Scalable Harmonisation of Complex Networks with Local Controllers

Miroslav Kárný, Randa Herzallah

***Abstract*—Computational and communication complexities call for distributed, robust, and adaptive control. This paper proposes a promising way of bottom up design of distributed control in which simple controllers are responsible for individual nodes. The overall behaviour of the network can be achieved by interconnecting such controlled loops in cascade control for example, and by enabling the individual nodes to share information about data with their neighbours without aiming at unattainable global solution. The problem is addressed by employing a fully probabilistic design, which can cope with inherent uncertainties, that can be implemented adaptively and which provide a systematic rich way to information sharing. The paper elaborates the overall solution, applies it to linear-Gaussian case and provides simulation results.**

I. Introduction

Complex dynamical systems formed by large ensembles of nodes interacting with a limited number of neighbouring nodes are essential in nature, technology and human societies. Controlling the dynamics of such a network is an important research, which is specifically considered here. The complexity and high dimensionality of a network often deny the opportunity of controlling the targeted enormous number of nodes in a centralised manner. Even a recent viable and effective approach of controlling a small fraction of the network nodes, known as pinning control [1], [2] has its inherent limits. Typically, the network need not be controllable with a technically feasible amount of centrally controlled nodes. Thus, it is worthwhile to inspect distributed adaptive control. The distributed solution admits to cope with the computational and communication complexities in large scale systems, when the noise, uncertainties and slow variations are respected by using probabilistic machinery. Width and variations of topology of complex networks make bottom-up design natural and the only fully scalable way. In this design method, simple controllers focus on individual nodes either completely independently or within various architectures such as cascade control. The desirable simplicity of controllers at individual nodes makes such architecture vulnerable to improper overall behaviour. Access to cheap computational and communication resources now allows the individual nodes to harmonise their acting by sharing information with their neighbours without aiming for unattainable global solutions.

M. Kárný is with the Department of Adaptive Systems, Institute of Information Theory and Automation, Czech Academy of Sciences, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic, http://www.utia.cas.cz/AS, school@utia.cas.cz

R. Herzallah is with NCRG, Aston University, Birmingham UK, r.herzallah@aston.ac.uk

Manuscript received 5 August 2015; revised 25 September 2015

The harmonisation can be supported by a message-passing. It is one of new decentralised methods for managing systems with large ensembles of interconnected nodes [3] where information is retrieved and disseminated in a consistent probabilistic fashion. The approach has emerged independently in a number of fields, including communications theory [4], artificial intelligence [5], and statistical physics [6]. However, the techniques and their potential generalisations have not yet been adequately introduced into the control community.

This paper develops a broadly applicable decentralised probabilistic adaptive control with an active passing of data-based as well as probabilistic "messages" that are exploitable without the need to increase complexity of the knowledge sharing nodes. It formulates the control of the large scale networks as a collection of smaller control problems, one for each connected node (subnetwork) in the system. The node-control problems are treated independently. Controllers can act asynchronously and autonomously and can be implemented individually. Knowledge sharing runs in the same mode. This brings additional advantages: a) nodes may follow individual aims and the network will (hopefully) stabilises at an acceptable compromise, b) design costs, which strongly limit top down decomposition of large scale problems into distributed solutions [7], [8], are low and do not limit scalability, c) designed controllers are randomised and naturally explorative, d) hierarchical solutions, possibly in pinning control style, can be simply created via set-point control of (selected) nodes.

This paper primarily proposes an adaptive controller applicable to each node. It controls a few outputs by its inputs while treating other available measurements as external variables. It recursively estimates parameters of a simple model while coping with inevitable approximation errors via stabilised forgetting [9], [10]. It exploits this model for control design via fully probabilistic design of controllers (FPD, [11], [12], [13]). In FPD, the optimal randomised controller is the minimiser of the Kullback-Leibler divergence [14] of the probability density (pd) describing closed loop dynamics to its ideal counterpart. Its relevance is due to its ability: i) to cope with stochastic nature of the controlled nodes, ii) to also adapt the ideal pd, expressing control aims, and iii) to use its unified probabilistic language for designing an efficient and well-grounded message-passing scheme, which does not force the knowledge sharing nodes to increase their complexity.

Section II formulates and solves the proposed adaptive control design for a single control node, which uses observed external variables. Its flexibility and computational simplicity form the main message brought. Section III deals with a network of such nodes and equips them with a simple way

of sharing knowledge contained in predictors they deal with. It opens a novel use of a recent methodology developed for knowledge elicitation [15]. Section IV applies the general methodology to linear Gaussian case and Section V illustrates it by simulations. Section VI provides concluding remarks.

II. Adaptive Control of Single Node with External Variables

A collection of control nodes is considered. Each node selects a sequence of real multivariate inputs[1] $u_t \in \boldsymbol{u_t}$, $t \in \boldsymbol{t} = \{1, \ldots, |\boldsymbol{t}|\}$, with the aim to influence real multivariate outputs $y_t \in \boldsymbol{y_t}$, $t \in \boldsymbol{t}$. The outputs are also influenced by multivariate observed external variables $x_{t-1} \in \boldsymbol{x_{t-1}}$. The relations of these random variables are modelled by a Markov-type probability density (pd[2]), by the system model

$$\begin{aligned} &\mathsf{M}(y_t, x_t | u_t, \ldots, u_1, y_{t-1}, \ldots, y_0, x_{t-1}, \ldots, x_0) \\ &\quad = \mathsf{M}(y_t | u_t, w_{t-1}) \mathsf{M}(x_t | x_{t-1}) \qquad (1) \\ &w_{t-1} \;=\; [y_{t-1}, x_{t-1}],\ t \in \boldsymbol{t},\ w_0 \text{ given.} \end{aligned}$$

The first factor $\mathsf{M}(y_t|u_t, w_{t-1})$ in (1) expresses the assumed Markovian dependence. The chosen form of the second factor $\mathsf{M}(x_t|x_{t-1})$ in (1) expresses the assumption (only approximately valid) that x_t are external variables with their inherent dynamics uninfluenced by the inputs u_t and outputs y_t.

The discussed node optimises the system inputs by using fully probabilistic design of decision strategies. FPD expresses the control objectives via pds (subscripted by $_\mathsf{I}$), which can be interpreted as factors of an ideal (desired) closed loop model

$$\begin{aligned} &\mathsf{C_I}(y_t, u_t, x_t | u_{t-1}, \ldots, u_1, y_{t-1}, \ldots, y_0, x_{t-1}, \ldots, x_0) \\ = \;& \mathsf{M_I}(y_t|w_{t-1}) \mathsf{S_I}(u_t|w_{t-1}) \mathsf{M}(x_t|x_{t-1}),\ t \in \boldsymbol{t}. \qquad (2) \end{aligned}$$

In (2), the factor $\mathsf{M}(x_t|x_{t-1})$ describing the desired behaviour of the external variable x_t equals to the corresponding counterpart in (1). This respects the "externality" of x_t and allows it to evolve uncontrollably.

With the given ideal closed loop model (2), FPD selects the optimal strategy $\mathsf{S_O}$ from the set $\mathbf{S}$ of randomised strategies, formed by sequences of randomised control laws[3] $\mathbf{S} = \{\mathsf{S}(u_t|w_{t-1}),\ t \in \boldsymbol{t}\}$, as follows

$$\mathsf{S_O} = \arg\min_{\mathsf{S} \in \mathbf{S}} \mathcal{D}(\mathsf{C_S} || \mathsf{C_I}). \qquad (3)$$

There, Kullback-Leibler divergence $\mathcal{D}$ (KLD, [14])

$$\mathcal{D}(\mathsf{H}||\mathsf{G}) = \int_{\boldsymbol{z}} \mathsf{H}(z) \ln(\mathsf{H}(z)/\mathsf{G}(z))\ \mathrm{d}z$$

measures proximity of pds H, G. In (3), FPD compares the closed loop model

$$\begin{aligned} \mathsf{C_S} \;&=\; \mathsf{C_S}(y_{|\boldsymbol{t}|}, \ldots, y_1, u_{|\boldsymbol{t}|}, \ldots, u_1, x_{|\boldsymbol{t}|}, \ldots, x_1 | w_0) \\ &=\; \prod_{t \in \boldsymbol{t}} \mathsf{M}(y_t|u_t, w_{t-1}) \mathsf{S}(u_t|w_{t-1}) \mathsf{M}(x_t|x_{t-1}), \end{aligned}$$

with the ideal closed loop model (over whole time span)

$$\begin{aligned} \mathsf{C_I} \;&=\; \mathsf{C_I}(y_{|\boldsymbol{t}|}, \ldots, y_1, u_{|\boldsymbol{t}|}, \ldots, u_1, x_{|\boldsymbol{t}|}, \ldots, x_1 | w_0) \\ &=\; \prod_{t \in \boldsymbol{t}} \mathsf{M_I}(y_t|x_{t-1}) \mathsf{S_I}(u_t|w_{t-1}) \mathsf{M}(x_t|x_{t-1}). \end{aligned}$$

FPD is taken as a ready methodology in this paper. Its details and axiomatic background are left aside. It suffices to recall that it densely extends the set of control problems that can be formulated and solved within Bayesian framework [13]. Note that a closely related and independently developed technique [17], [18] is referred to as KL control.

A. FPD with Observed External Variables

The next proposition specialises the general solution [12] of the optimisation (3) to the control design with observed external variables.

Proposition 1 (FPD with Observed External Variables): The optimal strategy in the FPD sense (3), for the system model (1) and the ideal closed loop model (2), consists of the optimal control laws $\mathsf{S_O}(u_t|w_{t-1})$, $w_{t-1} = (y_{t-1}, x_{t-1})$, $t \in \boldsymbol{t}$,

$$\mathsf{S_O}(u_t|w_{t-1}) = \frac{\mathsf{S_I}(u_t|w_{t-1}) \exp[-\omega(u_t, w_{t-1})]}{\underbrace{\int_{\boldsymbol{u_t}} \mathsf{S_I}(u_t|w_{t-1}) \exp[-\omega(u_t, w_{t-1})]\ \mathrm{d}u_t}_{\gamma(w_{t-1})}}$$

$$\begin{aligned} &\omega(u_t, w_{t-1}) = \\ &\int_{\boldsymbol{y_t}} \mathsf{M}(y_t|u_t, w_{t-1}) \ln\Big(\frac{\mathsf{M}(y_t|u_t, w_{t-1})}{\mathsf{M_I}(y_t|w_{t-1}) \bar{\gamma}(y_t, x_{t-1})}\Big) \mathrm{d}y_t \geq 0 \\ &\ln(\bar{\gamma}(y_t, x_{t-1})) = \int_{\boldsymbol{x_t}} \mathsf{M}(x_t|x_{t-1}) \ln(\gamma(y_t, x_t))\ \mathrm{d}x_t, \\ &\text{with} \quad \bar{\gamma}(\cdot),\ \gamma(\cdot) \leq 1. \qquad (4) \end{aligned}$$

The evaluations run in backward manner with $\gamma(y_{|\boldsymbol{t}|}, x_{|\boldsymbol{t}|}) = 1$.

