Maria Letizia Bertotti*, Amit K Chattopadhyay and Giovanni Modanese

Stochastic models with multiplicative noise for economic inequality and mobility

https://doi.org/10.1515/ijnsns-2017-0228
Received October 20, 2017; accepted March 16, 2021; published online April 12, 2021

Abstract: In this article, we discuss a dynamical stochastic model that represents the time evolution of income distribution of a population, where the dynamics develops from an interplay of multiple economic exchanges in the presence of multiplicative noise. The model remit stretches beyond the conventional framework of a Langevin-type kinetic equation in that our model dynamics is self-consistently constrained by dynamical conservation laws emerging from population and wealth conservation. This model is numerically solved and analysed to evaluate the inequality of income in correlation to other relevant dynamical parameters like the *mobility* M and the *total income* μ. Inequality is quantified by the Gini index G. In particular, correlations between any two of the mobility index M and/or the total income μ with the Gini index G are investigated and compared with the analogous quantities resulting from an additive noise model.

Keywords: economic inequality; multiplicative noise; social mobility; stochastic model.

1 Introduction

Statistical physics and kinetic theory approaches have been proposed in recent years for the description of economic exchanges and market societies; see for example [1–7]. In these approaches, individuals trading with each other are identified as particles or gas molecules which undergo collisions. The methods employed have proved useful also in such socio-economic contexts to investigate the emergence of macroscopic features from a whole of microscopic interactions. With this perspective, some mathematically founded market economy models, characterized by the ability to also incorporate taxation and redistribution processes, have been proposed and studied in [8, 9]. In these papers, society is equated to a system composed by a large number of heterogeneous individuals who exchange money through binary and other nonlinear interactions and are divided into a finite number n of income classes. The models are expressed by a system of n nonlinear ordinary differential equations of the kinetic-discretized Boltzmann type, involving transition probabilities relative to the jumps of individuals from a class to another. The specification of these probabilities and of the parameters which define the trading rules, including the tax rates pertaining to different income classes and other properties of the system, determines the dynamics. Collective features like the income profile and related indicators like the Gini index – a widespread measure of economic inequality – result from the interplay of a range of such interactions. Due to the presence of the mentioned transition probabilities, the models involve some randomness [10], but the differential equations governing the evolution of the fractions of individuals in the classes are deterministic.

***Corresponding author**: **Maria Letizia Bertotti**, Free University of Bozen-Bolzano, Faculty of Science and Technology, 39100 Bolzano, Italy, E-mail: MariaLetizia.Bertotti@unibz.it
Amit K Chattopadhyay, Aston University, Department of Mathematics, Birmingham B4 7ET, UK, E-mail: A.K.Chattopadhyay@aston.ac.uk
Giovanni Modanese, Free University of Bozen-Bolzano, Faculty of Science and Technology, 39100 Bolzano, Italy, E-mail: Giovanni.Modanese@unibz.it

In real world, however, the time evolution of an economic system is governed not only by fixed rules and parameters: it is subject to the effects of unpredictable perturbing factors as well. To consider the influence of these factors, we recently introduced a Langevin-type kinetic model [11], incorporating an Ito-type additive noise term into the set of dynamical equations. Some numerical simulations provided evidence of patterns consistent with those emerging in the deterministic problem [12], also in agreement with previously established empirical results [13–15]: in particular, they exhibited a negative correlation between economic inequality and social mobility. With reference to the case without income conservation, they reported a positive correlation between the Gini index and the total income. We regard this as a sign of reliability of the models. The noise additivity is a perceived drawback though, as it allows fluctuations of the class populations which are independent from the populations themselves, which is unrealistic, especially for the small populations of the classes with large income.

The goal of this paper is to overcome this limitation by considering instead a multiplicative noise term. This requires a much more subtle procedure than that proposed in [11], since we are dealing with a system of n Langevin equations, where n can be quite large, and the stochastic variables must satisfy dynamical constraints. In this sense, the procedure we implement has a more general technical value for the stochastic description of complex systems, extending beyond the specific applications described here.

The paper is organized as follows. In the next section, we introduce the structure of the Langevin-type kinetic model. Different choices for the construction of the noise term of this structure allow us to formulate different models. Here, in particular, we define two of them: one, in the first subsection, for which only conservation of the total population holds true, and another, in a second subsection, for which both conservation of total population and total income hold true. The features of the evolution in time of the solutions of these two models are discussed in Section 3. Attention is focused on the sign of the correlations between income inequality, mobility and total income under different conditions. If the total income μ is constant in time and not too large, the correlations between the Gini index and an indicator quantifying social mobility is negative. When income conservation does not hold true, the sign of the correlation between the total income and the Gini index can either be positive or negative, depending on the magnitude of μ. Section 4 summarizes these facts and some directions for future research and an appendix provides the proof of a claim contained in Subsection 2.1.

2 From a deterministic to a Langevin-type kinetic model

A simple model describing monetary exchanges between pairs of individuals in a society divided into n income classes can be formulated through a system of differential equations of the form[1]

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}(t) = \sum_{h,k=1}^{n} C_{hk}^{i} x_h(t) x_k(t) - \sum_{h,k=1}^{n} C_{ik}^{h} x_i(t) x_k(t), \quad i = 1, \ldots, n. \tag{1}$$

Here, $x_i(t)$ denotes the fraction of individuals which at time t belong to the ith class and the constant coefficients $C_{hk}^{i} \in [0, 1]$, such that $\sum_{i=1}^{n} C_{hk}^{i} = 1$ for any fixed h and k, expressing the probability that an individual of the hth class will belong to the ith class after a direct interaction with an individual of the kth class. An expression for these coefficients, valid for the case in which the average incomes of the n classes $0 < r_1 < r_2 < \cdots < r_n$ are given by

$$r_j = j \cdot \Delta r, \tag{2}$$

with $\Delta r > 0$, was first derived in [16] and then used also in [6, 8, 9].

In order to reproduce here this expression, we start and denote by $p_{h,k}$ for $h, k = 1, \ldots, n$ the probability that in an encounter between an individual of the hth class and one of the kth class, the one who pays is the

1 To avoid excessive technicalities, we do not include here the taxation and redistribution terms which were considered in [8, 9].

former one (also called an h-individual). We denote by $S \ll \Delta r$ the amount of money paid in a transaction. Any single interaction possibly causes a slight decrease or increase in the income of the involved individuals and, in turn, a slight variation in the class populations. In fact, these variations only affect the original and the neighbouring classes of the involved individuals, and the only possibly nonzero elements among the C^i_{hk} are easily found to be:

$$\begin{aligned} C^i_{i+1,k} &= p_{i+1,k}\,\frac{S}{\Delta r}, \\ C^i_{i,k} &= 1 - \,p_{k,i}\,\frac{S}{\Delta r} - \,p_{i,k}\,\frac{S}{\Delta r}, \\ C^i_{i-1,k} &= p_{k,i-1}\,\frac{S}{\Delta r}. \end{aligned} \tag{3}$$

Notice that the expression for $C^i_{i+1,k}$ in (3) holds true for $i \le n-1$ and $k \le n-1$, the second addendum of the expression for $C^i_{i,k}$ is effectively present only provided $i \le n-1$ and $k \ge 2$, while its third addendum is present only provided $i \ge 2$ and $k \le n-1$; and the expression for $C^i_{i-1,k}$ holds true for $i \ge 2$ and $k \ge 2$. This is functional to the meaningfulness of the indices and reflects the fact that in this model individuals of the first class never pay and individuals of the nth class never receive money (there are no poorer classes than the first one nor richer classes than the nth one).

