Finite element analysis of the statics of and vibrations in axisymmetric shells

Wajih Shahla
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FINITE ELEMENT ANALYSIS
OF
THE STATICS OF AND VIBRATIONS IN
AXISYMMETRIC SHELLS
To my mother and father,
sisters and brothers
whose encouragement and support
enabled me to pursue
the work that I enjoy.

A thesis submitted for
the degree of
Doctor of Philosophy

Department of Mechanical and Production Engineering
The University of Aston in Birmingham
August, 1985
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BY

WAJIH SHAHLA, B.Sc., M.Sc.

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Wajih Shahla
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Summary

The finite element process is now used almost
routinely as a tool of engineering analysis. From early
days, a significant effort has been devoted to developing
simple, cost effective elements which adequately fulfill
accuracy requirements.

In this thesis we describe the development and
application of one of the simplest elements available for
the statics and dynamics of axisymmetric shells. A semi-
analytic truncated cone stiffness element has been
formulated and implemented in a computer code: it has two
nodes with five degrees of freedom at each node,
circumferential variations in displacement field are
described in terms of trigonometric series, transverse
shear is accommodated by means of a penalty function and
rotary inertia is allowed for.

The element has been tested in a variety of
applications in the statics and dynamics of axisymmetric
shells subjected to a variety of boundary conditions.
Good results have been obtained for thin and thick shell
cases.

Keywords
AXISYMMETRIC SHELL, SEMI-ANALYTIC FINITE ELEMENT METHOD
PENALTY FUNCTION, TRANSVERSE SHEAR, ROTARY INERTIA
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NOMENCLATURE

The following list summarises the notations used throughout this thesis. Local constants and variables do not appear in this list, they are defined locally in the text where they appear. Rectangular matrices are indicated by [ ], and column vectors by { }. A dot over a symbol indicates differentiation with respect to time.

\( a \)  
outer radius of annular plate

\( b \)  
inner radius of annular plate

\([B],[B_1],[B_2]\)  
strain-nodal displacement matrices

\([\bar{B}],[\bar{B}_1],[\bar{B}_2]\)  
strain-nodal displacemten matrices after integrating with respect to \( \theta \)

\( C_n,C_m \)  
\( \cos n\theta, \cos m\theta \)

\( S_n,S_m \)  
\( \sin n\theta, \sin m\theta \)

\([C]\)  
damping matrix

\( D \)  
flexural rigidity

\( D_s \)  
shear rigidity

\([D]\)  
elasticity matrix

\( e \) (superscript)  
a typical element

\( E \)  
Young's Modulus for isotropic materials

\( E_f \)  
Young's Modulus for a fictitious element

\( E_i \)  
Young's Modulus in direction \( i \) for orthotropic materials

\( G \)  
Shear modulus for isotropic materials
$G_{ij}$  Shear modulus in i, j axes for orthotropic materials

$h$  thickness

$I$  second moment of inertia

$[I]$  identity matrix

$i, j$  node numbers for a typical element

$n[K]^e$  element stiffness matrix for a particular harmonic n

$n[K_1]^e, n[K_2]^e$  element stiffness matrices derived from the meridional and circumferential transverse shear energies, respectively, for a particular harmonic n

$n[K]$  the overall stiffness matrix for a particular harmonic n

$L$  length of a shell

$L_e, L$  length of an element

$L_f$  length of a fictitious element

$[L], [L_1], [L_2]$  strain-displacement operators

$m$  number of axial half waves

$m$ (subscript) $M_\phi, M_\Theta, M_\phi \Theta$

$[M]^e$  element mass matrix

$[M]$  overall mass matrix

$n$  number of circumferential waves

$n$ (subscript) any harmonic

$N_i, N_j$  shape functions at node i and j respectively
\[ [N] \] shape function matrix with respect to the meridional sense, \( s \) or \( n \)

\[ n[N_1], n[N_2] \] symmetric and antisymmetric shape functions with respect to the circumferential sense, \( \theta \)

\[ N_\phi, N_\theta, N_{\phi \theta} \]
\[ N_s, N_{\theta s}, N_{s \theta} \]
\[ n[p] \]

forces (per unit length) element force vector for a particular harmonic \( n \)

\[ n[p_d]^e \] element force vector from distributed load

\[ n[p_r]^e \] element force vector from nodal ring load

\[ n[p] \] overall force vector for a particular harmonic \( n \)

\[ n[q(s, \theta)]^e \] displacement field within an element for a particular harmonic \( n \)

\[ Q_\phi, Q_\theta \]
\[ Q_s, Q_{\theta s} \]

transverse shear forces (per unit length)

\( r \) the distance from a point on the middle surface to the axis of a shell

\( R \) radius of a cylindrical shell

\( R_1, R_2 \) smaller and larger radii of a conical shell

\( R_\phi, R_\theta \) radii of curvature

\( S \) meridional coordinate

\( T^e \) kinetic energy of an element

\( U^e \) strain energy of an element
nodal displacements for a particular element for a particular harmonic \( n \)
displacements (meridional (axial), circumferential, normal (radial))
axial coordinate
rotations (circumferential, meridional)
penalty number
frequency parameters of annular plates
non-dimensional frequency parameter
\[
\Delta = \frac{\rho R^2 (1-\nu^2) \omega^2}{E}
\]
strains at a distance \( \zeta \) from the middle surface
transverse shear strains
strains of the middle surface
element strain vector
local normal coordinate
local meridional coordinate
circumferential coordinate
coefficient of shear energy (=5/6)
the angle between a meridian and the axis of the shell
coordinates of a point of a shell
changes in curvature of the middle surface
frequency (Hz)
non-dimensional frequency parameter
\[
\chi^2 = \frac{\omega \sqrt{12 \rho (1-\nu^2) a^4}}{E h^2}
\]
element damping parameter
ν
Poisson's ratio

ν₁²
Poisson's ratio (ratio of contraction in 2 direction to elongation in 1 direction when loaded in 1 direction)

ρ
mass density

ρₐ
average mass density

[σ]ₑ
stress vector

σₑ,Ω
Potential energy of loads (element, system)

s,θ,ζ
local coordinates (meridional, circumferential, normal)

z,θ,r
global coordinates (axial, circumferential, radial)
INTRODUCTION

The analysis of problems in structural mechanics by means of digital computers has become extensive over the last two decades or so. Numerical methods of determining displacements, stress resultants, frequencies, mode shapes and dynamic response are of great importance as the number of practical problems that can be solved analytically from the governing equations is limited. The most used numerical methods are: the Rayleigh-Ritz method, the finite difference method, numerical integration procedures and the finite element method. All these methods have been applied to the analysis of shells of revolution; the present work is a contribution to knowledge in the area of finite elements which has proved very useful in practical applications and has rapidly become very popular for design work. It is now used almost exclusively as a tool of engineering analysis.

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CHAPTER 1

INTRODUCTION

The analysis of problems in structural mechanics by means of digital computation has become extensive over the last two decades or so. Numerical methods of determining displacements, stress resultants, frequencies, mode shapes and dynamic response are of great importance as the number of practical problems for which exact solutions can be obtained formally from the governing equations is limited. The most used numerical methods are: the Rayleigh-Ritz method, the finite difference method, numerical integration procedures and the finite element method. All those methods have been applied in the analysis of shells of revolution; the present work is a contribution to knowledge in the area of finite element which has proved very useful in practical applications and has rapidly become very popular for design work. It is now used almost routinely as a tool of engineering analysis.

The method is commonly thought to have its beginnings in an original paper by Turner et al [1]* but Clough [2]

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* Numbers in square brackets refer to references at the end of this thesis. However it is interesting to note that an earlier paper by R. Courant 'Variational methods for the solution of problems of equilibrium and vibrations', Bull. Am. Math. Soc., Vol. 49, pp. 1-23, January 1943, describes what is, essentially, the finite element process, but this was prior to the general availability of computers.
appears to be the first to use the term 'finite element', which implies in it a direct use of standard methodology applicable to discrete systems.

In the analysis of complex structures, by means of the finite element approach, the problem is transformed from one of determining the complicated mathematical functions pertinent for the continuum problem to one of subdividing the system into an assemblage of individual elements, whose behaviour is readily understood, and then rebuilding the original system from such elements to study its behaviour. The approach involves the approximation of a variational expression (functional) in terms of nodal variables of unknown magnitudes within each element. The stationary of the functional with respect to the unknowns yields the element characteristic equations in discrete (matrix) forms. The overall system matrices are obtained by an "assembly" process and the unknowns then determined by standard solution algorithms. Although the finite element problem involves much numerical calculations, the work consists mainly of matrix manipulation, and a digital computer is essential for the task. However, from early days, a significant effort has been devoted to developing simple, cost effective elements which adequately fulfill accuracy requirements.

The application of the finite element method to both the static and dynamic analysis of elastic structures is
well established. However, like most numerical methods, this method has been developed and applied initially to static problems and the majority of results available in the literature relates to static problems. For axisymmetric shells, the first application of the method was carried out by Grafton and Strome [3]. Percy et al. extended the axisymmetric deformation in reference [3] to the problem of non-axisymmetric loading [4]. Cubic shape functions, with respect to the meridional coordinate, were used to maintain continuity between adjacent elements. Zienkiewicz et al. [5] proposed a new element for axisymmetric deformation where bilinear shape functions for all components of displacement field were used. At Aston University, Richards and his M.Sc. student, Kumagai, extended the static analysis of reference [5] to the case of non-axisymmetric loads [6].

Based on these studies, the work reported here has, as its goal, the description of the development, application and evaluation of one of the simplest elements available for the statics and dynamics of axisymmetric shells. A semi-analytic truncated cone element has been formulated and implemented in a computer code; it has two nodes with five degrees of freedom at each node, the displacement field within each element is expressed by Fourier Series in the circumferential direction $\theta$ and by bilinear interpolation functions along the meridian $s$. The stiffness matrix is formulated by combining the direct
function and shear strain energies by means of the penalty function approach; this provides the introduction of transverse shear and rotary inertia into the analysis and at the same time enforce the required continuity between adjacent elements. The elemental mass matrices are derived by classical means from the kinetic energy associated with the admissible displacement states as was first described by Archer [7], (it is of interest to note here that the general presentation of the results given as $[M]^e = \int_N [N]^T \rho [N] \, dv$ is due to Zienkiewicz and Cheung [8]). The final equations are obtained by applying Hamilton's principle to solve the dynamical problem. The equations of motion of an element of the shell, the stress resultant-strain relations, the strain-displacement relations and the strain energy expression, given in both Chapter 2 and 5, are based on the theory of Novozhilov [9].

A general outline is given, in Chapter 2, of the necessary relations in axisymmetric shell theory. It begins with a brief review of some basic assumptions made by different authors and continues with a special discussion of axisymmetric shell relations based on the Novozhilov theory. The equation of motion of cylindrical shells are given at the end of the chapter.

The displacement-based finite element method is introduced in Chapter 3. A survey of elements used in the
analysis of shells is given in Chapter 4. It is shown, briefly, how linear shell problems could be formulated and solved using different general types of finite element with a variety of displacement interpolation. Considerable effort is shown to have been directed at finding good elements for the solution of shell structures. For shells of revolution, the survey shows that it is usually idealised by axisymmetric elements bounded by surfaces of constant meridional coordinate s. The circumferential variation of all components of displacement, stress, etc. may conveniently be expanded in terms of harmonic components so that it is only necessary to obtain solutions for the meridional variation of these quantities for each harmonic.

The purpose of Chapter 5 is to present the development of the semi-analytic truncated cone element for the analysis of axisymmetric shells. The difficulty in imposing $C_1$ continuity is shown to be side-stepped by means of penalty functions. The shear deformation, which is a very important feature in thick shell situations, is shown to be introduced by the omission of the statement, in general shell theory, that after deformation the normals remain normal to the deformed middle surface. Accounting for the orthotropic elastic properties, where the axes of orthotropy are coincident with the principal axes of the shells, included in the analysis. At the end of Chapter 5, element stiffness and mass matrices are
given explicitly.

To apply the finite element, described in Chapter 5, to the analysis of actual shell problems, a computer program, developed for a desk top machine, is described in Chapter 6. The program generates (from basic input data describing the geometry, material properties, number of elements, number of nodes and loading of the shell) the elemental stiffness, mass and load matrices and assembles them into gross matrices corresponding to the linear undamped equations of motion of the complete shell. These equations are solved for static deflections of the shell, and hence stress resultants, and for natural vibration modes and frequencies. Flow charts of the program and sub-programs are given at the end of Chapter 6.

The element accuracy is demonstrated in Chapter 7 on a variety of thick as well as thin shell problems in which both static and dynamic situations are considered. Axisymmetric shells subjected to a variety of boundary conditions are presented. In static analysis, a series of examples representing different types of axisymmetric shells have been chosen to show the adequacy of the element representing the shell geometry, i.e., the generator shape, variation in thickness, etc.; the element ability to respond to different types of loading and its usefulness in solving thin and thick shell problems. In dynamic analysis, again the adequacy of the element in
determining natural modes and frequencies for a variety of axisymmetric shells, (cylindrical shells, conical shells, annular plates and some combinations of these) is demonstrated. The results are compared with available exact solutions, with some other finite element solutions based on different types of elements and with results obtained by the application of other numerical methods (finite differences, the Rayleigh-Ritz method and numerical integration procedures).

Discussion and conclusions about the analysis presented in this thesis are given in Chapter 8.
CHAPTER TWO

THEORY OF PLASTIC SHELLS OF REVOLUTION

Introduction

Shell structures are used in many branches of technology such as structural engineering, mechanical engineering, shipbuilding, chemical engineering, aerospace engineering, nuclear engineering, and others. The theory of shells, which is in existence in a considerable amount, is therefore an important subject in structural engineering—whereas in thin plate theory the differential equations of motion is universally agreed upon, the same cannot be said for thin shell theory. Differences arise from different approximations and different points in a derivation where a given approximation is introduced.

The most commonly referred to shell theories are those by Love [10], Donnell [11], Timoshenko [12], Slijker [7], Kassai [14], Vlasov [15], Föppl [16], Goldenveizer [17], and Novoshilov [18]. All of these theories result from Love's "first approximation" and apply to shells of arbitrary curvature.

The work of these authors differ from one another essentially in the formulation of strain-displacement and stress-strain relations. On the one hand, one has waves
CHAPTER 2

THEORY OF ELASTIC SHELLS OF REVOLUTION

2.1 Introduction

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The more commonly referred to shell theories are those by Love [10], Donnell [11], Timoshenko [12], Reissner [13], Naghdi [14], Vlasove [15], Flügge [16], Goldenveizer [17], and Novoshilov [9]. All of these theories result from Love's "first approximation" and apply to shells of arbitrary curvature.

The work of these authors differ from one another essentially in the formulation of strain-displacement and stress-strain relations. On the one hand, one has stress
resultants and couples representing the state of stress in
the shell, and on the other hand one has expressions for
the strains of the stressed shell depending on the
deforation of the middle surface. Depending on what
assumptions are made, these strain-displacement relations
may appear in a variety of forms, differing from each
other only through terms which are small compared to the
principal terms which are common to all forms.

Love [10] made the following approximations in his
classical linear theory of thin shells:

(1) The thickness of the shell is small compared with
the smallest radius of curvature of its reference
surface.

(2) Strains and displacements are small. Hence
second-order terms in the strain-displacement
relations may be neglected in comparison with
first-order terms.

(3) The transverse normal stress is small compared
with the other normal stress components and may be
neglected.

(4) Normals to the undeformed middle surface remain
straight and do not change in length.
These four assumptions consistute what Love called his "first approximation" shell theory. The approximations are almost universally accepted in the derivation of linear thin shell theories.

Flügge [16] has obtained stress-strain relations by retaining assumptions (2), (3) and (4) but without the use of assumption (1). Reissner [13], Kraus [18] and Naghdi [14] have included in their theories, in addition to transverse shear and rotary inertia, other higher order effects as, for instance, the consideration of transverse normal stresses and the argument that straight lines normal to shell reference surface not only do not remain normal but also do not remain straight. These additions result in more complicated equation systems.

The complete bibliography of the subject is beyond the scope of this thesis. For an exposition of the general theory of shells, readers are advised to look at references mentioned so far in this chapter, where they can find more references there.

The derivation in general curvilinear systems follows, in general, the derivation of Love's thin shell equation as first shown by Reissner [19]. The transverse normal stress will be ignored here whilst the shear strains $\varepsilon_{\phi z}$ and $\varepsilon_{\theta z}$ will be included in the analysis so that a general theory for shells of revolution results
that is equivalent to the Timoshenko beam [20] and Mindlin plate theory [21].

