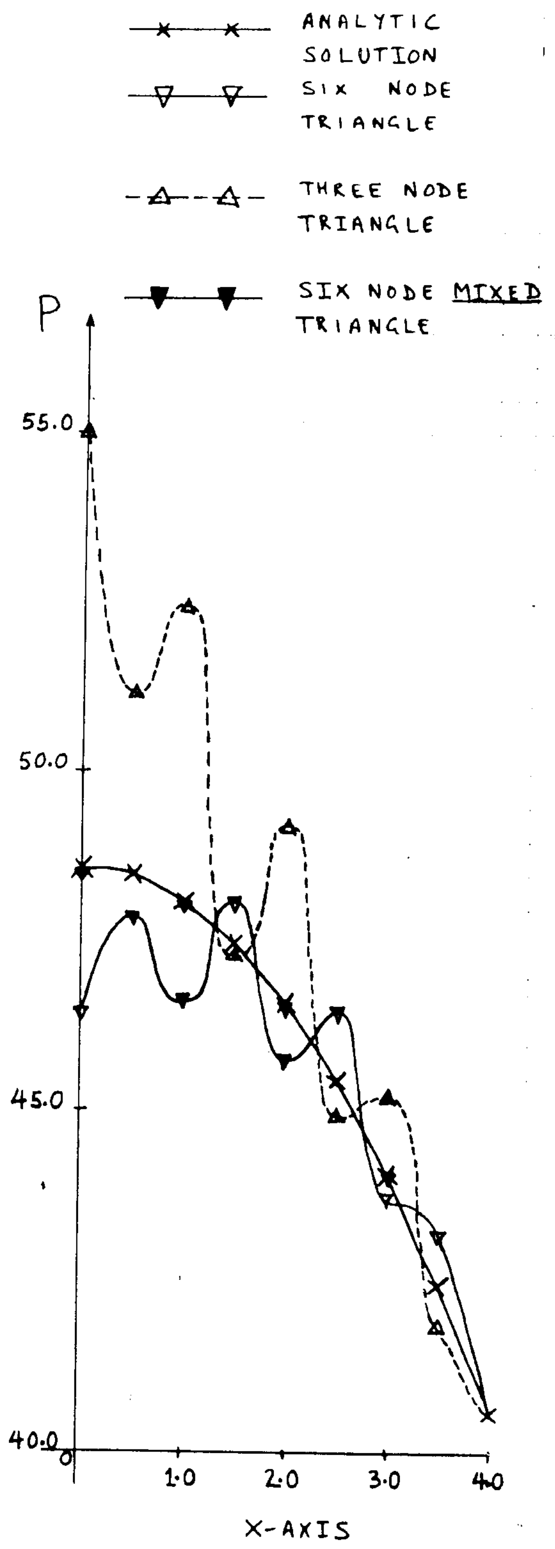
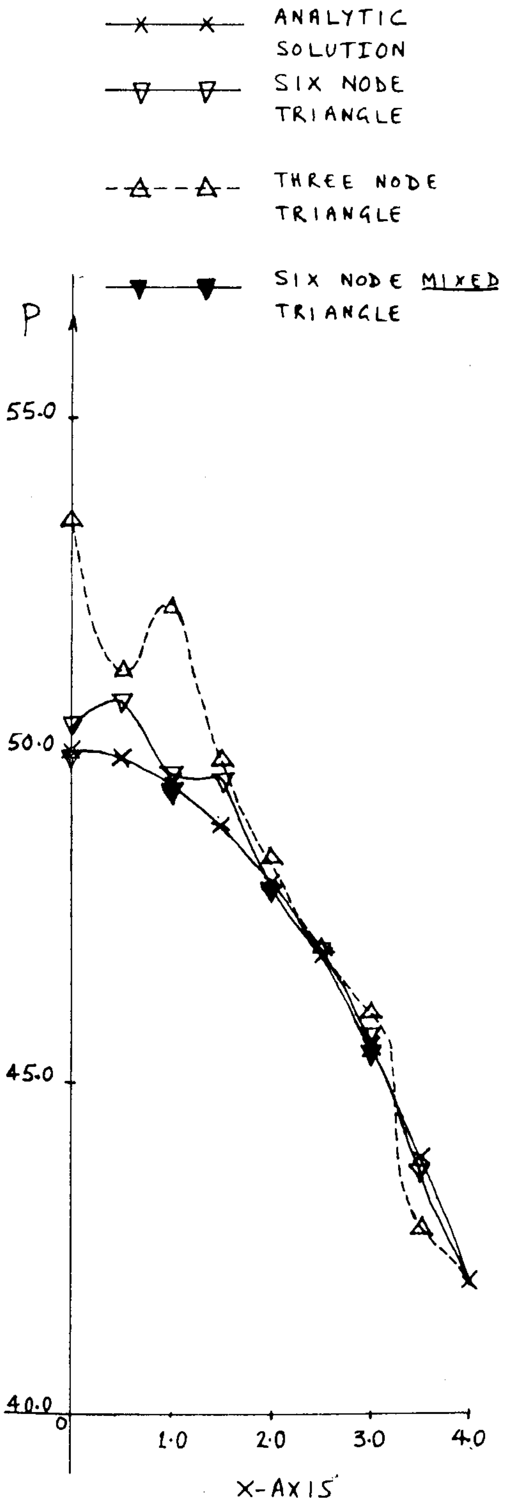


FIG 6.11

PRESSURE DISTRIBUTION ALONG Y=0 FOR Re=25 USING THE TRADITIONAL SCHEMES



PRESSURE DISTRIBUTION ALONG Y=1.6 FOR Re=25 USING THE TRADITIONAL SCHEMES

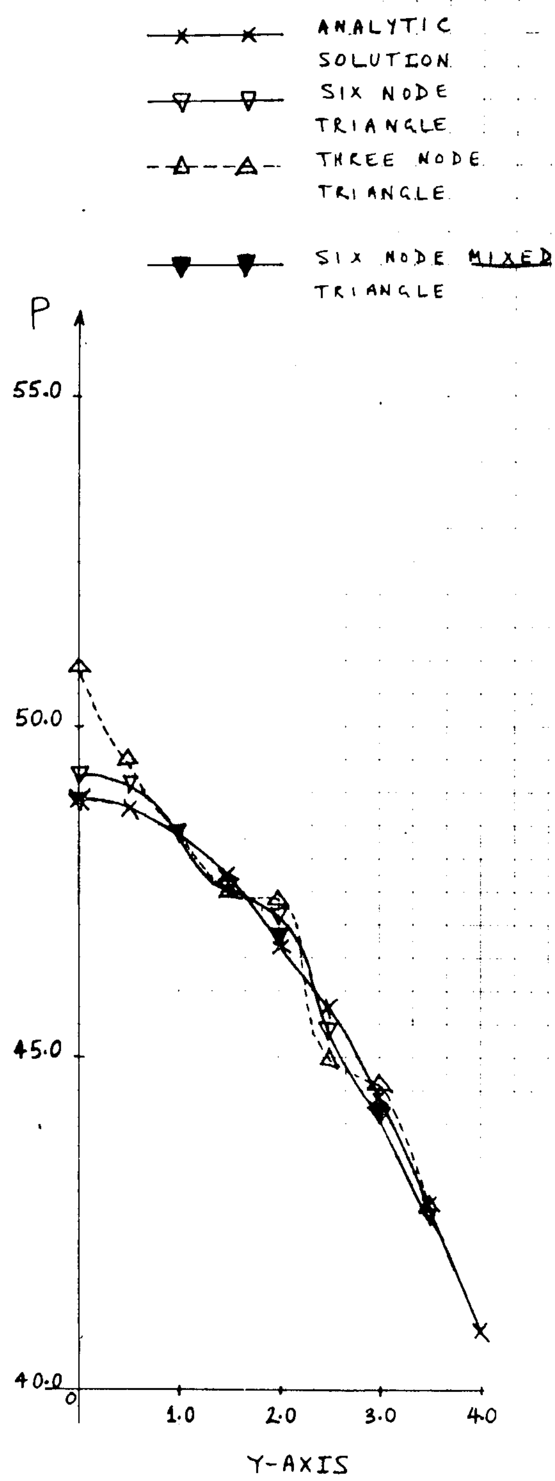


FIG 6.12

It has already been mentioned that a rigorous justification of mixed interpolation is not available. However, Taylor and Hood (20) used mixed interpolation and Galerkin's approach to obtain numerical solutions to the slider bearing problem and also flow past a cylinder. In this paper they gave a heuristic explanation to justify the used of mixed interpolation. Although their argument is solely confined to polynomial type trial functions but it is worth repeating here to see if it offers a solution for the new finite element scheme presented in this Chapter.

In (20) two requirements are posed in connection with a weighted residual process applied to coupled equations.

(i) The maximum order of error associated with the residual of each variable must be equal.

(ii) The residuals arising from each equation must be weighted according to the maximum error occurring in each equation.

The first condition may be illustrated by considering an example. Suppose that in a pair of partial differential equations, the variables are ϕ and ψ, in which the highest space derivative of ϕ is of order n, and ψ of order n+m. A Taylor expansion of the shape function within each element shows that to achieve the same order of accuracy for each variable, the polynomial shape function for ψ must be m orders higher than that for ϕ. Suppose that the interpolation field for ϕ is a polynomial of order n, then the accuracy of the interpolation field will be to $O(h^{n+1})$, and if ψ is similarly chosen to be a polynomial of order n+m then the accuracy will be of $O(h^{n+m+1})$, where h is the largest dimension in an element. On substitution of these polynomial fields into the differential equations the residual from both the ϕ and ψ fields will have an accuracy to O(h).

The second criterion is concerned with the weighting function allocated to each residual. Supposing that the residual from the first equation is accurate to $O(h^2)$ and that of the second to O(h), then the first equation should be weighted with a polynomial of one order less than the second equation, in an attempt to assign a consistent accuracy to all the equations.

That then is literally the argument presented in support of mixed interpolation by Taylor and Hood (20).

It is now readily seen how this applies to the Navier Stokes equations. The momentum equations contain second order derivatives of velocity, and first order of pressure. Thus to obtain the same order of error from each variable, parabolic elements might be used to depict variations in velocity, and linear elements for pressure. The residuals resulting from this choice of element are O(h) in the momentum equations and $O(h^2)$ in the continuity equation. Thus the momentum equations should be weighted with a parabolic shape function and the continuity equation by a linear shape function, in accordance with criterion (ii).

Let us now summarise the main requirements of mixed interpolation. These are -

(i) The trial function for velocities and pressure should be different. Furthermore the trial function for velocity should always be of a higher order than the trial function for pressure.

(ii) The continuity equation is weighted with the shape function for pressure but the momentum equations are weighted with the shape function for velocity. This essentially means that the continuity equation is not imposed at all the nodes of an element.

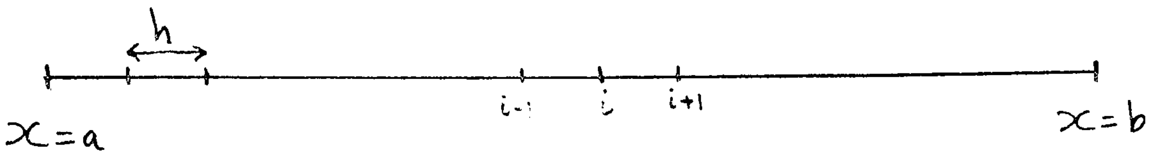
The new finite element scheme presented in this Chapter nearly satisfies the above requirements. However, it does not satisfy part of the requirement (ii) since with the new scheme the continuity equation is imposed at all the nodes. The latter point may be elucidated with reference to the following system of ordinary differential equations, viz:

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \qquad (6.7.1)$$

$$\frac{d^2u}{dx^2} - k_1\frac{du}{dx} = R_1\frac{dP}{dx} \qquad (6.7.2)$$

$$\frac{d^2v}{dx^2} - k_2\frac{dv}{dx} = R_2\frac{dP}{dx} \qquad (6.7.3)$$

Suppose this sytem of differential equations is to be solved over the interval (a,b) with a constant step length h as shown below



Over a typical element assume u, v and p are linear (i.e. = a+bx). Discretizing (6.7.1) - (6.7.3) using Galerkin's criterion gives the following finite element equations at a pivotal point i

$$(u_{i+1}-u_{i-1}) + (v_{i+1}-v_{i-1}) = 0 \qquad (6.7.4)$$

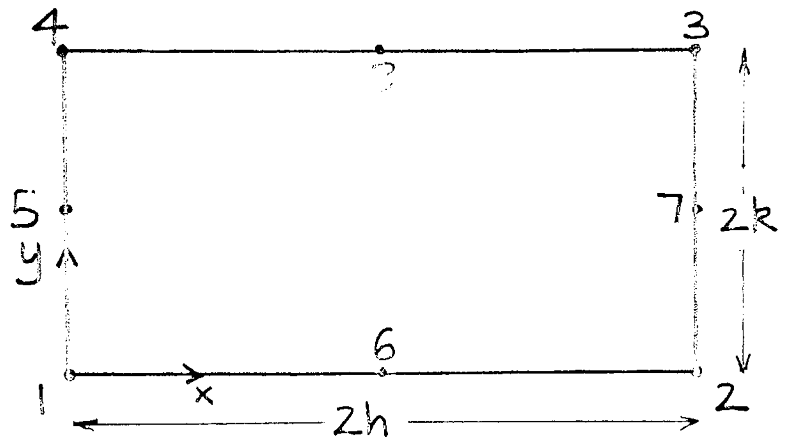
$$\left(1+\frac{k_1h}{2}\right)u_{i-1}-2u_i+\left(1-\frac{k_1h}{2}\right)u_{i+1}+\frac{h}{2}R_1\left(P_{i+1}-P_{i-1}\right) = 0 \qquad (6.7.5)$$

$$\left(1+\frac{k_2h}{2}\right)v_{i-1}-2v_i+\left(1-\frac{k_2h}{2}\right)v_{i+1}+\frac{h}{2}R_2\left(P_{i+1}-P_{i-1}\right) = 0 \qquad (6.7.6)$$

It is seen from (6.7.5) and (6.7.6) that P_i does not enter the finite element equations. This implies that when the totality of the finite element equations are solved for the values of u, v and p at the nodes it will be found that the values of p at alternate nodes are nonsensical. This is in fact the case. This simple example does throw some light on the need for requirement (ii).

6.8 AN EIGHT NODE MIXED ELEMENT

The new finite element scheme can be adapted to yield an accurate pressure field by using the ideas introduced in the last section. To this end consider the eight node rectangular element shown below



Where node 5 is the midpoint of side 14 etc.

By virtue of the results from the last section the trial function for velocities and pressure are taken as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \sum_{i=1}^{8} a_i f_i(x,y) \qquad (6.8.1)$$

$$p = \sum_{i=1}^{4} b_i g_i(x,y) \qquad (6.8.2)$$

This will mean that the pressure is interpolated at the four corner nodes of the rectangle but the velocity is interpolated at all the eight nodes. In addition the continuity equation will be weighted with the shape function for pressure. The only remaining problem now is to choose the functions $f_i(x,y)$ and $g_i(x,y)$ so that the resulting finite element scheme will be stable for practical Reynolds numbers. From the previous work it is required that the functions $f_i(x,y)$ should be particular solutions of the partial differential equation viz.

$$\nabla^2\phi - \lambda\frac{\partial\phi}{\partial x} - \mu\frac{\partial\phi}{\partial y} = 0 \qquad (6.8.3)$$

The only difficulty here is to decide which eight particular solutions are to be chosen from the infinite number available.

Some experiments with ordinary differential equations opened up other possibilities for the functions $f_i(x,y)$. It was shown in Chapter 4 that for the equation viz.

$$\frac{d^2y}{dx^2} - \lambda\frac{dy}{dx} = 0 \qquad (6.8.4)$$

the trial function $y = A+Be^{\lambda x}$ yields solutions for all $|\lambda|$. However, if this trial function is augmented with a polynomial it continues to yield stable solutions for large $|\lambda|$. For example it was found that the trial function $y = A+Be^{\lambda x}+Cx^2$ results in a stable finite element scheme for (6.8.4). These findings suggested that the trial function for velocities on our eight node mixed element may be taken as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = a_1 + a_2 e^{\lambda x} + a_3 e^{\mu y} + a_4 e^{\lambda x + \mu y} + \sum_{j=5}^{8} a_j f_j(x,y) \qquad (6.8.5)$$

Where the first four components of the trial function are solutions of (6.8.3). The functions $f_j(x,y)$ $(j = 5,6,7,8)$ are polynomials. Now traditionally the trial function for an eight node rectangle is of the form

$$\phi = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 + a_1 x^2 y + a_8 xy^2 \qquad (6.8.6)$$

The question is that from the eight polynomials in (6.8.6) which ones should be included in (6.8.5)? From requirement (i) (see 6.7) it is necessary to choose $f_j(x,y)$ $(j = 5,6,7,8)$ to correspond to the higher order polynomials in (6.8.6). Thus for the velocities choose the following trial function viz.