Concerning a choice for the $p_{h,k}$, since also the possibility of encounters without any payment exists, these coefficients are required to satisfy $0 \le p_{h,k} \le 1$ and $p_{h,k} + p_{k,h} \le 1$. As in previously quoted papers, we take in the following

$$p_{h,k} = \min\,\{r_h, r_k\}/4r_n, \tag{4}$$

with the exception of the terms $p_{j,j} = r_j/2r_n$ for $j = 2, \dots, n-1$, $p_{h,1} = r_1/2r_n$ for $h = 2, \dots, n$, $p_{n,k} = r_k/2r_n$ for $k = 1, \dots, n-1$, $p_{1,k} = 0$ for $k = 1, \dots, n$ and $p_{hn} = 0$ for $h = 1, \dots, n$.

We emphasize that the choice of the coefficients (3) is forced, if conservation of total income has to hold true for all $t \ge 0$ once it holds true for $t = 0$ (see the proof of Theorem 4.2 in [16]). In contrast, there is a certain degree of arbitrariness in the choice of p_{hk}. The specific formula (4) is suggested by the observation that usually poor individuals pay and receive less than rich ones. And the special definition of the coefficients with indices $h, k = 1$ or n is motivated by the same reasons given above for the C^i_{hk}.

To take now into account also the occurrence of random perturbations, a Langevin-type kinetic model [17] can be constructed as a system of stochastic equations of the form

$$\mathrm{d}x_i = D^{(1)}_i(x)\mathrm{d}t + \sum_{j=1}^{n} D^{(2)}_{ij}(x)\xi_j\sqrt{\Gamma\,\mathrm{d}t}, \quad i = 1, \dots, n, \tag{5}$$

in which the first term on the right hand side in Eq. (5) represents the "deterministic" contribution and the second term corresponds to noise. The interpretation of Eq. (5) is as follows. The first term describes direct money exchanges, ruled by norms, and behavioral attitudes which are the same for individuals belonging to the same class. The second term represents uncertainties randomly occurring, which also affect the change in the population distribution.

In the following, we take the operator $D^{(1)}_i$ as in (1),

$$D^{(1)}_i(x) = \sum_{h,k} C^i_{hk}x_h x_k - \sum_{h,k} C^h_{ik}x_i x_k,$$

i.e. we take $D^{(1)}_i$ to mimic that component of the models in [6, 8] which just describes the direct monetary exchanges without taxation and redistribution. As for the stochastic part, the ξ_i denote n independent Gaussian stochastic variables and Γ denotes the noise amplitude. The form of the operator $D^{(2)}_{ij}$ depends on the conservation requirements to which we want the model to obey.

Before specifying the announced models, we explicitly point out that births, deaths and other variations in the total number of individuals are not significant for the problem under investigation and over the period of interest here. Accordingly, it is quite natural to assume that the size of the population remains constant in time. Such constancy is guaranteed provided

$$\sum_{i=1}^{n} \mathrm{d}x_i = 0,$$

and equivalently (by (5)) provided

$$\sum_{i=1}^{n} D_i^{(1)}(x)\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}^{(2)}(x)\xi_j\sqrt{\Gamma\,\mathrm{d}t} = 0.$$

2.1 Multiplicative noise with conserved total population

It is easy to check that $\sum_{i=1}^{n} D_i^{(1)}(x)\mathrm{d}t = 0$ for any x. This implies in particular that $\sum_{i=1}^{n} x_i(t)$ is constant in time if the noise term $\sum_{j=1}^{n} D_{ij}^{(2)}(x)\xi_j\sqrt{\Gamma\,\mathrm{d}t}$ in Eq. (5) is absent. We normalize initial conditions $x_i(0)$ for $i = 1, \dots, n$ so as to have $\sum_{i=1}^{n} x_i(0) = 1$. Then, we have $\sum_{i=1}^{n} x_i(t) = 1$ for all $t \geq 0$ if the noise term in Eq. (5) is absent (this conservation property together with well posedness, existence of a unique solution for any time for the "deterministic" system has been proved in [16]).

Enforcing total population conservation for Eq. (5), we must also have

$$\sum_{i,j} D_{ij}^{(2)}(x)\xi_j = 0, \tag{6}$$

for any choice of $\{\xi_j\}$. A way to fulfill condition (6) together with a proportionality condition between the random variations in the class populations and the population themselves is to define, starting from the random ξ_i, new variations $\xi_i' = x_i\xi_i - x_i\sum_k x_k\xi_k$, or in matrix form

$$\xi_i' = \sum_j D_{ij}^{(2)}(x)\xi_j,$$

with

$$D_{ij[\text{pop-const}]}^{(2)}(x) = \begin{cases} x_i\,(1-x_i)\,, & \text{if } i = j \\ -x_i x_j, & \text{if } i \neq j. \end{cases} \tag{7}$$

The formula (7) provides an operator $D_{ij[\text{pop-const}]}^{(2)}$ which allows to construct, starting from random variables, a multiplicative noise term compatible with the conservation of the total population ("pop-const"). A proof of this, based on an iterative procedure, can be found in the Appendix. Here, we observe (see the Appendix) that in the following, as in [6, 8, 12], the total population is normalized to 1.

On the other hand, we emphasize that allowing a variation of the total income related to noise amounts to consider a society which also interacts in a stochastic way with the "external world": capital inflow or outflow is possibly thought to occur for example due to import–export of goods, incoming–outgoing of tourism, investment and stock trading.

2.2 Multiplicative noise with conserved total population and income

Alternatively, we may consider a closed system for which we also require conservation of the total income $\mu = \sum_i r_i x_i$. (Notice that, due to the normalization to 1 of the population, the total income coincides with the average income of the population itself.) We then point out that from now on we restrict attention on values of μ satisfying

$$r_1 < \mu < r_n. \tag{8}$$

Thanks to the inequalities (8), the possible occurrence can be excluded of situations in which all individuals belong to the poorest or to the richest income class. Similar odd cases are not of interest if one wants to deal with realistic situations. In other words, taking μ as in (8) does not represent a strong assumption.