2.2 Geometry of the Middle Surface

The position of a point of a shell of revolution is given by the coordinates $\phi$ (measured from the axis of the shell), $\theta$ (measured from a line of intersection of the parallel surface with the meridian plane, both planes passing through the point), and $\zeta$ (measured along the outward normal to the middle surface, on the middle surface $\zeta = 0$) as shown in Fig. 2.1.

For a shell of revolution the two principal radii of curvature, which determine the shape of the shell, are its meridian and parallel circles; therefore, $R_\phi$ will be considered as the radius of curvature of the meridian, and $R_\theta$ as the second radius of curvature which is the length of the intercept of the normal to the middle surface between the surface and the axis of the shell. $R_\phi$ and $R_\theta$ are functions of $\phi$.

Instead of $R_\theta$, it is convenient to use the distance $r$ from a point on the middle surface to the axis of the shell; from Fig. 2.2 it follows:

$$r = R_\theta \sin \phi$$ (2.1)
Fig. 2.1 Middle surface of a shell of revolution
Fig. 2.2 Shell element
\[ dr = ds_\phi \cos \phi \quad (2.2) \]
\[ dz = ds_\phi \sin \phi \quad (2.3) \]
\[ ds_\phi = R_\phi d\phi \quad (2.4) \]
\[ ds_\theta = r \, d\theta \quad (2.5) \]

so that:
\[ \frac{dr}{d\phi} = R_\phi \cos \phi \quad (2.6) \]

2.3 The Constitutive Equations of the Linear Theory

It has been noted by Truesdell and Toupin [22] that for general field theories any set of constitutive equations must satisfy certain mathematical principles. With particular reference to shell theory, Gol'denveizer [23] and Naghdi [14] have noted that from amongst these general principles there are three which have a particular significance:

1. **Consistency.** Any set of constitutive equations should be consistent with the principles of energy and equilibrium.

2. **Rigid body displacement invariance.** The equations...
should remain invariant under rigid body displacements, which require that such displacements give rise to zero strain energy.

(3) Coordinate invariance. The equations should be stated by a rule which holds equally well in all coordinate systems.

The constitutive equations of shells of revolution of elastic, homogeneous, linear and isotropic materials are presented here.

2.3.1 Strain-displacement relations

As shown in Fig. 2.1, the three components of displacement at a point of the middle surface are \( u, v \) and \( w \), which represent the meridian, circumferential and normal displacement respectively; \( \beta \) and \( \alpha \) represent the total angular rotations, in the meridional and circumferential directions respectively, including the angular rotation due to shear.

Introducing the assumption that straight lines normal to the reference surface remain straight lines after deflections, even if they are not any longer normal, the displacements \( u, v, w \) anywhere along the normal to the middle surface can be calculated from
\[ u(\phi, \theta, \zeta) = u(\phi, \theta) + \zeta \beta \quad (2.7a) \]

\[ v(\phi, \theta, \zeta) = v(\phi, \theta) + \zeta \alpha \quad (2.7b) \]

and leaves for the normal deflection

\[ w(\phi, \theta, \zeta) = w(\phi, \theta) \quad (2.7c) \]

The strains at a distance \( \zeta \) from the middle surface may be written in the form

\[ \varepsilon_\phi(\zeta) = \varepsilon_\phi^0 + \zeta \chi_\phi \]

\[ \varepsilon_\theta(\zeta) = \varepsilon_\theta^0 + \zeta \chi_\theta \quad (2.8) \]

\[ \gamma_{\phi\theta}(\zeta) = \gamma_{\phi\theta}^0 + \zeta \chi_{\phi\theta} \]

where the membrane strains are

\[ \varepsilon_\phi^0 = \frac{1}{R_\phi} \left( \frac{\partial u}{\partial \phi} + w \right) \]

\[ \varepsilon_\theta^0 = \frac{1}{r} \left( u \cos \phi + \frac{3v}{3\theta} + w \sin \phi \right) \quad (2.9) \]

\[ \gamma_{\phi\theta}^0 = \frac{1}{r} \left( \frac{\partial u}{\partial \theta} - v \cos \phi \right) + \frac{1}{R_\phi} \frac{3v}{3\phi} \]

and the curvature terms become
\[ x_\phi = \frac{1}{R_\phi} \frac{3q}{\partial \phi} \]

\[ x_\theta = \frac{1}{r} \left( \frac{3q}{\partial \theta} + \beta \cos \phi \right) \quad (2.10) \]

\[ 2x_{\phi \theta} = \frac{1}{r} \left( \frac{3\beta}{\partial \theta} - \alpha \cos \phi \right) + \frac{1}{R_\phi} \frac{3q}{\partial \phi} \]

where

\[ \beta = \frac{1}{R_\phi} (u - \frac{3w}{\partial \phi}) \quad (2.11) \]

\[ \alpha = \frac{1}{r} (v \sin \phi - \frac{3w}{\partial \theta}) \quad (2.12) \]

In addition, one now has transverse shear strain expressions

\[ \varepsilon_{\phi \zeta} = \beta - \frac{1}{R_\phi} (u - \frac{3w}{\partial \phi}) \quad (2.13a) \]

\[ \varepsilon_{\theta \zeta} = \alpha - \frac{1}{r} (v \sin \phi - \frac{3w}{\partial \theta}) \quad (2.13b) \]

Mathematically, the development differs from Love’s theory in that here \( \beta \) and \( \alpha \) are treated as unknowns while in Love’s theory relationships for \( \beta \) and \( \alpha \) are developed in terms of the displacements \( u, v \) and \( w \), utilising zero transverse shear strain conditions. According to equations (2.13a,b), \( \varepsilon_{\phi \zeta} \) and \( \varepsilon_{\theta \zeta} \) are the shear strains on the reference surface.
2.3.2 **Stress-strain relations**

The stress components of shell are given by Hooke's law as

\[\sigma_\phi = \frac{E}{1-\nu^2} (\varepsilon_\phi + \nu \varepsilon_\theta) \quad (2.14a)\]

\[\sigma_\theta = \frac{E}{1-\nu^2} (\varepsilon_\theta + \nu \varepsilon_\phi) \quad (2.14b)\]

\[\tau_{\phi\theta} = G \gamma_{\phi\theta} = \frac{E}{2(1+\nu)} \gamma_{\phi\theta} \quad (2.14c)\]

\[\tau_{\phi\zeta} = G \gamma_{\phi\zeta} \quad (2.14d)\]

\[\tau_{\theta\zeta} = G \gamma_{\theta\zeta} \quad (2.14e)\]

Note that \(\tau_{\phi\zeta}\) and \(\tau_{\theta\zeta}\) are the values at the middle surface only since, the free surfaces cannot support such shear stresses, they have to diminish the zero as these free surfaces are approached. The average values of \(\tau_{\phi\zeta}\) and \(\tau_{\theta\zeta}\) denoted \(\tau^a_{\phi\zeta}\) and \(\tau^a_{\theta\zeta}\) are therefore

\[\tau^a_{\phi\zeta} = \kappa' \tau_{\phi\zeta} = \kappa' G \gamma_{\phi\zeta} \quad (2.15)\]

\[\tau^a_{\theta\zeta} = \kappa' \tau_{\theta\zeta} = \kappa' G \gamma_{\theta\zeta} \quad (2.16)\]

where \(\kappa'\) is less than unity.

Now, from equations (2.14a - c) one can formulate the
moment and force resultants per unit length of middle surface; and they become

\[ N_\phi = \frac{Eh}{1-\nu^2} (\varepsilon^\Omega_\phi + \nu \varepsilon^\Omega_\theta) \]  

(2.17a)

\[ N_\theta = \frac{Eh}{1-\nu^2} (\varepsilon^\Omega_\theta + \nu \varepsilon^\Omega_\phi) \]  

(2.17b)

\[ N_{\phi\theta} = N_{\theta\phi} = \frac{1-\nu}{2} \frac{Eh}{1-\nu^2} \gamma^\Omega \]  

(2.17c)

\[ M_\phi = \frac{Eh^3}{12(1-\nu^2)} (\chi_\phi + \nu \chi_\theta) \]  

(2.17d)

\[ M_\theta = \frac{Eh^3}{12(1-\nu^2)} (\chi_\theta + \nu \chi_\phi) \]  

(2.17e)

\[ M_{\phi\theta} = M_{\theta\phi} = \frac{1-\nu}{2} \frac{Eh^3}{12(1-\nu^2)} \gamma_\phi \]  

(2.17f)

The integrals of equations (2.14d,e) over the element faces being simply defined as shear forces which are eventually obtained from the moment equations, one can now define the shear force per unit length of the middle surface in terms of \( u, v, w, \beta \) and \( \alpha \):

\[ Q_{\phi\zeta} = \tau^\alpha_{\phi\zeta} \cdot h = \kappa'Gh \gamma_{\phi\zeta} = \kappa'Gh[\beta - \frac{1}{R_\phi}(u - \frac{\partial w}{\partial \phi})] \]  

(2.18)

\[ Q_{\theta\zeta} = \tau^\alpha_{\theta\zeta} \cdot h = \kappa'Gh \gamma_{\theta\zeta} = \kappa'Gh[\alpha - \frac{1}{r}(v \sin \phi - \frac{\partial w}{\partial \phi})] \]  

(2.19)
2.3.3 The equations of motion

The consistency between the strain energy integrands and the equations of motion requires that the equations of motion are derivable from an energy principle by means of a variational procedure. One variational principle which may be invoked is Hamilton's principle [24,25] which may be written as:

$$\delta \int_{t_1}^{t_2} (T - V) \, dt = 0$$  \hspace{1cm} (2.20)

That is, the variation of the time integral between given time limits of the difference between the kinetic energy $T$ and potential energy $V$ must vanish, (i.e. virtual displacements are zero at $t_1$ and $t_2$).

The time derivations, $\dot{\phi}$ and $\ddot{\phi}$ in the kinetic energy expression, are not neglected because the rotary inertia effects are considered here. The equations of motion become, therefore,

$$-\frac{\partial}{\partial \phi}(rN_\phi) - R_\phi \frac{\partial N_\theta}{\partial \theta} + R_\phi \cos \phi N_\phi - rQ_\phi \zeta$$

$$+ rR_\phi \rho \ddot{\phi} = rR_\phi \dot{P}_\phi$$  \hspace{1cm} (2.21)

$$-\frac{\partial}{\partial \phi}(rN_\phi) - R_\phi \frac{\partial N_\theta}{\partial \theta} - R_\phi \cos \phi N_\phi - R_\phi \zeta \sin \phi$$

$$+ rR_\phi \rho \ddot{\phi} = rR_\phi \dot{P}_\theta$$  \hspace{1cm} (2.22)
\[-\frac{3}{3\phi} (rQ_{\phi}) - R_{\phi} \frac{\partial Q_{\phi}}{\partial \theta} + rN_{\phi} + R_{\phi} N_{\phi} \sin \phi \]

\[+ r R_{\phi} \omega = r R_{\phi} \phi \zeta \]

\[\frac{3}{3\phi} (rM_{\phi}) + R_{\phi} \frac{\partial M_{\phi}}{\partial \theta} - R_{\phi} \cos \phi M_{\phi} - r R_{\phi} Q_{\phi} \zeta \]

\[- rR_{\phi} (\rho h^3/12) \phi = 0 \]

\[\frac{3}{3\phi} (rM_{\phi}) + R_{\phi} \frac{\partial M_{\phi}}{\partial \theta} + R_{\phi} \cos \phi M_{\phi} - r R_{\phi} Q_{\phi} \zeta \]

\[- rR_{\phi} (\rho h^3/12) \alpha = 0 \]

For circular cylindrical shells, as special case of shells of revolution, the equations of motion become \[26],

\[- r \frac{\partial N_s}{\partial s} - \frac{\partial N_{\phi}}{\partial \theta} + r \rho h = r P_s \]

\[- r \frac{\partial N_{\phi}}{\partial s} - \frac{\partial N_{\phi}}{\partial \theta} + Q_{\phi} = r P_\phi \]

\[- r \frac{\partial Q_{\phi}}{\partial s} - \frac{\partial Q_{\phi}}{\partial \theta} + N_{\phi} + r \rho h = r P_{\phi} \]

\[r \frac{\partial M_s}{\partial s} + \frac{\partial M_{\phi}}{\partial \theta} - r Q_{\phi} \zeta - (\rho h^3/12) \phi = 0 \]

\[r \frac{\partial M_{\phi}}{\partial s} + \frac{\partial M_{\phi}}{\partial \theta} - r Q_{\phi} \zeta - (\rho h^3/12) \alpha = 0 \]

where the strains for cylindrical shells are defined as
\[-\frac{3}{2} (rQ_{\phi \zeta}) - R_{\phi} \frac{3Q_{\phi \theta}}{3\theta} + rN_{\phi} + R_{\phi} N_{\theta} \sin \phi \]

\[+ r R_{\phi} \rho h w = r R_{\phi} P_{\zeta} \]  

(2.23)

\[\frac{3}{2} (rM_{\phi \zeta}) + R_{\phi} \frac{3M_{\phi \theta}}{3\theta} - R_{\phi} \cos \phi M_{\theta} - r R_{\phi} Q_{\phi \zeta} \]

\[- r R_{\phi} (\rho h^3/12) \beta = 0 \]  

(2.24)

\[\frac{3}{2} (rM_{\phi \theta}) + R_{\phi} \frac{3M_{\phi \theta}}{3\theta} + R_{\phi} \cos \phi M_{\phi \theta} - r R_{\phi} Q_{\phi \zeta} \]

\[- r R_{\phi} (\rho h^3/12) \alpha = 0 \]  

(2.25)

For circular cylindrical shells, as special case of shells of revolution, the equations of motion become [26],

\[-r \frac{3N_{s \theta}}{3s} - \frac{3N_{s \theta}}{3\theta} + rphu = rP_{s} \]  

(2.26)

\[-r \frac{3Q_{s \theta}}{3s} - \frac{3Q_{s \theta}}{3\theta} + rQ_{\theta \zeta} + rphv = rP_{\theta} \]  

(2.27)

\[-r \frac{3Q_{s \zeta}}{3s} - \frac{3Q_{s \zeta}}{3\theta} + N_{\theta} + rphw = rP_{\zeta} \]  

(2.28)

\[r \frac{3M_{s \theta}}{3s} + \frac{3M_{s \theta}}{3\theta} - rQ_{\phi \zeta} - (\rho h^3/12) \beta = 0 \]  

(2.29)

\[r \frac{3M_{s \theta}}{3s} + \frac{3M_{s \theta}}{3\theta} - rQ_{\theta \zeta} - (\rho h^3/12) \alpha = 0 \]  

(2.30)

where the strains for cylindrical shells are defined as
\[ \varepsilon^{s}_{\theta} = \frac{3u}{3s}, \quad \varepsilon^{\phi}_{\theta} = \frac{1}{r} (w + \frac{3v}{3\theta}), \quad \gamma^{\phi}_{\theta} = \frac{1}{r} \frac{3u}{3\theta} + \frac{3v}{3s}, \]

\[ x^{s}_{\theta} = \frac{3\beta}{3s}, \quad x^{\phi}_{\theta} = \frac{1}{r} \frac{3\alpha}{3\theta}, \quad x^{s}_{\phi} = \frac{3\alpha}{3s} + \frac{1}{r} \frac{3\beta}{3\theta}, \]

\[ \varepsilon^{s}_{\xi} = \frac{3w}{3s} + \beta, \quad \varepsilon^{\phi}_{\xi} = \frac{1}{r} \frac{3w}{3\theta} - \frac{v}{r} + \alpha \quad (2.31a-h) \]

Eliminating all the stress resultants from the equations of motion for the cylindrical shell by equations (2.17), we obtain a set of five differential equations and five unknowns: \( u, v, w, \beta \) and \( \alpha \). One can write

\[ [L] \quad \{q\} = \{0\} \quad (2.32) \]

\[
\begin{bmatrix}
    u \\
    v \\
    \{q\} = \begin{bmatrix}
        a_{11} & a_{12} & a_{13} & 0 & 0 \\
        a_{21} & a_{22} & a_{23} & 0 & a_{25} \\
        a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
        0 & 0 & a_{43} & a_{44} & a_{45} \\
        0 & a_{52} & a_{53} & a_{54} & a_{55}
    \end{bmatrix}
\]

where,

\[ a_{11} = \frac{Eh}{1-v^2} \cdot r \left[ \frac{3^2}{3s^2} + \frac{(1-v)/2r^2}{3\theta^2} \right] - rph \frac{\partial^2}{\partial t^2} \quad (2.33,2.34) \]

\[ a_{22} = \frac{Eh}{1-v^2} \cdot r \left[ \frac{1}{2} (1-v) \frac{\partial^2}{\partial s^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] - (\kappa'Gh/a) - rph \frac{\partial^2}{\partial t^2} \quad (2.35) \]
\[ a_{33} = \frac{Eh}{r(1-v^2)} - r\kappa'Gh\left[\frac{a^2}{\partial s^2} + \frac{1}{r^2}\frac{a^2}{\partial \theta^2}\right] + rp\frac{a^2}{\partial t^2} \quad (2.37) \]

\[ a_{44} = \frac{Eh^3}{12(1-v^2)} \cdot \frac{r}{r(1-v^2)} \left[\frac{1}{2}\frac{a^2}{\partial s^2} + \frac{1}{r^2}\frac{a^2}{\partial \theta^2}\right] - r\kappa'Gh \]

\[ -(rp^3/12) \frac{a^2}{\partial t^2}, \quad (2.38) \]

\[ a_{55} = \frac{Eh^3}{12(1-v^2)} \cdot \frac{r}{r(1-v^2)} \left[\frac{1}{2}\frac{a^2}{\partial s^2} + \frac{1}{r^2}\frac{a^2}{\partial \theta^2}\right] - r\kappa'Gh \]

\[ -(rp^3/12) \frac{a^2}{\partial t^2}, \quad (2.39) \]

\[ a_{12} = a_{21} = \frac{Eh}{2(1-v^2)} (1+v) \frac{a}{\partial s \partial \theta} \quad (2.40) \]

\[ a_{13} = a_{31} = \frac{\nu Eh}{(1-v^2)} \frac{a}{\partial s}, \quad (2.41) \]

\[ a_{23} = a_{32} = \frac{Eh}{r(1-v^2)} \cdot \frac{a}{\partial \theta} + (\kappa'Gh/\tau) \frac{a}{\partial \theta} \quad (2.42) \]

\[ a_{34} = a_{43} = -r\kappa'Gh \frac{a}{\partial s}, \quad (2.43) \]

\[ a_{25} = a_{52} = \kappa'Gh, \quad (2.44) \]

\[ a_{35} = a_{53} = -\kappa'Gh \frac{a}{\partial \theta} \quad (2.45) \]

\[ a_{45} = a_{54} = \frac{Eh^3}{24(1-v^2)} (1+v) \frac{a^2}{\partial s \partial \theta} \quad (2.46) \]

This is almost identical to the Donnel-Mushtari theory, generalised to take account of the shear deformation, as given by Leissa [27], the difference is
that the terms $a_{25} = a_{52}$ are zero in the Donnel-Mushtari-Leissa theory, but exist in this theory.