Proof Let us assume that we already optimised over control laws for time moments starting after time $t \in \boldsymbol{t}$ up to the horizon $|\boldsymbol{t}|$. The achieved minimum is assumed to be of the form $-\ln(\gamma(y_t, x_t)) = -\ln(\gamma(w_t)) \geq 0$. For time $|\boldsymbol{t}|$, this form is valid with $\gamma(w_{|\boldsymbol{t}|}) = 1$. We perform an inductive step for a time $t \leq |\boldsymbol{t}|$ by optimising over the control law $\mathsf{S}(u_t|w_{t-1})$. The part of the partially minimised KLD influenced by this control law has the form

$$\begin{aligned} \mathsf{R}(w_{t-1}) = &\int_{\boldsymbol{y_t}} \int_{\boldsymbol{u_t}} \int_{\boldsymbol{x_t}} \mathsf{M}(y_t|u_t, w_{t-1}) \mathsf{S}(u_t|w_{t-1}) \mathsf{M}(x_t|x_{t-1}) \\ &\times \quad \ln\left(\frac{\mathsf{M}(y_t|u_t, w_{t-1}) \mathsf{S}(u_t|w_{t-1})}{\mathsf{M_I}(y_t|w_{t-1}) \mathsf{S_I}(u_t|w_{t-1}) \gamma(y_t, x_t)}\right) \mathrm{d}y_t \mathrm{d}u_t \mathrm{d}x_t \end{aligned}$$

The form of the ideal closed loop model (2), definitions of symbols in (4), Fubini theorem on multiple integration and normalisation of pds imply the next optimised part of KLD

$$\begin{aligned} &\mathsf{R}(w_{t-1}) \\ = \;& -\ln(\gamma(w_{t-1})) + \int_{\boldsymbol{u_t}} \mathsf{S}(u_t|w_{t-1}) \ln\left(\frac{\mathsf{S}(u_t|w_{t-1})}{\mathsf{S_O}(u_t|w_{t-1})}\right) \mathrm{d}u_t. \end{aligned}$$

The last term is conditional KLD, which is minimised for equal arguments and it is zero when the equality is achieved. This demonstrates the optimality of $\mathsf{S_O}$ and describes the backward evolution of the function $\gamma(w_{t-1})$. For the inductively

[1] A set of values of a variable z is denoted $\boldsymbol{z}$. It is specified when needed.

[2] Pd is Radon-Nikodým derivative [16] with respect to a dominating measure – either Lebesgue or counting one – denoted $\mathrm{d}\bullet$. Different functions denoted by the same letter are distinguished by identifiers of their arguments and possibly by an additional decoration, often being the set to which concerns.

[3] Individual admissible control laws are described by pds $\mathsf{S}(u_t|w_{t-1})$ having their supports on admissible sets of inputs $\boldsymbol{u_t}$ and conditioned by the available knowledge. For the system model (1) and the ideal closed loop model (2), the knowledge of w_{t-1} suffices.

assumed $\gamma(y_t, x_t) = \gamma(w_t) \leq 1$, the function $\bar{\gamma}(y_t, x_{t-1}) \leq 1$ and thus the function $\omega(u_t, w_{t-1}) \geq 0$. Thus, $\gamma(w_{t-1}) \leq 1$. This completes the inductive step. □

Remarks 1:

- The function $-\ln(\gamma(y_t, x_t)) = -\ln(\gamma(w_t))$ corresponds with the value function in dynamic programming [19].
- It is important to see the role of the model $\mathsf{M}(x_t|x_{t-1})$ describing external variables. It just maps $\ln(\gamma(y_t, x_t)) = \ln(\gamma(w_t))$ on $\ln(\bar{\gamma}(y_t, x_{t-1}))$ by averaging. This makes the design computationally undemanding even for high-dimensional external variables, cf. Section IV.

B. Bayesian Estimation in Exponential Family

FPD relies on availability of the system model (1). In the considered adaptive context, it is obtained via recursive Bayesian estimation [20] of a parametric model. The permanent estimation allows us to rely on simple models describing the modelled system locally. This motivates the use of system models from exponential family (EF, [21]). This is essentially the only family admitting a finite dimensional sufficient statistic [22] and consequently the permanent non-approximated estimation. It is recalled here.

The model relating a predicted multivariate real variable $\delta_t \in \boldsymbol{\delta_t}$, $\delta_t \in \{y_t, x_t\}$ to a multivariate explanatory variable $\psi_t \in \{(u_t, w_{t-1}), x_{t-1}\}$ parameterised by a finite-dimensional $\Theta \in \boldsymbol{\Theta}$ belongs to EF if it is described by a pd of the form

$$\mathsf{M}(\delta_t|\Theta, \psi_t) = \exp\left\langle \mathsf{A}(\Psi_t), \mathsf{B}(\Theta)\right\rangle, \Psi_t = [\delta_t, \psi_t], \quad (5)$$

where data Ψ_t enters the multivariate function A, dimension of which makes the scalar product $\langle \mathsf{A}, \mathsf{B}\rangle$ with the multivariate function $\mathsf{B}(\Theta)$ well defined.

Bayesian estimation evolves the posterior pd $\mathsf{P}(\Theta|K_t)$, which is the pd of the unknown parameter Θ conditioned on knowledge K_t. The evolving knowledge K_t compresses prior knowledge K_0 and data observed up to the time t, $K_t = (\Psi_t, K_{t-1})$, $t \in \boldsymbol{t}$. The parameter $\Theta \in \boldsymbol{\Theta}$ is unknown to the considered controllers, i.e.

$$\begin{aligned} & \mathsf{S}(u_t|\Theta, K_{t-1}) = \mathsf{S}(u_t|K_{t-1}) \quad (6)\\ \Leftrightarrow\ & \mathsf{P}(\Theta|K_{t-1}) = \mathsf{P}(\Theta|K_{t-1}, u_t) = \mathsf{P}(\Theta|K_{t-1}, \psi_t). \end{aligned}$$

Under these natural conditions of control [20], the evolution of the posterior pd is driven by the Bayes rule written for EF

$$\begin{aligned} \mathsf{P}(\Theta|K_t) &= \frac{\exp\left\langle \bar{V}_t, \mathsf{B}(\Theta)\right\rangle \mathsf{P}(\Theta|K_0)}{\bar{\mathsf{J}}(\bar{V}_t)} \\ &= \frac{\exp\left\langle \bar{V}_{t-1} + \mathsf{A}(\Psi_t), \mathsf{B}(\Theta)\right\rangle \mathsf{P}(\Theta|K_0)}{\bar{\mathsf{J}}(\bar{V}_{t-1} + \mathsf{A}(\Psi_t))} \\ \bar{\mathsf{J}}(\bar{V}) &= \int_{\boldsymbol{\Theta}} \exp\left\langle \bar{V}, \mathsf{B}(\Theta)\right\rangle \mathsf{P}(\Theta|K_0)\, \mathrm{d}\Theta. \quad (7) \end{aligned}$$

$\bar{V}_t = \bar{V}_{t-1} + \mathsf{A}(\Psi_t)$ is the sufficient statistic[4] of the fixed and finite dimension of A. The recursion starts with $\bar{V}_0 = 0$. The evaluation needs a prior pd $\mathsf{P}(\Theta|K_0)$ quantifying prior knowledge K_0 about Θ. Without a substantial decrease of flexibility, the conjugate prior pd $\mathsf{P}(\Theta|K_0) \propto \exp\langle V_0, \mathsf{B}(\Theta)\rangle$, [23], is considered further on. It has the form mimic to the likelihood of EF and simplifies (7) to the form

$$\begin{aligned} \mathsf{P}(\Theta|K_t) &= \mathsf{P}(\Theta|V_t) = \frac{\exp\langle V_t, \mathsf{B}(\Theta)\rangle}{\mathsf{J}(V_t)},\ V_t = \bar{V}_t + V_0 \\ V_t &= V_{t-1} + \mathsf{A}(\Psi_t),\ V_0 \text{ chosen a priori,} \\ \mathsf{J}(V) &= \int_{\boldsymbol{\Theta}} \exp\langle V, \mathsf{B}(\Theta)\rangle\, \mathrm{d}\Theta. \quad (8) \end{aligned}$$

This estimation exactly provides the model of δ_t as the predictor, i.e. the pd

$$\mathsf{F}(\delta_t|K_{t-1}, \psi_t) = \mathsf{F}(\delta_t|V_{t-1}, \psi_t) = \frac{\mathsf{J}(V_{t-1} + \mathsf{A}(\Psi_t))}{\mathsf{J}(V_{t-1})}. \quad (9)$$

This form exploits the natural conditions of control (6).

For the parametric system model in EF, the predictor (9) can formally be used in Proposition 1 as the system model. It leads to dual control problem [24] or, in the more recent vocabulary, the problem of approximate dynamic programming [25]. We avoid its complexity by adopting certainty-equivalence approximation of the predictive pd (9), i.e. by taking

$$\mathsf{F}(\delta_t|V_{t-1}, \psi_t) \approx \mathsf{M}(\delta_t|\hat{\Theta}_{t-1}, \psi_t). \quad (10)$$

There, $\hat{\Theta}_{t-1}$ is a point estimate of $\Theta \in \boldsymbol{\Theta}$ selected according to the pd $\mathsf{P}(\Theta|V_{t-1})$, say, its expected value or its maximiser.

The used system model is intentionally simple to keep computational demands low. Thus, it is inevitably approximate. Paper [9] has shown that stabilised forgetting [10] is the proper tool for preventing a permanent accumulation of approximation errors. Within EF, it modifies the updating of the value of the sufficient statistic V_t to

$$V_t = \phi_t(V_{t-1} + \mathsf{A}(\Psi_t)) + (1 - \phi_t)V_{t-1} = V_{t-1} + \phi_t \mathsf{A}(\Psi_t). \quad (11)$$

The forgetting factor $\phi_t \in [0, 1]$ entering (11) is in [9] selected heuristically. An asymptotic analysis of the corresponding weighted Bayes rules [26] leads to the following, better motivated, choice used further on

$$\phi_t = \frac{\mathsf{J}(V_{t-1} + \mathsf{A}(\Psi_t))^2}{\mathsf{J}(V_{t-1} + 2\mathsf{A}(\Psi_t))}, \quad (12)$$

which coincides with the recommendation in [26] when taking the predictive pd in measured data – after using them in updating – as the predictor ideally fitting them. A detailed justification of (12) is out of scope of this paper. Importantly, the forgetting factor (12) is always in the range $[0, 1]$. It approaches 1 if the posterior pd $\mathsf{P}(\Theta|K_{t-1})$ is concentrated and the observation δ_t is close to its point prediction.