In addition to (6), a further condition has now to be imposed, i.e.,

$$\sum_{i,j} r_i D_{ij}^{(2)}(x)\xi_j = 0 \tag{9}$$

for any choice of $\{\xi_j\}$. In order to construct a diffusion matrix satisfying both (6) and (9), we begin by observing that, given a vector $x = (x_1, \dots, x_n)$ with $x_i > 0$ for all i, and n positive constants r_i ($n \geq 3$), from any vector $\eta_0 = (\eta_{01}, \dots, \eta_{0n})$ with $|\eta_{0i}| \leq 1$ for all i, a new vector $\bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_n)$ may be obtained, which satisfies the estimates

$$|\bar{\eta}_i| \leq x_i \quad \text{for } i = 1, \dots, n$$

and the two conditions

$$\sum_i \bar{\eta}_i = 0, \quad \text{and} \quad \sum_i r_i \bar{\eta}_i = 0. \tag{10}$$

What is important here is finding an algorithm for the construction of such a vector $\bar{\eta}$.

Toward this end, we begin by associating with η_0 a vector

$$\eta = (\eta_1, \dots, \eta_n) = \left(\frac{x_1 \eta_{0,1}}{C}, \dots, \frac{x_n \eta_{0,n}}{C} \right), \tag{11}$$

where $C \geq 1$ is a constant to be determined in the following. We want then to transform the vector η to a perturbed vector $\bar{\eta} = \eta + A\eta$, with components

$$\bar{\eta}_i = \eta_i + \sum_{j=1}^{n} a_{ij}\, \eta_j \quad \text{for } i = 1, \dots, n \tag{12}$$

satisfying the conservation conditions (10). Inserting (12) in (10), we find (keeping also the arbitrariness of $\eta_{0,i}$ into account) that conditions (10) become

$$1 + \sum_{j=1}^{n} a_{ji} = 0, \quad \text{and} \quad r_i + \sum_{j=1}^{n} a_{ji}\, r_j = 0 \tag{13}$$

for $i = 1, \dots, n$. It is convenient choosing the matrix A in the set of tridiagonal matrices. Indeed, with this choice the variation of the ith component when passing from η to $\bar{\eta}$ only involves η_{i-1}, η_i, and η_{i+1} and conditions (13) read[2]

$$1 + \sum_{j=i-1}^{i+1} a_{ji} = 0, \quad \text{and} \quad r_i + \sum_{j=i-1}^{i+1} a_{ji}\, r_j = 0 \tag{14}$$

for $i = 1, \dots, n$. Formulas (14) express $2n$ constraints which the $3n - 2$ elements a_{ij} of the matrix A have to satisfy.[3] We then minimize the function of the $3n - 2$ variables a_{ji},

$$f = \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} {a_{ji}}^2$$

2 Here and henceforth only indexed terms with meaningful indices are to be considered present. For example, if $i = 1$, one has $\sum_{j=i-1}^{i+1} a_{ji} = a_{11} + a_{21}$.

3 Recall that $n \geq 3$ here, which is a quite natural assumption for the problem at hand.

subject to the $2n$ constraints (14). To this end, we introduce Lagrange multipliers λ_i and μ_i for $i = 1, \dots, n$, and consider the Lagrangian

$$L = \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} {a_{ji}}^2 + \sum_{i=1}^{n} \lambda_i \left(1 + \sum_{j=i-1}^{i+1} a_{ji}\right) + \sum_{i=1}^{n} \mu_i \left(r_i + \sum_{j=i-1}^{i+1} a_{ji} r_j\right).$$

The search for critical points of L (as a function of the variables a_{ji}, λ_i and μ_i) yields in particular, after straightforward calculations,

$$a_{ji} = \frac{N_i r_i r_j + T_i - R_i r_i - R_i r_j}{R_i^2 - N_i T_i}, \tag{15}$$

for $i = 1, \dots, n, j = i-1, i, i+1$ (the remaining a_{ji} being equal to zero), where

$$N_1 = 2, \quad N_i = 3 \quad \text{for } i = 2, \dots n-1, \quad N_n = 2,$$

and

$$R_i = \sum_{k=i-1}^{i+1} r_k \quad \text{and} \quad T_i = \sum_{k=i-1}^{i+1} r_k^2.$$

In view of the linearity of r_j in j as formulated in Eq. (2), it can be easily seen that the matrix A with elements as in (15) takes the form

$$A = \begin{bmatrix}
-1 & -1/3 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & -1/3 & -1/3 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & -1/3 & -1/3 & -1/3 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/3 & -1/3 & -1/3 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & -1/3 & -1/3 & -1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1/3 & -1/3 & -1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -1/3 & -1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & -1/3 & -1
\end{bmatrix}.$$

We observe now that applying the transformation (12) with the matrix A just found, we get

$$\frac{\bar{\eta}_i}{x_i} = \frac{\eta_i}{x_i} + \frac{\sum_{j=i-1}^{i+1} a_{ij}\, \eta_j}{x_i} \quad \text{for } i = 1, \dots, n, \tag{16}$$

namely,

$$\begin{aligned}
\frac{\bar{\eta}_1}{x_1} &= -\frac{1}{3C} \frac{x_2}{x_1} \eta_{0,2}, \\
\frac{\bar{\eta}_2}{x_2} &= \frac{2}{3C} \eta_{0,2} - \frac{1}{3C} \frac{x_3}{x_2} \eta_{0,3}, \\
\frac{\bar{\eta}_i}{x_i} &= \frac{2}{3C} \eta_{0,i} - \frac{1}{3C} \frac{x_{i-1}}{x_i} \eta_{0,i-1} - \frac{1}{3C} \frac{x_{i+1}}{x_i} \eta_{0,i+1}, \quad \text{for } i = 3, \dots, n-2, \\
\frac{\bar{\eta}_{n-1}}{x_{n-1}} &= \frac{2}{3C} \eta_{0,n-1} - \frac{1}{3C} \frac{x_{n-2}}{x_{n-1}} \eta_{0,n-2}, \\
\frac{\bar{\eta}_n}{x_n} &= -\frac{1}{3C} \frac{x_{n-1}}{x_n} \eta_{0,n-1}.
\end{aligned}$$

For the choice of the constant C appearing here and in (11), we first calculate

$$M_{\text{minus}} = \max_{i=2,\dots n} \left\{ \frac{x_i}{x_{i-1}} \right\} \quad \text{and} \quad M_{\text{plus}} = \max_{i=1,\dots n-1} \left\{ \frac{x_i}{x_{i+1}} \right\}, \tag{17}$$

and set

$$\Omega = \max\left\{1, M_{\text{minus}}, M_{\text{plus}}\right\}. \tag{18}$$

Then, we fix the constant C in (11) to be equal to $\frac{4}{3}\,\Omega$. Hence, according to (11), we associate to a randomly chosen vector η_0 the vector

$$\eta = (\eta_1, \dots \eta_n) = \left(\frac{3}{4}\,\frac{x_1\eta_{0,1}}{\Omega}, \dots, \frac{3}{4}\,\frac{x_n\eta_{0,n}}{\Omega}\right). \tag{19}$$