For shells of revolution, which consist of complete latitude circles, the surface loads may be represented to be periodic with respect to $\theta$ with a period of $2\pi$, and they can be assumed to be of the form

$$\{P_\phi, P_\zeta, M_\phi\} = \sum_{n=0}^{\infty} \{P_{\phi n}, P_{\zeta n}, M_{\phi n}\}\{\cos n\theta \over \sin n\theta\}$$

$$\{P_\theta, M_\phi\} = \sum_{n=0}^{\infty} \{P_{\phi n}, M_{\phi n}\}\{\sin n\theta \over \cos n\theta\} \tag{2.47}$$

where the variables with subscript $n$ depend only on $s$, the meridional coordinate, and each integral value of $n$ in equation (2.47) can be regarded as one Fourier component in a general Fourier series expansion of arbitrary periodic surface loads.

Assuming the displacements and rotations as

$$u = \sum_{n=0}^{\infty} \left( u_n(\phi) \cos n\theta + \bar{u}_n(\phi) \sin n\theta \right) \tag{2.48}$$

$$v = \sum_{n=0}^{\infty} \left( v_n(\phi) \sin n\theta + \bar{v}_n(\phi) \cos n\theta \right) \tag{2.49}$$

$$w = \sum_{n=0}^{\infty} \left( w_n(\phi) \cos n\theta + \bar{w}_n(\phi) \sin n\theta \right) \tag{2.50}$$

$$\beta = \sum_{n=0}^{\infty} \left( \beta_n(\phi) \cos n\theta + \bar{\beta}_n(\phi) \sin n\theta \right) \tag{2.51}$$
\[
\alpha = \sum_{n=0}^{\infty} \left( a_n(\phi) \sin \theta + \bar{a}_n(\phi) \cos \theta \right)
\]

we have an infinite number of ordinary differential equations.

Analytical solution for shells are limited to very simple cases and not applicable, in general, to shell problems encountered in practice which are characterised by some form of structural irregularities, which directly affect the analysis procedure, these include

1. Branches and intersections.
2. Stiffeners, including those with eccentricities.
3. Wide variation in shell thickness.
4. Openings, cut-outs or reinforcements in the structure, leading to anisotropic or at least orthotropic behaviour.

2.4 Concluding Remarks

From the foregoing presentation and discussion of the equations of motion for shells of revolution, it is seen that the need for numerical solutions is inevitable.

The finite element approach has become, perhaps, the most powerful tool of engineering analysis available. It has been adopted for the present work and a brief account
of its application to shell problems in general is given in Chapter 3. The new formulation of a relatively simple, but effective semi-analytical element, suitable for axisymmetric shells is presented in Chapter 5.
CHAPTER THREE

FINITE ELEMENT ANALYSIS FOR SHELLS

Introduction

It was mentioned in Chapter 2 that available analytical solutions to shell structural problems are limited in scope and in general do not apply to arbitrary shapes, load conditions, material stiffening and supported conditions, etc. Moreover, and many other aspects of practical design. The finite element method has consequently come to the fore as an approach to shell structural analysis because of its facility to deal with these complications; it has thus become the most powerful and widely used approximate approach to the design analysis of shell structures.

A number of different variational principles are in use which lead to different finite element formulations. The three widely used variational principles are the total potential energy, the complementary energy, and the Reissner variational principle.

In the minimum potential energy formulation, the only field variables are the element displacements. The use of this principle leads to displacement-based compatible
CHAPTER 3

FINITE ELEMENT ANALYSIS FOR SHELLS

3.1 Introduction

It was mentioned in Chapter 2 that available analytical solutions to shell structural problems are limited in scope and in general do not apply to arbitrary shapes, load conditions, irregular stiffening and supported conditions, cutouts, and many other aspects of practical design. The finite element method has consequently come to the fore as an approach to shell structural analysis because of its facility to deal with these complications; it has thus become the most powerful and widely used approximate approach to the design analysis of shell structures.

A number of different variational principles are in use, which lead to different finite element formulations. The three widely used variational principles are the total potential energy, the complementary energy, and the Reissner variational principle.

In the minimum potential energy formulation, the only field variables are the element displacements. The use of this principle leads to displacement-based compatible
finite element models.

In the complementary energy principle, the field variables are stresses, instead of displacements; use of this principle results in stress-based equilibrium finite element models.

A more general variational principle is that of Reissner, in which the primary field variables are both displacements and stresses. The application of this principle leads to finite element discretizations referred to as mixed models.

The displacement-based finite element formulation is the one which is adopted throughout this thesis.

3.2 General Formulation

The displacement-based finite element technique is an application of the theorem of minimum total potential energy. The essential steps in the solution may be summarised as follows:

1. The structure is idealised as an assemblage of a discrete number of finite elements, which are defined by the location of their nodal points. The engineer has to decide the number, shape, size and configuration of the elements in such a way that the original structure is
simulated as closely as possible.

There have been three distinct approaches to the finite element representation of shells:

(a) in faceted form using flat elements,
(b) via curved shell elements,
(c) by means of three-dimensional elements.

In idealisation of the shell by finite elements, a geometrical simplification of replacing the curved shell by an assemblage of flat elements occupied, in large measure, the formative years of the finite element method. In the division of an arbitrary shell into flat elements only triangular elements can be used [28]. Although the concept of the use of such elements in the analysis was suggested as early as 1961 by Green et al. [29], the successes of such analyses was hampered by the lack of a good stiffness matrix for triangular plate elements in bending [30,31]. Some shells, for example those with general cylindrical shapes, can be well represented by flat elements of rectangular or quadrilateral shape. With good stiffness matrices available for such elements the progress here has been more satisfactory [32].

A simpler approach can be adopted for dealing with axisymmetric shells, where the geometry, material
properties, and such dependent variables as displacements can all be expressed in terms of one independent space coordinate. In this the elements are simple conical frusta [3-6].

In spite of the limitations and shortcomings of the flat elements, for many practical purposes the flat element approximation gives very adequate answers and indeed permits an easy coupling with edge beam and rib members, a facility sometimes not present in curved element formulation. Indeed, in many practical problems the structure is in fact composed of flat surfaces, at least in part, and these can be simply reproduced.

As an alternative, curved shell elements, whose formulation required a new examination of modes of element geometric representation, strain-displacement equations and assumed displacement fields, could be used. By 1969 a large number of curved thin shell finite elements had appeared in the literature [33-36]. The application of these elements was limited to shallow shell situations and to those in which shear deformation was neglected.

The use of three-dimensional elements for the purpose of finite element modelling of nuclear structures has been widespread. With a straightforward use of the three-dimensional concept, however, certain difficulties are encountered. In the first place, the retention of three
degrees of freedom at each node leads to large stiffness coefficients for relative displacements along an edge corresponding to the shell thickness. This presents numerical problems and inevitably, leads to ill-conditioned equations when shell thicknesses become small compared with the other dimensions in the element [37]. In the second place, the use of several nodes across the shell thickness ignores the well-known fact that even for thick shells the normals to the middle surface remain practically straight after deformation [28]. Thus, an unnecessarily high number of degrees of freedom has to be carried, involving penalties of computer time. Certain assumptions must be invoked if curved thin shell solutions are to be achieved by this mode of representation.

The basic assumption to be made when the 20-node brick element in Fig. 3.1d is employed in curved thin shell analysis are that no strain occurs across the element thickness and that the direct (ξ- and η- direction) strains vary linearly in that direction. This enables elimination of the nodal points in the middle surface and results in the equality of radial displacements \( w \) of corresponding points on the upper and lower surfaces of the element. Also, the ξ- and η- direction displacements of corresponding nodal points on the top and bottom surfaces can be transformed into angular displacements \( \theta_\xi \) and \( \theta_\eta \). Thus, the element node point displacements are \( u, v, w, \theta_\xi \) and \( \theta_\eta \) at each of eight locations.
Fig. 3.1 Different types of shell elements

(a) representation of flat plate element for shell analysis

(b) idealization of shell by truncated cone elements

(c) curved shell elements
(d) parabolic isoparametric hexahedron element (20 node points)

degeneration to a shell element (8 node points)

Fig. 3.1 (continue)
(a) an arbitrary shell
(b) an assemblage of triangular elements

Fig. 3.2 a simplified representation of a discretized shell problem
Fig. 3.3 Finite element mesh using different elements to analyze a problem.
Transverse shear deformation is permitted, because the angular displacements are not tied to the slope of the middle surface.

Numerical evidence has shown that the above assumptions are not sufficient to recapture the behaviour of curved thin shell structures. The most popular and most effective scheme, to account for the above difficulty, involves the use of lower-order numerical integration of the strain energy of the element.

In Fig. 3.1 some types of elements are shown, whilst Fig. 3.2 shows a simplified representation of a discretised problem.

In some cases, two or more types of elements are needed to approximate a particular problem as shown in Fig. 3.3.

2. In the finite element method, structural discretisation of continuous elastic systems is based on the fundamental relationship \( [q]^e = [N] [u]^e \) which relates the displacement \( [q]^e \) within an element to its nodal displacements \( [u]^e \). The components of \( [N] \) are prescribed functions of position, called displacement models or shape functions, which are chosen to approximate the actual or exact distribution of the displacements over each element. It is generally not possible to select a
displacement function that can represent exactly the actual variation of displacements in the element. Hence, the basic approximations of the finite element method are introduced at this stage.

A displacement function can be expressed in various forms, such as polynomials and trigonometric functions. Since polynomials offer ease in mathematical manipulations, they have been employed commonly in finite element applications.

The general polynomial form of a two-dimensional displacement model can be written as

\[
\begin{align*}
\mathbf{u}(x,y) &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + \ldots + a_m y^n \\
\mathbf{v}(x,y) &= a_{m+1} + a_{m+2} x + a_{m+3} y + a_{m+4} x^2 + a_{m+5} xy + \ldots + a_{2m} y^n
\end{align*}
\]  

(3.1)

where the coefficients of the polynomial, the \( a \)'s, are known as generalised coordinates or generalised displacement amplitudes. The number of terms retained in the polynomial determines the shape of the displacement models whereas the magnitudes of the generalised coordinates govern the amplitude. The general polynomial form of displacement models in equation (3.1) can be
truncated at any desired degree to give constant, linear, quadratic, cubic or higher order patterns. For example, in a plane strain (Fig. 3.4) we can write in a matrix form

\[
\{q(x,y)\}^e = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \{a\} = [\phi] \{a\}
\]

(3.2)

expressing the nodal displacements in terms of the generalised coordinates, we obtain

\[
[u]^e = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = [A] \{a\}
\]

(3.3)

inverting equation (3.3) to give

\[
\{a\} = [A]^{-1} \{u\}^e
\]

(3.4)

where \([A]^{-1}\) is a displacement transformation matrix. When equation (3.4) is substituted into equation (3.2) we can
As discussed in Section 3.4, the formulation of the semianalytic
conservative shell element, developed later in Chapter 5, utilize
symmetric terms are used to describe circumferential
variations of a displacement.

For each element, we evaluate certain characteristics which are usually expressed as a stiffness
matrix $[K]$ and an applied load vector $[p]$. 

Fig. 3.4 Plane strain triangle

\begin{equation}
\begin{aligned}
\mathbf{q}^b &= (x_1, y_1, u(x_1, y_1), v(x_1, y_1))^T \\
\mathbf{q}^b &= (x_2, y_2, u(x_2, y_2), v(x_2, y_2))^T \\
\mathbf{q}^b &= (x_3, y_3, u(x_3, y_3), v(x_3, y_3))^T
\end{aligned}
\end{equation}
eliminate the generalised coordinates and obtain

\[ \{q\}^e = [\psi][A]^{-1}\{u\}^e_o = [N]\{u\}^e_o \quad (3.5) \]

Equation (3.5) expresses the displacements \(\{q\}^e\) at any point \(P\) (Fig. 3.4) within the element in terms of the displacements of the nodes, \(\{u\}^e_o\). One limitation of generalised coordinate displacement models is that it is not always possible to obtain the inverse of matrix \([A]\) \(\text{[38]}\). From equation (3.5), it is apparent that if we can directly construct the matrix \([N]\), we can avoid the necessity of computing and inverting the matrix \([A]\).

Indeed, it is possible to avoid this inversion by selecting interpolation functions as the basis for the displacement model. An interpolation function, also known as a shape function, is a function which has unit value at one nodal point, and zero value at all other nodal points. We commonly use shape functions which are polynomials and select the order of the polynomial to satisfy the requirements discussed in Section 3.3.

However, in the formulation of the semi-analytic axisymmetric shell element, developed later in Chapter 5, trigonometric terms are used to describe circumferential variations of a displacement.

3. For each element we evaluate certain characteristics which are usually expressed as a stiffness matrix \([K]\)^e and an applied load vector \([p]\)
respectively. Use of the minimum of total potential energy principle then leads to a system of simultaneous equations in the general discrete form

\[ [K] \{u\} = \{P\} \quad (3.6) \]

for linear situations.

The solution of equation (3.6), after imposing the displacement of boundary conditions, yields all element nodal displacements and rotations \( \{u\} \) and, hence, the strains and the stresses.

4. When the loads are time dependent, inertia forces need to be considered, i.e. a truly dynamic problem needs to be solved. Using d'Alembert's principle [24] we can simply include the element inertia forces. When Hamilton principle is applied, the resulting equations of motion take the form

\[ [M]\ddot{\{u\}} + [K]\{u\} = \{P\} \quad (3.7) \]

Energy is dissipated during vibration, which in vibration analysis, is usually taken account of by introducing velocity dependent damping forces, the equations of motion are, in this case,

\[ [M]\ddot{\{u\}} + [C]\dot{\{u\}} + [K]\{u\} = \{P(t)\} \quad (3.8) \]
3.3 Convergence

For the finite element method we obtain upper bounds to the true solution. In order to assure convergence to a valid result by mesh refinement, three conditions must be met.

The first rule states that the displacement function chosen should be such that it does not permit straining of an element to occur when the nodal displacements are caused by a rigid body displacement.

The second states that the displacement function must include the constant strain states of the element.

The third rule states that a certain degree of continuity must be maintained within and across the element boundaries.