C. Parametric System Model and Ideal Closed Loop Model

The recalled estimation is applied to the system model parameterised by multivariate parameters $(\Theta, \Theta_{\boldsymbol{x}}) \in (\boldsymbol{\Theta}, \boldsymbol{\Theta_x})$

$$\begin{aligned} & \mathsf{M}(y_t, x_t|\Theta, \Theta_{\boldsymbol{x}}, u_t, w_{t-1}) \\ = \ & \mathsf{M}(y_t|\Theta, u_t, w_{t-1})\mathsf{M}(x_t|\Theta_{\boldsymbol{x}}, x_{t-1}). \quad (13) \end{aligned}$$

The independent parametrisation respects the external nature of x_t.

[4] Recall, *sufficient* statistic comprises all knowledge on Θ brought by data.

The call for simplicity of parameter estimation makes us to assume that both factors of the system model (13) are in EF. The first one is

$$\begin{aligned}\mathsf{M}(y_t|\Theta,u_t,w_{t-1}) &= \exp\langle \mathsf{A}(\Psi_t),\mathsf{B}(\Theta)\rangle\\ \psi_t &= [u_t,w_{t-1}],\ \Psi_t=[y_t,\psi_t]\end{aligned}$$

and its conjugate posterior pd $\mathsf{P}(\Theta|K_t)=\mathsf{P}(\Theta|V_t)$ is determined by the sufficient statistic $V_t=V_{t-1}+\mathsf{A}(\Psi_t)$, V_0 chosen a priori. The second one modelling the external variables in EF

$$\begin{aligned}\mathsf{M}(x_t|\Theta_{\boldsymbol{x}},x_{t-1}) &= \exp\langle \mathsf{A}_{\boldsymbol{x}}(\Psi_{t;\boldsymbol{x}}),\mathsf{B}_{\boldsymbol{x}}(\Theta_{\boldsymbol{x}})\rangle\\ \psi_{t;\boldsymbol{x}} &= x_{t-1},\ \Psi_{t;\boldsymbol{x}}=[x_t,x_{t-1}]\end{aligned}$$

deals with its choice of functions $\mathsf{A}_{\boldsymbol{x}}$, $\mathsf{B}_{\boldsymbol{x}}$, data $\Psi_{t;\boldsymbol{x}}=[x_t,x_{t-1}]$ and unknown parameter $\Theta_{\boldsymbol{x}}$ described by the conjugate posterior pd $\mathsf{P}(\Theta_{\boldsymbol{x}}|K_t)=\mathsf{P}(\Theta_{\boldsymbol{x}}|V_{t;\boldsymbol{x}})$ with the sufficient statistic $V_{t;\boldsymbol{x}}=V_{t-1;\boldsymbol{x}}+\mathsf{A}_{\boldsymbol{x}}(\Psi_{t;\boldsymbol{x}})$, $V_{0;\boldsymbol{x}}$ chosen a priori.

Remarks 2:

- The parameter estimation concerning a multivariate predicted variable, say, $y_t'=[y_{t;1},\dots,y_{t;\ell_{\boldsymbol{y}}}]$, $'$ is transposition, can be reduced to parallel estimation of single variate parametric models: the chain rule for pds implies

$$\begin{aligned}&\mathsf{M}(y_t|\Theta,u_t,w_{t-1}) \qquad (14)\\ =\ &\prod_{i=1}^{\ell_{\boldsymbol{y}}}\mathsf{M}(y_{t;i}|\Theta_i,y_{t;i+1},\dots,y_{t;\ell_{\boldsymbol{y}}},u_t,w_{t-1}).\end{aligned}$$

  The factors in the right-hand side of (14) predict scalars and have to deal with collection of explanatory variables u_t,w_{t-1} extended by $y_{t;i+1},\dots,y_{t;\ell_{\boldsymbol{y}}}$. The parameters Θ_i are constituents of Θ. Their choice allows us to introduce a structural prior knowledge about independence of some predicted entries. This is especially important for external variables. Their rough model often neglects mutual dependencies of the current and delayed entries of $x_{t;i}$, $x_{t;j}$, $x_{t-1;i}$, $x_{t-1;j}$, $j\neq i$ and assumes

$$\mathsf{M}(x_t|\Theta_{\boldsymbol{x}},x_{t-1})=\prod_{i=1}^{\ell_{\boldsymbol{x}}}\mathsf{M}(x_{t;i}|\Theta_{\boldsymbol{x}i},x_{t-1;i}). \qquad (15)$$

  The simplification (15) reduces estimation computational load and it is at least partially compensated by adaptivity (recursive learning with forgetting) and by the knowledge sharing discussed in Section III-B.
- The errors caused by certainty-equivalence approximation of the predictor (10) are also counteracted by adaptivity.
- The lack of active exploration connected with the certainty-equivalence approximation is a hard and open problem [27]. Its systematic treatment goes beyond the scope of this paper. We conjecture that the randomised nature of the FPD-optimal control laws, Proposition 1, diminishes the lack of an intentional exploration.
- The difficult and important choice of the ideal pd, falling into the general problem of preference elicitation [28], is treated here marginally. For the elaborated classical aim of pushing the output y_t to an externally supplied set-point $y_{t;s}$, which is embedded into x_t, the following system-model-dependent choice is meaningful

$$\mathsf{C}_{\mathsf{I}}(y_t,u_t|w_{t-1})=\mathsf{M}_{\mathsf{I}}(y_t|w_{t-1})\mathsf{S}_{\mathsf{I}}(u_t|w_{t-1}). \qquad (16)$$

  There $\mathsf{M}_{\mathsf{I}}(y_t|w_{t-1})=\mathsf{M}(y_t|u_{t;s},w_{t-1})$
  $u_{t;s}\in \mathrm{Arg}\max_{u_t\in\boldsymbol{u_t}}\mathsf{M}(y_{t;s}|u_t,w_{t-1})$

  and $\mathsf{S}_{\mathsf{I}}(u_t|w_{t-1})$ is a flat pd concentrating its mass on the set $\boldsymbol{u_t}$ of desired inputs. This makes the ideal closed loop model potentially reachable. The use of the recursively estimated system model in (16) then also adapts the ideal closed loop model. For related discussions, see [29], [30].

III. Unlimited Network of Control Nodes

A. Considered Network of Controllers

The control node described in Section II in fact acts within a network of interacting nodes of the same type. They may differ in explanatory variables and thus in character and dimensions of unknown parameters and possibly even in functions A, B, $\mathsf{A}_{\boldsymbol{x}},\mathsf{B}_{\boldsymbol{x}}$ defining specific members of EF.

The targeted size of the network and disparity of local aims prevent a global joint optimisation. Individual nodes share part of the data with a limited (small) number of their neighbours. Without a message passing, each node selects the input u_t and tries to influence the output y_t that it is responsible for. The other observed data, including inputs and outputs optimised locally by neighbours, is modelled and treated by the specific node as external variables in x_t.

If the mutual influence of locally optimised nodes is weak enough, it may happen that the whole network will behave well. Generally, however, incompletely compatible aims and non-harmonised dynamics cause emergent behaviours, which are very far from the desired ones (up to instability).

Question arises, whether it is possible to allow neighbours to systematically share additional information, which harmonises their acting but does not force individual nodes to go beyond the model they handle. In other words, the sharing of additional information does not force nodes to go towards the infeasible global model and global optimisation of the network. This constraint reduces the harmonisation to information sharing, which improves description of unknown parameters entering parametric models. This direction is elaborated here.

The sharing of information between nodes is asynchronous and distributed over the network within various overlapping groups of neighbours. This practically makes the sharing fully scalable with respect to the network size. At the same time, it allows us to consider information sharing merely for a pair of nodes indexed by $\rho\in\{\alpha,\beta\}$. Each node has its elements used for constructing and using adaptive controllers. Recall that they consist of optional inputs $u_{t;\rho}\in\boldsymbol{u_{t;\rho}}$, optimised outputs $y_{t;\rho}\in\boldsymbol{y_{t;\rho}}$, the related set points $y_{t;s\rho}\in\boldsymbol{y_{t;\rho}}$, external variables $x_{t;\rho}\in\boldsymbol{x_{t;\rho}}$, the parametric system models in EF

$$\mathsf{M}(y_{t;\rho}|\Theta_\rho,u_{t;\rho},w_{t-1;\rho})\mathsf{M}(x_{t;\rho}|\Theta_{\boldsymbol{x}\rho},x_{t-1;\rho}),$$

and, importantly, the conjugate pds describing the unknown parameters, for nodes $\rho\in\{\alpha,\beta\}$,

$$\mathsf{P}(\Theta_\rho|K_{t;\rho})=\mathsf{P}(\Theta_\rho|V_{t;\rho}),\ \mathsf{P}(\Theta_{\boldsymbol{x}\rho}|K_{t;\rho})=\mathsf{P}(\Theta_{\boldsymbol{x}\rho}|V_{t;\boldsymbol{x}\rho}).$$

Neighbours, by their definition, share a non-void part δ_t of data vectors $\Psi_{t;\rho}, \Psi_{t;\boldsymbol{x}\rho}$ and each node has disposal models of their future occurrences, i.e. pds

$$\mathsf{F}(\delta_t|K_{t-1;\rho}) = \mathsf{F}(\delta_t|K_{t-1;\rho}, \psi_{t;\rho}). \tag{17}$$

Indeed, these pds are the output predictors if outputs are part of δ_t, or they are the designed randomised control laws (property of FPD) if inputs are part of δ_t, or they are predictors of external variables included in δ_t.

The nodes are assumed to be uninformed about system models or control laws used by their neighbours as the extent of possible options is too broad even when dealing with EF only. Thus, they cannot directly share information about their parameters. Thus, the predictors of common data δ_t (17) offer the only affordable way for improving neighbours' models without increasing their complexity.

B. Sharing of Knowledge Brought by Predictors

Question arises how to extract the information contained in a probabilistic data model for correcting description of unknown parameters. Exactly this question was answered in the context of knowledge elicitation and led to the definite proposal summarised in Proposition 2 below. The derivation of the sharing formula (18) below is simple but it needs machinery, which goes beyond the scope of this paper. It is presented in [31]. Let us note that it was proposed by the first author of [15] and successfully applied in [32]. Loosely, it follows from an application of minimum cross-entropy principle [33], [34] and its generalisation [35] allowing non-linear constraints on pds to be optimised according to this principle.

Proposition 2 (How Data Pd Modifies Pd of Parameter): Let us consider a fixed knowledge, K, determining the pd $\mathsf{M}(\delta|\Theta, \psi)$ of a finite-dimensional data $\delta \in \boldsymbol{\delta}$, conditioned on a finite-dimensional parameter $\Theta \in \boldsymbol{\Theta}$ and explanatory variables in $\psi \in \boldsymbol{\psi}$. Let the pd of the parameter Θ under the knowledge K be $\mathsf{P}(\Theta|K) = \mathsf{P}(\Theta|K, \psi)$, cf. (6). Let $\mathsf{F}(\delta) = \mathsf{F}(\delta|\text{external knowledge})$ be an externally supplied pd describing data δ. Then, the description of unknown parameter respecting this information is the pd

$$\mathsf{P}(\Theta|\mathsf{F}, K) = \frac{\mathsf{P}(\Theta|K)\exp[\mu\int_{\boldsymbol{\delta}}\mathsf{F}(\delta)\ln(\mathsf{M}(\delta|\Theta,\psi))\mathrm{d}\delta]}{\int_{\boldsymbol{\Theta}}\mathsf{P}(\Theta|K)\exp[\mu\int_{\boldsymbol{\delta}}\mathsf{F}(\delta)\ln(\mathsf{M}(\delta|\Theta,\psi))\mathrm{d}\delta]\mathrm{d}\Theta}, \tag{18}$$

where the optional scalar $\mu > 0$ informally expresses amount of *informative* data items used for creating the pd $\mathsf{F}(\delta)$.