Now, applying to η the transformation (12) with the $a'_{ij}s$ as in (15), we get

$$\begin{aligned}
\frac{\bar\eta_1}{x_1} &= \frac{3}{4}\left(-\frac{1}{3}\,\frac{x_2}{x_1}\,\frac{1}{\Omega}\,\eta_{0,2}\right),\\
\frac{\bar\eta_2}{x_2} &= \frac{3}{4}\left(\frac{2}{3}\,\frac{1}{\Omega}\,\eta_{0,2} - \frac{1}{3}\,\frac{x_3}{x_2}\,\frac{1}{\Omega}\,\eta_{0,3}\right),\\
\frac{\bar\eta_i}{x_i} &= \frac{3}{4}\left(\frac{2}{3}\,\frac{1}{\Omega}\,\eta_{0,i} - \frac{1}{3}\,\frac{x_{i-1}}{x_i}\,\frac{1}{\Omega}\,\eta_{0,i-1} - \frac{1}{3}\,\frac{x_{i+1}}{x_i}\,\frac{1}{\Omega}\,\eta_{0,i+1}\right), \quad \text{for } i = 3, \dots, n-2,\\
\frac{\bar\eta_{n-1}}{x_{n-1}} &= \frac{3}{4}\left(\frac{2}{3}\,\frac{1}{\Omega}\,\eta_{0,n-1} - \frac{1}{3}\,\frac{x_{n-2}}{x_{n-1}}\,\frac{1}{\Omega}\,\eta_{0,n-2}\right),\\
\frac{\bar\eta_n}{x_n} &= \frac{3}{4}\left(-\frac{1}{3}\,\frac{x_{n-1}}{x_n}\,\frac{1}{\Omega}\,\eta_{0,n-1}\right),
\end{aligned}$$

which in turn implies

$$\begin{aligned}
\left|\frac{\bar\eta_1}{x_1}\right| &\le \frac{1}{4}\,|\eta_{0,2}| \le 1,\\
\left|\frac{\bar\eta_2}{x_2}\right| &\le \frac{2}{4}\,|\eta_{0,2}| + \frac{1}{4}\,|\eta_{0,3}| \le 1,\\
\left|\frac{\bar\eta_i}{x_i}\right| &\le \frac{2}{4}\,|\eta_{0,i}| + \frac{1}{4}\,|\eta_{0,i-1}| + \frac{1}{4}\,|\eta_{0,i+1}| \le 1, \quad \text{for } i = 3, \dots, n-2,\\
\left|\frac{\bar\eta_{n-1}}{x_{n-1}}\right| &\le \frac{2}{4}\,|\eta_{0,n-1}| + \frac{1}{4}\,|\eta_{0,n-2}| \le 1,\\
\left|\frac{\bar\eta_n}{x_n}\right| &\le \frac{1}{4}\,|\eta_{0,n-1}| \le 1.
\end{aligned}$$

In conclusion, the vector $\bar\eta$ satisfies the conservation conditions given in Eq. (10) as well as the estimates $|\bar\eta_i| \le x_i$ for $i = 1, \dots, n$.

We can now summarize all this and provide in the following proposition an expression for the sought algorithm.

Proposition 1. *Given a vector $x = (x_1, \dots, x_n)$ with $x_i > 0$ for all i, and n positive constants r_i ($n \ge 3$), from any $\eta_0 = (\eta_{01}, \dots, \eta_{0n})$ with $|\eta_{0i}| \le 1$ for all i, a new vector $\bar\eta = (\bar\eta_1, \dots, \bar\eta_n)$ which satisfies $|\bar\eta_i| \le x_i$ for $= 1, \dots, n$ and the two conditions* (10) *may be obtained as the vector with components* (12)*, in which η is given by* (11) *(and, in turn, $C = \frac{4}{3}\Omega$ with Ω as in* (18)*), and the elements a_{ji} of the linear transformation matrix are as in* (15).

In order to construct from the stochastic variable ξ a multiplicative noise term satisfying conservation of population and income, one can discretize time and repeatedly iterate, as illustrated below, the procedure of Proposition 1. We emphasize that a warning as discussed in the next lines is in order here.

At each step, say at each time t_k with $k = 0, 1, 2, \dots$, a vector ξ is picked whose components ξ_i for $i = 1, \dots, n$ are Gaussian random numbers ranging from -1 to 1. Here, ξ plays the role of η_0 in Proposition 1. The vector $x = (x_1, \dots, x_n)$ of Proposition 1 is given at the beginning of the process, i.e., at time t_0, by a stationary distribution x_{eq} (reached in the long run) of the “deterministic” system (1), whereas at subsequent steps, i.e., at time t_k with $k = 1, 2, \ldots$, it is given by the solution $x(t_k)$ of the system (5), or of the system (1), according to the criterion described next. There are two possibilities: either $x_i > 0$ for all $i = 1, \ldots, n$ or there exists at least an index value i^* such that x_{i^*} vanishes. In fact it is highly improbable that the second alternative occurs. Nevertheless, we take it too into consideration. A control loop in the algorithm checks which of the two possibilities holds true. Accordingly, the procedure to be applied is as follows.

1. If at time t_k it is $x_i > 0$ for all $i = 1, \ldots, n$, one calculates Ω according to Eqs. (17) and (18) and then defines, by applying the formula (19) with this value of Ω, an "intermediate" vector η. Then, one applies to η the transformation (12) with the a'_{ij}s as in (15). In this way one obtains, as Proposition 1 shows, a vector whose components are proportional to the classes populations and which, when inserted in the Eq. (5), guarantees both population and total income conservation ("pop-inc-const"). This vector can be denoted by
$$D^{(2)}_{[\text{pop}-\text{inc}-\text{const}]}(x)\xi. \tag{20}$$
Numerical solutions of (5) can be found by calculating (20), inserting the noise term
$$D^{(2)}_{ij[\text{pop}-\text{inc}-\text{const}]}(x)\xi_j\sqrt{\Gamma\,\mathrm{d}t}$$
into the Eq. (5) and getting the corresponding solution $x(t_{k+1})$. If $x_i(t_{k+1}) > 0$ for all $i = 1, \ldots, n$ and all $k \in \mathbf{N}$, one repeats all this over and over again.
2. If for some integer k and some index i^*, the component $x_{i^*}(t_k)$ vanishes, i.e., denoting $t_k = t^*$ one has $x_{i^*}(t^*) = 0$, then one lets only the system (1) evolve, without adding any noise up to when $x_i > 0$ for all $i = 1, \ldots, n$. From then on, the algorithm described in 1 has to be applied again. To give an insight as to why the re-establishment of the situation with all $x_i > 0$ is to be expected, we argue as follows.