In the case of strains being defined by first derivatives, as in plane stress and plane strain, only the displacements have to be continuous. If, however, as in plate and shell problems, the strains are defined by second derivatives of deflections, first derivatives of these have also to be continuous.
Elements which meet both the first and second requirements are known as complete. Formulations which satisfy the third condition above are called compatible or conforming. The isoparametric concept enables us to formulate elements of any order which satisfy the completeness and compatibility requirements. In isoparametric elements both geometry and displacements of the element are described in terms of the same parameters and are of the same order. There are two essential ingredients of the isoparametric approach; natural coordinate systems and interpolation displacement models. A local coordinate system is one that is defined for a particular element and not necessarily for the entire structure; the coordinate system for the entire body is called the global coordinate system.

Physically, compatibility assures that no gaps occur between elements when the assemblage is loaded. When only translational degrees of freedom are defined at the element nodes, only continuity in the displacements must be preserved; when rotational degrees of freedom are also defined at the element nodes it is necessary to satisfy element continuity in the corresponding first displacement derivatives. This is a consequence of the kinematic assumption on the displacements over the depth of the shell bending element. The requirements of compatibility are difficult to satisfy in shell analysis [39]. However, there exist many examples of bending elements that do not
maintain slope continuity for coarse meshes, but, nevertheless converge monotonically to exact results by mesh refinement.

3.4 Penalty Function Methods in Finite Element Analysis

Standard finite element discretisation of physical problems has, on occasions, to be supplemented by imposing constraints on the variational principle governing the problem. The procedure is found useful in imposing constraints implicit in the functional or to impose required inter-element continuity. One possible way of imposing such constraints is to apply the Lagrangian multiplier method, first used in this context for the development of plate elements [40] and extended later to thin shells [41]. Two disadvantages of using this method are:

(a) the number of unknowns is increased by the number of Lagrangian multipliers;
(b) the system of equations become indefinite [28].

To overcome the disadvantages mentioned above some researchers [42,43] have attempted to eliminate the Lagrangian multipliers on an element level; the basic idea was to replace the multipliers by the respective Euler equations resulting from the principle of stationary
value of a modified potential energy. However, as was pointed out in reference [44] the proposed elimination is not applicable within the framework of the finite element method.

As an alternative procedure of imposing constraints and at the same time avoiding both difficulties mentioned in connection with Lagrange multipliers we are going to introduce the penalty function method. Briefly, considering the problem of making a functional \( \pi \) stationary subject to the unknown \( \{q\} \) obeying some set of constraint equations \( C(q) = 0 \), we can introduce this constraint by forming another functional

\[
\bar{\pi} = \pi + a_1 \int_\Omega C^T(q) C(q) \, d\Omega
\]

in which \( a_1 \) is a so-called penalty number. Imposing the stationary condition to \( \bar{\pi} \) yields the constrained solution.

As already mentioned, the main motivation for imposing constraints comes from two sources:

(a) their implicit requirement in the functional;
(b) the continuity requirement imposed on the trial functions.

The application procedure under the first heading will not be discussed here but as examples of this we have the
complementary energy formulation with equilibrium imposed as a constraint. A second example is the introduction of a penalty term to suppress any compressibility allowed by an arbitrary displacement variation, it is in fact equivalent to the use of a high Poisson's ratio for the study of fluids [45].

While the imposition of \( C_0 \) continuity presents little difficulties in the finite element process, \( C_1 \) continuity is difficult to achieve in piecewise defined fields of two dimensions. Difficulties in introducing the desired continuity were encountered and many researchers have been tempted to rephrase the problem using an independent interpolation of slopes and displacements and imposing the relation between these as a constraint.

This approach is that which has been adopted in this work and to clarify ideas we shall consider here the problem of beam deflection (the shell element discussed in Chapter 5 is an extension of the process given here). The total potential energy of the beam is given by

\[
\delta = \frac{1}{2} \int_0^L \frac{EI}{E} \left( \frac{d^2w}{dx^2} \right)^2 \, dx - \int_0^L w \, P \, dx \tag{3.9}
\]

when using this the need for \( C_1 \) continuity is implied by the second derivative of \( w \) existing in the functional (it actually exists in the strain expression). If strain is defined in terms of rotation as
\[ \varepsilon = \frac{d\theta}{dx} \]

then, if \( \theta \) is considered to vary independently, a constraint relating the deflection and slope has to be imposed. That is

\[ C(w, \theta) = \frac{dw}{dx} - \theta = 0 \quad (3.10) \]

Introducing this constraint by the penalty number method we have thus to minimise

\[ \overline{F} = \frac{1}{2} \int_0^L D \left( \frac{d\theta}{dx} \right)^2 dx - \int_0^L wP \, dx + \frac{1}{2} \int_0^L \alpha_1 \left( \frac{dw}{dx} - \theta \right)^2 dx \]

\[ = \pi + \frac{1}{2} \int_0^L \alpha_1 \left( \frac{dw}{dx} - \theta \right)^2 dx \quad (3.11) \]

where \( D = \frac{Eh^3}{12} \), \( w \) is the lateral deflection of the beam, \( p \) is the applied load, and \( \alpha_1 \) is a large number called penalty number. The factor \( \frac{1}{2} \) is merely introduced for convenience. The second term of (3.11) can be identified as the shear energy for a beam in which the shear strain is given by

\[ \varepsilon_s = \frac{dw}{dx} - \theta \neq 0 \quad (3.12) \]

where \( \alpha_1 \) here represents the shear rigidity given as \( \alpha_1 = \kappa Gh \) (\( \kappa \) accounts for a correction due to non-uniform shear distribution). Thus the solution
with $\alpha_1^*$ has now a physical meaning, as representing the deflections of the beam with shear distortion.

The constrained functional of equation (3.11) leads to equations of the form

$$([K] + \alpha_1 [K_1]) \{q\} = \{p\}$$  \hspace{1cm} (3.13)

where $[K]$ is derived from the original functions and $[K_1]$ from the constraints; $\{q\} = [w_i, \theta_i]^T$. When $\alpha_1$ is large (in a very thin beam) equation (3.13) degenerates

$$[K_1]\{q\} = -\frac{\{p\}}{\alpha_1} + 0$$

and $\{q\} = 0$ unless the matrix $[K_1]$ is singular; the use of a single point integration will assure the singularity of $[K_1]$ as we will see in Chapter 5.

3.5 Solution of Equilibrium Equations in Static Analysis

In this section, we are concerned with the solution of the simultaneous linear equations that arise in static analysis, that equations are given by

$$[K]\{u\} = \{P\}$$  \hspace{1cm} (3.6)

Essentially, there are two different classes of methods
for the solution of equation (3.6):

(a) direct solution techniques, and
(b) iterative solution methods.

The most effective direct solution techniques currently used are basically applications of Gauss elimination. The basic procedure of the Gauss elimination solution is to reduce the coefficient matrix of the equations (the [K] matrix) to an upper triangular matrix form, by subtracting multiples of the equations from each other, from which the unknown displacements {u} can be calculated by a back-substitution. The basic Gauss solution scheme can be applied to almost any set of simultaneous linear equations, however, the effectiveness in finite element analysis depends on the specific properties of the finite element stiffness matrix: symmetry, positive definiteness, and bandedness. Fortunately, in the analysis of displacement-based finite element systems, all diagonal elements of the coefficient stiffness matrix are positive at all times in the solution, which makes an application of the Gauss elimination procedure very effective. (This property is not necessarily preserved when the stiffness matrices, are derived using hybrid or mixed formulations.)

In Cholesky factorisation, as an application of the basic Gauss elimination procedure, the symmetric stiffness matrix is decomposed as follows:
\[ [K] = [L] [L]^T \tag{3.14} \]

where

\[
[L] = \begin{bmatrix}
\lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} & \cdots & \lambda_{n,1} \\
\lambda_{2,1} & \lambda_{2,2} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots \\
\lambda_{n,1} & \lambda_{n,2} & \cdots & \cdots & \lambda_{n,n}
\end{bmatrix} \tag{3.15}
\]

and then to solve \([K][u] = [P]\) we solve \([L][z] = [P]\) by forward reduction to find \([z]\), then \([L]^T[u] = [z]\) by backward substitution to find \([u]\).

The Cholesky factorisation is only suitable for the solution of positive definite systems. The decomposition is used effectively in the transformation of a generalised eigenproblem to the standard form as we will see in the next section.

In practical finite element analysis using a direct solution scheme, the Gauss solution algorithm is nearly always used in one form or the other, because the procedure is cost-effective and numerically sufficiently stable. However, a direct solution can also be achieved using other techniques. The Givens factorisation and the
Householder factorisation are two additional direct solution methods.

In the iterative solution of \([K]\{u\} = \{P\}\) it is necessary to use an initial estimate for the displacements \(\{u\}\) say \(\{u\}\) which, if no better value is known, may be a null vector. A basic disadvantage of an iterative solution is that the time of solution can only be estimated very approximately, because the number of iterations required for convergence depends on the condition number of the \([K]\) and whether effective acceleration factors are used. Iterative solutions will not be employed throughout this thesis and full details of the methods mentioned in this section or in the next two sections can be found in references [46 - 48].

3.6 Solution Methods for Free Vibration Equations

For free vibration, with damping neglected, equation (3.8) becomes

\[
[M]\ddot{\{u\}} + [K]\{u\} = \{0\}
\]  (3.16)

The solution to equation (3.16) can be postulated to be of the form

\[
\{u\} = \{Q\} \cos \omega t
\]  (3.17)
where \( \{Q\} \) is a vector of order \( n \), \( t \) the time variable, and \( \omega \) a constant identified to represent the frequency of vibration (rad/sec) of the vector \( \{Q\} \).

Substituting equation (3.17) into (3.16), we obtain the generalised eigenproblem, from which \( \{Q\} \) and \( \omega \) must be determined

\[
[K]\{Q\} = \omega^2[M]\{Q\} \quad \text{or} \\
[K]\{Q\} = \lambda[M]\{Q\}
\]

(3.18)

The solution of equation (3.18) yields \( n \) natural frequencies and corresponding mode shapes \( \{Q_1\}, \{Q_2\}, \ldots, \{Q_n\} \). Each eigenpair \( (\lambda_i, \{Q_i\}) \) satisfies equation (3.18) i.e. we have

\[
[K]\{Q_i\} = \lambda_i[M]\{Q_i\}, \quad i=1, \ldots, n
\]

(3.19)

The significance of equation (3.19) should be well understood; the equation says that if we establish a vector \( \lambda_i[M]\{Q_i\} \) and use it as a load vector \( \{P\} \) in the equation \( [K]\{u\} = \{P\} \) then \( \{u\} = \{Q_i\} \). This thought may immediately suggest the use of static solution algorithms for the calculation of an eigenvector. Equation (3.19) also shows that an eigenvector is only defined within a multiple of itself; i.e. we also have

\[
[K]\{CQ_i\} = \lambda_i[M]\{CQ_i\}
\]

(3.20)
where C is a non-zero constant.

An important relation which the eigenvectors satisfy is that of M-orthonormality; i.e. we have

\[ \{ Q_i \}^T [M] \{ Q_j \} = \delta_{ij} \quad (3.21) \]

premultiplying equation (3.19) by \( [Q]^T \) and using the condition in equation (3.21), we obtain

\[ \{ Q_i \}^T [K] \{ Q_j \} = \lambda_i \delta_{ij} \quad (3.22) \]

meaning that the eigenvectors are also K-orthogonal.

The solution of eigenvalues and corresponding eigenvectors requires, in general, much more computer effort than the solution of static equilibrium equations, so that using effective methods can be subdivided into four groups, corresponding to which basic property is used as the basis of the solution algorithm.

The vector iteration methods make up the first group, in which the basic property used is that

\[ [K] \{ Q_i \} = \lambda_i [M] \{ Q_i \} \quad (3.23) \]

The transformation methods constitute the second group,
using that

\[ [Q]^T[K][Q] = \Lambda \]  \hspace{1cm} (3.24) \]

\[ [Q]^T[M][Q] = [I] \]  \hspace{1cm} (3.25) \]

where \([Q] = [Q_1, \ldots, Q_n]\) and \(\Lambda = \text{diag} (\lambda_i)\),

\(i_1, \ldots, i_n\).

The solution methods of the third group are

polynomial techniques which operate on the fact that

\[ P(\lambda_i) = 0 \quad \text{where} \]

\[ P(\lambda) = \det ([K] - \lambda[M]) \]  \hspace{1cm} (3.26) \]

The solution method of the fourth group employ the sturm

sequence property of the characteristic polynomials

\[ P(\lambda) = \det ([K] - \lambda[M]) \]  \hspace{1cm} (3.27) \]

and

\[ p(r)(\lambda(r)) = \det ([K]^r[r] - \lambda(r)[M]^r[r]) \]  \hspace{1cm} (3.28) \]

where \(p(r)(\lambda(r))\) is the characteristic polynomial of the

rth associated constraint problem corresponding to

\([K][Q] = \lambda[M][Q] \).
A number of solution algorithms have been developed within each of the above four groups of solution methods [46-48]. However, only the second group will be considered in this thesis. A few basic additional points should be noted, before presenting the solution techniques of the second group. It is important to realise that all solution methods must be iterative in nature because, basically, solving the eigenvalue problem

\[ [K][Q] = \lambda [M][Q] \]

is equivalent to calculating the roots of the polynomial \( P(\lambda) \) which has order equal to the order of \([K]\) and \([M]\). Although iteration is needed in the solution of an eigenpair \((\lambda_i, [Q_i])\) it should be noted that once one member of the eigenpair has been calculated, we can obtain the other member without further iteration. It may also be helpful if the solution method identifies at an early stage of solution a badly conditioned system.

In the Householder-QR-Inverse iteration solution the generalised eigenproblem in equation (3.18) can be reduced to the standard form \([K][Q] = \lambda [Q]\), then by employing Householder transformation to reduce the matrix \([K]\) into tridiagonal form we employ QR iteration to get the eigenvalues. Using inverse iteration, the required eigenvectors of the traditional matrix are calculated and transformed to obtain the eigenvectors of \([K]\).
Assuming that $[M]$ is positive definite, we can transform the generalised eigenproblem given in equation (3.18) by using a decomposition of $[M]$ of the form

$$[M] = [L][L]^T$$

substituting for $[M]$ into equation (3.18) we obtain

$$[K][Q] = \lambda [L][L]^T[Q]$$

premultiplying both sides of equation (3.29) by $[L]^{-1}$ and defining a vector $[\tilde{Q}] = [L]^T[Q]$ we obtain the standard eigenproblem

$$[\tilde{K}][\tilde{Q}] = \lambda [\tilde{Q}]$$

where

$$[\tilde{K}] = [L]^{-1}[K][L]^{-1}$$

for convenience we will write (3.30) in the form

$$[K][Q] = \lambda [Q]$$

The next step is the Householder reduction of $[K]$ in (3.32) to the tridiagonal form

$$[K]_{k+1} = [P]^T_k [K]_k [P]_k; \quad k = 1, \ldots, n-2$$
using \([K]_1 = [K]\) where the \([P]_k\) are Householder transformation matrices. The QR iteration is applied then to the tridiagonal matrix.

3.7 Solution Methods for Forced Vibration Equations

It has been shown that the equations of motion for a finite element system are written as

\[
[M]\ddot{\{u\}} + [C]\dot{\{u\}} + [K]\{u\} = \{P(t)\} \tag{3.8}
\]

where \(\{P(t)\}\) is the force input, which may take the various forms shown in Fig. 3.5.

The damping force is proportional to the velocity \(\dot{\{u\}}\) and oppose the motion. (A dot over a symbol indicates differentiation with respect to time.)

There are basically two methods of solving these equations: direct integration methods and mode superposition methods.

In direct integration the equations in (3.8) are integrated using a numerical step-by-step procedure, the term "direct" meaning that prior to the numerical integration, no transformation of the equations into a different form is carried out. In essence, direct
Fig. 3.5 Types of force input that can occur in the solution of the forced equations of motion
numerical integration is based on two ideas:

(1) the response is obtained at a series of sequential time intervals apart;

(2) a variation of displacements, velocities, and accelerations within each time interval is assumed.

It should be noted here that the total cost of analysis is approximately inversely proportional to the magnitude of time step used.

A commonly employed integration scheme which is unconditionally stable (i.e. however large the time step length used in the time integration, the predicted response remains bounded) is the Wilson θ method [49].

In the mode superposition method the equations of motion are transformed into a more effective form. The variables are transformed from the original set \{\mathbf{u}\} to a new set \{\mathbf{X}\}:

\[
\mathbf{u}(t) = [Q] \mathbf{X}(t)
\] (3.33)

the objective of the transformation is to obtain new system stiffness, mass and damping matrices which have a smaller bandwidth than the original system matrices; the
transformation matrix should be selected accordingly.