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = a_1 + a_2 e^{\lambda x} + a_3 e^{\mu y} + a_4 e^{\lambda x + \mu y} + a_5 x^2 + a_6 y^2 + a_7 x^2 y + a_8 xy^2 \qquad (6.8.7)$$

and for pressure choose the traditional trial function for a four node rectangle as this contains lower order polynomials

$$p = b_1 + b_2 x + b_3 y + b_4 xy \qquad (6.8.8)$$

Alternatively (6.8.7) and (6.8.8) may be written

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \sum_{j=1}^{8} N_j(x,y) \begin{Bmatrix} u_j \\ v_j \end{Bmatrix} \tag{6.8.9}$$

$$p = \sum_{j=1}^{4} M_j(x,y) p_j \tag{6.8.10}$$

The shape functions $M_j(x,y)$ are well known. Using the "shape function formula" the shape function for velocity are given by equation (6.8.11) shown on the next page. The actual shape functions may be found in Appendix 4.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{2\lambda h} & e^{2\lambda h} & 1 & 1 & e^{\lambda h} & e^{2\lambda h} & e^{\lambda h} \\ 1 & 1 & e^{2\mu k} & e^{2\mu k} & e^{\mu k} & 1 & e^{\mu k} & e^{2\mu k} \\ 1 & e^{2\lambda h} & e^{2(\lambda h+\mu k)} & e^{2\mu k} & e^{\mu k} & e^{\mu k} & e^{2\lambda h+\mu k} & e^{\lambda h+2\mu k} \\ 0 & 4h^2 & 4h^2 & 0 & 0 & h^2 & 4h^2 & h^2 \\ 0 & 0 & 4k^2 & 4k^2 & k^2 & 0 & k^2 & 4k^2 \\ 0 & 0 & 8h^2k & 0 & 0 & 0 & 4h^2k & 2h^2k \\ 0 & 0 & 8hk^2 & 0 & 0 & 0 & 2hk^2 & 4hk^2 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\lambda x} \\ e^{\mu y} \\ e^{\lambda x+\mu y} \\ x^2 \\ y^2 \\ x^2y \\ xy^2 \end{bmatrix} \qquad (6.8.11)$$

6.9 FINITE ELEMENT FORMULATION USING THE EIGHT NODE MIXED ELEMENT

The Navier Stokes equations are written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.9.1)$$

$$\nabla^2 u - \lambda\frac{\partial u}{\partial x} - \mu\frac{\partial u}{\partial y} = Re\,\frac{\partial p}{\partial x} \qquad (6.9.2)$$

$$\nabla^2 v - \lambda\frac{\partial v}{\partial x} - \mu\frac{\partial v}{\partial y} = Re\,\frac{\partial p}{\partial y} \qquad (6.9.3)$$

where

$$\lambda = Re * u \qquad (6.9.4)$$

$$\mu = Re * v \qquad (6.9.5)$$

Applying Galerkin's method to descretize the above equations using the eight node mixed element yields the following finite element equations -

$$\sum_e \sum_j \left[a^e_{ij} u_j + b^e_{ij} v_j\right] = 0 \qquad (6.9.6)$$

$$\sum_e \sum_j \left[\beta^e_{ij}\right] u_j + Re \sum_e \sum_j \left[k_{ij}\right] p_j = \sum_e f^e_i \qquad (6.9.7)$$

$$\sum_e \sum_j \left[\beta^e_{ij}\right] v_j + Re \sum_e \sum_j \left[\ell^e_{ij}\right] p_j = \sum_e g^e_i \qquad (6.9.8)$$

where the coefficients are defined below: -

$$a^e_{rs} = \iint_e M_r \frac{\partial N_s}{\partial x} dxdy \quad ; \qquad b^e_{rs} = \iint_e M_r \frac{\partial N_s}{\partial y} dxdy \quad ;$$

$$c^e_{ts} = \iint_e N_t \frac{\partial N_s}{\partial x} dxdy \quad ; \qquad d^e_{ts} = \iint_e N_t \frac{\partial N_s}{\partial y} dxdy \quad ;$$

$$e^e_{ts} = \iint_e \left(\frac{\partial N_t}{\partial x} \frac{\partial N_s}{\partial x} + \frac{\partial N_t}{\partial y} \frac{\partial N_s}{\partial y} \right) dxdy$$

$$\beta^e_{ts} = \bar{\lambda} c^e_{ts} + \bar{\mu} d^e_{ts} + e^e_{ts}$$

$$k^e_{tr} = \iint_e N_t \frac{\partial M_r}{\partial x} dxdy; \qquad \ell^e_{tr} = \iint_e N_t \frac{\partial M_r}{\partial y} dxdy$$

$$f^e_t = \oint N_t \frac{\partial u}{\partial n} ds \quad ; \qquad g^e_t = \oint N_t \frac{\partial v}{\partial n} ds$$

$$r = 1,2,3,4$$

$$t,s = 1,2,3,\ldots,8.$$

λ and μ have been approximated as constants $\bar{\lambda}$ and $\bar{\mu}$ resspectively over element e as described in Chapter 5. Thus -

$$\overline{\lambda} = \frac{1}{8} \text{Re} \sum_{j=1}^{8} \overline{u}_j \qquad (6.9.9)$$

$$\overline{\mu} = \frac{1}{8} \text{Re} \sum_{j=1}^{8} \overline{v}_j \qquad (6.9.10)$$

$\overline{u}_j$ etc. being the value of u at node j from the previous iteration.

All the coefficients defined above are listed in Appendix 4.

6.10 NUMERICAL EXAMPLE

The Hiemenz flow problem was solved using the "eight node mixed element" for various values of the Reynolds number. The finite element grid used is shown in Figure (6.13). The pressure distribution along y =const for Re = 1 is depicted in Figure (6.14). It is seen that the pressure distribution obtained using the eight node mixed element compares very well with the analytic solution as seen in Figure (6.15) and Figure (6.16). The velocity distribution was also found to compare very well with the analytic solution.

For large Reynolds numbers the iterative process for the non-linear finite element equations failed to converge. However, it is found that if the partial differential equation, viz.

FINITE ELEMENT MESH FOR THE NAVIER STOKES EQUATIONS USING THE EIGHT NODE MIXED RECTANGLE

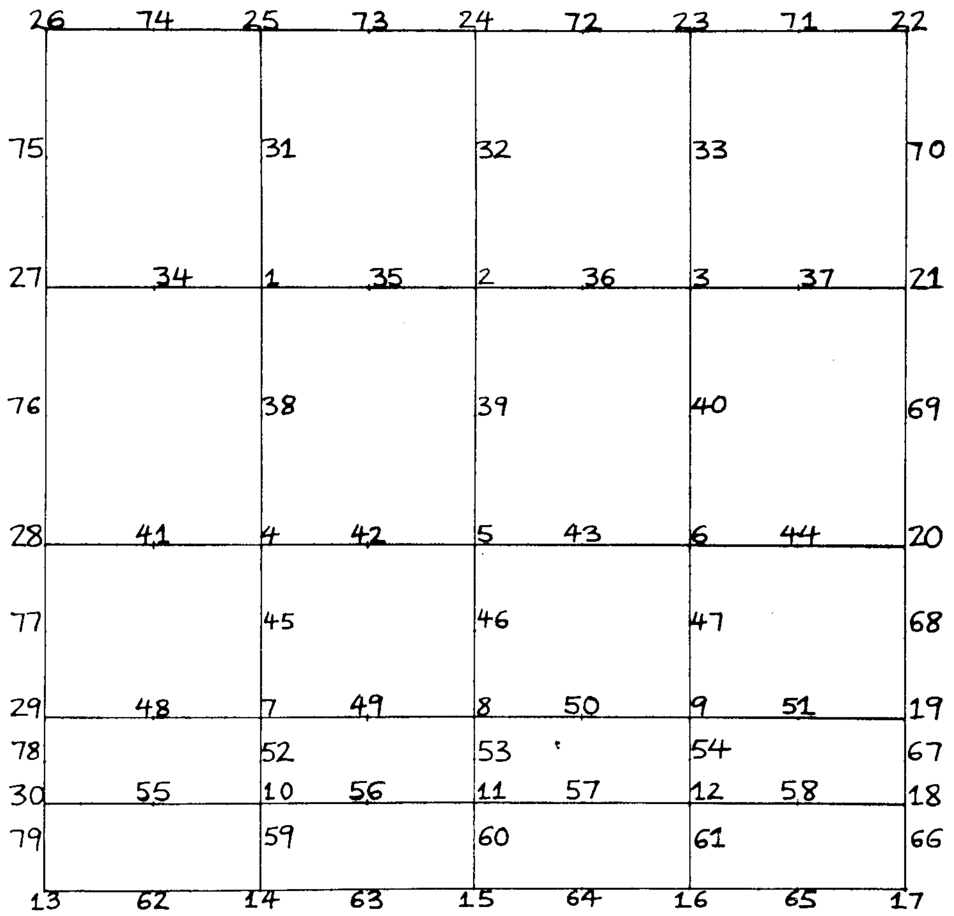
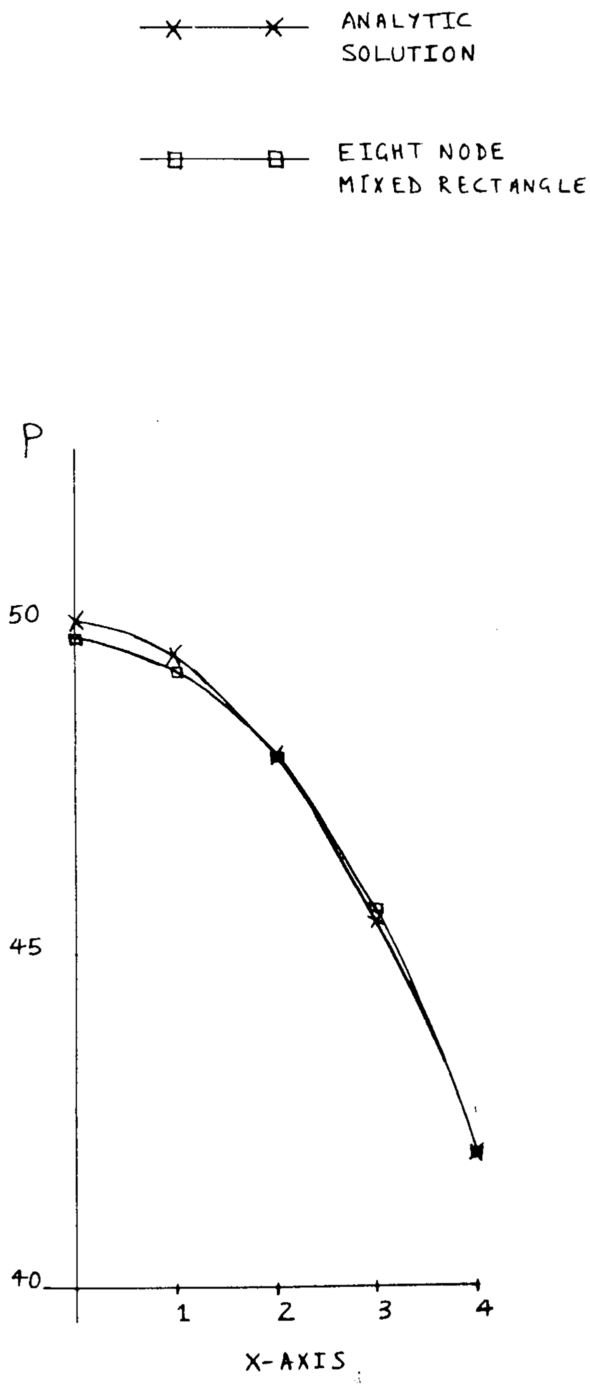