Specialisation of (18) to is straightforward and appealing.

Proposition 3 (How Data Pd Modifies Pd of Parameter in EF): Let us consider the parametric model in EF, $\mathsf{M}(\delta|\Theta, \psi) = \exp\langle \mathsf{A}(\Psi), \mathsf{B}(\Theta)\rangle$, and the conjugate pd,

$$\begin{aligned}\mathsf{P}(\Theta|K) &= \mathsf{P}(\Theta|V) = \exp\langle V, \mathsf{B}(\Theta)\rangle / \mathsf{J}(V)\\ \mathsf{J}(V) &= \int_{\boldsymbol{\Theta}} \exp\langle V, \mathsf{B}(\Theta)\rangle \ \mathrm{d}\Theta.\end{aligned}$$

Then, the pd $\mathsf{P}(\Theta|\mathsf{F}, K) = \mathsf{P}(\Theta|\mathsf{F}, V)$ (18) is also conjugate

$$\begin{aligned}\mathsf{P}(\Theta|\mathsf{F}, V) &= \mathsf{P}(\Theta|\bar{V}) = \frac{\exp\langle \bar{V}, \mathsf{B}(\Theta)\rangle}{\mathsf{J}(\bar{V})}\\ \bar{V} &= V + \mu\bar{\mathsf{A}}(\psi) \ \text{ with } \ \bar{\mathsf{A}}(\psi) = \int_{\boldsymbol{\delta}} \mathsf{A}(\delta, \psi)\mathsf{F}(\delta)\ \mathrm{d}\delta\\ \mathsf{J}(\bar{V}) &= \int_{\boldsymbol{\Theta}} \exp\langle \bar{V}, \mathsf{B}(\Theta)\rangle \ \mathrm{d}\Theta.\end{aligned} \tag{19}$$

Remarks 3:

- The proposed sharing of knowledge is directly applicable to any node pair at any time moment.
- The operation (19) behaves similarly as updating by observed data but takes into account its uncertainty assigned by the pd $\mathsf{F}(\delta)$. The processed information is corrupted by errors caused by approximate nature of shared predictors $\mathsf{F}(\delta)$. Thus, the use of stabilised forgetting (11) with the optimised factor (12), after its application is a must.
- Processing of the predictive pd corresponds to information updated by single data item, which hints to choosing $\mu = 1$. The choice can be varied when the information about the number of processed data since the last predictor sharing is available. Any finite choice $\mu \geq 1$ is acceptable as the subsequent forgetting tailors it.

C. Message Passing within the Supported Networks

The presented concept of control nodes interacting with their neighbours is extremely flexible: i) topologically – neighbours are simply those nodes, which share some external variables; the contemporary information technology makes spatial relations of secondary importance; ii) in acting – each node acts almost as being alone using its measurement and its input-selection time schedules; iii) design – individual nodes can be designed and implemented at various time moments.

These features follow from the fact that a node α deals with:

$u_{t;\alpha}$, system input, which is chosen *solely* by the node α – this is the only strict network-induced constraint, which has to be respected.

$y_{t;\alpha}$, system output, which is optimised by the node α, typically solely, but possibly by other nodes $\beta, \gamma, \ldots$ Conflicts may arise in the latter case and it is useful to avoid it again by respecting potential neighbours in the network.

$x_{t;\alpha}$, external variables, consisting of the node-dependent selection of the following variables

- $x_{t;\alpha s}$, external variables originating in the system with which node α interacts,
- $\delta_{t;\rho}$, which is a part of inputs of $u_{t;\rho}$ generated by neighbouring nodes $\rho = \beta, \gamma, \ldots$ or a part of the outputs $y_{t;\rho}$ optimised by them or a part of external variables $x_{t;\rho}$ predicted by them. This signal can be complemented by a predictor $\mathsf{F}_\rho(\delta_{t;\rho}|K_{t-1;\rho})$, which is exploited according to Proposition 2. Whenever $\delta_{t;\rho}$ is modelled by the node α within exponential family $\mathsf{M}_\alpha(\delta_{t;\rho}|\Theta_\alpha, \psi_{t;\alpha}) = \exp\langle \mathsf{A}_\alpha(\delta_{t;\rho}, \psi_{t;\alpha}), \mathsf{B}_\alpha(\Theta_\alpha)\rangle$ learnt within conjugate family, it suffices to pass the predictive moment $\bar{\mathsf{A}}_\alpha(\psi_{t;\alpha}) \equiv \int_{\boldsymbol{\delta}_{t;\rho}} \mathsf{A}_\alpha(\delta_{t;\rho}, \psi_{t;\alpha})\mathsf{F}_\rho(\delta\delta_{t;\rho})\ \mathrm{d}\delta\delta_{t;\rho}$, cf. Proposition 3. Here, the abundant use of indices α, ρ stresses the node-related origin of treated entities.

IV. Application to Linear Gaussian Case

In this section, the general methodology is applied to linear Gaussian system model and Gaussian ideal pd. This case is: i) the FPD counterpart of the widely used classical linear-quadratic control design (including its model predictive version [36]), which forms the firm basis in solving more complex problems by relying on linearisation; ii) solvable without additional approximations.

Hereafter, domains $\boldsymbol{y_t}$, $\boldsymbol{u_t}$, $\boldsymbol{x_t}$ coincide with multivariate real spaces but this fact is not stressed by notation. For clarity, y_t, u_t, x_t are treated as column vectors, i.e. $w_t = [y_t', x_t']'$.

A. FPD with Observed External Variables

The assumed system model (1) has the next first factor[5]

$$\begin{aligned}\mathsf{M}(y_t|u_t, w_{t-1}) &= \mathsf{N}_{y_t}(\mathbb{A}w_{t-1} + \mathbb{B}u_t, \mathbb{R}\mathbb{R}') \qquad (20)\\ \mathbb{A} &= [\mathbb{A}_{\boldsymbol{y}}, \mathbb{A}_{\boldsymbol{x}}] \text{ while with } ||z||^2 = z'z\\ \mathsf{N}_y(\hat{y}, \mathbb{R}\mathbb{R}') &= |2\pi\mathbb{R}\mathbb{R}'|^{-0.5}\exp\big[-0.5||\mathbb{R}^{-1}(y-\hat{y})||^2\big],\end{aligned}$$

where the matrices $\mathbb{A}$, $\mathbb{B}$ and the regular square-root $\mathbb{R}$ of covariance matrix are appropriately sized.

The second factor in (1), modelling external variables x_t, is

$$\mathsf{M}(x_t|x_{t-1}) = \mathsf{N}_{x_t}(\mathbb{C}x_{t-1}, \mathbb{R}_{\boldsymbol{x}}\mathbb{R}_{\boldsymbol{x}}'). \qquad (21)$$

It is given by the matrix $\mathbb{C}$, and the square-root $\mathbb{R}_{\boldsymbol{x}}$ of covariance matrix. The factors of the ideal pd (2) are

$$\begin{aligned}\mathsf{M}_{\mathsf{I}}(y_t|u_t, w_{t-1}) &= \mathsf{M}_{\mathsf{I}}(y_t|w_{t-1}) = \mathsf{N}_{y_t}(\mathbb{A}_{\mathsf{I}}w_{t-1} + \mathbb{B}_{\mathsf{I}}u_t, \mathbb{R}\mathbb{R}')\\ \mathsf{S}_{\mathsf{I}}(u_t|w_{t-1}) &= \mathsf{N}_{u_t}(\mathbb{D}_{\mathsf{I}}w_{t-1}, \mathbb{R}_{\mathsf{I}\boldsymbol{u}}\mathbb{R}_{\mathsf{I}\boldsymbol{u}}') \text{ with } |\mathbb{R}_{\mathsf{I}\boldsymbol{u}}| \neq 0\\ \mathbb{A}_{\mathsf{I}} &= [\mathbb{A}_{\mathsf{I}\boldsymbol{y}}, \mathbb{A}_{\mathsf{I}\boldsymbol{x}}], \quad \mathbb{D}_{\mathsf{I}} = [\mathbb{D}_{\mathsf{I}\boldsymbol{y}}, \mathbb{D}_{\mathsf{I}\boldsymbol{x}}]. \qquad (22)\end{aligned}$$

Remarks 4:

- The involved matrices are known in the design phase due to the use of the certainty-equivalence approximation.
- The matrices in the discussed pds are assumed to meet an algebraic condition guaranteeing existence and uniqueness of the optimal control laws, cf. Lemma 1.
- $\mathbb{A}_{\mathsf{I}}w_{t-1}$ defines the expected value of an externally generated set point $y_{t;s}$ of y_t. The set point $y_{t;s}$ does not depend on u_t, w_{t-1} and it is included into the vector x_t of external variables. Its expectation, however, may depend on w_{t-1}, especially, when the construction (16) of this ideal factor is used.
- The use of $\mathbb{R}\mathbb{R}'$ from the system model (20) as the ideal covariance follows from the recommended choice (16). The matrices $\mathbb{D}_{\mathsf{I}}$ and $|\mathbb{R}_{\mathsf{I}\boldsymbol{u}}| \neq 0$ are chosen so that inputs pushing the output to its set point $y_{t;s} \in \boldsymbol{y_t}$ maintain a high probability within a desired subset of $\boldsymbol{u_t}$.

Application of Proposition 1 to linear-Gaussian case uses:

Lemma 1 (Operations on Quadratic Form):

1) **Expected quadratic form:** Let $\hat{\delta} = \mathsf{E}[\delta]$ be expectation of a vector $\delta \in \boldsymbol{\delta}$ and $\mathbb{W}\mathbb{W}'$ its covariance. Then, expectation of the quadratic form $||\mathbb{U}\delta + v||^2$, determined by a given matrix $\mathbb{U}$ and a vector v, has the form

$$\mathsf{E}\big[||\mathbb{U}\delta + v||^2\big] = ||\mathbb{U}\hat{\delta} + v||^2 + \mathrm{tr}[\mathbb{U}\mathbb{W}\mathbb{W}'\mathbb{U}'],$$

where $\mathrm{tr}[\bullet]$ is matrix trace.