It can be shown that if the transformation matrix \([Q]\) is established using the free undamped vibration mode shape vectors, the equations in terms of the transformed variables are uncoupled. Therefore, the equations of motion reduce to \(n\) equations of the form

\[
\ddot{X}_i + 2\omega_i \xi_i \dot{X}_i + \omega_i^2 X_i = \frac{P_i}{m_i}
\]  

(3.34)

The \(i\)th typical equation in (3.34) is the equation of motion of a single degree of freedom problem with \(m_i\) mass and stiffness \(\omega_i^2\), \(\xi_i\) is a modal damping parameter.

The solution to each equation can then be obtained by evaluating the Duhamel integral and the response of the system to prescribed time-dependent forces is obtained as a sum of contributions from individual modes. These solutions are then transformed back in terms of the original variables and a general solution of the problem is obtained \([50, 51]\).

If the damping of the system is assumed to be of a restricted form which does not introduce modal coupling \([52]\) the equation of motion can be solved economically by the mode superposition method. However, the step-by-step integration presented earlier may give a more efficient solution in cases where a large number of modes
participate in the response.

3.8 Evaluation of the Stiffness, Mass and Damping Matrices

So far, we have not shown how the stiffness, mass, and damping matrices can be generated using the finite element technique. However, the process will be shown, in this section, in brief.

3.8.1 Element stiffness matrix

The relationship between the strains and displacements of an element can be written in matrix relation as

\[
\{\varepsilon\}^e = [L] \{q\}^e
\]  \hspace{1cm} (3.35)

where \([L]\) is a suitable linear operator; but the displacements \([q]\) at any point within the element should be described in terms of nodal displacements \([u]\) of a particular element, i.e.,

\[
\{q\}^e = [N] \{u\}^e
\]  \hspace{1cm} (3.36)

where \([N] = [N_1, N_2, \ldots, N_n]\) are called shape functions.

Inserting equation (3.36) into (3.35) to get
\[ [\varepsilon]^e = [L][N][u]^e_0 = [B][u]^e_0 \] (3.37)

Now, the strain energy of an element is given as

\[ U^e = \frac{1}{2} \int_{\text{Vol}} [\varepsilon]^e_T[D][\varepsilon] \ d(\text{vol}) \]

\[ U^e = \frac{1}{2}[u]^e_0 [B]^T[D][B] [u]^e_0 \int_{\text{Vol}} \]

\[ U^e = \frac{1}{2}[u]^e_0 [K]^e [u]^e_0 \] (3.38)

where

\[ [K]^e = \int_{\text{Vol}} [B]^T[D][B] \ d(\text{vol}) \] (3.39)

is the element stiffness matrix.

### 3.8.2 Element mass matrix

In a similar manner the kinetic energy of an element can be written as

\[ T^e = \frac{1}{2} \int_{\text{Vol}} \{\dot{u}\}^e_T:\{\dot{u}\}^e \ d(\text{vol}) \]

\[ = \frac{1}{2} \{\dot{u}\}^e_0 [N]^T:\{\dot{u}\}^e_0 \int_{\text{Vol}} \]

\[ = \frac{1}{2} \{\dot{u}\}^e_0 [M]^e:\{\dot{u}\}^e_0 \] (3.40)

where

\[ [M]^e = \int_{\text{Vol}} [N]^T:\{\rho\} [N] \ d(\text{vol}) \] (3.41)

is the element mass matrix.
Once this stage has been reached the assembly of the element stiffness and mass matrices can follow the standard way.

3.8.3 Damping matrix

The element damping matrix can be expressed as

\[
[C]^e = \int_v [N]^T [\mu] [N] \, d(\text{vol})
\]  

(3.42)

but, in practice, it is difficult to determine, for general finite element assemblage, the element damping parameters as the knowledge of the viscous matrix \([\mu]\) is lacking. For this reason the matrix \([C]\) is in general not assembled from element damping matrices, but is constructed using the mass and stiffness matrices of the complete element assemblage.

It is often assumed that the damping matrix is a linear combination of stiffness and mass matrices, i.e.

\[
[C] = \alpha [M] + \beta [K]
\]

where \(\alpha\) and \(\beta\) are determined experimentally [53]. Other generalisations of the damping matrix are possible [52, 54].
CHAPTER FOUR

A perusal of the literature devoted to finite element techniques reveals the very considerable effort directed at finding good elements for the solution of shell structures. These efforts can be classified into three groups: flat, curved and isoparametric solid elements. Nevertheless, for practical problems no one type of element has yet been to be preeminent (34). Distinction is made between elements which are limited to axisymmetric shells, such as conical frusta, and those which are of more general form, such as triangles and quadrilaterals, identified throughout the survey.

The application of the finite element method to the analysis of shells of revolution, using conical frusta as the finite elements, was first presented by Grafton and Streeter (3). The element possesses six degrees of freedom, three in each nodal circle, two components of displacements, $u$ and $w$, and a rotation $\beta$. In this a direct approach via displacement functions, which satisfy the end conditions and the relationship between the change in slope and displacements, i.e., $\beta=(dw/dn) \cos \phi -(du/dn)$ with respect to the meridional coordinate $\phi$. Grafton and Streeter introduced further approximation by approximating...
CHAPTER 4

SURVEY OF LITERATURE ON SHELL ELEMENTS

A perusal of the literature devoted to finite element techniques reveals the very considerable effort directed at finding good elements for the solution of shell structures. These elements can be classified into three groups: flat, curved and isoparametric solid elements. Nevertheless, for practical problems no one type of elements has yet been to be preeminent [34]. Distinction between elements which are limited to axisymmetric shells, such as conical frusta, and those which are of more general form, such as triangles and quadrilaterals, is identified throughout the survey.

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all of the integrals, in evaluating the stiffness matrix, by the trapezoidal rule. The same idealization has been used, with some refinements, in the derivation of the element stiffness matrix, by Popov et al [55].

An extension to the case of asymmetrical deformations, which was suggested in [3] was elaborated by Percy et al [4]. The generalized displacements of a nodal ring are the coefficients of the terms of finite Fourier Series representations of $u, v, w$ and $\beta$ as functions of the circumferential coordinate $\theta$, where $u, v, w$ are the axial, tangential, and radial displacements of a point of the nodal circle, and $\beta$ is the rotation in a plane containing the axis of the shell, of the meridian in the shell surface that passes through the point. The element possesses eight degrees of freedom. The in-plane displacements were approximated by a linear function and the normal displacements by a third degree polynomial of the meridian coordinate. Some of the integrals were evaluated numerically as their analytical form was ill-conditioned. It was pointed out that the conical frustum representation of deep shells can yield residual bending moments in an area where essentially only membrane forces should exist.

Jones and Strome [56] stated that the formulation in [3], which was extended in [4], has failed to provide satisfactory results in many problems. In order to remedy difficulties experienced with the conical shell element,
the shell was approximated by double-curved elements, in which there is continuity of the slope at every point of the shell, and the slope and radius of elements are matched with those of the actual shell. It was stated that the use of simple polynomial expressions is unsuitable for a doubly curved element; special terms corresponding to constant axial and radial displacements and a constant rotation of each element were introduced.

In Stricklin et al [57] several changes have been made to the matrix displacement method that both eliminate the residual bending moment and reduce the number of stations needed for an accurate solution. The major modifications to the method of ref.[4] were: 1. the incorporation of an accurate representation of the shell geometry, 2. the proper treatment of the singularity at the apex, 3. the consistent lumping of distributed loads, and 4. a revised method for computing the stress resultants.

Improvements in the ring finite element analysis, for shells of revolution, obtained by extending the polynomials representing the displacements were illustrated by Webster [58]. The displacements of the middle surface of the curved elements were represented by truncated power series in the meridional coordinate. The coefficients of the terms in the power series and the natural frequencies were found by finding the stationary values of the Hamiltonian subjected to the constraints
required to satisfy the boundary conditions at the ends of the shell and the continuity conditions at the junctions between shell elements. The simplest displacement functions for the tangential and meridional displacements contained two terms and there were four terms in the simplest normal displacement function. The results for the natural frequencies for three shells given in the paper indicated that there are considerable advantages in representing a shell by a few elements with displacement functions extended beyond that required to satisfy continuity between elements rather than using a large number of elements with the simplest displacement functions.

Two complicated elements, for the solution of shells of revolution, have been presented by Fonder [59]. The first is a 24 d.o.f., a doubly-curved quadrilateral element, whose edges were taken along the circumferential and meridian lines. Bilinear and bicubic polynomial expressions were chosen for the u, v, and w displacements respectively; the 24 constants, therefore, can be found in terms of \( \frac{\partial w}{\partial s}, \frac{\partial w}{\partial \theta} \) and \( \frac{\partial^2 w}{\partial s \partial \theta} \) at the four corner nodes. Numerical integration was performed over the element area; a grid of 4 x 4 Gauss points was found necessary for this purpose.

Since the explicit addition of rigid modes, to the element, proved unsuccessful, an alternative procedure was to improve the recovery in the limit of rigid modes by taking bicubic interpolation polynomials for all three
curvilinear displacements \( u, v \) and \( w \). This led to the 48 degrees of freedom element with 12 unknowns at each of the four corners: \( u, \frac{du}{ds}, \frac{d^2u}{d\theta d\phi}, v, \frac{dv}{ds}, \frac{d^2v}{d\theta d\phi}, \frac{dw}{ds}, \frac{d^2w}{d\theta d\phi}, \frac{d^2w}{d\phi d\theta} \).

This element can be considered a generalisation of the Bogner, Fox and Schmit cylindrical element [60]. It ensures continuity of displacements, their first derivatives and some of their second derivatives. Therefore, this element is supercompatible. Because of the second derivatives degrees of freedom, which cannot be rotated, it cannot accommodate intersecting shells but only smooth surfaces. Numerical integration was used again; grids of \( 6 \times 6, 5 \times 5, 4 \times 4 \) and \( 3 \times 3 \) Gauss points were successively tried for different types of shells of revolution (circular plate, cylinder, cone, sphere). The comparison of the resulting stiffness matrices was based on their 48 eigenvalues rather than their 48 \( \times \) 48 terms. It showed identical results for \( 6 \times 6, 5 \times 5 \) and \( 4 \times 4 \) grids, whatever the type of shell and element size. Important changes were obtained when going down from the \( 4 \times 4 \) to the \( 3 \times 3 \) grid so the former was adopted.

To overcome the difficulties that arise in satisfying the necessary continuity of slopes at interfaces and in the inability of such formulations to account for shear deformations, a fairly obvious artifice of abandoning the Kirchhoff hypothesis in favour of a bending theory which
includes shear deformations could be adopted. Such elements have been proposed and used in thick shell situations by Melosh [61], Utku [33] and others [37, 62-65]. It was discovered, however, that in thin shell situations the new approach gives such a large stiffness matrix as to make the use of such elements quite uneconomical, and several devices for improvement were suggested. Utku [33] and Martin [65] for instance, used in this context a substitution of constants arrived at by an intuitive reasoning. Such procedures are not easy to generalise and may indeed lead to non-convergent results. A more acceptable alternative was that of constraining the element to obey Kirchhoff conditions at a discreet number of points. This idea introduced by Wempner and co-workers [62, 63], has been elaborated further by others [64, 66]. While now convergence is achieved and the process can be generalized to any element, its practical application was difficult [67]. Some improvements over this have been proposed by Fried [68]. On the other hand, reduced integration by Zienkiewicz et al [67] and Pawsey and Clough [69] utilizes a lower order of numerical integration and has proved to be successful in relaxation of constraints on the transverse shear strains.

An accurate quadrilateral element for thick and thin shells has been developed by Zienkiewicz et al [67]. This element possesses eight nodes — four corner and four midside — and it has as its degrees of freedom three displacements of the middle plane and two rotations of the
normal about axes parallel to it at each node. The element was derived in earlier an publication [70] from a full three-dimensional isoparametric form by prescription of linear displacement variation across the thickness and suppression of strain energy due to stresses normal to its middle plane. Two-by-two Gaussian quadrature is an essential requirement for the good performance of the element and has proved to be very successful in relaxation of constraints on the transverse shear strains. This integration was carried out originally [70] using two Gauss points in the transverse direction and either $3 \times 3$ or $4 \times 4$ mesh of Gauss points in the middle plane depending on whether a parabolic or cubic type of element was used. In this original form, excellent results were obtained for thick plates and shells, but complete failure occurred in bending when thin sections were considered.

We have seen that the continuity requirements imposed by Kirchhoff assumption, which reduces the number of independent variables in the variational statement but introduces higher order derivatives in the formulation of shell elements, has prevented the development of simple and natural elements. Thus, the interelement compatibility inevitably leads to extensive algebraic operations in the formation of the basic element stiffness coefficients and consequently to large storage requirements and computational time. Furthermore, the conforming elements have suffered drawbacks in practical engineering purposes where shells with varying
thicknesses, with stiffeners or with branches are met. The difficulty arises where it is wrong to require strain continuity in such cases. Therefore, the search for simple and accurate elements, with less algebraic operations and small storage requirements and computational time, has been in progress.

Many users of finite element computer programs find a basic four-node quadrilateral element, particularly appealing due to its simplicity. This appeal might become even greater when non-linear applications are undertaken. In the non-linear regime - and especially in non-linear dynamics - computational cost is the prime concern. Due to frequent reformulations of tangent stiffnesses, complicated element routines can lead to exorbitant computational expenditures. A simpler element of competitive accuracy becomes quite desirable under such circumstances. Other factors in non-linear analysis support this viewpoint. For example, the accuracy level attainable in non-linear problems is often severely limited due to an uncertainty of non-linear material characterisations. Thus, it makes little sense to apply elements which are only marginally more accurate than simpler elements, since the confidence level of the overall analysis may be affected only negligibly.

In a paper dealing with the problems of plates Hughes et al [71] developed what is believed to be the simplest
effective bending element proposed so far. The element is a four-node quadrilateral with the basic three degrees-of-freedom per node. The element shape functions are bilinear for transverse displacement and rotations. The shear 'locking' associated with such low order functions in application to thin plates was alleviated by splitting the shear and bending energies and using one point quadrature on the shear term. The simplicity of the element lends itself to concise and efficient computer implementation.

Zienkiewicz et al [5] applied an almost identical process of [71] to the generation of an equivalent axisymmetric element family with particular attention being given to its lowest linear member. A two-noded, straight element which includes shear deformation effects was presented and shown to be extremely efficient in the static analysis of axisymmetric shells. A single point of numerical integration was essential for its success when applied to thin shells where the results compare favourably with those achieved with more complex curved elements.

At Aston university, Richards and his Msc student Kumagai extended the static analysis of [5] to the case of non-axisymmetric loads [6]. Their program can handle axisymmetric deformation and the deformation which is symmetric with respect to the diametral plane $\theta=0$. 

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The simple formulation of the element, first proposed by Zienkiewicz et al., extended by Richards and Kumagai to the non-axisymmetric loading cases made it attractive to extend the formulation to dynamic analysis which has been done in this thesis with a more careful study of the role of penalty numbers in the formulation of element characteristics.
CHAPTER FIVE:

A SIMPLE

SEMI-ANALYTIC ELEMENT

FOR AXISYMMETRIC SHELLS

Several axisymmetric shell elements have previously been derived taking advantage of the fact that geometry and material properties are constant with respect to one co-ordinate direction, namely the circumferential coordinate, \( \theta \). It was found in reference [4] that acceptable accuracy could be obtained with a reasonable number of elements (429 elements plus), but it was thought that a better compromise between accuracy and computing effort might be achieved if improved element stiffness matrices were used allowing the same accuracy of solution to be achieved with fewer elements. More than a decade later, Glinko and others proposed this point by introducing a very simple and economical element with bilinear interpolation for all components of displacement field, for static analysis of axisymmetric deformation of
CHAPTER 5

5.1 Introduction

In the previous two chapters it was shown, briefly, how linear shell problems could be formulated and solved using different general types of finite element with a variety of displacement interpolations. In this Chapter a specialised formulation, for a semi-analytic element, for axisymmetric shells will be presented.

Several axisymmetric shell elements have previously been derived taking advantage of the fact that geometry and material properties are constant with respect to one co-ordinate direction, namely the circumferential coordinate, \( \theta \), [3 - 5]. It was found in reference [4] that acceptable accuracy could be obtained with a reasonable number of elements (35 elements plus), but it was thought that a better compromise between accuracy and computing effort might be achieved if improved element stiffness matrices were used allowing the same accuracy of solution to be achieved with fewer elements. More than a decade later Zienkiewicz et al. [5] proved this point by introducing a very simple and economical element (with bilinear interpolation for all components of displacement field) for static analysis of axisymmetric deformation of
shells of revolution. Transverse shear strain was introduced, by way of a penalty function, to improve the accuracy of the solution with fewer elements. It must be mentioned here that the formulation of the axisymmetric shell element introduced in reference [5] is equivalent to that of the simple beam element, dealing with plate problems, which was introduced by Hughes et al. [71].