FIG 6·13

PRESSURE DISTRIBUTION ALONG Y=0 FOR Re=1 USING THE EIGHT NODE MIXED RECTANGLE



PRESSURE DISTRIBUTION ALONG Y=1.6 FOR Re=1 USING THE EIGHT NODE MIXED RECTANGLE

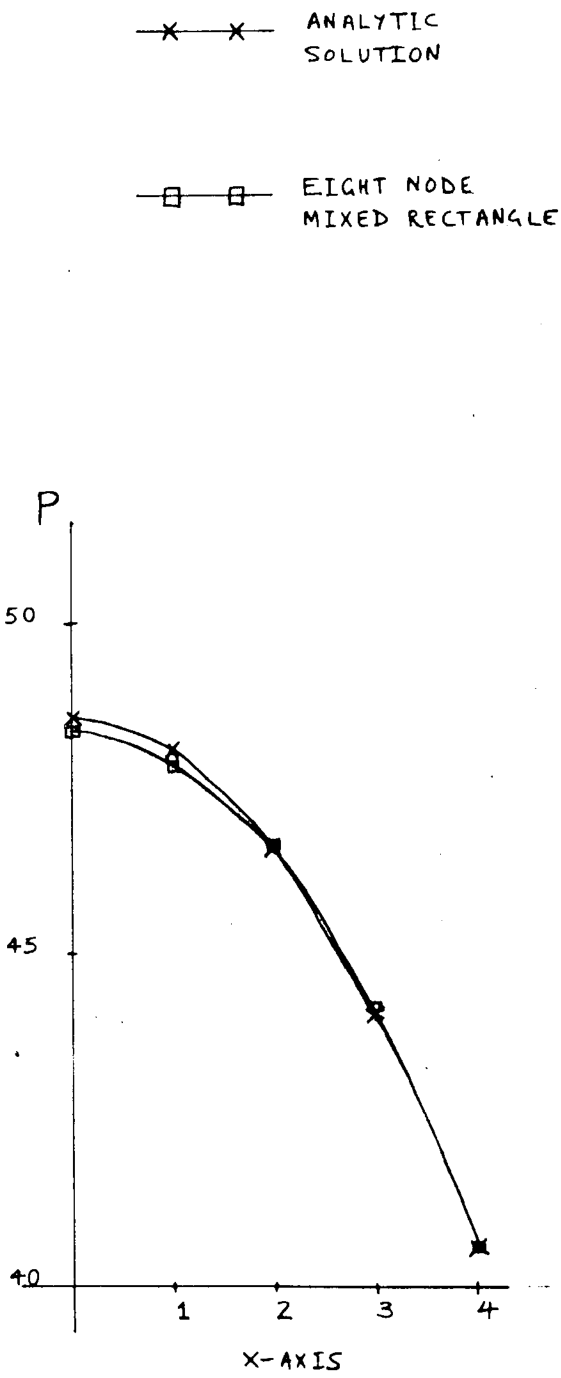


FIG 6.14

DISTRIBUTION OF u ALONG Y=1·6 FOR Re=1 USING THE EIGHT NODE MIXED RECTANGLE

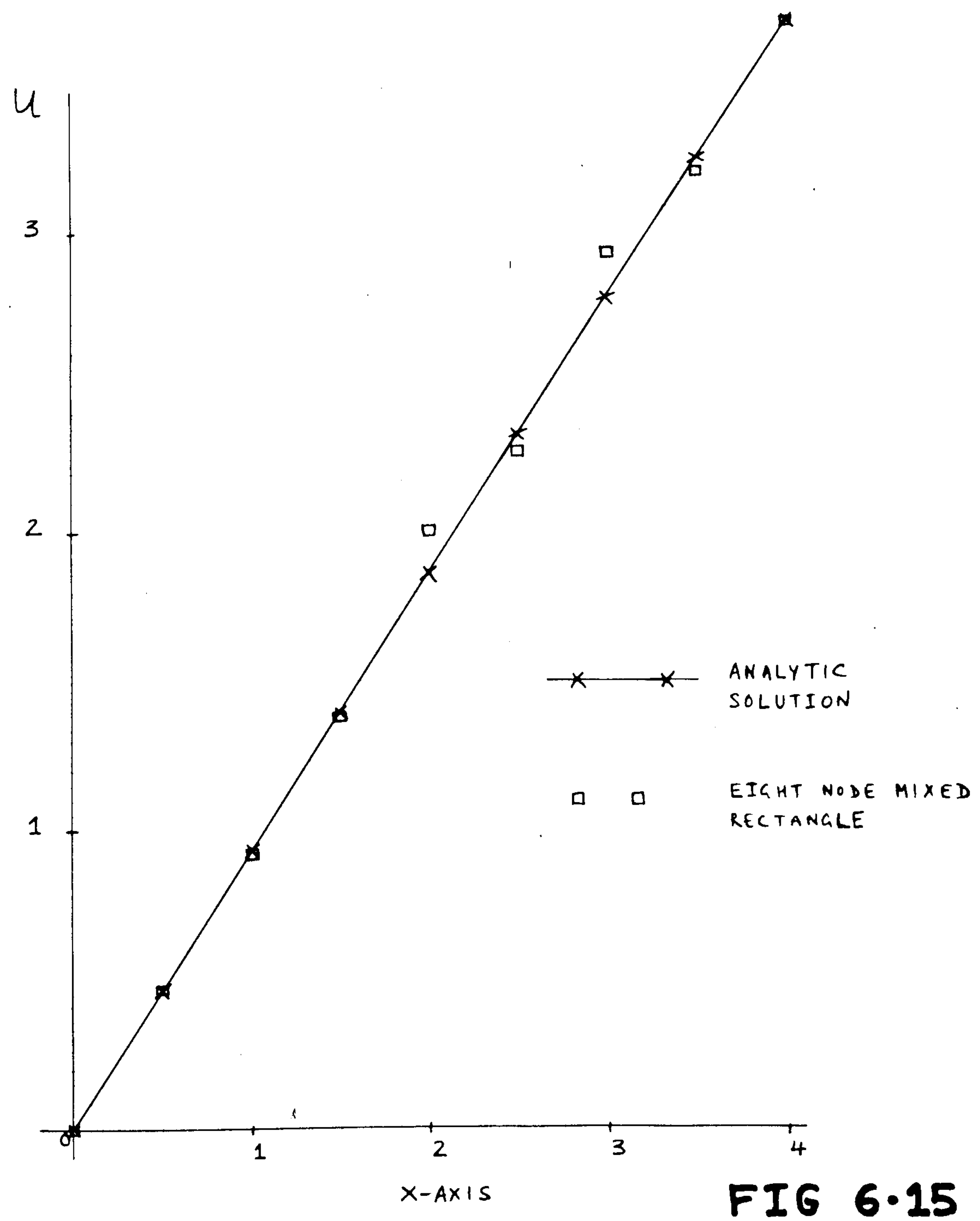


FIG 6·15

DISTRIBUTION OF V ALONG
Y = 1·6 FOR R_e = 1 USING THE
EIGHT NODE MIXED RECTANGLE

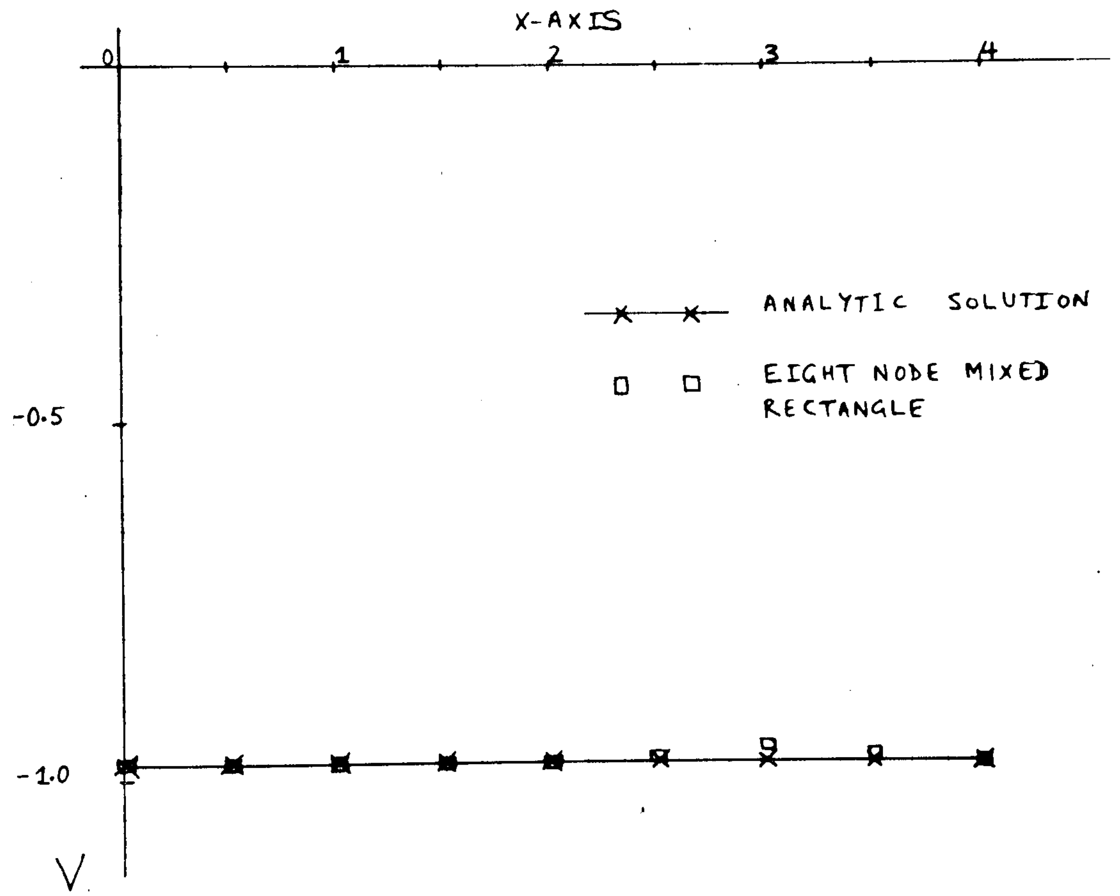


FIG 6·16

$$\nabla^2\phi - \lambda\phi_x - \mu\phi_y = 0 \qquad (6.10.1)$$

is solved using an eight node rectangular element with the trial function

$$\phi = A+Be^{\lambda x}+Ce^{\mu y}+De^{\lambda x+\mu y}+Ex^2+Fy^2+Gx^2y+Hxy^2 \qquad (6.10.2)$$

then stability for large $|\lambda|$ and $|\mu|$ is not obtained. This indicates that the trial functions for the eight-node mixed element are probably not suitable from the point of view of obtaining solutions to the Navier Stokes equation for practical Reynolds numbers. But this work does clearly show the strategy which has to be employed in order to get correct answers for both the velocity and pressure field simultaneously.

6.11 SUMMARY AND CONCLUSIONS

In this Chapter a new finite element scheme has been presented for the Navier Stokes equations. Numerical results were obtained for the "Hiemenz flow" problem for which a semi-analytical solution is available. For low values of the Reynolds numbers the new finite element scheme produced acceptable results for velocity although in particular instances the pressure was subject to inaccuracies. For large Reynolds numbers ($\simeq 10^4$) the iterative process for the non-linear finite element equations does not converge but the results indicated that the new scheme was probably stable.

The author then discusses the concept of mixed interpolation and proposes a scheme which yields accurate results for both velocity and pressure fields simultaneously.

In this thesis we are mainly interested in establishing a stable finite element scheme for the Navier Stokes equations. Consequently the author decided to by-pass the convergence problem in order to make some statement about the stability of the new scheme. The next Chapter goes on to show that the new finite element scheme presented in this Chapter is inherently more stable than the traditional polynomial schemes.

CHAPTER SEVEN

STABILITY ASPECTS OF THE NEW FINITE ELEMENT SCHEME

7.1 INTRODUCTION

One obvious way to establish that the new scheme is stable is to persevere with the Navier Stokes equations and try to obtain convergence for practical Reynolds numbers. Unfortunately the elements of the global matrix are rather complicated. This virtually rules out an analytical approach to the convergence problem. The alternative is an empirical approach. The latter approach would involve considerable numerical experimentation and is necessarily very time consuming.

Ideally it would be best to investigate the stability of the new scheme independentally of the convergence problem. If the scheme proves to be stable then all further efforts can be concentrated on convergence.

To this end we merely have to study the system of linear partial differential equations which are solved by the new scheme in each iteration of the Navier Stokes equations. These equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7.1.1)$$

$$\nabla^2 u - \lambda\frac{\partial u}{\partial x} - \mu\frac{\partial u}{\partial y} = R\frac{\partial p}{\partial x} \qquad (7.1.2)$$

$$\nabla^2 v - \lambda\frac{\partial v}{\partial x} - \mu\frac{\partial v}{\partial y} = R\frac{\partial p}{\partial y} \qquad (7.1.3)$$

where λ, μ and R are taken to be constants. For convenience the equations (7.1.1) - (7.1.3) will be called the "quasi Navier Stokes equations". To validate the numerical results from the new scheme it is first necessary to obtain an analytic solution to the quasi Navier Stokes equations.

7.2 AN ANALYTIC SOLUTION TO THE QUASI NAVIER STOKES EQUATIONS

A solution of the quasi Navier Stokes equations may be found by looking for a solution in the form

$$u = g(x)f'(y) \quad ; \quad v = -g'(x)f(y)$$
$$p = G(x)F(y)$$

Where g(x), G(x), f(y) and F(y) are unknown functions, which are to be chosen suitably.

Now

$$\frac{\partial u}{\partial x} = g'(x)f'(y) \quad ; \quad \frac{\partial v}{\partial x} = -g''(x)f(y)$$

$$\frac{\partial u}{\partial y} = g(x)f''(y) \quad ; \quad \frac{\partial v}{\partial y} = -g'(x)f'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = g''(x)f'(y) \quad ; \quad \frac{\partial^2 v}{\partial x^2} = - g'''(x)f(y)$$

$$\frac{\partial^2 u}{\partial y^2} = g(x)f'''(y) \quad ; \quad \frac{\partial^2 v}{\partial y^2} = -g'(x)f''(y)$$

$$\frac{\partial p}{\partial x} = G'(x)f(y) \quad ; \quad \frac{\partial p}{\partial y} = G(x)F'(y)$$

Notice that (7.1.1) is identically satisfied. Equations (7.2.2) and (7.2.3) give respectively

$$g''f' + gf''' - \lambda g'f' - \mu gf'' = RG'F \tag{7.2.1}$$

$$-g'''f - g'f'' + \lambda g''f + \mu g'f' = RGF' \tag{7.2.2}$$

(7.2.1) rearranges to

$$f''' - \mu f'' + \left(\frac{g''-\lambda g'}{g}\right)f' = \frac{RG'F}{g} \tag{7.2.3}$$

choose $g(x)$ such that

$$\frac{g''-\lambda g'}{g} = -\frac{kg'}{g}$$

k being a constant, i.e.

$$g'' + (k-\lambda)g' = 0$$

or

$$\underline{g(x) = e^{-(k-\lambda)x}} \tag{7.2.4}$$

Equation (7.2.3) now becomes

$$f''' - \mu f'' + k(k-\lambda)f' = RG'Fe^{(k-\lambda)x} \qquad (7.2.5)$$

Substituting for g(x) in (7.2.2) gives after simplification

$$f'' - \mu f' + k(k-\lambda)f = \frac{RGF'e^{(k-\lambda)x}}{(k-\lambda)} \qquad (7.2.6)$$

Thus it has been shown that with g(x) given by (7.3.4) equations (7.1.2) and (7.1.3) reduce to equations (7.2.5) and (7.2.6). These are two independent equations containing three unknown functions f(y), F(y) and G(x). This means that one of these functions is at our disposal.

Now as the left hand sides of (7.2.5) and (7.2.6) are functions of y so the right hand sides must also be functions of y. This may be ensured if G(x) is chosen as

$$\underline{G(x) = (k-\lambda)e^{-(k-\lambda)x}} \qquad (7.2.7)$$

Equations (7.2.5) and (7.2.6) now becomes

$$f''' - \mu f'' + k(k-\lambda)f' = -R(k-\lambda)^2F \qquad (7.2.8)$$

$$f'' - \mu f' + k(k-\lambda)f = RF' \qquad (7.2.9)$$

These are two equations for f(y) and F(y). Eliminating f(y) gives

$$F'' + (k-\lambda)^2F = 0$$

i.e.

$$\underline{F(y) = A\cos(k-\lambda)y + B\sin(k-\lambda)y} \qquad (7.2.10)$$

A and B being arbitrary constants.

Substituting for F(y) into (7.2.9) gives

$$f''-\mu f'+k(k-\lambda)f = R(k-\lambda)[B\cos(k-\lambda)y-A\sin(k-\lambda)y] \qquad (7.2.11)$$

Assuming $\mu^2-4k(k-\lambda) > 0$ (which can always be ensured by choosing k suitably) and defining

$$p_1 = \frac{\mu + \sqrt{\mu^2-4k(k-\lambda)}}{2} \qquad (7.2.12)$$

$$p_2 = \frac{\mu - \sqrt{\mu^2-4k(k-\lambda)}}{2} \qquad (7.2.13)$$

The solution of (7.2.11) may now be written

$$\underline{f(y) = Ce^{p_1 y} + De^{p_2 y}}$$

$$\underline{+ \frac{R}{(\lambda^2+\mu^2)}[(\lambda B-\mu A)\cos(k-\lambda)y-(B\mu+A\lambda)\sin(k-\lambda)y]} \qquad (7.2.14)$$

where C and D are arbitrary constants.

Hence for u, v and p we have

$$u = e^{-(k-\lambda)x}\,[C p_1 e^{p_1 y} + D p_2 e^{p_2 y}$$

$$-\frac{R(k-\lambda)}{(\lambda^2+\mu^2)}\,\{(\lambda B-\mu A)\sin(k-\lambda)y + (B\mu+A\lambda)\cos(k-\lambda)y\}] \tag{7.2.15}$$

$$v = (k-\lambda)e^{-(k-\lambda)x}[Ce^{p_1 y} + De^{p_2 y}$$

$$+\frac{R}{(\lambda^2+\mu^2)}\,\{(\lambda B-\mu A)\cos(k-\lambda)y-(B\mu+A\lambda)\sin(k-\lambda)y\}] \tag{7.2.16}$$

$$p = (k-\lambda)e^{-(k-\lambda)x}[A\cos(k-\lambda)y+B\sin(k-\lambda)y] \tag{7.2.17}$$

where A, B, C and D are arbitrary constants at our disposal.

7.3 FINITE ELEMENT FORMULATIONS OF THE QUASI NAVIER STOKES EQUATIONS

The new finite element scheme applied to the quasi Navier Stokes equations yields the following finite element equations:

$$\sum_e \sum_j A^e_{ij}u_j + B^e_{ij}v_j = 0 \tag{7.3.1}$$

$$\sum_e \sum_j \left[\alpha^e_{ij}\right] u_j + R \sum_e \sum_j \left[K^e_{ij}\right] p_j = \sum_e F^e_i \tag{7.3.2}$$

$$\sum_e \sum_j \left[\alpha^e_{ij}\right] v_j + R \sum_e \sum_j \left[L^e_{ij}\right] p_j = \sum_e G^e_i \qquad (7.3.3)$$

The various coefficients are defined below

$$A^e_{ij} = \iint_e M_i \frac{\partial N_j}{\partial x} dxdy \quad ; \qquad B^e_{ij} = \iint_e M_i \frac{\partial N_j}{\partial y} dxdy \quad ;$$

$$C^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial x} dxdy \quad ; \qquad D^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial y} dxdy$$

$$E^e_{ij} = \iint_e \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) dxdy$$

$$\alpha^e_{ij} = \lambda C^e_{ij} + \mu D^e_{ij} + E^e_{ij}$$

$$K^e_{ij} = \iint_e N_i \frac{\partial M_j}{\partial x} dxdy \quad ; \qquad L^e_{ij} = \iint_e N_i \frac{\partial M_j}{\partial y} dxdy$$

$$F^e_i = \oint N_i \frac{\partial u}{\partial n} ds \quad ; \qquad G^e_i = \oint N_i \frac{\partial v}{\partial n} ds$$

Where of course the trial functions for velocities and pressure for the new scheme are

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = A + Be^{\lambda x} + Ce^{\mu y} + De^{\lambda x + \mu y}$$

$$p = a + bx + cy + dxy$$

over a four node rectangle. Or

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \sum_j N_j \begin{Bmatrix} u_j \\ v_j \end{Bmatrix}$$

$$p = \sum_j M_j p_j$$

The coefficients defined above are the same as those given in (6.3) except that λ and μ are constants in this case.

7.4 NUMERICAL EXAMPLE

A solution of the quasi Navier Stokes equations was derived in (7.2). The constants appearing in that solution are now chosen -

Choose

$$(k-\lambda) = \quad ; \quad A = \frac{1}{\pi} \quad ; \quad B = 0$$

write

$$\alpha = \frac{\mu \pm \sqrt{\mu^2 - 4\pi(\pi+\lambda)}}{2}$$

Assuming that λ , μ and R are numbers of the same order of absolute magnitude, then if $\mu < 0$, choose negative sign and if $\mu > 0$ choose positive sign. Then if $|\mu| \gg \pi$ and if $|\mu| \sim |\lambda|$ we shall have $\alpha \sim \mu$.

Choose C or D to be zero and the other to be β/R, where will be determined below. Then the solutions to the quasi Navier Stokes equations i.e.(7.2.15) - (7.2.17) will become

$$u = e^{-\pi x}\left[\frac{\beta\alpha}{R}e^{\alpha y} + \frac{R}{\lambda^2+\mu^2}\left\{\mu\sin\pi y - \lambda\cos\pi y\right\}\right]$$

$$v = e^{-\pi x}\left[\frac{\beta\pi}{R}e^{\alpha y} + \frac{R}{\lambda^2+\mu^2}\left\{\mu\cos\pi y + \lambda\sin\pi y\right\}\right]$$

$$p = e^{-\pi x}\cos\pi y$$

When $\mu > 0$, $\alpha > 0$ and $e^{\alpha y}$ will increase with increasing y. Choose β so that $u(0,\frac{1}{2}) = 1$.

When $\lambda > 0$, $\alpha > 0$ and $e^{\alpha y}$ will decrease with increasing y. Choose β so that $u(0,0) = 1$.