2) **Square-root-based completion of quadratic form:** Let us consider the sum of quadratic forms of $\ell_{\boldsymbol{u}}$, $\ell_{\boldsymbol{y}}$, and $\ell_{\boldsymbol{x}}$ vectors $u \in \boldsymbol{u}$, $y \in \boldsymbol{y}$, $x \in \boldsymbol{x}$, weighted by given weighing matrices $\mathbb{F}_{\boldsymbol{u}}$, $\mathbb{F}_{\boldsymbol{y}}$, $\mathbb{F}_{\boldsymbol{x}}$, $\mathbb{G}_{\boldsymbol{u}}$, $\mathbb{G}_{\boldsymbol{y}}$, $\mathbb{G}_{\boldsymbol{x}}$, $\mathbb{H}_u$, $\mathbb{H}_y$, $\mathbb{H}_x$ of appropriate dimensions. Let squares in $u \in \boldsymbol{u}$ and $y \in \boldsymbol{y}$ be completed. Then, there is an orthogonal matrix $\mathbb{T}$ guaranteeing the identity

$$\begin{aligned}&||\mathbb{F}_{\boldsymbol{u}}u + \mathbb{F}_{\boldsymbol{y}}y + \mathbb{F}_{\boldsymbol{x}}x||^2 + ||\mathbb{G}_{\boldsymbol{u}}u + \mathbb{G}_{\boldsymbol{y}}y + \mathbb{G}_{\boldsymbol{x}}x||^2\\ &\quad + ||\mathbb{H}_{\boldsymbol{u}}u + \mathbb{H}_{\boldsymbol{y}}y + \mathbb{H}_{\boldsymbol{x}}x||^2 \qquad (23)\\ &\quad = ||\mathbb{L}_{\boldsymbol{u}}u + \mathbb{L}_{\boldsymbol{y}}y + \mathbb{L}_{\boldsymbol{x}}x||^2 + ||\mathbb{E}_{\boldsymbol{y}}y + \mathbb{E}_{\boldsymbol{x}}x||^2\\ &\mathbb{T}\mathbb{Z} = \mathbb{T}\begin{bmatrix}\mathbb{F}_{\boldsymbol{u}} & \mathbb{F}_{\boldsymbol{y}} & \mathbb{F}_{\boldsymbol{x}}\\ \mathbb{G}_{\boldsymbol{u}} & \mathbb{G}_{\boldsymbol{y}} & \mathbb{G}_{\boldsymbol{x}}\\ \mathbb{H}_{\boldsymbol{u}} & \mathbb{H}_{\boldsymbol{y}} & \mathbb{H}_{\boldsymbol{x}}\end{bmatrix} = \begin{bmatrix}\mathbb{L}_{\boldsymbol{u}} & \mathbb{L}_{\boldsymbol{y}} & \mathbb{L}_{\boldsymbol{x}}\\ 0 & \mathbb{E}_{\boldsymbol{y}} & \mathbb{E}_{\boldsymbol{x}}\\ 0 & 0 & \mathbb{U}_{\boldsymbol{x}}\end{bmatrix},\end{aligned}$$

where $\mathbb{L}_{\boldsymbol{u}}$ is square $(\ell_{\boldsymbol{u}}, \ell_{\boldsymbol{u}})$ upper triangular matrix and $\mathbb{E}_{\boldsymbol{y}}$ is square $(\ell_{\boldsymbol{y}}, \ell_{\boldsymbol{y}})$ matrix. Such $\mathbb{T}$ exists if the initial $\ell_{\boldsymbol{u}} + \ell_{\boldsymbol{y}}$ columns of the matrix $\mathbb{Z}$ have full rank.

Proof

ad 1. It can be verified by direct evaluations.

ad 2. This square-root completion of squares in u and y was used for control purposes since the eighties, e.g. [37]. It exploits invariance of quadratic norms to rotations made by an orthogonal matrix $\mathbb{T}$, ($\mathbb{T}' = \mathbb{T}^{-1}$). Elementary rotations or QR algorithm [38] are examples of making the efficient matrix (block) triangularisation represented by (23). □

Proposition 4 (Linear Gaussian FPD with External Variables): Let us define $(\ell_{\boldsymbol{y}}, \ell_{\boldsymbol{w}})$ matrix $[\mathbb{E}_{|\boldsymbol{t}|;\boldsymbol{y}}, \mathbb{E}_{|\boldsymbol{t}|;\boldsymbol{x}}] = 0$ and perform the following iterations for $\tau = |\boldsymbol{t}|, |\boldsymbol{t}| - 1, \ldots, t$ consisting of triangularisations of $\mathbb{Z}_\tau =$

$$\begin{bmatrix}\mathbb{R}^{-1}(\mathbb{B} - \mathbb{B}_{\mathsf{I}}) & \mathbb{R}^{-1}(\mathbb{A}_{\boldsymbol{y}} - \mathbb{A}_{\mathsf{I}\boldsymbol{y}}) & \mathbb{R}^{-1}(\mathbb{A}_{\boldsymbol{x}} - \mathbb{A}_{\mathsf{I}\boldsymbol{x}})\\ \mathbb{R}_{\mathsf{I}\boldsymbol{u}}^{-1} & \mathbb{R}_{\mathsf{I}\boldsymbol{u}}^{-1}(\quad -\mathbb{D}_{\mathsf{I}\boldsymbol{y}}) & \mathbb{R}_{\mathsf{I}\boldsymbol{u}}^{-1}(\quad -\mathbb{D}_{\mathsf{I}\boldsymbol{x}})\\ \mathbb{E}_{\tau;\boldsymbol{y}}\mathbb{B} & \mathbb{E}_{\tau;\boldsymbol{y}}\mathbb{A}_{\boldsymbol{y}} & \mathbb{E}_{\tau;\boldsymbol{y}}\mathbb{A}_{\boldsymbol{x}} + \mathbb{E}_{\tau;\boldsymbol{x}}\mathbb{C}\end{bmatrix} \qquad (24)$$

by an orthogonal transformation $\mathbb{T}_\tau$ giving

$$\mathbb{T}_\tau\mathbb{Z}_\tau = \begin{bmatrix}\mathbb{L}_{\tau;\boldsymbol{u}} & \mathbb{L}_{\tau;\boldsymbol{y}} & \mathbb{L}_{\tau;\boldsymbol{x}}\\ 0 & \mathbb{E}_{\tau-1;\boldsymbol{y}} & \mathbb{E}_{\tau-1;\boldsymbol{x}}\\ 0 & 0 & \mathbb{U}_{\tau-1;\boldsymbol{x}}\end{bmatrix}. \qquad (25)$$

Then, the optimal strategy in the FPD sense (3), for the system model (20), (21) and the ideal pd (22) given by known matrices involved, is determined by the optimal control laws $\mathsf{S}_{\mathsf{O}}(u_t|w_{t-1}) = \mathsf{S}_{\mathsf{O}}(u_t|y_{t-1}, x_{t-1})$, $t \in \boldsymbol{t}$,

$$\begin{aligned}&\mathsf{S}_{\mathsf{O}}(u_t|y_{t-1}, x_{t-1}) \qquad (26)\\ &= \mathsf{N}_{u_t}\big(-\mathbb{L}_{t;\boldsymbol{u}}^{-1}(\mathbb{L}_{t;\boldsymbol{y}}y_{t-1} + \mathbb{L}_{t;\boldsymbol{x}}x_{t-1}),\ \mathbb{L}_{t;\boldsymbol{u}}^{-1}(\mathbb{L}_{t;\boldsymbol{u}}^{-1})'\big).\end{aligned}$$

Proof By induction for $\tau = |\boldsymbol{t}|, \ldots, t$, we shall show that

$$-2\ln(\gamma(y_\tau, x_\tau)) = ||\mathbb{E}_{\tau;\boldsymbol{y}}y_\tau + \mathbb{E}_{\tau;\boldsymbol{x}}x_\tau||^2 + h_\tau$$

given by $(\ell_{\boldsymbol{y}}, \ell_{\boldsymbol{y}})$ matrix $\mathbb{E}_{\tau;\boldsymbol{y}}$, $(\ell_{\boldsymbol{y}}, \ell_{\boldsymbol{x}})$ matrix $\mathbb{E}_{\tau;\boldsymbol{x}}$ and a data-independent offset h_τ. The starting value $\ln(\gamma(y_{|\boldsymbol{t}|}, x_{|\boldsymbol{t}|})) = 0$ has this form for $\mathbb{E}_{|\boldsymbol{t}|;\boldsymbol{y}} = 0$, $\mathbb{E}_{|\boldsymbol{t}|;\boldsymbol{x}} = 0$ and $h_{|\boldsymbol{t}|} = 0$, cf. the starting value in Proposition 1.

For $\tau \leq |\boldsymbol{t}|$, the definition of the function $\omega(u_\tau, y_{t-1}, x_{t-1})$ (4) and Lemma 1 imply the form of the exponent of the

[5]The set subscript at matrices indicates the vector variable by which the (sub)matrix is multiplied.

optimal control law, which – after completion squares in its exponent with respect to u_t – defines $\gamma(y_{t-1}, x_{t-1})$.

$$
\begin{aligned}
&-\omega(u_\tau, y_{\tau-1}, x_{\tau-1}) + \ln(\mathsf{S}_{\mathsf{I}}(u_\tau|y_{\tau-1}, x_{\tau-1}))\\
&= 0.5||\underbrace{\mathbb{R}^{-1}(\mathbb{B}-\mathbb{B}_{\mathsf{I}})}_{\mathbb{F}_{\boldsymbol{u}}} u_\tau + \underbrace{\mathbb{R}^{-1}(\mathbb{A}_{\boldsymbol{y}}-\mathbb{A}_{\mathsf{I}\boldsymbol{y}})}_{\mathbb{F}_{\boldsymbol{y}}} y_{\tau-1}\\
&+\underbrace{\mathbb{R}^{-1}(\mathbb{A}_{\boldsymbol{x}}-\mathbb{A}_{\mathsf{I}\boldsymbol{x}})}_{\mathbb{F}_{\boldsymbol{x}}} x_{\tau-1}||^2 + h\\
&+0.5||\underbrace{\mathbb{R}^{-1}_{\mathsf{I}\boldsymbol{u}}}_{\mathbb{G}_{\boldsymbol{u}}} u_\tau - \underbrace{\mathbb{R}^{-1}_{\mathsf{I}\boldsymbol{u}}\mathbb{D}_{\mathsf{I}\boldsymbol{y}}}_{\mathbb{G}_y} y_{\tau-1} - \underbrace{\mathbb{R}^{-1}_{\mathsf{I}\boldsymbol{u}}\mathbb{D}_{\mathsf{I}\boldsymbol{x}}}_{\mathbb{G}_x} x_{\tau-1}||^2\\
&+0.5||\underbrace{\mathbb{E}_{\tau;\boldsymbol{y}}\mathbb{B}}_{\mathbb{H}_{\tau;\boldsymbol{u}}} u_\tau + \underbrace{\mathbb{E}_{\tau;\boldsymbol{y}}\mathbb{A}_{\boldsymbol{y}}}_{\mathbb{H}_{\tau;\boldsymbol{y}}} y_{\tau-1} + \underbrace{(\mathbb{E}_{\tau;\boldsymbol{y}}\mathbb{A}_{\boldsymbol{x}} + \mathbb{E}_{\tau;\boldsymbol{x}}\mathbb{C})}_{\mathbb{H}_{\tau;\boldsymbol{x}}} x_{\tau-1}||^2\\
&= 0.5||\mathbb{L}_{\tau;\boldsymbol{u}}u_\tau + \mathbb{L}_{\tau;\boldsymbol{y}}y_{\tau-1} + \mathbb{L}_{\tau;\boldsymbol{x}}x_{\tau-1}||^2 +\\
&\quad 0.5||\mathbb{E}_{\tau-1;\boldsymbol{y}}y_{\tau-1} + \mathbb{E}_{\tau-1;\boldsymbol{x}}x_{\tau-1}||^2 + 0.5||\mathbb{U}_{\tau-1;\boldsymbol{x}}x_{\tau-1}||^2.
\end{aligned}
$$