There have been many curved elements [72 - 74] which include transverse shear strain. The element by Venkatarana and Rao [72] employs 20 freedoms and uses cubic polynomials for all the generalised displacements and rotations whereas the 9 freedoms Mohr element [73] with three nodes, employs quadratic interpolation. There have been other types of doubly curved axisymmetric elements, without the introduction of transverse shear, for the solution of shells of revolution [36, 56, 58]. One of these [56] was expected to remedy difficulties experienced in references [3] and [4]. Webster [58] represented the displacements of the middle surface by truncated power series in the meridional coordinate and proved that there are considerable advantages in representing a shell by a few elements with displacement functions extended beyond that required to satisfy continuity between elements rather than using a large number of elements with the simplest displacement functions. Green et al. [36] arrived at a similar conclusion.
In this investigation a specialised formulation is presented overcoming some of the difficulties mentioned so far. The difficulty in imposing $C_1$ continuity on the shape functions and accounting for thick shell problems have resulted in alternative approach to the problems in which these difficulties are side-stepped. Two basic alternatives are present:

(1) imposition of continuity as a constraint using penalty functions. This method was first introduced by Zienkiewicz et al. [5] for the static analysis of axisymmetric shells subjected to axisymmetric loads. The analysis was later extended to include $\theta$-symmetric deformation [6].

(2) complete reformations of the problem in terms of relationships and energy functionals requiring only $C_0$ continuity.

The complete shell is idealised into a series of two noded, straight elements joining nodal circles. The statement, in general shell theory, that after deformation the normals remain normal to the deformed middle surface has been omitted. This omission permits the shell to experience shear deformation, which is a very important feature in thick shell situations. The whole process is then extended to dynamic analysis. Rotary inertia is
included in the kinetic energy expression. With these modifications an efficient tool for analysing axisymmetric shells becomes available. The element accuracy is demonstrated in Chapter 7 on a variety of thick, as well as thin, shell problems in which both static and dynamic situations are considered.

5.2 Element Characteristics

Figure 5.1 shows the general element geometry and the global (at node i) and local (at node j) freedoms of the element. The external faces of the element are straight and the joining circle sections across the thickness are generated by straight lines. The element has a one-dimensional isoparametric form. The element with two nodes, i and j, posses ten degrees of freedom; five at each node; the axial, radial and circumferential displacements of the middle surface and the rotations (including rotations due to shear) of the normal along the meridional and circumferential directions.

Taking the local dimensionless coordinate in the middle surface, $\eta$, to vary from $-1$ (at node i) to $+1$ (at node j) the displacement field within an element could be described in terms of nodal displacements as

$$\{q(s)\}^e = [N] \{u\}^o_e \quad (5.1)$$
where \( \mathbf{X}_k \) is the \( 3 \times 5 \) identity matrix.

The equations of the element are as shown in Eq. 3.1, and in vector form for the element (i.e., the \( \mathbf{X}_k \) matrix is obtained by applying 100 times the right-hand side of Eq. 3.1). In addition, we will redefine the angle \( \alpha \) as shown in Fig. 3.1 instead of the definition given in Chapter 3. Then

![Diagram of truncated cone shell element configuration](#)

**Fig. 5.1** Truncated cone shell element configuration
where

\[ [N] = \begin{bmatrix} [I] N_i, & [I] N_j \end{bmatrix} \] (5.2)

and where \([I]\) is a 5x5 identity matrix. The shape functions are defined as

\[
N_i = \left(1 + n n_i \right)/2, \quad N_j = \left(1 + n n_j \right)/2
\]

\[
 n = (2s/L) - 1, \quad dn = (2/L)ds
\]

\[
n = n_i = -1, \quad n = n_j = +1
\] (5.3)

The appropriate midsurface strain-displacement relations for the element, are derived after adjusting equations (2.9 - 2.16) to suite a straight element (i.e., \(R_\phi = \infty\) and \(R_\phi \partial_\phi = \partial s\) or \(\partial / R_\phi \partial_\phi = \partial / \partial s\)). In addition, we will redefine the angle \(\phi\) as shown in Fig. 5.1 instead of the definition for it given in Chapter 2. Then

\[
(\varepsilon_{s})_o = \frac{\partial u}{\partial s}\] (5.4)

\[
(\varepsilon_\theta)_o = (u \sin \phi + \frac{\partial v}{\partial \phi} + w \cos \phi)/r\] (5.5)

\[
(\varepsilon_{s\theta})_o = \frac{\partial u}{\partial \theta} - v \sin \phi)/r + \frac{\partial v}{\partial s}\] (5.6)

\[
\chi_s = -\frac{\partial^2 w}{\partial s^2}\] (5.7)
\[ \chi_\theta = \frac{\cos \phi}{r^2} \cdot \frac{\partial v}{\partial \theta} - \frac{\sin \phi}{r}, \quad \frac{\partial w}{\partial s} - \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2} \quad (5.8) \]

\[ \chi_s \theta = 2\left( -\frac{\sin \phi \cos \phi}{r^2} \cdot v + \frac{\cos \phi}{r} \cdot \frac{\partial v}{\partial s} + \frac{\sin \phi}{r^2} \cdot \frac{\partial w}{\partial \theta} - \frac{1}{r} \cdot \frac{\partial^2 w}{\partial s \partial \theta} \right) \quad (5.9) \]

The transverse shear strains with respect to the meridional and tangential directions, respectively, are given as

\[ \varepsilon_{s \zeta} = \beta - \frac{\partial w}{\partial s} \quad (5.10) \]

\[ \varepsilon_{\theta \zeta} = \alpha - \frac{1}{r} \left( \frac{\partial w}{\partial \theta} - v \cdot \cos \phi \right) \quad (5.11) \]

in which \( \beta \) and \( \alpha \) are, respectively, the total rotations of the normal along the meridional and tangential directions,

\[ \beta = \frac{\partial w}{\partial s} \quad (5.12) \]

\[ \alpha = \frac{1}{r} \left( \frac{\partial w}{\partial \theta} - v \cdot \cos \phi \right) \quad (5.13) \]

When the displacement field of the middle surface of the shell (subjected to the action of arbitrary external load) is expanded in Fourier series in the circumferential sense then it can be written as follows:
\[ n\{q(s, \theta)\}^e = \sum_{n=0}^{\infty} n\{N_1\}^e_{s} + \sum_{n=1}^{\infty} n\{N_2\}^e_{a} \]  

(5.14)

where the subscripts \(s\) and \(a\) designate, respectively, the \(\theta\)-symmetric and the \(\theta\)-antisymmetric components of \(n\{q(s, \theta)\}^e\), and the diagonal shape function matrices, with respect to \(\theta\), \([N_1]\) and \([N_2]\) are defined as follows

\[ n\{N_1\} = [cn, sn, cn, sn, cn, sn] \]  

(5.15)

\[ n\{N_2\} = [sn, cn, sn, sn, cn, cn] \]  

(5.16)

where

\[ cn = \cos n\theta, \text{ and } sn = \sin n\theta \]

Combining equations (5.1) and (5.14) we obtain

\[ n\{q(s, \theta)\}^e = \sum_{n=0}^{\infty} n\{N_1\}[N] n\{u\}^e_{os} \]

\[ + \sum_{n=1}^{\infty} n\{N_2\}[N] n\{u\}^e_{oa} \]  

(5.17)

The aim of equations (5.14) - (5.17) is to eliminate the circumferential coordinate by expanding \(u, v, w, \beta\) and \(\alpha\) in Fourier series, where \(n\) corresponds to any harmonic number.
For the sake of simplicity, in the sequel we will drop the subscript s and a. The development for both sets of harmonics, symmetric and antisymmetric, are entirely similar and the following development will cover both cases. Matrices developed in the following sections will be distinguished when it is necessary.

5.3 Strain Energy - Derivation of the Element Stiffness

The strain energy of an element of the shell, excluding shear, is expressed in terms of the strains and changes in curvatures of the middle surface as

$$ U_e = \frac{1}{2} \int_v \{\varepsilon\}^e [D]\{\varepsilon\}^e \, d(vol) $$

(5.18)

where \( d(vol) \) is the elemental volume which can be written in the form

$$ d(vol) = h \, d(area) = h \, (rd\theta ds) = hr \frac{L}{2} \, d\theta \, d\eta $$

where \( ds = \frac{L}{2} \, d\eta \) is substituted from equation (5.3).

Here, \( h \), the thickness of the element, is included in the elastic matrix, \([D]\), which is defined in an isotropic material as \([28]\)
\[
[D] = \frac{Eh}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
& & \frac{1-\nu}{2} & 0 & 0 & 0 \\
& & \frac{h^2}{12} & \frac{\nu h^2}{12} & 0 & 0 \\
& & & & \frac{h^2}{12} & 0 \\
& & & & & \frac{h^2}{12} \cdot \frac{1-\nu}{2}
\end{bmatrix}
\]

\text{(5.19)}

In equation (5.18) the flexural strains have second order derivatives, so that \( C_1 \) continuity is required. In order to avoid the introduction of slope continuity, we will use equations (5.12) - (5.13) so that strains are defined by first derivatives and, therefore, the displacements only have to be continuous. Equations (5.7) - (5.9) can now be written in the form

\[
\chi_s = -\frac{3\gamma}{\delta s}
\]

\[
\chi_\theta = -\frac{1}{r} (\beta \sin \phi + \frac{3\alpha}{3\theta})
\]

\[
\chi_s \theta = 2 (\frac{\sin \phi}{r} \cdot \alpha - \frac{3\alpha}{\delta s})
\]

The vector \( \{\varepsilon\}^e \), in a truncated cone element, is defined, therefore, in matrix form as
\[
\{\varepsilon\}^e = \left\{ \begin{array}{c}
\varepsilon_s \\
\varepsilon_\theta \\
\varepsilon_{s\theta} \\
\chi_s \\
\chi_\theta \\
\chi_{s\theta}
\end{array} \right\} = \left\{ \begin{array}{c}
\frac{3u}{\partial s} \\
\frac{1}{r} \left( u \cdot \sin \phi + \frac{\partial v}{\partial \theta} + w \cdot \cos \phi \right) \\
\frac{1}{r} \left( \frac{3u}{\partial \theta} - v \cdot \sin \phi \right) + \frac{3v}{\partial s} \\
- \frac{\partial \theta}{\partial s} \\
- \frac{1}{r} \left( \beta \sin \phi + \frac{\partial \alpha}{\partial \theta} \right) \\
2 \left( \frac{\sin \phi}{r} \cdot \alpha - \frac{\partial \alpha}{\partial s} \right)
\end{array} \right\}
\]

(5.20)

and where the linear operator \([L]\) is defined as

\[
[L] = \begin{bmatrix}
\frac{3}{\partial s} & 0 & 0 & 0 & 0 \\
\frac{\sin \phi}{r} & \frac{1}{r} \frac{\partial}{\partial \theta} & \frac{\cos \phi}{r} & 0 & 0 \\
\frac{1}{r} \frac{\partial}{\partial \theta} & \frac{3}{\partial s} - \frac{\sin \phi}{r} & 0 & 0 & 0 \\
0 & 0 & 0 & - \frac{3}{\partial s} & 0 \\
0 & 0 & 0 & - \frac{\sin \phi}{r} & - \frac{1}{r} \frac{\partial}{\partial \theta} \\
0 & 0 & 0 & 0 & \frac{2 \sin \phi}{r} - 2 \frac{3}{\partial s}
\end{bmatrix}
\]

(5.21)

The strains within an element can be written, therefore, in the final form as

\[
\{\varepsilon\}^e = \sum_{n=0}^{\infty} [L] [N]_n [N]_n \{u\}_n^e
\]

(5.22)

where

\[
\{\varepsilon\}^e = \sum_{n=0}^{\infty} [B] [N]_n \{u\}_n^e
\]

The internal strain energy of an element can be written in the form
\[ n^*[B] = [L] \, n^*[N_1] \, [N] \]  

(5.23)

is the strain nodal displacement matrix.

Substituting equation (5.23) into equation (5.18) we obtain

\[
\mathbf{u}^e = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} n^i(u)^e \mathbf{T} \left[ \int_0^1 \int_0^{2\pi} n^*[B] \mathbf{T}[D] \mathbf{m}[B] \, \mathrm{d}\theta \, \mathrm{d}n \right] \, m^i(u)^e
\]

(5.24)

Carrying out the matrix multiplications we shall get terms containing the submultiple

\[
\begin{bmatrix}
\text{cn} \\
\text{sn}
\end{bmatrix}
\begin{bmatrix}
\text{cm} \\
\text{sm}
\end{bmatrix}
\text{[function of } n \text{]}
\]

Integrating that over the range \(0 < \theta < 2\pi\), and noting the following integral formulae

\[
\int_0^{2\pi} \text{cn} \, \text{cm} \, d\theta = \begin{cases}
\pi & n = m = 0 \\
0 & n = m \neq 0 \\
0 & n \neq m
\end{cases}
\]

\[
\int_0^{2\pi} \text{sn} \, \text{sm} \, d\theta = \begin{cases}
\pi & n = m = 0 \\
0 & n = m \neq 0 \\
0 & n \neq m
\end{cases}
\]

(5.25)

\[
\int_0^{2\pi} \text{cn} \, \text{sm} \, d\theta = 0,
\]

the internal strain energy of an element can be written in the form
\[ U^e = \frac{1}{2} \sum_{n=0}^{\infty} n\{u\}_0^e n[K]^e n\{u\}_0^e \]  \hspace{1cm} (5.26)

where the nth element stiffness matrix is

\[ n[K]^e = \frac{a \cdot r \cdot L}{2} \int_{-1}^{1} [B]^T [D] [B] \, dn \]  \hspace{1cm} (5.27)

At this stage, special cases of equation (5.27) should be mentioned where \([B]\) here is of the same form as was given in equation (5.23) after integrating with respect to \(\theta\).

Case 1. Axisymmetric shells under axisymmetric loads (n=0)

In this case, the displacement vector for each node contains only axial and radial displacements as well as the meridional rotation, so that the nodal displacement vector \(\{u\}_0^e\) for an element is a 6x1 and the identity matrix in equation (5.2) is 3x3. It follows that matrix \([B]\) and hence \([B]\) is a 4x6 matrix, since it is a product of 4x3, 3x3 and 3x6 matrices (equation (5.23)). The element stiffness matrix is a 6x6 matrix, since it is a product of 6x4, 4x4 and 4x6 matrices (equation (5.27)). Finally, in equation (5.27) \(\pi\) should be replaced by 2\(\pi\).
Case 2. Axisymmetric shells subjected to loads symmetric with respect to the plane $\theta = 0$ ($n=1$ to $\infty$).

In this case the nodal displacement vector $n[u]_o^e$ is a $10\times1$ vector

$$n[u]_o^e = [u_i, v_i, w_i, \beta_i, \alpha_i, u_j, v_j, w_j, \beta_j, \alpha_j]^T$$

(5.28)

The identity matrix in equation (5.2) is a $5\times5$. It follows that the element stiffness matrix is a $10\times10$.

Case 3. Axisymmetric shells subjected to loads antisymmetric with respect to the plane $\theta = 0$.

In this case, the nodal displacement vector $n[u]_{oa}^e$ is a $10\times1$ vector as well, but because it is due to different load vector from that in Case 2, the vector

$$n[u]_{oa}^e = [u_i, v_i, w_i, \beta_i, \alpha_i, u_j, v_j, w_j, \beta_j, \alpha_j]^T$$

(5.29)

is different from that in equation (5.28). For this case $[N_1]$, in equation (5.23), should be replaced by $[N_2]$ which was defined in equation (5.16), or in other words, we should take the second sum in equation (5.17) for this case instead of the first sum which was taken for Case 2. The dimensions of the matrices are the same as in Case 2.
It should be noted that the formulations are uncoupled in \( n \) and hence, an analysis for each harmonic can be carried out separately.