The corresponding solutions are

(a) <u>If $\mu > 0$</u>

$$u = e^{-\pi x}\left[\left(1 - \frac{\mu R}{\lambda^2+\mu^2}\right)e^{\alpha(y-\frac{1}{2})} + \frac{R}{(\lambda^2+\mu^2)}\left\{\mu\sin\pi y - \lambda\cos\pi y\right\}\right]$$

$$v = e^{-\pi x}\left[\frac{\pi}{\alpha}\left(1 - \frac{\mu R}{\lambda^2\ \mu^2}\right)e^{\alpha(y-\frac{1}{2})} - \frac{R}{(\lambda^2+\mu^2)}\left\{\mu\cos\pi y + \lambda\sin\pi y\right\}\right]$$

$$p = e^{-\pi x}\cos\pi y$$

$$\alpha = \frac{\mu + \sqrt{\mu^2 - 4\pi(\pi+\lambda)}}{2}$$

(b) If $\mu < 0$

$$u = e^{-\pi x}\left[\left(1+\frac{\lambda R}{\lambda^2+\mu^2}\right)e^{\alpha y}+\frac{R}{\lambda^2+\mu^2}\left\{\mu\sin\pi y-\lambda\cos\pi y\right\}\right]$$

$$v = e^{-\pi x}\left[\frac{\pi}{\alpha}\left(1+\frac{\lambda R}{\lambda^2+\mu^2}\right)e^{\alpha y}-\frac{R}{\lambda^2+\mu^2}\left\{\mu\cos\pi y+\lambda\sin\pi y\right\}\right]$$

$$p = e^{-\pi x}\cos\pi y$$

$$\alpha = \frac{\mu - \sqrt{\mu^2 - 4\pi(\pi+\lambda)}}{2}$$

It will be noticed that with this choice of constants the exponential terms occurring in the solution are easily computed for large $|\lambda|$ and $|\mu|$. Also u and v vary rapidly with $|\lambda|$ and $|\mu|$ which is reminiscent of the Navier Stokes equations.

The quasi Navier Stokes equations were solved on the region

$$0 < x < 1$$
$$0 < y < 0.5$$

using

(i) Three node triangle

(ii) Six node triangle

(iii) The new finite element scheme. (For convenience we shall refer to this element as the "exponential element".)

For various values of λ, μ and R. The analytic solution used is that given in (a). Two sets of meshes were employed. The course mesh was (4 x 4) and the five mesh was (8 x 8). On the boundary of the rectangular region u, v and p were specified. In addition it was found necessary to specify p at an internal point.

The results for u and v are illustrated graphically for the coarse mesh in Figure (7.1) - Figure (7.3). It is seen that for small $|\lambda|$, $|\mu|$ and R there is not much discrepancy between the various elements. For large $|\lambda|$ and $|\mu|$ the traditional elements breakdown completely but the "exponential element" continues to give good agreement.

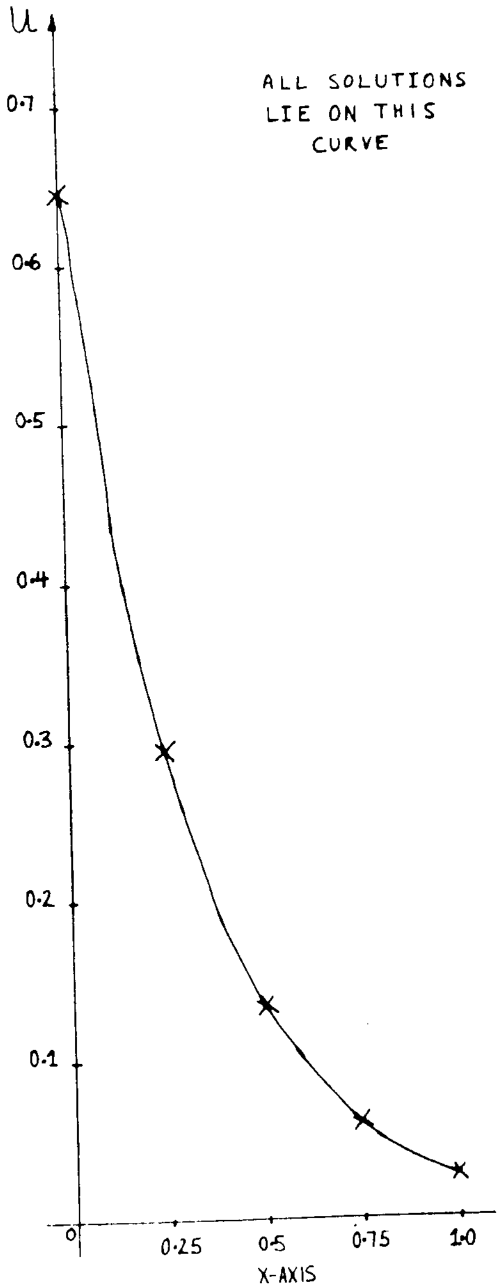
The results for p are illustrated graphically in Figure (7.4) and Figure (7.5).

For small $|\lambda|$, $|\mu|$ and R good results were obtained for p using the "exponential element". As $|\lambda|$, $|\mu|$ and R were allowed to increase the accuracy in p deteriorated slightly. However the accuracy could always be improved by refining the mesh as shown in Figure (7.5).

Finally it must be said that it was expected that for any given grid size the numerical results for p would be inferior to the corresponding results for u and v. This is so because the discussion of mixed interpolation as presented in the last Chapter is clearly also applicable to the quasi Navier Stokes equations.

DISTRIBUTION OF u ON THE COARSE GRID ALONG Y=0.25 FOR THE QUASI NAVIER STOKES EQUATIONS.

$\lambda=-3\,;\,\mu=3\,;\,R=1$



DISTRIBUTION OF V ON THE COARSE GRID ALONG Y=0.25 FOR THE QUASI NAVIER STOKES EQUATIONS

$\lambda=-3\,;\,\mu=3\,;\,R=1$

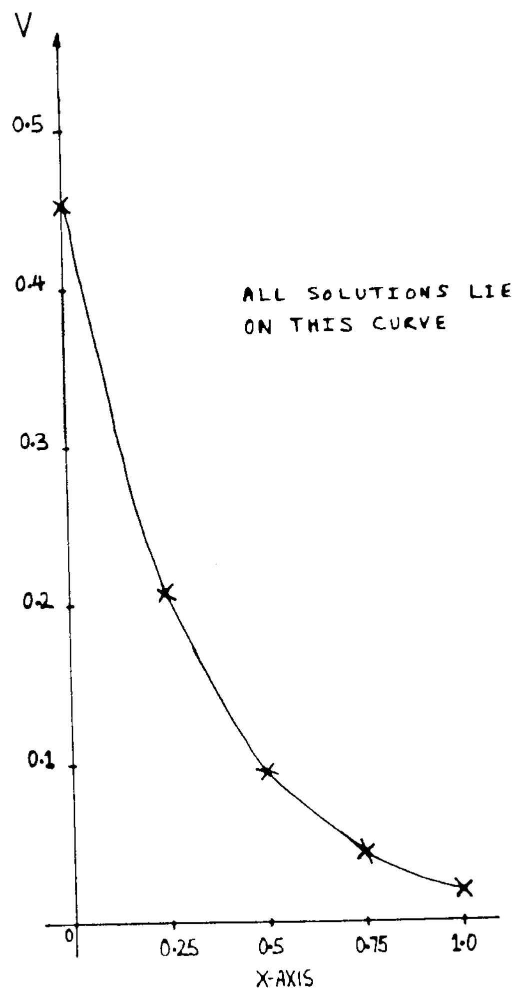


FIG 7.1

DISTRIBUTION OF U ON THE COARSE GRID ALONG Y=0.25 FOR THE QUASI NAVIER STOKES EQUATIONS

$\lambda = \mu = 10^4 \; ; \; R = 8 \times 10^3$

DISTRIBUTION OF V ON THE COARSE GRID ALONG Y=0.25 FOR THE QUASI NAVIER STOKES EQUATIONS

$\lambda = \mu = 10^4 \; ; \; R = 8 \times 10^3$

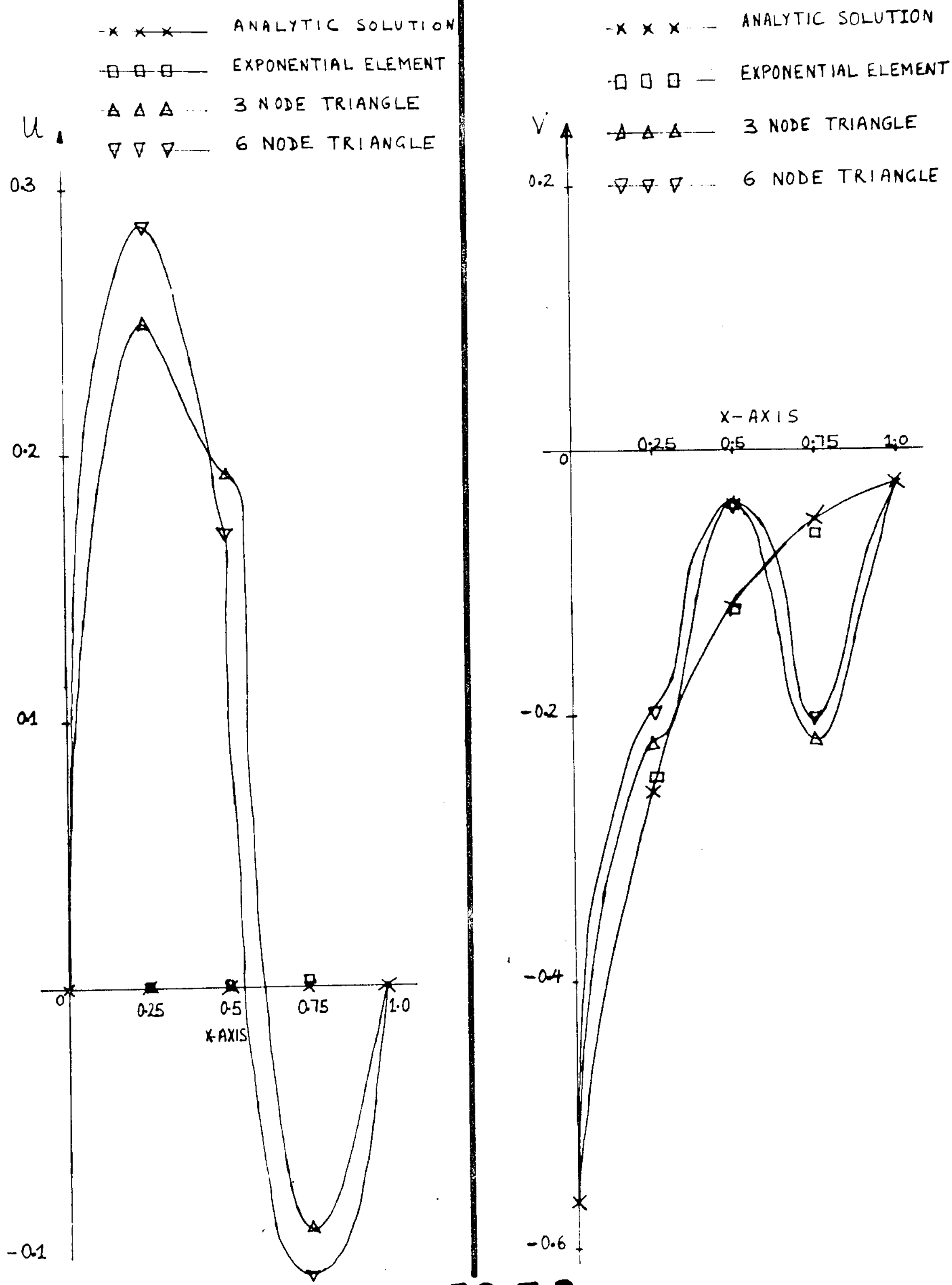
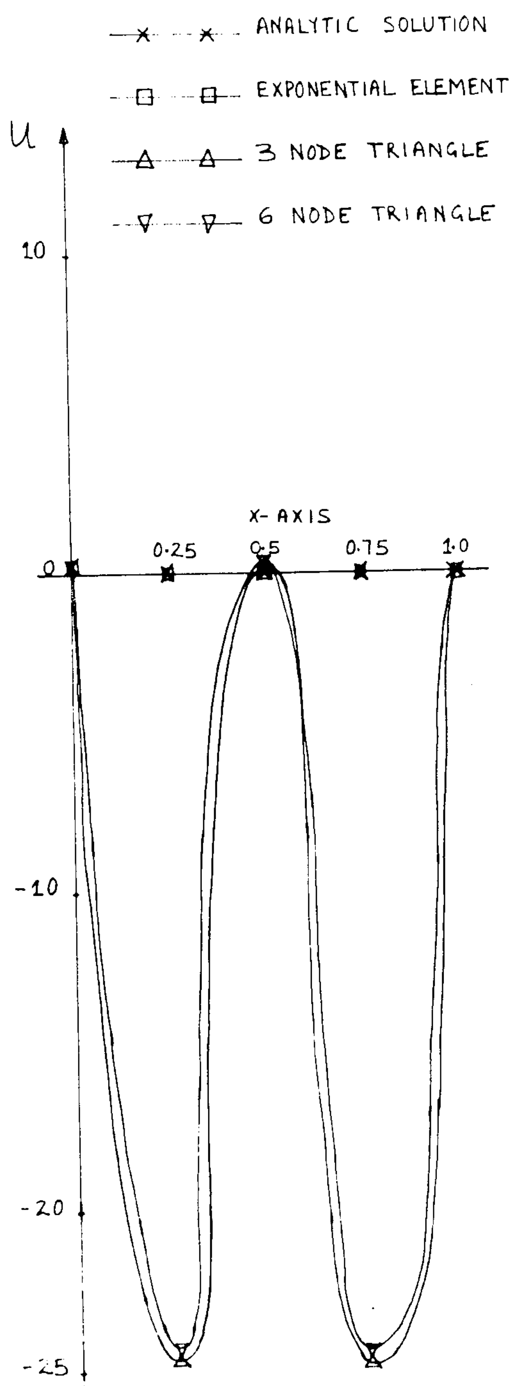