This confirms the form of the optimal control law as well as of the assumed form of $\gamma(y_\tau, x_\tau)$. Notice that factors given by h_τ and $||\mathbb{U}_{\tau-1;\boldsymbol{x}}x_{\tau-1}||^2$ cancels in the definition of the optimal control law and does not enter $\gamma(y_\tau, x_\tau)$. □

Remarks 5:

- From complexity view point, it is important to notice that
  – the triangularisation runs on $\ell_{\boldsymbol{u}} + \ell_{\boldsymbol{y}}$ columns and $(\ell_{\boldsymbol{y}} + 2\times\ell_{\boldsymbol{u}})$ rows of $(\ell_{\boldsymbol{y}} + 2\times\ell_{\boldsymbol{u}})\times(\ell_{\boldsymbol{u}} + \ell_{\boldsymbol{w}})$ matrix,
  – inversions are only needed once and for small-sized triangular matrices $\mathbb{R}^{0.5}$, $\mathbb{R}^{0.5}_{\mathsf{I}}$ and $\mathbb{L}_{t;u}$,
  – square-root form of the covariance of the optimal control law (26) makes sampling from this pd simple.
- The recursions are equivalent to the Riccati equation corresponding to linear systems with external variables and quadratic criterion whose weight are inversions of covariance matrices of the ideal pd, cf. [11].
- The resulting controller is randomised and specific actions should be sampled from its pd: the control quality is slightly worse than using input equal to expected value as it respects constraints on entropy of the controller [13] but randomisation makes the controller explorative.

B. Parameter Estimation

In the following, we exploit the possibility to deal with predicting the scalar variable δ_t, see (14). Its linear Gaussian model casts into the EF form as follows

$$
\begin{aligned}
&\mathsf{M}(\delta_t|\Theta, \psi_t) = (2\pi r)^{-0.5}\exp[-0.5r^{-1}(\delta_t - \theta'\psi_t)^2]\\
=\;& \exp\Big\{1\times(-0.5\ln(2\pi r))\\
+\;& \mathrm{tr}\left[\Psi_t\Psi_t'\left(-0.5r^{-1}\begin{bmatrix}-1\\ \theta\end{bmatrix}\begin{bmatrix}-1,\ \theta'\end{bmatrix}\right)\right]\Big\}, \qquad (27)
\end{aligned}
$$

where data vector $\Psi_t = [\delta_t, \psi_t']'$ and the unknown parameter Θ consists of the vector of coefficients θ and noise variance r.

Proposition 5 (Estimation for Linear Gaussian Model): The form (27) corresponds to (5) with $\mathsf{A}(\Psi_t) = (1, \Psi_t\Psi_t')$

$$
\begin{aligned}
\mathsf{B}(\Theta) &= \left(-0.5\ln(2\pi r), -0.5r^{-1}\begin{bmatrix}-1\\ \theta\end{bmatrix}\begin{bmatrix}-1,\ \theta'\end{bmatrix}\right)\\
V_t &= (\nu_t, \mathbb{V}_t),\ \nu_t = \nu_{t-1} + 1\\
\mathbb{V}_t &= \mathbb{V}_{t-1} + \Psi_t\Psi_t',\quad \nu_0,\ \mathbb{V}_0 \text{ chosen a priori.}
\end{aligned}
$$

The conjugate prior pd is Gauss-inverse-gamma pd. It is proper iff $\mathbb{V}_0 > 0$ and $\nu_0 > 0$ when takes the next form, with $\ell_{\boldsymbol{\theta}}$ equal to the number of θ coefficients and $t \geq 0$,

$$
\begin{aligned}
\mathsf{P}(\theta, r|\nu_t, \mathbb{V}_t) &= \frac{1}{r^{0.5(\nu_t+\ell_{\boldsymbol{\theta}}+2)}\mathsf{J}(\nu_t, \mathbb{V}_t)}\\
&\times\quad \exp\left\{-0.5r^{-1}\Big[-1,\ \theta'\Big]\mathbb{V}_t\begin{bmatrix}-1\\ \theta\end{bmatrix}\right\}\\
&= \frac{1}{r^{0.5(\nu_t+\ell_{\boldsymbol{\theta}}-2)}\mathsf{J}(\nu_t, \mathbb{V}_t)}\\
&\times\quad \exp\left\{-0.5r^{-1}[(\theta-\hat\theta_t)'\mathbb{W}_t^{-1}(\theta-\hat\theta_t) + \nu_t\hat r_t]\right\}\\
\mathbb{V} &= \begin{bmatrix}\mathbb{V}_{\boldsymbol{\delta}} & \mathbb{V}'_{\boldsymbol{\delta\psi}}\\ \mathbb{V}_{\boldsymbol{\delta\psi}} & \mathbb{V}_{\boldsymbol{\psi}}\end{bmatrix} \text{ with scalar } \mathbb{V}_{\boldsymbol{\delta}} \text{ defines}\\
\hat\theta &= \mathbb{V}_{\boldsymbol{\psi}}^{-1}\mathbb{V}_{\boldsymbol{\delta\psi}},\ \mathbb{W} = \mathbb{V}_{\boldsymbol{\psi}}^{-1}, \hat r = \frac{\mathbb{V}_{\boldsymbol{\delta}} - \mathbb{V}'_{\boldsymbol{\delta\psi}}\mathbb{V}_{\boldsymbol{\psi}}^{-1}\mathbb{V}'_{\boldsymbol{\delta\psi}}}{\nu}\\
\mathsf{J}(\nu_t, \mathbb{V}_t) &= \hat r_t^{-0.5\nu_t}\,|\mathbb{V}_{t;\boldsymbol{\psi}}|^{-0.5}\,\Gamma(0.5\nu_t)(0.5\nu_t)^{-0.5\nu_t}(2\pi)^{0.5\ell_{\boldsymbol{\psi}}}\\
\Gamma(z) &= \int_0^\infty v^{z-1}\exp(-v)\,\mathrm{d}v < \infty \quad \text{for } z > 0
\end{aligned}
$$

$\mathsf{F}(\delta|\psi, \nu_t, \mathbb{V}_t)$ is Student distribution with moments

$$
\begin{aligned}
&\mathsf{E}[\delta|\psi, \nu_t, \mathbb{V}_t] = \hat\theta_t'\psi,\quad \mathsf{variance}[\delta|\psi, \nu_t, \mathbb{V}_t] = \hat{\mathsf{r}}_t(1+\zeta_t)\\
&\zeta_t = \psi'\mathbb{W}\psi. \qquad (28)
\end{aligned}
$$

Remarks 6:

- The equivalent expression of the sufficient statistic $\nu, \mathbb{V}$ via $\nu, \hat\theta, \mathbb{W}, \hat r$ connect the discussed estimation with recursive least squares [20]. The algorithm implemented using factorised version of $\mathbb{V}$ makes it numerically robust and simplifies evaluation of the normalisation factor J needed for selecting forgetting factor ϕ_t (12).
- The form of the normalising factor J can be found in [20], too. There it is shown that the factorised version of recursive estimation [20] makes the evaluation of $\hat r_t$ and $|\mathbb{V}_{t;\psi}|$ computationally cheap and, as said, robust.
- The statistic values $\hat\theta$, $\hat r$ are maximum a posteriori point estimates of θ, r. They serve for the certainty-equivalence-based approximation of the system model (10).

C. Sharing of Knowledge Brought by Predictors

Sharing of knowledge brought by predictors consists of a simple specialisation of Proposition (3) to linear Gaussian case with conjugate Gaussian-inverse-gamma posterior pd.

Proposition 6 (Proposition 3 for Linear Gaussian Model): Let us consider that node β uses the linear Gaussian parametric model (27) and the conjugate Gaussian-inverse-gamma pd (28) determined by statistics $\nu_\beta, \mathbb{V}_\beta$. Then, this pd corrected by the predictor offered by node α with moments

$$
\hat\delta_\alpha = \mathsf{E}_\alpha[\delta],\ \mathsf{variance}_\alpha(\delta) = \hat{\mathsf{r}}_{\boldsymbol{\delta}\alpha}
$$

is also conjugate Gaussian-inverse-gamma pd determined by

$$
\begin{aligned}
\bar\nu_\beta &= \nu_\beta + \mu \qquad (29)\\
\bar{\mathbb{V}}_\beta &= \mathbb{V}_\beta + \mu\begin{bmatrix}\hat\delta_\alpha\\ \psi_\beta\end{bmatrix}[\hat\delta_\alpha, \psi_\beta] + \mu\begin{bmatrix}\hat r_{\boldsymbol{\delta}\alpha} & 0\\ 0 & 0\end{bmatrix}
\end{aligned}
$$

If the predictor results from learning of the linear Gaussian system model used by node α then the moments are

$$\begin{aligned}\hat{\delta}_\alpha &= \hat{\theta}'_\alpha\psi_\alpha = \mathbb{V}'_{\boldsymbol{\delta\psi}\alpha}\mathbb{V}^{-1}_{\boldsymbol{\psi}\alpha}\psi_\alpha\\ \hat{r}_{\boldsymbol{\delta}\alpha} &= \frac{\mathbb{V}_{\boldsymbol{y}\alpha} - \mathbb{V}'_{\boldsymbol{\delta\psi}\alpha}\mathbb{V}^{-1}_{\boldsymbol{\psi}\alpha}\mathbb{V}_{\boldsymbol{\delta\psi}\alpha}}{\nu_\alpha}(1+\zeta_\alpha)\\ \zeta_\alpha &= \psi'_\alpha\mathbb{V}^{-1}_{\boldsymbol{\psi}\alpha}\psi_\alpha,\end{aligned}$$

where the used symbols are defined in Proposition 5. If the predictor coincides with the control law (26) ($\delta_\alpha = u_\alpha$) then

$$\hat{\delta}_\alpha = -\mathbb{L}^{-1}_{t-1;\boldsymbol{u}\alpha}\mathbb{L}_{t;\boldsymbol{w}\alpha}w_{t-1;\alpha},\ \hat{r}^{-1}_{\boldsymbol{\delta}\alpha} = \mathbb{L}'_{t-1;\boldsymbol{u}\alpha}\mathbb{L}_{t-1;\boldsymbol{u}\alpha},$$

where the coefficients and the covariance are the final values obtained from iterations described by Proposition 4 and $w_{t-1;\alpha}$ contain data to be used for generating the predicted $\delta_\alpha = u_\alpha$.