First of all, we want to exclude the cases (both of which are equilibria for the system (1)), for which all individuals belong to the poorest class or to the richest class. Since the value of the total income with which the former case is compatible is $\mu = r_1$ whereas for the latter case it is $\mu = r_n$, the assumption (8) guarantees that these cases cannot occur, thereby assuring "moderate income" remit.

We then observe that exploiting the fact that $x_{i^*}(t^*) = 0$ one gets from (1),
$$\mathrm{d}x_{i^*}(t^*) = \sum_{h\neq i^*}\sum_{k\neq i^*} C^{i^*}_{hk}x_h(t^*)x_k(t^*)\,\mathrm{d}t \geq 0. \tag{21}$$
It is of course possible that other x_i in addition to x_{i^*} vanish at time t^*. Then, let m be the smallest positive integer such that
$$x_{i^*-m}(t^*) \neq 0 \quad \text{or} \quad x_{i^*+m}(t^*) \neq 0 \tag{22}$$
holds true. Such a number certainly exists. Assume, without loss of generality, the second of the two inequalities (22) to hold true. The other case can be handled similarly. Now, either $i^* + m < n$ or $i^* + m = n$ holds true.

– If $i^* + m < n$, observing that $C^{i-1}_{ii} > 0$ (as also $C^{i+1}_{ii} > 0$) provided $1 < i < n$, we conclude that $C^{i^*+m-1}_{i^*+m\,,\,i^*+m} > 0$ and hence
$$\mathrm{d}x_{i^*+m-1}(t^*) \geq C^{i^*+m-1}_{i^*+m\,,\,i^*+m}\,x^2_{i^*+m}(t^*)\,\mathrm{d}t > 0.$$
Consequently, $x_{i^*+m-1}(t^* + 1) > 0$. Iterating the procedure m times, one obtains
$$x_{i^*}(t^* + m) > 0.$$
– If $i^* + m = n$, we know, in view of (8), that there exists a positive integer p, satisfying $1 \leq i^* - p$, such that $x_{i^*-p}(t^*) \neq 0$. If $i^* - p > 1$, then, similarly as above, one notices that $\frac{\mathrm{d}}{\mathrm{d}t}x_{i^*-p+1}(t^*) \geq C^{i^*-p+1}_{i^*-p\,,\,i^*-p}\,x^2_{i^*-p}(t^*) > 0$, from which $x_{i^*-p+1}(t^* + 1) > 0$ and then $x_{i^*}(t^* + p) > 0$ follows. If $i^* - p = 1$, then one may exploit the fact that $C^2_{1n} > 0$ and $\frac{d}{dt}x_{i^*-p+1}(t^*) \geq C^2_{1n}\,x_1(t^*)x_n(t^*) > 0$ to be reconduced to the case just dealt with.

By repeating, if necessary, the procedure here illustrated, one ends up with $x_i(t_{k+q}) > 0$ for all $i = 1, \dots, n$, for some $q \in \mathbf{N}$.

3 Simulation results

To investigate the stochastic processes of the two models designed in Section 2, we fixed $n = 10$, $r_1 = 10$, $\Delta r = 10$ and $\Gamma = 0.001$. We numerically solved Eq. (5) and took the average of various quantities over a large number of stochastic realizations. Of course, no equilibria have to be expected in the present case.

Before drawing some conclusions, we need to recall the definition – more precisely, a variant of it, suitable for the present case – of an indicator of social mobility introduced in [12]. This indicator, which expresses the collective probability of class advancement of all classes from the 2nd to the $(n-1)$th one, is given by

$$M = \frac{1}{(1 - x_1 - x_n)} \frac{S}{\Delta r} \sum_{i=2}^{n-1} \sum_{k=1}^{n} p_{k,i} x_k x_i. \tag{23}$$

The origin of the expression in (23) is the following one: the probability of an individual of the ith class to move to the upper class as a consequence of encounters with the other individuals is expressed by

$$\frac{S}{\Delta r} \sum_{k=1}^{n} p_{ki} x_k.$$

An averaged class probability of the ith class population for $2 \le i \le n-1$ to move to the upper class can then be described by

$$\frac{1}{(1 - x_1 - x_n)} \frac{S}{\Delta r} \sum_{k=1}^{n} p_{ki} x_k x_i,$$

where individuals of the poorest and richest classes are not counted so as to avoid possible boundary effects. Taking the sum of these class probabilities over the indices $2 \le i \le n-1$, the collective averaged probability in (23) is then obtained.

We calculated the value of M in a succession of equally spaced instants $\{t_j\}$ along the evolution in time of several solutions of Eq. (5). As well, in correspondence to the same instants, we calculated the Gini index G. This coefficient was introduced by the Italian statistician Corrado Gini a century ago. It takes values in $[0, 1]$ and it is defined as a ratio, having the numerator given by the area between the Lorenz curve of a distribution and the uniform distribution line, and the denominator given by the area of the region under the uniform distribution line.

A significant finding concerns the sign of the correlation between G and M, namely between economic inequality and social mobility. For values of the total income μ which are not too large when total income is conserved, and which are neither too large nor too small when total income is not conserved, the statistical value of the sign of the correlation between G and M turns out to be negative.

The values of μ under consideration are reasonable in a realistic perspective (see, e.g., [18]) because they are compatible with a distribution of individuals in which most of the population belongs to the low-middle classes. If $n = 10$ and the values of r_i for $i = 1, \dots, n$ are linearly growing from $r_1 = 10$ to $r_{10} = 100$, values of $\mu \le 30$ in the conservative case and $\mu \in [24, 30]$ in the non-conservative case meet this criterion. We also stress here that the noise amplitude $\Gamma = 0.001$ is such that the variation of the total income μ taking place in the non-conservative case is compatible with GDP variation values occurring in real world (at least in the countries where we live, Italy and UK).

The negativity of the correlation which we get is in agreement with a great deal of empirical data [13, 14] and provides evidence of some robustness against random perturbations of the corresponding property established for systems without noise in [12]. A few samples of correlations R_{GM} (Gini and mobility index) are given in Table 1 for the case with constant total income and in Table 2 for the case with varying total income.

The correlations were obtained as averages of 50 realizations, each over 5000 integration steps. In these samples, three initial conditions – the same in Tables 1 and 2 – compatible with values of the total income equal to 24.5, 27 and 29.5, respectively, are considered (Figure 1 displays the initial condition corresponding to the asymptotic stationary distribution for the system without noise (1) with $\mu = 27$). And for each of these initial conditions, three different average results are reported.