FIG 7.2

DISTRIBUTION OF u ON THE COARSE GRID ALONG Y=0.375 FOR THE QUASI NAVIER STOKES EQUATIONS

$\lambda = \mu = 10^4$; $R = 8 \times 10^3$



DISTRIBUTION OF V ON THE COARSE GRID ALONG Y=0.375 FOR THE QUASI NAVIER STOKES EQUATIONS

$\lambda = \mu = 10^4$; $R = 8 \times 10^3$

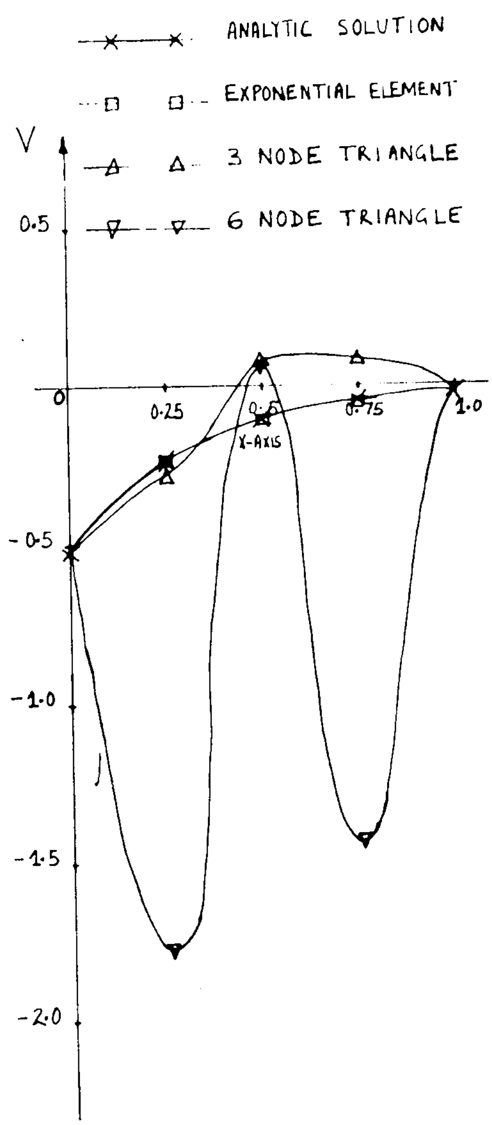


FIG 7.3

DISTRIBUTION OF P ALONG Y=0.25 FOR THE QUASI NAVIER STOKES EQUATIONS

$\lambda=-3; \mu=3; R=1$

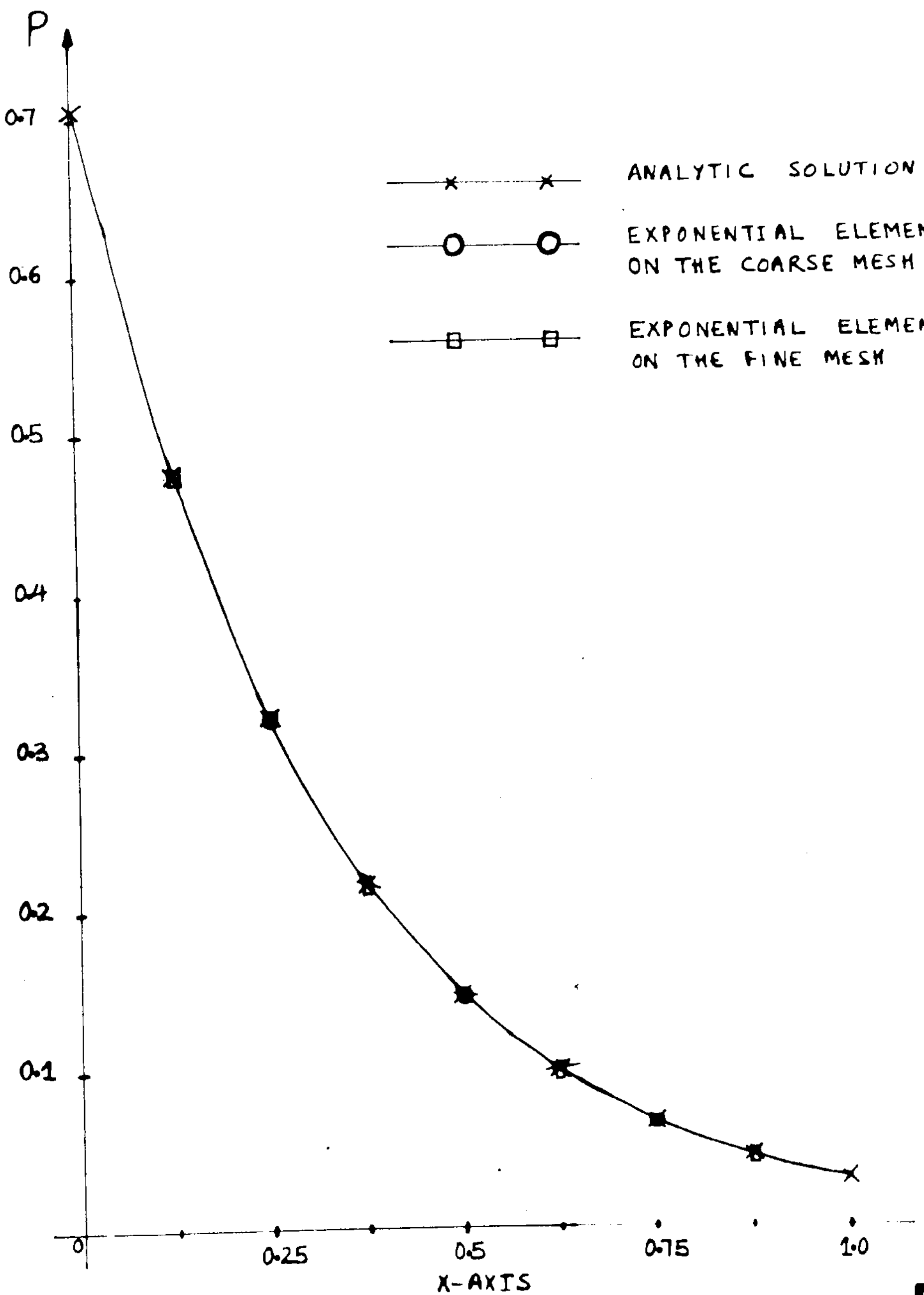


FIG 7.4

DISTRIBUTION OF P ALONG Y = 0.25 FOR THE QUASI NAVIER STOKES EQUATIONS

$\lambda = \mu = 10^4$; $R = 8 \times 10^3$

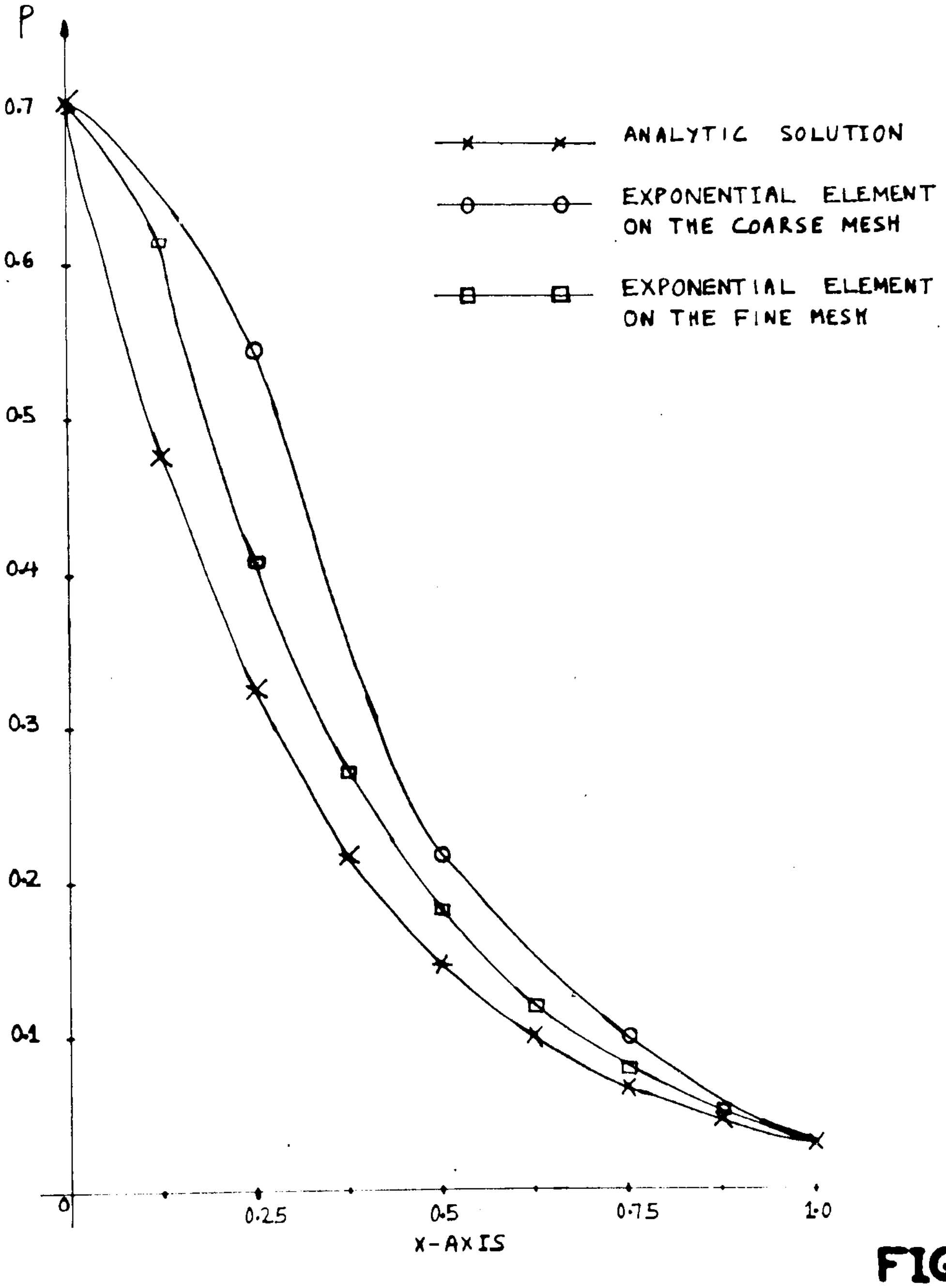


FIG 7.5

7.5 SUMMARY AND CONCLUSIONS

The system of partial differential equations which are solved by the new scheme in each iteration of the solution procedure for the Navier Stokes equations are very similar to the "quasi Navier-Stokes equations", viz

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7.5.1)$$

$$\nabla^2 u - \lambda\frac{\partial u}{\partial x} - \mu\frac{\partial u}{\partial y} = R\,\frac{\partial p}{\partial x} \qquad (7.5.2)$$

$$\nabla^2 v - \lambda\frac{\partial v}{\partial x} - \mu\frac{\partial v}{\partial y} = R\,\frac{\partial p}{\partial y} \qquad (7.5.3)$$

where λ,μ and R are constants.

In this Chapter an analytic solution has been obtained for the quasi Navier-Stokes equations. A striking feature of the analytic solution for u and v is that it varies rapidly with λ and μ, something very reminiscent of the Navier Stokes equations. The numerical results showed that for small $|\lambda|$, $|\mu|$ and R there was not much discrepancy between the new schemes and traditional schemes. For large $|\lambda|$, $|\mu|$ and R the traditional schemes break down completely but the new scheme continues to give good agreement.

7.6 FUTURE WORK

The work presented in this thesis opens up a wide field of research in both numerical mathematics and numerical fluid dynamics. Further scope of research in each of these areas is briefly outlined below -

(i) Numerical Mathematics

Investigate the derivation of finite element schemes using the technique described in Chapters 4 and 5 for differential operators other than those discussed in this thesis. It seems that if the first derivatives in a differential equation are significant then the new schemes are probably 'better' than the traditional schemes.

(ii) Numerical Fluid Dynamics

There is still a real need for a reliable numerical method for the solution of the Navier-Stokes equations. A new finite element scheme for the Navier-Stokes equations has been presented in this thesis. The numerical results obtained using the new scheme indicate very strongly that the scheme is stable for practical Reynolds numbers. However, further numerical experimentation is required to obtain convergence for high Reynolds numbers.

It would also be worthwhile investigating schemes which employ mixed interpolation. One such scheme was presented in Chapter 6.

APPENDIX I

A DISCUSSION ON THE STABILITY OF SOME FINITE ELEMENT SCHEMES

AI.1 AN EXAMPLE

To illustrate an aspect of the stability problem, consider the equation

$$\frac{d^2y}{dx^2} - \lambda\frac{dy}{dx} = 0 \qquad \text{(AI.1.1)}$$

The finite element equations at a pivotal point i corresponding to both the linear and exponential elements are -

(i) Linear Element

$$(1+\frac{\lambda h}{2})y_{i-1} - 2y_i + (1-\frac{\lambda h}{2})y_{i+1} = 0 \qquad \text{(AI.1.2)}$$

(ii) Exponential Element

$$e^{\lambda h}y_{i-1} - (1+e^{\lambda h})y_i + y_{i+1} = 0 \qquad \text{(AI.1.3)}$$

Now the solution of (AI.1.1) is given by

$$y = A + Be^{\lambda x} \qquad \text{(AI.1.4)}$$

The behaviour of the exponential term of (AI.1.4) is illustrated in figure (AI.1) below

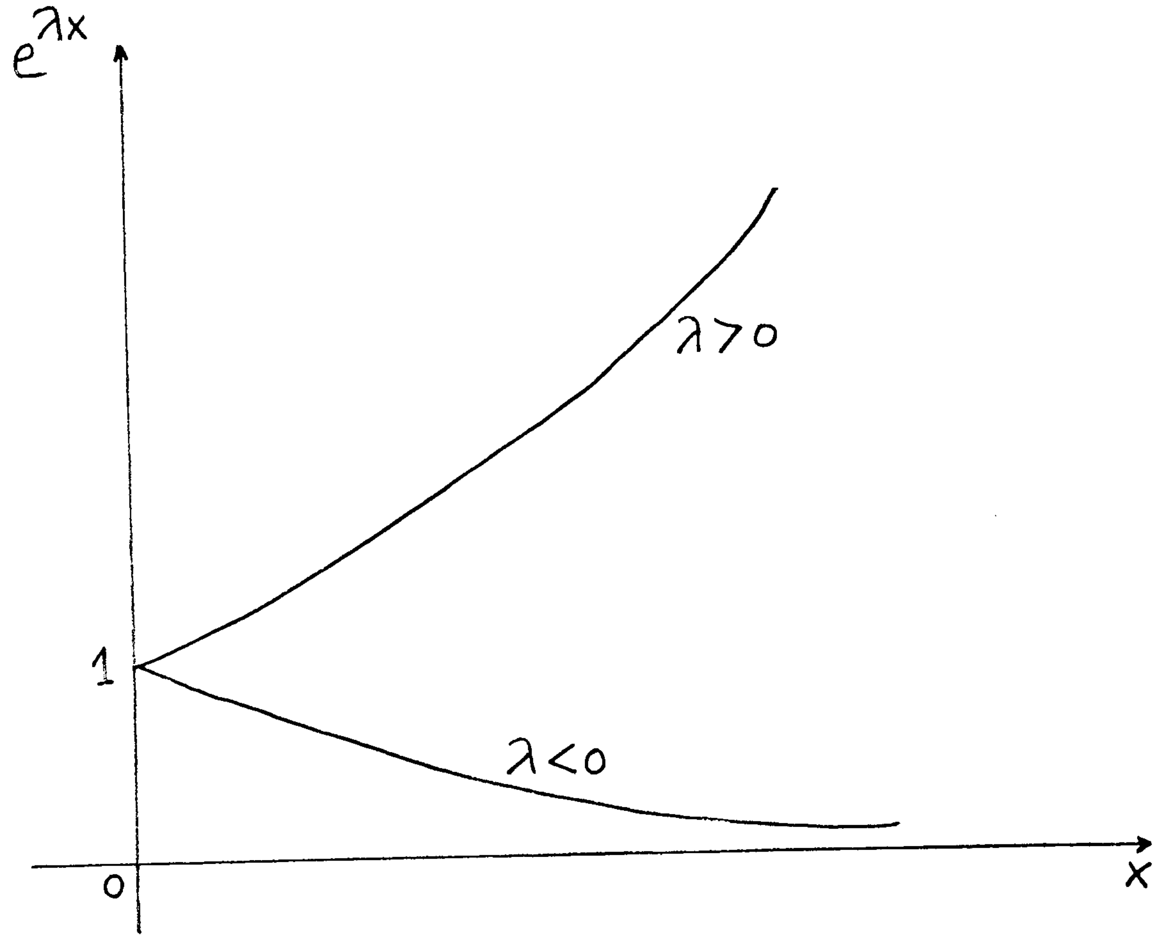


Figure (AI.1)

For positive λ the function increases, and for negative λ, the function decays.

The very least that should be expected of the finite element solutions is that they behave monotonically as $e^{\lambda x}$ for $\lambda > 0$ and $\lambda < 0$.

Each solution will be analysed in turn

(i) (AI.1.2) yields the solution

$$y_i = A + B \left(\frac{2+\lambda h}{2-\lambda h}\right)^i \qquad (AI.1.5)$$

If the behaviour of the exponential term of (AI.1.5) is analysed, it is seen that it only displays the correct monotonic behaviour for $\lambda < 0$ and $\lambda > 0$ if the condition

$$h < \left|\frac{2}{\lambda}\right| \qquad (AI.1.6)$$

is satisfied.

If $\lambda > 0$ and $\lambda h = 2$ then the solution of the difference equation becomes unbounded. If $\lambda < 0$ and $\lambda h = 2$ then the only solution of the difference equation is a constant, and thus two independent boundary conditions could not be fitted.

If (AI.1.6) is contravened by the opposite inequality, then the solution becomes oscillatory, and then although a solution can be found it is useless. Hence (AI.1.6) is the condition for stability of the difference equation.

(ii) It is easily verified that $y = A+Be^{\lambda x}$ satisfies the difference equation (Al.1.3). This means that the difference equation using the exponential element has the same solution as the analytic solution to the differential equation. It is also worth noting that the difference equation for linear elements is contained in the difference equation for the exponential element. To see this multiply (AI.1.3) by

$$e^{-\frac{\lambda h}{2}}$$

to give

$$e^{\frac{\lambda h}{2}} y_{i-1} - 2\cosh\left(\frac{\lambda h}{2}\right) y_i + e^{-\frac{\lambda h}{2}} y_{i+1} = 0$$

Expanding the coefficients of the above equation and neglecting second and higher powers of λ gives

$$\left(1+\frac{\lambda h}{2}\right) y_{i-1} - 2y_i + \left(1-\frac{\lambda h}{2}\right) y_{i+1} = 0$$

But this is precisely the difference equation obtained using linear elements.

To summarise these results there is -

(i) A condition of stability always exists when using linear elements, and if $|\lambda|$ becomes very large, then the condition (AI.1.6) would make linear elements computationally infeasible.

(ii) The exponential element yields a difference equation which has an identical solution to that of the differential equation. The finite element scheme obtained using the exponential element is thus unconditionally stable.

From these conditions, it is reasonable to infer that if an equation like

$$\frac{d^2y}{dx^2} - f(x,y)\frac{dy}{dx} = 0$$

where f(x,y) takes both large positive and large negative values, is to be solved using polynomial elements (in particular linear elements), then a stability condition

$$h < \frac{1}{|f(x,y)|_{max}} \qquad \text{(AI.1.7)}$$

must be applied. In the Navier-Stokes equations this precise situation holds where

$$f(x,y) = ReU$$

U being a flow velocity and Re the Reynolds number. Thus, the condition (AI.1.7) gives the approximate requirement for two dimensional flow

$$Max(h,k) < \frac{1}{|ReU|} \qquad \text{(AI.1.8)}$$

In many situations (gas flows), $Re \simeq 10^6$, the condition (AI.1.8) makes the application of polynomial elements to many fluid flow problems completely infeasible.