Remarks 7:

- The verification of Proposition 6 is straightforward. The attention is to be paid for distinguishing data processed by the predictor α from those used in estimation by node β.
- The formula (29) is intuitively appealing as it replaces the unavailable δ by its prediction made by the neighbour α but at the same time it respects precision of this prediction: if $\hat{r}_{\boldsymbol{\delta}\alpha}$ is large then the corrected pd gets the large $\hat{r}_{\boldsymbol{\delta}\beta}$. It can be interpreted as an addition of data $\Psi' = [\sqrt{\hat{r}_{\boldsymbol{\delta}\alpha}}, 0]$ with the predicted variable unrelated to explanatory variables.
- The application of the stabilised forgetting can lead to a complete suppression of the correction resulting from the predictor when it does not improve prediction quality.

V. On Experiments

The first part provides illustrative simulation results indicating the use of the presented theory. The second one just summarises experience, we gained from extensive (for the space sake unreported) simulation experiments.

A. Illustrative Example

This section describes simulation of interactions of a pair interacting nodes controlling a linearised version of coupled map lattice (CML) with periodic boundary conditions. CML is disturbed by white zero mean Gaussian noise κ_{t+1} with covariance matrix $0.001\mathbb{I}$. Its un-controlled dynamics is described by $X_{t+1} = \mathbb{A}_S X_t + \kappa_{t+1}$, where

$$\mathbb{A}_S = \eta\begin{bmatrix} 1-2\varepsilon & \varepsilon & 0 & \dots & \varepsilon\\ \varepsilon & 1-2\varepsilon & \varepsilon & \dots & 0\\ 0 & \varepsilon & 1-2\varepsilon & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \varepsilon & 0 & 0 & \dots & 1-2\varepsilon\end{bmatrix} \quad (30)$$

is the $\ell_{\boldsymbol{X}} \times \ell_{\boldsymbol{X}}$ Jacobian matrix, ε is the coupling strength, $\eta = \frac{\partial f(z)}{\partial z}\ |_{z=z^\star}$, $z^\star = 1 - 1/a$ is the homogeneous steady state of the lattice, and $f(z)$ is a logistic local map, a nonlinear function with parameter a that describes the nonlinear dynamical behavior of CML,

$$\begin{aligned} & z^j_{t+1} = F(z^{j-1}_t, z^j_t, z^{j+1}_t)\\ = \ & f[(1-2\varepsilon)z^j_t + \varepsilon(z^{j-1}_t + z^{j+1}_t)] + \kappa^j_{t+1}, \end{aligned} \quad (31)$$

where $j = 1, 2, \dots, \ell_{\boldsymbol{X}}$ label the lattice sites z^j, and $\ell_{\boldsymbol{X}}$ is the system size. For detailed description of the CML, the readers are referred to [39] where CML has been used to illustrate theoretical developments for probabilistic pinning control of complex dynamical networks.

The reported pair of experiments compares the proposed adaptive distributed probabilistic control with the global probabilistic pinning control [39]. In these experiments, the lattice is initiated by $X = X_0$ and the control aim is to keep X_t as close as possible to the origin. The parameters of the CML are taken to be $a = 3.0$, $\ell_{\boldsymbol{X}} = 4$, and $\varepsilon = 0.33$, yielding

$$\mathbb{A}_S = \begin{bmatrix} -0.34 & -0.33 & 0 & -0.33\\ -0.33 & -0.34 & -0.33 & 0\\ 0 & -0.33 & -0.34 & -0.33\\ -0.33 & 0 & -0.33 & -0.34\end{bmatrix}. \quad (32)$$

In the first experiment, the presented theory is applied. The control of the four-dimensional lattice is treated as a pair of control tasks, one for each node. Node α takes $X_{t+1;1} = y_{t+1;1\alpha}$, $X_{t+1;2} = y_{t+1;2\alpha}$ and takes $X_{t+1;3} = x_{t+1;1\alpha}$ as an external variable. Hence, the system model (1) of node α has factors (the simulated system does not imply $\mathbb{C}_\rho$-entries),

$$\begin{aligned} \mathsf{M}_\alpha(y_t|u_t, w_{t-1}) &= \mathsf{N}_{y_t}(\mathbb{A}_\alpha w_{t-1} + \mathbb{B}_\alpha u_t, \mathbb{RR}'),\\ \mathbb{A}_\alpha &= \begin{bmatrix} -0.34 & -0.33 & 0\\ -0.33 & -0.34 & -0.33\end{bmatrix}, \quad \mathbb{B}_\alpha = \begin{bmatrix} 1\\ 1\end{bmatrix},\\ \mathsf{M}_\alpha(x_t|x_{t-1}) &= \mathsf{N}_{x_t}(\mathbb{C}_\alpha x_{t-1}, \mathbb{R}_{\boldsymbol{x}}\mathbb{R}'_{\boldsymbol{x}})\\ \mathbb{C}_\alpha &= \begin{bmatrix} 0 & 0 & c_{3,3}\end{bmatrix}. \end{aligned} \quad (33)$$

Node β is responsible for $X_{t+1;3} = y_{t+1;1\beta}$ and $X_{t+1;4} = y_{t+1;2\beta}$. It identifies $X_{t+1;1} = x_{t+1;1\beta}$, $X_{t+1;2} = x_{t+1;2\beta}$ and the first input $U_{t;1} = x_{t+1;3\beta}$, i.e. treats them as external signals. Hence, model (1) of node β has two factors

$$\begin{aligned} \mathsf{M}_\beta(y_t|u_t, w_{t-1}) &= \mathsf{N}_{y_t}(\mathbb{A}_\beta w_{t-1} + \mathbb{B}_\beta u_t, \mathbb{RR}'),\\ \mathbb{A}_\beta &= \begin{bmatrix} -0.34 & -0.33 & 1 & 0 & -0.33\\ -0.33 & -0.34 & 1 & -0.33 & 0\end{bmatrix},\ \mathbb{B}_\beta = \begin{bmatrix} 1\\ 1\end{bmatrix}\\ \mathsf{M}(x_t|x_{t-1}) &= \mathsf{N}_{x_t}(\mathbb{C}_\beta x_{t-1}, \mathbb{R}_{\boldsymbol{x}}\mathbb{R}'_{\boldsymbol{x}}),\\ \mathbb{C}_\beta &= \begin{bmatrix} 0 & 0 & c_{u_1,u_1} & c_{u_1,1} & c_{u_1,2}\\ 0 & 0 & c_{1,u_1} & c_{1,1} & c_{1,2}\\ 0 & 0 & c_{2,u_1} & c_{2,1} & c_{2,2}\end{bmatrix}. \end{aligned}$$

The entries of the matrices $\mathbb{A}_\rho$, $\mathbb{B}_\rho$ $\mathbb{C}_\rho$, $\rho \in \{\alpha, \beta\}$, are assumed to be unknown to controllers (except those zero entries which are enforced by treating some signals as external variables). They are (on line) recursively estimated using the Bayesian technique recalled in Section IV-B.

The typical resulting trajectories are in Figures 1, which confirm that in spite of the crude approximation adopted by the distributed controllers the global behavior of the overall closed loop is satisfactory as seen from the comparative experiment.

In a comparative experiment, one controller is designed using the probabilistic pinning control methodology, where the length 4 lattice is controlled using two control signals that are placed next to each other at the sides of the lattice [39], thus yielding the following controlled version of CML,

$$\begin{aligned} X_{t+1} &= \mathbb{A}_S X_t + \mathbb{B}_S U_{t+1} + \kappa_{t+1},\ \text{where}\\ \mathbb{A}_S &\text{ is given by (32) and } \mathbb{B}_S = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}' \end{aligned} \quad (34)$$

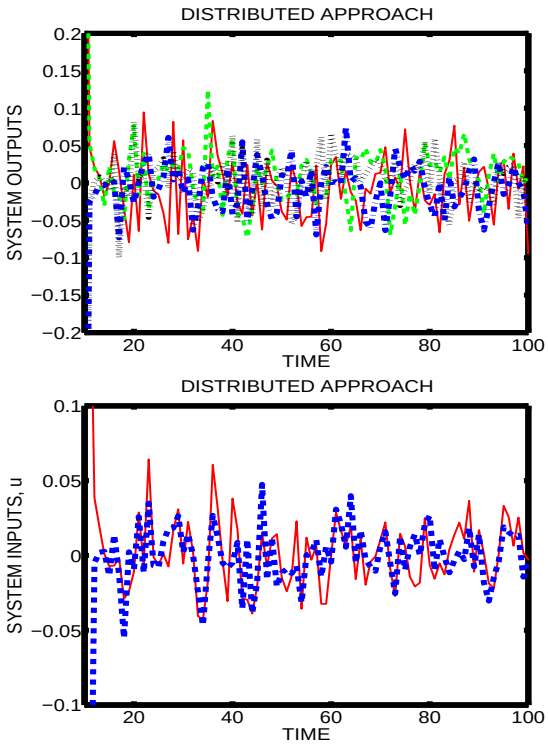

Fig. 1. Outputs (up) and inputs (bellow) of a non chaotic coupled map lattice with, $\ell_{\boldsymbol{X}} = 4$, $a = 3$, and $\varepsilon = 0.33$ resulting from the distributed adaptive fully probabilistic controller.

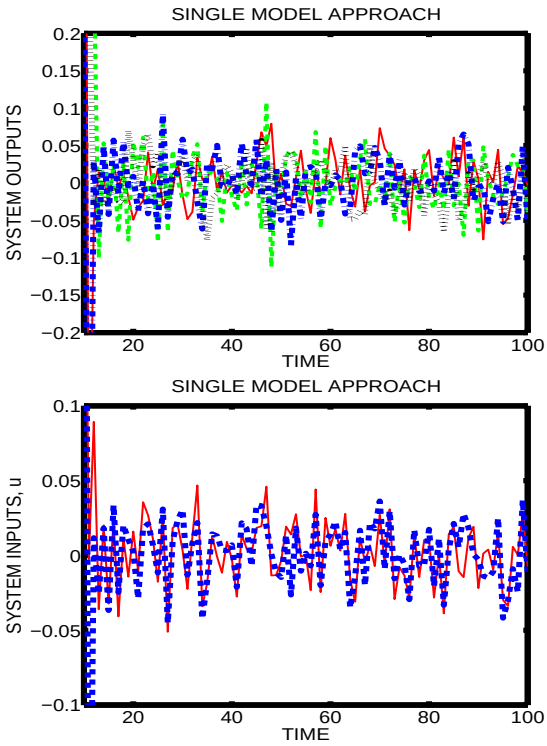

Fig. 2. Outputs (up) and inputs (bellow) of a non chaotic coupled map lattice with, $\ell_{\boldsymbol{X}} = 4$, $a = 3$, and $\varepsilon = 0.33$ for adaptive controller based on the single model.

and where κ_{t+1} is a Gaussian noise with covariance matrix $0.001\mathbb{I}$, and $U_{t+1} = [u_{1,t+1}, u_{1,t+1}]'$ is the vector of control inputs. The parameters of the lattice given in (34) are assumed to be unknown and recursively estimated. The typical resulting trajectories are in Figures 2. The global solution even achieves a slightly worse quality than that of the distributed solution. This can be intuitively expected as the distributed probabilistic controllers estimate less parameters than the global pinning controller. The difference diminishes with the diminishing noise term, κ_{t+1} in (30).