We stress here that the distributions we obtain after the 5000 integration steps remain in fact quite "close" to the distributions from which they evolve, which are equilibria if noise is absent. We measure the "closeness" by calculating in correspondence to each realization the average of the squared difference between 5000 values x_i of each component of the distribution attained during evolution and the corresponding initial value $x_i(0)$; in addition, we calculate the standard deviation σ_{x_i} of each component x_i. We find that the differences $\hat{x}_i - x_i(0)$ take values whose order of magnitude typically are between 10^{-5} and 10^{-7}, the σ_{x_i} take values whose order of magnitude typically oscillate between 10^{-4} and 10^{-6}, whereas the values of the relative standard variations σ_{x_i}/x_i typically are of the order of 10^{-4}.

A further issue which one can explore in the non-conservative case is the correlation $R_{G\mu}$ between the Gini index and the total income. A difference comes out in this respect, depending on whether the noise is additive or multiplicative: whereas the value of $R_{G\mu}$ provided by the numerical simulations is positive in the first case, it turns out to be sometimes negative and sometimes positive in the second one, depending on the

Table 1: Correlations R_{GM} (Gini and mobility index) computed in nine cases in which total income μ is conserved, with noise amplitude $\Gamma = 0.001$. Averages of 50 realizations, each of 5000 integration steps.

μ	R_{GM}	R_{GM}	R_{GM}
24.5	-0.980 ± 0.002	-0.984 ± 0.001	-0.983 ± 0.002
27.0	-0.967 ± 0.003	-0.970 ± 0.003	-0.968 ± 0.003
29.5	-0.913 ± 0.007	-0.923 ± 0.008	-0.920 ± 0.007

Table 2: Correlations R_{GM} (Gini and mobility index) and $R_{G\mu}$ (Gini index and total income) computed in nine cases in which total income μ is not conserved, with noise amplitude $\Gamma = 0.001$. Averages of 50 realizations, each of 5000 integration steps.

$\mu(0)$	R_{GM}	R_{GM}	R_{GM}
24.5	-0.150 ± 0.061	-0.204 ± 0.056	-0.220 ± 0.062
27.0	-0.276 ± 0.064	-0.475 ± 0.051	-0.450 ± 0.052
29.5	-0.610 ± 0.044	-0.611 ± 0.034	-0.605 ± 0.047
$\mu(0)$	$R_{G\mu}$	$R_{G\mu}$	$R_{G\mu}$
24.5	0.096 ± 0.061	0.043 ± 0.059	0.045 ± 0.063
27.0	-0.068 ± 0.067	-0.271 ± 0.059	-0.239 ± 0.058
29.5	-0.465 ± 0.052	-0.443 ± 0.043	-0.466 ± 0.054

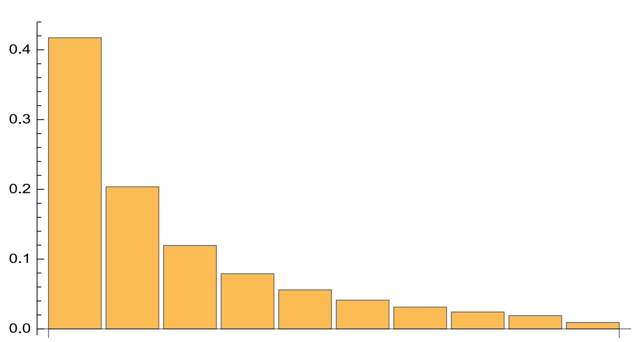


Figure 1: The asymptotic stationary solution of the "deterministic" system with constant total income $\mu = 27$. The height of each bar in the histogram represents the fraction of individuals in the corresponding income class. As described in the text, the distributions we get after the 5000 integration steps remain in fact quite "close" to the distributions from which they evolve.

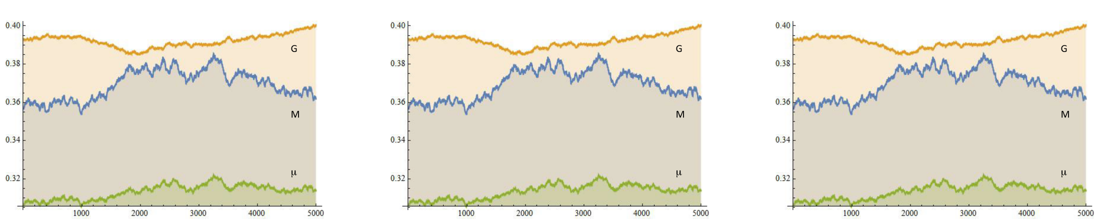
value of the initial total income μ. An intuitive argument for a possible explanation of the positive sign in the additive case is as follows: in the presence of additive noise the variations in the rich classes are typically much larger (with respect to those in the low and middle classes) than when the noise is multiplicative. This causes larger variations in the total income. Since increases of μ mainly affect the richer classes, this brings about an increase of inequality, i.e., of G. Yet, we do not have an explanation for the behavior of the correlation $R_{G\mu}$ in the multiplicative case. It has also to be noticed that the values reported in Table 2 display a great variability (and possibly, even no meaningfulness) of $R_{G\mu}$, when the total income is not fixed. We notice however that a strong positive correlation $R_{M\mu}$ between mobility and total income comes out of the realizations. A few samples of that are reported in Table 3. Also, from the three panels in Figure 2 displaying time series of G, M and μ the negativity of the correlation between G and M and the positivity of the correlation between M and μ is clearly visible.

As can be seen in Tables 2 and 3, the correlations between G, M and μ depend on the value of μ. Since the values of μ and G at equilibrium are mutually related, the correlations can also be seen as functions of G. (Note that in the range of μ and G considered here the relation between μ and G can be well approximated from the deterministic solutions as $G = -0.000448\mu^2 + 0.0276\mu - 0.0146$.) In order to further check this dependence, we ran simulations over 100 cycles varying μ, the results of which are shown in Figure 3. μ is varied approximately between $21 < \mu < 28$, corresponding for G to $0.36 < G < 0.41$. Each simulation consists of 50 stochastic realizations, each over 5000 steps and starting from the same equilibrium configuration; the solid circles in the plot represent the simulation data.

Figure 3A shows that the $M-G$ correlation is positive in the interval $0.36 < G < 0.38$; for $G > 0.38$, the aforementioned correlation becomes negative. Therefore, according to our model, the "Great Gatsby law", which states that the correlation between inequality and economic mobility is negative, strictly holds for $G > 0.38$. This is actually a range representing the pre-taxation values of G that includes most industrialized countries.