APPENDIX TWO

STIFFNESS MATRICES FOR THE SINGLE ELLIPTIC PARTIAL DIFFERENTIAL EQUATION USING POLYNOMIAL ELEMENTS

A2.1

A typical element α^e_{ij} of the stiffness matrix for the partial differential equation, viz.

$$\nabla^2\phi - \lambda\frac{\partial\phi}{\partial x} - \mu\frac{\partial\phi}{\partial y} = 0 \qquad \text{(A2.1.1)}$$

is comprised of the following coefficients -

$$C^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial x} dxdy \quad ; \quad D^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial y} dxdy \ ;$$

$$E^e_{ij} = \iint_e \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy$$

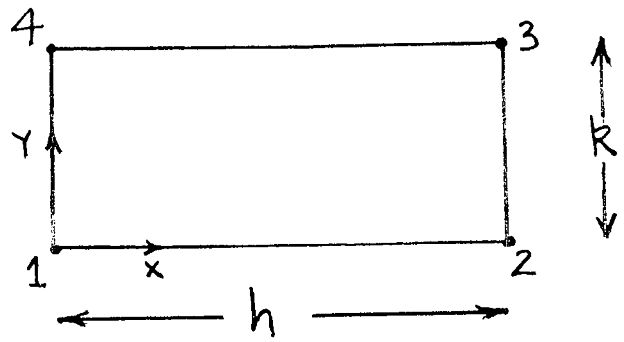
where

$$\alpha^e_{ij} = \lambda C^e_{ij} + \mu D^e_{ij} + E^e_{ij}$$

The coefficients C^e_{ij}, D^e_{ij} and E^e_{ij} for a number of traditional elements are given below:

(i) Four Node Rectangle

For the four node rectangle shown below



The coefficients C^e_{ij}, D^e_{ij} and E^e_{ij}: -

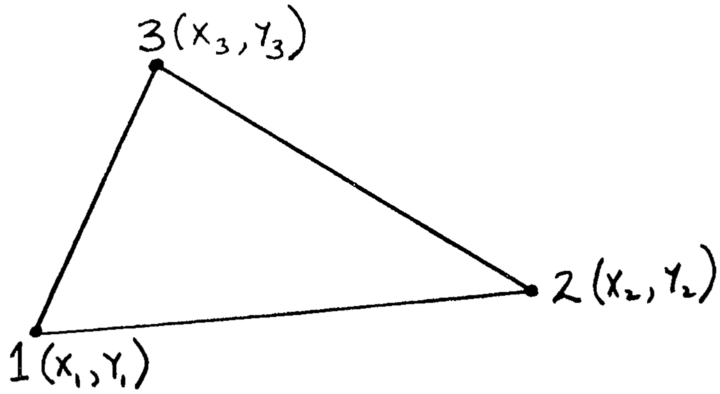
$$C^e_{11} = -\frac{k}{6}\ ; \quad C^e_{12} = -C^e_{11}\ ; \quad C^e_{13} = -\tfrac{1}{2}C^e_{11}\ ; \quad C^e_{14} = \tfrac{1}{2}C^e_{11}$$
$$C^e_{21} = C^e_{11}\ ; \quad C^e_{22} = -C^e_{11}\ ; \quad C^e_{23} = -\tfrac{1}{2}C^e_{11}\ ; \quad C^e_{24} = \tfrac{1}{2}C^e_{11}$$
$$C^e_{31} = \tfrac{1}{2}C^e_{11}\ ; \quad C^e_{32} = -\tfrac{1}{2}C^e_{11}\ ; \quad C^e_{33} = -C^e_{11}\ ; \quad C^e_{34} = C^e_{11}$$
$$C^e_{41} = \tfrac{1}{2}C^e_{11}\ ; \quad C^e_{42} = -\tfrac{1}{2}C^e_{11}\ ; \quad C^e_{43} = -C^e_{11}\ ; \quad C^e_{44} = C^e_{11}$$

$$D^e_{11} = -\frac{h}{6}\ ; \quad D^e_{12} = \tfrac{1}{2}D^e_{11}\ ; \quad D^e_{13} = -\tfrac{1}{2}D^e_{11}\ ; \quad D^e_{14} = -D^e_{11}$$
$$D^e_{21} = \tfrac{1}{2}D^e_{11}\ ; \quad D^e_{22} = D^e_{11}\ ; \quad D^e_{23} = -D^e_{11}\ ; \quad D^e_{24} = -\tfrac{1}{2}D^e_{11}$$
$$D^e_{31} = \tfrac{1}{2}D^e_{11}\ ; \quad D^e_{32} = D^e_{11}\ ; \quad D^e_{33} = -D^e_{11}\ ; \quad D^e_{34} = -\tfrac{1}{2}D^e_{11}$$
$$D^e_{41} = D^e_{11}\ ; \quad D^e_{42} = \tfrac{1}{2}D^e_{11}\ ; \quad D^e_{43} = -\tfrac{1}{2}D^e_{11}\ ; \quad D^e_{44} = -D^e_{11}$$

$$E^e_{11} = \frac{(h^2+k^2)}{3hk}\ ; \quad E^e_{12} = \frac{(h^2-2k^2)}{6hk}\ ; \quad E^e_{13} = -\tfrac{1}{2}E^e_{11}\ ; \quad E^e_{14} = \frac{(k^2-2h^2)}{6hk}$$
$$E^e_{21} = E^e_{12}\ ; \quad E^e_{22} = E^e_{11}\ ; \quad E^e_{23} = E^e_{14}\ ; \quad E^e_{24} = E^e_{11}$$
$$E^e_{31} = E^e_{13}\ ; \quad E^e_{32} = E^e_{23}\ ; \quad E^e_{33} = E^e_{11}\ ; \quad E^e_{34} = E^e_{12}$$
$$E^e_{41} = E^e_{14}\ ; \quad E^e_{42} = E^e_{24}\ ; \quad E^e_{43} = E^e_{34}\ ; \quad E^e_{44} = E^e_{11}$$

(ii) Three Node Triangle

For the three node triangle shown below



define

$$b_1 = y_2 - y_3 \; ; \quad b_2 = y_3 - y_1 \; ; \quad b_3 = y_1 - y_2$$

$$c_1 = x_3 - x_2 \; ; \quad c_2 = x_1 - x_3 \; ; \quad c_3 = x_2 - x_1$$

$$\Delta = \text{area of triangle 123.}$$

Also define

$$\gamma^e_{ij}(P_1, P_2, P_3) \; \{i,j = 1,2,3\}.$$

$$\gamma^e_{11} = \frac{P_1}{6} \; ; \quad \gamma^e_{12} = \frac{P_2}{6} \; ; \quad \gamma^e_{13} = \frac{P_3}{6}$$

$$\gamma^e_{21} = \gamma^e_{11} \; ; \quad \gamma^e_{22} = \gamma^e_{12} \; ; \quad \gamma^e_{23} = \gamma^e_{13}$$

$$\gamma^e_{31} = \gamma^e_{11} \; ; \quad \gamma^e_{32} = \gamma^e_{12} \; ; \quad \gamma^e_{33} = \gamma^e_{13}$$

The coefficients C^e_{ij}, D^e_{ij}, and E^e_{ij} are

$$C^e_{ij} = \gamma^e_{ij}(b_1, b_2, b_3) \qquad i,j = 1,2,3$$

$$D^e_{ij} = \gamma^e_{ij}(c_1, c_2, c_3) \qquad i,j = 1,2,3$$

and

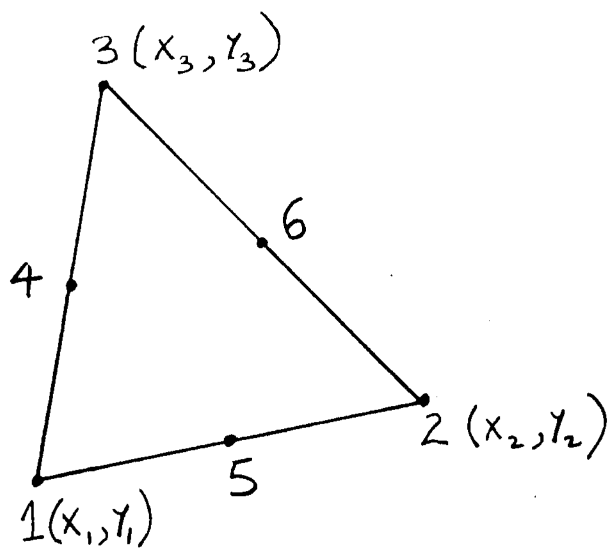
$$E^e_{11} = \frac{(b_1^2+c_1^2)}{4\Delta} \quad ; \quad E^e_{12} = \frac{(b_1 b_2+c_1 c_2)}{4\Delta} \quad ; \quad E^e_{13} = \frac{(b_1 b_3+c_1 c_3)}{4\Delta}$$

$$E^e_{21} = E^e_{12} \quad ; \quad E^e_{22} = \frac{(b_2^2+c_2^2)}{4\Delta} \quad ; \quad E^e_{23} = \frac{(b_2 b_3+c_2 c_3)}{4\Delta}$$

$$E^e_{31} = E^e_{13} \quad ; \quad E^e_{32} = E^e_{23} \quad ; \quad E^e_{33} = \frac{(b_3^2+c_3^2)}{4\Delta}$$

(iii) <u>Six Node Triangle</u>

For the six node triangle shown below:



define

$b_1 = y_2 - y_3 \quad ; \quad b_2 = y_3 - y_1 \quad ; \quad b_3 = y_1 - y_2$

$c_1 = x_3 - x_2 \quad ; \quad c_2 = x_1 - x_3 \quad ; \quad c_3 = x_2 - x_1$

Δ = area of triangle 123

Also define

$O_{ij}(P_1, P_2, P_3) \qquad \{i,j = 1,2,3,\ldots,6\}$

$$\theta^e_{11} = \frac{P_1}{15} \; ; \quad \theta^e_{12} = -\frac{P_2}{30} \; ; \quad \theta^e_{13} = -\frac{P_3}{30} \; ; \quad \theta^e_{14} = \frac{(2P_3-P_1)}{30}$$

$$\theta^e_{15} = \frac{(2P_2-P_1)}{30} ; \quad \theta^e_{16} = -\frac{(P_2+P_3)}{30} ; \quad \theta^e_{21} = -\frac{P_1}{30} \; ; \quad \theta^e_{22} = \frac{P_2}{15}$$

$$\theta^e_{23} = -\frac{P_3}{30} \; ; \quad \theta^e_{24} = -\frac{(P_1+P_3)}{30} ; \quad \theta^e_{25} = \frac{2(P_1-P_2)}{30} ; \quad \theta^e_{26} = \frac{2(P_3-P_2)}{30}$$

$$\theta^e_{31} = -\frac{P_1}{30} \; ; \quad \theta^e_{32} = -\frac{P_2}{30} \; ; \quad \theta^e_{33} = \frac{P_3}{15} \; ; \quad \theta^e_{34} = \frac{(2P_1-P_3)}{30}$$

$$\theta^e_{35} = -\frac{(P_1+P_2)}{30} ; \quad \theta^e_{36} = \frac{(2P_2-P_3)}{30} \; ; \quad \theta^e_{41} = \frac{P_1}{10} \; ; \quad \theta^e_{42} = -\frac{P_2}{30}$$

$$\theta^e_{43} = \frac{P_3}{10} \; ; \quad \theta^e_{44} = \frac{4(P_1+P_3)}{15} ; \quad \theta^e_{45} = \frac{2(P_1+2P_2)}{15} \quad \theta^e_{46} = \frac{2(2P_2+P_3)}{15}$$

$$\theta^e_{51} = \frac{P_1}{10} \; ; \quad \theta^e_{52} = \frac{P_2}{10} \; ; \quad \theta^e_{53} = -\frac{P_3}{30} \; ; \quad \theta^e_{54} = \frac{2(P_1+2P_3)}{15}$$

$$\theta^e_{55} = \frac{4(P_1+P_2)}{15} \quad \theta^e_{56} = \frac{2(P_2+2P_3)}{15} ; \quad \theta^e_{61} = -\frac{P_1}{30} \; ; \quad \theta^e_{62} = \frac{P_2}{10}$$

$$\theta^e_{63} = \frac{P_3}{10} \; ; \quad \theta^e_{64} = \frac{2(2P_1+P_3)}{15} ; \quad \theta^e_{65} = \frac{2(2P_1+P_2)}{15} ; \theta^e_{66} = \frac{4(P_2+P_3)}{15}$$

The coefficients C^e_{ij}, D^e_{ij} and E^e_{ij} are

$$C^e_{ij} = \theta^e_{ij}(b_1, b_2, b_3) \qquad i,j = 1,2,\ldots 6$$

$$D^e_{ij} = \theta^e_{ij}(c_1, c_2, c_3) \qquad i,j = 1,2,\ldots 6$$

and

$E^e_{11} = \frac{(b_1{}^2+c_1{}^2)}{4\Delta}$; $E^e_{12} = -\frac{(b_1b_2+c_1c_2)}{12\Delta}$; $E^e_{13} = -\frac{(b_1b_3+c_1c_3)}{12\Delta}$

$E^e_{14} = \frac{(b_1b_3+c_1c_3)}{3\Delta}$; $E^e_{15} = \frac{(b_1b_2+c_1c_2)}{3\Delta}$; $E^e_{16} = 0$

$E^e_{21} = E^e_{12}$; $E^e_{22} = \frac{(b_2{}^2+c_2{}^2)}{4\Delta}$; $E^e_{23} = -\frac{(b_2b_3+c_2c_3)}{12\Delta}$

$E^e_{24} = 0$; $E^e_{25} = \frac{(b_1b_2+c_1c_2)}{3\Delta}$; $E^e_{26} = \frac{(b_2b_3+c_2c_3)}{3\Delta}$

$E^e_{31} = E^e_{13}$; $E^e_{32} = E^e_{23}$; $E^e_{33} = \frac{(b_3{}^2+c_3{}^2)}{4\Delta}$

$E^e_{34} = \frac{(b_1b_3+c_1c_3)}{3\Delta}$; $E^e_{35} = 0$; $E^e_{36} = \frac{(b_2b_3+c_2c_3)}{3\Delta}$

$E^e_{41} = E^e_{14}$; $E^e_{42} = E^e_{24}$; $E^e_{43} = E^e_{34}$

$$E^e_{44} = \frac{2(b_1{}^2+b_1b_3+b_3{}^2+c_1{}^2+c_1c_3+c_3{}^2)}{3\Delta}$$

$$E^e_{45} = \frac{(b_1{}^2+b_1b_2+b_1b_3+2b_2b_3+c_1{}^2+c_1c_2+c_1c_3+c_1c_3+2c_2c_3)}{3\Delta}$$

$$E^e_{46} = \frac{(b_3{}^2+b_3b_1+b_3b_2+2b_1b_2+c_3{}^2+c_3c_1+c_3c_2+2c_1c_2)}{3\Delta}$$

$E^e_{51} = E^e_{15}$; $E^e_{52} = E^e_{25}$; $E^e_{53} = E^e_{35}$; $E^e_{54} = E^e_{45}$

$$E^e_{55} = \frac{2(b_1{}^2+b_1b_2+b_2{}^2+c_1{}^2+c_1c_2+c_2{}^2)}{3\Delta}$$

$$E^e_{56} = \frac{(b_2{}^2+b_1b_2+b_2b_3+2b_1b_3+c_2{}^2+c_1c_2+c_2c_3+2c_1c_3)}{3\Delta}$$

$E^e_{61} = E^e_{16}$; $E^e_{62} = E^e_{26}$; $E^e_{63} = E^e_{36}$; $E^e_{64} = E^e_{46}$;

$E^e_{65} = E^e_{56}$

$$E^e_{66} = \frac{2(b_2{}^2+b_2b_3+b_3{}^2+c_2{}^2+c_2c_3+c_3{}^2)}{3\Delta}$$

APPENDIX THREE

A NOTE ON THE DERIVATION OF TRADITIONAL FINITE ELEMENT SCHEMES FOR THE NAVIER STOKES EQUATIONS

A3.1

The Navier Stokes equations in non-dimensional form are -

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(A3.1.1)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{Re}\nabla^2 u \qquad \text{(A3.1.2)}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{Re}\nabla^2 v \qquad \text{(A3.1.3)}$$

Galerkins criterion can be used to discretize the Navier Stokes equations to obtain the finite element equations. There are two cases to be considered.