B. Simulation Experience

The experience listed below comes from experiments with: i) stabilisation of the coupled map lattice (30) for various $\ell_{\boldsymbol{X}}$, a, ε with various noise realisations; ii) another extensively simulated high-dimensional linear system, referred to as Flock.

Matrices of Flock were chosen to imitate linear, stochastically disturbed movement of a flock controlled by acceleration (deceleration) of individual agents among several tens of simulated ones. Each was described by a position and velocity. The control objective, expressed by individual ideal model, was to keep the same velocity as the right neighbour while keeping a distance from it. The most right-hand side agent aimed to follow externally supplied position. All agents, thus dealt with two-dimensional output, scalar input and two-dimensional external variables (neighbour's position and speed).

The observations we feel worth sharing are:

- The general solution mostly worked very satisfactorily.
- Stability of whole network is *not* guaranteed but instability was recorded quite rarely. The adaptive specification of penalties inherent to the proposed methodology (cf. the discussion near (16)) seems to be responsible for it.
- The data-dependent forgetting worked as expected: its average values have exponentially approached unity and individual values stay there even in long runs (several thousands of simulation steps). This has contributed to the (mostly) satisfactory behaviour of the whole network.
- Randomised nature of the constructed controller indeed helped to move parameter estimates into meaningful areas. This has allowed us to have extremely short open-loop learning period (at most several tens was sufficient).
- Sharing of probabilistic information did help in the achieved quality but, for the made simulations, the improvements were more minor than we expected (any reasonable statistical test would take them as insignificant).

VI. Concluding Remarks

Complexity of networks of interacting control nodes in current society and technology, together with the quest for improving their behaviour, makes the addressed problem extremely important. The *systematic overall solution* for an important and widely met class of distributed control problems is the main contribution of the paper. It is achieved by: i) the adopted use of observed signals as external variables; ii) the use of local adaptive controllers with built-in forgetting mechanism; iii) strictly respecting limited evaluation abilities of local controllers; iv) the novel use of exploiting external

data predictors for correcting parameter estimation; v) the full exploitation of probabilistic machinery enhanced by the adopted fully probabilistic design of controllers.

The proposed solution is directly applicable to controlled Markov chains (Markov Decision Processes [40]). It is expected to be feasible for the general cases of mixed discrete and continuous data. The solution can be used as a building block of hierarchical, possibly pinning, set point control.

The missing analysis of stability, quality and emergent network behaviour is the main gap to be filled in. Also, an explanation of the weaker-than-expected contribution of predictor-based knowledge sharing is to be inspected. The optimistic hypothesis that the control was too much successful even without it has to be tested.

In spite of open problems, the achieved state of development and available experimental evidence make it worth of putting a further effort into the control-design direction described in the paper. The extreme application potential width is the decisive reason for a further development of our solution. It suits Industry 4.0 [41], which approaches the production processes as the complex cyber-physical systems with control networks organised in the way advocated our paper. Similarly, control of town traffic via traffic lights [42], energy intelligent buildings [43], naturally distributed markets, etc. are technically ready for advantageous use of the presented concept.

Acknowledgements

This research has been supported by GAČR 13-13502S.

REFERENCES

[1] A. Das and F. Lewis, "Distributed adaptive control for synchronization of unknown nonlinear networked systems," *Automatica*, vol. 46, pp. 2014–2021, 2010.

[2] M. Porfiri and M. di Bernardo, "Criteria for global pinning controllability of complex networks," *Automatica*, vol. 44, pp. 3100–3106, 2008.

[3] L. Zdeborová, A. Decelle, and M. Chertkov, "Message passing for optimization and control of a power grid: Model of a distribution system with redundancy," *Phys. Rev. E*, vol. 80:046112, 2009.

[4] R. Gallager, *Low-Density Parity Check Codes*. M.I.T. Press, Cambridge, MA, 1963.

[5] J. Pearl, *Probabilistic Reasoning in Intelligent Systems*. Morgan Kaufman, San Mateo, CA, 1988.

[6] G. Parisi, M. Mézard, and M. Virasoro, *Spin Glass Theory and Beyond*. World Scientific, Singapore, 1987.

[7] L. Bakule, "Decentralized control: An overview," *Annual Reviews in Control*, vol. 32, no. 1 (2008), pp. 87–98, 2008.

[8] L. Bakule, "Decentralized control: Status and outlook," *Annual Reviews in Control*, vol. 38, no. 1, pp. 71–80, 2014.

[9] M. Kárný, "Approximate Bayesian recursive estimation," *Information Sciences*, vol. 289, pp. 100–111, 2014.

[10] R. Kulhavý and M. B. Zarrop, "On a general concept of forgetting," *Int. J. of Control*, vol. 58, no. 4, pp. 905–924, 1993.

[11] M. Kárný, "Towards fully probabilistic control design," *Automatica*, vol. 32, no. 12, pp. 1719–1722, 1996.

[12] M. Kárný and T. V. Guy, "Fully probabilistic control design," *Systems & Control Letters*, vol. 55, no. 4, pp. 259–265, 2006.

[13] M. Kárný and T. Kroupa, "Axiomatisation of fully probabilistic design," *Information Sciences*, vol. 186, no. 1, pp. 105–113, 2012.

[14] S. Kullback and R. Leibler, "On information and sufficiency," *Annals of Mathematical Statistics*, vol. 22, pp. 79–87, 1951.

[15] J. Kracík and M. Kárný, "Merging of data knowledge in Bayesian estimation," in *Proc. of the Second Int. Conference on Informatics in Control, Automation and Robotics*, J. Filipe, J. A. Cetto, and J. L. Ferrier, Eds. Barcelona: INSTICC, 2005, pp. 229–232.

[16] M. Rao, *Measure Theory and Integration*. NY: John Wiley, 1987.

[17] P. Guan, M. Raginsky, and R. Willett, "Online Markov decision processes with KullbackLeibler control cost," *IEEE Trans. on Automatic Control*, 2014.

[18] E. Todorov, "Linearly-solvable Markov decision problems," in *Advances in Neural Inf. Processing*, B. Schölkopf and et al, Eds. NY: MIT Press, 2006, pp. 1369 – 1376.

[19] D. Bertsekas, *Dynamic Programming and Optimal Control*. US: Athena Scientific, 2001.

[20] V. Peterka, "Bayesian system identification," in *Trends and Progress in System Identification*, P. Eykhoff, Ed. Oxford: Pergamon Press, 1981, pp. 239–304.

[21] O. Barndorff-Nielsen, *Information and exponential families in statistical theory*. New York: Wiley, 1978.

[22] R. Koopman, "On distributions admitting a sufficient statistic," *Trans. of American Mathematical Society*, vol. 39, p. 399, 1936.

[23] J. Berger, *Statistical Decision Theory and Bayesian Analysis*. New York: Springer, 1985.

[24] A. Feldbaum, "Theory of dual control," *Autom. Remote Control*, vol. 21, no. 9, 1960.

[25] J. Si, A. Barto, W. Powell, and D. Wunsch, Eds., *Handbook of Learning and Approximate Dynamic Programming*. Danvers: Wiley-IEEE Press, May 2004.

[26] M. Kárný, K. Macek, and T. Guy, "Lazy fully probabilistic design of decision strategies," in *11th International Symposium on Neural Networks*, ser. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), 2014, no. 8866, pp. 140–149.

[27] P. Kumar, "A survey on some results in stochastic adaptive control," *SIAM J. Control and Applications*, vol. 23, pp. 399–409, 1985.

[28] L. Chen and P. Pu, "Survey of preference elicitation methods," Human Computer Interaction Group Ecole Politechnique Federale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland, Tech. Rep. IC/2004/67, 2004.

[29] M. Kárný and T. Guy, "Preference elicitation in fully probabilistic design of decision strategies," in *Proc. of the 49th IEEE Conference on Decision and Control*, 2010.

[30] M. Kárný and T. Guy, "On the origins of imperfection and apparent non-rationality," in *Decision Making: Uncertainty, Imperfection, Deliberation and Scalability*, T. Guy, M. Kárný, and D. Wolpert, Eds. Springer, 2014, pp. 57–91.

[31] A. Quinn, M. Kárný, and T. Guy, "Fully probabilistic design of hierarchical Bayesian models," *Information Sciences*, 2015, under review.

[32] M. Kárný, A. Bodini, T. Guy, J. Kracík, P. Nedoma, and F. Ruggeri, "Fully probabilistic knowledge expression and incorporation," *Statistics and Its Interface*, vol. 7, no. 4, pp. 503–515, 2014.

[33] J. V. Campenhout and T. Cover, "Maximum entropy and conditional probability," *IEEE Tran. on Inf. Theory*, vol. 27, no. 4, pp. 483–489, 1981.

[34] J. Shore and R. Johnson, "Axiomatic derivation of the principle of maximum entropy & the principle of minimum cross-entropy," *IEEE Tran. on Inf. Th.*, vol. 26, no. 1, pp. 26–37, 1980.

[35] M. Kárný and T. Guy, "On support of imperfect Bayesian participants," in *Decision Making with Imperfect Decision Makers*, T. Guy, M. Kárný, and D. Wolpert, Eds. Berlin: Springer, 2012, vol. 28, Intelligent Systems Reference Library.

[36] S. Qin and T. Badgwell, "A survey of industrial model predictive control technology," *Control Engineering Practice*, vol. 11, no. 7, pp. 733–764, 2003.

[37] V. Peterka, "Control of uncertain processes: applied theory and algorithms," *Kybernetika*, vol. 22, 1986, supplement to No. 3, 4 ,5, 6.

[38] G. Golub and C. V. Loan, *Matrix Computations*. Johns Hopkins University Press, 2012.

[39] R. Herzallah, "Probabilistic synchronisation of pinning control," *International Journal of Control*, vol. 85, no. 11, pp. 1708–1716, 2012.

[40] H. Mine and S. Osaki, *Markovian Decision Processes*. New York: Elsevier, 1970.

[41] M. Brettel, N. Friederichsen, M. Keller, and M. Rosenberg, "How virtualization, decentralization and network building change the manufacturing landscape: An industry 4.0 perspective," *International Journal of Mechanical, Aerospace, Industrial, Mechatronic and Manufacturing Engineering*, vol. 8, no. 1, pp. 71–80, 2014.

[42] K. M. Yousef, J. N. Al-Karaki, and A. Shatnawi, "Intelligent traffic light flow control system using wireless sensors networks," *Journal Of Information Science And Engineering*, vol. 26, pp. 753–768, 2010.

[43] T. Nguyen and M. Aiello, "Energy intelligent buildings based on user activity: A survey," *Energy and Buildings*, vol. 56, pp. 244–257, 2013.