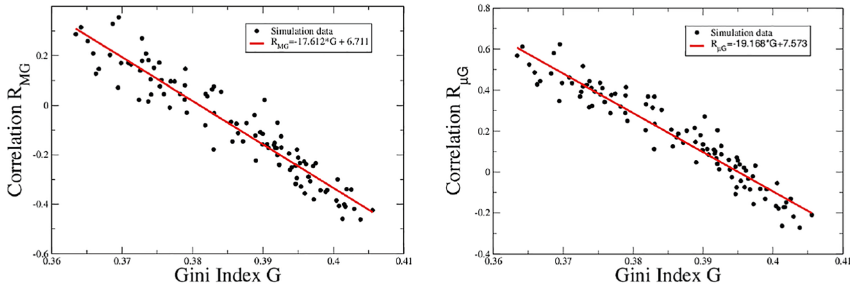
Table 3: Correlations $R_{M\mu}$ (mobility index and total income) computed in five cases with different values of the initial total income μ. Again, noise amplitude Γ is equal to 0.001 and averages are taken out of 50 realizations, each of 5000 integration steps.

$\mu(0)$	$R_{M\mu}$
22.0	0.951 ± 0.007
24.5	0.950 ± 0.006
27.0	0.960 ± 0.006
29.5	0.972 ± 0.005
32.0	0.981 ± 0.004



(a) G vs t (b) M vs t (c) μ vs t

Figure 2: Samples of time series of G, M and μ in three cases with $\mu(0)$ equal to 24.5, 27 and 29.5, respectively. The values of M are here multiplied by 800 and those of μ are divided by 80 so as to obtain a clearer comparison. In particular, a negative correlation between G and M, as well as a positive correlation between M and μ are clearly visible.



(a) R_{MG} versus G (b) $R_{\mu G}$ versus G

Figure 3: (a) Correlation between the total income μ and the Gini index G. (b) Correlation between the mobility M and the Gini index G.

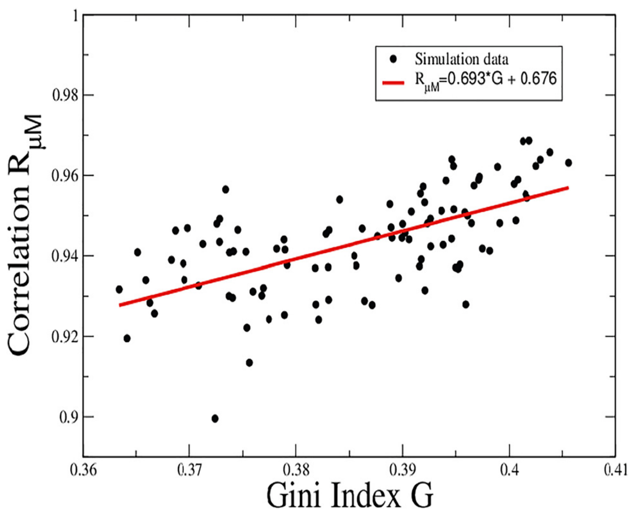


Figure 4: Correlation between the total income μ and the mobility M.

Figure 3b shows that the $\mu - G$ correlation is positive in the interval $0.36 < G < 0.395$ but gets negative thereafter. We thus identify a window of values for G for which the influx of wealth to the system contributes in decreasing inequality.

Finally, Figure 4 shows that the total income μ and mobility M always have a strong positive correlation which shows a slow increase with increasing G. This could be understood from an established thermodynamic allusion: in a canonical ensemble, for any reasonable definition of mobility, we expect a strong positive correlation between mobility and temperature; and in turn temperature variations will be strongly correlated with the variations in the free energy (corresponding to income in our case).

4 Conclusion

In this article, we proposed two different models to analyse the time evolution of income distribution resulting from multiple economic exchanges, in the presence of a multiplicative noise (abiding the Ito formulation). The presence of noise causes a continuous dynamical adjustment of the income distribution, which nevertheless stays reasonably close to the asymptotic steady state that it would have reached in the absence of noise. By ensemble averaging over a large set of stochastic realizations, we observed the emergence of correlations between the Gini inequality index G and a suitably defined mobility index M. The $G - M$ correlation is generally negative, becoming positive only in a small range of values of G, which do not correspond to any industrialized country (Figure 3A). The $G - \mu$ correlation can be both positive or negative (Figure 3b), while the $M - \mu$ correlation is always positive and close to 1 (Figure 4). In other words, mobility and inequality are mostly negatively correlated, mobility and total income are always positively correlated, and finally inequality and total income are positively correlated when inequality is low, and vice versa.

Probably, a more realistic model should involve a weighted combination of both additive and multiplicative stochastic perturbation. Indeed, certain events act as additive noise, whereas others are more properly represented by multiplicative noise. Economics modeling is replete both with examples of application of additive noise [19, 20] and multiplicative noise [21, 22, 23].

Correlated noise like Ornstein-Uhlenbeck, see, e.g., [24], could be considered as well, which is one of our ongoing research projects. More complicated noise structures, resembling power-law scaling have found popular applications in cognition science [25], another possibility for future investigation.

A further extension of the models developed in [11] and here could involve studying the impact of the coefficients C^i_{hk} themselves changing with the income distribution. Finally, it would be of great interest to investigate the dependence of the entire dynamical process, both on the amplitude as also on the nature of the noise distribution, as alluded to in some of the earlier references in other fields.

Author contribution: All the authors have accepted responsibility for the entire content of this submitted manuscript and approved submission.
Research funding: This research was funded in part by the Free University of Bozen-Bolzano through the project DEFENSSE.
Conflict of interest statement: The authors declare no conflicts of interest regarding this article.

Appendix

Proof of $\sum\limits_{i,j} D^{(2)}_{ij}(x)\xi_j = 0$ for $D^{(2)}_{ij}$ as in (7).

The solutions of Eq. (5) are found based on a discretization of time, i.e., considering times t_k with $k = 0, 1, 2, \ldots$ and

$$x_i(t_{k+1}) = x_i(t_k) + D^{(1)}_i(x(t_k))\,(t_{k+1} - t_k) + \sum_{j=1}^{n} D^{(2)}_{ij}(x(t_k))\xi_j \sqrt{\Gamma\,(t_{k+1} - t_k)},$$

for $i = 1, \ldots, n$.

If $t_0 = 0$ and $\sum_{i=1}^n x_i(t_0) = 1$, recalling that $\sum_{i=1}^n D^{(1)}_i(x)\mathrm{d}t = 0$ for any x we have,

$$\sum_{i=1}^{n} x_i(t_1) = 1 + \sum_{i=1}^{n} \sum_{j=1}^{n} D^{(2)}_{ij}(x(t_0))\xi_j \sqrt{\Gamma\,(t_1 - t_0)}.$$

We show next that in fact $\sum_{i=1}^n x_i(t_1) = 1$.