(i) Common Interpolation

By common interpolation it is meant that the same trial function is to be used for the variables u, v and p. Galerkin's discretization process leads to the following finite element equations.

$$\sum_{e} \sum_{j} \left[A^e_{ij} u_j + B^e_{ij} v_j \right] = 0 \qquad \text{(A3.1.4)}$$

$$\sum_{e} \sum_{j} \sum_{k} \left(C^e_{ijk} u_j u_k + D^e_{ijk} u_j v_k \right) + \frac{1}{Re} \sum_{e} \sum_{j} E^e_{ij} u_j + \sum_{e} \sum_{j} A^e_{ij} p_j = \sum_{e} F^e_i$$

(A3.1.5)

$$\sum_{e} \sum_{j} \sum_{k} \left(C^e_{ijk} u_k v_j + D^e_{ijk} v_j v_k \right) + \frac{1}{Re} \sum_{e} \sum_{j} E^e_{ij} v_j + \sum_{e} \sum_{j} B^e_{ij} p_j = \sum_{e} G^e_i$$

(A3.1.6)

The various coefficients are defined below

$$A^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial x} \, dxdy \quad ; \quad B^e_{ij} = \iint_e N_i \frac{\partial N_j}{\partial y} dxdy$$

$$C^e_{ijk} = \iint_e N_i \frac{\partial N_j}{\partial x} N_k dxdy \quad ; \quad D^e_{ijk} = \iint_e N_i \frac{\partial N_j}{\partial y} N_k dxdy$$

$$E^e_{ij} = \iint_e \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy$$

$$F^e_i = \oint N_i \frac{\partial u}{\partial n} ds \quad ; \quad G^e_i = \oint N_i \frac{\partial v}{\partial n} ds.$$

All the coefficients defined above may be calculated analytically for polynomial elements. The non-linear finite element equations may be solved by the Newton Raphson method. An alternative method is to linearise equations (A3.1.4) - (A3.1.6). This gives the following linear algebraic equations to be solved within each iteration.

$$\sum_e \sum_j \left[A^e_{ij} u_j + B^e_{ij} v_j \right] \qquad (A3.1.7)$$

$$\sum_e \sum_j \beta^e_{ij} u_j + \sum_e \sum_j A^e_{ij} p_j = \sum_e F^e_i \qquad (A3.1.8)$$

$$\sum_e \sum_j \beta^e_{ij} v_j + \sum_e \sum_j B^e_{ij} p_j = \sum_r G^e_i \qquad (A3.1.9)$$

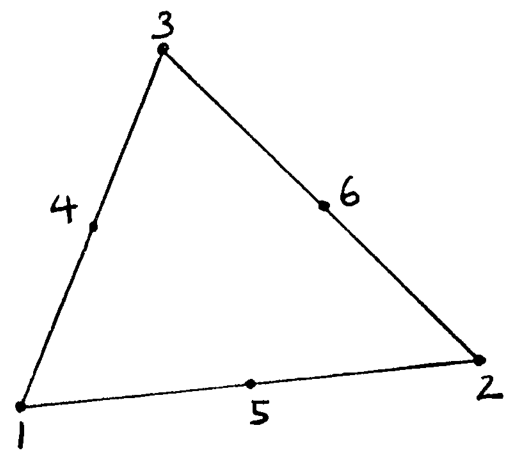
where

$$\beta^e_{ij} = \sum_k \left(C^e_{ijk} \bar{u}_k + D^e_{ijk} \bar{v}_k \right) + \frac{1}{Re} E^e_{ij} \qquad (A3.1.10)$$

Where $\bar{u}_k$ and $\bar{v}_k$ are the values of u and v respectively at node k from the prevous iteration.

(ii) Mixed Interpolation

The essential point of this formulation is that the continuity equation is weighted with a linear shape function but the two momentum equations are weighted with quadratic shape functions. The element to be used is the six node triangle. The pressure is interpolated at the corner nodes only but the velocities are interpolated at all the nodes. Thus over the six node triangle shown below



$$u = \sum_{j=1}^{6} N_j u_j \quad ; \quad v = \sum_{j=1}^{6} N_j v_j \quad ; \quad p = \sum_{j=1}^{3} L_j p_j$$

N_j - quadratic ; L_j - linear

Galerkins process applied to the Navier Stokes equations using mixed interpolation gives the following finite element equations

$$\sum_{e} \sum_{j} a_{ij}^{e} u_j + b_{ij}^{e} v_j = 0. \qquad \text{(A3.1.11)}$$

$$\sum_e \sum_j \sum_k \left(C^e_{ijk} u_j u_k + D^e_{ijk} u_j v_k \right) + \frac{1}{Re} \sum_e \sum_j E^e_{ij} u_j + \sum_e \sum_j g^e_{ij} p_j = \sum_e F^e_i \qquad \text{(A3.1.12)}$$

$$\sum_e \sum_j \sum_k \left(C^e_{ijk} u_k v_j + D^e_{ijk} v_j v_k \right) + \frac{1}{Re} \sum_e \sum_j E^e_{ij} v_j + \sum_e \sum_j h^e_{ij} p_j = \sum_e G^e_i \qquad \text{(A3.1.13)}$$

The various coefficients are defined below

$$a^e_{ij} = \iint_e L_i \frac{\partial N_j}{\partial x} dxdy \quad ; \quad b^e_{ij} = \iint_e L_i \frac{\partial N_j}{\partial y} dxdy$$

$$C^e_{ijk} = \iint_e N_i \frac{\partial N_j}{\partial x} N_k dxdy \quad ; \quad D^e_{ijk} = \iint_e N_i \frac{\partial N_j}{\partial y} N_k \, dxdy$$

$$E^e_{ij} = \iint_e \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy$$

$$g^e_{ij} = \iint_e N_i \frac{\partial L_j}{\partial x} dxdy \quad ; \quad h^e_{ij} = \iint_e N_i \frac{\partial L_j}{\partial y} dxdy$$

$$F^e_i = \oint N_i \frac{\partial u}{\partial n} ds \quad ; \quad G^e_i = \oint N_i \frac{\partial v}{\partial n} ds$$

All the coefficients defined above are easily calculated analytically using area coordinates. Once again linearising equations (A3.1.11) - (A3.1.13) gives the following linear algebraic equations to be solved within each iteration.

$$\sum_e \sum_j \left[a^e_{ij} u_j + b^e_{ij} v_j \right] = 0 \qquad \text{(A3.1.14)}$$

$$\sum_e \sum_j \beta^e_{ij} u_j + \sum_e \sum_j g^e_{ij} p_j = \sum_e F^e_i \qquad \text{(A3.1.15)}$$

$$\sum_e \sum_j \beta^e_{ij} v_j + \sum_e \sum_j h^e_{ij} p_j = \sum_e G^e_i \qquad \text{(A3.1.16)}$$

where β^e_{ij} is given by (A3.1.10).

The author has written a series of Fortran subroutines which compute the element stiffness matrices for the Navier Stokes equations.

APPENDIX FOUR

SHAPE FUNCTIONS AND STIFFNESS MATRIX FOR THE EIGHT NODE MIXED ELEMENTS

The shape functions $N_i(x,y)$ $(i = 1,2,\ldots,8)$ for a trial function of the form

$$\phi = a_1+a_2e^{\lambda x}+a_3e^{\mu y}+a_4e^{\lambda x+\mu y}+a_5x^2+a_6y^2+a_7x^2y+a_8xy^2$$

over the eight node rectangular element shown below -

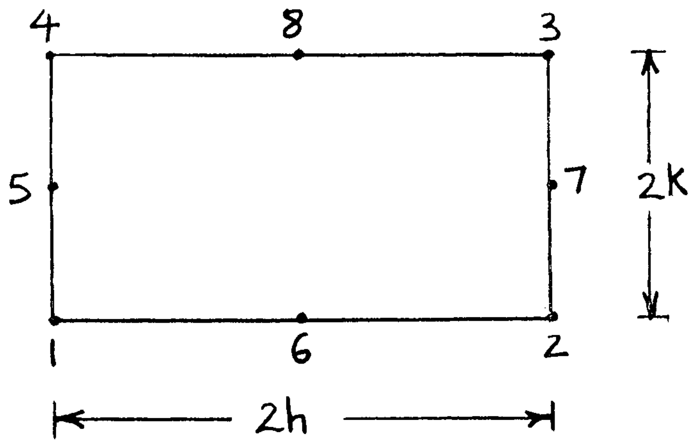


Figure (A4.1)

$$N_i(x,y) = \alpha_{i1}+\alpha_{i2}e^{\lambda x}+\alpha_{i3}e^{\mu y}+\alpha_{i4}e^{\lambda x+\mu y}+\alpha_{i5}x^2+\alpha_{i6}y^2+\alpha_{i7}x^2y$$

$$+ \alpha_{i8}xy^2$$

$i = 1,2,\ldots, 8.$

The sixty-four coefficients α_{ij} are given below. First define the following quantities: -

$$\delta_1 = e^{\lambda h} \quad ; \quad \delta_2 = e^{\mu k} \quad ; \quad \delta_3 = (5+\delta_1+\delta_2-3\delta_1\delta_2)$$

$$\delta_4 = (\delta_1-3) \quad ; \quad \delta_5 = (\delta_2-3) ; \quad \delta_6 = 1-\delta_1{}^2$$

$$\delta_7 = 1-\delta_2{}^2 \quad ; \quad \delta_8 = 1-\delta_1 \quad ; \quad \delta_9 = 1-\delta_2$$

$$\delta_{10} = 1+\delta_1 \quad ; \quad \delta_{11} = 1+\delta_2$$

Then

$$\alpha_{11} = \frac{(1-\delta_2{}^2\delta_6)}{\delta_6\delta_7} + \frac{4\delta_2}{\delta_5\delta_7} + \frac{3}{\delta_1\delta_8} + \frac{4(\delta_1+\delta_2+2\delta_1\delta_2)}{\delta_3\delta_6\delta_7}$$

$$\alpha_{12} = (-1)\left[\frac{3}{\delta_4\delta_8} + \frac{5}{\delta_3\delta_8\delta_9}\right]$$

$$\alpha_{13} = (-1)\left[\frac{(4\delta_2\delta_6+\delta_1{}^2\delta_5)}{\delta_5\delta_6\delta_7} + \frac{4(\delta_1+\delta_2+2\delta_1\delta_2)}{\delta_3\delta_6\delta_7}\right]$$

$$\alpha_{14} = \frac{5}{\delta_3\delta_8\delta_9} \quad ; \quad \alpha_{15} = -\frac{\delta_1}{h^2\delta_4} \quad ; \quad \alpha_{16} = -\frac{\delta_2}{k^2\delta_5}$$

$$\alpha_{17} = \frac{(2\delta_2-3\delta_1-\delta_1\delta_2)}{2h^2k\delta_3} \quad ; \quad \alpha_{18} = \frac{(2\delta_1-3\delta_2-\delta_1\delta_2)}{2hk^2\delta_3}$$

$$\alpha_{21} = \frac{1}{\delta_4\delta_8} \quad \frac{1}{\delta_3\delta_8\delta_9} \quad ; \quad \alpha_{22} = -\alpha_{21}$$

$$\alpha_{23} = \frac{1}{\delta_3\delta_8\delta_9} \quad ; \alpha_{24} = -\alpha_{23} \quad ; \quad \alpha_{25} = -\frac{1}{h^2\delta_4}$$

$$\alpha_{26} = 0 \quad ; \quad \alpha_{27} = -\frac{(3+\delta_2-2\delta_1\delta_2)}{2h^2k\delta_3}$$

$$\alpha_{28} = -\frac{(\delta_1\delta_2-\delta_2-2)}{2hk^2\delta_3}$$

$$\alpha_{31} = -\frac{3}{\delta_3\delta_8\delta_9} \quad ; \quad \alpha_{32} = -\alpha_{31}$$

$$\alpha_{33} = \alpha_{32} \quad ; \quad \alpha_{34} = \alpha_{31}$$

$$\alpha_{35} = \alpha_{36} = 0 \quad ; \quad \alpha_{37} = \frac{(1-\delta_2+2\delta_1)}{2h^2k\delta_3}$$

$$\alpha_{38} = \frac{(1-\delta_1+2\delta_2)}{2hk^2\delta_3}$$

$$\alpha_{41} = \frac{(4\delta_6-\delta_1{}^2\delta_5)}{\delta_5\delta_6\delta_7} + \frac{4(1-\delta_1\delta_2)}{\delta_3\delta_6\delta_7} ; \quad \alpha_{42} = \frac{1}{\delta_3\delta_8\delta_9}$$

$$\alpha_{43} = -\alpha_{41} \quad ; \quad \alpha_{44} = -\alpha_{42}$$

$$\alpha_{45} = 0 \quad ; \quad \alpha_{46} = \frac{1}{k^2\delta_5}$$

$$\alpha_{47} = \frac{(2+\delta_1\delta_9)}{2h^2k\delta_3} \quad ; \quad \alpha_{48} = \frac{\delta_1(2\delta_2-1)-3}{2hk^2\delta_3}$$

$$\alpha_{51} = -4\left[\frac{1}{\delta_5\delta_9} + \frac{1}{\delta_3\delta_8\delta_9}\right] \quad ; \quad \alpha_{52} = \frac{4}{\delta_3\delta_8\delta_9}$$

$$\alpha_{53} = -\alpha_{51} \quad ; \quad \alpha_{54} = -\alpha_{52}$$

$$\alpha_{55} = 0 \quad ; \quad \alpha_{56} = \frac{\delta_{11}}{k^2\delta_5}$$

$\alpha_{57} = -\frac{\delta_{11}\delta_8}{h^2k\delta_3}$; $\alpha_{58} = -\frac{\delta_4\delta_{11}}{2hk^2\delta_3}$

$\alpha_{61} = -4\left[\frac{1}{\delta_4\delta_8} + \frac{1}{\delta_3\delta_8\delta_9}\right]$; $\alpha_{62} = -\alpha_{61}$

$\alpha_{63} = \frac{4}{\delta_3\delta_8\delta_9}$; $\alpha_{64} = -\alpha_{63}$

$\alpha_{65} \quad \frac{\delta_{10}}{2\delta_4}$; $\alpha_{66} = 0$

$\alpha_{67} = -\frac{\delta_5\delta_{10}}{2^2k\delta_3}$; $\alpha_{68} = -\frac{\delta_9\delta_{10}}{hk^2\delta_3}$

$\alpha_{71} = \frac{4}{\delta_3\delta_8\delta_9}$; $\alpha_{72} = \alpha_{73} = -\alpha_{71}$

$\alpha_{74} = \alpha_{71}$; $\alpha_{75} = \alpha_{76} = 0$

$\alpha_{77} = \frac{\delta_8\delta_{11}}{h^2k\delta_3}$; $\alpha_{78} = \frac{\delta_4\delta_{11}}{2hk^2\delta_3}$

$\alpha_{81} = \frac{4}{\delta_3\delta_8\delta_9}$; $\alpha_{82} = \alpha_{83} = -\alpha_{81}$

$\alpha_{84} = \alpha_{81}$; $\alpha_{85} = \alpha_{86} = 0$

$\alpha_{87} = \frac{\delta_5\delta_{10}}{2h^2k\delta_3}$; $\alpha_{88} = \frac{\delta_9\delta_{10}}{hk^2\delta_3}$

The shape functions $M_j(x,y)$ ($j = 1,2,3,4$) for a trial function of the form

$$\phi = a + bx + cy + dxy$$

over the four node rectangle 1234 shown in Figure (A4.1) are of the form

$$M_j(x,y) = \beta_{j1}+\beta_{j2} + \beta_{j3}y + \beta_{j4}xy \qquad (j = 1,2,3,4)$$

The sixteen coefficients β_{ij} are given below: -

$$\beta_{11} = 1 \;;\; \beta_{12} = -\frac{1}{2h} \;;\; \beta_{13} = -\frac{1}{2k} \;;\; \beta_{14} = \frac{1}{4hk} \;;$$
$$\beta_{21} = 0 \;;\; \beta_{22} = \beta_{12} \;;\; \beta_{23} = 0 \;;\; \beta_{24} = -\beta_{14}$$
$$\beta_{31} = 0 \;;\; \beta_{32} = 0 \;;\; \beta_{33} = 0 \;;\; \beta_{34} = \beta_{14}$$
$$\beta_{41} = 0 \;;\; \beta_{42} = 0 \;;\; \beta_{43} = \frac{1}{2k} \;;\; \beta_{44} = -\beta_{14}$$

For convenience we now define the quantities $\tau_i(\xi,\eta)$ $\{i = 1,2,...,9\}$ as follows:

$$\tau_1(\xi,\eta) = 2\eta \;;\quad \tau_2(\xi,\eta) = 2\eta^2 \;;\quad \tau_3(\xi,\eta) = \frac{8}{3}\eta^2;\quad \tau_4(\xi,\eta) = 4\eta^2$$
$$\tau_5(\xi,\eta) = \frac{32}{5}\eta^5 \;;\quad \tau_6(\xi,\eta) = \frac{1}{\xi}(e^{2\xi\eta}-1) \;;\quad \tau_7(\xi,\eta) = \frac{1}{2\xi}(e^{4\xi\eta}-1)$$
$$\tau_8(\xi,\eta) = \frac{1}{\xi^2}[e^{2\xi\eta}(2\xi\eta-1)+1\;]$$
$$\tau_9(\xi,\eta) = \frac{2}{\xi^3}[(2\xi^2\eta^2-2\xi\eta+1)e^{2\xi\eta}-1]$$

Let

$$\varepsilon_i = \tau_i(\lambda,h)$$
$$\varepsilon_i' = \tau_i(\mu,k)$$
$$i = 1,2,...,9$$

All the coefficients (as defined in 6.9) necessary to compute the element stiffness matrix for the Navier Stokes equations using the 8-node mixed element are given below in terms of α_{ij}, β_{ij}, ε_i and ε_i'.