5.3.1 Transverse shear - Penalty function matrices

In the study of the transverse shear effects, the expression corresponding to the strain energy in equation (5.18) must be modified properly, e.g. adding the terms corresponding to the shear energy. Introducing the shear strain energies, derived from meridional and circumferential transverse shear strains \( \varepsilon_{s\xi} \) and \( \varepsilon_{\theta\xi} \) which were given in equations (5.10) and (5.11) respectively, these are

\[
u_1^e = \frac{1}{2} \int_V \varepsilon_{s\xi} D_s \varepsilon_{s\xi} \, d(\text{vol}) \quad (5.30)
\]

\[
u_2^e = \frac{1}{2} \int_V \varepsilon_{\theta\xi} D_s \varepsilon_{\theta\xi} \, d(\text{vol}) \quad (5.31)
\]

The total potential energy, for an element, is given therefore as

\[
\pi = \frac{1}{2} \int_V \{\varepsilon\}^T [D] \{\varepsilon\}^e d(\text{vol}) \\
+ \frac{1}{2} \int_V \varepsilon_{s\xi} D_s \varepsilon_{s\xi} d(\text{vol}) + \frac{1}{2} \int \varepsilon_{\theta\xi} D_s \varepsilon_{\theta\xi} \, d(\text{vol}) \\
- \int \text{area} \, n \begin{bmatrix} \{p\}^T \\ \{q\}^e \end{bmatrix} d(\text{area}) \quad (5.32)
\]
comparing this with equation (3.6) we can write

$$
\pi_1 = \frac{1}{2} \int \{\varepsilon\}^T [D] \{\varepsilon\} \, d(\text{vol}) - \int n^T p \, q \, d(\text{area})
$$

(5.33)

$$
\pi_2 = \frac{1}{2} \left[ \int \varepsilon_{s\xi} D_s \varepsilon_{s\xi} \, d(\text{vol}) + \int \varepsilon_{s\varphi} D_s \varepsilon_{s\varphi} \, d(\text{vol}) \right]
$$

(5.34)

The first of these includes only in-plane and flexural strain energy whilst the second involves the shear strain energy terms only.

For thin shell $D_s$, the shear rigidity which is defined for isotropic materials as $D_s = \kappa G h = \kappa \frac{E h}{2(1+v)}$, tends to become very large and can be identified simply as $2a_1$-penalty number of equation (3.11) introduced to enforce the Kirchhoff constraints ($\varepsilon_{s\xi} = 0$, $\varepsilon_{s\varphi} = 0$), $\kappa = \frac{5}{6}$ is a correction factor to allow for non-uniform shear stress distribution. To explain, in more detail, why $D_s$ tends to become very large, in the case of very thin shells, let us write one typical term of the strain energy (a bending term say) and compare it with a shear energy term.

The bending term can be written as

$$
U_b = \frac{1}{2} \frac{E h^3}{12} \int \text{area} \, (\varepsilon_b)^2 \, d(\text{area})
$$
and the shear term can be written as

$$U_s = \frac{1}{2} \frac{Eh^3}{12} \int \text{area} \ (\varepsilon_s)^2 \ d(\text{area})$$

the total energy \( U = U_b + U_s \) can be written, therefore, as

$$U = \frac{1}{2} \frac{Eh^3}{12} \left[ \int (\varepsilon_b)^2 d(\text{area}) + \frac{12G}{E} \frac{1}{h^2} \int (\varepsilon_s)^2 d(\text{area}) \right]$$

(5.35)

minimising the total energy of equation (5.35) a typical stiffness term results and can be written in the form

$$K = \frac{Eh^3}{12} \left[ K_b + \frac{12G}{E} \left( \frac{1}{h^2} \right) K_s \right]$$

(5.36)

as \( h \) becomes very small the second term of equation (5.36) tends to become very large, and hence the penalty number, so that the discontinuity, in thin shell situations, is minimised.

Now, before generating the transverse shear matrices, let us look at the physical meaning of the Penalty factor \( a_1 \). We have seen that in the case of the present element with linear interpolation for the displacement field and 10 degrees of freedom we have only \( C^1 \) continuity along inter-element boundary circles. To ensure that no gaps will occur between elements, or in other words to satisfy \( C^1 \) continuity for the transverse displacement \( w \).
and indeed for the circumferential displacement \( v \), two additional constraint conditions (along each inter-element boundary) have been introduced (equation (5.10) and (5.11)) by means of the penalty function approach. Therefore, we can regard the penalty factor \( \alpha_1 \) as the stiffness of a rotational spring connecting neighbouring elements along their inter-element boundary circles. The higher its value the smaller is the discontinuities.

On discretization of equation (5.32) and subsequent minimisation, an equation system of the form

\[
[n[K]^e + \alpha_1 (n[K_1]^e + n[K_2]^e)] n[q]^e = n[p]
\]

(5.37)

is obtained, where \( n[K]^e \) is the element stiffness matrix which was derived in equation (5.27), while \( n[K_1]^e \) and \( n[K_2]^e \) are the element stiffness matrices derived from meridional and circumferential transverse shear energy respectively. The thin shell solution is then obtained as \( \alpha_1 \rightarrow \infty \).

Clearly, unless the matrices \( n[K_1]^e \), \( n[K_2]^e \) are singular such a solution will be over-constrained and an unrealistic answer with \( |q| \rightarrow 0 \) will be obtained. We shall find later that single point integration is one way of providing such a singularity of the matrices.
Recalling equations (5.30) and (5.31) we can express the shear strains in terms of the nodal displacements as

$$\varepsilon_{s\xi} = [0, 0, \frac{\partial}{\partial s}, -1, 0] \{q\}^e$$

$$= [L_1] \{N_1\} [N] \{u\}_0^e$$

$$= n\{B_1\} \{u\}_0^e$$  \hspace{1cm} (5.38)$$

$$\varepsilon_{\theta\xi} = [0, \frac{\cos \theta}{r}, \frac{1}{r} \frac{\partial}{\partial \theta}, 0, -1] \{q\}^e$$

$$= [L_2] \{N_1\} [N] \{u\}_0^e$$

$$= n\{B_2\} \{u\}_0^e$$  \hspace{1cm} (5.39)$$

where $n\{B_1\}$ and $n\{B_2\}$ are both $1 \times 10$ matrices.

Substituting equations (5.38) and (5.39) into equations (5.30) and (5.31) respectively, $U_1^e$ and $U_2^e$ can be written in the form

$$U_1^e = \frac{1}{2} \int \sum_{n=0}^{\infty} \int_{m=0}^{\infty} \{u\}_0^e \{T\} \left[ \frac{1}{2} \int \int_{0}^{2\pi} \int_{0}^{n[B_1]^T} D_s m [B_1] \frac{\xi L}{2} d\theta d\xi \right] \{u\}_0^e$$

$$= \frac{1}{2} \int \sum_{n=0}^{\infty} \int_{m=0}^{\infty} \{u\}_0^e \{T\} \left[ \frac{1}{2} \int \int_{0}^{2\pi} \int_{0}^{n[B_2]^T} D_s m [B_2] \frac{\xi L}{2} d\theta d\xi \right] \{u\}_0^e$$  \hspace{1cm} (5.40)$$

Following an argument similar to that leading up to
equation (5.26), equations (5.40) and (5.41) can be written, finally, in the form

\[ U^e_1 = \frac{1}{2} \sum_{n=0}^{\infty} n[u]^e_T n[K_1]^e n[u]^e \]

(5.42)

\[ U^e_2 = \frac{1}{2} \sum_{n=0}^{\infty} n[u]^e_T n[K_2]^e n[u]^e \]

(5.43)

where

\[ n[K_1]^e = \frac{\pi L r}{2} D_s \int_{-1}^{1} [B_1]^T [B_1] \ dn \]

(5.44)

\[ n[K_2]^e = \frac{\pi L r}{2} D_s \int_{-1}^{1} [B_2]^T [B_2] \ dn \]

(5.45)

It should be noted here that for the axisymmetric loading case (i.e., \( n=0 \)), there will be no shear strains in the circumferential direction, i.e., \( n[K_2]^e = 0 \) while \( \pi \) in equation (5.44) should be replaced by \( 2\pi \).

5.4 Potential Energy of Applied Loads - Force Vector

Distributed loads on elements and loads applied along the nodal rings will be considered here (similar process could be applied for body forces, initial strains, etc.).

Let

\[ n[p]^e = \sum_{p} p_i^e \]

\[ n[p]^e_0 = n[p]^e_0 \]
define the applied nodal forces (or those which are equivalent statically to the distributed loads on the element, equivalent in the sense that the work done during any incremental deformation approximates the work done by the actual loading). Each of the forces \( \{P_i\}^e \) must contain the same number of components as the corresponding nodal displacements \( \{u_i\}^e \) and be ordered in the appropriate corresponding directions. For instance, a typical nodal force is

\[
\begin{bmatrix}
n \{P_i\}^e \end{bmatrix} = \begin{bmatrix} n[P_{u_i}, P_{v_i}, P_{w_i}, P_{\beta_i}, P_{\alpha_i}]^T \end{bmatrix}
\]

where

\[
\begin{align*}
P_{u_i} &= \sum_{n=0}^{\infty} P_u \cos n\theta \\
P_{v_i} &= \sum_{n=0}^{\infty} P_v \sin n\theta \\
P_{w_i} &= \sum_{n=0}^{\infty} P_w \cos n\theta \\
P_{\beta_i} &= \sum_{n=0}^{\infty} P_{\beta} \cos n\theta \\
P_{\alpha_i} &= \sum_{n=0}^{\infty} P_{\alpha} \sin n\theta
\end{align*}
\]

(5.46)

with the components \( P_{u_i}, P_{v_i}, P_{w_i}, P_{\beta_i} \) and \( P_{\alpha_i} \) correspond to the directions of \( u, v, w, \beta \) and \( \alpha \) at node \( i \), respectively. Loads applied along the nodal ring and distributed loads can be written, therefore, as
\[ \Sigma_{n=0}^{\infty} n[N_1] n[Pr]^e \]  
\[ \Sigma_{n=0}^{\infty} n[N_1] n[Pr]^e \]  
\[ \Sigma_{n=0}^{\infty} n[N_1] n[Pd]^e \] 

where \( n[Pr]^e \) and \( n[Pd]^e \), which has five components each, are the coefficients of cosine Fourier series.

The potential energy of the applied loads on an element can be written as

\[ \Omega^e = - \int_0^{2\pi} n[q] e^T \left( \sum_{n=0}^{\infty} n[Pr]^e n[N_1] \right) r d\theta \]

\[ - \int_{\text{area}} n[q] e^T \left( \sum_{n=0}^{\infty} n[Pd]^e n[N_1] \right) d \text{(area)} \]  

(5.49)

Now, substituting the first part of equation (5.17) into equation (5.49) to obtain

\[ \Omega^e = - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m[u]^e_0 \left( \int_0^{2\pi} [N]^T_m [N_1]^T n[N_1] r d\theta \right) n[Pr]^e \]

\[ + \left( \int_{-1}^{1} \int_0^{2\pi} [N]^T_m [N_1] n[N_1] \frac{L}{2} r d\theta d\omega \right) n[Pd]^e \]  

(5.50)

integrating equation (5.50) with respect to \( \theta \) and by making use of the integral formulas in equation (5.25)

\[ \Omega^e = - \sum_{n=0}^{\infty} n[u]^e_o \left[ n[Pr]^e_o + n[Pd]^e_o \right] \]

\[ = - \sum_{n=0}^{\infty} n[u]^e_o n[P]^e_o \]  

(5.51)
where

\[ n\{P\}_e^e = n\{P_r\}_e^o + n\{P_d\}_e^o \]

and

\[ n\{P_r\}_o^e = r\pi [N]^T \Sigma n\{P_r\}_e^e \]  \hspace{1cm} (5.52)

\[ n\{P_d\}_o^e = \left( \int_{-1}^{1} [N]^T \frac{L}{\pi} rdn \right) \Sigma n\{P_d\}_e^e \]  \hspace{1cm} (5.53)

In axi-symmetric case \((n=0)\) the problem reduces to only three variables (each node) and \(\pi\) in equations (5.52) and (5.53) should be replaced by \(2\pi\).

In antisymmetric case, the same form as equation (5.46) with \(\sin n\theta\) for \(\cos n\theta\) and \(\cos n\theta\) for \(\sin n\theta\) must be considered. The second part of equation (5.17) must be substituted into equation (5.49) whilst \([N_1]\) in both equations (5.47) and (5.48) should be replaced by \([N_2]\) before they are substituted in equation (5.49). Similar operations follow as before.

5.5  **Kinetic Energy - Derivation of the Element Mass Matrix**

An approximation to the mass matrix, consistent with the approximate element stiffness matrix, may be obtained by using the same displacement function, this time the nodal displacements are replaced by nodal velocities (we
assume a velocity field \( \dot{q}^e \) for the element and express this in terms of the nodal velocities \( \dot{u}^e_0 \).

The kinetic energy of the moving shell surface is calculated by integrating the kinetic energy of translations and rotations over the element - deformed in the assumed manner. We can write

\[
T^e = \frac{1}{2} \int_v \{q\}^e T [\rho] \{\dot{q}\}^e d(\text{vol})
\]

\[
= \frac{1}{2} \int_{-1}^{1} \int_0^{2\pi} \{q\}^e T [\rho] \{\dot{q}\}^e \frac{L}{2} r h d\theta dn \tag{5.54}
\]

Then, if we express \( \{q\}^e \) as

\[
\{q\}^e = \sum_{n=0}^{\infty} [N^1] [N] \{\dot{u}\}^e_0 \]

\[
= \sum_{n=0}^{\infty} [G^1] \{\dot{u}\}^e_0 \tag{5.55}
\]

where \([N^1]\) and \([N]\) have been defined before, and \([\rho]\) the mass density matrix, which is diagonal, is defined as

\[
[\rho] = [\rho, \rho, \rho, \rho h^2/12, \rho h^2/12]
\]

then the kinetic energy expression (5.54) becomes

\[
T^e = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \{\dot{u}\}^e_0 [G^1]^T [\rho] [G^1] \frac{L}{2} rh d\theta dn \{u\}^e_0 \tag{5.56}
\]
carrying out the matrix multiplications, and then integrating over the range $0 < \phi < 2\pi$ (with integral formulas defined in equation (5.25) kept in one mind) we get finally

$$T^e = \frac{1}{2} \sum_{n=0}^{\infty} n ! \{u\}^e_0 [M]^e n \{u\}^e_0$$ \hspace{1cm} (5.57)

where

$$[M]^e = \frac{\pi \cdot L \cdot r \cdot h}{2} \int_{-1}^{1} [N]^T \rho [N] \, d\eta$$ \hspace{1cm} (5.58)

is the consistent element mass matrix.

Unlike $n[K]^e$, $[M]^e$ is independent of $n$ for $n \neq 1$, and its form for $n=0$ is obtained simply by replacing $\pi$ in equation (5.58) by $2\pi$ (when $n=0$ consistent variables should be taken according to that mentioned for $n[K]^e$ before).

Accordingly, the mass matrix is calculated only once; this value serves all of the harmonics.

5.6 **Synthesis of System Matrices**

The generation of element matrices complete the analysis for an individual element. By using a local coordinate system, the element matrices are independent of geometry. The next task is to assemble the element
matrices so that a full solution for the whole structure can be achieved.

Before assembling the element matrices, one problem has yet to be solved: evaluation of the element matrices integrals.

Recalling equation (5.27)

\[
[K]^e = \frac{1}{2} \int_{-1}^{1} [B]^T [D] [B] \, d\eta
\]

we can see that the product \([B][D][B]\) has to be evaluated first, and the terms of the resulting matrix have to be integrated over the range -1 to +1 of direction \(\eta\).

The above integral can be written in the form

\[
[K]^e = \int_{-1}^{1} f(\eta) \, d\eta
\]

this integral is in practice evaluated numerically using the Gaussian rule

\[
\int_{-1}^{1} f(\eta) \, d\eta = \sum_{i} w_i f(\eta_i)
\]

Here the summation is extended over all \(i\) (the specified integration points), \(w_i\) are weighting factors, and \(f(\eta_i)\) are the matrices evaluated at the points specified. The same process is also applied when
evaluating load and mass matrices integrals.

Again, before an assembly of the structure matrices can be attempted, it is necessary to transform the coordinates of the displacement and force components (if the force components are specified in local coordinates rather than in global coordinates) from a local level to a global one. After the assembly of the element matrices is completed the boundary conditions of the structure must be specified before any solution can be achieved.

We shall discuss all these matters in more detail when we write about computer algorithms in the next chapter.

5.7 Vibrations of Orthotropic Shells

The work carried out so far has been based on the assumption that the materials behave in an isotropic manner. This section is concerned with orthotropic shells of revolution.

One example of using orthotropic shell equation is in the representation of a shell which is stiffened by longitudinal beam-like elements (stringers) and/or circumferential rings. Stiffeners are often thought of as giving the shell material orthotropic character and are, as a consequence, often described by way of an effective
stiffness matrix in contrast to the more exact way of thinking of the shell and the stiffeners as individual structural elements with dynamic characters that are joined along lines.