Indeed, if the operator $D^{(2)}_{ij}$ is as in (7) in Subsection 2.1, applying it to any random vector ξ, namely for any choice of $\{\xi_j\}$, we get

$$\begin{aligned}
\sum_{i,j} D^{(2)}_{ij}(x)\xi_j &= \sum_j \sum_{i\neq j} D^{(2)}_{ij}(x)\xi_j + \sum_j D^{(2)}_{jj}(x)\xi_j \\
&= \sum_j \sum_{i\neq j} - x_i x_j\, \xi_j + \sum_j x_j\left(1 - x_j\right)\xi_j \\
&= -\sum_j x_j\, \xi_j \sum_{i\neq j} x_i + \sum_j x_j\left(1 - x_j\right)\xi_j \\
&= -\sum_j x_j\, \xi_j \left(\sum_i x_i - x_j\right) + \sum_j x_j\, \xi_j - \sum_j x_j^2\, \xi_j \\
&= -\sum_j x_j\, \xi_j \sum_i x_i + \sum_j x_j^2\, \xi_j + \sum_j x_j\, \xi_j - \sum_j x_j^2\, \xi_j \\
&= \sum_j x_j\, \xi_j \left(1 - \sum_i x_i\right).
\end{aligned}$$

Then, in particular,

$$\sum_{i,j} D^{(2)}_{ij}(x(t_0))\xi_j = \sum_j x_j(t_0)\,\xi_j\left(1 - \sum_i x_i(t_0)\right) = 0,$$

the last equality being true because by assumption the quantity in parentheses vanishes.

At this point iteration of this procedure shows that $\sum_{i=1}^n x_i(t_k) = 1$ and $\sum_{i,j} D^{(2)}_{ij}(x(t_k))\xi_j = 0$ hold true for any $k = 1, 2, \ldots$ too. The claim then follows in view of the arbitrariness of the t_k's.

References

[1] A. Chatterjee and B. K. Chakrabarti, "Kinetic exchange models for income and wealth distributions," *Eur. Phys. J. B*, vol. 60, pp. 135–149, 2007.

[2] B. Düring, D. Matthes, and G. Toscani, "Kinetic equations modelling wealth redistribution: a comparison of approaches," *Phys. Rev. E*, vol. 78, p. 05613, 2008.

[3] S. Sinha and B. K. Chakrabarti, "Towards a physics of economics," *Physics News (Bulletin of the Indian Physics Association)*, vol. 39, pp. 33–46, 2009.

[4] V. M. Yakovenko and J. Barklry Rosser Jr., "Colloquium: statistical mechanics of money, wealth, and income," *Rev. Mod. Phys.*, vol. 81, pp. 1703–1725, 2009.

[5] M. Patriarca, E. Heinsalu, and A. Chakraborti, "Basic kinetic wealth-exchange models: common features and open problems," *Eur. Phys. J. B*, vol. 73, pp. 145–153, 2010.

[6] M. L. Bertotti and G. Modanese, "From microscopic taxation and redistribution models to macroscopic income distributions," *Physica A*, vol. 390, pp. 3782–3793, 2011.

[7] M. Patriarca and A. Chakraborti, "Kinetic exchange models: from molecular physics to social science," *Am. J. Phys.*, vol. 81, no. 8, pp. 618–623, 2013.

[8] M. L. Bertotti and G. Modanese, "Exploiting the flexibility of a family of models for taxation and redistribution," *Eur. Phys. J. B*, vol. 85, pp. 1–10, 2012.

[9] M. L. Bertotti and G. Modanese, "Micro to macro models for income distribution in the absence and in the presence of tax evasion," *Appl. Math. Comput.*, vol. 244, pp. 836–846, 2014.

[10] M. Aoki and H. Yoshikawa, *Reconstructing Macroeconomics*, Cambridge, Cambridge University Press, 2007.

[11] M. L. Bertotti, A. K. Chattopadhyay, and G. Modanese, "Stochastic effects in a discretized kinetic model of economic exchange," *Physica A*, vol. 471, pp. 724–732, 2017.

[12] M. L. Bertotti and G. Modanese, "Economic inequality and mobility in kinetic models for social sciences," *Eur. Phys. J. Spec. Top.*, vol. 225, no. 10, pp. 1945–1958, 2016.

[13] D. Andrews and A. Leigh, "More inequality, less social mobility," *Appl. Econ. Lett.*, vol. 16, pp. 1489–1492, 2009.

[14] M. Corak, "Income inequality, equality of opportunity, and intergenerational mobility," *J. Econ. Perspect.*, vol. 27, pp. 79–102, 2013.

[15] R. Wilkinson and K. Pickett, *The Spirit Level. Why Equality is Better for Everyone*, London, Penguin Books, 2010.

[16] M. L. Bertotti, "Modelling taxation and redistribution: a discrete active particle kinetic approach," *Appl. Math. Comput.*, vol. 217, pp. 752–762, 2010.

[17] H. Risken, *The Fokker-Planck Equation*, Berlin, Springer-Verlag, 1984.

[18a] Pew Research Center analysis of data from the World Bank PovcalNet database (Center for Global Development version available on the Harvard Dataverse Network) and the Luxembourg Income Study database), 2015, https://www.pewresearch.org/global/interactives/global-population-by-income.

[18b] R. Kochhar, From 2001 to 2011, Nearly 700 Million Step Out of Poverty, but Most Only Barely, 2015, https://www.pewresearch.org/global/ 2015/07/08/a-global-middle-class-is-more-promise-than-reality.

[19] A. K. Chattopadhyay, "Role of fluctuations in membrane models: thermal versus nonthermal," *Phys. Rev. E*, vol. 84, p. 032101, 2011.

[20] A. Dechant, A. Baule, and S.-I. Sasa, "Gaussian white noise as a resource for work extraction," *Phys. Rev. E*, vol. 95, p. 032132, 2017.

[21] R. N. Mantegna and H. E. Stanley, "Scaling behaviour in the dynamics of an economic index," *Nature*, vol. 376, p. 46, 1995.

[22] L. Laloux, P. Cizeau, J.-P. Bouchaud, and M. Motters, "Noise dressing of financial correlation matrices," *Phys. Rev. Lett.*, vol. 83, p. 1467, 1999.

[23] V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. N. Amaral, and H. E. Stanley, "Universal and non-universal properties of cross correlations in financial time series," *Phys. Rev. Lett.*, vol. 83, p. 1471, 1999.

[24] A. K. Chattopadhyay and E. C. Aifantis, "Stochastically forced dislocation density distribution in plastic deformation," *Phys. Rev. E*, vol. 94, p. 022139, 2016.
[25] E. J. Wagenmakers, S. Farrell, and R. Ratcliff, "Estimation and interpretation of $1/f^{\alpha}$ noise in human cognition," *Psychon. Bull. Rev.*, vol. 11, no. 4, p. 579, 2004.