(i) $$a_{rs}^{e} = \iint_{e} M_r \frac{\partial N_s}{\partial x}\, dxdy$$

$$\begin{aligned} a_{rs}^{e} = {} & \beta_{r1}(\lambda\alpha_{s2}\varepsilon_6\varepsilon_1' + \lambda\alpha_{s4}\varepsilon_6\varepsilon_6' + 2\alpha_{s5}\varepsilon_2\varepsilon_1' + 2\alpha_{s7}\varepsilon_2\varepsilon_2' + \alpha_{s8}\varepsilon_3'\varepsilon_1) \\ & + \beta_{r2}(\lambda\alpha_{s2}\varepsilon_8\varepsilon_1' + \lambda\alpha_{s4}\varepsilon_8\varepsilon_6' + 2\alpha_{s5}\varepsilon_3\varepsilon_1' + 2\alpha_{s7}\varepsilon_3\varepsilon_2' + \alpha_{s8}\varepsilon_2\varepsilon_3') \\ & + \beta_{r3}(\lambda\alpha_{s2}\varepsilon_2'\varepsilon_6 + \lambda\alpha_{s4}\varepsilon_6\varepsilon_8' + 2\alpha_{s5}\varepsilon_2\varepsilon_2' + 2\alpha_{s7}\varepsilon_2\varepsilon_3' + \alpha_{s8}\varepsilon_4'\varepsilon_1) \\ & + \beta_{r4}(\lambda\alpha_{s2}\varepsilon_8\varepsilon_2' + \lambda\alpha_{s4}\varepsilon_8\varepsilon_8' + 2\alpha_{s5}\varepsilon_3\varepsilon_2' + 2\alpha_{s7}\varepsilon_3\varepsilon_3' + \alpha_{s8}\varepsilon_2\varepsilon_4') \end{aligned}$$

$r = 1,2,3,4 \quad ; \quad s = 1,2,\ldots 8$

(ii) $$b_{rs}^{e} = \iint_{e} M_r \frac{\partial N_s}{\partial y}\, dxdy$$

$$\begin{aligned} b_{rs}^{e} = {} & \beta_{r1}(\mu\alpha_{s3}\varepsilon_6'\varepsilon_1 + \mu\alpha_{s4}\varepsilon_6\varepsilon_6' + 2\alpha_{s6}\varepsilon_2'\varepsilon_1 + \alpha_{s7}\varepsilon_2\varepsilon_1' + 2\alpha_{s8}\varepsilon_2\varepsilon_2') \\ & + \beta_{r2}(\mu\alpha_{s3}\varepsilon_2\varepsilon_6' + \mu\alpha_{s4}\varepsilon_8\varepsilon_6' + 2\alpha_{s6}\varepsilon_2\varepsilon_2' + \alpha_{s7}\varepsilon_4\varepsilon_1' + 2\alpha_{s8}\varepsilon_3\varepsilon_2') \\ & + \beta_{r3}(\mu\alpha_{s3}\varepsilon_8'\varepsilon_1 + \mu\alpha_{s4}\varepsilon_6\varepsilon_8' + 2\alpha_{s6}\varepsilon_3'\varepsilon_1 + \alpha_{s7}\varepsilon_3\varepsilon_2' + 2\alpha_{s8}\varepsilon_2\varepsilon_3') \\ & + \beta_{r4}(\mu\alpha_{s3}\varepsilon_2\varepsilon_8' + \mu\alpha_{s4}\varepsilon_8\varepsilon_8' + 2\alpha_{s6}\varepsilon_2\varepsilon_3' + \alpha_{s7}\varepsilon_4\varepsilon_2' + 2\alpha_{s8}\varepsilon_3\varepsilon_3') \end{aligned}$$

$r = 1,2,3,4 \quad ; \quad s = 1,2,\ldots 8$

(iii) $$c^e_{ts} = \iint_e N_t \frac{\partial N_s}{\partial x} dxdy$$

$$c^e_{ts} = \alpha_{t1}(\lambda\alpha_{s2}\varepsilon_6\varepsilon'_1+\lambda\alpha_{s4}\varepsilon_6\varepsilon'_6+2\alpha_{s5}\varepsilon_2\varepsilon'_1+2\alpha_{s7}\varepsilon_2\varepsilon'_2+\alpha_{s8}\varepsilon'_3\varepsilon_1)$$

$$+ \alpha_{t2}(\lambda\alpha_{s2}\varepsilon_7\varepsilon'_1+\lambda\alpha_{s4}\varepsilon_7\varepsilon'_6+2\alpha_{s5}\varepsilon_2\varepsilon'_6+2\alpha_{s7}\varepsilon'_8\varepsilon_2+\alpha_{s8}\varepsilon'_9\varepsilon_1)$$

$$+ \alpha_{t3}(\lambda\alpha_{s2}\varepsilon_6\varepsilon'_6+\lambda\alpha_{s4}\varepsilon_6\varepsilon'_7+2\alpha_{s5}\varepsilon_2\varepsilon'_6+2\alpha_{s7}\varepsilon'_8\varepsilon_2+\alpha_{s8}\varepsilon'_9\varepsilon_1)$$

$$+ \alpha_{t4}(\lambda\alpha_{s2}\varepsilon_7\varepsilon'_6+\lambda\alpha_{s4}\varepsilon_7\varepsilon'_7+2\alpha_{s5}\varepsilon_8\varepsilon'_6+2\alpha_{s7}\varepsilon_8\varepsilon'_8+\alpha_{s8}\varepsilon_6\varepsilon'_9)$$

$$+ \alpha_{t5}(\lambda\alpha_{s2}\varepsilon_9\varepsilon'_1+\lambda\alpha_{s4}\varepsilon_9\varepsilon'_6+2\alpha_{s5}\varepsilon_4\varepsilon'_1+2\alpha_{s7}\varepsilon_4\varepsilon'_2+\alpha_{s8}\varepsilon_3\varepsilon'_3)$$

$$+ \alpha_{t6}(\lambda\alpha_{s2}\varepsilon_6\varepsilon'_3+\lambda\alpha_{s4}\varepsilon'_9\varepsilon_6+2\alpha_{s5}\varepsilon_2\varepsilon'_3+2\alpha_{s7}\varepsilon_2\varepsilon'_4+\alpha_{s8}\varepsilon'_5\varepsilon_1)$$

$$+ \alpha_{t7}(\lambda\alpha_{s2}\varepsilon_9\varepsilon'_2+\lambda\alpha_{s4}\varepsilon_9\varepsilon'_8+2\alpha_{s5}\varepsilon_4\varepsilon'_2+2\alpha_{s7}\varepsilon_4\varepsilon'_3+\alpha_{s8}\varepsilon_3\varepsilon'_4)$$

$$+ \alpha_{t8}(\lambda\alpha_{s2}\varepsilon_8\varepsilon'_3+\lambda\alpha_{s4}\varepsilon_8\varepsilon'_9+2\alpha_{s5}\varepsilon_3\varepsilon'_3+2\alpha_{s7}\varepsilon_3\varepsilon'_4+\alpha_{s8}\varepsilon_2\varepsilon'_5)$$

$$s,t = 1,2,\ldots,8$$

(iv) $$d^e_{ts} = \iint_e N_t \frac{\partial N_s}{\partial y} dxdy$$

$$\begin{aligned}
d^e_{ts} = {} & \alpha_{t1}(\mu\alpha_{s3}\varepsilon'_6\varepsilon_1+\mu\alpha_{s4}\varepsilon_6\varepsilon'_6+2\alpha_{s6}\varepsilon_6\varepsilon'_2+\alpha_{s7}\varepsilon_9\varepsilon'_1+2\alpha_{s8}\varepsilon_8\varepsilon'_2) \\
& + \alpha_{t2}(\mu\alpha_{s3}\varepsilon_6\varepsilon'_6+\mu\alpha_{s4}\varepsilon_7\varepsilon'_6+2\alpha_{s6}\varepsilon'_8\varepsilon_1+\alpha_{s7}\varepsilon_3\varepsilon'_6+2\alpha_{s8}\varepsilon_2\varepsilon'_8) \\
& + \alpha_{t3}(\mu\alpha_{s3}\varepsilon'_7\varepsilon_1+\mu\alpha_{s4}\varepsilon_6\varepsilon'_7+2\alpha_{s6}\varepsilon'_8\varepsilon_1+\alpha_{s7}\varepsilon_3\varepsilon'_6+2\alpha_{s8}\varepsilon_2\varepsilon'_8) \\
& + \alpha_{t4}(\mu\alpha_{s3}\varepsilon_6\varepsilon'_7+\mu\alpha_{s4}\varepsilon_7\varepsilon'_7+2\alpha_{s6}\varepsilon_6\varepsilon'_8+\alpha_{s7}\varepsilon_9\varepsilon'_6+2\alpha_{s8}\varepsilon_8\varepsilon'_8) \\
& + \alpha_{t5}(\mu\alpha_{s3}\varepsilon_3\varepsilon'_6+\mu\alpha_{s4}\varepsilon_9\varepsilon'_6+2\alpha_{s6}\varepsilon_3\varepsilon'_2+\alpha_{s7}\varepsilon_5\varepsilon'_1+2\alpha_{s8}\varepsilon_4\varepsilon'_2) \\
& + \alpha_{t6}(\mu\alpha_{s3}\varepsilon'_9\varepsilon_1+\mu\alpha_{s4}\varepsilon'_9\varepsilon_6+2\alpha_{s6}\varepsilon'_4\varepsilon_1+\alpha_{s7}\varepsilon_3\varepsilon'_3+2\alpha_{s8}\varepsilon_2\varepsilon'_4) \\
& + \alpha_{t7}(\mu\alpha_{s3}\varepsilon_3\varepsilon'_8+\mu\alpha_{s4}\varepsilon_9\varepsilon'_8+2\alpha_{s6}\varepsilon_3\varepsilon'_3+\alpha_{s7}\varepsilon_5\varepsilon'_2+2\alpha_{s8}\varepsilon_4\varepsilon'_3) \\
& + \alpha_{t8}(\mu\alpha_{s3}\varepsilon_2\varepsilon'_9+\mu\alpha_{s4}\varepsilon_8\varepsilon'_9+2\alpha_{s6}\varepsilon_2\varepsilon'_4+\alpha_{s7}\varepsilon_4\varepsilon'_3+2\alpha_{s8}\varepsilon_3\varepsilon'_4)
\end{aligned}$$

$$s,\ t = 1,2,3,\ldots,8.$$

(v) $$e^e_{ts} = \iint_e \left(\frac{\partial N_t}{\partial x}\frac{\partial N_s}{\partial x} + \frac{\partial N_t}{\partial y}\frac{\partial N_s}{\partial y} \right) dxdy$$

$$\begin{aligned}
e^e_{ts} = & \lambda\alpha_{t2}(\lambda\alpha_{s2}\varepsilon_7\varepsilon'_1+\lambda\alpha_{s4}\varepsilon_7\varepsilon'_6+2\alpha_{s5}\varepsilon_8\varepsilon'_1+2\alpha_{s7}\varepsilon_8\varepsilon'_2+\alpha_{s8}\varepsilon_6\varepsilon'_3) \\
& +\lambda\alpha_{t4}(\lambda\alpha_{s2}\varepsilon_7\varepsilon'_6+\lambda\alpha_{s4}\varepsilon_7\varepsilon'_7+2\alpha_{s5}\varepsilon_8\varepsilon'_6+2\alpha_{s7}\varepsilon_8\varepsilon'_8+\alpha_{s8}\varepsilon_6\varepsilon'_9) \\
& +2\alpha_{t5}(\lambda\alpha_{s2}\varepsilon_8\varepsilon'_1+\lambda\alpha_{s4}\varepsilon_8\varepsilon'_6+2\alpha_{s5}\varepsilon_3\varepsilon'_1+2\alpha_{s7}\varepsilon_3\varepsilon'_2+\alpha_{s8}\varepsilon_2\varepsilon'_3) \\
& +2\alpha_{t7}(\lambda\alpha_{s2}\varepsilon_8\varepsilon'_2+\lambda\alpha_{s4}\varepsilon_8\varepsilon'_8+2\alpha_{s5}\varepsilon_3\varepsilon'_2+2\alpha_{s7}\varepsilon_3\varepsilon'_3+\alpha_{s8}\varepsilon_2\varepsilon'_4) \\
& +\alpha_{t8}(\lambda\alpha_{s2}\varepsilon_6\varepsilon'_3+\lambda\alpha_{s4}\varepsilon_6\varepsilon'_9+2\alpha_{s5}\varepsilon'_3\varepsilon_2+2\alpha_{s7}\varepsilon_2\varepsilon'_4+\alpha_{s8}\varepsilon'_5\varepsilon_1) \\
& +\mu\alpha_{t3}(\mu\alpha_{s3}\varepsilon'_7\varepsilon_1+\mu\alpha_{s4}\varepsilon_6\varepsilon'_7+2\alpha_{s6}\varepsilon'_8\varepsilon_1+\alpha_{s7}\varepsilon_3\varepsilon'_6+2\alpha_{s8}\varepsilon_2\varepsilon'_8) \\
& +\mu\alpha_{t4}(\mu\alpha_{s3}\varepsilon_6\varepsilon'_7+\mu\alpha_{s4}\varepsilon_7\varepsilon'_7+2\alpha_{s6}\varepsilon_6\varepsilon'_8+\alpha_{s7}\varepsilon_9\varepsilon'_6+2\alpha_{s8}\varepsilon_8\varepsilon'_8) \\
& +2\alpha_{t6}(\mu\alpha_{s3}\varepsilon'_8\varepsilon_1+\mu\alpha_{s4}\varepsilon_6\varepsilon'_8+2\alpha_{s6}\varepsilon'_3\varepsilon_1+\alpha_{s9}\varepsilon_3\varepsilon'_2+2\alpha_{s8}\varepsilon_2\varepsilon'_3) \\
& +\alpha_{t7}(\mu\alpha_{s3}\varepsilon_3\varepsilon'_6+\mu\alpha_{s4}\varepsilon_9\varepsilon'_6+2\alpha_{s6}\varepsilon_3\varepsilon'_2+\alpha_{s7}\varepsilon_5\varepsilon'_1+2\alpha_{s8}\varepsilon_4\varepsilon'_2) \\
& +2\alpha_{t8}(\mu\alpha_{s3}\varepsilon_2\varepsilon'_8+\mu\alpha_{s4}\varepsilon_8\varepsilon'_8+2\alpha_{s6}\varepsilon_2\varepsilon'_3+\alpha_{s7}\varepsilon_4\varepsilon'_2+2\alpha_{s8}\varepsilon_3\varepsilon'_3)
\end{aligned}$$

$$s,t = 1,2,\ldots,8$$

(vi) $$k^e_{tr} = \iint_e \frac{\partial M_r}{\partial x} dxdy$$

$$k^e_{tr} = \beta_{r2}(\alpha_{t1}\varepsilon_1\varepsilon_1' + \alpha_{t2}\varepsilon_6\varepsilon_1' + \alpha_{t3}\varepsilon_6'\varepsilon_1 + \alpha_{t4}\varepsilon_6\varepsilon_6' + \alpha_{t5}\varepsilon_3\varepsilon_1'$$
$$+ \alpha_{t6}\varepsilon_3'\varepsilon_1 + \alpha_{t7}\varepsilon_3\varepsilon_2' + \alpha_{t8}\varepsilon_2\varepsilon_3')$$
$$+ \beta_{r4}(\alpha_{t1}\varepsilon_2'\varepsilon_1 + \alpha_{t2}\varepsilon_2'\varepsilon_6 + \alpha_{t3}\varepsilon_8'\varepsilon_1 + \alpha_{t4}\varepsilon_6\varepsilon_8' + \alpha_{t5}\varepsilon_2'\varepsilon_3$$
$$+ \alpha_{t6}\varepsilon_4'\varepsilon_1 + \alpha_{t7}\varepsilon_3\varepsilon_3' + \alpha_{78}\varepsilon_2\varepsilon_4')$$

$$r = 1,2,3,4 \quad ; \qquad t = 1,2,\ldots,8.$$

(vii) $$\ell^e_{tr} = \iint_e N_t \frac{\partial M_r}{\partial y} dxdy$$

$$\ell^e_{tr} = \beta_{r3}(\alpha_{t1}\varepsilon_1\varepsilon_1' + \alpha_{t2}\varepsilon_6\varepsilon_1' + \alpha_{t3}\varepsilon_6'\varepsilon_1 + \alpha_{t4}\varepsilon_6\varepsilon_6' + \alpha_{t5}\varepsilon_3\varepsilon_1'$$
$$+ \alpha_{t6}\varepsilon_3'\varepsilon_1 + \alpha_{t7}\varepsilon_3\varepsilon_2' + \alpha_{t8}\varepsilon_2\varepsilon_3')$$
$$+ \beta_{r4}(\alpha_{t1}\varepsilon_2\varepsilon_1' + \alpha_{t2}\varepsilon_8\varepsilon_1' + \alpha_{t3}\varepsilon_2\varepsilon_6' + \alpha_{t4}\varepsilon_8\varepsilon_6' + \alpha_{t5}\varepsilon_4\varepsilon_1'$$
$$+ \alpha_{t6}\varepsilon_2\varepsilon_3' + \alpha_{t7}\varepsilon_4\varepsilon_2' + \alpha_{t8}\varepsilon_3\varepsilon_3')$$

$$r = 1,2,3,4 \quad ; \qquad t = 1,2,\ldots,8.$$

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