If there exists a regular pattern of relatively closely spaced stiffening elements, their contribution to the wall stiffness of the shell might be modelled by an averaging of their extensional and bending rigidities over length equal to the local spacing between them. This representation can be accurately made for the purpose of determining free vibration frequencies and mode shapes (but not stress resultants [27]). Thus, the actual wall is treated as if it were orthotropic. When the stiffeners are so far apart then they cannot be averaged over the shell surface but the structure must be represented as a combination of shell elements and stiffener elements each having its own equations of motion and coupled to each other by equations of continuity. The standard approach in finite element analysis is to assume that the cross section of the stiffener does not deform but it translates and rotates in a fashion compatible with the shell to which it is attached. Alnajafi and Warburton [75] have investigated the natural frequencies and mode shapes of thin circular cylindrical shells with stiffening rings; each stiffening ring was treated as a discrete element. When the rings and/or stringers can be "smeared out" along the shell to yield a single equivalent orthotropic shell,
the problem will be considered here; while the discrete stiffener element will not be considered in this work.

Another example of using orthotropic shell equations is in composite shell structures where, for example, the structure is combined of multilayers, made of different materials, which makes it possible to adapt the material properties to a desired design rather than the design to the material.

Laminated composite engineering materials are employed in increasing volume and in more and more diverse fields because

(1) they combine the properties of their component parts to obtain composite properties which may be new or unique, or

(2) they make it easier or less costly to obtain certain properties than is possible with single-component materials, and

(3) as the following few sentences [76] explain, the great interest in mechanics of heterogeneous system which arose in the engineering and scientific community during the last half century: "Demands on materials imposed by today's advanced technologies have become so diverse and severe
that they often cannot be met by single-component materials acting alone. It is frequently necessary to combine several materials into a composite to which each constituent not only contributes its share, but whose combined action transcends the sum of the individual properties, and provides new performance unattainable by the constituents acting alone. Space vehicles, heat shields, rocket propellants, deep submergence vessels, buildings, vehicles for water and land transport, aircraft, pressure tanks and many others impose requirements that are best met, and in many instances met only, by composite materials."

The stress-strain equations of an orthotropic shell in terms of the principal coordinates of the middle surface of the shell, are given by

\[ \sigma_\phi = \frac{E_\phi}{1-\nu_\phi \cdot \nu_{\theta \phi}} (\varepsilon_\phi + \nu_\phi \varepsilon_\theta) \]  
\[ \sigma_\theta = \frac{E_\theta}{1-\nu_\phi \cdot \nu_{\theta \phi}} (\varepsilon_\theta + \nu_\theta \varepsilon_\phi) \]  
\[ \tau_{\phi \theta} = G_{\phi \theta} \varepsilon_{\phi \theta} \]  
\[ \tau_{\phi \zeta} = G_{\phi \zeta} \varepsilon_{\phi \zeta} \]  
\[ \tau_{\theta \zeta} = G_{\theta \zeta} \varepsilon_{\theta \zeta} \]
In general the axes of orthotropy are not coincident with the $\phi$ and $\theta$ directions. Such a situation could arise, for example, in the case of a filament-wound shell. In this case, the stresses referred to the natural axes $\phi$ and $\theta$ can be transformed into stresses referred to the axes of orthotropy. The complicating effect of orthotropy stems from the fact that the number of independent elastic constants has risen into six. The seven elastic constants $E_{\phi}^*, E_{\theta}^*, \nu_{\phi\theta}^*, \nu_{\phi\phi}^*, G_{\phi\theta}^*, G_{\phi\phi}^*$ and $G_{\theta\theta}^*$ are not all independent; symmetry consideration require that

$$\nu_{\theta\phi} E_{\phi} = \nu_{\phi\theta} E_{\theta} \quad (5.60)$$

The transformation of the stresses from the natural axes into the axes of orthotropy and the calculation of the elastic constants, for laminated composites, can be done as outlined in reference [77].

Now, one can formulate the stress resultants per unit length of the middle surface; they become

$$N_\phi = \frac{E_{\phi} h}{1 - \nu_{\phi\theta}^* \nu_{\phi\phi}^*} (\varepsilon_{\phi} + \nu_{\theta\phi}^* \varepsilon_{\theta}^*) \quad (5.61a)$$

$$N_\theta = \frac{E_{\theta} h}{1 - \nu_{\phi\theta}^* \nu_{\phi\phi}^*} (\varepsilon_{\theta} + \nu_{\phi\theta}^* \varepsilon_{\phi}^*) \quad (5.61b)$$

$$N_{\phi\theta} = N_{\theta\phi} = G_{\phi\theta}^* h \varepsilon_{\theta\theta}^* \quad (5.61c)$$
\[
M_\theta = \frac{E_\phi \cdot h^3}{12\left(1 - \nu_\phi \cdot \nu_{\theta \phi}\right)} \left(x_\phi + \nu_{\theta \phi} \cdot x_\theta\right) \quad (5.61d)
\]
\[
M_\phi = \frac{E_\theta \cdot h^3}{12\left(1 - \nu_\theta \cdot \nu_{\theta \phi}\right)} \left(x_\theta + \nu_{\phi \theta} \cdot x_\phi\right) \quad (5.61e)
\]
\[
M_{\phi \theta} = M_{\theta \phi} = G_{\phi \theta} \cdot \frac{h^3}{12} \cdot x_{\phi \theta} \quad (5.61f)
\]

The elastic property matrix \([D]\), which was defined for isotropic shells in equation (5.19), can be redefined for orthotropic axisymmetric shells as:

\[
[D] = \begin{bmatrix}
1 & \nu_{\theta \phi} & 0 & 0 & 0 & 0 \\
\nu_{\phi \theta} & \frac{E_\theta}{E_\phi} & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \frac{G_{\phi \theta} \cdot h}{d} & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \frac{h^2}{12} & \frac{\nu_{\phi \theta} \cdot h^2}{d} & 0 \\
\cdot & \cdot & \cdot & \cdot & \frac{E_\theta \cdot h^2}{12} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{G_{\phi \theta} \cdot h^3}{12 \cdot d}
\end{bmatrix}
\]

\[\text{(5.62)}\]

where \[d = \frac{E_\phi \cdot h}{1 - \nu_\phi \cdot \nu_{\theta \phi}}\]

The rest of the formulations are carried out in a similar manner discussed so far in this chapter. One point should be mentioned here, that is the mass density of composite materials is taken as the average of the mass densities of the constituents.
Carrying out the matrix multiplications one now can write the element stiffness matrices for orthotropic axisymmetric shells explicitly as follows: (for convenience, and to be consistent with the computer code, the following simplification will be made

\[ E_\phi = E_1, \quad E_\theta = E_2, \quad \nu_{\phi \theta} = \nu_1, \quad \nu_{\theta \phi} = \nu_2, \quad d = \frac{E_1 h}{1 - \nu_1 \nu_2} \]

\[ F_1 = \pi r L, \quad D_2 = \frac{1 - \nu_1 \nu_2}{2(1 + \sqrt{\nu_1 \nu_2})}, \quad E = \frac{E_2}{E_1} \]

\[ n[K]^e = F_1 d n[K]^e \quad (5.63) \]

where

\[ K_{1,1} = \frac{1}{L^2} (n_i \times n_i) + \frac{\nu_2}{2 L r} \sin \phi (n_i \times n_i) + \frac{1}{4r^2} (E \sin^2 \phi + D_2 n^2) \]

\[ K_{1,2} = n[(D_2 + E) \cdot \frac{\sin \phi}{4r^2} + \frac{\nu_2}{2 L r} \eta_i - \frac{D_2}{2 L r} \eta_i] \]

\[ K_{1,3} = \frac{(E \cdot \sin \phi)}{4r^2} + \frac{\nu_2}{2 L r} \eta_i \cos \phi \]

\[ K_{1,4} = K_{1,5} = 0 \]

\[ K_{1,6} = \frac{1}{L^2} (n_i \times n_j) + \frac{\nu_2}{2 L r} \sin \phi (n_i \times n_j) \]
\[ K_{1,7} = n[(D_2+E) \cdot \frac{\sin \phi}{4r^2} + \frac{v_2}{2Lr} \cdot n_i - \frac{D_2}{2Lr} \cdot n_j] \]

\[ K_{1,8} = K_{1,3} \]

\[ K_{1,9} = K_{1,10} = 0 \]

\[ K_{2,2} = \frac{1}{4r^2} \left( E \cdot n^2 + D_2 \cdot \sin^2 \phi \right) \]

\[ + \frac{D_2}{L^2} (n_i \times n_i) - \frac{D_2}{2Lr} (n_i + n_i) \]

\[ K_{2,3} = \frac{E \cdot n \cdot \cos \phi}{4r^2} \]

\[ K_{2,4} = K_{2,5} = 0 \]

\[ K_{2,6} = n[(D_2+E) \cdot \frac{\sin \phi}{4r^2} + \frac{v_2}{2Lr} \cdot n_j - \frac{D_2}{2Lr} \cdot n_i] \]

\[ K_{2,7} = \frac{1}{4r^2} \left( E \cdot n^2 + D_2 \cdot \sin^2 \phi \right) \]

\[ + \frac{D_2}{L^2} (n_i \times n_j) - \frac{D_2}{2Lr} (n_i + n_j) \]

\[ K_{2,8} = K_{2,3} \]

\[ K_{2,9} = K_{2,10} = 0 \]

\[ K_{3,3} = \frac{E \cdot \cos^2 \phi}{4r^2} \]
\[ K_{3,4} = K_{3,5} = 0 \]

\[ K_{3,6} = \left( \frac{E \cdot \sin \phi}{4r^2} + \frac{\nu_2}{2Lr} \eta_j \right) \cos \phi \]

\[ K_{3,7} = K_{2,3} \]

\[ K_{3,8} = K_{3,3} \]

\[ K_{3,9} = K_{3,10} = 0 \]

\[ K_{4,4} = \frac{h^2}{12} \left[ \frac{E \cdot \sin \phi}{r^2} + \frac{1}{L^2} (\eta_i \times \eta_i) \right. \]

\[ \left. + \frac{\nu_2 \cdot \sin \phi}{2Lr} (\eta_i + \eta_i) \right] \]

\[ K_{4,5} = \frac{h^2}{12} \left( \frac{n \cdot E \cdot \sin \phi}{r^2} + \frac{n \cdot \nu_2}{2Lr} \eta_i \right) \]

\[ K_{4,6} = K_{4,7} = K_{4,8} = 0 \]

\[ K_{4,9} = \frac{h^2}{12} \left[ \frac{E \cdot \sin \phi}{4r^2} + \frac{1}{L^2} (\eta_i \times \eta_j) \right. \]

\[ \left. + \frac{\nu_2 \cdot \sin \phi}{2Lr} (\eta_i + \eta_j) \right] \]

\[ K_{4,10} = K_{4,5} \]

\[ K_{5,5} = \frac{h^2}{12} \left[ \frac{E \cdot n^2}{4r^2} + D_2 \left( \frac{\sin \phi}{r^2} - \frac{4 \cdot \sin \phi}{Lr} \cdot \eta_i \right) \right. \]

\[ \left. + \frac{4}{L^2} (\eta_i \times \eta_i) \right] \]
\[ \begin{align*}
K_{5,6} &= K_{5,7} = K_{5,8} = 0 \\
K_{5,9} &= \frac{h^2}{12} \left( \frac{n.E.\sin \phi}{4r^2} + \frac{n.v_2}{2Lr} \cdot n_j \right) \\
K_{5,10} &= \frac{h^2}{12} \left\{ \frac{E.n^2}{4r^2} + D_2 \left[ \frac{\sin^2 \phi}{r^2} + \frac{2 \sin \phi}{Lr} (n_i \cdot n_j) \right] + \frac{4}{L^2} (n_i \times n_j) \right\} \\
K_{6,6} &= \frac{1}{L^2} (n_j \times n_j) + \frac{v_2 \sin \phi}{2Lr} (n_j + n_j) \\
&+ \frac{1}{4r^2} (E.\sin^2 \phi + D_2.n^2) \\
K_{6,7} &= n[(D_2 + E) \frac{\sin \phi}{4r^2} + \frac{v_2}{2Lr} n_j - \frac{D_2}{2Lr} n_j] \\
K_{6,8} &= K_{6,6} \\
K_{6,9} &= K_{6,10} = 0 \\
K_{7,7} &= \frac{1}{4r^2} (E.n^2 + D_2 \sin^2 \phi) + \frac{D_2}{L^2} (n_j \times n_j) \\
&- \frac{D_2 \sin \phi}{2Lr} (n_j + n_j) \\
K_{7,8} &= K_{7,3} \\
K_{7,9} &= K_{7,10} = 0 \\
K_{8,8} &= K_{3,3}
\end{align*} \]
\[ K_{9,9} = \frac{h^2}{12} \left[ \frac{E \sin^2 \phi}{4r^2} + \frac{1}{L^2} (\eta_j \times \eta_j) \right] \]
\[ + \frac{v_2}{2Lr} (\eta_j \times \eta_j) \]
\[ K_{9,10} = \frac{h^2}{12} \left( \frac{N_{E \sin \phi}}{4r^2} + \frac{n \cdot v_2}{2Lr} \cdot \eta_j \right) \]
\[ K_{10,10} = \frac{h^2}{12} \left[ \frac{En^2}{4r^2} + D_2 \left( \frac{\sin^2 \phi}{r^2} - \frac{4 \sin \phi}{Lr} \right) \eta_j \right] \]
\[ + \frac{4}{L^2} (\eta_j \times \eta_j) \]

The element stiffness matrices \( n[K_1]^e \) and \( n[K_2]^e \) which were called the penalty function matrices will be defined in the computer code as

\[ n[K_1]^e = n[K_b]^e, \quad n[K_2]^e = n[K_a]^e \]

where

\[ n[K_a]^e = \kappa D_s \pi r L n[K_a]^e \quad (5.64) \]
\[ n[K_b]^e = \kappa D_s \pi r L n[K_b]^e \quad (5.65) \]

where the non-zero elements of \( n[K_a]^e \) and \( n[K_b]^e \) are

\[ K_{a2,2} = K_{a2,7} = K_{a7,7} = \frac{\cos^2 \phi}{4r^2} \]
\[ K_{a2,3} = K_{a2,8} = K_{a7,8} = \frac{n \cdot \cos \phi}{4r^2} \]
\[ K_{a2,5} = K_{a2,10} = K_{a7,10} = \frac{\cos \phi}{4r} \]
\[ K_{a3,3} = K_{a3,8} = K_{a8,8} = \frac{n^2}{4r^2} \]
\[ K_{a3,5} = K_{a3,10} = K_{a8,10} = \frac{n}{4r} \]
\[ K_{a5,5} = K_{a10,10} = \frac{1}{4} \]
\[ K_{b3,3} = \frac{1}{L^2} (n_i \times n_i), \quad K_{b8,8} = \frac{1}{L^2} (n_j \times n_j) \]
\[ K_{b3,4} = K_{b3,9} = K_{b4,9} = K_{b8,9} = - \frac{1}{2L} n_i \]
\[ K_{b3,8} = \frac{1}{L^2} (n_i \times n_j) \]
\[ K_{b4,4} = K_{b4,9} = K_{b9,9} = \frac{1}{4} \]

with symmetry observed in \( [K]^e \), \( [K_a]^e \) and \( [K_b]^e \).

For the element mass matrix, the explicit form can be written as

\[ [M]^e = \pi \cdot r \cdot L \cdot h \cdot \rho_a \ [M_o]^e \quad (5.66) \]

where the non-zero elements are:

\[ M_{o1,1} = M_{o2,2} = M_{o3,3} = N_i^2 \]
\[ M_{01,6} = M_{02,7} = M_{03,8} = N_i N_j \]

\[ M_{04,4} = M_{05,5} = N_i^2 \cdot \frac{h^2}{12} \]

\[ M_{04,9} = M_{05,10} = N_i N_j \frac{h^2}{12} \]

\[ M_{06,6} = M_{07,7} = M_{08,8} = N_j^2 \]

\[ M_{09,9} = M_{10,10} = N_j^2 \cdot \frac{h^2}{12} \]
CHAPTER SIX

Computer Implementation

Introduction

The development of an effective computer program for asymmetric shell analysis requires a knowledge of three scientific disciplines: structural mechanics, numerical analysis, and computer science. A most important aspect of a computer program is, however, the ease with which it can be modified and extended.

In the preceding chapter, the characteristic equations of the simple truncated cone element were described and established, together with the overall system matrices required to solve both the static and dynamic problems. Gaussian quadrature was adopted because of its accuracy and economical saving. More economical saving is achieved by adopting a single point numerical integration.

In this chapter, we describe the implementation of the information presented in Chapter 5 in the form of a structural computer program. The top-down approach is applied both in the documentation and structure of the program. By this it is implied that the problem will be attacked from a global to a particular point of view. The